QUASISTATIC EVOLUTION PROBLEMS FOR NONHOMOGENEOUS ELASTIC PLASTIC MATERIALS

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ABSTRACT. The paper studies the quasistatic evolution for elastoplastic materials when the yield surface depends on the position in the reference configuration. The main results are obtained when the yield surface is continuous with respect to the space variable. The case of piecewise constant dependence is also considered. The evolution is studied in the framework of the variational formulation for rate independent problems developed by Mielke. The results are proved by adapting the arguments introduced for a constant yield surface, using some properties of convex valued semicontinuous multifunctions. A strong formulation of the problem is also obtained, which includes a pointwise version of the plastic flow rule. Some examples are considered, which show that strain concentration may occur as a consequence of a nonconstant yield surface

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CONTENTS

1. INTRODUCTION

The aim of this paper is to study quasistatic evolution problems in small strain associative elastoplasticity for nonhomogeneous materials. More precisely, in the spirit of [5], we consider the case of a material whose plastic response is governed by the Prandtl-Reuss flow rule, without hardening (perfect plasticity); differently from [5], however, the elasticity tensor and the yield surface are allowed to change pointwise. This is a situation which may occur for example in composite materials, when each of the components present a different elastoplastic response. In the first part of the paper, indeed, we extend the results proved in [5], to this more general setting, under suitable conditions.

The problem is formulated as follows in a domain $\Omega \subset \mathbb{R}^n$. The linearized strain Eu, that is the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $Eu = e + p$, where e and p respectively represent the elastic and plastic strain. The stress σ is determined only by e, through the formula $\sigma = \mathbb{C}e$, where $\mathbb{C} := \mathbb{C}(x)$ is the elasticity tensor. At each point $x \in \Omega$, σ is constrained to lie in a prescribed subset $\mathbb{K}$ of the space $\mathbb{M}^{n\times n}_{sym}$ of $n\times n$ symmetric matrices, whose boundary $\partial\mathbb{K}$ is referred to as the yield surface.

Given a time-dependent body force $f(t,x)$, the classical formulation of the quasistatic evolution problem in a time interval $[0,T]$ consists in finding functions $u(t,x)$, $e(t,x)$, $p(t,x)$, $\sigma(t,x)$ satisfying the following conditions for every $t \in [0,T]$ and every $x \in \Omega$:

(cf1) additive decomposition: $Eu(t,x) = e(t,x) + p(t,x)$,
(cf2) constitutive equation: $\sigma(t,x) = \mathbb{C}(x)e(t,x)$,
(cf3) equilibrium: $-\mathrm{div}_x\, \sigma(t,x) = f(t,x)$,
(cf4) stress constraint: $\sigma(t,x) \in \mathbb{K}$
(cf5) associative flow rule: $\dot{p}(t,x) \in N_{\mathbb{K}(x)}(\sigma_D(t,x))$,

where the colon denotes the scalar product between matrices and $N_{\mathbb{K}(x)}$ is the normal cone to $\mathbb{K}(x)$. The problem is supplemented by initial conditions at time $t = 0$ and by boundary conditions for $t \in [0,T]$, $x \in \partial\Omega$, of the form $u(t,x) = w(t,x)$ on a portion Γ_0 of the boundary, and $\sigma(t,x)\nu(x) = g(t,x)$ on the complementary portion Γ_1, where $\nu(x)$ is the outer unit normal to $\partial\Omega$, $w(t,x)$ is the prescribed displacement on Γ_0, and $g(t,x)$ is the prescribed surface force on Γ_1. Alternatively, one can consider the so-called Dirichlet-Periodic problem: generally in this case the domain Ω is a cube and a periodic condition is prescribed on the portion Γ_1 of the boundary.

In this paper, we consider the case where $\mathbb{K}$ is a cylinder of the form $\mathbb{K} = K(x) + \mathbb{R}I$, where I is the identity matrix and $K(x)$ is a convex compact subset of $\mathbb{M}^{n\times n}_D$, the space of trace free $n\times n$ symmetric matrices as in the model of Tresca and von Mises (see, e.g., [9]). We shall suppose that there exist two balls $B_m(0)$, $B_M(0)$ such that $B_m(0) \subseteq K(x) \subseteq B_M(0)$ for every $x \in \Omega \cup \Gamma_0$. If we introduce the support function

$$H(x,\xi) := \sup_{\zeta \in K(x)} \xi : \zeta\,,$$

the flow rule (cf5) can be written in the equivalent forms (see Section 2):

(cf5′) flow rule in primal formulation: $\sigma_D(t,x) \in \partial_\xi H(x, \dot{p}(t,x))$,
(cf5″) maximal dissipation: $H(x, \dot{p}(t,x)) = \sigma_D(t,x) : \dot{p}(t,x)$,

where $\sigma_D(t,x)$ denotes the deviator of $\sigma(t,x)$ (see definition in the following section of the paper) and ∂_ξ is the subdifferential with respect to ξ.

An approximation of quasistatic evolution problems of this type is obtained by solving a finite number of incremental variational problems. The time interval $[0,T]$ is divided into k subintervals by means of points

$$0 = t^0_k < t^1_k < \cdots < t^{k-1}_k < t^k_k = T\,,$$

and the approximate solution u^i_k, e^i_k, p^i_k at time t^i_k is defined, inductively, as a minimizer of the incremental problem

$$\min_{(u,e,p)\in A(w(t^i_k))} \{\mathcal{Q}(e) + \mathcal{H}(p - p^{i-1}_k) - \langle \mathcal{L}(t^k_i)|u\rangle\}\,, \tag{1.1}$$

where

$$\mathcal{Q}(e) := \tfrac{1}{2}\int_\Omega \mathbb{C}e(x) : e(x)\,dx\,, \qquad \mathcal{H}(p) := \int_\Omega H(x, p(x))\,dx\,,$$
$$\langle \mathcal{L}(t)|u\rangle := \int_\Omega f(t,x)\,u(x)\,dx + \int_{\Gamma_1} g(t,x)\,u(x)\,d\mathcal{H}^{n-1}(x)\,, \tag{1.2}$$

$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and $A(w(t))$ is defined, at this stage of the discussion, as the set of triples (u,e,p), with $Eu(x) = e(x) + p(x)$ for every $x \in \Omega$, such that u satisfies the prescribed Dirichlet boundary condition at time t, i.e., $u(x) = w(t,x)$ for every $x \in \Gamma_0$. Finally, the stress at time t^i_k is obtained as $\sigma^i_k(x) := \mathbb{C}(x)e^i_k(x)$.

Since $\mathcal{H}$ has linear growth, problem (1.1) has, in general, no solution in Sobolev spaces: a suitable functional space, for a weak formulation of the problem, proves then to be the space $BD(\Omega)$ of functions with bounded deformation, whose theory was developed in [14], [8], [13]; the plastic strain p belongs to the space $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ of $\mathbb{M}^{n\times n}_D$-valued bounded Borel measures on $\Omega\cup\Gamma_0$. In the weak formulation of problem (1.1) the functional $\mathcal{H}(p)$ is defined as

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma_0} H(x, p/|p|)\,d|p|\,, \tag{1.3}$$

where $p/|p|$ is the Radon-Nikodym derivative of the measure p with respect to its variation $|p|$, while the boundary conditions are suitably relaxed, leading to a weaker definition of $A(w(t))$ (see Section 2). Functionals of this type were first studied by Y.G.Reshetnyak, who investigated their properties of w^*-lower semicontinuity and continuity in the space of bounded Radon measures.

The problem of continuous-time quasistatic evolution in the functional framework $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, can be solved, provided a uniform safe-load condition is satisfied, by means of the introduction of the following weak definition (see also the work of Suquet [11] for a different approach): a quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ which satisfies the following conditions:

(qs1) global stability: for every $t\in[0,T]$ we have $(u(t),e(t),p(t)) \in A(w(t))$ and
$$\mathcal{Q}(e(t)) - \langle\mathcal{L}(t)|u(t)\rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t)|v\rangle$$
for every $(v,\eta,q) \in A(w(t))$;

(qs2) energy balance: the function $t\mapsto p(t)$ from $[0,T]$ into $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t\in[0,T]$
$$\mathcal{Q}(e(t)) + \mathcal{D}_\mathcal{H}(p;0,t) - \langle\mathcal{L}(t)|u(t)\rangle = \mathcal{Q}(e(0)) - \langle\mathcal{L}(0)|u(0)\rangle +$$
$$+ \int_0^t \{\langle\sigma(s)|E\dot w(s)\rangle - \langle\mathcal{L}(s)|\dot w(s)\rangle - \langle\dot{\mathcal{L}}(s)|u(s)\rangle\}\,ds\,,$$
where $\sigma(t) := \mathbb{C}e(t)$, dots denote time derivatives, the first brackets $\langle\cdot|\cdot\rangle$ in the integral denote the scalar product in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, while the other brackets $\langle\cdot|\cdot\rangle$ are defined as in (1.2).

Here for every time interval $[s,t]$ contained in $[0,T]$ $\mathcal{D}_\mathcal{H}(p;0,t)$ represents the dissipation associated with $\mathcal{H}$, defined by

$$\mathcal{D}_\mathcal{H}(p;s,t) := \sup\Big\{\sum_{j=1}^N \mathcal{H}(p(t_j) - p(t_{j-1})) : \ s = t_0 \le t_1 \le \cdots \le t_N = t, N\in\mathbb{N}\Big\}.$$

In the first part of the present paper we are going to slightly generalize results of [5], about the existence and regularity properties of quasistatic evolutions: while the explicit dependence on x of the elasticity tensor does not introduce changes in the proofs, some effort has to be made for the functional $\mathcal{H}$. Relying on the ideas of the theory of measurable compact convex multifunctions developed in [4] and using the above mentioned results by Reshetnyak, it is possible to show that, if the map $x \mapsto K(x)$ is lower semicontinuous, the functional $\mathcal{H}$ defined in (1.3) is lower semicontinuous too, but coercivity of the incremental problems and the same Euler conditions as in [5] are not always guaranteed; however they are still true if the multimap is continuous, or in some piecewise constant cases, for example when the convex sets are given by the Von Mises' condition; in these situations, also the construction of a precise representative satisfying a strong formulation of the problem is still valid.

In the second part of the paper we focus on one dimensional problems, which appear also in the solution of multidimensional Dirichlet periodic problems as in the case of simple shear. In dimension one the assumption on the stress constraint can be weakened, assuming the multifunction $x \mapsto K(x)$ is only lower semicontinuous. Some interesting phenomena can be observed in this case when the force does not depend on time; the plastic deformation tends to be concentrated on a finite number of points (strain localization), except in some degenerate situations; a complete qualitative study can be easily done in this case, showing also an explicit formula for solutions in terms of the data. These results can also be used to discuss the uniqueness of the solutions; moreover, they provide examples where, indipendently of the regularity of the data, the elastic part of the solution has at best a Lipschitz dependence on the time variable.

2. Notation and preliminary results

2.1. *Mathematical preliminaries.*

For what concerns definitions and notatins about measures, matrices and functions of bounded deformation we refer to [5], Section 2. All the Borel measures are tacitly understood to be extended to the corresponding completion of the σ-algebra of Borel sets.

Multifunctions. Let X and Y be two sets. A multifunction is a map $\varphi : X \to 2^Y$, that is a map from X to the subsets of Y.
According to [4] , if $(X, \mathcal{C})$ is a measurable space and Y is a finite-dimensional Hilbert space we say this map to be $\mathcal{C}-$measurable $\Longleftrightarrow$

$$\forall\, U \text{open},\ \varphi^-(U) \in \mathcal{C}$$

where

$$\varphi^-(U) := \{x \in X |\, \varphi(x) \cap U \neq \emptyset\}.$$

If moreover X is a topological space, we say this map is lower semicontinuous $\Longleftrightarrow$

$$\forall\, U \text{open},\ \varphi^-(U) \,\text{is open}$$

We say this map is upper semicontinuous $\Longleftrightarrow$

$$\forall\, U \text{open}, \text{the set}\ : \{x \in X |\, \varphi(x) \subseteq U\} \,\text{is open}.$$

We say the map φ is continuous if it is upper and lower semicontinuous.
The following equivalence can be easily proved:

Proposition 2.1. *Let $\varphi : X \to 2^Y$ a map from X to the compact subsets of Y. Then φ is lower semicontinuous (resp. upper semicontinuous) $\Longleftrightarrow$*

$$\forall \varepsilon > 0\ \exists U \ni\ x \textit{ open t.c. } \forall y \in U : \ \varphi(x) \subseteq \varphi(y) + \varepsilon B$$

(resp.

$$\forall \varepsilon > 0\ \exists U \ni\ x \textit{ open t.c. } \forall y \in U : \ \varphi(y) \subseteq \varphi(x) + \varepsilon B).$$

Here B denotes the closed unitary ball of Y.

An important problem when dealing with multifunctions is the existence of a selection, that is a map $\sigma\colon X \to Y$ with the property: $\sigma(x) \in \varphi(x) \forall x \in X$. The following result, due to E.Michael, besides its intrinsic relevance, will be useful for the purpose of the paper:

Theorem 2.2 (Continuous selections of lower semicontinuous multifunctions)**.** *If X is a separable metric space, then every lower semicontinuous function φ from X to the non-empty, convex, closed subsets of Y admits a sequence (σ_n) of continuous selections such that*

$$\forall x \in X: \ \varphi(x) = \overline{\{\sigma_n(x)\}}.$$

Proof. See for instance [7] for the existence of a continuous selection; the rest of the proof follows with the same argument as in [4], Theorem III.7. □

Remark 2.3 (Monotone approximation by continuous multifunctions)**.** Under the assumptions of the previous theorem we easily construct a sequence φ_j of continuous multifunctions such that:

$$\varphi_j(x) \nearrow \varphi(x) \ \ \forall x \in X$$

by putting, for all $j \in \mathbb{N}$:

$$\varphi_j(x) := \overline{Co\{\sigma_n(x) |\, 1 \leq n \leq j\}},$$

where as usual CoA denotes the smallest convex set containing a given set A and (σ_n) is a sequence of continuous selections as in the previous theorem. If in addition there exists $m > 0$ such that

$$B(0, m) \subseteq \varphi(x) \ \ \forall x \in X$$

we can also assume that

$$\forall j \in \mathbb{N} \ \ B(0, m) \subseteq \varphi_j(x) \ \ \forall x \in X$$

by putting:

$$\varphi_j(x) := \overline{Co\, B(0, m) \cup \{\sigma_n(x) |\, 1 \leq n \leq j\}}.$$

Given a closed convex nonempty subset C of a normed space V we define its indicator function δ_C as

$$\delta_C(v) = \begin{cases} 0 \ \ if \ \ v \in C \\ +\infty \ \ if \ \ v \notin C \end{cases}$$

and the normal cone N_C to C at $u \in C$ as:

$$N_C(u) = \{v^* \in V^* \ \ : \ \ \langle v^*, v - u\rangle \leq 0\} \forall v \in V.$$

It is easy to see that δ_C is a convex proper l.s.c. function.

2.2. *Mechanical preliminaries.*

The reference configuration. Throughout the paper Ω is a *bounded connected open set* in $\mathbb{R}^n$ with C^2 *boundary*. We suppose that the boundary $\partial\Omega$ is partitioned into two disjoint open sets Γ_0, Γ_1 and their common boundary $\partial\Gamma_0 = \partial\Gamma_1$ (topological notions refer here to the relative topology of $\partial\Omega$); we assume

$$\Gamma_0 \neq \emptyset \tag{2.1}$$

$$\partial\Gamma_0 = \partial\Gamma_1 \text{ is } C^2 \text{ regular} \tag{2.2}$$

that is: for every $x \in \partial\Gamma_0 = \partial\Gamma_1$ there exists a C^2 diffeomorphism defined in an open neighbourhood of x in $\mathbb{R}^n$ which maps $\partial\Omega$ to an $(n-1)$-dimensional plane and $\partial\Gamma_0 = \partial\Gamma_1$ to an $(n-2)$-dimensional plane.

On Γ_0 a Dirichlet boundary condition will be prescribed. This will be done by assigning a function $w \in H^{1/2}(\Gamma_0; \mathbb{R}^n)$, or, equivalently, a function $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, whose trace on Γ_0 (also denoted by w) is the prescribed boundary value. The set Γ_1 will be the part of the boundary on which the traction is prescribed.

Every function $u \in BD(\Omega)$ has a *trace* on $\partial\Omega$, still denoted by u, which belongs to $L^1(\partial\Omega;\mathbb{R}^n)$. If u_k, $u \in BD(\Omega)$, $u_k \to u$ strongly in $L^1(\Omega;\mathbb{R}^n)$, and $\|Eu_k\|_1 \to \|Eu\|_1$, then $u_k \to u$ strongly in $L^1(\partial\Omega;\mathbb{R}^n)$ (see [13, Chapter II, Theorem 3.1]). Moreover, there exists a constant $C>0$, depending on Ω and Γ_0, such that

$$\|u\|_{1,\Omega} \le C\,\|u\|_{1,\Gamma_0} + C\,\|Eu\|_{1,\Omega} \tag{2.3}$$

(see [13, Proposition 2.4 and Remark 2.5]).

The space $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, which is the dual of $C_0(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, can be identified with the space of functions in $C(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ vanishing on $\overline{\Gamma}_1$. The duality product is defined by

$$\langle \tau|\mu\rangle := \int_{\Omega\cup\Gamma_0} \tau : d\mu := \sum_{ij} \int_{\Omega\cup\Gamma_0} \tau_{ij}\, d\mu_{ij} \tag{2.4}$$

for every $\tau=(\tau_{ij}) \in C(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ and every $\mu=(\mu_{ij}) \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$.

The set of admissible stresses. Let $K\colon \Omega\cup\Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a continuous or piecewise constant multivalued map from $\Omega\cup\Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$, which will play the role of a constraint on the deviatoric part of the stress. We assume that there exist two constants m and M, with $0<m\le M<\infty$, such that

$$\{\xi\in\mathbb{M}^{n\times n}_D : |\xi|\le m\} \subset K(x) \subset \{\xi\in\mathbb{M}^{n\times n}_D : |\xi|\le M\}\ \forall x\in\Omega\cup\Gamma_0. \tag{2.5}$$

It is convenient to introduce the convex closed set

$$\mathcal{K}_D(\Omega) := \{\tau\in L^2(\Omega;\mathbb{M}^{n\times n}_D) : \tau(x)\in K(x) \text{ for a.e. } x\in\Omega\}\,.$$

The *set of admissible stresses* is defined by

$$\mathcal{K}(\Omega) := \{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \sigma_D\in\mathcal{K}_D(\Omega)\}\,.$$

For fixed x the *support function* $H(x,\cdot)\colon \mathbb{M}^{n\times n}_D \to [0,+\infty[$ of $K(x)$ is given by

$$H(x,\xi) := \sup\nolimits_{\zeta\in K(x)} \xi:\zeta\,. \tag{2.6}$$

It turns out that $H(x,\cdot)$ is the conjugate function of the indicator function $\delta_{K(x)}$, hence it is convex; moreover, it is positively homogeneous of degree one. In particular it satisfies the triangle inequality

$$H(x,\xi+\zeta) \le H(x,\xi) + H(x,\zeta)\,.$$

From (2.5) it follows that

$$m|\xi| \le H(x,\xi) \le M|\xi| \tag{2.7}$$

for every $(x,\xi)\in\Omega\cup\Gamma_0\times\mathbb{M}^{n\times n}_D$.
It follows that:

Proposition 2.4. *Let $K\colon \Omega\cup\Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a lower semicontinuous multivalued map from $\Omega\cup\Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$. Assume (2.5). Then the function $H(x,\xi)$ defined by (2.6) is lower semicontinuous from $\Omega\cup\Gamma_0\times\mathbb{M}^{n\times n}_D$ to $[0,+\infty[$.*

Proof. Fix ξ in $\mathbb{M}^{n\times n}_D$ and consider the function $H(\cdot,\xi)$. Let $t\in\mathbb{R}$ such that $H(x,\xi)>t$. This means that $K(x)$ meets the open set $U := \{\zeta\in\mathbb{M}^{n\times n}_D : \zeta\colon\xi>t\}$, hence there is an open neighborhood A of x such that, $K(y)\cap U\neq\emptyset$ for every $y\in A$.This in turn implies that $H(y,\xi)>t$ for every $y\in A$, i.e. the function $H(\cdot,\xi)$ is lower semicontinuous. Since, for fixed x, using (2.5) and for instance [1, Lemma 13.2.1] the function $H(x,\cdot)$ is lipschitzian uniformly with respect to x, the function H is lower semicontinuous in the product space. □

For every $\mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ let $\mu/|\mu|$ be the Radon-Nikodym derivative of μ with respect to its variation $|\mu|$. We introduce the nonnegative Radon measure $H(\mu) \in M_b(\Omega \cup \Gamma_0)$ defined by $H(\mu) := H(\cdot, \mu/|\mu|)|\mu|$, i.e.,

$$H(\mu)(B) := \int_B H(x, \mu/|\mu|(x))\, d|\mu| \tag{2.8}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$. Finally, we consider the functional $\mathcal{H}\colon M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H(x, \mu/|\mu|(x))\, d|\mu|\,. \tag{2.9}$$

Since H is lower semicontinuous and positively 1-homogeneous in the second variable it follows from Reshetnyak's lower semicontinuity theorem (see [2], Theorem 2.38) that $\mathcal{H}$ is lower semicontinuous on $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ with respect to weak* convergence. It follows from the properties of H that $\mathcal{H}$ satisfies the triangle inequality, i.e.,

$$\mathcal{H}(\lambda + \mu) \le \mathcal{H}(\lambda) + \mathcal{H}(\mu) \tag{2.10}$$

for every $\lambda, \mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$.

The elasticity tensor. At each point $x \in \Omega$, let $\mathbb{C}(x)$ be the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C}(x)\colon \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$. The orthogonal subspaces $\mathbb{M}^{n\times n}_D$ and $\mathbb{R}I$ are assumed to be invariant under $\mathbb{C}(x)$ for every x. This is equivalent to saying that there exist a symmetric positive definite linear operator $\mathbb{C}_D(x)\colon \mathbb{M}^{n\times n}_D \to \mathbb{M}^{n\times n}_D$ and $\kappa(x) > 0$ such that

$$\mathbb{C}(x)\xi := \mathbb{C}_D(x)\xi_D + \kappa(x)(\operatorname{tr}\xi)I \tag{2.11}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$. Note that when $\mathbb{C}(x)$ is isotropic we have $\mathbb{C}(x)\xi = 2\mu(x)\xi_D + \kappa(x)(\operatorname{tr}\xi)I$; here $\mu(x) > 0$ is the shear modulus and $\kappa(x)$ is the modulus of compression, so that our assumptions are satisfied. We shall suppose that $\mathbb{C}(x)$ is uniformly positive definite, and we will require uniform boundedness of the norms of the operators $\mathbb{C}(x)$, which is to say that, if $Q(x,\cdot)\colon \mathbb{M}^{n\times n}_{sym} \to [0, +\infty[$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(x,\xi) := \tfrac{1}{2}\mathbb{C}(x)\xi : \xi = \tfrac{1}{2}\mathbb{C}_D(x)\xi_D : \xi_D + \tfrac{\kappa(x)}{2}(\operatorname{tr}\xi)^2\ . \tag{2.12}$$

there exist $0 < \alpha < \beta$ not depending on x such that

$$\alpha|\xi|^2 \le Q(x,\xi) \le \beta|\xi|^2 \tag{2.13}$$

for every $x \in \Omega$ and $\xi \in \mathbb{M}^{n\times n}_{sym}$.

It is convenient to introduce the quadratic form $\mathcal{Q}\colon L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ defined by

$$\mathcal{Q}(e) := \int_\Omega Q(e)\, dx \tag{2.14}$$

for every $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. It is well known that $\mathcal{Q}$ is lower semicontinuous on $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ with respect to weak convergence.

The prescribed boundary displacements. For every $t \in [0,T]$ the *boundary displacement* $w(t)$ is prescribed in the space $H^1(\mathbb{R}^n; \mathbb{R}^n)$. This choice is motivated by the fact that it is preferable not to impose "discontinuous" boundary data, so that, if the displacement develops sharp discontinuities, this is due only to energy minimization.

We assume also that the function $t \mapsto w(t)$ is absolutely continuous from $[0,T]$ into $H^1(\mathbb{R}^n; \mathbb{R}^n)$, so that the time derivative $t \mapsto \dot w(t)$ belongs to $L^1([0,T]; H^1(\mathbb{R}^n; \mathbb{R}^n))$ and its strain $t \mapsto E\dot w(t)$ belongs to $L^1([0,T]; L^2(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym}))$.

Body and surface forces. For every $t \in [0,T]$ the *body force* $f(t)$ belongs to the space $L^n(\Omega; \mathbb{R}^n)$ and the *surface force* $g(t)$ acting on Γ_1 belongs to $L^\infty(\Gamma_1; \mathbb{R}^n)$. By hypothesis, the functions $t \mapsto f(t)$ and $t \mapsto g(t)$ are absolutely continuous from $[0,T]$ into

$L^n(\Omega;\mathbb{R}^n)$ and $L^\infty(\Gamma_1;\mathbb{R}^n)$, respectively, so that the time derivative $t \mapsto \dot{f}(t)$ belongs to $L^1([0,T];L^n(\Omega;\mathbb{R}^n))$, the weak* limit

$$\dot{g}(t) := w^*\text{-}\lim_{s\to t} \frac{g(s)-g(t)}{s-t}\,,$$

exists for a.e. $t \in [0,T]$, and $t \mapsto \|\dot{g}(t)\|_\infty$ belongs to $L^1([0,T])$ (see [5], Theorem 7.1).

Throughout the paper we will assume also the following *uniform safe-load condition*: there exist a function $t \mapsto \varrho(t)$ from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and a constant $\alpha > 0$ such that for every $t \in [0,T]$

$$-\mathrm{div}\varrho(t) = f(t) \;\text{ a.e. on } \Omega\,, \qquad [\varrho(t)\nu] = g(t) \text{ on } \Gamma_1\,, \tag{2.15}$$

and

$$\varrho_D(t,x) + \xi \in K(x) \tag{2.16}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| \le \alpha$. In these formulas $\varrho_D(t,x)$ denotes the value of $\varrho_D(t)$ at $x \in \Omega$, and the trace $[\varrho(t)\nu]$ of $\varrho(t)\nu$ on Γ_1 is interpreted in the sense of (2.25) below. We assume also that the functions $t \mapsto \varrho(t)$ and $t \mapsto \varrho_D(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, respectively, so that the time derivative $t \mapsto \dot{\varrho}(t)$ belongs to $L^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$,

$$\frac{\varrho_D(s)-\varrho_D(t)}{s-t} \;\rightharpoonup\; \dot{\varrho}_D(t) \tag{2.17}$$

weakly* in $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ for a.e. $t \in [0,T]$, and $t \mapsto \|\dot{\varrho}_D(t)\|_\infty$ belongs to $L^1([0,T])$ (see [5], Theorem 7.1).

2.3. *Stress and strain.*

Admissible displacements and strains. Given a displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ satisfy the equalities

$$Eu = e + p \quad \text{in } \Omega\,, \tag{2.18}$$

$$p = (w-u)\odot\nu\,\mathcal{H}^{n-1} \quad \text{on } \Gamma_0\,. \tag{2.19}$$

Therefore we have $e = E^a u - p^a$ a.e. on Ω and $p^s = E^s u$ on Ω. Since $\mathrm{tr}\,p = 0$, it follows from (2.18) that $\mathrm{div}\,u = \mathrm{tr}\,e \in L^2(\Omega)$ and from (2.19) that

$$(w-u)\cdot\nu = 0 \;\; \mathcal{H}^{n-1} - \text{a.e.} \quad \text{on } \Gamma_0\,. \tag{2.20}$$

The *stress* $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is defined by

$$\sigma := \mathbb{C}e = \mathbb{C}_D e_D + \kappa\,\mathrm{tr}\,e\,. \tag{2.21}$$

The *stored elastic energy* is given by

$$\mathcal{Q}(e) = \int_\Omega Q(e)\,dx = \tfrac{1}{2}\langle\sigma|e\rangle\,. \tag{2.22}$$

Given $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *set of admissible displacements and strains* for the boundary datum w on Γ_0 is denoted by $A(w)$: it is defined as the set of all triples (u,e,p), with $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, satisfying (2.18) and (2.19).
In the case $n=1$ which I shall deal with in the last section of the paper, Ω will be an open interval (a,b) in $\mathbb{R}$, and $u \in BV([a,b])$, $e \in L^2(\Omega)$, $p \in M_b(\Omega\cup\Gamma_0)$ will satisfy:

$$u' = e + p \quad \text{in } \Omega\,, \tag{2.23}$$

$$p = (u(a)-w(a))\delta_a + (w(b)-u(b))\delta_b \quad \text{on } \{a,b\}\,, \tag{2.24}$$

where the boundary values are taken in the sense of traces.
We shall also use *the space* $\Pi_{\Gamma_0}(\Omega)$ *of admissible plastic strains*, defined as the set of all $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ for which there exist $u \in BD(\Omega)$, $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying (2.18) and (2.19), i.e., $(u,e,p) \in A(w)$.

The traces of the stress. If $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^n)$, then the distribution $[\sigma\nu]$ on $\partial\Omega$ defined by

$$\langle[\sigma\nu]|\psi\rangle_{\partial\Omega} := \langle \operatorname{div}\sigma|\psi\rangle + \langle\sigma|E\psi\rangle \tag{2.25}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^n)$ belongs to $H^{-1/2}(\partial\Omega;\mathbb{R}^n)$ (see, e.g., [13, Theorem 1.2, Chapter I]). The normal and tangential parts of $[\sigma\nu]$ are respectively defined by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu\,, \qquad [\sigma\nu]_\nu^\perp := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu\,. \tag{2.26}$$

Since $\nu \in C^1(\partial\Omega;\mathbb{R}^n)$, we have that $[\sigma\nu]_\nu$, $[\sigma\nu]_\nu^\perp \in H^{-1/2}(\partial\Omega;\mathbb{R}^n)$. If, in addition, $\sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, then $[\sigma\nu]_\nu^\perp \in L^\infty(\partial\Omega;\mathbb{R}^n)$ and

$$\|[\sigma\nu]_\nu^\perp\|_{\infty,\partial\Omega} \le \tfrac{1}{\sqrt{2}}\|\sigma_D\|_\infty \tag{2.27}$$

(see [13, Lemma 7.2]).

Stress-strain duality. Let

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n),\, \sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\}\,.$$

If $\sigma \in \Sigma(\Omega)$, then $\sigma \in L^r(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $r<\infty$ by [13, Proposition 7.1]. For the definition of the bounded Radon measure $[\sigma_D : E_D u]$ we refer to [8] and [13], while for what concerns the bounded Radon measure $[\sigma_D : E^s_D u]$, when $u \in BD(\Omega)$ with $\operatorname{div} u \in L^2$, and the duality between $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$ we refer to [5], Section 2.3: here some useful properties are collected for the reader's convenience. The measure $[\sigma_D : p]$ does not depend on the choice of u, e, and w. It satisfies

$$\begin{aligned} &[\sigma_D : p]^a = \sigma_D : p^a \quad \text{a.e. on } \Omega\,, && [\sigma_D : p]^s = [\sigma_D : E^s_D u] \qquad \text{on } \Omega\cup\Gamma_0\,,\\ &|[\sigma_D : p]| \le \|\sigma_D\|_\infty |p| \quad \text{on } \Omega\cup\Gamma_0\,, && |[\sigma_D : p]^s| \le \|\sigma_D\|_\infty |p^s| \quad \text{on } \Omega\cup\Gamma_0\,. \end{aligned} \tag{2.28}$$

Moreover

$$[\psi\sigma_D : p] = \psi[\sigma_D : p] \qquad \text{in } \Omega\cup\Gamma_0 \tag{2.29}$$

for every $\psi \in C^1(\overline{\Omega})$ and

$$\langle[\sigma_D : p]|\varphi\rangle = \langle\varphi\,\sigma_D|p\rangle \tag{2.30}$$

for every $\sigma \in C^1(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C^1(\overline{\Omega})$, where the duality used in the right-hand side is defined in (2.4). By approximation, using the continuity properties collected in (2.28), (2.30) holds also for every $\sigma \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C(\overline{\Omega})$. Therefore, for every $\sigma \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $p \in \Pi_{\Gamma_0}(\Omega)$

$$[\sigma_D : p] = \sigma_D : p \qquad \text{on } \Omega\cup\Gamma_0\,, \tag{2.31}$$

where the right-hand side denotes the measure defined by

$$(\sigma_D : p)(B) := \int_B \sigma_D : dp := \sum_{ij}\int_B \sigma_{ij}\,dp_{ij} \tag{2.32}$$

for every Borel set $B \subset \Omega\cup\Gamma_0$.

If $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$, and $(\sigma_k)_D$ is bounded in $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$,

$$\langle[(\sigma_k)_D : p]|\varphi\rangle \to \langle[\sigma_D : p]|\varphi\rangle \tag{2.33}$$

for every $\varphi \in C(\overline{\Omega})$.

Finally, for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define

$$\begin{aligned} \langle\sigma_D|p\rangle &:= [\sigma_D : p](\Omega\cup\Gamma_0)\\ &= \langle\sigma_D|p^a\rangle + \langle\sigma_D|E^s_D u\rangle + \langle[\sigma\nu]_\nu^\perp|w-u\rangle_{\Gamma_0}\\ &= \langle\sigma_D|E_D u\rangle - \langle\sigma_D|e_D\rangle + \langle[\sigma\nu]_\nu^\perp|w-u\rangle_{\Gamma_0}\,, \end{aligned} \tag{2.34}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ satisfy (2.18) and (2.19).

In [5] the following integration by parts formula for stresses $\sigma \in \Sigma(\Omega)$ and displacements $u \in BD(\Omega)$, involving the elastic and plastic strains e and p, is proved.

Proposition 2.5 (Integration by parts)**.** *Let $\sigma \in \Sigma(\Omega)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, and let $(u,e,p) \in A(w)$, with $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$. Assume that $-\operatorname{div}\sigma = f$ a.e. on Ω and $[\sigma\nu] = g$ on Γ_1. Then*

$$\langle \sigma_D | p\rangle + \langle \sigma | e - Ew\rangle = \langle f | u - w\rangle + \langle g | u - w\rangle_{\Gamma_1}\,, \tag{2.35}$$

where $\langle\cdot|\cdot\rangle_{\Gamma_1}$ denotes the duality pairing between $L^\infty(\Gamma_1;\mathbb{R}^n)$ and $L^1(\Gamma_1;\mathbb{R}^n)$. Moreover

$$\begin{aligned}\langle [\sigma_D : p] | \varphi\rangle + \langle \sigma : (e - Ew) | \varphi\rangle + \langle \sigma | (u-w)\odot\nabla\varphi\rangle = \\ = \langle f | \varphi(u-w)\rangle + \langle g | \varphi(u-w)\rangle_{\Gamma_1}\end{aligned} \tag{2.36}$$

for every $\varphi \in C^1(\overline{\Omega})$.

It is useful for our purposes to get insight of the behavior of $[\sigma_D : p]$ on a C^2 orientable hypersurface S contained in Ω. First of all, we need a notion of trace. We fix an orientation on S: locally S splits Ω into two disjoint open sets Ω_+ and Ω_- determined by the convention that the oriented normal ν_S to S is inner to Ω_-. Regarding S as a subset of $\partial\Omega_+$ and $\partial\Omega_-$ we can define, as in the previous section, the distributions $[\sigma\nu_S]^+$, $[\sigma\nu_S]^-$ and the normal and tangential parts of the two distributions, which belong to $H^{-1/2}_{loc}(S;\mathbb{R}^n)$ as well. But condition $\operatorname{div}\sigma \in L^n$ easily yelds:

$$[\sigma\nu_S]^+ = [\sigma\nu_S]^- \text{ in } H^{-1/2}_{loc}(S;\mathbb{R}^n) \tag{2.37}$$

and we denote this common value by $[\sigma\nu_S]$. We then have the following formula, completely analogous to that we already know in the case where S is part of the boundary of Ω.

Proposition 2.6. *Let $S \subset \Omega$ an orientable C^2 hypersurface, $\sigma \in \Sigma(\Omega), p \in \Pi_{\Gamma_0}(\Omega)$, $u \in BD(\Omega)$, such that there exist $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying (2.18) and (2.19), i.e., $(u,e,p) \in A(w)$. Then $[\sigma\nu_S]^\perp_\nu \in L^\infty(S;\mathbb{R}^n)$ and*

$$[\sigma_D : p]\lfloor S = [\sigma\nu_S]^\perp_\nu \cdot (u^+ - u^-)\mathcal{H}^{n-1}\lfloor S \tag{2.38}$$

where u^+ and u^- are the traces on S of u.

Proof. By ([13, Chapter 2, Lemma 7.1]), we can take a sequence $(\sigma_m) \subset C^\infty(\bar{\Omega};\mathbb{M}^{n\times n}_{sym})$ such that:

$$\begin{aligned}&\sigma_m \to \sigma \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\\ &\operatorname{div}\sigma_m \to \sigma \text{ in } L^n(\Omega;\mathbb{R}^n)\\ &||(\sigma_m)_D||_\infty \le ||\sigma_D||_\infty;\end{aligned}$$

from the definitions we easily get:

$$[(\sigma_m)\nu_S] \to [\sigma\nu_S] \text{ in } H^{-1/2}_{loc}(S);$$

the same reasonings as in the case when $S = \partial\Omega$ (see [13, Chapter 2, Remark 7.2]), actually show that:

$$[(\sigma_m)\nu_S]^\perp_\nu \overset{*}{\rightharpoonup} [\sigma\nu_S]^\perp_\nu,\ w^* \text{ in } L^\infty(S;\mathbb{R}^n) \tag{2.39}$$

so that the right-hand side in (2.38) is well-defined. Now

$$p\lfloor S = Eu\lfloor S = (u^+ - u^-)\odot\nu_S\mathcal{H}^{n-1}\lfloor S \tag{2.40}$$

and the condition $\operatorname{tr} p = 0$ implies

$$(u^+ - u^-)\cdot\nu_S = 0 \;\; \mathcal{H}^{n-1} - \text{a.e.} \quad \text{on } S\,. \tag{2.41}$$

By (2.31) and (2.41), formula (2.38) holds for the functions σ_m, hence we immediately conclude by (2.39). ∎

3. Quasistatic evolution

3.1. *Properties of the functional $\mathcal{H}$ and study of the minimum problem.*

The relevant differences between our situation and the one studied in [5] come only by the definition of the functional $\mathcal{H}$. Clearly, in fact, by (2.13), the assumptions on the elasticity tensor allow us to repeat the same arguments as in [5] about the quadratic term $\mathcal{Q}$. In this sections, instead, we show the connections between the duality (2.34) and the functional $\mathcal{H}$; only under certain conditions they will prove to be analogous to the case studied in [5]. We start by the following approximation result.

Lemma 3.1. *Let $K\colon \Omega \cup \Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a continuous multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$, which satisfies (2.5). Fix $\varepsilon > 0$. Let $\sigma \in \Sigma(\Omega)$ such that $\sigma(x) \in \mathbb{K}$ for a.e. $x \in \Omega$. Then there exist a sequence $(\sigma_k) \subset C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}) \cap \Sigma(\Omega)$ such that $(\sigma_k)_D$ is uniformly bounded in $L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ and:*

- (i) *$\sigma_k \to \sigma$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$*
- (ii) *$\operatorname{div} \sigma_k \to \operatorname{div} \sigma$ strongly in $L^n(\Omega; \mathbb{R}^n)$*
- (iii) *for every $\varepsilon > 0$ and every compact subset C of $\Omega \cup \Gamma_0$ there exists $k_0 = k_0(\varepsilon, C)$ such that $(\sigma_k)_D(x) \in K(x) + \varepsilon B$ for every $x \in C$, $k \geq k_0$.*

Proof. Taking the functions

$$\sigma_R(x) := \frac{1}{\mathcal{L}^n(B(x,R) \cap \Omega)} \int_{B(x,r)\cap\Omega} \sigma(y)\, dy$$

it only suffices to show that there exist R sufficiently small, only depending on ε and C, such that $(\sigma_R)_D(x) \in K(x) + \varepsilon B$ for every $x \in C$. Since the measure $\frac{\mathcal{L}^n}{\mathcal{L}^n(B(x,R)\cap\Omega)}$ is a probability measure on $B(x,R) \cap \Omega$ and a.e. in $B(x,R) \cap \Omega$ the function $\sigma_D(y)$ belongs to the fixed convex closed set $Z_{R,x} := \overline{Co \bigcup_{y\in B(x,R)} K(y)}$ it is well-known that, for every $x \in C$, $(\sigma_R)_D(x) \in Z_{R,x}$. By an usual uniform continuity argument, one has that there exist $\bar{R} > 0$, only depending on ε and C, such that for every $y \in B(x, \bar{R})$, $K(y) \subset K(x) + \varepsilon B$; this implies that for all $R \leq \bar{R}$, $Z_{R,x} \subset K(x) + \varepsilon B$, this is to say $(\sigma_R)_D(x) \in K(x) + \varepsilon B$ for every $x \in C$. Observe that this $\bar{R}$ does not depend on σ.

□

We can now prove the analogous of proposition 2.4 of [5], in the continuous case.

Proposition 3.2. *Let $K\colon \Omega \cup \Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a continuous multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$, which satisfies (2.5). Then, for every $p \in \Pi_{\Gamma_0}$:*

$$H(p) \geq [\sigma_D \colon p\,] \quad on\ \Omega \cup \Gamma_0 \tag{3.1}$$

Proof. Let $\varphi \in C(\bar{\Omega})$, $\varphi \geq 0$. We fix $\varepsilon > 0$ and a compact set $C \subset \Omega \cup \Gamma_0$ such that $|p|(\Omega \cup \Gamma_0 \setminus C) < \varepsilon$; considering the sequence σ_k defined as in the previous lemma (we omit to relabel subsequences), for every $k \in \mathbb{N}$, for every $x \in C$, we get that there exists $\zeta_{k,x} \in K(x)$ such that $|(\sigma_k)_D(x) - \zeta_{k,x}| < \varepsilon$, and so, by the Cauchy-Schwarz inequality:

$$(\sigma_k)_D(x) \colon (p/|p|)(x) \leq H(x, (p/|p|)(x)) + \varepsilon;$$

moreover, there exists a positive constant Z such that

$$||(\sigma_k)_D||_\infty \leq Z$$

for every $k \in \mathbb{N}$. Then we get, by (2.32) and the previous inequalities:

$$\begin{aligned}\langle [(\sigma_k)_D : p\,] | \varphi \rangle \qquad &= \int_C \varphi (\sigma_k)_D : \frac{p}{|p|} \;\; d|p| + \int_{\Omega \cup \Gamma_0 \setminus C} \varphi \; (\sigma_k)_D : \frac{p}{|p|} \;\; d|p| \\ &\leq \int_C \varphi \, H(x, \frac{p}{|p|}(x)) \;\; d|p| + \; \varepsilon \int_C \varphi \, d\,|p| + \; \varepsilon Z ||\varphi||_\infty \\ &\leq \langle H(p) | \varphi \rangle + \varepsilon \int_{\Omega \cup \Gamma_0} \varphi \;\; d\,|p| + \varepsilon Z ||\varphi||_\infty ,\end{aligned}$$

where the last inequality is trivially true by nonnegativeness of the integrands. As by (2.33),

$$[(\sigma_k)_D : p\,] \rightharpoonup^* [\sigma_D : p\,]$$

when k goes to $+\infty$, we obtain,

$$\langle [\sigma_D : p\,] | \varphi \rangle \leq \langle H(p) | \varphi \rangle + \varepsilon \int_{\Omega \cup \Gamma_0} \varphi \;\; d\,|p| + \varepsilon Z ||\varphi||_\infty$$

and letting ε to 0 we have $H(p) \geq [\sigma_D : p\,]$.

□

Concerning the case of a piecewise constant constraint $K(x)$ we assume that there exist a finite collection $(\Omega_i)_{i=1}^m$ of open subsets of Ω, a finite family $(K_i)_{i=1}^m$ of compact convex subsets of $\mathbb{M}_D^{n\times n}$ and a closed subset N of $\overline{\Omega}$ such that:

$$K_1 \subseteq K_2 \subseteq ... \subseteq K_m \tag{3.2}$$

$$\Omega \subset \bigcup_{i=1}^m \bar{\Omega}_i, \; \Omega_i \text{ pairwise disjoint} \tag{3.3}$$

$$(\partial\Omega_i \cap \partial\Omega_j \cap \Omega) \backslash N \text{ is a } C^2 \text{ hypersurface} \tag{3.4}$$

$$(\partial\Omega_i \cap \partial\Omega) \backslash N \text{ is a } C^2 \text{ hypersurface} \tag{3.5}$$

$$K(x) \equiv K_i \text{ if } x \in \Omega_i \tag{3.6}$$

$$K(x) \equiv K_{i\wedge j} \text{ if } x \in (\partial\Omega_i \cap \; \partial\Omega_j \cap \Omega) \backslash N \tag{3.7}$$

$$K(x) \equiv K_i \text{ if } x \in \partial\Omega_i \cap \Gamma_0 \backslash N \tag{3.8}$$

$$\mathcal{H}^{n-1}(N) = 0 \tag{3.9}$$

where $i \wedge j = \min\{i, j\}$. By suitably redefining $K(x)$ on N, we may assume that $K(x)$ is lower semicontinuous in $\Omega \cup \Gamma_0$.

Remark 3.3. Condition (3.2) is easily satisfied for example if the sets K_i are given by the von Mises'condition. Condition (3.4), as N is closed, implies that if $x \in (\partial\Omega_i \cap \partial\Omega_j \cap \Omega) \backslash N$ one has $\nu_{\Omega_i}(x) = -\nu_{\Omega_j}(x)$, as Ω_i and Ω_j are disjoint and so if $m \geq 3$, $(\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k \cap \Omega) \subseteq N$ whenever $i < j < k$. Similarly, by condition (3.5), we easily have that $(\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega) \subseteq N$ whenever $i \neq j$.

Proposition 3.4. *Let $K : \Omega \cup \Gamma_0 \to 2^{\mathbb{M}_D^{n\times n}}$ be a piecewise constant multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}_D^{n\times n}$, which satisfies (2.5), (3.2)-(3.9). Then, for every $p \in \Pi_{\Gamma_0}$:*

$$H(p) \geq [\sigma_D : p\,] \;\; \text{on } \Omega \cup \Gamma_0. \tag{3.10}$$

Proof. For each $1 \leq i < j \leq m$ let $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$. By the above remark, for each $1 \leq i < j \leq m$, $\Gamma_{ij} \backslash N$ are disjoint C^2 hypersurfaces contained in Ω. Fix $i < j$ and a relatively compact open subset S of $\Gamma_{ij} \backslash N$: we have $K(y) \equiv K_i$ for every $y \in \Gamma_{ij}$. We can then regard S as the Dirichlet part of the boundary of a C^2 open set $\Omega_i^{'} \subseteq \Omega_i$ oriented by the exterior normal. By Lemma 2.3 in [5] , we can take a sequence $(\sigma_k) \subset C^\infty(\bar{\Omega}_i'; \mathbb{M}^{n\times n}_{sym})$ such that:

$$\begin{aligned}
&\sigma_k \to \sigma \text{ in } L^2(\Omega_i'; \mathbb{M}^{n\times n}_{sym})\\
&\operatorname{div} \sigma_k \to \sigma \text{ in } L^n(\Omega_i'; \mathbb{R}^n)\\
&(\sigma_k)_D(x) \in K_i \text{ for every } x \in \bar{\Omega}_i'.
\end{aligned}$$

By [13, Chapter 2, Remark 7.2] and (2.37) we have:

$$[(\sigma_k)\nu_S]^\perp_\nu \overset{*}{\rightharpoonup} [\sigma\nu_S]^\perp_\nu,\ w^* \text{ in } L^\infty(S; \mathbb{R}^n) \tag{3.11}$$

and so, as, by (2.31), the functions σ_k easily satisfy

$$H(p) \geq [(\sigma_k)_D \colon p\ ] \ \ on\ S$$

we get that

$$H(p) \geq [\sigma_D \colon p\ ] \ \ on\ S$$

thanks to (3.11) and (2.38). This at once implies

$$H(p) \geq [\sigma_D \colon p\ ] \ \ on \bigcup_{1\leq i<j\leq m} \Gamma_{ij}$$

and then we conclude by applying Proposition 2.4. in [5] to each of the connected components of the open set $\Omega \cup \Gamma_0 \setminus (\bigcup_{1\leq i<j\leq m} \Gamma_{ij} \cup N)$, as the map K is constant on each of them. $\square$

Remark 3.5. Easy examples show that the required inequality is not true in the case of a piecewise constant multivalued map K, which does not satisfy (3.2), except in the case $n = 1$ where each function $\sigma \in \Sigma(\Omega)$ has a continuous representative by usual imbeddings, and so the required inequality trivially follows from (2.31) in the general case of a lower semicontinuous multivalued map K. As (3.1) is crucial in proving the Euler conditions related to the incremental minimum problems the results of [5], except in the onedimensional case, can be extended only assuming the conditions on K given in Proposition 3.2 or in Proposition 3.4.

It is well known that in the case when the multifunction K is constant, by [6] (Theorem 4) and [13] (Chapter II, Lemma 5.2) the following formula holds, for every $\mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ and every Borel set $B \subset \Omega \cup \Gamma_0$:

$$H(\mu)(B) = \sup\left\{\int_B \tau d\mu \colon \tau \in C_0(\Omega\cup\Gamma_0) \cap \mathcal{K}_D(\Omega)\right\}; \tag{3.12}$$

this will be useful in the next theorem, which is proven in the general case of a lower semicontinuous multivalued map.

Theorem 3.6. *Let $K \colon \Omega \cup \Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a lower semicontinuous multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$, which satisfies (2.5). Then, for every $\mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$:*

$$\mathcal{H}(\mu) = \sup\left\{\int_{\Omega\cup\Gamma_0} \tau d\mu \colon \tau \in C_0(\Omega\cup\Gamma_0) \cap \mathcal{K}_D(\Omega)\right\}; \tag{3.13}$$

Proof. The "$\geq$" inequality is trivial by (2.31).
The proof is given in two steps. First we suppose that the map is continuous. We assume that $\Gamma_0 = \partial\Omega$: this is not restrictive, because otherwise we can proceed by inner approximation with smooth sets Ω_k whose boundary is contained in $\Omega \cup \Gamma_0$.

Let $\varepsilon > 0$ be fixed. By the compactness of $\bar{\Omega}$ and standard properties of bounded Radon measures we can find a finite family of pairwise disjoint open sets $(Q_i)_{i=1}^{j(\varepsilon)}$ such that:

$$\bar{\Omega} \subseteq \bigcup_{i=1}^{j(\varepsilon)} \bar{Q}_i \tag{3.14}$$

$$K(x) \subseteq K(y) + \varepsilon B \quad \forall x, y \in Q_i \cap \bar{\Omega}, \ \forall 1 \leq i \leq j(\varepsilon); \tag{3.15}$$

$$|\mu|(\partial Q_i \cap \bar{\Omega}) = 0. \tag{3.16}$$

In particular, (3.15) easily yields:

$$|H(x,\xi) - H(y,\xi)| < \varepsilon|\xi| \quad \forall x, y \in Q_i \cap \bar{\Omega}, \ \forall 1 \leq i \leq j(\varepsilon), \ \forall \xi \in \mathbb{M}^{n\times n}_D. \tag{3.17}$$

We choose $X_i \in Q_i \cap \bar{\Omega}$: we can find, by (3.12), functions $\tau_i \in C_0(Q_i \cap \bar{\Omega}; \mathbb{M}^{n\times n}_D)$ such that:

$$\int_{Q_i\cap\bar{\Omega}} \tau_i \, d\mu \geq \int_{Q_i\cap\bar{\Omega}} H(X_i, \mu/|\mu|(x)) \, d|\mu| - \frac{\varepsilon}{j(\varepsilon)}$$

and such that

$$\tau_i(x) \in K(X_i)$$

for every x in $Q_i \cap \bar{\Omega}$. Since the mapping K is continuous, putting

$$\varphi_i(x) := \Pi_{K(x)}(\tau_i(x)),$$

where $\Pi_{K(x)}$ is the canonical projection on the closed convex set $K(x)$, the following properties are verified, thanks to (3.15):

$$\begin{aligned}
&\varphi_i \in C_0(Q_i \cap \bar{\Omega}; \mathbb{M}^{n\times n}_D) \\
&|\varphi_i(x) - \tau_i(x)| < \varepsilon \ \text{ for every } x \in Q_i \cap \bar{\Omega}, \ \text{for every } 1 \leq i \leq j(\varepsilon) \\
&\varphi_i(x) \in K(x) \ \text{ for every } x \in Q_i \cap \bar{\Omega}, \ \text{for every } 1 \leq i \leq j(\varepsilon);
\end{aligned}$$

We now easily have:

$$\int_{Q_i\cap\bar{\Omega}} \varphi_i \, d\mu \geq \int_{Q_i\cap\bar{\Omega}} H(X_i, \mu/|\mu|(x)) \, d|\mu| - \frac{\varepsilon}{j(\varepsilon)} - \varepsilon|\mu|(Q_i \cap \bar{\Omega});$$

so, if we put

$$\varphi(x) := \begin{cases} \varphi_i(x) & if\ x \in Q_i \cap \bar{\Omega} \\ 0 & if\ x \in \bar{\Omega} \setminus \bigcup_{i=1}^{j(\varepsilon)} Q_i \end{cases}$$

which is still continuous since the Q_i's are a finite collection and the functions φ_i vanish on the interfaces, we get an admissible function verifying

$$\int_{\bar{\Omega}} \varphi \, d\mu \geq \Big(\sum_{i=1}^{j(\varepsilon)} \int_{Q_i\cap\bar{\Omega}} H(X_i, \mu/|\mu|(x)) \, d|\mu|\Big) - \varepsilon - \varepsilon|\mu|(\bar{\Omega}).$$

From (3.17), since $|\mu/|\mu|| = 1$ $|\mu|$-a.e., we get:

$$\int_{\bar{\Omega}} \varphi \, d\mu \geq \int_{\bar{\Omega}} H(x, \mu/|\mu|(x)) \, d|\mu| - \varepsilon - 2\varepsilon|\mu|(\bar{\Omega})$$

and this implies:

$$\mathcal{H}(\mu) \leq \sup\left\{ \int_{\bar{\Omega}} \tau d\mu \colon \tau \in C_0(\Omega \cup \Gamma_0) \cap \mathcal{K}_D(\Omega) \right\},$$

as required.
By considering a continuous monotone approximation as in Remark 2.3 the result can be easily extended to the case of a lower semicontinuous multivalued map. $\square$

Remark 3.7. Actually, the regularity of admissible functions in (3.13) can be improved: it suffices in the previous proof to take a suitable uniform approximation in $C_0^\infty(\Omega \cup \Gamma_0))$ of the (admissible) function $(1-\varepsilon)\varphi$ to get:

$$\mathcal{H}(\mu) = \sup \left\{ \int_{\Omega \cup \Gamma_0} \tau d\mu \colon \tau \in C_0^\infty(\Omega \cup \Gamma_0) \cap \mathcal{K}_D(\Omega) \right\}. \tag{3.18}$$

As a corollary of the previous theorems, we have:

Corollary 3.8. *Let $K \colon \Omega \cup \Gamma_0 \to 2^{\mathbb{M}_D^{n\times n}}$ be a continuous multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}_D^{n\times n}$, which satisfies (2.5), or a piecewise constant multivalued map from $\Omega \cup \Gamma_0$ to the closed convex subsets of $\mathbb{M}_D^{n\times n}$, which satisfies (2.5), (3.2)-(3.9). Assume that $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and that there exists $\rho \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ such that $[\rho\nu] = g$ on Γ_1. Then, for every $p \in \Pi_{\Gamma_0}$:*

$$\mathcal{H}(p) = \sup \left\{ \langle \sigma_D | p \rangle \colon \sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega),\ [\sigma\nu] = g \text{ on } \Gamma_1 \right\}. \tag{3.19}$$

Proof. Apply the same argument of [5], Proposition 2.4., using Propositions 3.2 and 3.4, and (3.18). □

It is now possible to prove the existence of the minima of the incremental problems exactly the same way as in [5]. The data are the current values $p_0 \in \Pi_{\Gamma_0}$ of the plastic strain and the updated values $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, and $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$, of the boundary displacement and of the body and surface forces. The total load $\mathcal{L} \in BD(\Omega)'$ is defined by:

$$\langle \mathcal{L} | u \rangle := \langle f | u \rangle + \langle g | u \rangle_{\Gamma_1} \tag{3.20}$$

for every $u \in BD(\Omega)$. The minimum problem to be solved is then:

$$\min_{(u,e,p)\in A(w)} \{ \mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle \mathcal{L} | u \rangle \} \tag{3.21}$$

and the following uniform safe-load condition is assumed: there exists $\varrho \in L^2(\Omega; \mathbb{M}_{sym}^{n\times n})$ and $\alpha > 0$ such that:

$$-\mathrm{div}\varrho = f \text{ a.e on } \Omega, \quad [\varrho\nu] = g \text{ on } \Gamma_1, \tag{3.22}$$

and

$$\varrho_D(x) + \xi \in K(x) \tag{3.23}$$

for a.e. $x \in \Omega$, and for every $\xi \in \mathbb{M}_D^{n\times n}$ with $|\xi| \leq \alpha$. As shown in [5], the minimum problem (3.21) is equivalent to

$$\min_{(u,e,p)\in A(w)} \{ \mathcal{Q}(e) - \langle \varrho | e \rangle + \mathcal{H}(p - p_0) - \langle \varrho_D | p - p_0 \rangle \} \tag{3.24}$$

in the sense that they have the same solutions. The following theorem holds:

Theorem 3.9. *Let $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, and $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$, and let $\mathcal{L}$ be defined by (3.20). Assume (2.1), (2.2), (3.22), (3.23). Then, the minimum problem (3.21) has a solution.*

Proof. Thanks to (3.19), it is possible to apply the same arguments as in [5], Theorem 3.3. □

Also the same Euler conditions, summarized here for the reader's convenience, hold.

Theorem 3.10. *Assume (2.1), (2.2). Let $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, and $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$, let $(u, e, p) \in A(w)$, let $\sigma := \mathbb{C}e$ and let $\mathcal{L}$ be defined by (3.20). The following conditions are then equivalent:*

(a) *(u, e, p) is a solution of (3.21) with $p_0 = p$;*
(b) $-\mathcal{H}(q) \leq \langle \sigma | \eta \rangle - \langle \mathcal{L} | v \rangle \leq \mathcal{H}(q)$ for every $(v, \eta, q) \in A(0)$;
(c) $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\mathrm{div}\sigma = f$ a.e on Ω, $[\sigma\nu] = g$ on Γ_1.

Proof. See [5], Theorem 3.6. □

Remark 3.11. As in [5], Remark 3.9, if (u_1, e_1, p_0) and (u_2, e_2, p_0) are solutions to problem (3.21) with the same data, then $u_1 = u_2$ and $e_1 = e_2$ a.e. on Ω.

3.2. *Quasistatic evolution.*

Thanks to Propositions 3.2 and 3.4, and to (3.18) the same results as in [5] about existence and regularity of quasistatic evolutions hold in our case. Here they are collected for the reader's convenience. Through all of this Section it is understood that the stress constraint is given by a continuous multifunction satisfying (2.5), or piecewise constant satisfying (2.5), (3.2)- (3.9). We now consider time-dependent absolutely continuous boundary conditions $w(t)$, as well as absolutely continuous body and surface forces $f(t)$ and $g(t)$; also the uniform safe-load condition (2.15)-(2.17) is assumed. For every $t \in [0,T]$ the total load $\mathcal{L}(t) \in BD(\Omega)'$ is defined by:

$$\langle \mathcal{L}(t)|u\rangle := \langle f(t)|u\rangle + \langle g(t)|u\rangle_{\Gamma_1} \tag{3.25}$$

for every $u \in BD(\Omega)$. Considering the separable Banach space $Y := C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, whose dual X is given by $X = M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ and the closed convex set $\mathcal{K}_1 = \mathcal{K}_D(\Omega) \cap C_0(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, by (3.13) the functional $\mathcal{H}$ can be regarded as the support function of $\mathcal{K}_1$, that is:

$$\mathcal{H}(x) = \sup_{y\in\mathcal{K}_1}\langle x|y\rangle \ \ \forall x \in X; \tag{3.26}$$

this gives another proof of the w^*-lower semicontinuity of $\mathcal{H}$. By (2.7) the following bounds hold:

$$m||x|| \le \mathcal{H}(x) \le M||x|| \ \ \forall x \in X, \tag{3.27}$$

where $m,\ M$ are defined as in Section 2.1.

Given an absolutely continuous function $p : [0,T] \to X$ the $\mathcal{H}$-variation of p in the time interval $[s,t]$, which will play the role of the dissipation in $[s,t]$, will be defined as:

$$\mathcal{D}_{\mathcal{H}}(p; s, t) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}(p(t_j) - p(t_{j-1})) : \ s = t_0 \le t_1 \le \cdots \le t_N = t, N \in \mathbb{N} \Big\}.$$

By the w^*-lower semicontinuity of $\mathcal{H}$ it easily follows that:

$$\mathcal{D}_{\mathcal{H}}(p; a, b) \le \mathcal{D}_{\mathcal{H}}(p_k; a, b) \tag{3.28}$$

whenever $p_k(t) \to p(t)$ in the w^* -topology, for every $t \in [a,b]$; the following theorem, proved in ([5], Theorem 7.1) introduces the definition of $\dot{p}(t)$ we will use in the rest of the paper:

Theorem 3.12. *Given an absolutely continuous function $p : [0,T] \to X$, the weak*-limit*

$$\dot{p}(t) := w^* - \lim_{s\to t}\frac{p(s) - p(t)}{s - t} \tag{3.29}$$

exists for a.e. $t \in [0,T]$, and

$$\mathcal{H}(\dot{p}(t)) = \lim_{s\to t}\mathcal{H}\left(\frac{p(s) - p(t)}{s - t}\right). \tag{3.30}$$

Moreover, the function $t \mapsto \mathcal{H}(\dot{p}(t))$ is measurable and

$$\mathcal{D}_{\mathcal{H}}(p; a, b) = \int_a^b \mathcal{H}(\dot{p}(t))\, dt \tag{3.31}$$

for every $a,\ b \in [0,T]$ with $a \le b$.

The definition of quasistatic evolution is the same as in [5], that is:

Definition 3.13. A *quasistatic evolution* is a function $t \mapsto (u(t),\ e(t),\ p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ satisfying:

(qs1) global stability: for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t) | u(t) \rangle \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t) | v \rangle$$

for every $(v, \eta, q) \in A(w(t))$;

(qs2) energy balance: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \mathcal{L}(t) | u(t) \rangle = \mathcal{Q}(e(0)) - \langle \mathcal{L}(0) | u(0) \rangle + \\ + \int_0^t \{ \langle \sigma(s) | E\dot{w}(s) \rangle - \langle \mathcal{L}(s) | \dot{w}(s) \rangle - \langle \dot{\mathcal{L}}(s) | u(s) \rangle \} \, ds \,,$$

where $\sigma(t) := \mathbb{C}e(t)$.

By the results previously obtained there's actually nothing new to prove about the existence and regularity of quasistatic evolutions, in the sense that the same arguments as in [5] lead to the following:

Theorem 3.14. *Assume (2.1), (2.2),(2.12), (2.13), and that the maps $t \mapsto w(t)$, $t \mapsto f(t)$, $t \mapsto g(t)$ are absolutely continuous. If the stress constraint is given by a continuous multifunction satisfying (2.5), or piecewise constant satisfying (2.5), (3.2)-(3.9), and the safe-load conditions (2.15)-(2.17) hold, given a triple $(u_0, e_0, p_0) \in A(w(t))$ satisfying:*

$$\mathcal{Q}(e_0) - \langle \mathcal{L}(0) | u_0 \rangle \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p_0) - \langle \mathcal{L}(0) | u \rangle \tag{3.32}$$

for every $(u, e, p) \in A(w(0))$, there exist a quasistatic evolution $t \mapsto (u(t),\, e(t),\, p(t))$ such that $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$.
Moreover, the functions $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto u(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $BD(\Omega)$ respectively and the functions $t \mapsto e(t)$, $t \mapsto \sigma(t)$ are uniquely determined by the initial conditions.

Proof. Apply the same arguments as in [5], Theorem 4.5, Theorem 5.2, and Theorem 5.9 □

Remark 3.15. By the previous theorem and Theorem 3.12, condition (qs2) in the definition of quasistatic evolution is actually equivalent to:

(qs2') For a.e. $t \in [0,T]$

$$\langle \sigma(t) | \dot{e}(t) \rangle + \mathcal{H}(\dot{p}(t)) = \langle \sigma(t) | E\dot{w}(t) \rangle - \langle \mathcal{L}(t) | \dot{w}(t) \rangle + \langle \mathcal{L}(t) | \dot{u}(t) \rangle.$$

Moreover, as shown in [5], Remark 5.4, if the data are Lipschitz continuous in time, the functions $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto u(t)$ are Lipschitz continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $BD(\Omega)$ respectively.

3.3. *Strong formulation and precise definition of the stress.*

Also in this case the construction done in [5], Theorem 6.4, of a so-called *precise representative* of the stress satisfying a pointwise formulation of the classical flow rule, holds; namely, given an absolutely continuous mapping $t \mapsto (e(t), p(t), u(t))$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \times BD(\Omega)$ this formulation will be expressed by the inclusion:

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \in N_{K(x)}(\sigma_D(t,x)) \;\; for \; |\dot{p}(t)| - a.e. \, x \in \Omega \cup \Gamma_0 \tag{3.33}$$

where $N_{K(x)}$ denotes the normal cone to the closed convex set $K(x)$ and $\frac{\dot{p}(t)}{|\dot{p}(t)|}$ is the Radon-Nikodym derivative of $\dot{p}(t)$ with respect to its variation $|\dot{p}(t)|$, which is a function defined $|\dot{p}(t)| - a.e.$ in $\Omega \cup \Gamma_0$. We only have to check the measurability and the boundedness of this representative: they are guaranteed by the following simple lemma, which we state in the general case when K is only lower semicontinuous.

Lemma 3.16. *Let $K\colon \Omega\cup\Gamma_0 \to 2^{\mathbb{M}^{n\times n}_D}$ be a lower semicontinuous multivalued map from $\Omega\cup\Gamma_0$ to the closed convex subsets of $\mathbb{M}^{n\times n}_D$, which satisfies (2.5). Let $\xi\colon \Omega\cup\Gamma_0 \to \mathbb{M}^{n\times n}_D$ be a bounded Borel function. Then, for every $\mu\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ the map:*

$$x\mapsto (\partial_\xi)_0 H(x,\xi(x)),$$

where $(\partial_\xi)_0 H(x,\xi)$ denotes the element of $\partial_\xi H(x,\xi)$ with minimum norm, is μ-measurable and bounded.

Proof. It easily follows from [4], Theorem III.41, that if the multifunction:

$$x\mapsto \partial_\xi H(x,\xi(x)) \tag{3.34}$$

is μ-measurable, so is the map:

$$x\mapsto (\partial_\xi)_0 H(x,\xi(x)),$$

hence it only suffices to prove (3.34). Take an open set $U\subset\mathbb{M}^{n\times n}_D$: denoting by $G(x,\zeta)$ the Borel function from the product space $(\Omega\cup\Gamma_0)\times U$ into $\mathbb{R}$ defined by:

$$G(x,\zeta) := H(x,\xi(x)) + \delta_{K(x)}(\zeta) - \zeta\colon \xi(x),$$

we easily get, by well-known properties of the subdifferential (see for instance [1], Propisition 9.5.1), that:

$$\{x\in X | \partial_\xi H(x,\xi(x))\cap U\neq\emptyset\} = \Pi_{\Omega\cup\Gamma_0}\left\{(x,\zeta)|\ G(x,\zeta)=0\right\},$$

where $\Pi_{\Omega\cup\Gamma_0}$ is the projection on the first factor: by the Projection Theorem ([4], theorem III.23), this is an universally measurable set. Finally, the boundedness of the map $x\mapsto (\partial_\xi)_0 H(x,\xi(x))$ follows at once from the inclusion $\partial_\xi H(x,\xi)\subseteq K(x)$ which is a trivial consequence of the definitions. $\square$

We have the following:

Theorem 3.17. *Assume (2.1), (2.2),(2.12), (2.13), that the maps $t\mapsto w(t)$, $t\mapsto f(t)$, $t\mapsto g(t)$ are absolutely continuous and that (2.15)-(2.17) hold. Assume that the stress constraint is given by a continuous multifunction satisfying (2.5), or piecewise constant satisfying (2.5), (3.2)-(3.9). Let $t\mapsto (u(t),e(t),p(t))$ be a function from $[0,T]$ into $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ $\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, let $\sigma(t):=\mathbb{C}e(t)$, and let $\mu(t):=\mathcal{L}^n+|\dot p(t)|$. Then $t\mapsto (u(t),e(t),p(t))$ is a quasistatic evolution if and only if*

- (e) *$t\mapsto (u(t),e(t),p(t))$ is absolutely continuous and*
  - (e1) *for every $t\in[0,T]$ we have $(u(t),e(t),p(t))\in A(w(t))$, $\sigma(t)\in\Sigma(\Omega)\cap\mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t)=f(t)$ a.e. on Ω, and $[\sigma(t)\nu]=g(t)$ on Γ_1,*
  - (e2) *for a.e. $t\in[0,T]$ there exists $\hat\sigma_D(t)\in L^\infty_{\mu(t)}(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ such that*

$$\hat\sigma_D(t)=\sigma_D(t) \qquad \mathcal{L}^n\text{-a.e. on } \Omega, \tag{3.35}$$

$$[\sigma_D(t):\dot p(t)] = \Big(\hat\sigma_D(t):\frac{\dot p(t)}{|\dot p(t)|}\Big)\,|\dot p(t)| \qquad \text{on } \Omega\cup\Gamma_0, \tag{3.36}$$

$$\frac{\dot p(t)}{|\dot p(t)|}(x)\in N_{K(x)}(\hat\sigma_D(t,x)) \qquad \text{for } |\dot p(t)|\text{-a.e. } x\in\Omega\cup\Gamma_0, \tag{3.37}$$

where $\hat\sigma_D(t,x)$ denotes the value of $\hat\sigma_D(t)$ at the point x.

Proof. By Lemma 3.16, one can repeat the same construction as in [5], Theorem 6.4; the proof then follows with the same argument. $\square$

Remark 3.18. By Remark 3.5, in the special case of the dimension $n=1$ (and related problems) the results of this section hold with the weaker assumption of a lower semicontinuous multivalued map K; there is no need of constructing a precise representative as all admissible σ have a continuous representative by canonical imbeddings.

4. One-dimensional problems

For the definition of the *set of admissible displacements and strains* in dimension $n = 1$, we refer to Section 2.2; moreover we recall that in this case it is always assumed $\sigma = \sigma_D$.

4.1. *The case of simple shear.*

Before studying one-dimensional problems, here is a typical situation when the solution to the quasistatic evolution problem leads to the solution of a one-dimensional one. Let $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$, let $\Gamma_0 = (\{-\frac{1}{2}\} \times (-\frac{1}{2}, \frac{1}{2})) \cup (\{\frac{1}{2}\} \times (-\frac{1}{2}, \frac{1}{2}))$, and $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}_0$; Dirichlet conditions are prescribed only on Γ_0, while periodic ones (in the second coordinate) are assumed on Γ_1. this problem will be referred as (DP) (Dirichlet-periodic problem). The elasticity tensor $\mathbb{C}$ is assumed to be isotropic (see Section 2.2). In the following proposition x and y respectively denote the first and the second coordinate in $\mathbb{R}^2$,

Proposition 4.1. *Assume that the boundary displacement is of the form*

$$\hat{w}(t, x, y) = \sqrt{2}w(t, x)\mathbf{e_2},$$

the applied load is of the form

$$\hat{f}(t, x, y) = \frac{1}{\sqrt{2}} f(t, x)\mathbf{e_2},$$

the constraint on the deviatoric part is given by:

$$K(x, y) := \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} | \sqrt{\alpha^2 + \beta^2} \le k_1(x) \right\}$$

where $k_1(x)$ is a strictly positive lower semicontinuous function only depending on the first coordinate. Assume also that the initial conditions are of the form:

$$\begin{aligned} &\hat{u}_0(x, y) = \sqrt{2}u_0(x) \\ &\hat{e}_0(x, y) = M(e_0(x)) \\ &\hat{p}_0 = M(p_0 \otimes \mathcal{L}^1) \textit{ in } \Omega \\ &\hat{p}_0 = (\hat{w}_0 - \hat{u}_0) \odot \nu \textit{ on } \Gamma_0 . \end{aligned}$$

where (u_0, e_0, p_0) is an admissible triple for the boundary displacement w_0, the constraint $K_1(x) := [-k_1(x), k_1(x)]$ and M is defined for every $\alpha \in \mathbb{R}$ as follows:

$$M(\alpha) := \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ \frac{\alpha}{\sqrt{2}} & 0 \end{pmatrix}.$$

Then every quasistatic evolution satisfying (DP) conditions, and the given boundary and initial conditions is of the form:

$$\begin{aligned} &\hat{u}(t, x, y) = \sqrt{2}u(t, x) \\ &\hat{e}(t, x, y) = M(e(t, x)) \\ &\hat{p}(t) = M(p_0 \otimes \mathcal{L}^1) \textit{ in } \Omega \\ &\hat{p}(t) = (\hat{w}(t) - \hat{u}(t)) \odot \nu \textit{ on } \Gamma_0 , \end{aligned}$$

where $(u(t), e(t), p(t))$ is an admissible triple for the one-dimensional Dirichlet problem on the interval $(-\frac{1}{2}, \frac{1}{2})$ with boundary displacement $w(t)$ verifying:

(Cf1) $\frac{d}{dx}u(t) = e(t) + p(t)$,
(Cf2) $\sigma(t) = 2\mu\, e(t)$,
(Cf3) $-\frac{d}{dx}\,\sigma(t) = f(t)$,
(Cf4) $|\sigma(t, x)| \le k_1(x)$
(Cf5) $(\xi - \sigma(t, x)) : \frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \le 0$ *for* $|\dot{p}(t)|$*-a.e.* $x \in [-\frac{1}{2}, \frac{1}{2}]$, *for every* $\xi \in [-k_1(x), k_1(x)]$.

Proof. It is easily seen that if $(u(t),\, e(t),\, p(t))$ is an admissible triple, so is $(\hat{u}(t),\, \hat{e}(t),\, \hat{p}(t))$. In our hypothesis, $\hat{\sigma} = \hat{\sigma}_D$, so the constraint is easily satisfied and so is the equilibrium equation, since $-\mathrm{div}\, \sigma_D(t,x,y) = f(t,x.y)$ whenever (Cf3) holds.
Frome the definition of $\hat{p}(t)$ it easily follows that:

$$\frac{\dot{\hat{p}}(t)}{|\dot{\hat{p}}(t)|}(x,y) = M(\frac{\dot{p}(t)}{|\dot{p}(t)|}(x));$$

this equality is also true on the boundary Γ_0. Hence, for fixed $(x,y) \in \Omega \cup \Gamma_0$, taken $\hat{\xi} \in K(x,y)$:

$$(\hat{\xi} - \hat{\sigma}(t,x,y)) : \frac{\dot{\hat{p}}(t)}{|\dot{\hat{p}}(t)|}(x,y) = \sqrt{2}(\hat{\xi}_{12} - \hat{\sigma}_{12}(t,x,y))\frac{\dot{p}(t)}{|\dot{p}(t)|}(x),$$

that is to say, with easy computations:

$$(\hat{\xi} - \hat{\sigma}(t,x,y)) : \frac{\dot{\hat{p}}(t)}{|\dot{\hat{p}}(t)|}(x,y) = (\sqrt{2}\hat{\xi}_{12} - \sigma_{12}(t,x))\frac{\dot{p}(t)}{|\dot{p}(t)|}(x);$$

as the stress constraint implies, for every $\hat{\xi} \in K(x,y)$:

$$|\sqrt{2}\hat{\xi}_{12}| \leq k_1(x),$$

by (Cf5) we conclude that, $|\dot{\hat{p}}(t)|$-a.e.

$$(\hat{\xi} - \hat{\sigma}(t,x,y)) : \frac{\dot{\hat{p}}(t)}{|\dot{\hat{p}}(t)|}(x,y) \leq 0$$

and so $(\hat{u}(t),\, \hat{e}(t),\, \hat{p}(t))$ is a quasistatic evolution according to Theorem 3.17.
Vice versa, since $\hat{\sigma}(t)$ is uniquely determined by the initial conditions, so it certainly is of the required form, it only suffices to verify that, given $\hat{q}(t)$ such that $(\hat{u}(t),\, \hat{e}(t),\, \hat{q}(t))$ is a quasistatic evolution, there exists $q_1(t) \in M_b([-\frac{1}{2}, \frac{1}{2}])$ such that $q(t) = M(q_1(t) \otimes \mathcal{L}^1)$: once this is achieved infact, straightforward computations assure that (Cf1)-(Cf5) are verified. The constraint $K(x,y)$ is a sphere in the space of deviatoric matrices, so it is well known that:

$$N_{K(x,y)}(\hat{\sigma}(t,x,y)) = \{\lambda \hat{\sigma}(t,x,y) \,|\, \lambda \geq 0\}\,;$$

by Theorem 3.17, we deduce that, for a.e. t, there exists $g(t,x,y) \in L^1_{|\dot{\hat{q}}(t)|}$ such that:

$$\frac{\dot{\hat{q}}(t)}{|\dot{\hat{q}}(t)|}(x,y) = M(g(t,x,y));$$

this implies by absolute continuity, taking into account the initial conditions, that:

$$\hat{q}(t) = M(\nu(t))$$

for a suitable $\nu(t) \in M_b(\Omega \cup \Gamma_0)$, and that:

$$Eu(t) = M(\sigma(t,x) + \nu(t)).$$

This implies that there exists two functions $u_1(t),\, u_2(t) \in BV([-\frac{1}{2}, \frac{1}{2}])$ such that:

$$\hat{u}(t,x,y) = u_1(t,y)\mathbf{e_1} + u_2(t,x)\mathbf{e_2};$$

but (2.20), since the normal to Γ_0 is parallel to $\mathbf{e_1}$ implies that:

$$u_1(y) = 0 \;\; \mathcal{L}^1 - a.e. \quad \text{in}\, (-\frac{1}{2}, \frac{1}{2}).$$

From this the conclusion easily follows. □

4.2. *The autonomous case.*

We first study a one-dimensional Dirichlet problem where the force does not depend on time: in this case a complete qualitative study, as it will be shown, is possible. The data of the problem are an open interval in $\mathbb{R}$ (only to simplify notation, this interval will be $(0,1)$),the stress constraint

$$K(x) := [-\alpha(x), \beta(x)] \tag{4.1}$$

where α, β are strictly positive lower semicontinuous functions, and the boundary displacement $w(t,x)$: from now on, we define $w_0(t) := w(t,0)$ and similarly $w_1(t) := w(t,1)$. These two functions are absolutely continuous with respect to time. For simplicity, the function μ in the elasticity tensor will be taken constant and equal to 1. According to Theorem 3.17, the problem to be solved will be then:

$$\frac{d}{dx}u(t) = \sigma(t) + p(t), \tag{4.2}$$

$$-\frac{d}{dx}\sigma(t) = f \tag{4.3}$$

$$\sigma(t,x) \in K(x) \tag{4.4}$$

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \in N_{K(x)}(\sigma(t,x)) \qquad \text{for } |\dot{p}(t)|\text{-a.e. } x \in [0,1]; \tag{4.5}$$

in fact, since the elastic part is in the 1-dimensional case absolutely continuous in space by canonical imbeddings, the continuous representative is a precise representative in the sense of Theorem 3.17. We easily have that:

$$N_{K(x)}(\sigma(t,x)) = \begin{cases} [0,+\infty) & \text{if } \sigma(t,x) = \beta(x) \\ \{0\} & \text{if } \alpha(x) < \sigma(t,x) < \beta(x) \\ (-\infty, 0] & \text{if } \sigma(t,x) = \alpha(x) \end{cases} \tag{4.6}$$

We assume for simplicity that

$$\sigma_0(x) := \sigma(0,x) \in (-\alpha(x), \beta(x))$$

for every x and that $p_0 \equiv 0$; most of the reasonings later developed will also work, however, with different initial data, with slight adaptations. The initial displacement can be recovered from σ_0 by integration, taking into account the boundary conditions. The function σ_0 provides a safe-load solution at any time, so quasistatic evolution exists in every time interval $[0,T]$ with $T > 0$: from now on, T will be fixed. By (4.3) the elastic part will be given by:

$$\sigma(t,x) = \sigma_0(x) + c(t) \tag{4.7}$$

for a suitable function $c(t)$ to be determined in the sequel, and, by continuity in time, at small times the solution will be purely elastic, that is to say, taking into account the boundary conditions

$$c(t) = h(t) := w_1(t) - w_0(t) - \int_0^1 \sigma_0(x)\, dx. \tag{4.8}$$

In fact, from (4.5), since, given a convex closed set C $N_C(\xi) = \{0\}$ whenever ξ is an interior point of C, no plastic deformation can appear as long as $\sigma(t,x)$ is in the interior of $K(x)$ at any point. Now if for every $t \in [0,T]$,

$$-(\min_{[0,1]}(\alpha(x) + \sigma_0(x))) < h(t) < (\min_{[0,1]}(\beta(x) - \sigma_0(x)))$$

the solution remains purely elastic and it is given by (4.7) and (4.8). Plastic deformation may instead occur when, putting

$$\nu := \min_{[0,1]}(\beta(x) - \sigma_0(x)), \tag{4.9}$$

$$\lambda := \min_{[0,1]}(\sigma_0(x) + \alpha(x)), \tag{4.10}$$

either $\inf\{t \in [0,T] | h(t) = \nu\}$ or $\inf\{t \in [0,T] | h(t) = -\lambda\}$ are finite; they cannot coincide, as both λ, ν are strictly positive, so it is not restrictive to assume for example that

$$t_1 := \inf\{t \in [0,T] | h(t) = \nu\} < \inf\{t \in [0,T] | h(t) = -\lambda\}$$

(the discussion of the other case is similar). First of all, we make some general remarks.

An interesting case is when the minimum values defining λ, ν are attained at a finite number of points of $[0,1]$: in this case the plastic deformation will be necessarily *concentrated*. In fact, let $\{x_i\}_{i=1}^m$ be the set of points such that

$$\beta(x_i) - \sigma_0(x_i) = \nu$$

and $\{y_j\}_{j=1}^n$ the set of points such that

$$\sigma_0(y_j) + \alpha(y_j) = \lambda.$$

Clearly, by (4.4), the function $c(t)$ defined in (4.7), must satisfy:

$$-\lambda \leq c(t) \leq \nu; \tag{4.11}$$

so, whenever $x \notin \{x_i\}_{i=1}^m \cup \{y_j\}_{j=1}^n$, by lower semicontinuity of the functions involved there exists an open neighborhood U_x of x such that, for every $y \in U_x$:

$$-\alpha(y) < \sigma_0(y) - \lambda \leq \sigma_0(y) + c(t) \leq \sigma_0(x) + \nu < \beta(y);$$

from (4.5), since, given a convex closed set C $N_C(\xi) = \{0\}$ whenever ξ is an interior point of C, it is easy to conclude that:

$$\frac{\dot{p}(t)}{|\dot{p}(t)|} = 0 \quad |\dot{p}(t)|\text{-a.e. in } U_x,$$

hence $\operatorname{supp}\dot{p}(t) \subseteq \{x_1, ..., x_m\} \cup \{y_1, ..., y_n\}$: since this set is finite (this is the only point where it is exactly needed), one has that there exist suitable functions φ_i, ψ_j such that:

$$\langle f, \dot{p}(t)\rangle = \sum_{i=1}^m \varphi_i(t)\, f(x_i) + \sum_{j=1}^n \psi_j(t)\, f(y_j) \text{ for every } f \in C([0,1])$$

and by the w^*-absolute continuity of the map $t \mapsto p(t)$, one gets:

$$\langle f, p(t)\rangle = \sum_{i=1}^m (\int_0^t \varphi_i(s)\, ds)\, f(x_i) + \sum_{j=1}^n (\int_0^t \psi_j(s)\, ds)\, f(y_j) \text{ for every } f \in C([0,1]$$

hence:

$$p(t) = \sum_{i=1}^m (\int_0^t \varphi_i(s)\, ds)\delta_{x_i} + \sum_{j=1}^n (\int_0^t \psi_j(s)\, ds)\delta_{y_j}. \tag{4.12}$$

Given a quasistatic evolution $(u(t), \sigma(t), p(t))$, we define:

$$P(t) = p(t)([0,1]); \tag{4.13}$$

we observe that P is absolutely continuous and that:

$$\dot{P}(t) = \dot{p}(t)([0,1]). \tag{4.14}$$

By (4.2), taking into account the relaxed boundary conditions and (4.7),(4.8) one gets the equality $\sigma(t,x) = \sigma_0(x) + h(t) - P(t)$, so $P(t)$ is uniquely determined by $\sigma(t)$.

We claim that there exists a right neighborhood $[t_1, t_2]$ of t_1 such that, for every t in this neighborhood:

$$P(t) = r(t) := \max_{s \in [t_1, t]} h(s) - \nu \tag{4.15}$$

$$\sigma(t,x) = \sigma_0(x) + h(t) - r(t). \tag{4.16}$$

To prove the claim, we define $\tilde{\sigma}(t,x)$ as the right-hand side of (4.16) and observe that it suffices to find an absolutely continuous function $t \mapsto (\tilde{u}(t), \tilde{p}(t))$ such that $\tilde{p}(t)([0,1]) = r(t)$ and the triple $(\tilde{u}(t), \tilde{\sigma}(t), \tilde{p}(t))$ satisfies (4.2)-(4.5).

Clearly (4.3) is verified by $\tilde{\sigma}(t)$: the definition of $r(t)$ yields

$$\tilde{\sigma}(t,x) \leq \beta(x)$$

and by continuity in time, in a suitable neighborhood of t_1, we will have:

$$\tilde{\sigma}(t,x) \geq -\alpha(x).$$

Regarding (4.5), we choose $\tilde{p}(t)$ defined by (4.12), with φ_i replaced by $\tilde{\varphi}_i$ and $\psi_j \equiv 0$ (we already know that in our hypothesis no plastification appear at these times at points y_j). It then suffices to check that there is, for every $1 \leq i \leq m$, for every $t \in [t_1, t_2]$ a choice of $\tilde{\varphi}_i(t)$ compatible with the condition:

$$\dot{r}(t) = \sum_{i=1}^{m} \tilde{\varphi}_i(t) \tag{4.17}$$

such that, for every $1 \leq i \leq m$,

$$\frac{\dot{\tilde{p}}(t)}{|\dot{\tilde{p}}(t)|}(x_i) \in N_{K(x_i)}(\tilde{\sigma}(t,x_i)). \tag{4.18}$$

Since r is absolutely continuous and nondecreasing, $\dot{r}(t) \geq 0$: moreover,

$$r(t) > h(t) - \nu \Rightarrow \dot{r}(t) = 0; \tag{4.19}$$

indeed, under this assumption, r is constant in a neighborhood of t. In this case, the choice:

$$\tilde{\varphi}_i(t) = 0 \; \forall \, 1 \leq i \leq m$$

is clearly compatible with both (4.17) and (4.18). The only other possible case is $r(t) = h(t) - \nu$; in this case, by (4.9), $\tilde{\sigma}(t,x_i) = \beta(x_i)$: (4.17) and (4.18), by (4.6), are easily satisfied by putting:

$$\tilde{\varphi}_i(t) \geq 0, \;\; \sum_{i=1}^{m} \tilde{\varphi}_i(t) = \dot{r}(t).$$

It is now easy to verify that:

$$u(t,x) = \int_0^x \sigma_0(y)\,dy + (h(t) - r(t))x + \tilde{p}(t)([0,x)) + w_0(t)$$

is a distributional primitive of $\tilde{\sigma} + \tilde{p}$ satisfying the relaxed boundary conditions, so the claim is proved. Observe that, by the uniqueness of σ the conditions on $\tilde{\varphi}_i$ are also necessary. We also can get rid of t_1 by putting:

$$r(t) = \max\nolimits_{s\in[0,t]}(h(s) - \nu)^+. \tag{4.20}$$

The solutions found in this way obviously change form if at a certain time, still labeled by t_2, one has:

$$\min\nolimits_{x\in[0,1]}\sigma(t_2,x) = -\lambda$$

that is to say:

$$h(t_2) - r(t_2) = -\lambda.$$

At this time a concentrated plastic deformation can occur at points y_j's.

We claim that there exists a right neighborhood $[t_2, t_3]$ where the elastic part of the solution is given by:

$$\sigma(t,x) = \sigma_0(x) + h(t) - r(t) - s(t), \tag{4.21}$$

where $s(t)$ is defined as:

$$s(t) := \min\nolimits_{s\in[0,t]}[-(h(s) - r(s) + \lambda)^-]; \tag{4.22}$$

as before, we get rid of t_2 in the definition of $s(t)$ by noticing that, at times smaller than t_2 $h(s) - r(s) + \lambda$ is strictly positive, hence at these times $s(t) = 0$. In general, it turns out that $s(t)$ is a non-positive, nonincreasing, absolutely continuous function. Noticing that t_3, if we assume the claim, can be defined as:

$$t_3 = \inf\{t \in [0,T] : h(t) - r(t) - s(t) \geq \nu\}, \tag{4.23}$$

with the convention $t_3 = T$ whenever the right-hand side is the empty set, one also has, by the nonpositiveness of the function s, with an easy application of (4.19), that:

$$\dot{r}(t) = 0 \, \forall t \in [t_2, t_3) : \tag{4.24}$$

in particular, equation:

$$\sum_{i=1}^{m} \varphi_i(t) = \dot{r}(t)$$

is still verified (no new plastification is possible at points x_i at these times, so we are forced to put $\varphi_i(t) = 0$ for every $t \in [t_2, t_3)$). The claim can now be proved with the same reasonings as before, leading to the following possible choices of ψ_j:

$$\psi_j(t) \leq 0, \ \ \sum_{j=1}^{n} \psi_j(t) = \dot{s}(t).$$

Clearly, the procedure can be further iterated:if $t_3 < T$, at time t_3 new plastification can appear at points x_i: one should then define a new function $r_1(t)$ as:

$$r_1(t) = \max_{s\in[0,t]} \left[(h(s) - r(s) - s(s) - \nu)\right]^+$$

and then proceed as before.

These results can be summarized as follows:

Theorem 4.2. *Let $T > 0$ be fixed, let $\Omega = (0,1)$, let $f(t,x) \equiv f(x)$, let the stress constraint be $K(x) := [-\alpha(x), \beta(x)]$ where α, β are strictly positive lower semicontinuous functions, and the boundary displacement $w(t)$ be an absolutely continuous function from $[0,T]$ into $H^1((0,1))$, let the stress tensor be equal to 1. Suppose that*

$$\sigma_0(x) := \sigma(0,x) \in (-\alpha(x), \beta(x))$$

for every x, and that

$$t_1 := \inf\{t \in [0,T] | h(t) = \nu\} < (T \wedge \inf\{t \in [0,T] | h(t) = -\lambda\}),$$

where $h(t)$ is defined as in (4.8) and

$$\nu := \min_{[0,1]} (\beta(x) - \sigma_0(x)), \ \ \lambda := \min_{[0,1]} (\sigma_0(x) + \alpha(x)). \tag{4.25}$$

Suppose that the values λ, ν are attained at a finite number of points in the interval $[0,1]$, and let $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ be the set of points such that

$$\beta(x_i) - \sigma_0(x_i) = \nu$$

and

$$\sigma_0(y_j) + \alpha(y_j) = \lambda.$$

Let $r(t)$, $s(t)$, t_3 as in (4.20), (4.22), (4.23) respectively: then, for every quasistatic evolution $(u(t), \sigma(t), p(t))$ in the time interval $[0, t_3)$ one has:

$$\sigma(t,x) = \sigma_0(x) + h(t) - r(t) - s(t) \tag{4.26}$$

$$p(t) = \sum_{i=1}^{m} (\int_0^t \varphi_i(s)\, ds)\delta_{x_i} + \sum_{j=1}^{n} (\int_0^t \psi_j(s)\, ds)\delta_{y_j} \tag{4.27}$$

where the functions φ_i, ψ_j respectively satisfy

$$\varphi_i(t) \geq 0, \quad \sum_{i=1}^{m} \varphi_i(t) = \dot{r}(t) \tag{4.28}$$

$$\psi_j(t) \leq 0, \quad \sum_{j=1}^{n} \psi_j(t) = \dot{s}(t). \tag{4.29}$$

Viceversa, to every pair $(\sigma(t), p(t))$ defined as before, corresponds a quasistatic evolution.

Remark 4.3. The solution is then unique whenever it is purely elastic or when $m = n = 1$; elsewhere, we have always infinitely many solutions. For other examples of non-uniqueness of the solutions, see [11], or [12]. Moreover, as we already knew, lipschitzianity in the time variable of the boundary displacement (in this case, this is the only time-dependent datum, as both the force and the safe-load solution only depend on the space variable) ensures lipschitzianity in time of the quasistatic evolutions, but Theorem 4.2 is also an example showing that this is the best dependence to be a priori assumed on the solutions; in fact, whatever the regularity of the boundary displacement is, the functions r, s of the previous theorem are in general not better than lipschitzian, and so are obviously the explicit solutions found in this particular case.

4.3. *The nonautonomous case.*

This last, very simple example is a case where a solution can be explicitly found but, due to the time dipendence of the force, we cannot observe a concentration phenomenon, even if the situation is somewhat similar to that considered in Theorem 4.2: the effect of a time-dependent force is in fact, as it is intuitive, that the points where plasticity appears "move" along the body, giving rise to a diffuse plastic deformation.

Example 4.4. Let $\Omega := (0, 3)$, consider a boundary displacement $w(t, x) := tx$ and an applied load of the form $f(t, x) := -t$. The stress constraint is $K(x) := [-1, k(x)]$ where:

$$k(x) := 3 + (x - 1)^2 \tag{4.30}$$

and the initial condition is the null triple. We consider the quasistatic evolution problem in the time interval $[0, \frac{5}{2}]$: a correspondant safe-load solution is for example $\varrho(t, x) = tx - \frac{2}{3}$. We easily have $\sigma(t, x) = tx + c(t)$ and at small times $c(t) \equiv 0$. Plastic deformation may occur at time when the function tx meets the yield surface at at least one point, and this is easily equivalent to say that, for fixed t, the function:

$$d(t, x) = 3 + (x - 1)^2 - tx$$

has minimum value 0 in the space variable. The function d, for fixed t takes its minimum value at the point $x = \frac{t+2}{2}$: the smallest time when the minimum value is 0 is then, by a direct computation, $t = 2$. At this time a plastic deformation appears at the point $x = 2$. We claim that for $t \in (2, \frac{5}{2}]$ one has $c(t) = 3 - \frac{t^2}{4} - t$. We assume the claim: with this assumption one has, by (4.5), that at every time $t \in [2, \frac{5}{2}]$ only at the point $x = \frac{t+2}{2}$ (which is always in Ω) a new plastic deformation appears, so we have:

$$\dot{p}(t) = \alpha(t)\delta_{\frac{t+2}{2}} \tag{4.31}$$

for a function α to be determined. Taking a test function φ, by absolute continuity and by the initial conditions one has:

$$\begin{aligned}\langle p(t),\, \varphi\rangle &= \int_2^t \frac{d}{ds}\langle p(s),\, \varphi\rangle\, ds \\ &= \int_2^t \alpha(s)\varphi(\frac{t+2}{2})\, ds \\ &= \int_2^{\frac{t+2}{2}} 2\alpha(2(x-1))\varphi(x)\, dx\end{aligned}$$

with an easy change of variables, so the claim implies that:

$$p(t) = 2\alpha(2(x-1))\chi_{[2,\frac{t+2}{2}]}(x)\,\mathcal{L}^1, \tag{4.32}$$

that is a diffuse measure. Finally as the boundary conditions imply:

$$p(t)([0,3]) = -c(t) \tag{4.33}$$

by deriving the formula:

$$\int_2^{\frac{t+2}{2}} 2\alpha(2(x-1))\, dx = \frac{t^2}{4} + t - 3 \tag{4.34}$$

one gets:

$$\alpha(t) = \frac{t}{2} + 1$$

hence

$$\begin{aligned}p(t) &= 2x\chi_{[2,\frac{t+2}{2}]}(x)\,\mathcal{L}^1 \\ \dot{p}(t) &= (\frac{t}{2}+1)\,\delta_{\frac{t+2}{2}}.\end{aligned} \tag{4.35}$$

With this choice of $p(t)$ the claim can now a posteriori be easily verified. By the uniqueness of σ one can also infer that this is the only solution to the considered problem (the above reasonings show that if we assume that σ is of the claimed form, $\dot{p}(t)$ necessarily verifies (4.35)): we thus have a diffuse plastic deformation as announced; at every fixed time, instead, the w^*- derivative of the measure $p(t)$ is concentrated.

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References

[1] Attouch H., Buttazzo G., Michaille G. Variational analysis in Sobolev e BV spaces. MPS-SIAM series on optimization, 2006.

[2] Ambrosio L., Fusco N., Pallara D..: Functions of bounded variation and free discontinuity problems. Oxford University Press, Oxford, 2000

[3] Anzellotti G.: On the extremal stress and displacement in Hencky plasticity. *Duke Math. J.* **51** (1984), 133-147.

[4] Castaing C., Valadier M.: Convex Analysis and measurable multifunctions. Springer-verlag, Berlin, 1977

[5] Dal Maso G., De Simone A., Mora M.G.: Quasistatic evolutions problems for linearly elastic-plastic perfect materials *Arch. Rational Mech. Anal.* **180** (2006), 237-291.

[6] Goffman C., Serrin J.: Sublinear functions of measures and variational integrals. *Duke Math. J.* **31** (1964), 159-178.

[7] Michael, E.: Continuous selections I, *Ann. of Math.* **63** (1956), 361-382.

[8] Kohn R.V., Temam R.: Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.* **10** (1983), 1-35.

[9] Lubliner J.: Plasticity theory. Macmillan Publishing Company, New York, 1990.

[10] Rudin W.: Real and Complex Analysis. McGraw-Hill, New York, 1966.

[11] Suquet, P.: Sur les équations de la plasticité: existence et regularité des solutions. *J. Mécanique*, **20** (1981), 3-39.
[12] Suquet, P.: Discontinuities and plasticity. *Nonsmooth mechanics and applications* (ed J.J. Moreau, P.D. Panagiotopoulos), (1988), 278-340, Springer.
[13] Temam R.: Mathematical problems in plasticity. Gauthier-Villars, Paris, 1985. Translation of Problèmes mathématiques en plasticité. Gauthier-Villars, Paris, 1983.
[14] Temam R., Strang G.: Duality and relaxation in the variational problem of plasticity. *J. Mécanique*, **19** (1980), 493-527.

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