Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces

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1 Introduction

The development of Geometric Measure Theory in metric spaces is still in a very early stage (but see [1], [2], [6], [8], [11], [19], [22], [30], [31], [36], [34], [38]), when compared to the relevant progress made by the Sobolev space theory in the last decade, see for instance the survey book by Hajłasz and Koskela [24]. In this paper we give a contribution in this direction, proving some fine properties of sets of finite perimeter in Ahlfors regular spaces in which a weak $(1,1)$-Poincaré inequality holds.

Specifically, we assume that (X,d) is a complete metric space and $\mu:\mathcal{B}(X)\to[0,\infty]$ is a Borel measure satisfying

$$a\varrho^k \le \mu(B_\varrho(x)) \le \tilde{a}\varrho^k \qquad \forall x\in X,\ \ \varrho\in(0,\operatorname{diam}X) \tag{1.1}$$

for suitable positive constants a, $\tilde{a}$, with $k>1$. Moreover, we assume the existence of constants $C_P\ge 0$ and $\lambda\ge 1$ such that

$$\int_{B_\varrho(x)} |u(y)-u_{x,\varrho}|\,d\mu(y) \le C_P\varrho\int_{B_{\lambda\varrho}(x)} |\nabla u|(y)\,d\mu(y) \tag{1.2}$$

whenever $u:X\to\mathbf{R}$ is a locally Lipschitz function and $|\nabla u|$ is an upper gradient of u according to Heinonen and Koskela [25]. Examples of spaces supporting a (weak) $(1,1)$-Poincaré inequality are Riemannian manifolds with lower bounds on the Ricci curvature, Carnot–Carathéodory groups and more generally (in the case of doubling spaces) Carnot–Carathéodory spaces associated to smooth (or locally Lipschitz) vector fields satisfying Hörmander's condition (see [3], [5], [16], [15], [20], [26], [29], [35], [36], [40] but the list is far from being exhaustive). Quite surprisingly, Laakso recently proved in [28] that for any $k\in(1,\infty)$ there exists a Ahlfors regular space of dimension k admitting a weak $(1,1)$-Poincaré inquality.

BV functions and sets of finite perimeter in the metric measure space (X,d,μ) can be defined by a relaxation procedure analogous to the one used for the definition of Sobolev spaces: namely $u\in L^1_{\rm loc}(X,\mu)$ belongs to $BV_{\rm loc}(X,\mu)$ if for any bounded open set $A\subset X$ there exists a sequence $(u_h)\subset \mathrm{Lip}_{\rm loc}(A)$ converging to u in $L^1_{\rm loc}(A)$ and satisfying

$$\limsup_{h\to\infty}\int_A |\nabla u_h|\,d\mu<\infty.$$

The least possible $\limsup$ above defines the variation $V(u,A)$ of u in A. For $E\subset X$, we say that E has locally finite perimeter in X if $P(E,A)<\infty$ for any bounded open set $A\subset X$, where $P(E,A)=V(\chi_E,A)$ (χ_E denotes the characteristic function of E). As shown in Example 3.2, *BV* functions in Carnot–Carathéodory spaces (see [17], [20] and [7]) fit exactly in this framework.

The functional properties of *BV* functions and sets of finite perimeter have been studied by Miranda in [31]; we recall in Theorem 3.3 the main results we need from that paper. Here, having in mind the classical De Giorgi rectifiability theorem for Euclidean sets of finite perimeter [10] (also called Caccioppoli sets, after the pioneering work [4]), we tackle the following two problems:

(a) representation of the perimeter by the Hausdorff $(k-1)$-dimensional measure $\mathcal{H}^{k-1}$;

(b) density bounds on volume and perimeter.

Concerning (a), we prove in Theorem 4.2 that

$$P(E,A)=\int_{A\cap\partial^* E}\theta\,d\mathcal{H}^{k-1}\qquad\text{for all } A\subset X \text{ open}\tag{1.3}$$

where $\partial^* E$ is the essential boundary of E, i.e. the set of points where the volume density of E is neither 0 nor 1. Moreover, the function θ is uniformly bounded from below (depending on $(k,a,\tilde{a},C_P,\lambda)$). The proof of this fact strongly depends on a relative isoperimetric inequality which can be deduced from (1.2) and a Sobolev emdedding theorem of Hajłasz and Koskela (see Theorem 5.1 in [24]).

The proof of volume bounds

$$\liminf_{\varrho\downarrow 0}\min\left\{\frac{\mu(B_\varrho(x)\cap E))}{\mu(B_\varrho(x))},\frac{\mu(B_\varrho(x)\setminus E)}{\mu(B_\varrho(x))}\right\}\geq\tau(k,a,\tilde{a},C_P,\lambda)>0\quad \mathcal{H}^{k-1}\text{-a.e. in } \partial^* E\tag{1.4}$$

requires a more sophisticated analysis and is carried on in Theorem 4.3. One of the technical difficulties, already pointed out in [19], is that no Vitali theorem holds for a generic measure ν in an Ahlfors regular space (X,d,μ). We get rid of this technical difficulty proving (as we did for the mass of metric currents in [2]) that the perimeter measure has indeed some special properties. Indeed, since we know that this measure is absolutely continuous with respect to $\mathcal{H}^{k-1}$ (see the simple proof of this fact in Lemma 4.1), we can use a Vitali-type covering theorem well adapted to the Hausdorff measures, stated in Theorem 2.1.

Another important ingredient in the proof of (1.4) is the fact that any set E of finite perimeter, when seen at small scales around $P(E,\cdot)$-a.e. point, behaves as a quasi-minimizer of the perimeter, i.e., the perimeter increases in a controlled way under local perturbations (see Proposition 4.4 for a precise statement). This is an instance of a general phenomenon, related to local, additive and lower semicontinuous energies, first used (to the author's knowledge) by Federer and Fleming in [14] and then, in a more explicit form, in [41]. This general principle appears also in Cheeger's recent work [6] on Rademacher theorem in doubling spaces with a weak $(1,1)$-Poincaré inequality (1.2).

Point (a) and (b) actually are only a part of De Giorgi's program. Indeed, he was also able to prove that $\partial^* E$ is countably $\mathcal{H}^{k-1}$-rectifiable, i.e., that $\mathcal{H}^{k-1}$-almost all of $\partial^* E$ can be covered by a sequence of C^1 embedded hypersurfaces of $\mathbf{R}^k$. In our setting, as a byproduct of (1.4) and the relative isoperimetric inequality, we obtain that the perimeter measure is a.e. asymptotically doubling, i.e.

$$\limsup_{\varrho\downarrow 0}\frac{P(E,B_{2\varrho}(x))}{P(E,B_\varrho(x))}<\infty\qquad\text{for } P(E,\cdot)\text{-a.e. } x\in X.$$

This implies, by Theorem 2.8.17 of [13], that the spherical differentiation theory can be done using the perimeter measure, thus initiating De Giorgi's approach (based on a blow-up procedure) to rectifiability.

Indeed, the doubling result of this paper is used by Franchi, Serapioni and Serra Cassano to prove that, for any set E of finite perimeter in the Heisenberg group $\mathbf{H}_n$, the essential boundary $\partial^* E$ is $\mathbf{H}$-rectifiable, i.e. $\mathcal{H}^{2n+1}$-almost all of $\partial^* E$ can be covered by a sequence of C^1-hypersurfaces. In this setting, a C^1 hypersurface is understood as a noncritical level sets of an intrinsinc C^1 function $f : \mathbf{H}_n \to \mathbf{R}$ (see also [1], [8], [19], [37] for an explaination of why the canonical rectifiability concept is not suitable in this situation). Moreover, the constant τ in (1.4) is $1/2$ and so the lim inf is a limit and the density of E is $1/2$. This task requires in [19] the development of new analytic tools in the Heisenberg group, as the implicit function theorem and a suitable version of the Whitney extension theorem. Actually the author's initial interest in this topic was motivated by [19], where this rectifiability result was first proved for a particular class of sets of finite perimeter.

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2 Notation and preliminary results

We denote the open ball $\{y \in X : d(x,y) < \varrho\}$ by $B_\varrho(x)$ and, with a slight abuse of notation, the closed ball $\{y \in X : d(x,y) \leq \varrho\}$ by $\overline{B}_\varrho(x)$. We use the notation $\mathcal{B}(X)$ for the Borel σ-algebra of X and $\mathrm{Lip}_{\mathrm{loc}}(X)$ for the space of real valued Lipschiz functions on bounded subsets of X. More generally, whenever $A \subset X$ is an open set, by $u \in L^1_{\mathrm{loc}}(A)$ we mean that $u \in L^1(C)$ for any bounded and closed set $C \subset A$; the same convention applies to other functions spaces, convergence, and so on.

The *Hausdorff α-dimensional measure* in X will be denoted by $\mathcal{H}^\alpha$. We recall (see for instance [39]) that $\mathcal{H}^\alpha(B)$ is defined for any set $B \subset X$ by $\sup_{\delta>0} \mathcal{H}^\alpha_\delta(B)$, where

$$\mathcal{H}^\alpha_\delta(B) := \frac{\omega_\alpha}{2^\alpha} \inf \left\{ \sum_{i\in I} [\mathrm{diam}(B_i)]^\alpha : \ \mathrm{diam}\,(B_i) < \delta, \ B \subset \bigcup_{i\in I} B_i \right\}$$

and $\omega_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$ (here $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}\,ds$ is the Euler Γ function) is the Lebesgue measure of the unit ball of $\mathbf{R}^\alpha$ if α is an integer.

For $E \subset X$, we denote by $E^c = X \setminus E$ the complement of E and by χ_E the characteristic function of E. If E is a Borel set, we denote the volume $\mu(E \cap B_\varrho(x))$ of E in $B_\varrho(x)$ by $m_E(x,\varrho)$. Moreover, $\partial^* E$ stands for the *essential boundary* of E, i.e. $x \in \partial^* E$ if

$$\text{neither } \lim_{\varrho\downarrow 0} \frac{m_E(x,\varrho)}{\mu(B_\varrho(x))} = 0 \qquad \text{nor } \lim_{\varrho\downarrow 0} \frac{m_{E^c}(x,\varrho)}{\mu(B_\varrho(x))} = 0.$$

We will use a classical covering theorem (see for instance Theorem 1.10 of [12]) well adapted to the Hausdorff measures.

Theorem 2.1 (Vitali covering theorem) *Let (X,d) be a metric space. Let $\mathcal{F}$ a family of closed balls and $K \subset X$ be such that:*

(i) for any $x \in K$ and any $\delta > 0$ the set

$$\left\{\varrho \in (0,\delta) : \ \overline{B}_\varrho(x) \in \mathcal{F}\right\}$$

is not empty;

(ii) there exist $\alpha > 0$ and a positive finite measure ν in $(X, \mathcal{B}(X))$ such that

$$\nu\left(\overline{B}_\varrho(x)\right) \geq \varrho^\alpha \qquad \forall\, \overline{B}_\varrho(x) \in \mathcal{F}.$$

Then, there exists a disjoint collection $\mathcal{G} \subset \mathcal{F}$ such that

$$\mathcal{H}^\alpha\left(K \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0. \tag{2.1}$$

A simple and well known consequence of the above covering theorem (see for instance [13], 2.10.19) is the following implication

$$\limsup_{\varrho\downarrow 0} \frac{\nu(B_\varrho(x))}{\omega_\alpha \varrho^\alpha} \geq t \quad \forall x \in B \qquad \Longrightarrow \qquad \nu(B) \geq t\mathcal{H}^\alpha(B). \tag{2.2}$$

for any locally finite measure ν in X and any $B \in \mathcal{B}(X)$. Letting $t \uparrow \infty$ in (2.2) we obtain also

$$\limsup_{\varrho\downarrow 0} \frac{\nu(B_\varrho(x))}{\omega_\alpha \varrho^\alpha} < \infty \qquad \mathcal{H}^\alpha\text{-a.e. in } X. \tag{2.3}$$

Now we recall some basic facts about upper gradients and Poincaré inequalities. We say that a metric measure space (X,d,μ) supports a *weak (1,1)-Poincaré inequality* if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that

$$\int_{B_\varrho(x)} |u(y) - u_{x,\varrho}|\, d\mu(y) \leq C_P \varrho \int_{B_{\lambda\varrho}(x)} |\nabla u|(y)\, d\mu(y) \tag{2.4}$$

for any ball $B_\varrho(x)$ with $\varrho < \operatorname{diam} X$ and any $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$. Here

$$u_{x,\varrho} := \frac{1}{\mu(B_\varrho(x))} \int_{B_\varrho(x)} u(y)\, d\mu(y)$$

is the mean value of u in $B_\varrho(x)$ and $|\nabla u|$ is any *upper gradient* of u according to Heinonen and Koskela (see [25]). By a remarkable result of Cheeger (see [6], Theorem 6.1), under assumption (1.1) and (2.4) the formula

$$|\nabla u|(x) := \limsup_{\varrho\downarrow 0} \frac{1}{\varrho} \sup_{y\in B_\varrho(x)} |u(y) - u(x)|$$

provides a minimal upper gradient of u whenever $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$ (i.e. any other upper gradient is larger than $|\nabla u|$ μ-a.e. in X).

We say that (X,d,μ) supports a $(1,1)$-Poincaré inequality if (2.4) holds with $\lambda = 1$.

3 Sets of finite perimeter

In this section we recall the main properties of sets of finite perimeter which will be useful in the sequel. According to [31] (see also [17], [20] and [7]) we define the class of sets of finite perimeter and the perimeter measure by a relaxation procedure, using as energy functional the L^1 norm of the (minimal) upper gradient.

Definition 3.1 (Perimeter) *Let $E \in \mathcal{B}(X)$ and $A \subset X$ open. The perimeter of E in A, denoted by $P(E, A)$, is defined by*

$$P(E,A) := \inf \left\{ \liminf_{h\to\infty} \int_A |\nabla u_h| \, d\mu : \ (u_h) \subset \mathrm{Lip}_{\mathrm{loc}}(A), \ \ u_h \to \chi_E \ \ in \ L^1_{\mathrm{loc}}(A) \right\}.$$

We say that E has finite perimeter in X if $P(E, X) < \infty$.

More generally, the *variation* $V(u, A)$ of $u \in L^1_{\mathrm{loc}}(X)$ in A can be defined by

$$V(u,A) := \inf \left\{ \liminf_{h\to\infty} \int_A |\nabla u_h| \, d\mu : \ (u_h) \subset \mathrm{Lip}_{\mathrm{loc}}(A), \ \ u_h \to u \ \text{ in } L^1_{\mathrm{loc}}(A) \right\}.$$

Accordingly $BV_{\mathrm{loc}}(X)$ denotes the space of all functions $u \in L^1_{\mathrm{loc}}(X)$ such that $V(u, A) < \infty$ for any bounded open set $A \subset X$. Notice that $P(E, A) = V(\chi_E, A)$.

Example 3.2 (Carnot–Carathéodory spaces) Let $\Omega \subset \mathbf{R}^n$ be an open connected set, let $Y_1, \ldots, Y_k$ be locally Lipschitz vector fields defined in Ω, and let ρ be the Carnot–Carathéodory distance induced by (Y_i). Then, assuming that $\rho(x, y) < \infty$ whenever x, $y \in \Omega$, the function

$$|Yu| := \left(|Y_1 u|^2 + \ldots + |Y_k u|^2\right)^{1/2}$$

is a minimal upper gradient of u whenever $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \rho)$ (see the discussion in §11.2 of [24]). Then, the definitions of BV functions and sets of finite perimeter adopted in [17], [20] and [7] are equivalent to the one adopted here (with $X = \Omega$, $d = \varrho$ and μ equal to the Lebesgue measure). We have also $V(u, A) = |D_Y u|(A)$, where $Du = (D_{Y_1} u, \ldots, D_{Y_k} u)$ is the distributional derivative of u along Y, i.e.

$$\int_\Omega \phi Y_i u \, dx = -\int_\Omega Y_i \phi \, dD_{Y_i} u \qquad \forall \phi \in C^\infty_c(\Omega), \ i = 1, \ldots, k.$$

In particular this equivalence holds for sets of finite perimeter in the Heisenberg group, whose fine properties are discussed in [19].

The following properties easily follow from the definition of perimeter (see [31]). In (a) and in the sequel we use the notation $E\Delta F$ for the symmetric difference of E and F.

(a) (locality) $P(E, A) = P(F, A)$ whenever $(E\Delta F) \cap A$ is μ-negligible;

(b) (lower semicontinuity) $E \mapsto P(E,A)$ is lower semicontinuous with respect to the $L^1_{\rm loc}(A)$ topology;

(c) (subadditivity) $P(E\cup F,A)+P(E\cap F,A)\leq P(E,A)+P(F,A)$;

(d) (complementation) $P(E,A)=P(E^c,A)$.

By (c) and (d) it follows that

$$\max\{P(E\cup F,A),P(E\cap F,A),P(E\setminus F,A)\}\leq P(E,A)+P(F,A). \tag{3.1}$$

Theorem 3.3 *Let E be a set of finite perimeter in X. Then:*

(i) the set function $A\mapsto P(E,A)$ is the restriction to the open subsets of X of a finite Borel measure $P(E,\cdot)$ in X, defined by

$$P(E,B):=\inf\{P(E,A):\ A\supset B,\ A \text{ open}\}\qquad \forall B\in\mathcal{B}(X);$$

(ii) if (X,d,μ) supports the weak $(1,1)$-Poincaré inequality (2.4), the following relative isoperimetric inequality holds:

$$\min\{m_E(x,\varrho),m_{E^c}(x,\varrho)\}\leq C_I\left[P(E,B_{2\lambda\varrho}(x))\right]^{k/(k-1)}; \tag{3.2}$$

(iii) for any $u\in {\rm Lip}_{\rm loc}(X)$ the following coarea formula holds:

$$\int_{\mathbf{R}}P(\{u>t\},B)\,dt=V(u,B)\leq\int_B|\nabla u|\,d\mu\qquad \forall B\in\mathcal{B}(X). \tag{3.3}$$

Proof. Properties (i), (ii), (iii) are proved in [31]. We repeat, for the reader's convenience, only the proof of (ii). By a well known result of Hajłasz and Koskela (see [24], Theorem 5.1), (2.4) and (1.1) imply a weak $(1^*,1)$-Poincaré inequality, i.e.

$$\left(\int_{B_\varrho(x)}|u(y)-u_{x,\varrho}|^{k/(k-1)}\,d\mu(y)\right)^{(k-1)/k}\leq C\int_{B_{2\lambda\varrho}(x)}|\nabla u|\,d\mu$$

for any $u\in{\rm Lip}(B_{2\lambda\varrho}(x))$. Taking into account the definition of $P(E,B_{2\lambda\varrho}(x))$, and noticing that by a truncation argument we need only to consider sequences (u_h) converging to χ_E in $L^1(B_{2\lambda\varrho}(x))$, we obtain

$$\left(\int_{B_\varrho(x)}|\chi_E(y)-(\chi_E)_{x,\varrho}|^{k/(k-1)}\,d\mu(y)\right)^{(k-1)/k}\leq CP(E,B_{2\lambda\varrho}(x)).$$

Then, (3.2) follows with $C_I=(2C)^{k/(k-1)}$. □

Remark 3.4 By a similar argument, we have

$$\min\{m_E(x,\varrho), m_{E^c}(x,\varrho)\} \le C_I \left[P(E, B_\varrho(x))\right]^{k/(k-1)} \tag{3.4}$$

whenever (X,d,μ) supports a $(1,1)$-Poincaré inequality. In the Heisenberg case, this inequality was proved first in [35].

Finally, we will need the following canonical relation between perimeter and derivative of volume. The proof uses the lower semicontinuity of the variation, an approximation argument and the fact, still proved in [31], that also for $u \in BV_{\mathrm{loc}}(X)$ the variation $V(u,\cdot)$ is the trace on open sets of a locally finite measure.

Lemma 3.5 (Localization) *Let E be a set of finite perimeter in X and $x \in X$. For a.e. $\varrho > 0$ the set $E \setminus B_\varrho(x)$ has finite perimeter in X and satisfies*

$$P(E \setminus B_\varrho(x), \partial B_\varrho(x)) \le \frac{d}{dr} m_E(x,r)\bigg|_{r=\varrho}.$$

PROOF. Let $\varrho > 0$ be such that $t \mapsto m_E(x,t)$ is differentiable at $t = \varrho$ (in particular $\mu(\partial B_\varrho(x)) = 0$). By approximation with Lipschitz functions, the following inequality

$$V(uf, B_{\varrho+\delta}(x)) \le \int_{B_{\varrho+\delta}(x)} |u||\nabla f|\, d\mu + \int_{\overline{B}_{\varrho+\delta}} |f|\, dV(u,\cdot)$$

holds whenever $u \in BV_{\mathrm{loc}}(X)$, $\delta > 0$ and $f \in \mathrm{Lip}_{\mathrm{loc}}(X)$. For $\epsilon > 0$, let $f_\epsilon(y) = 1$ for $d(y,x) \ge \varrho + \epsilon$, $f_\epsilon(y) = 0$ for $d(y,x) \le \varrho$ and $f_\epsilon(y) = (d(y,x) - \varrho)/\epsilon$ in the remaining cases. Insert $f = f_\epsilon$ and $u = \chi_E$ in the inequality above and let $\epsilon \downarrow 0$ to obtain

$$P(E \setminus B_\varrho(x), B_{\varrho+\delta}(x)) \le \liminf_{\epsilon \downarrow 0} \frac{m_E(x,\varrho+\epsilon) - m_E(x,\varrho)}{\epsilon} + P(E, \overline{B}_{\varrho+\delta} \setminus \overline{B}_\varrho(x)).$$

Letting $\delta \downarrow 0$ the proof is achieved. □

4 Representation of perimeter and doubling property

Throughout this section we assume that (X,d,μ) fulfils (1.1), that the weak $(1,1)$-Poincaré inequality (2.4) holds, and that E is a set of finite perimeter in X. As a consequence, the relative isoperimetric inequality (3.2) holds. We also notice that (1.1) implies that any ball $B_\varrho(x)$ has finite $\mathcal{H}^k$-measure (less than $\omega_k \varrho^k / a$) and therefore, for x fixed, we have $\mathcal{H}^{k-1}(\partial B_\varrho(x)) < \infty$ with at most countably many exceptions (see [13], Theorem 2.10.25).

We start by proving the absolute continuity of $P(E,\cdot)$ with respect to $\mathcal{H}^{k-1}$; in the proof of this fact only (1.1) is involved.

Lemma 4.1 (Absolute continuity) *We have $P(E,B) = 0$ whenever $B \in \mathcal{B}(X)$ is $\mathcal{H}^{k-1}$-negligible.*

PROOF. Assuming with no loss of generality that B is a compact set, for any $\epsilon > 0$ we can cover B by a finite number of balls B_i^ϵ of radius r_i^ϵ and center x_i^ϵ such that $\sum_i (r_i^\epsilon)^{k-1} < \epsilon$. By (3.3) with $u = d(\cdot, x_i^\epsilon)$ and $A = B_{2r_i^\epsilon}(x_i^\epsilon)$ we can find concentric balls $\hat{B}_i^\epsilon \supset B_i^\epsilon$ with radius at most $2r_i^\epsilon$ such that

$$P(\hat{B}_i^\epsilon, X) \le 2^k \tilde{a} (r_i^\epsilon)^{k-1}$$

Denoting by $A_\epsilon \supset B$ the union of the balls $\hat{B}_i^\epsilon$, by locality and subadditivity of perimeter we get

$$P(E \cup A_\epsilon, X) = P(E \cup A_\epsilon, X \setminus B) \le P(E, X \setminus B) + 2^k \tilde{a} \epsilon.$$

Since $\mu(A_\epsilon) \le 2^k \tilde{a} \epsilon^{k/(k-1)} \to 0$, passing to the limit as $\epsilon \downarrow 0$ the lower semicontinuity of perimeter gives

$$P(E, X) \le P(E, X \setminus B)$$

whence $P(E, B) = 0$. □

Now we prove that $P(E, \cdot)$ is representable by integration of $\mathcal{H}^{k-1}$ on $\partial^* E$; moreover, at $\mathcal{H}^{k-1}$-a.e. point of $\partial^* E$ we have lower bounds on $m_E(x, \varrho)$ and $m_{E^c}(x, \varrho)$ for arbitrarily small radii ϱ. The volume lower bounds will be improved in Theorem 4.3.

Theorem 4.2 (Hausdorff representation of perimeter) *The measure $P(E, \cdot)$ is concentrated on the set*

$$\Sigma_c := \left\{ x : \limsup_{\varrho \downarrow 0} \varrho^{-k} \min\{m_E(x, \varrho), m_{E^c}(x, \varrho)\} \ge c \right\} \subset \partial^* E$$

with $c > 0$ depending only on $(k, a, \tilde{a}, \lambda, C_I)$. Moreover $\partial^* E \setminus \Sigma_c$ is $\mathcal{H}^{k-1}$-negligible, $\mathcal{H}^{k-1}(\partial^* E) < \infty$ and

$$P(E, B) = \int_{B \cap \partial^* E} \theta \, d\mathcal{H}^{k-1} \qquad \forall B \in \mathcal{B}(X)$$

for some Borel function $\theta : X \to [c', \infty)$, with $c' = (c/C_I)^{(k-1)/k} / \omega_{k-1}$.

PROOF. We prove that $P(E, K) = 0$ for any compact set $K \subset X \setminus \Sigma_c$. By Egorov theorem we can assume the existence of $r_0 > 0$ such that

$$\min\{m_E(x, \varrho), m_{E^c}(x, \varrho)\} < c\rho^k \qquad \forall x \in K, \ \varrho \in (0, r_0).$$

We define

$$\underline{m}_E(x, \varrho) := \frac{2}{\varrho} \int_{\varrho/2}^{\varrho} m_E(x, \tau) \, d\tau$$

and notice that $\underline{m}_E(x, \varrho) \le m_E(x, \varrho) \le \underline{m}_E(x, 2\varrho)$ and that $\varrho \mapsto \underline{m}_E(x, \varrho)$ is continuous. By a continuity argument either $\underline{m}_E(x, \varrho) < c\varrho^k$ in $(0, r_0)$ or $\underline{m}_{E^c}(x, \varrho) < c\varrho^k$ in $(0, r_0)$ and not both, provided $c < a/2^{k+2}$. Hence, possibly splitting K in two parts and replacing E by E^c, we can assume that $\underline{m}_E(x, \varrho) < c\varrho^k$ in $(0, r_0)$, hence $m_E(x, \varrho) < 2^k c\varrho$ in $(0, r_0/2)$.

Let $r \in (0, r_0/4)$ and let $x_1, \ldots, x_n \in K$ be recursively chosen in such a way that $d(x_i, x_j) \geq r$ for $i \neq j$ and $K \subset \cup_i B_r(x_i)$. We can find $\rho_i \in (r, 2r)$ such that $\mathcal{H}^{k-1}(\partial B_{\rho_i}(x_i))$ is finite and

$$r \frac{d}{d\varrho} m_E(x, \rho)\Big|_{\varrho=\varrho_i} \leq m_E(x, 2r) \leq C_I \left[P(E, B_{2\lambda r}(x_i))\right]^{k/(k-1)}.$$

We can also choose recursively ρ_i in such a way that $\mathcal{H}^{k-1}(\partial B_{\rho_i}(x_i) \cap \partial B_{\rho_j}(x_j)) = 0$ whenever $i \neq j$. By Lemma 3.5 and (3.2) we get

$$P(E \setminus B_{\rho_i}(x_i), \partial B_{\rho_i}(x_i)) \leq \frac{1}{r} m_E(x, 2r) \leq 4c^{1/k} C_I^{(k-1)/k} P(E, B_{2\lambda r}(x_i)).$$

Now we estimate the overlapping of the balls $B_{2\lambda r}(x_i)$. Let $x \in X$ be in all balls $B_{2\lambda r}(x_i)$, $i \in J$. Taking into account that $d(x_i, x_j) \geq r$, we obtain that the balls $B_{r/2}(x_i)$ are pairwise disjoint and, for $i \in J$, contained in $B_{(2\lambda+1)r}(x)$, hence the cardinality of J is at most $\xi = 2^k (2\lambda + 1)^k \tilde{a}/a$.

Then, setting $A_r = \cup_i B_{2\lambda r}(x_i)$, subadditivity and locality of perimeter give

$$\begin{aligned}
P(E \setminus A_r, X) &= P(E \setminus A_r, X \setminus A_r) = P(E, X \setminus \overline{A}_r) + P(E \setminus A_r, \partial A_r) \\
&\leq P(E, X \setminus K) + \sum_{i=1}^n P(E \setminus A_r, \partial B_{\rho_i}(x_i)) \\
&\leq P(E, X \setminus K) + \sum_{i=1}^n P(E \setminus B_{\rho_i}(x_i), \partial B_{\rho_i}(x_i)) + P(\bigcup_{j \neq i} B_{\rho_j}(x_i), \partial B_{\rho_i}(x_i)) \\
&\leq P(E, X \setminus K) + 4c^{1/k} C_I^{(k-1)/k} \xi P(E, A_r).
\end{aligned}$$

In the last line we have used Lemma 4.1 and the fact that $\partial B_{\rho_i}(x_i) \cap \partial B_{\rho_j}(x_j)$ is $\mathcal{H}^{k-1}$-negligible for $i \neq j$. Since A_r is contained in the $2r$-neighbourhood of K and

$$\mu(E \cap A_r) \leq \sum_{i=1}^n m_E(x_i, 2r) \leq 4rc^{1/k} C_I^{(k-1)/k} \xi P(E, X) \to 0,$$

passing to the limit as $r \downarrow 0$ we obtain

$$P(E, X) \leq P(E, X \setminus K) + 4c^{1/k} C_I^{(k-1)/k} \xi P(E, K).$$

Thus, $P(E, K) = 0$ provided $4c^{1/k} C_I^{(k-1)/k} \xi < 1$.

This proves that $P(E, \cdot)$ is concentrated on Σ_c. By (3.2) we get

$$\limsup_{\varrho \downarrow 0} \frac{P(E, B_\varrho(x))}{\varrho^{k-1}} \geq (c/C_I)^{(k-1)/k}$$

hence (2.2) gives $\omega_{k-1} P(E, B) \geq (c/C_I)^{(k-1)/k} \mathcal{H}^{k-1}(B)$ for any Borel set $B \subset \Sigma_c$. By Lemma 4.1 and Radon–Nikodým theorem it follows that that

$$P(E, B) = \int_{B \cap \Sigma_c} \theta \, d\mathcal{H}^{k-1} \qquad \forall B \in \mathcal{B}(X)$$

for some Borel function $\theta : X \to [c', \infty)$.

It remains to prove that $\partial^* E \setminus \Sigma_c$ is $\mathcal{H}^{k-1}$-negligible. By (2.2) with $\nu = P(E, \cdot)$ we know that $P(E, B_\varrho(x)) = o(\varrho^{k-1})$ $\mathcal{H}^{k-1}$-a.e. in $X \setminus \Sigma_c$, because ν is concentrated on Σ_c. The relative isoperimetric inequality gives

$$\min\{\underline{m}_E(x,\varrho), \underline{m}_{E^c}(x,\varrho)\} = o(\varrho^k) \qquad \text{for } \mathcal{H}^{k-1}\text{-a.e. } x \in X \setminus \Sigma_c$$

and, by a continuity argument, either $\underline{m}_E(x,\varrho) = o(\varrho^k)$ or $\underline{m}_{E^c}(x,\varrho) = o(\varrho^k)$ (thus $x \notin \partial^* E$) for $\mathcal{H}^{k-1}$-a.e. $x \in X \setminus \Sigma_c$. $\square$

Now we prove a lower density estimate for both perimeter and area and, as a consequence, the asymptotic doubling property. The main ingredient in the proof is the asymptotic quasi minimality stated in Proposition 4.4 below, which provides a lower bound for the derivative of $\varrho \mapsto [m_E(x,\varrho)]^{1/k}$.

Theorem 4.3 *The measure $P(E, \cdot)$ satisfies*

$$\infty > \limsup_{\varrho \downarrow 0} \frac{P(E, B_\varrho(x))}{\varrho^{k-1}} \geq \liminf_{\varrho \downarrow 0} \frac{P(E, B_\varrho(x))}{\varrho^{k-1}} > \tau_1 \qquad \textit{for } P(E,\cdot)\textit{-a.e. } x \in X \tag{4.1}$$

$$\liminf_{\varrho \downarrow 0} \varrho^{-k} \min\{m_E(x,\varrho), m_{E^c}(c,\varrho)\} > \tau_2 \qquad \textit{for } P(E,\cdot)\textit{-a.e. } x \in X \tag{4.2}$$

with τ_1, $\tau_2 > 0$ depending only on $(k, a, \tilde{a}, \lambda, C_I)$.

PROOF. The upper estimate in (4.1) follows by (2.3) and Lemma 4.1.

In the proof of the lower estimate in (4.2) we can assume that (X,d) is a *length* space, i.e., that any pair of points x, $y \in X$ can be connected by a rectifiable curve of length $d(x,y)$. Indeed, by [38] (see also the appendix in [6]) Ahlfors regularity and weak $(1,1)$-Poincaré inequality imply that (X,d) is quasi-convex, i.e. there exists a constant C depending only on a, $\tilde{a}$, C_I and λ such that any pair of points x, $y \in X$ can be connected by a rectifiable curve of length at most $Cd(x,y)$. Hence, being the statement of the theorem bi-Lipschitz invariant, we can simply replace d by the geodesic metric

$$\tilde{d}(x,y) := \inf\left\{\sum_{i=1}^{n-1} d(x_{i+1}, x_i) : \ x_1 = x, \ x_n = y\right\}$$

associated to d. Since any ball in a length space is a John domain, from Corollary 9.8 in [24] we obtain the $(1^*,1)$-Poincaré inequality. In particular, by Remark 3.4, the relative isoperimetric inequality (3.4) follows.

We fix a positive number d such that

$$d < \min\left\{c, (\alpha/k)^k, a/2\right\},$$

with $\alpha^{-1} = 2C_I^{(k-1)/k}$. By Proposition 4.4 below, we need only to prove the lower estimate in (4.1) for any compact set $K \subset X$ where the following property holds: there exists $\varrho_0 > 0$

such that for any $x \in K$ and a.e. $\varrho \in (0, \varrho_0)$, the volume bounds $a\varrho^k/2 \geq m_E(x,\varrho) > d\varrho^k$ imply

$$P(E, B_\varrho(x)) \leq 2P(E \setminus B_\varrho(x), \partial B_\varrho(x)).$$

In particular, the relative isoperimetric inequality (3.4) gives

$$\begin{aligned} \frac{d}{d\varrho} m_E(x,\varrho) &\geq P(E \setminus B_\varrho(x), \partial B_\varrho(x)) \geq \frac{1}{2} P(E, B_\varrho(x)) \\ &\geq \alpha \left[m_E(x,\varrho)\right]^{(k-1)/k} \end{aligned} \tag{4.3}$$

for a.e. $\varrho \in (0, \varrho_0)$, provided $a\varrho^k/2 \geq m_E(x,\varrho) > d\varrho^k$. By Theorem 4.2 we can also assume that

$$\limsup_{\varrho \downarrow 0} \frac{m_E(x,\varrho)}{\varrho^k} \geq c \qquad \forall x \in K. \tag{4.4}$$

Fix a positive number $\beta \in \left(d^{1/k}, \min\{c^{1/k}, \alpha/k, (a/2)^{1/k}\}\right)$ and, for $x \in K$ fixed, consider the function

$$h(\varrho) := \min\left\{[m_E(x,\varrho)]^{1/k} - \beta\varrho, (a/2)^{1/k}\varrho - \beta\varrho\right\} \qquad \varrho \in (0, \varrho_0).$$

Then, a simple computation based on (4.3) shows that, for a.e. $\varrho \in (0, r_0)$, $h(\varrho) > 0$ implies $h'(\varrho) \geq \gamma$ with

$$\gamma := \min\left\{\frac{\alpha}{k} - \beta, (a/2)^{1/k} - \beta\right\} > 0.$$

Since (4.4) gives

$$\limsup_{\varrho \downarrow 0} \frac{h(\varrho)}{\varrho} \geq (c^{1/k} - \beta) > 0$$

we can find $(\varrho_i) \downarrow 0$ such that $h(\varrho_i) > 0$, hence the monotonicity of h gives

$$h(\varrho) \geq h(\varrho_i) + \gamma(\varrho - \varrho_i) \geq \gamma(\varrho - \varrho_i) \qquad \forall \varrho \in [\varrho_i, \varrho_0).$$

Letting $i \to \infty$ we infer $h(\varrho) \geq \gamma\varrho$ in $(0, \varrho_0)$.

Since x and K are arbitrary, we have proved that

$$\liminf_{\varrho \downarrow 0} \frac{m_E(x,\varrho)}{\varrho^k} \geq (\gamma + \beta)^k = \min\left\{\frac{\alpha^k}{k^k}, \frac{a}{2}\right\} \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X.$$

A similar argument with E^c in place of E gives a density lower bound for the volume of E^c; these two estimates imply (4.2) and, in conjunction with the relative isoperimetric inequality (3.4), give the lower bound in (4.1). $\square$

Proposition 4.4 (Asymptotic quasi-minimality) *Assume that the relative isoperimetric inequality (3.4) holds. Let $d \in (0, a/2)$ and $M > 1$. Then, for $P(E, \cdot)$-a.e. $x \in X$ there exists $\varrho_x > 0$ such that, for a.e. $\varrho \in (0, \varrho_x)$, the volume bounds*

$$a\varrho^k/2 \geq m_E(x, \varrho) > d\varrho^k$$

imply

$$P(E, B_\varrho(x)) \leq M P(E \setminus B_\varrho(x), \partial B_\varrho(x)).$$

PROOF. Let $\mathcal{B}$ be the family of all balls $B_\varrho(x)$ of finite perimeter such that

(a) $P(E, \partial B_\varrho(x)) = 0$;

(b) $a\varrho^k/2 \geq m_E(x, \varrho) > d\varrho^k$;

(c) $P(E, B_\varrho(x)) > M P(E \setminus B_\varrho(x), \partial B_\varrho(x))$.

Notice that, for x given, the condition in (a) is fulfilled with at most countably many exceptions. Let $B = \cap_h B_h$, where B_h is the set of points x such that

$$\left\{\varrho \in (0, 2^{-h}) : \text{ (b) and (c) hold } \right\}$$

has strictly positive measure. Since

$$L_h := \left\{(x, \varrho) \in X \times (0, \infty) : \ \varrho < 2^{-h} \text{ and (b), (c) hold } \right\}$$

is a Borel subset of $X \times (0, \infty)$ (we leave the simple but tedious proof of this fact to the reader), and since

$$B = \bigcap_{h=1}^{\infty} \left\{x : \int_0^\infty \chi_{L_h}(x, \tau)\, d\tau > 0\right\}$$

we obtain that B is a Borel set as well.

We will prove that $P(E, K) = 0$ for any compact set $K \subset B$. To this aim, for any $\delta > 0$ we consider the family

$$\mathcal{F} = \left\{\overline{B}_\varrho(x) : \ x \in K, \ \varrho \in (0, \delta), \ \ B_\varrho(x) \in \mathcal{B}\right\}.$$

Notice that, for any ball $\overline{B}_\varrho(x) \in \mathcal{F}$, (b) and the relative isoperimetric inequality (3.4) give $P(E, B_\varrho(x)) \geq (d/C_I)^{(k-1)/k} \varrho^{k-1}$. In particular, by the inclusion $K \subset B$, $\mathcal{F}$ fulfils the assumptions (i), (ii) of Theorem 2.1.

Hence, by applying Theorem 2.1 (with $\alpha = k - 1$ and ν equal to a constant multiple of $P(E, \cdot)$), we can find a disjoint family of balls $(\overline{B}_{\varrho_i}(x_i))_{i \in I} \subset \mathcal{F}$ such that $\cup_i \overline{B}_{\varrho_i}(x_i)$ contains $\mathcal{H}^{k-1}$-almost all (hence $P(E, \cdot)$ almost all) of K. By condition (a), the open set $A_\delta = \cup_i B_{\varrho_i}(x_i)$ satisfies $P(E, K \setminus A_\delta) = 0$.

Let $J \subset I$ be a finite family and let A_J be the union of the balls $B_{\varrho_i}(x_i)$, $i \in J$. By locality and subadditivity of perimeter we get

$$
\begin{aligned}
P(E \setminus A_J, X) &= P(E \setminus A_J, X \setminus A_J) = P(E \setminus A_J, X \setminus \overline{A}_J) + P(E \setminus A_J, \partial A_J) \\
&= P(E, X \setminus \overline{A}_J) + P(E \setminus A_J, \partial A_J) \\
&\leq P(E, X \setminus A_J) + \sum_{i,\, j \in J} P(E \setminus B_{\varrho_i}(x_i), \partial B_{\varrho_j}(x_j)) \\
&= P(E, X \setminus A_J) + \sum_{i \in J} P(E \setminus B_{\varrho_i}(x_i), \partial B_{\varrho_i}(x_i)) \\
&\leq P(E, X \setminus A_J) + M^{-1} P(E, A_\delta).
\end{aligned}
$$

Letting $J \uparrow I$ and using the lower semicontinuity of perimeter we infer

$$
P(E \setminus A_\delta, X) \leq P(E, X \setminus A_\delta) + \frac{1}{M} P(E, A_\delta) \leq P(E, X \setminus K) + \frac{1}{M} P(E, A_\delta).
$$

Since

$$
\mu(A_\delta) \leq \tilde{a} \sum_i \varrho_i^k \leq \tilde{a}\delta \sum_i \varrho_i^{k-1} \leq \frac{\tilde{a}\delta C_I^{(k-1)/k}}{d^{(k-1)/k}} P(E, X)
$$

letting $\delta \downarrow 0$ and using again the lower semicontinuity of perimeter we obtain $P(E, X) \leq P(E, X \setminus K) + P(E, K)/M$, hence $P(E, K) = 0$. $\square$

Corollary 4.5 (Asymptotic doubling property) *The measure $P(E, \cdot)$ is a.e. asymptotically doubling, i.e.*

$$
\limsup_{\varrho \downarrow 0} \frac{P(E, B_{2\varrho}(x))}{P(E, B_\varrho(x))} < \infty \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X.
$$

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