Quantitative stability for the first Dirichlet eigenvalue in Reifenberg flat domains in $\mathbb{R}^N$.

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Abstract

In this paper we prove that if Ω and Ω' are close enough for the complementary Hausdorff distance and their boundaries satisfy some geometrical and topological conditions then

$$|\lambda_1 - \lambda_1'| \leq C|\Omega \triangle \Omega'|^{\frac{\alpha}{N}}$$

where λ_1 (resp. λ_1') is the first Dirichlet eigenvalue of the Laplacian in Ω (resp. Ω') and $|\Omega \triangle \Omega'|$ is the Lebesgue measure of the symmetric difference.

1 Introduction

In this paper we prove a stability result for the first Dirichlet eigenvalue of the Laplacian in some bounded open sets in $\mathbb{R}^N$. More precisely, we estimate the difference

$$|\lambda_1 - \lambda_1'| \leq C|\Omega \triangle \Omega'|^{\frac{\alpha}{N}} \tag{1.1}$$

where λ_1 (resp. λ_1') is the first Dirichlet eigenvalue of the Laplacian in Ω (resp. Ω'), and $|\Omega \triangle \Omega'| := |\Omega \backslash \Omega' \cup \Omega \backslash \Omega'|$ is Lebesgue measure of the symmetric difference between Ω and Ω'.

Stability results for the eigenvalues were studied a lot in the last two decades (see [4, 9, 11]) and have many applications, for instance in shape optimization problems (see [5, 10]). On the other hand, as far as the authors know, estimates with a precise quantitative bound as (1.1) were only recently investigated ([3, 1, 19]), and always for regular domains, $C^{1,1}$ or at least bi-Lipschitz domains. We would like to mention also [17] where a weaker inequality than (1.1) is proved for a very large class of domains for instance bounded connected John sets with a "twisting external cone condition". The proof of Pang [17] uses a Brownian motion and is based on estimates on the Poisson kernel. In this paper we present a simpler proof in the case of Reifenberg-flat domains.

More precisely, we seek some geometrical conditions to impose on the domains in order to guarantee that (1.1) is true. What we obtain is that a "strong"-Reifenberg-flat boundary is sufficient. In particular, domains with cracks are not permitted. Roughly, in terms of regularity, such domains have boundaries which are well approximated by hyperplanes at every scales (see Definition 1). This is weak enough to permit Hölderian spirals or snowflake-like boundaries (in particular it is weaker than Lipschitz domains) but at the same time the geometry of a Reifenberg-flat set is sufficiently under control in order to make some thin estimates. We refer the reader to [6, 15, 16] for some earlier works on the analysis of operators in Reifenberg-flat domains.

Notice that one could expect (1.1) to be true with $\alpha = 1$ for the case of Lipschitz domains. In this paper, since we work with Reifenberg flat domains we only get (1.1) with $\alpha < 1$ which is optimal in our class of Domains.

It is worth mentioning that (1.1) cannot be true without assuming any kind of regularity on the boundary of the domains. For example in $\mathbb{R}^2$, the domains $\Omega := B(0,1)\backslash\{x_1 = 0, x_2 \leq 0\}$ and $\Omega' := B(0,1)$ are such that $|\Omega \triangle \Omega'| = 0$ but clearly $\lambda_1 \neq \lambda_1'$. Inequality (1.1) can either not be true without adding some topological assumptions. Indeed, the Lipschitz domains $\Omega := B(0,1)\backslash\{x_1 = 0\}$ and $\Omega' := B(0,1)$ are again such that $|\Omega \triangle \Omega'| = 0$ but $\lambda_1 \neq \lambda_1'$.

We denote d_H the Hausdorff distance, namely for two compact sets A and B

$$d_H(A,B) := \sup_{x\in A} dist(x,B) + \sup_{y\in B} dist(y,A).$$

In this paper we consider the case of Reifenberg flat Domains which are defined as follows.

Definition 1. *An (ε, r_0)-Reifenberg-flat domain $\Omega \subset \mathbb{R}^N$ is an open and bounded set such that for each $x \in \partial\Omega$ and for any $r \leq r_0$, $\Omega \cap B(x,r)$ is connected and there exists a hyperplane $P(x,r)$ containing x which satisfies*

$$\frac{1}{r} d_H(\partial\Omega \cap B(x,r), P(x,r) \cap B(x,r)) \leq \varepsilon. \tag{1.2}$$

Remark 2. This last definition is significant only when ε is small enough, less than 1/2 say. For technical reasons, in the sequel we will always assume $\varepsilon < \varepsilon_0 \leq 10^{-2}$ where ε_0 is the one given by the Theorem of Reifenberg [18]. Under this assumption, the topological disk Theorem of Reifenberg [18] applies (see also Theorem 1.1. of [7]) and says that the boundary of our Reifenberg flat domains is locally the bi-Hölderian image of a $N-1$ dimensional disk. As a consequence, all our results are still valid for a Lipschitz domain with sufficiently small constant with respect to ε_0. On the other hand the assumption $\varepsilon \leq \varepsilon_0$ could be weaken in $\varepsilon \leq 10^{-2}$ by adding topological assumptions (see Remark 6).

Next we present our main result concerning the Dirichlet eigenvalues.

Theorem 3. *Let Ω be an (ε, r_0)-Reifenberg flat domain in $\mathbb{R}^N$ such that*

$$0 < \mathcal{H}^{N-1}(\partial\Omega) = L < +\infty.$$

Let B be a ball such that $10B$ is contained in Ω and let γ_1 be the first eigenvalue of B. Then for every $\alpha < 1$ and for every $M > L$ there is a constant C depending on α, N, $|\Omega|$, γ_1, r_0 and M such that the following holds. Let Ω' be an (ε, r_0)-Reifenberg-flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega') \leq M$ and let λ_1 (resp. λ_1') be the first eigenvalue for the Dirichlet Laplacian in Ω (resp. Ω'). If

$$d_H(\Omega'^c, \Omega^c) \leq C^{-1}$$

then

$$|\lambda_1 - \lambda_1'| \leq C|\Omega \triangle \Omega'|^{\frac{\alpha}{N}}.$$

The proof relies on a different approach than the technics in [1] and [17]. The principal idea is to obtain some estimates on the behavior of eigenfunctions near the boundary and combine them with the Min-Max principle using a good extension Lemma to compare two functions defined on different domains.

The paper is organized as follows. Section 2 is devoted to some preliminary results, especially a covering lemma and a geometrical fact saying that $d_H(\Omega'^c, \Omega^c)$ and $|\Omega \triangle \Omega'|^{\frac{1}{N}}$ are equivalent for two Reifenberg flat domains. Next in Section 3 we prove some boundary estimates for both eigenfunctions and their gradients near the boundary of a Reifenberg flat domain. The more difficult part is to control the gradient. We first prove a decay result on balls centered at the boundary and then use the covering lemma to estimate the gradient in a region close to the boundary. In Section 4 we prove an extension result for functions in $H^1_0(\Omega)$. This extension lemma is a powerful tool which is used to compare two Dirichlet eigenvalues. In Section 5 we remark that the extension Lemma implies a γ-convergence result from which we automatically obtain the stability for Dirichlet eigenvalues. Finally Section 6 contains the proof of Theorem 3 using the Min-Max principle.

Acknowledgements. Part of this work was done while the first author was affiliated with the research center Ennio De Giorgi and he wishes to thank all the members of the center and the Scuola Normale Superiore of Pisa for their hospitality.

Notations :

Δ : the Laplacian operator.

$\mathcal{H}^N$: the Hausdorff measure of dimension N.

$|A|$: the Lebesgue measure of the borel set A.

$C_0^\infty(\Omega)$: the C^∞ functions with compact support on Ω.

$W^{1,p}(\Omega)$: the Sobolev space of L^p functions whose derivatives are in L^p.

$W_0^{1,p}(\Omega)$: the adherence of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

$H^1(\Omega)$: the space $W^{1,2}(\Omega)$.

$H_0^1(\Omega)$: the space $W_0^{1,2}(\Omega)$.

$d_H(A,B)$: the Hausdorff distance between the sets A and B.

χ_A : the characteristic function associated to the set A.

2 Preliminary

We start by giving some useful and classical facts about the Dirichlet eigenvalue problem for the Laplacian, that can be found for instance in [8] (see page 214).

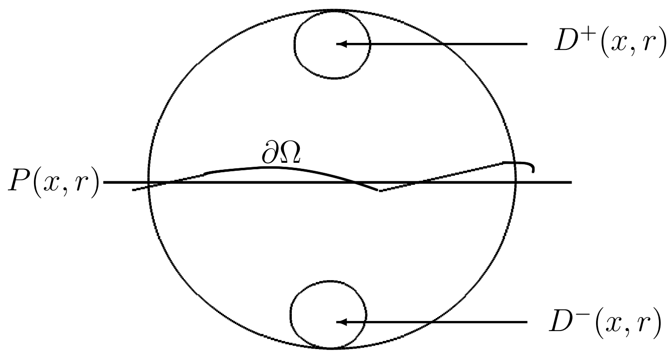
Proposition 4. *Let Ω be a domain in $\mathbb{R}^N$. Then $-\Delta$ has a countably infinite discrete set of eigenvalues, whose eigenfunctions span $H_0^1(\Omega)$. Moreover each eigenfunction v belongs to $L^\infty(\Omega)$ and we have*

$$\|v\|_\infty \leq C(n, |\Omega|)\|v\|_2.$$

For a Reifenberg-flat domain Ω and for any ball $B(x,r)$ centered at $\partial\Omega$ and radius $r \leq r_0$, let us define the sets $D^\pm(x,r)$ in the following way. Let $P(x,r)$ be the hyperplane given by the definition of Reifenberg flatness of Ω. Denote by $z^\pm(x,r)$ the two points that lie at distance $3r/4$ from $P(x,r)$ and whose orthogonal projection on $P(x,r)$ is equal to x. Then we set

$$D^\pm(x,r) := B(z^\pm(x,r), r/4) \tag{2.1}$$

as in the following picture.



We have the following useful fact regarding the sets $D^{\pm}$.

Lemma 5. *Let Ω be an (ε, r_0)-Reifenberg flat domain. Then for all $x \in \partial\Omega$ and $r < r_0/2$, the balls $D^+(x,r)$ and $D^-(x,r)$ lie in different connected components of $B(x,r)\backslash\partial\Omega$.*

Proof. This can be seen as a consequence of the topological disk Theorem of Reifenberg [18]. Actually one could also prove it directly without using the whole result of Reifenberg but just the very beginning of Reifenberg's construction, but in our situation we find it convenient to simply apply the Theorem. More precisely, we use the statement of Theorem 1.1. in [7] (which holds for $N \neq 3$ for the case of hyperplanes) that gives for every $r < r_0$ and $x \in \partial\Omega$ a hyperplane P through x and a continuous homeomorphism $f : B(x, \frac{3}{2}r) \to f(B(x, \frac{3}{2}r)) \subset B(x, 2r)$ such that

$$B(x,r) \subset f(B(x, \frac{3}{2}r)) \subset B(x, 2r) \tag{2.2}$$

$$\partial\Omega \cap B(x,r) \subset f(P \cap B(x, \frac{3}{2}r)) \subset \partial\Omega \cap B(x, 2r). \tag{2.3}$$

Now if we denote by ν any normal vector to P and consider

$$P^+ := \{x \in \mathbb{R}^N; x \cdot \nu > 0\} \quad P^- := \{x \in \mathbb{R}^N; x \cdot \nu < 0\},$$

it is clear from (2.2) and (2.3) that $\partial\Omega$ separates the domains $f(P^{\pm} \cap B(x, \frac{3}{2}r))$ and in particular the sets $D^{\pm}(x,r)$. $\square$

Remark 6. Lemma 5 is the only reason we assume $\varepsilon \leq \varepsilon_0$ where ε_0 is the one given by the Theorem of Reifenberg [18]. All the results of this paper remains true assuming only $\varepsilon \leq 10^{-2}$ and adding the separating property of Lemma 5 in the Definition of Reifenberg-flat domains. On the other hand one can find an explicit value for ε_0 in [18] depending on dimension and for instance equals to 10^{-15} in dimension 3 (see [7]). So assuming $\varepsilon < \varepsilon_0$ is not restrictive.

Remark 7. Since in our definition of Reifenberg flat domains we assume $\Omega \cap B(x,r)$ to be connected for every $x \in \partial\Omega$ and $r < r_0$, Lemma 5 implies in addition that one ball among $D^{\pm}(x,r)$ lies in Ω while the other one lies in Ω^c. Thanks to this fact, any boundary $\partial\Omega$ of a Reifenberg-flat domain *separates* $\mathbb{R}^N$ as in the Definition 2.1 of [16]. In other words our Definition of Reifenberg-flat domains is equivalent to the one considered in [16].

Remark 8. An obvious consequence of Lemma 5 is that any Reifenberg-flat domain has a *twisting external cone condition* as in the Assumption (AIII) of [17].

The following Lemma will be useful to obtain our main result.

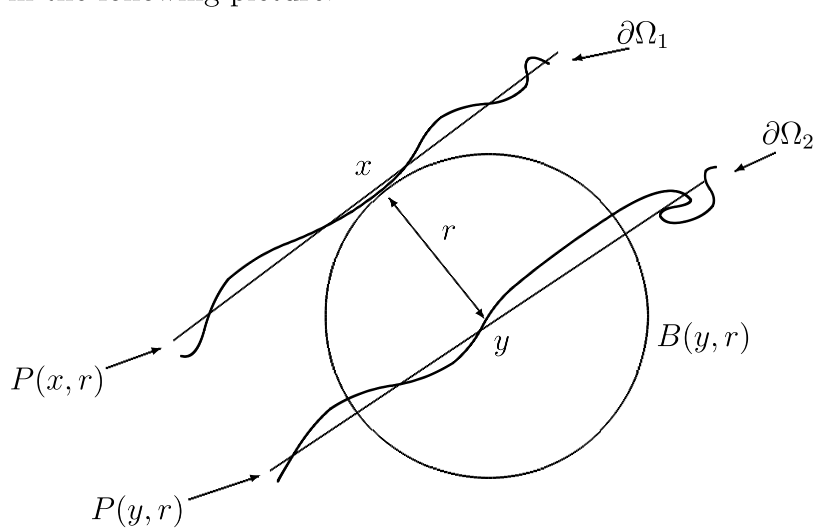
Lemma 9. *Let Ω_1 and Ω_2 be two (ε, r_0)-Reifenberg flat domains such that $d_H(\Omega_1^c, \Omega_2^c) \leq r_0/2$. Then*

$$d_H(\Omega_1^c, \Omega_2^c) \leq C|\Omega_1 \triangle \Omega_2|^{\frac{1}{N}}$$

where C depends only on N.

Proof. Let $x \in \partial\Omega_1$, be such that $r := dist(x, \partial\Omega_2)$ is maximum, and let $y \in \partial\Omega_2$ be such that $dist(x, \partial\Omega_2) = d(x,y) = r$. Let us set $D_1^{\pm} := D^{\pm}(x,r)$ and $D_2^{\pm} := D^{\pm}(y,r)$ as being the balls defined in (2.1). Under our assumptions we know that only one of $D_1^{\pm}$ lies in Ω_1 and only one of $D_2^{\pm}$ lies in Ω_2. Let us simply denote D_i those two balls.

Now by the definition of y, we know that $B(y,r) \cap \partial\Omega_1$ is empty. In particular, the two "approximating" hyperplanes $P(x,r)$ and $P(y,r)$ are almost parallel (with error less than $2.\varepsilon \leq 2.10^{-2}$) as in the following picture.



Then it is not difficult to show, considering also a similar situation in $B(x, 3r)$ and $B(y, 3r)$ with the corresponding selection of domains $D_i(3r) \in \{D^{\pm}(x, 3r), D^{\pm}(y, 3r)\}$, that whatever the positions of the D_i and $D_i(3r)$ with respect to the lines $P(x, 3r)$ and $P(y, 3r)$, one can always find a ball of radius equivalent to r that lies in the symmetric difference of Ω_1 and Ω_2. We conclude the proof by exchanging the role of Ω_1 and Ω_2 and using the same argument. $\square$

Finally we end this section with the following elementary covering lemma.

Lemma 10. *Let $\Omega \subset \mathbb{R}^N$ be an (ε, r_0)-Reifenberg-flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega) = L < +\infty$. Then for every $r < r_0/2$ we can extract among $\{B(x, r)\}_{x\in\partial\Omega}$ a subfamily of at most $L/(C_N r^{N-1})$ balls that forms a covering of $\bigcup_{x\in\partial\Omega} B(x, \frac{8}{10}r)$ where C_N is a dimensional constant. Moreover, for all x we have that*

$$\sharp\{i; x \in B_i\} \leq C \tag{2.4}$$

where C is again a dimensional constant.

Proof. Since $r < r_0$, we have that

$$d_H(\partial\Omega \cap B(x, r), P(x, r) \cap B(x, r)) \leq 10^{-2}r. \tag{2.5}$$

We also known that $\partial\Omega$ separates $D^+(x, r)$ from $D^-(x, r)$ and since the set of minimal $\mathcal{H}^{N-1}$ area having this property and satisfying (2.5) is the corresponding part of the hyperplane, we deduce that there exists a dimensional constant C_N such that for all $x \in \partial\Omega$ and all $r \leq r_0$

$$\mathcal{H}^{N-1}(\partial\Omega \cap B(x, r)) \geq C_N r^{N-1}.$$

Now let $B(x_i, r_i)$, be a subfamily of $\{B(x, r)\}_{x\in\partial\Omega}$ indexed by $i \in I$, maximal for the property that $\frac{1}{10}B_i \cap \frac{1}{10}B_j = \emptyset$. Using this fact (2.4) comes from a classical geometric argument in $\mathbb{R}^N$. Now we claim that $\sharp I$ is finite. Indeed, since $\frac{1}{10}B_i$ are disjoint balls we have

$$L \geq \mathcal{H}^{N-1}(\cup_{i\in I}\partial\Omega \cap \frac{1}{10}B_i) \geq \sharp I C_N r^{N-1} 10^{1-N}$$

thus

$$\sharp I \leq \frac{10^{N-1}L}{C_N r^{N-1}}.$$

Finally, it remains to prove that the family $\{B_i\}_{i\in I}$ forms a covering of $\bigcup_{x\in\partial\Omega} B(x, \frac{8}{10}r)$. Let $y \in \bigcup_{x\in\partial\Omega} B(x, \frac{8}{10}r)$ and let $x \in \partial\Omega$ be such that $y \in B(x, \frac{8}{10}r)$. Then by the maximality of the $\{B_i\}$, there exists a point $z \in \frac{1}{10}B_i \cap B(x, r/10)$. Then if x_i denotes the center of B_i, we have

$$d(y, x_i) \le d(y, x) + d(x, z) + d(z, x_i) \le \frac{8}{10}r + \frac{2}{10}r = r$$

which proves that $y \in B(x_i, r)$. □

3 Estimates close to the boundary for some functions in the Sobolev space

3.1 A Sobolev inequality at the boundary

We will need the following boundary version of the classical Sobolev inequality when Ω is a Reifenberg flat domain.

Proposition 11. *Let Ω be an (ε, r_0)-Reifenberg flat domain in $\mathbb{R}^N$ and $u \in W_0^{1,p}(\Omega)$ for some $p \ge 1$. Then for all $x \in \partial\Omega$ and $r \le r_0$ we have*

$$\|u\|_{L^p(B(x,r)\cap\Omega)} \le Cr\|\nabla u\|_{L^p(B(x,br)\cap\Omega)}$$

where $C := C(p, N)$ and $b := b(N)$.

Proof. The proof is a small modification of the classical proof of the Sobolev inequality that we will write here with full details for the convenience of the reader.

Without loss of generality, we may assume that $u \in C_0^1(\Omega)$, x is the origin and that $P(x, r)$ is the hyperplane $\{x_1 = 0\}$. We shall prove that

$$\|u\|_{L^p(\Omega\cap Q(x,r))} \le Cr\|\nabla u\|_{L^p(\Omega\cap Q(x,r))} \tag{3.1}$$

where $Q(x, r)$ is a cube centered at x, and with faces orthogonal to the axis of $\mathbb{R}^N$. Observe that (3.1) implies the desired inequality with constant b coming from the comparison between cubes and euclidian balls in $\mathbb{R}^N$.

By changing the orientation of x_1 we can assume that $Q(x, r) \cap \Omega$ (which is connected by our assumptions) contains the upper part $Q(x, r) \cap \{x_1 > \frac{r}{2}\}$.

It is clear that for any $u \in C_0^1(\Omega)$,

$$|u(x)| \le \int_{-\infty}^{x_1} |D_1 u| \,\mathrm{d}x_1 \le \int_{-\infty}^{r} |D_1 u| \,\mathrm{d}x_1.$$

Integrating over x_1 we obtain

$$\int_{-\infty}^{r} |u(x)| \, \mathrm{d}x_1 \leq 2r \int_{-\infty}^{r} |D_i u| \, \mathrm{d}x_1.$$

Now integrating the last inequality between $-r$ and r successively over each variable $x_2, ..., x_N$ we get

$$\begin{aligned}\int_{Q(x,r)\cap\Omega} |u(x)| \, \mathrm{d}x_1 &\leq 2r \int_{Q(x,r)\cap\Omega} |D_1 u| \, \mathrm{d}x \\ &\leq 2r \int_{Q(x,r)\cap\Omega} \|Du\| \, \mathrm{d}x.\end{aligned}$$

Then (3.1) follows if we apply this last inequality to u^p and use the Hölder's inequality. $\square$

Corollary 12. *Let Ω be an (ε, r_0)-Reifenberg flat domain in $\mathbb{R}^N$ and for $\delta \leq r_0/2$ set*

$$A_\delta := \Omega_1 \cap \{d(x, \partial\Omega_1) \leq \delta\}.$$

Then for any function $u \in W_0^{1,p}(\Omega)$ we have

$$\Big(\int_{A_\delta} |u|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} \leq C\delta\Big(\int_{A_{2b\delta}} |\nabla u|^p \, \mathrm{d}x\Big)^{\frac{1}{p}}$$

where b is the dimensional constant of Proposition 11.

Proof. Let $\{B_i\}_{i\in I}$ be the subfamily of balls $\{B(x, 2\delta)\}_{x\in\partial\Omega_1}$ given by Lemma 10. Then

$$\Omega_1 \cap \{x; d(x, \partial\Omega_1) \leq \delta\} \subset \bigcup_{x\in\partial\Omega_1} B(x, \frac{16}{10}\delta) \subset \bigcup_{i\in I} B_i.$$

Moreover the covering is bounded by a dimensional constant C. Then,

$$\int_{A_\delta} |u|^p \, \mathrm{d}x \leq \sum_{i\in I} \int_{B_i} |u|^p \, \mathrm{d}x$$

and using Proposition 11, together with the fact that the B_i are centered at $\partial\Omega_1$, we obtain

$$\begin{aligned}\int_{A_\delta} |u|^p \, \mathrm{d}x &\leq C \sum_{i\in I} \delta^p \int_{bB_i} |\nabla u|^p \, \mathrm{d}x \\ &\leq C\delta^p \int_{A_{2b\delta}} |\nabla u|^p \, \mathrm{d}x\end{aligned}$$

which proves the Corollary. $\square$

3.2 Boundary estimate on the gradient of eigenfunctions

Proposition 13. *Let Ω be an (ε, r_0)-Reifenberg flat domain in $\mathbb{R}^N$, and let u be an eigenfunction for the Dirichlet Laplacian in Ω, associated to the eigenvalue λ. Then for every $\beta > 0$ there is a constant C_0 depending on $N, |\Omega|$ and β such that for every $x \in \partial\Omega$ and for all $r \leq r_0$, we have that*

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2 \,\mathrm{d}x \leq C_0 \lambda \|u\|_{L^2(\Omega)} \left(\frac{r}{r_0}\right)^{N-\beta}. \tag{3.2}$$

Proof. For a given $\beta > 0$, define

$$a := 2^{\frac{2}{\beta}}. \tag{3.3}$$

Without loss of generality we assume that $r_0 = 1$ and $\|u\|_2 = 1$. Now let $x \in \partial\Omega$. We will obtain the appropriate decay by showing that for $k \in \mathbb{N}$ and a specific selection of the constant C_1 we have

$$\int_{B(x,a^{-k})\cap\Omega} |\nabla u|^2 dx \leq C_1 \lambda a^{-k(N-\beta)}. \tag{3.4}$$

We will prove (3.4) inductively. It is clear that (3.4) is true for $k = 0$ if $C_1 \geq 1$.

Suppose now that (3.4) is true for k and denote by v the "harmonic" replacement of u in $S_k := B(x, a^{-k}) \cap \Omega$; that is a harmonic function $v \in H^1(S_k)$ which satisfies $u - v \in H^1_0(S_k)$. Such a function v can be obtained by minimizing the Dirichlet integral, and since u is a competitor we have that

$$\int_{S_k} |\nabla v|^2 dx \leq \int_{S_k} |\nabla u|^2 dx \leq C_1 \lambda a^{-k(N-\beta)}$$

by the inductive hypothesis. On the other hand $w := |\nabla v|^2$ is subharmonic therefore by the classical mean value inequality applied to w we have

$$\int_{S_{k+1}} |\nabla v|^2 dx \leq a^{-N} \int_{S_k} |\nabla v|^2 dx,$$

that is

$$\int_{S_{k+1}} |\nabla v|^2 dx \leq C_1 \lambda a^{-N} a^{-k(N-\beta)}. \tag{3.5}$$

Now we want to estimate $\int_{S_k} |\nabla(u-v)|^2 \, \mathrm{d}x$. Notice that for all k, u is the unique solution of the problem

$$\begin{cases} -\Delta w = \lambda w \text{ in } S_k \\ w - u \in H_0^1(S_k) \end{cases}$$

therefore u is minimizing the energy

$$\frac{1}{2}\int_{S_k} |\nabla w|^2 \, \mathrm{d}x - \int_{S_k} \lambda u w \, \mathrm{d}x$$

among all functions w such that $u - w \in H_0^1(S_k)$. Therefore we deduce that

$$\frac{1}{2}\int_{S_k} |\nabla u|^2 - \lambda \int_{S_k} |u|^2 \quad \leq \quad \frac{1}{2}\int_{S_k} |\nabla v|^2 - \lambda \int_{S_k} uv$$

hence

$$\begin{aligned} \int_{S_k} |\nabla u|^2 - \int_{S_k} |\nabla v|^2 \quad &\leq \quad 2\lambda(\int_{S_k} |u|^2 - \int_{S_k} uv) \\ &\leq \quad C\lambda |S_k| \|u\|_\infty^2 \\ &\leq \quad \lambda C(N, |\Omega|) a^{-kN} \\ &\leq \quad \lambda C_2 a^{-kN} \end{aligned} \tag{3.6}$$

where $|v|$ was estimated in terms of $\|u\|_\infty$ by the maximum principle and $\|u\|_\infty \leq C(N, |\Omega|)$ by Proposition 4.

Now since v is harmonic in S_k and $u - v \in H_0^1(S_k)$, we deduce that ∇v and $\nabla(u-v)$ are orthogonal in $L^2(S_k)$ thus (3.6) and Pythagoras inequality imply

$$\int_{S_k} |\nabla u - \nabla v|^2 = \int_{S_k} |\nabla u|^2 - \int_{S_k} |\nabla v|^2 \leq \lambda C_2 a^{-kN}. \tag{3.7}$$

Gathering (3.5) and (3.7) together we complete the induction as follows:

$$\begin{aligned} \int_{S_{k+1}} |\nabla u|^2 dx \quad &\leq \quad 2\int_{S_{k+1}} |\nabla v|^2 + 2\int_{S_{k+1}} |\nabla(u-v)|^2 dx \\ &\leq \quad 2C_1 \lambda a^{-N} a^{-k(N-\beta)} + 2\lambda C_2 a^{-kN} \\ &\leq \quad C_1 \lambda a^{-(k+1)(N-\beta)} \end{aligned}$$

where the last inequality holds by the definition of a and provided for instance

$$C_1 \geq C_2 2^{\frac{2(N+1)}{\beta}}. \tag{3.8}$$

Now to finish the proof, since (3.4) is true, for every $r < 1$ one can find an integer k such that

$$r \leq a^{-k} < ar$$

thus

$$\int_{B(x,r)} |\nabla u|^2 \, \mathrm{d}x \leq \int_{B(x,a^{-k})} |\nabla u|^2 \, \mathrm{d}x \leq C_1 \lambda a^{-k(N-\beta)} \leq C_1 \lambda (ar)^{N-\beta}$$

so the Proposition is true with $C_0 := a^{N-\beta} C_1$. □

A consequence of the above proposition is the following.

Corollary 14. *Let Ω be an (ε, r_0)-Reifenberg flat domain in $\mathbb{R}^N$ such that $0 < \mathcal{H}^{N-1}(\partial\Omega_1) = L < +\infty$. Then for any $\alpha < 1$ there is a constant $C_1 := C_1(|\Omega|, r_0, N, \alpha)$ such that for any eigenfunction u for the Dirichlet Laplacian associated to the eigenvalue λ in Ω and for any $\delta \leq r_0/2$ we have*

$$\int_{\Omega_1 \cap \{d(x,\partial\Omega_1) \leq \delta\}} |\nabla u|^2 \, \mathrm{d}x \leq C_1 \lambda L \|u\|_{L^2(\Omega)} \delta^\alpha.$$

Proof. We argue as in Corollary 12. Let $\{B_i\}_{i \in I}$ be the subfamily of balls $\{B(x, 2\delta)\}_{x \in \partial\Omega_1}$ given by Lemma 10. We know that

$$\sharp I \leq L/(2^{N-1} C_N \delta^{N-1})$$

and that

$$\Omega_1 \cap \{x; d(x, \partial\Omega_1) \leq \delta\} \subset \bigcup_{x \in \partial\Omega_1} B(x, \frac{16}{10}\delta) \subset \bigcup_{i \in I} B_i.$$

Moreover the covering is bounded by a dimensional constant C. Then,

$$\int_{\Omega_1 \cap \{d(x,\partial\Omega_1) \leq \delta\}} |\nabla u|^2 \, \mathrm{d}x \;\; \leq \;\; \sum_{i \in I} \int_{B_i} |\nabla u|^2 \, \mathrm{d}x$$

and using Proposition 13 together with the fact that the B_i are centered at $\partial\Omega_1$ we obtain

$$\begin{aligned}
\int_{\Omega_1 \cap \{d(x,\partial\Omega_1) \leq \delta\}} |\nabla u|^2 \, \mathrm{d}x &\leq C\lambda \|u\|_{L^2(\Omega)} \sum_{i \in I} \delta^{N-1+\alpha} \\
&\leq C\lambda \|u\|_{L^2(\Omega)} \sharp I \delta^{N-1+\alpha} \\
&\leq CL\lambda \|u\|_{L^2(\Omega)} \delta^\alpha
\end{aligned}$$

where $C = C(r_0, N, \alpha, |\Omega|)$ and the proof is complete. □

4 An extension Lemma

In our approach we need the following extension Lemma for Sobolev functions in Reifenberg-flat domains. The proof relies on a Whitney extension which is now quite common. A first result of this kind is probably due to P. Jones in [12] which has been used by several authors, particulary by scientists working on quasiconformal maps (see for instance [2] and references therein). We would like to mention that [15] contains a Lemma very close to the following one but for Neumann extensions and for domains with cracks. One can also find a similar extension Lemma used together with a stopping time argument to prove some thin convergence results in [13], or a regularity result in [14].

Lemma 15. *Let Ω_1 and Ω_2 be two (ε, r_0)-Reifenberg flat domains such that*

$$d_H(\Omega_1^c, \Omega_2^c) \leq \delta \leq (100b)^{-1} r_0.$$

Set

$$A_\delta := \{x; d(x, \partial\Omega_1) \leq \delta\}.$$

Then for any $v \in W_0^{1,p}(\Omega_1)$ there exists a function $\tilde{v} \in W_0^{1,p}(\Omega_2)$ such that $v = \tilde{v}$ in $\Omega_1 \backslash A_{2\delta}$ and

$$\|\tilde{v}\|_{L^p(\Omega_2)} \leq \|v\|_{L^p(\Omega_1)} \tag{4.1}$$

$$\|\nabla \tilde{v}\|_{L^p(\Omega_2)} \leq \|\nabla v\|_{L^p(\Omega_1)} + C\|\nabla v\|_{L^p(A_{4b\delta})}. \tag{4.2}$$

Proof. Let $\{B_i\}_{i\in I}$ be the subfamily of balls $\{B(x, 2\delta)\}_{x\in\partial\Omega_1}$ given by Lemma 10. We will denote by x_i the center of B_i and r_i its radius. Since $\Omega_1 \triangle \Omega_2 \subset \bigcup_{i\in I} B_i$, to define a function $\tilde{v} \in W^{1,2}(\Omega_2)$, it is sufficient to define an extension of v in $\Omega_2 \cap \bigcup_{i\in I} B_i$.

For all i, define a function $\varphi_i \in C_c^1(5B_i)$, such that $\varphi_i = 1$ in $2B_i$, $|\nabla\varphi| \leq \delta^{-1}$ and let φ_0 be a function that is equal to 1 in $\Omega_1 \backslash \bigcup_{i\in I} 4B_i$, $\varphi_0 = 0$ in $\bigcup_{i\in I} 2B_i$ and $\varphi_0 + \sum_{j\in J}\varphi_j \geq 1$ in $\Omega_1 \cup \bigcup_{j\in J} 5B_i$. Moreover, we can assume that for all $x \in 4B_i \backslash 2B_i$, $|\nabla\varphi_0(x)| \leq \delta^{-1}$. Indeed, such a function φ_0 can be obtained by setting

$$\varphi_0(x) := \prod_{i\in I} l(d(x, x_i)/\delta)$$

where l is a Lipschitz function equal to 0 in $[0, 2]$, equal to 1 in $[4, +\infty)$ and $l'(x) \leq 1$. Finally, define

$$\theta_i := \frac{\varphi_i}{\varphi_0 + \sum_{i\in I}\varphi_i} \quad \text{ for } i \in I \cup \{0\}.$$

This allows us to obtain a partition of the unity in $\Omega_1 \cup \bigcup_{i\in I} 5B_i$.

Next we simply define $\tilde{v}$ by

$$\tilde{v}(x) := \theta_0(x)v(x) \tag{4.3}$$

in such a way that $\tilde{v}(x)$ vanishes on $\cup_{i\in I} 2B_i \supset \partial\Omega_2$. We claim that $\tilde{v} \in W_0^{1,p}(\Omega_2)$ and that (4.1), (4.2) are satisfied. The first estimate (4.1) comes directly from the fact that $\theta_0(x) \leq \chi_{\Omega_1}$. So we only have to prove (4.2), which will also imply that $\tilde{v} \in W^{1,p}(\Omega_2)$.

We have that

$$\nabla\tilde{v}(x) = v(x)\nabla\theta_0(x) + \theta_0(x)\nabla v(x)$$

thus

$$\begin{aligned} \|\nabla\tilde{v}(x)\|_{L^p(\Omega_2)} &\leq \|\nabla v(x)\chi_{supp(\theta_0)}\|_{L^p(\Omega_2)} + \|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)} \\ &\leq \|\nabla v(x)\|_{L^p(\Omega_1)} + \|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)} \end{aligned}$$

therefore it is enough to prove that

$$\|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)} \leq C\|\nabla v(x)\|_{L^p(A)} \tag{4.4}$$

with

$$A := \Omega_1 \cap \bigcup_{i\in I} 4bB_i \subset A_{4b\delta}.$$

On the other hand, from the construction of θ_0 we have

$$|\nabla\theta_0(x)| \leq \sum_{i\in I} \chi_{4bB_i}(x)\delta^{-1}. \tag{4.5}$$

Therefore, since the sum in (4.5) is locally finite we conclude that

$$\begin{aligned} \|v(x)\nabla\theta_0(x)\|^p_{L^p(\Omega_2)} &\leq \int\limits_{\Omega_2} \Big|v(x)\sum_{j\in J}\chi_{4B_i}(x)\delta^{-1}\Big|^p \\ &\leq C\sum_{i\in I}\delta^{-p}\int\limits_{4B_i}|v(x)|^p. \end{aligned} \tag{4.6}$$

Now since B_i is centered on $\partial\Omega_1$, from Proposition 11 we have

$$\int\limits_{4B_i} |v|^p \,\mathrm{d}x \leq C\delta^p \int\limits_{b4B_i} |\nabla u|^p \,\mathrm{d}x$$

so

$$\|v(x)\nabla\theta_0(x)\|^p_{L^p(\Omega_2)} \leq C\sum_{i\in I}\int\limits_{4B_i}|\nabla v(x)|^p \leq C\int\limits_{A_{4b\delta}}|\nabla v(x)|^p$$

which concludes the proof. $\square$

5 Mosco convergence and consequences

As in [15], the extension Lemma will imply the Mosco-convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ while Ω_n tends to Ω for the complementary Hausdorff distance. It is well known that this notion is equivalent to the γ-convergence of Ω_n to Ω which will in particular imply a stability result for eigenvalues. Actually this section will not be used in the proof of our main result but the authors would like to say a few words about those classical notions.

For $u \in H_0^1(\Omega)$ we will identify u and ∇u as function in $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N, \mathbb{R}^N)$ by extending them being zero outside Ω.

Definition 16 (Mosco-convergence)**.** *Let Ω_n and Ω be open subsets of $\mathbb{R}^N$. We say that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco if the following two properties hold:*

$(M1)$ *for every $u \in H_0^1(\Omega)$, there exists a sequence $u_n \in H_0^1(\Omega_n)$ such that u_n converges to u strongly in $L^2(\mathbb{R}^N)$ and ∇u_n converges to ∇u strongly in $L^2(\mathbb{R}^N, \mathbb{R}^N)$;*

$(M2)$ *if h_k is a sequence of indices converging to $+\infty$, u_k is a sequence such that $u_k \in H_0^1(\Omega_{h_k})$ for every k, and u_k converges weakly in $L^2(\mathbb{R}^N)$ to a function ϕ, while ∇u_k converges weakly in $L^2(\mathbb{R}^N, \mathbb{R}^N)$ to a function ψ, then $\varphi \in H_0^1(\Omega)$.*

The Mosco convergence is a great tool to study stability for elliptic problems. Indeed, for any bounded open set $\Omega \subset \mathbb{R}^N$ and any $f \in H^{-1}(\Omega)$ let us denote by $u_\Omega^f \in H_0^1(\Omega)$ the unique solution of the equation $-\Delta u = f$ in Ω.

Definition 17. *Let $D \subset \mathbb{R}^N$ be bounded. We say that the sequence of open sets $\Omega_n \subset D$ γ-converge to $\Omega \subset D$ if for any $f \in H^{-1}(D)$ we have that $u_{\Omega_n}^f$ strongly converges to u_Ω^f in $H_0^1(D)$.*

The following classical result shows the link between Mosco convergence and γ-convergence (see for instance Proposition 3.5.4 of [10]).

Proposition 18. *Ω_n γ-converges to Ω if and only if $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.*

In our situation, by the same argument as for Theorem 11 of [15], we can prove the following.

Theorem 19. *Let $r_0, \varepsilon > 0$ and let $\{\Omega_n\}_{n\in\mathbb{N}}$ and Ω be some (ε, r_0)-Reifenberg-flat domains. Assume that Ω_n converges to Ω for the complementary Hausdorff distance. Then $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.*

Proof. The proof is the same as for Theorem 11 in [15], using this time the extension Lemma for H^1_0 (Lemma 15). $\square$

A useful consequence of γ-convergence is the stability of eigenvalues, which is again very standard (see [10, 5]).

Proposition 20. *Let $r_0, \varepsilon > 0$ and let $\{\Omega_n\}_{n\in\mathbb{N}}$ and Ω be (ε, r_0)-Reifenberg-flat. Assume that Ω_n converges to Ω for the complementary Hausdorff distance. Then the k-th eigenvalue in Ω_n converges to the k-th eigenvalue in Ω.*

6 Quantitative Stability

We are now ready to prove Theorem 3. Notice that our theorem contains in particular a second proof of Proposition 20 for the case of the first eigenvalue.

Proof of Theorem 3. Let u_1 (resp. u_1') be an eigenfunction of unit norm associated to the first eigenvalue λ (resp. λ') in Ω (resp. Ω'). We denote by $\delta := d_H(\Omega^c, \Omega'^c)$. Let γ_1 be the first eigenvalue of the Laplacian in a ball contained in both Ω and Ω', in such a way that the inequality $\max(\lambda, \lambda') \leq \gamma_1$ holds by the monotonicity property for Dirichlet eigenvalues. We finally denote by $C := min(C_0, C_1)$ where C_0 and C_1 are the constants of Corollary 12 and Corollary 14, depending on N, α and $max(|\Omega|, |\Omega'|) \leq 10|\Omega|$.

We know that

$$\lambda := \inf_{u\in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \int_\Omega |\nabla u_1|^2.$$

Let $\tilde{u}_1' \in H^1_0(\Omega)$ be the extension of u_1' given by Lemma 15. Using Corollary 12 and Corollary 14 we obtain

$$\begin{aligned}
\lambda_1 = \int_\Omega |\nabla u_1|^2 \;&\leq\; \frac{\int_\Omega |\nabla \tilde{u}_1'|^2 \,\mathrm{d}x}{\int_\Omega |\tilde{u}_1'|^2 \,\mathrm{d}x} \\
&\leq\; \frac{\int_{\Omega'} |\nabla u_1'|^2 \,\mathrm{d}x + C\delta^\alpha}{1 - C\delta^\alpha} \\
&\leq\; \lambda_1' + \frac{C\delta^\alpha(1+\gamma_1)}{1 - C\delta^\alpha} \\
&\leq\; \lambda_1' + C\delta^\alpha
\end{aligned}$$

provided for instance that $\delta \leq (2C)^{-\alpha}$. Here C depends on γ_1, C_2, N, α, $|\Omega|$, and M. Then by the same argument and exchanging the role of λ_1 and λ_1' we get the desired inequality, namely

$$|\lambda_1 - \lambda_1'| \leq C\delta_1^\alpha.$$

We conclude the proof by observing that the estimate involving $|\Omega_1 \triangle \Omega_2|^{\frac{1}{N}}$ is a direct consequence of Lemma 9. $\square$

References

[1] G. Barbatis, V. I Burenkov, and P. D. Lamberti. Stability estimates for resolvents, eigenvalues and eigenfunctions of elliptic operators on variable domains. *Preprint.*

[2] J. Björn and N. Shanmugalingam. Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces. *J. Math. Anal. Appl.*, 332(1):190–208, 2007.

[3] V. I. Burenkov and P. D. Lamberti. Spectral stability of Dirichlet second order uniformly elliptic operators. *J. Differential Equations*, 244(7):1712–1740, 2008.

[4] V. I. Burenkov, P. D. Lamberti, and M. Lantsa de Kristoforis. Spectral stability of nonnegative selfadjoint operators. *Sovrem. Mat. Fundam. Napravl.*, 15:76–111, 2006.

[5] G. Buttazzo and G. Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.*, 122(2):183–195, 1993.

[6] S.-S. Byun and L. Wang. Elliptic equations with BMO nonlinearity in Reifenberg domains. *Adv. Math.*, 219(6):1937–1971, 2008.

[7] G. David, T. De Pauw, and T. Toro. A generalization of Reifenberg's theorem in $\mathbb{R}^3$. *Geom. Funct. Anal.*, 18(4):1168–1235, 2008.

[8] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[9] J. K. Hale. Eigenvalues and perturbed domains. In *Ten mathematical essays on approximation in analysis and topology*, pages 95–123. Elsevier B. V., Amsterdam, 2005.

[10] A. Henrot and M. Pierre. *Variation et optimisation de formes, Une analyse géométrique.* Mathématiques et Applications. Springer, 2005.

[11] D. Henry. *Perturbation of the boundary in boundary-value problems of partial differential equations*, volume 318 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005. With editorial assistance from Jack Hale and Antônio Luiz Pereira.

[12] P. W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.*, 147(1-2):71–88, 1981.

[13] A. Lemenant. Energy improvement for energy minimizing functions in the complement of generalized reifenberg-flat sets. *Preprint*, 2008.

[14] A. Lemenant. Regularity of the singular set for mumford-shah minimizers in $\mathbb{R}^3$ near a minimal cone. *Preprint*, 2008.

[15] A. Lemenant and E. Milakis. A stability result for nonlinear Neumann problems in Reifenberg flat domains in $\mathbb{R}^n$. *submitted paper*, 2009.

[16] E. Milakis and T. Toro. Divergence form operators in Reifenberg flat domains. *Mathematische Zeitschrift*, 2009.

[17] M. M. H. Pang. Approximation of ground state eigenvalues and eigenfunctions of Dirichlet Laplacians. *Bull. London Math. Soc.*, 29(6):720–730, 1997.

[18] E. R. Reifenberg. Solution of the Plateau Problem for m-dimensional surfaces of varying topological type. *Acta Math.*, 104:1–92, 1960.

[19] G. Savaré and G. Schimperna. Domain perturbations and estimates for the solutions of second order elliptic equations. *J. Math. Pures Appl. (9)*, 81(11):1071–1112, 2002.

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