Overall properties of a discrete membrane with randomly distributed defects

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1 Introduction

In this paper we study the problem of the description of the overall behaviour of a discrete membrane with defects. Our model mixes two extreme behaviours and is set in a variational framework. One extreme bahaviour is that of a 'strong membrane', consisting just in the finite-difference approximation of the Dirichlet integral. The defects are introduced as nonlinear pair interactions as in Blake and Zisserman's *weak membrane* model in Computer Vision [3]. This model can be translated into a simple non-linear finite-difference scheme (see Chambolle [10]) on a two-dimensional (in general, d-dimensional) grid. Its underlying discrete energy is of the form

$$E_\varepsilon(u) = \frac{1}{2} \sum_{z_1, z_2 \in \mathbb{Z}^2 \cap \frac{1}{\varepsilon} Q, |z_1 - z_2| = 1} \min\{(u(z_1) - u(z_2))^2, \varepsilon\},$$

where ε is the mesh size and u represents the vertical displacement in the case of a membrane (or the grey level of an output picture in the applications to Computer Vision). The energy density can be seen as a *truncated quadratic potential.* In terms of the finite difference $u(z_1) - u(z_2)/\varepsilon$ it can be rewritten as

$$\min\{(u(z_1) - u(z_2))^2, \varepsilon\} = \varepsilon^2 \min\Big\{\Big(\frac{u(z_1) - u(z_2)}{\varepsilon}\Big)^2, \frac{1}{\varepsilon}\Big\}.$$

Its interpretation is that the 'springs' of the discrete weak membrane behave linearly until the gradient (difference quotient) reaches the threshold $c_\varepsilon = 1/\sqrt{\varepsilon}$; after this threshold is reached, the spring is broken and the energy of the spring remains constant.

The overall behaviour of such a system can be expressed by an asymptotic study of this energy as $\varepsilon \to 0$. Note that in the case of a (strong) discrete membrane the underlying energy is simply

$$F_\varepsilon(u) = \frac{1}{2} \sum_{z_1, z_2 \in \mathbb{Z}^2 \cap \frac{1}{\varepsilon} Q, |z_1 - z_2| = 1} (u(z_1) - u(z_2))^2.$$

This is a convex energy and it is just a finite-difference approximation of the Dirichlet integral

$$F(u) = \int_Q |\nabla u|^2 \, dx. \tag{1.1}$$

In the case of the weak membrane we are not in a convex setting; in particular energies of the form E_ε cannot be seen as finite-approximations of integral energies defined on $H^1(Q)$. An asymptotic study of E_ε has been performed by Chambolle [11] using the language of Γ-convergence (see [5, 13]) and the techniques of Ambrosio and De Giorgi's SBV-spaces (see [14, 2]). In loose terms, the space where the limit energy is finite consists of functions u that are H^1 outside their set of discontinuities, which we denote by $S(u)$, and we have sufficient regularity to define a normal ν to $S(u)$. The Γ-limit of E_ε can be written in this space as

$$E(u) = \int_Q |\nabla u|^2\,dx + \int_{S(u)} |\nu|_1\,d\mathcal{H}^1, \tag{1.2}$$

where $|\nu|_1 = |(\nu_1, \nu_2)|_1 = |\nu_1|+|\nu_2|$ and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. The anisotropy in the 'jump part' of the energy is clearly due to the anisotropy of the lattice structures. This energy highlights that the weak membrane may indeed undergo fracture in a fashion similar to Griffith brittle fracture [16], but the fracture energy is anisotropic since cracks follow the microscopical pattern of the lattice. Note that E is an anisotropic version of the Mumford and Shah functional of Computer Vision [20] to whom the weak membrane model is deeply connected.

In our setting we consider a random mixture of the two types of springs (the precise definition through the introduction of i.i.d. random variables being found in Section 2), the strong springs with probability p and the weak springs with probability $1-p$, the case $p = 1$ corresponding to the discretization of the Dirichlet integral and the case $p = 0$ corresponding to the weak membrane. We show the appearance of two regimes: if p is larger than a *percolation threshold* p_c then the effect of the weak springs is negligible; more precisely, we obtain that the Γ-limit is almost surely the Dirichlet integral. Conversely, if $p < p_c$ then the limit functional is defined on an SBV space, with a surface energy density depending on p. In this case, the limit energy takes the form

$$E_p(u) = \int_Q |\nabla u|^2\,dx + \int_{S(u)} \lambda_p(\nu)\,d\mathcal{H}^1.$$

The result is deeply linked to percolation techniques. The proofs use arguments of Γ-convergence and geometric measure theory together with geometrical properties of percolation clusters. In the super-critical case we combine the compactness results in SBV and the rectifiability properties of the jump set $S(u)$ with the existence of many paths of strong connections to obtain that actually $S(u)$ is negligible. Conversely, in the sub-critical case we use the properties of the 'cluster of weak connections' to define the functions λ_p and to construct test functions that provide the upper bound. The value $\lambda_p(\nu)$ is defined through the asymptotic behaviour of the *chemical distance* (i.e., the distance on the weak cluster) between pair of points aligned with ν. A crucial tool in the proof of the lower bound is the technical Lemma 2.4 that ensures that paths whose length is strictly less than the chemical distance contain a 'substantial proportion' of strong connections. This result has been kindly provided to us by H. Kesten, and its proof is contained in the Appendix.

It must be remarked that the corresponding problem in the case of *deterministic* homogenization, where we prescribe the microscopical periodic arrangement of the springs, can be performed by following the localization methods of Γ-convergence. In this case, the knowledge of averaged quantities such as the percentage of strong and weak springs only does not allow any accurate description of the limit energy; in particular, under proper choices of the local geometry, we can obtain both energies (1.1) and (1.2). In fact, we may consider the 'extreme' periodic geometries of period N where all weak connections are placed on the boundary of the periodicity cell. If the fraction of weak connections exceeds $1/N$, then strong connections are completely isolated, and the arguments used by Chambolle [11] can be repeated to show that the limit is again the weak membrane energy. Conversely, if the positions of the weak and strong connections are reversed, then the strong connections constitute a connected frame that guarantees that the limit is defined on $H^1(Q)$, and hence, by a comparison argument it is the Dirichlet integral. Again note that we only need a fraction of $1/N$ strong connections to construct such an example (in dimension d actually only N^{1-d}). These considerations show that with fixed proportions of weak and strong connections different from 0 and 1 both strong and weak membrane models (and hence also the 'intermediate' models) can be obtained; in particular an analysis giving optimal bounds for the limit energies as that carried by Braides and Francfort [8] for the homogenization of conducting networks cannot provide additional information.

Besides the many interesting variants of the problem, some fundamental issues remain to be explored, as the asymptotical behaviour of λ_p as p approaches increasingly the percolation threshold: whether $\lambda_p \to +\infty$ and whether λ_p becomes 'isotropic' close to the critical level.

2 Notation and setup

We will deal with limits of discrete models with randomly distributed nearest-neighbour interactions giving rise to free-discontinuity energies. In this section we recall the necessary background of percolation theory, Γ-convergence and the theory of special functions of bounded variation.

$\mathcal{L}^d$ denotes the n-dimensional Lebesgue measure and $\mathcal{H}^k$ the k-dimensional Hausdorff measure. We also use the notation $|A| = \mathcal{L}^d(A)$. $B_\rho(x)$ is the open ball of centre x and radius ρ.

2.1 Special functions of bounded variation

For the general theory of functions of bounded variation we refer to [2]; here we just recall some definitions and results we shall use in the sequel. Let Q be an open subset of $\mathbb{R}^d$. We say that $u \in L^1(Q)$ is a *function of bounded variation* if its distributional first derivatives $D_i u$ are (Radon) measures with finite total variation in Q. This space will be denoted by $BV(Q)$. We use Du to indicate the vector-valued measure whose components are $D_i u$.

Let $u : Q \to \mathbb{R}$ be a Borel function. We say that $z \in \mathbb{R}$ is the *approximate limit* of u in x if for every $\varepsilon > 0$

$$\lim_{\rho\to 0^+} \rho^{-d} |\{y \in B_\rho(x) \cap Q : |u(y) - z| > \varepsilon\}| = 0\,.$$

We define the *jump set* $S(u)$ *of function* u as the subset of Q where the approximate limit of u does not exist. It turns out that $S(u)$ is a Borel set and $|S(u)| = 0$. If $u \in BV(Q)$, then $S(u)$ is *countably* $(n-1)$*-rectifiable*, i.e. $S(u) = N \cup \big(\bigcup_{i\in\mathbb{N}} K_i\big)$, where $\mathcal{H}^{d-1}(N) = 0$ and (K_i) is a sequence of compact sets, each contained in a C^1 hypersurface Γ_i. A *normal unit vector* ν_u to $S(u)$ exists $\mathcal{H}^{d-1}$-a.e. on $S(u)$, in the sense that, if $S(u)$ is represented as above then $\nu_u(x)$ is normal to Γ_i for $\mathcal{H}^{d-1}$-a.e. $x \in K_i$. Moreover, $\nu_u : S(u) \to S^{d-1}$ is a Borel function.

We say that a function $u \in BV(Q)$ is a *special function of bounded variation* if the singular part of Du is concentrated on $S(u)$; i.e., there exist $\phi \in (L^1(Q))^d$ and $\psi \in (L^1(Q, \mathcal{H}^{d-1} \llcorner S(u))^d$ such that $Du = \phi\,\mathcal{L}^d + \psi\,\mathcal{H}^{d-1} \llcorner S(u)$. We denote the space of the special functions of bounded variation by SBV(Q). Note that if $\psi = 0$ then $u \in W^{1,1}(Q)$. We denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure (the function ϕ above). ∇u turns out to be the *approximate differential of* u, in the sense that

$$\lim_{\rho\to 0^+} \rho^{-d} \int_{B_\rho(x)\cap Q} \frac{|u(y) - u(x) - \nabla u(x)\cdot(y-x)|}{|y-x|}\,dy = 0$$

for a.e. $x \in Q$.

A function $u : Q \to [-\infty, +\infty]$ is a a *generalized special function of bounded variation* if its truncations are in SBV(Q); i.e., for every $T > 0$ we have $u_T := -T \wedge (u \vee T) \in$ SBV(Q). This space is denoted by GSBV(Q). If $u \in$ GSBV(Q) and $|\{|u| = +\infty\}| = 0$ then $\nabla u = \lim_{T\to+\infty} \nabla u_T$ is defined a.e. and we set $S(u) = \bigcup_{T>0} S(u_T)$. Energies of the form

$$\int_Q f(\nabla u)\,dx + \int_{S(u)} g(\nu_u)\,d\mathcal{H}^{d-1}$$

with f, g non-negative Borel functions are then well defined on such functions.

2.2 Γ-convergence

We recall the definition of Γ-*convergence* of a sequence of functionals F_j defined on GSBV(Q): we say that (F_j) Γ-converges to F (on GSBV(Q) with respect to the convergence in measure) if for all $u \in$ GSBV(Q)

(i) (*lower bound*) for all sequences (u_j) converging to u in measure we have

$$F(u) \leq \liminf_j F_j(u_j);$$

(ii) (*upper bound*) there exists a sequence (u_j) converging to u in measure such that

$$F(u) \geq \limsup_j F_j(u_j)\,.$$

If (i) and (ii) hold then we write $F(u)=\Gamma\text{-}\lim_j F_j(u)$. We define the Γ-*lower limit* as

$$\Gamma\text{-}\liminf_j F_j(u)= \inf\{\liminf_j F_j(u_j) : u_j \to u\}$$

and the and Γ-*upper limit* as

$$\Gamma\text{-}\limsup_j F_j(u)= \inf\{\limsup_j F_j(u_j) : u_j \to u\},$$

respectively. Then (i) also reads as $F(u) \leq \Gamma\text{-}\liminf_j F_j(u)$ and (ii) as $F(u) \geq \Gamma\text{-}\limsup_j F_j(u)$.

We will say that a family (F_ε) Γ-converges to F if for all sequences (ε_j) of positive numbers converging to 0 the conditions (i) and (ii) above are satisfied with F_{ε_j} in place of F_j. The notation is modified accordingly. For an introduction to Γ-convergence we refer to [5, 13].

2.3 Discrete-to-continuous limits

The application of Γ-convergence to describe continuum limits of discrete systems has been used in different frameworks in various degrees of generality (see e.g. [5, 10, 11, 7, 9, 1]). Here we will deal with a simple situation of nearest-neighbour energies in $\mathbb{R}^2$.

With fixed $\varepsilon > 0$ we consider energies defined on functions parameterized on the lattice $\varepsilon\mathbb{Z}^2 \cap Q$, or equivalently, upon scaling, on $\mathbb{Z}^2 \cap \frac{1}{\varepsilon}Q$, of the form

$$E_\varepsilon(u) = \frac{1}{2} \sum_{z_1,z_2 \in \mathbb{Z}^2 \cap \frac{1}{\varepsilon}Q, |z_1 - z_2| = 1} \varepsilon^2 f_\varepsilon\Big(z_1, z_2, \frac{u(\varepsilon z_1) - u(\varepsilon z_2)}{\varepsilon}\Big). \tag{2.1}$$

with $u : \varepsilon\mathbb{Z}^2 \cap Q \to \mathbb{R}$. We identify each such $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}$ (extended to 0 outside $\varepsilon\mathbb{Z}^2 \cap Q$) with the piecewise-constant extension $u(x) = u(\varepsilon z)$ if $x \in \varepsilon z + [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^2$, $z \in \mathbb{Z}^2$.

We will say that E_ε Γ-*converge* to F if they Γ-converge as functionals defined on GSBV(Q) with the identification above (we set $E_\varepsilon(u) = +\infty$ if u is not a piecewise-constant function as above).

Example 2.1. 1) (*quadratic potentials*) In the trivial quadratic case $f_\varepsilon(i, j, z) = z^2$ then E_ε Γ-converge to the Dirichlet integral

$$F(u) = \int_Q |\nabla u|^2\, dx$$

on $H^1(Q)$ and $F(u) = +\infty$ if $u \in \text{GSBV}(Q) \setminus H^1(Q)$.

2) (*truncated quadratic potentials* – see [10, 11, 9]) If $f_\varepsilon(i, j, z) = \min\{z^2, \frac{1}{\varepsilon}\}$ then E_ε Γ-converge to the weak membrane energy

$$F(u) = \int_Q |\nabla u|^2\, dx + \int_{S(u)} |\nu|_1 d\mathcal{H}^1,$$

where $|\nu|_1 = |\nu_1| + |\nu_2|$.

3) In both cases the sequence is equi-coercive, in the sense that from a sequence of functions bounded in measure with $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty$ we may extract a subsequence converging to $u \in$ GSBV(Q).

2.4 Percolation models

The object of this paper is the asymptotic description of some models randomly mixing the two energy functionals in Example 2.1. Among the many ways to set up this problem in a stochastic framework, we will focus on the two percolation models described below. In both cases, a random choice is made, whether a bond (i.e., the link between two neighbouring points in the lattice) is 'strong' (i.e., the energy density between the two points is a quadratic potential) or 'weak' (i.e., it is a truncated quadratic potential as above).

2.4.1 Bond percolation model

The first type of percolation model considered in this work consists in simply assigning the label 'weak' or 'strong' to a bond with probability p and $1-p$, respectively, the choice being independent on distinct bonds. More precisely, this model is introduced as follows. Denote by $\hat{\mathbb{Z}^2}$ the set of middle points of the segments $[z_1, z_2]$, z_1, $z_2 \in \mathbb{Z}^2$, $|z_1 - z_2| = 1$, of the standard integer grid $\mathbb{Z}^2$. Notice that $\hat{\mathbb{Z}^2}$ forms the *dual grid* of $\mathbb{Z}^2$:

$$\hat{\mathbb{Z}^2} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2. \tag{2.2}$$

The notation $z_1(\hat{z})$, $z_2(\hat{z})$ is used for the endpoints of the segment containing $\hat{z}$. We may identify each point in $\hat{z} \in \hat{\mathbb{Z}^2}$ with the corresponding closed segment $[z_1(\hat{z}), z_2(\hat{z})]$, so that points in $\hat{\mathbb{Z}^2}$ are identified with bonds in $\mathbb{Z}^2$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\{\xi_{\hat{z}}\,,\ \hat{z} \in \hat{\mathbb{Z}^2}\}$ be a family of independent identically distributed (i.i.d.) random variables such that

$$\xi_{\hat{z}} = \begin{cases} \text{'weak'} & \text{with probability } 1-p, \\ \text{'strong'} & \text{with probability } p \end{cases} \tag{2.3}$$

In this way we associate to each bond of $\mathbb{Z}^2$ one of the labels 'weak' or 'strong'. For brevity in what follows we identify 'weak' and 'strong' with the values 0 and 1, respectively

Given a bounded Lipschitz domain $Q \subset \mathbb{R}^2$, we will investigate the asymptotic behaviour as $\varepsilon \to 0^+$ of the functionals

$$F_\varepsilon^{b,\omega}(u) = \sum_{\hat{z} \in \frac{1}{\varepsilon} Q \cap \hat{\mathbb{Z}^2}} \varepsilon \phi_{\hat{z}}^{\omega} \Big(\frac{(u(\varepsilon z_1(\hat{z})) - u(\varepsilon z_2(\hat{z})))^2}{\varepsilon} \Big),$$

where u is a function defined on $\varepsilon\mathbb{Z}^2 \cap Q$, and

$$\phi_{\hat{z}}^{\omega}(s) = \begin{cases} s, & \text{if } \ \xi_{\hat{z}}(\omega) = 1, \\ \min(s,1), & \text{otherwise.} \end{cases}$$

We will study the possible Γ-limits of $F_\varepsilon^{b,\omega}$, as $\varepsilon \to 0^+$, for different values of p in (2.3). Note that this functional is of the form (2.1) with $f_\varepsilon(z_1, z_2, w) = \frac{1}{\varepsilon} \phi^{\omega}_{\frac{z_1+z_2}{2}}(\varepsilon w^2)$.

2.4.2 Site percolation model

Another possible way of assigning a label 'weak' or 'strong' to bonds, is by instead randomly labelling the points of the lattice, and then assigning the label 'strong' to a bond only if both its endpoints are labelled as 'strong'.

A standard *site percolation model* in $\mathbb{Z}^2$ is formed by a collection of i.i.d. random variables ξ_z, $z \in \mathbb{Z}^2$, such that

$$\xi_z = \begin{cases} 0 \text{ ('weak')} & \text{with probability } 1-p, \\ 1 \text{ ('strong')} & \text{with probability } p. \end{cases} \tag{2.4}$$

Then we define a random function

$$\phi^{\omega}_{z_1,z_2}(s) = \begin{cases} 0 & \text{if } |z_1 - z_2| \neq 1, \\ s & \text{if } |z_1 - z_2| = 1 \text{ and } \xi_{z_1}(\omega) = \xi_{z_2}(\omega) = 1, \\ \min(s,1) & \text{otherwise.} \end{cases}$$

Given a bounded Lipschitz domain $Q \subset \mathbb{R}^2$, we introduce the functional

$$F^{s,\omega}_{\varepsilon}(u) = \sum_{z_1,\, z_2 \in \frac{1}{\varepsilon}Q \cap \mathbb{Z}^2} \varepsilon \phi^{\omega}_{z_1,z_2}\Big(\frac{(u(\varepsilon z_1) - u(\varepsilon z_2))^2}{\varepsilon}\Big),$$

where u is a function defined on $\varepsilon\mathbb{Z}^2 \cap Q$, and ε is a small positive parameter. We will study the possible Γ-limits of $F^{s,\omega}_{\varepsilon}$, as $\varepsilon \to 0^+$, for different values of p.

Note that the functional $F^{s,\omega}_{\varepsilon}$ is of the form (2.1) with $f_\varepsilon(z_1, z_2, w) = \frac{1}{\varepsilon}\phi^{\omega}_{z_1,z_2}(\varepsilon w^2)$. Note also that in the notation of (2.1) the points z_i with $|z_1 - z_2| \neq 1$ are not considered, so that we may regard the function ϕ_{z_1,z_2} as having only two possible forms.

2.5 Some results from percolation theory

In this section we recall a number of percolation theory results which are formulated in the form adapted for our needs. We refer to the book by Kesten [18] for their proof if not stated otherwise.

We first consider the bond percolation model. We introduce a terminology for *strong bonds*; i.e., those points $\hat{z} \in \hat{\mathbb{Z}}^2$ such that $\xi_{\hat{z}} = 1$. Keeping in mind the identification of $\hat{z}$ with $[z_1(\hat{z}), z_2(\hat{z})]$ we say that two strong points $\hat{z}$ and $\hat{z}'$ are *adjacent* if the corresponding two segments have an endpoint in common. A sequence of strong bonds $\hat{z}_1, \ldots, \hat{z}_k$ is said to be a *strong path* if any two consecutive points of this sequence are adjacent. In what follows we identify a strong path with the subset of $\mathbb{R}^2$ composed of the union of the corresponding segments. A subset A of $\hat{\mathbb{Z}}^2$ of strong points is said to be *connected* if for every two points $\hat{z}', \hat{z}''$ of A there exists a strong path as above such that $\hat{z}_j \in A$, $\hat{z}_1 = \hat{z}'$, $\hat{z}_k = \hat{z}''$. A maximum connected component of adjacent strong points is called a *strong cluster*.

Theorem 2.2 (percolation threshold)**.** *For any $p < p_c := 1/2$ all the strong clusters are almost surely (a.s.) finite, while for any $p > 1/2$ with probability one there is exactly one infinite strong cluster.*

The notation S^δ stands for a square of size δ, centered at the origin, whose sides are not necessary parallel to the coordinate axis. A path joining two opposite sides of a square (or, more generally, of a bar) will be called a *channel*. Our analysis relies essentially on the following statement.

Theorem 2.3 (channel property)**.** *Assume that $p > 1/2$* (supercritical mode). *Then there exist constants $c(p) > 0$ and $c_1(p) > 0$ such that a.s. for any δ, $0 < \delta \leq 1$ there is a large enough number $N_0 = N_0(\omega, \delta)$ such that for all $N > N_0$ and for any S^δ and $x_0 \in [0,1]^2$ the square $N(S^\delta + x_0)$ contains at least $c(p)\delta N$ disjoint strong channels which connect opposite sides of the square. Moreover, the length of each such a channel does not exceed $c_1(p)\delta N$.*

In the subcritical regime we need to introduce some terminology also for *weak bonds*; i.e., those points $\hat{z} \in \hat{\mathbb{Z}}^2$ such that $\xi_{\hat{z}} = 0$. In that case, we consider the shifted lattice $Z_b = \mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$ and notice that the set of middle points of its bonds coincides with $\hat{\mathbb{Z}}^2$ defined in (2.2). Thus, to each points $\hat{z} \in \hat{\mathbb{Z}}^2$ we can associate the corresponding bond in Z_b. If $\hat{z}$ is identified with the corresponding segment with endpoints in Z_b, then we may define the notion of *adjacent points* as for strong bonds. The notion of a *weak channel* and a *weak cluster* is modified accordingly.

For $p < 1/2$ there is a.s. a unique infinite weak cluster and the channel property stated above holds for the weak channels as well. Moreover, if we denote by T^ρ_ν the bar

$$T^\rho_\nu = \{x \in \mathbb{R}^2 \,:\, |\langle x, \nu\rangle| \leq \rho,\, 0 \leq \langle x, \nu^\perp\rangle \leq 1\},$$

where $0 < \rho < 1$, $\nu = (\nu_1, \nu_2) \in R^2$ is a unit vector and $\nu^\perp = (-\nu_2, \nu_1)$, then a.s. for sufficiently large N, in the bar NT^ρ_ν there is a weak channel that links the smaller sides of the bar. Denote by $L_N = L_N(\nu, \rho, p, \omega)$ the length of the shortest such channel ($L_N = +\infty$ if there is no such a channel), and

$$\lambda^b(\nu, p) = \operatorname*{esssup}_{\omega\in\Omega} \limsup_{\rho\to 0^+} \limsup_{N\to\infty} \frac{L_N(\nu, \rho, p, \omega)}{N} \tag{2.5}$$

According to [21, 15] for $p < 1/2$ we have $\lambda^b(\nu, p) \le c_2(p)$ with some constant $c_2(p)$. Conversely, it is easily seen that $\lambda^b(\nu, p) \ge \lambda^b(\nu, 1) = |\nu|_1$ for all p.

The following lemma (see the Appendix for a proof) expresses the fact that paths joining opposite sides of a bar with length less than the minimal one corresponding to that in the definition of λ^b, contain a substantial percentage of strong bonds.

Lemma 2.4. *Let $\eta > 0$ be fixed. Then there are $\rho \in (0, 1)$ and $\delta > 0$ such that a.s. there exists N_0 such that for all $N \ge N_0$ and all channels $\{\hat z_i\}$ of length L connecting the two shorter sides of NT^ρ_ν and with $L < (\lambda^b(\nu, p) - \eta)N$ we have $\#(\{i : \xi_{\hat z_i} = 1\}) \ge \delta(\eta)N$.*

We now consider site percolation. We introduce a terminology for *strong points*; i.e., those points $z \in \mathbb{Z}^2$ with $\xi_z = 1$. Two strong points z_1, $z_2 \in \mathbb{Z}^2$ are called *adjacent* if $|z_1 - z_2| = 1$. In this framework the corresponding notions of path, connectedness and cluster are defined as for bond percolation. The analogue of Theorem 2.2 reads as follows.

Theorem 2.5 (percolation threshold). *There is a critical value $p_c \approx 0.59$ such that for any $p < p_c$ all the strong clusters are almost surely (a.s.) finite, while for any $p > p_c$ with probability one there is exactly one infinite strong cluster.*

The statement of Theorem 2.3 remains valid as well, upon replacing the value 1/2 with the site percolation threshold p_c.

In the subcritical regime we need to introduce some terminology also for *weak points*; i.e. those $z \in \mathbb{Z}^2$ with $\xi_z = 0$. The situation is not symmetric to that of strong points, since in order to have a weak interaction between two points we only need that at least one of the two is a weak point. The notion of adjacent weak point is then modified; namely, we say that two weak vertices z_1 and z_2 are *adjacent* if $0 < |z_1 - z_2| \le \sqrt{2}$. The notion of a weak channel and a weak cluster is modified accordingly. Again, for $p < p_c$ there is a.s. a unique unbounded weak cluster and the channel property holds for weak channels. We may then define λ^s analogously to λ^b in (2.5). The statement of Lemma 2.4 also hold true for the site percolation model.

Note that, in contrast with bond percolation, in the case of the site percolation model the presence of a weak channel in some direction creates a topological obstacle for the existence of a strong transversal channel.

3 Main results

3.1 Bond percolation model

The structure of the Γ-limit functional depends crucially on whether $p < 1/2$ or $p > 1/2$ (recall that in dimension two the percolation threshold p_c is 1/2). We have the following result.

Theorem 3.1. (i) (supercritical regime) *Let $p > 1/2$. Then a.s.*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F^{b,\omega}_\varepsilon(u) = \begin{cases} \displaystyle\int_Q |\nabla u|^2 dx, & \text{if } u \in H^1(Q), \\ +\infty, & \text{otherwise} \end{cases}$$

on the space of GSBV *functions.*

(ii) (subcritical regime) *If $p < 1/2$ then a.s. we have*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon^{b,\omega}(u) = \int_Q |\nabla u|^2 dx + \int_{S(u)} \lambda^b(\nu(x),p) d\mathcal{H}^1$$

for all $u \in \mathrm{GSBV}(Q)$, where $\lambda^b(\nu,p)$ has been defined in (2.5).

Before proving the results above note the following coerciveness property.

Proposition 3.2 (equi-coerciveness and lower bound)**.** *Let $p \in [0,1]$ and $\omega \in \Omega$; then*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^{b,\omega}(u) \geq \int_Q |\nabla u|^2 \, dx + \int_{S(u)} |\nu|_1 d\mathcal{H}^1, \tag{3.1}$$

where $|\nu|_1 = |\nu_1| + |\nu_2|$. Moreover if (ε_j) converges to 0, then for all (u_j) bounded in measure such that $\sup_j F_{\varepsilon_j}^{b,\omega}(u_j) < +\infty$ there exists a subsequence of (u_j) converging in measure to some $u \in \mathrm{GSBV}(Q)$.

Proof. It suffices to remark that for all $p \in [0,1]$, $\omega \in \Omega$ and u we have $F_\varepsilon^{b,\omega}(u) \geq E_\varepsilon(u)$, where E_ε is as in Example 2.1(2). It suffices then to use the results recalled in items (2) and (3) in Example 2.1. □

Remark 3.3. Note that for all $p \in [0,1]$, $\omega \in \Omega$ and u we have $F_\varepsilon^{b,\omega}(u) \leq E_\varepsilon(u)$, where E_ε is as in Example 2.1(1). By taking the Γ-lim sup in this inequality, if $u \in H^1(Q)$ we get Γ-$\limsup_{\varepsilon\to 0} F_\varepsilon^{b,\omega}(u) \leq \int_Q |\nabla u|^2 \, dx$. By (3.1) it then follows that the Γ-limit always exists in $H^1(Q)$ and

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^{b,\omega}(u) = \int_Q |\nabla u|^2 \, dx \quad \text{for all } u \in H^1(Q). \tag{3.2}$$

Proof of theorem 3.1(i). We say that an element $\omega \in \Omega$ (or, equivalently a 'realization' of the medium) is *typical* if the statement of Theorem 2.3 holds. Fix a typical realization ω, and let $u^\varepsilon \to u$ be a sequence converging in measure and with bounded energy; i.e., $F_\varepsilon^{b,\omega}(u^\varepsilon) \leq C$. Recall that we identify discrete functions u^ε with the corresponding piecewise-constant functions of continuous argument: $u^\varepsilon(x) = u^\varepsilon(z(x))$ with $x \in \mathbb{R}^2$ and $z(x) \in \mathbb{Z}^2$ defined a.e. by $|z(x) - x| = \mathrm{dist}(x, \mathbb{Z}^2)$. For the extended function we keep the same notation u^ε. Note that $u \in \mathrm{GSBV}(Q)$ by Proposition 3.2. By Remark 3.3 to prove the thesis of the theorem it suffices to show that $u \in H^1(Q)$

Suppose that $u \notin H^1(Q)$, and denote by u_t^ε the truncation of u^ε at the level t:

$$u_t^\varepsilon(x) = \min\big(t, \max(-t, u^\varepsilon)\big).$$

For any $t > 0$ the sequence u_t^ε is of bounded energy since $F_\omega^{b,\varepsilon}(u_t^\varepsilon) \leq F_\omega^{b,\varepsilon}(u^\varepsilon)$, and $u_t^\varepsilon \to u_t \in \mathrm{SBV}(Q)$ in $L^2(Q)$. Under our assumptions, the function u_t does not belong to $H^1(Q)$ for sufficiently large t; that is, the discontinuity set $S(u_t)$ is not empty and $\mathcal{H}^1(S(u_t)) > 0$.

By the properties of $S(u_t)$ (see [2]) for $\mathcal{H}^1$-a.a. $x_0 \in S(u_t)$ there exist two numbers $u^+ = u^+(x_0)$ and $u^- = u^+(x_0)$, $u^+ \neq u^-$, and $\nu = \nu_u(x_0)$ such that, in the unit square S_ν centred in 0 and with one side parallel to ν, the following relation holds

$$\lim_{\eta\to 0^+} \int_{S_\nu} \Big| u_t\Big(\frac{x - x_0}{\eta}\Big) - u_0(x)\Big| dx \ = \ 0,$$

where

$$u_0(x) = \begin{cases} u^+ & \text{if } \ \langle x,\nu\rangle > 0, \\ u^- & \text{if } \ \langle x,\nu\rangle < 0. \end{cases}$$

Since u_t^ε converges in $L^2([0,1]^2)$ to u_t, as $\varepsilon\to 0^+$, this yields

$$\lim_{\eta\to 0^+}\lim_{\varepsilon\to 0^+}\int\limits_{S_\nu}\Big|u_t^\varepsilon\Big(\frac{x-x_0}{\eta}\Big)-u_0(x)\Big|dx \;=\; 0,$$

Denote $I^\pm=\{x\in S_\nu\,:\,\langle x,\nu\rangle=\pm 1/2\}$ the two sides of S_ν orthogonal to ν. Taking appropriate subsequences $\eta\to 0^+$ and $\varepsilon\to 0^+$, one may assume without loss of generality that

$$\lim_{\eta\to 0^+}\lim_{\varepsilon\to 0^+}\mathcal{H}^1\Big\{x\in I^\pm\,:\,\Big|u_t^\varepsilon\Big(\frac{x-x_0}{\eta}\Big)-u^\pm\Big|>\frac{u^+-u^-}{8}\Big\}=0.$$

We choose η and then ε_0 in such a way that

$$\mathcal{H}^1\Big\{x\in I^\pm\,:\,\Big|u_t^\varepsilon\Big(\frac{x-x_0}{\eta}\Big)-u^\pm(x)\Big|>\frac{u^+-u^-}{8}\Big\}<\frac{1}{8}c(p). \tag{3.3}$$

for $\varepsilon<\varepsilon_0$.

By Theorem 2.3 for small enough ε in the square $\eta S_\nu+x_0$ there are at least $c(p)\eta/\varepsilon$ disjoint strong channels that connect the sides I^- and I^+. Denote these channels by $K_1,K_2,\dots,K_M$, $M\geq c(p)\eta/\varepsilon$, and the vertices of each K_j by $z_1^j,z_2^j,\dots,z_{l_j}^j$ with $l_j\leq c_1(p)\eta/\varepsilon$ and $z_i^j\in\varepsilon\mathbb{Z}^2$.

Considering (3.3) and slightly reducing the constant $c(p)$ we may suppose that all the channels possess the following properties

$$u_t^\varepsilon(z_1^j)\leq u^-+\frac{u^+-u^-}{4},\qquad u_t^\varepsilon(z_{l_j}^j)\geq u^+-\frac{u^+-u^-}{4}.$$

Therefore, for each channel K_j one has

$$\begin{aligned}\frac{(u^+-u^-)^2}{4}\;&\leq\;(u_t^\varepsilon(z_{l_j}^j-u_t^\varepsilon(z_1^j))^2=\Big(\sum_{i=1}^{l_j-1}(u(z_{i+1}^j)-u(z_i^j))\Big)^2\\ &\leq\; l_j\sum_{i=1}^{l_j-1}(u_t^\varepsilon(z_{i+1}^j)-u_t^\varepsilon(z_i^j))^2\\ &\leq\; c_1(p)\frac{\eta}{\varepsilon}\sum_{i=1}^{l_j-1}(u_t^\varepsilon(z_{i+1}^j)-u_t^\varepsilon(z_i^j))^2.\end{aligned}$$

This implies the inequality

$$\sum_{i=1}^{l_j-1}(u_t^\varepsilon(z_{i+1}^j)-u_t^\varepsilon(z_i^j))^2\geq\frac{(u^+-u^-)^2}{4}\frac{\varepsilon}{\eta}\frac{1}{c_1(p)}.$$

Summing up over the channels gives

$$\sum_{j=1}^{M}\sum_{i=1}^{l_j-1}(u_t^\varepsilon(z_{i+1}^j)-u_t^\varepsilon(z_i^j))^2\geq\frac{(u^+-u^-)^2}{4}\frac{\varepsilon}{\eta}\frac{1}{c_1(p)}c(p)\frac{\eta}{\varepsilon}=\frac{(u^+-u^-)^2}{4}\frac{c(p)}{c_1(p)}.$$

Clearly, since the channels are disjoint, the total energy of the square $S_\nu^\eta+x_0$ is estimated from below as follows

$$\begin{aligned}\sum_{z_1,z_2\in(S_\nu^\eta+x_0)\cap\varepsilon\mathbb{Z}^2}\varepsilon\phi_{\hat z}\Big(\frac{(u^\varepsilon(z_1)-u^\varepsilon(z_2))^2}{\varepsilon}\Big)\;&\geq\;\sum_{j=1}^{M}\sum_{i=1}^{l_j-1}(u_t^\varepsilon(z_{i+1}^j)-u_t^\varepsilon(z_i^j))^2\\ &\geq\;\frac{(u^+-u^-)^2}{4}\frac{1}{c_1(p)}c(p).\end{aligned}$$

Since this estimate is η-independent, we may fix an arbitrary number N of distinct points of $S(u_t)$ as above with $(u^+ - u^-) \geq C > 0$, and repeat the arguments above on disjoint squares centered at these points. We then obtain the estimate for the total energy

$$F_\varepsilon^{b,\omega}(u^\varepsilon) \geq N\frac{C^2}{4}\frac{c(p)}{c_1(p)},$$

which contradicts our assumption that the energy is bounded. Hence we must have $\mathcal{H}^1(S(u_t)) = 0$ for all t, which implies that $u \in H^1(Q)$. □

Remark 3.4. The statement (i) of Theorem 3.1 remains valid in any dimension $d \geq 2$. Indeed, the assertions of Theorems 2.2 and 2.3 hold true for any $d \geq 2$, and the proof of item (i) above can be easily rearranged so that it applies for any $d \geq 2$. Notice that for $d > 2$ we have $p_c < 1/2$.

Proof of theorem 3.1(ii). We first prove the liminf inequality. It suffices to deal with $u \in SBV(Q)$ since by comparison the limit energy is $+\infty$ outside $GSBV(Q)$ and it is not restrictive to suppose that $u \in L^\infty(Q)$ by a truncation argument.

Let $u_\varepsilon \to u$ be such that $F_\varepsilon^{b,\omega}(u_\varepsilon) \leq c < +\infty$. We will give an estimate on the energy related to u_ε concentrating at a.e. point in $S(u)$. We fix $R > 0$ and $\gamma > 0$; we will show that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F^{b,\omega}(u_\varepsilon) \geq \int_{S(u)\cap\{u^+-u^-|>1/R\}} (\lambda^b(\nu_u(x), p) - \gamma)d\mathcal{H}^1. \tag{3.4}$$

From this estimate the complete liminf inequality is easily recovered.

For γ as above let $\delta = \delta(\gamma/2) > 0$ be defined by Lemma 2.4. Consider $x \in S(u)$ and $\rho > 0$. We denote by $T_{\rho,k}(x)$ the bar

$$T_{\rho,k}(x) = \{y \in \mathbb{R}^2 \,:\, |\langle(y-x), \nu_u(x)\rangle| < \frac{1}{2}k\rho, |\langle x, \nu^\perp\rangle| < \frac{1}{2}\rho\},$$

and by

$$T_{\rho,k}^{\pm}(x) = \{y \in \mathbb{R}^2 \,:\, |\langle(y-x), \nu_u(x)\rangle| \leq \frac{1}{2}k\rho, \langle x, \nu^\perp\rangle = \pm\frac{1}{2}\rho\},$$

$$T_{\rho,k}^{\pm,\perp}(x) = \{y \in \mathbb{R}^2 \,:\, \langle(y-x), \nu_u(x)\rangle = \frac{1}{2}k\rho, |\langle x, \nu^\perp\rangle| \leq \frac{1}{2}\rho\}$$

its four sides. We choose $k = \delta/16$.

We choose x and ρ such that

1) $|u^+(x) - u^-(x)| \geq \dfrac{1}{R}$;
2) $\rho < \dfrac{\delta}{4}\dfrac{1}{R^2}\dfrac{1}{\lambda^b(\nu_u(x), p)}$;
3) $\mathcal{H}^1\Big(\Big\{x \in T_{\rho,k}^{\pm}(x) : |u^+(x) - u^-(x)| > \dfrac{1}{8R}\Big\}\Big) \leq \dfrac{\delta}{8}\rho$;
4) $u_\varepsilon \to u$ in $L^1(\partial T_{\rho,k}(x))$.

In order to simplify the notation, it is not restrictive to suppose that $x = 0$. We scale our problem by the factor $N = \frac{\rho}{\varepsilon}$ to the bar $T_{N,k}(0)$ and the corresponding $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$.

Let $I = I(\varepsilon)$ be the set of all $\hat{z} \in T_{N,k}(0)$ such that

$$\text{i) } \xi_{\hat{z}} = 0; \qquad \text{ii) } |u_\varepsilon(\varepsilon z_1(\hat{z})) - u_\varepsilon(\varepsilon z_2(\hat{z}))| \geq \sqrt{\varepsilon}.$$

Suppose that

$$\#(I(\varepsilon)) < (\lambda^b(\nu_u(x), p) - \gamma)N. \tag{3.5}$$

We then construct strong channels as follows. Let

$$J_0 = \bigcup\{S \text{ closed unit cube with integer vertices, } S \cap T_{N,k}^{-,\perp}(0) \neq \emptyset\},$$

$$J_i^\perp = \bigcup_{\hat z\in J_i} \hat z^\perp$$

(we use the general notation $A^\perp = \bigcup_{\hat z\in A}\hat z^\perp$),

$$\widetilde J_i^\perp = J_i^\perp \cup \{\hat z : \exists \text{ path in } I^\perp \text{ connecting } \hat z^\perp \text{ and } J_i^\perp\},$$

$$J_{i+1} = \bigcup\{S \text{ closed unit cube with integer vertices, } S\cap \widetilde J_i^\perp \neq \emptyset\}.$$

We finally define K_i as the unique path in $\partial J_i \cap T_{N,k}(0)$ joining $T^+_{N,k}(0)$ and $T^-_{N,k}(0)$, and M as the maximal index such that

$$K_{M-1}\cap T^{+,\perp}_{N,k}(0) = \emptyset.$$

By construction, $K_1,\dots,K_M$ are disjoint strong channels connecting $T^+_{N,k}(0)$ and $T^-_{N,k}(0)$. Moreover, for each element in $(K_i)^\perp$ there exists a path with exactly i elements in the complement of $I^\perp$ connecting it with J_0.

We conclude that there exists a path in $\hat Z^\perp$ connecting $T^{-,\perp}_{N,k}(0)$ and $T^{+,\perp}_{N,k}(0)$ with exactly M elements from the complement of $I^\perp$. From Lemma 2.4 we then deduce that $M > \delta N$.

Note that in order to estimate the limit of $F^{b,\omega}_\varepsilon(u_\varepsilon, T_{\rho,k}(x))$ we may set $\xi^\varepsilon_{\hat z} = 1$ for all $\hat z \notin I(\varepsilon)$. We can then repeat the argument in the proof of theorem 3.1 and conclude that

$$\liminf_{\varepsilon\to 0^+} F^{b,\omega}_\varepsilon(u_\varepsilon, T_{\rho,k}(x)) \geq C,$$

with $C = C(\gamma, R)$. Since this inequality may hold only for a finite number of x, we conclude that (3.5) must fail for almost all x, so that

$$\#(I(\varepsilon)) \geq (\lambda^b(\nu_u(x), p) - \gamma)N,$$

from which we deduce that

$$\liminf_{\varepsilon\to 0^+} F^{b,\omega}_\varepsilon(u_\varepsilon, T_{\rho,k}(x)) \geq (\lambda^b(\nu_u(x), p) - \gamma)\rho,$$

and inequality (3.4).

We now prove the limsup inequality. By the density results by Cortesani and Toader [12] (see also [6]) the space $GSBV(Q)$ can also be seen as the closure of piecewise-smooth functions in the sense that for all $u \in GSBV(Q)$ with $\nabla u \in L^2(Q)$ and $\mathcal H^1(S(u)) < +\infty$ there exists a sequence of finite families of disjoint closed segments (P^i_j) and functions u_j that are C^∞ and Lipschitz outside $\bigcup_i P^i_j$ converging to u in measure such that

$$\lim_j\Big(\int_Q f(\nabla u_j)\,dx + \int_{S(u_j)} g(\nu_{u_j})d\mathcal H^1\Big) = \int_Q f(\nabla u)\,dx + \int_{S(u)} g(\nu_u)d\mathcal H^1 \tag{3.6}$$

for all continuous f, g with $|f(z)| \leq C(1+|z|^2)$.

Using this observation, it suffices to prove the limsup inequality of Theorem 3.1(ii) only for piecewise-smooth functions as above. In fact, by the lower semicontinuity of the Γ-upper limit (see e.g. [5] Section 1.7), from (3.6), applied to $f(w) = |w|^2$ and $g(w) = \lambda^b(w,p)$, we get

$$\begin{aligned}
\Gamma\text{-}\limsup_{\varepsilon\to 0^+} F^{b,\omega}_\varepsilon(u) &\leq \liminf_j\Big(\Gamma\text{-}\limsup_{\varepsilon\to 0^+} F^{b,\omega}_\varepsilon(u_j)\Big)\\
&\leq \lim_j\Big(\int_Q |\nabla u_j|^2\,dx + \int_{S(u_j)} \lambda^b(\nu_{u_j}, p)\,d\mathcal H^1\Big)\\
&= \int_Q |\nabla u|^2\,dx + \int_{S(u)} \lambda^b(\nu_u, p)d\mathcal H^1
\end{aligned}$$

as desired.

We then have to construct a recovery sequence for the Γ-limsup for a piecewise-smooth function. Since the construction is performed by modifying the target function u close to each segment of $S(u)$, we may deal with $S(u)$ composed of only one segment. It is no restriction to consider only the case $S(u) = [-1, 1] \times \{0\}$. Note that the traces of u on both sides of $S(u)$ are well-defined Lipschitz functions. We can also suppose that

$$u(x^1, x^2) = u^+(x^1) \quad \text{on} \quad [-1-\eta, 1+\eta] \times (0, \eta),$$
$$u(x^1, x^2) = u^-(x^1) \quad \text{on} \quad [-1-\eta, 1+\eta] \times (-\eta, 0),$$

for some $\eta > 0$.

Fix arbitrary small $\delta > 0$. Then a.s. for all sufficiently small $\varepsilon > 0$ in the rectangle $R^\eta_\delta = [-1-\eta, 1+\eta] \times [-\delta, \delta]$ there is a weak channel K_ε connecting the left and the right sides of the rectangle, of length no greater than $2\lambda^b(e_1, p)(1+\eta) + o(1)$, where $o(1)$ tends to zero as $\varepsilon \to 0$.

We define

$$\tilde{u}^\varepsilon = \begin{cases} u(x) & \text{if } x \notin R^\eta_\delta, \\ u^+(x) & \text{if } x \text{ belongs to the closure of the connected component of} \\ & \quad R^\eta_\delta \text{ containing the upper side,} \\ u^-(x) & \text{otherwise.} \end{cases}$$

Let u^ε be the discretization of $\tilde{u}^\varepsilon$. Then after a straightforward computation we have

$$\begin{aligned} F^{b,\omega}_\varepsilon(u^\varepsilon) &\le \int_Q |\nabla u|^2 dx + 2\lambda^b(e_2, p)(1+\eta) + o(1) \\ &= \int_Q |\nabla u|^2 dx + \lambda^b(e_2, p)\mathcal{H}^1(s(u))(1+\eta) + o(1), \end{aligned}$$

with $o(1) \to 0$ as $\varepsilon \to 0$. Since η and δ are arbitrary positive numbers, this implies the desired bound. □

3.2 Site percolation model

The results below are basically the same as those obtained for the bond percolation model. We only formulate these results and leave their proof to the reader.

In the supercritical regime $p > p_c$ the structure of the Γ-limit of $F^{s,\omega}_\varepsilon$ is described by the following statement

Theorem 3.5. (i) (supercritical regime) *Let $p > p_c$. Then a.s.*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F^{s,\omega}_\varepsilon(u) = \begin{cases} \int_Q |\nabla u|^2 dx, & \text{if } u \in H^1(Q), \\ +\infty, & \text{otherwise.} \end{cases}$$

The limit functional is defined on GSBV(Q).

(ii) (subcritical regime) *If $p < p_c$ then a.s. we have*

$$\Gamma\text{-}\lim_{\varepsilon \to 0^+} F^{s,\omega}_\varepsilon(u) = \int_Q |\nabla u|^2 dx + \int_{S(u)} \lambda^s(\nu(x), p) d\mathcal{H}^1$$

on GSBV(Q), *where $\lambda^s(\nu, p)$ has been defined as in* (2.5).

4 Appendix

This appendix contains a proof of Lemma 2.4 kindly provided by H. Kesten. It relies on the following large deviation result for Bernoulli percolation at $p > p_c$ (see [15]). We denote by $D(x, y)$ the *chemical distance* between x and y for $x, y \in \mathbb{Z}^d$, i.e. the length of the shortest weak path which connects x and y, and by $\|\cdot\|$ the norm $\|x\| = \sum_{i=1}^{d} |x_i|$. Differently from the previous sections we will use the terminology 'black' and 'white' bonds in place of 'strong' and 'weak' bonds, respectively.

Proposition 4.1 ([15])**.** *There exists a norm $\mu(x)$ such that for all $\varepsilon > 0$*

$$\limsup_{\|x\|\to\infty} \frac{\log P\{\mathbf{0} \leftrightarrow x, \frac{D(\mathbf{0},x)}{\mu(x)} \notin (1-\varepsilon, 1+\varepsilon)\}}{\|x\|} < 0, \tag{4.1}$$

where $y \leftrightarrow x$ stands for the event that both x and y are elements of the infinite cluster.

We need a number of definitions. Let

$$\begin{aligned}\mathcal{E}(v, N, \delta, \eta) = \{ &\text{ there exists a path from } \mathbf{0} \text{ to } Nv \text{ of length} \leq N(\alpha - \delta) \\ &\text{and containing at most } \eta N \text{ black edges} \}.\end{aligned}$$

Here $v \in \mathbb{R}^d \setminus \mathbf{0},\ \alpha = \mu(v)$. Further, let M be some large integer and $\mathbf{i} = (i_1, i_2, \dots, i_d) \in \mathbb{Z}^d$, and define

$$B(\mathbf{i}) = B_M(\mathbf{i}) = \prod_{k=1}^{d} [i_k - M, i_k + M]$$

(a closed cube of size $2M$ with center at $\mathbf{i}$). Now let $\mathbf{w} = w(0) = \mathbf{0}, w(1), \cdots, w(r)$ be a sequence of vertices in $\mathbb{Z}^d$ for which

$$w(j+1) \in \partial B(w(j)),\ \text{ or equivalently } \|w(j+1) - w(j)\| = M,\ 0 \leq j \leq r-1 \tag{4.2}$$

(∂B denotes the topological boundary of B), and

$$[Nv] := (\text{nearest lattice point to } Nv) \in B(w(r)). \tag{4.3}$$

In addition, let $\mathcal{S}$ be a subsequence of $0, 1, \dots, r$, say $\mathcal{S} = \{j_1, \dots, j_t\}$ with $0 \leq j_1 < j_2 < \cdots < j_t \leq r\}$ and define the event

$$\mathcal{F}(\mathbf{w}, \mathcal{S}) = \{\text{there exists a path } \sigma \text{ on } \mathbb{Z}^d \text{ with the properties (4.3)–(4.7) below}\} :$$

$$\begin{aligned}&\sigma \text{ is a self-avoiding path from } \mathbf{0} \text{ to } [Nv] \text{ which successively} \\ &\qquad \text{visits the vertices } w(0), w(1), \dots, w(r).\end{aligned} \tag{4.4}$$

Let $\sigma = \sigma(0), \sigma(1), \dots, \sigma(m)$ and $w(j) = \sigma(s(j))$ with $s(0) = 0 < s(1) < \cdots < s(r)$. Define further $w(r+1) = [Nv]$ and $s(r+1) = m$. Then

$$\begin{aligned}&\text{the piece } (\sigma(s(j)), \sigma(s(j)+1), \dots, \sigma(s(j+1)-1) \text{ of } \sigma \text{ is contained} \\ &\qquad \text{in the interior of } B(w(j)),\ 0 \leq j \leq r.\end{aligned} \tag{4.5}$$

For $j = r$ we even require that $\sigma(s(r+1)) = w(r+1) = [Nv]$ lies in the interior of $B(w(r))$. For any path π on $\mathbb{Z}^d$ denote its length, i.e., the number of edges in π, by $|\pi|$. Then

$$m = |\sigma| \leq N(\alpha - \delta). \tag{4.6}$$

$$\text{all edges of } \sigma \text{ except possibly those in } \cup_{j\in\mathcal{S}} B(w(j)) \text{ are white.} \tag{4.7}$$

$$|\mathcal{S}| := \text{ cardinality of } \mathcal{S} \le \eta N. \tag{4.8}$$

We claim that (4.2) and (4.6) imply that

$$r \le \frac{N\alpha}{M}. \tag{4.9}$$

To see this, note that the piece of σ from $w(j)$ to $(w(j+1)$ contains at least $\|w(j+1)-w(j)\| = M$ edges, for $0 \le j \le r-1$. This shows that $N(\alpha - \delta) \ge m = |\sigma| \ge rM$. Thus (4.9) holds.

Next we note that once $w(j)$ with $j < r$ is given, there are at most $K_1 := 2d(2M+1)^{d-1}$ possible choices for $w(j+1)$, because of (4.2). As a consequence, the number of sequences $w(0), \dots, w(r)$ and subsequences $\mathcal{S}$ of $\{0, \dots, r\}$ for which $\mathcal{F}(\mathbf{w}, \mathcal{S})$ can occur is at most $[2K_1]^{N\alpha/M}$.

Proposition 4.2.

$$\mathcal{E}(v, N, \delta, \eta) \subset \bigcup_{(w,\mathcal{S})} \mathcal{F}(\mathbf{w}, \mathcal{S}), \tag{4.10}$$

where $(\mathbf{w}, \mathcal{S})$ *runs over the pairs for which* $\mathcal{F}(\mathbf{w}, \mathcal{S})$ *is possible. (Thus this union contains at most* $[2K_1]^{N\alpha/M}$ *elements).*

Proof. Assume that $\mathcal{E}(v, N, \delta, \eta)$ occurs. By definition, there then exists a self-avoiding path σ from $\mathbf{0}$ to $[Nv]$ with $|\sigma| \le N(\alpha - \delta)$ which contains at most ηN black edges. We now show that we can choose $\mathbf{w}$ and $\mathcal{S}$ such that (4.2)–(4.8) all hold. To this end take $s(0) = 0$ and $w(0) = \sigma(s(0)) = \sigma(0) = \mathbf{0}$. Then for $j \ge 0$ such that $s(j)$ and $w(j)$ have already been determined such that $w(j) = \sigma(s(j))$, take

$$\begin{aligned} s(j+1) &= \inf\{k > s(j) : \|\sigma(k) - \sigma(s(j))\| = M\} \\ &= \inf\{k > s(j) : \|\sigma(k) - w(j)\| = M\}. \end{aligned} \tag{4.11}$$

We stop the process at the first index r such that $\|\sigma(k) - \sigma(s(r))\| < M$ for all $k > s(r)$ and take $s(r+1) = |\sigma|$, so that $w(r+1) = \sigma(|\sigma|) = [Nv]$, the endpoint of σ. With this choice (4.2)–(4.5) are immediate. (For (4.5) note that $\|\sigma(k) - w(j)\| < M$ for $s(j) \le k < s(j+1)$, by virtue of (4.11).) Also (4.6) holds by assumption. To get (4.7) and (4.8) we take

$$\begin{aligned} \mathcal{S} \;=\; &\{0 \le j \le r : \text{there is a black edge in the piece} \\ &\sigma(s(j)), \sigma(s(j)+1), \dots \sigma(s(j+1)) \text{ of } \sigma\}. \end{aligned} \tag{4.12}$$

(4.7) follows, because any black edge of σ belongs to exactly one of the pieces $\sigma(s(j)), \sigma(s(j)+1), \dots \sigma(s(j+1))$ with $0 \le j \le r$, and this piece is contained in $B(w(j))$ by (4.5). Since σ contains at most ηN black edges, by assumption, (4.8) is also immediate. □

We next define the events

$$\mathcal{G}(\mathbf{i}) = \mathcal{G}_M(\mathbf{i}) = \{\text{all edges in } B_M(\mathbf{i}) \text{ are white}\}$$

and

$$\mathcal{H}(\mathbf{w}, \mathcal{S}) = \bigcap_{j \in \mathcal{S}} \mathcal{G}(w(j)).$$

The following proposition is a kind of converse of Proposition 4.2. Note that $\mathcal{E}(N, v, \delta, 0)$ is the event that there exists a white path of length at most $N(\alpha - \delta)$ from $\mathbf{0}$ to $[Nv]$.

Proposition 4.3. *For each pair* $(\mathbf{w}, \mathcal{S})$ *for which* (4.3)–(4.8) *hold, it is the case that*

$$\mathcal{F}(\mathbf{w}, \mathcal{S}) \cap \mathcal{H}(\mathbf{w}, \mathcal{S}) \subset \mathcal{E}(v, N, \delta, 0). \tag{4.13}$$

Proof. Assume that $\mathcal{F}(\mathbf{w},\mathcal{S}) \cap \mathcal{H}(\mathbf{w},\mathcal{S})$ occurs, and let σ be a self-avoiding path from $\mathbf{0}$ to $[Nv]$ for which (4.4)–(4.8) hold. Let $\sigma = (\sigma(0), \sigma(1), \dots, \sigma(m))$. Let $\mathcal{S} = \{0 \le j_1 < \cdots < j_t\}$ with $t \le \eta N$. Then σ runs from $\mathbf{0}$ to $[Nv]$ and it contains only white edges. Indeed, by (4.7) there are no black edges in σ except possibly edges in $B(w(j))$ for some $j \in \mathcal{S}$. However, for $j \in \mathcal{S}$ there are no black edges in $B(w(j))$ either, because we assumed that $\mathcal{G}(w(j))$ occurs. Finally, the length of σ is at most $N(\alpha - \delta)$, by virtue of (4.6). Thus $\mathcal{E}(v, N, \delta, 0)$ does indeed occur. □

We next compare the probabilities of $\mathcal{F}(\mathbf{w},\mathcal{S})$ and of $\mathcal{F}(\mathbf{w},\mathcal{S}) \cap \mathcal{H}(\mathbf{w},\mathcal{S})$.

Proposition 4.4. *Let*

$$K_2 = d2M(2M+1)^{d-1}.$$

Then, for each pair $(\mathbf{w},\mathcal{S})$ *for which* (4.3)–(4.8) *hold, it is the case that*

$$P\{\mathcal{F}(\mathbf{w},\mathcal{S})\} \le p^{-K_2\eta N} P\{\mathcal{F}(\mathbf{w},\mathcal{S}) \cap \mathcal{H}(\mathbf{w},\mathcal{S})\}. \tag{4.14}$$

Proof. The number of edges in any $B_M(\mathbf{i})$ equals K_2. Therefore,

$$\text{the number of edges in } \cup_{j\in\mathcal{S}} B(w(j)) \le \eta N K_2. \tag{4.15}$$

Now notice that for given $(\mathbf{w},\mathcal{S})$, the event $\mathcal{F}(\mathbf{w},\mathcal{S})$ is increasing (in the white edges). Therefore, if $\mathcal{F}(\mathbf{w},\mathcal{S})$ occurs in some sample point, then $\mathcal{F}(\mathbf{w},\mathcal{S})$ still occurs if we change any collection of edges to white. Now let $\mathcal{B}$ be an increasing event $\mathcal{B}$ and fix the configuration outside a given edge edge e. Then either $\mathcal{B}$ occurs only if e is white, or $\mathcal{B}$ occurs no matter what the color of e is. In the former case, the conditional probability of $\mathcal{B}$ (given the configuration outside e) is the same as the conditional probability of $\mathcal{B} \cap \{e \text{ is white}\}$. In the latter case, the conditional probability of $\mathcal{B}$ equals 1 and the conditional probability of $\mathcal{B} \cap \{e \text{ is white}\}$ equals p. By averaging over the configuration outside e this shows that

$$P\{\mathcal{B}\} \le \frac{1}{p} P\{\mathcal{B}, e \text{ is white}\}.$$

The inequality (4.14) now follows by successively applying this argument to each edge in $\cup_{j\in\mathcal{S}} B(w(j))$ and taking (4.15) into account. □

The following final proposition proves Lemma 2.4.

Proposition 4.5. *For* $\delta > 0$ *there exists an* $\eta > 0$ *and a* $K_3 > 0$ *such that for all large* N

$$P\{\mathcal{E}(v, N, \delta, \eta\} \le \exp[-K_3 N]. \tag{4.16}$$

Proof. Fix $\delta > 0$. By Proposition 4.1 there exists some constant $K_4 = K_4(\delta) > 0$ such that

$$P\{\mathcal{E}(v, N, \delta, 0\} \le \exp[-K_4 N]$$

for all large N. Proposition 4.3 then tells us that for each permissible $(\mathbf{w},\mathcal{S})$ it also holds

$$P\{\mathcal{F}(\mathbf{w},\mathcal{S}) \cap \mathcal{H}(\mathbf{w},\mathcal{S})\} \le \exp[-K_4 N]$$

for all large N. Then, by Proposition 4.4,

$$P\{\mathcal{F}(\mathbf{w},\mathcal{S})\} \le p^{-K_2\eta N} \exp[-K_4 N]$$

for all large N. Since K_2 depends on M only, we can for given δ and M choose η so small that

$$K_2 \eta \log \frac{1}{p} \le \frac{K_4}{2}. \tag{4.17}$$

For such a choice we even have

$$P\{\mathcal{F}(\mathbf{w},\mathcal{S})\} \leq \exp[-K_4 N/2]$$

for all large N. Note that these estimates are uniform in the choice of $(\mathbf{w},\mathcal{S})$ which satisfy (4.1)–(4.8). By Proposition 4.2 this finally gives

$$\begin{aligned} P\{\mathcal{E}(v,N,\delta,\eta)\} &\leq \sum_{(\mathbf{w},\mathcal{S})} \exp[-K_4 N/2] \\ &\leq [2K_1]^{N\alpha/M} \exp[-K_4 N/2] \\ &= [4d(2M+1)^{d-1}]^{N\alpha/M} \exp[-K_2 N/2] \end{aligned}$$

for large N. We can therefore first choose M so large that

$$[4d(2M+1)^{d-1}]^{\alpha/M} \leq \exp[K_2/4],$$

and then η so small that (4.17) holds. Result (4.16) with $K_3 = K_4/4$ then follows. □

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