On partially and globally overdetermined problems of elliptic type

Alberto Farina[1,2]
Enrico Valdinoci[1,3,4]

April 16, 2011

Abstract

We consider some elliptic PDEs with Dirichlet and Neumann data prescribed on some portion of the boundary of the domain and we obtain rigidity results that give a classification of the solution and of the domain.

In particular, we find mild conditions under which a partially overdetermined problem is, in fact, globally overdetermined: this enables to use several classical results in order to classify all the domains that admit a solution of suitable, general, partially overdetermined problems.

These results may be seen as solutions of suitable inverse problems – that is to say, given that an overdetermined system possesses a solution, we find the shape of the admissible domains.

Models of these type arise in several areas of mathematical physics and shape optimization.

Keywords: Dirichlet and Neumann overdetermined boundary conditions, rigidity results, classification of admissible domains, inverse problems, free boundaries.

2010 Mathematics Subject Classification: 35J25, 49N45, 35J60, 35J67, 49R05.

1 Introduction

The purpose of this paper is to prove some rigidity results for overdetermined problems arising in elliptic PDEs – namely, assuming both Dirichlet and Neumann conditions gives very strict constraints on both the solution and the domain, leading, quite often, to a complete classification. We refer to [Ser71, Wei71, BCN97] for classical results on overdetermined problems.

[1]LAMFA – CNRS UMR 6140 – Université de Picardie Jules Verne – Faculté des Sciences – 33, rue Saint-Leu – 80039 Amiens CEDEX 1, France

[2]Email: alberto.farina@u-picardie.fr

[3]Università di Roma Tor Vergata – Dipartimento di Matematica – via della ricerca scientifica, 1 – I-00133 Rome, Italy

[4]Email: enrico@math.utexas.edu

In particular, it was proved in [Ser71] that if, in a smooth, bounded, connected domain Ω, there exists a smooth solution u of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 \text{ and } \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \tag{1}$$

then Ω is a ball and u is radially symmetric about the center of such ball. As usual, here above, ν denotes the exterior unitary normal vector on $\partial\Omega$ and $c \in \mathbb{R}$ is any fixed constant.

An alternative, striking proof of some of the results of [Ser71] was given in [Wei71]. The results in [Ser71] have been of outstanding importance, both for the applications and for the development of a fruitful mathematical theory.

Indeed, according to [Ser71], the study of problem (1) was motivated by a question of R. L. Fosdick about fluid mechanics. In this setting, the model in (1) describes a viscous incompressible fluid moving through a straight pipe of given planar cross section Ω: then, u represents the flow velocity of the fluid, and $\partial_\nu u$ is the tangential stress on the pipe. So, the conclusion of [Ser71] is that the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section.

Other models from physics are also referable to (1), e.g. in the linear theory of torsion of a solid straight bar (see pages 109–119 in [Sok56]). In this framework, the result of [Ser71] states that when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of the position if and only if the bar has a circular cross section.

Also, (1) can be related to a lower dimension obstacle problem (the so called Signorini problem, see for instance [Fre77]) and so to the fractional Laplacian and to Dirichlet-to-Neumann operators. See also [DSV10] for an overdetermined problem in a fully nonlinear case. Overdetermined boundary conditions also arise naturally in free boundary problems, where the variational structure imposes suitable conditions on the separation interface: see, e.g., page 109 in [AC81].

Besides the very many applications, [Ser71] also made available to the mathematical community the (up to now classical) moving plane method, which is a very flexible version of the reflection method of [Ale56], and from which many fundamental results originated, such as the ones in [GNN79].

Then, in [BCN97], the analogue of (1) in unbounded domains was considered, in connection with the regularity theory of some free boundary problems. In this case, the "typical" nonlinearity f taken into account is the bistable nonlinearity $f(u) = u - u^3$, which reduces the PDE in (1) to the Allen-Cahn equation. Under suitable assumptions on Ω (such as, $\partial\Omega$ being a globally Lipschitz epigraph with some control at infinity), it was proved in [BCN97] that if (1) admits a smooth, bounded solution, then Ω is a halfspace – in a sense, while the results of [Ser71] on bounded domains reduce Ω via a rotational symmetry, the one in [BCN97] on unbounded domains perform a planar symmetry.

Some of the results in [BCN97] have been extended in [FV10a] in low dimensions, by dropping some of the conditions at infinity. Also, the planar symmetry results of [BCN97] may be related to a famous question posed by [DG79] (see [FV09] for a recent review on this topic). It was also asked in [BCN97] whether

$$\text{the existence of a bounded, smooth solution of (1) in a smooth domain } \Omega \\ \text{whose complement is connected implies that } \Omega \text{ is either a halfspace, a ball or a cylinder} \\ \text{(or the complement of one these regions).} \tag{2}$$

It has been recently shown in [Sic10] that the answer to such a question is negative in dimension 3 or higher when f is linear (as far as we know, the question is still open in dimension 2, and in any dimension if f is nonlinear, e.g. if f is bistable). We refer to [FV10a] for some related results, and to the forthcoming Theorem 14 here for a further remark.
The main feature of (1) is that it is an overdetermined problem, i.e. both the Dirichlet and the Neumann data are given on $\partial\Omega$ (and this is the main reason that excludes many nasty solutions). A natural variant of the problem consists in a partial overdetermination, that is in considering a boundary value problem whose overdetermined prescription occurs only on a portion of the boundary $\Gamma \subseteq \partial\Omega$, such as, for instance,

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma. \end{cases} \tag{3}$$

These partially overdetermined problems arise quite naturally, for instance, in shape optimization – e.g., in the minimization problem of the first Dirichlet eigenvalue of $-\Delta$ among all open sets constrained to lie in a given box and also with a given volume, in the minimization problem of the second Dirichlet eigenvalue among all planar convex domains of given area, in the problem of minimizing the Dirichlet energy of domains with prescribed volume and confined in a planar box, etc., see [Hen06]. In fact, we can not attempt to list an exhaustive bibliography on overdetermined and partially overdetermined elliptic problems: we refer to [Ser71, Wei71, WCG95, BCN97, HP05, FG08, FGLP09] and the bibliography therein for basic references and also to Section 2 of [FG08] for detailed physical motivations and relations, among other subjects, to fluid and solid mechanics, thermodynamics and electrostatics.
A particular case of partially overdetermined problem is the one in which Ω in (3) is a cone, and so the Neumann prescription cannot be given in the whole of $\partial\Omega$ but, at most, outside the vertex of the cone (in this case $\partial\Omega \setminus \Gamma$ would consist of just one point). We refer to [FV10b] for the study of this problem in dimension 2 and 3 (see also Corollary 13 in what follows).
Here, the problems we consider may be divided into two categories, namely the problems whose overdetermined prescription occurs on a well-known portion of the boundary, and the ones which possess rotational symmetry. Here below, we describe in detail the results obtained, whose proofs will follow in the subsequent sections.
We remark that our results may be extended to uniformly elliptic linear operators on Riemannian manifolds. From now on, we will suppose $n \geq 2$.

1.1 A Unique Continuation Principle

For clearly stating our results, it is convenient to recall the following, classical notation – see, e.g., page 22 in [DZ01]:

Definition 1. *Let $\Omega \subset \mathbb{R}^n$ be open, nonempty, and $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. We say that Ω is of class C^k if, for any $x \in \partial\Omega$,*

$$\begin{aligned} &\text{there exist an open set } U^{(x)} \subset \mathbb{R}^n, \text{ with } x \in U^{(x)}, \\ &\text{and a bijective map } g^{(x)} \in C^k(U^{(x)}, B_1), \text{ with inverse map } h^{(x)} \in C^k(B_1, U^{(x)}), \\ &\qquad \text{such that} \\ &U^{(x)} \cap \Omega = h^{(x)}\big(B_1 \cap \{x_n > 0\}\big), \\ &U^{(x)} \cap (\partial\Omega) = h^{(x)}\big(B_1 \cap \{x_n = 0\}\big), \\ &\text{and } U^{(x)} \setminus \overline{\Omega} = h^{(x)}\big(B_1 \cap \{x_n < 0\}\big). \end{aligned} \tag{4}$$

We also use the following notation:

Definition 2. *Let $\mathcal{M} \subset \mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. We say that $\mathcal{M}$ is a C^k-hypersurface if it is nonempty, connected and there exists an open set $\mathcal{O} \subset \mathbb{R}^n$ of class C^k such that $\mathcal{M} = \partial\mathcal{O}$.*

Definition 3. *Let $\Omega \subset \mathbb{R}^n$ be open, $\Gamma \subseteq \partial\Omega$ and $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. We say that Γ is a C^k-subset of $\partial\Omega$ if Γ is nonempty, open in the natural topology of $\partial\Omega$, and (4) holds for any $x \in \Gamma$.*

Definition 4. *Let $\Gamma \subseteq \mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. We say that Γ is a C^k-subset if there exists an open set $\Omega \subset \mathbb{R}^n$ such that Γ is a C^k-subset of $\partial\Omega$.*

The following result, which is an enhancement of the classical Unique Continuation Principle, uses Definition 3 and is one of the cornerstones of our analysis (Definitions 1, 2 3 and 4 will also be exploited in Theorem 6). Roughly speaking, it says that if two solutions have the same Dirichlet and Neumann datum on a portion of the boundary, then they must coincide.

Theorem 5. *Let $\epsilon_o > 0$, and $\Omega \subset \mathbb{R}^n$ be open and connected. Let Γ be a C^1-subset of $\partial\Omega$, with exterior normal ν, and let $p_o \in \Gamma$.*
Let $u^{(1)}$, $u^{(2)} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap W^{2,2}\big(\Omega \cap B_{\epsilon_o}(p_0)\big)$ solve

$$\Delta u^{(\ell)} + f(x, u^{(\ell)}, \nabla u^{(\ell)}) = 0 \qquad \text{in } \Omega, \text{ for } \ell = 1, 2.$$

Suppose that f is locally Lipschitz in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.
If $u^{(1)}(x) = u^{(2)}(x)$ and $\partial_\nu u^{(1)}(x) = \partial_\nu u^{(2)}(x)$ for any $x \in \Gamma$, then $u^{(1)}(x) = u^{(2)}(x)$ for any $x \in \Omega$.

We remark that the only regularity needed in Theorem 5 is of first derivative type (on Γ and f). This will allow us to obtain results similar to the ones of [FG08] in a non-analytic setting (see, e.g., Corollary 7 and Theorem 10 below; see also Appendix A for obtaining results related to [FG08] from ours).

1.2 From partially to globally overdetermined problems

Now, we deal with problems which are overdetermined only in a portion of the boundary (see [FG08, FGLP09]).
The next result gives a condition under which a partially overdetermined problem is, in fact, globally overdetermined. In this case, the reason for recalling Definitions 1, 2, 3 and 4 is that

we will deal with domains that are not necessarily connected or bounded, that may have several connected components, and that may not be smooth. We consider this result as the major contribution of this paper, since it holds under very mild hypotheses and it has a wide range of applications – for instance, reducing to a globally overdetermined problem will allow us to use several classical results in order to classify all the domains that admit a solution of suitable, general, partially overdetermined problems.

Theorem 6. *Let f be locally Lipschitz in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and let Ω, $\tilde{\Omega} \subset \mathbb{R}^n$ be open sets.*
Let Γ be a C^1-subset of $\partial\Omega$. Let $\mathcal{C}$ be the connected component of $\partial\Omega$ that contains Γ.
Suppose that there exists a C^2-hypersurface $\mathcal{M}$, with exterior[1] *normal ν, such that $\Gamma \subseteq \mathcal{M}$ and $\Omega \cup (\mathcal{C} \cap \mathcal{M}) \subset \tilde{\Omega}$.*
Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be C^1 in a neighborhood of $\mathcal{M}$ and $\psi \in C^0(\mathcal{M})$, with

$$\psi(x) - \partial_\nu \phi(x) \neq 0 \text{ for any } x \in \mathcal{M} \setminus \Gamma. \tag{5}$$

Assume that for any $P \in \mathcal{M} \setminus \Gamma$ there exist $r(P) > 0$ and

$$u^{(P)} \in C^2(B_{r(P)}(P)) \tag{6}$$

which solves

$$\begin{cases} \Delta u^{(P)} + f(x, u^{(P)}, \nabla u^{(P)}) = 0 & \text{in } B_{r(P)}(P), \\ u^{(P)} = \phi & \text{on } \mathcal{M} \cap B_{r(P)}(P), \\ \partial_\nu u^{(P)} = \psi & \text{on } \mathcal{M} \cap B_{r(P)}(P). \end{cases} \tag{7}$$

Let $\epsilon_o > 0$, $p_o \in \Gamma$, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \cap W^{2,2}(\Omega \cap B_{\epsilon_o}(p_o)) \cap C^1(\tilde{\Omega})$ be a solution of a problem that is partially overdetermined on $\partial\Omega$ by the following equations:

$$\begin{cases} \Delta u + f(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ \partial_\nu u = \psi & \text{on } \Gamma. \end{cases} \tag{8}$$

Then

$$\mathcal{C} = \mathcal{M} \tag{9}$$

and

$$u = \phi \text{ and } \partial_\nu u = \psi \text{ on } \mathcal{C}. \tag{10}$$

Concerning the statement of Theorem 6, and in particular the boundary conditions in (8), we remark that

$$\begin{array}{c} u \text{ is supposed to solve an overdetermined problem only in } \Omega, \\ \text{not in the larger set } \tilde{\Omega}. \end{array} \tag{11}$$

[1]We observe that there is a slight abuse of terminology here, because the interior/exterior normal is not determined by $\mathcal{M}$ itself, but also by the choice of the set $\mathcal{O}$ for which $\mathcal{M} = \partial\mathcal{O}$ in Definition 2. Indeed, one could replace $\mathcal{O}$ by $\mathbb{R}^n \setminus \overline{\mathcal{O}}$ and reverse the direction of ν. On the other hand, once the choice of $\mathcal{O}$ is fixed once and for all, we can also think that the outward direction is given, and this prescribes ν along $\mathcal{M}$, and therefore, along Γ.

A natural, possible choice is also to consider ν to be defined on $\mathcal{M}$ as the normal that extends the exterior normal of Γ with respect to the open set Ω.

Moreover, there is no sign assumption on either u on f, so they are allowed, in principle, to change sign.
Furthermore, we stress that the Neumann datum of problem (8) is not necessarily given on the whole of $\partial\Omega$ (in any case, as far as we know, Theorem 6 is new even in this simpler case).
In this spirit, condition (5) may be seen as a nondegeneracy (or transversality) conditions between the Dirichlet and Neumann data. Such condition cannot, in general, be dropped (see Appendix B). Similarly, the condition that Γ is open in the relative topology of $\partial\Omega$ cannot be dropped (see Appendix C).
At a first glance, Theorem 6 may look pretty technical[2]. On the other hand, its statement may be interpreted, roughly speaking, saying that if the problem is partially overdetermined on some portion Γ of $\partial\Omega$ and we can locally solve an overdetermined problem on some smooth $\mathcal{M}$ that contains Γ, than the portion of the boundary on which the problem is overdetermined is the whole boundary $\partial\Omega$ – or, better to say, the connected component $\mathcal{C}$ of it that contains Γ.

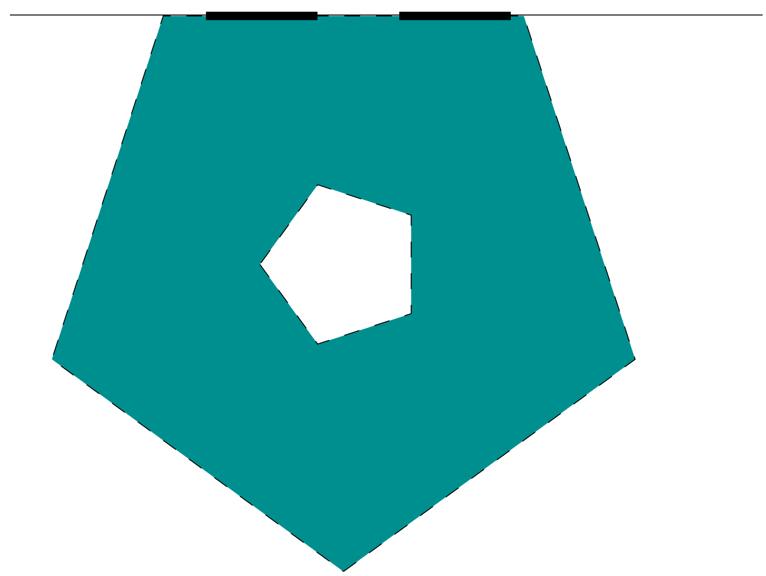

Figure 1: *Drawing what Theorem 6 prevents*

Indeed, the statement of Theorem 6 is somewhat depicted in Figure 1, where Ω is the grey area enclosed by the two pentagons, Γ is represented as the union of the thick solid (open) segments, and $\mathcal{M}$ as the thin solid straight line. Then, $\mathcal{C}$ is the boundary of the bigger pentagon and $\partial\Omega$ is the union of the boundaries of the two pentagons. In such a situation, (9) says that Figure 1 cannot hold, since the boundary of the bigger pentagon should agree with the solid straight line on its top.
In this spirit, Theorem 6 states that the Neumann boundary condition is propagated from a subset Γ of the boundary to the whole of its connected component $\mathcal{C}$, via a smooth hypersurface $\mathcal{M}$.

[2] We could have done worse though. In fact, it is enough, in Theorem 6, to have the hypotheses stated with $(\mathcal{C} \cap \mathcal{M}) \setminus \Gamma$, instead of $\mathcal{M} \setminus \Gamma$. Even if this assumption would be weaker, we think it is not really "fair", since, in principle, one does not know $\mathcal{C}$ from the beginning (one would like to reconstruct it!).

The assumption that $\Omega \cup \mathcal{M} \subset \tilde{\Omega}$ in Theorem 6 is taken in order to ensure some very mild regularity up to the boundary. We stress that $\partial\Omega$ is not assumed to be smooth outside Γ: in particular its wild behavior may not allow, a-priori, to deduce by elliptic estimates any regularity information on u near $\partial\Omega \setminus \Gamma$.
Concerning this boundary regularity issue, we think that is quite important to avoid unnecessary regularity assumptions on $\partial\Omega \setminus \Gamma$ in Theorem 6, not only for the sake of generality, but also because of the applications that Theorem 6 has. For instance, if the overdetermined system arises in a free boundary context, it is a very delicate issue to establish the regularity of $\partial\Omega$ in general, and, in fact, singularities may occur: see, e.g. [AC81, DSJ09].
Of course, particular cases of Theorem 6 are when f depends only on u (viz, the case of a semilinear PDE), when $\partial\Omega$ is connected (hence $\mathcal{C} = \partial\Omega$), or when the Dirichlet and Neumann data are constants (say, $\phi := 0$ and $\psi := c$, in this case (5) reads $c \in \mathbb{R} \setminus \{0\}$). As far as we know, Theorem 6 is new even in these simpler cases.
In spite of its technical flavor, we think that it is good to have Theorem 6 stated in such a general form, since, in this way, several results of this paper may be traced back to it, so this general form of Theorem 6 has a somewhat unifying purpose.
We also observe that if one knows from the beginning that $\partial\Omega$ is C^2 and $u \in C^2(\overline{\Omega})$, then Theorem 6 simplifies, because there is no need to involve $\tilde{\Omega}$ in its statement (indeed, the existence of such $\tilde{\Omega}$ is warranted by classical extension results, see [GT01]; in fact one can take $\tilde{\Omega} := \mathbb{R}^n$, recall (11), and deduce Corollary 7 from Theorem 6). In fact, the assumption that $\partial\Omega$ is C^2 is quite natural and compatible with the statement of Theorem 6 a posteriori, since, by (9), $\mathcal{C}$ agrees with $\mathcal{M}$, which is C^2. In light of these observations, we think it is worth to state this simplified case in a separate result as follows:

Corollary 7. *Let f be locally Lipschitz in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ be of class C^2.*
Let $\Gamma \subseteq \partial\Omega$ be nonempty and open in the natural topology of $\partial\Omega$.
Let $\mathcal{C}$ be the connected component of $\partial\Omega$ that contains Γ.
Suppose that there exists a C^2-hypersurface $\mathcal{M}$, with exterior normal ν, such that $\Gamma \subseteq \mathcal{M}$.
Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be C^1 in a neighborhood of $\mathcal{M}$ and $\psi \in C^0(\mathcal{M})$, with

$$\psi(x) - \partial_\nu \phi(x) \neq 0 \text{ for any } x \in \mathcal{M} \setminus \Gamma, \tag{12}$$

and assume that for any $P \in \mathcal{M} \setminus \Gamma$ there exist $r(P) > 0$ and $u^{(P)} \in C^2(B_{r(P)}(P))$ which solves

$$\begin{cases} \Delta u^{(P)} + f(x, u^{(P)}, \nabla u^{(P)}) = 0 & \text{in } B_{r(P)}(P), \\ u^{(P)} = \phi & \text{on } \mathcal{M} \cap B_{r(P)}(P), \\ \partial_\nu u^{(P)} = \psi & \text{on } \mathcal{M} \cap B_{r(P)}(P). \end{cases} \tag{13}$$

Let $u \in C^2(\overline{\Omega})$ be a solution of

$$\begin{cases} \Delta u + f(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \\ \partial_\nu u = \psi & \text{on } \Gamma. \end{cases}$$

Then $\mathcal{C} = \mathcal{M}$, and $u = \phi$ and $\partial_\nu u = \psi$ on $\mathcal{C}$.

We believe that Theorem 6 and Corollary 7 are quite powerful, and they allow to use many results, such as the ones in [FV10a], that deal with fully overdetermined problems. For example, we obtain from Corollary 7 the following result on the partially overdetermined Allen-Cahn equation on globally Lipschitz, analytic epigraphs:

Corollary 8. *Let $n \le 3$ and $c \in \mathbb{R}$. Let $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be an analytic and globally Lipschitz function. Let U be an open subset of $\mathbb{R}^{n-1}$,*

$$\begin{aligned} &\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \Psi(x')\} \\ \text{and} \qquad &\Gamma := \{(x', x_n) \in U \times \mathbb{R} \text{ s.t. } x_n = \Psi(x')\}. \end{aligned}$$

Let ν be the exterior normal to $\partial\Omega$. Suppose that there exists $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfying

$$\begin{cases} \Delta u + u - u^3 = 0 \text{ and } u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ \partial_\nu u = c \text{ on } \Gamma. \end{cases}$$

Then, we have that $\Omega = \mathbb{R}^{n-1} \times (0, +\infty)$ up to isometry and

$$u(x_1, \dots, x_n) = \tanh\left(\frac{x_n}{\sqrt{2}}\right) \qquad \text{for any } (x_1, \dots, x_n) \in \Omega.$$

Also, we will make use of Theorem 6 to prove the forthcoming Theorem 10, in which the shape of the domain will be reconstructed from a "nice" portion of the boundary on which the problem is overdetermined.

1.3 Partially overdetermined problems in "nice" domains

Now, we point out the following result for analytic domains, which may be seen as a refinement of the classical Cauchy-Kowaleskaya Theorem in a setting convenient for our purposes:

Theorem 9. *Let $\Omega \subset \mathbb{R}^n$ be open and connected, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve*

$$\sum_{i,j=1}^{n} a_{i,j}(x, u, \nabla u)\partial_{ij} u + f(x, u, \nabla u) = 0 \qquad \text{in } \Omega. \tag{14}$$

Suppose that a_{ij} and f are analytic, and that there exist $\Lambda \ge \lambda > 0$ such that

$$\lambda \le \sum_{i,j=1}^{n} a_{ij}(x, r, p)\xi_i(x)\xi_j(x) \le \Lambda \qquad \text{for any } \xi \in \mathrm{S}^{n-1},\ x \in \Omega,\ r \in \mathbb{R},\ p \in \mathbb{R}^n. \tag{15}$$

Let Γ be a C^ω-subset of $\partial\Omega$, and let ν be its exterior normal.
Then:

(i). there exist $\Omega_\star \subseteq \Omega$ and $u_\star \in C^\omega(\Omega_\star) \cap C^1(\overline{\Omega_\star})$ such that $\partial\Omega_\star \cap \partial\Omega = \Gamma$ and $u_\star$ is a solution of (14) in $\Omega_\star$, with $u_\star(x) = u(x)$ and $\partial_\nu u_\star(x) = \partial_\nu u(x)$ for any $x \in \Gamma$;

(ii). $u(x) = u_\star(x)$ *for any* $x \in \Omega_\star$*;*

(iii). if $u_\star$ *can be extended to a solution of* (14) *in the whole of* Ω*, then* $u(x) = u_\star(x)$ *for any* $x \in \Omega$*.*

We remark that the ellipticity condition in (15) is classical and it ensures to locally deal with many standard operators – e.g., the Laplacian, the mean curvature operator, etc.
Now, we consider domains which contain pieces of spheres, or hyperplanes, or cylinders, on which the problem is overdetermined, and we reconstruct both the solution and the domain, according to the following result:

Theorem 10. *Let* $c \in \mathbb{R} \setminus \{0\}$*. Let* $\Omega \subset \mathbb{R}^n$ *be connected and of class* C^1*.*
Let $\Gamma \subseteq \partial\Omega$ *be nonempty and open in the natural topology of* $\partial\Omega$*, with exterior normal* ν*.*
Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *solve*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma. \end{cases} \tag{16}$$

with f *locally Lipschitz.*

- *Suppose that* Γ *agrees with a portion of sphere* ∂B_1*, i.e. that there exist* $p_o \in \mathbb{R}^n$ *and* $r_o > 0$ *for which*

$$\Gamma := B_{r_o}(p_o) \cap (\partial\Omega) = B_{r_o}(p_o) \cap (\partial B_1). \tag{17}$$

  Then[3] *u has rotational symmetry. Moreover, one of the following four possibilities holds:* $\Omega = B_1$*,* $\Omega = \mathbb{R}^n \setminus \overline{B_1}$*,* $\Omega = B_1 \setminus \overline{B_{1-\kappa}}$ *or* $\Omega = B_{1+\kappa} \setminus \overline{B_1}$*, for some* $\kappa > 0$*.*

---

[3]The statement of Theorem 10 can be made more explicit in the following way. If (17) holds, let $\tau > 0$ and $u_\star \in C^2((1-\tau, 1+\tau))$ be the local solution of the Cauchy problem

$$\begin{cases} u_\star''(r) + \dfrac{n-1}{r} u_\star'(r) = -f(u_\star(r)) \text{ for any } r \in (1-\tau, 1+\tau) \\ u_\star(1) = 0 \text{ and } u_\star'(1) = c. \end{cases} \tag{18}$$

Then, the rotational symmetry of u means that $u(x) = u_\star(|x|)$ for any $x \in \Omega$.
Similarly, if (21) holds, let $\tau > 0$ and $u_\star \in C^2((-\tau, \tau))$ be the local solution of the Cauchy problem

$$\begin{cases} u_\star''(r) = -f(u_\star(r)) \text{ for any } r \in (-\tau, \tau), \\ u_\star(0) = 0 \text{ and } u_\star'(0) = c. \end{cases} \tag{19}$$

Then, the planar symmetry of u means that $u(x) = u_\star(x_n)$ for any $x \in \Omega$.
Analogously, if (22) holds, let $\tau > 0$ and $u_\star \in C^2((1-\tau, 1+\tau))$ be the local solution of the Cauchy problem

$$\begin{cases} u_\star''(r) + \dfrac{k-1}{r} u_\star'(r) = -f(u_\star(r)) \text{ for any } r \in (1-\tau, 1+\tau) \\ u_\star(1) = 0 \text{ and } u_\star'(1) = c. \end{cases} \tag{20}$$

Then, the cylindrical symmetry of u means that $u(x_1, \dots, x_n) = u_\star(|(x_1, \dots, x_k)|)$ for any $x \in \Omega$.
We remark that the existence of the solutions $u_\star$ in either (18), or (19), or (20), is warranted by the standard ODE existence and uniqueness theory, since f is locally Lipschitz.

- *Suppose that Γ agrees with a portion of hyperplane $\{x_n = 0\}$, i.e. that there exist $p_o \in \mathbb{R}^n$ and $r_o > 0$ for which*

$$\Gamma := B_{r_o}(p_o) \cap (\partial\Omega) = B_{r_o}(p_o) \cap \{x_n = 0\}, \tag{21}$$

Then u has planar symmetry. Moreover, one of the following four possibilities holds: $\Omega = \{x_n > 0\}$, $\Omega = \{x_n < 0\}$, $\Omega = \{0 < x_n < \kappa\}$ or $\Omega = \{-\kappa < x_n < 0\}$, for some $\kappa > 0$.

- *Analogously, suppose that Γ agrees with a portion of cylinder $\partial B'_1 \times \mathbb{R}^{n-k}$, where $1 \le k \le n-1$ and B'_1 is the k-dimensional, unit ball, i.e. assume that there exist $p_o \in \mathbb{R}^n$ and $r_o > 0$ for which*

$$\Gamma := B_{r_o}(p_o) \cap (\partial\Omega) = B_{r_o}(p_o) \cap \big((\partial B'_1) \times \mathbb{R}^{n-k}\big). \tag{22}$$

Then u has cylindrical symmetry. Moreover, one of the following four possibilities holds: $\Omega = B'_1 \times \mathbb{R}^{n-k}$, $\Omega = (\mathbb{R}^k \setminus \overline{B'_1}) \times \mathbb{R}^{n-k}$, $\Omega = (B'_1 \setminus \overline{B'_{1-\kappa}}) \times \mathbb{R}^{n-k}$ or $\Omega = (B'_{1+\kappa} \setminus \overline{B'_1}) \times \mathbb{R}^{n-k}$, for some $\kappa > 0$.

Classical simple examples of Theorem 10 are when $f = -1$ and $u_\star(r) := (|x|^2 - 1)/(2n)$ in case (17) and when $f = 0$ and $u_\star(r) := r$ in case (17). Of course, in these cases, the domain Ω may be classified with more precision, since it follows that either $\Omega = B_1$ or $\Omega = \mathbb{R}^n \setminus \overline{B_1}$ if (17) holds, and either $\Omega = \{x_n > 0\}$ or $\Omega = \{x_n < 0\}$ if (21) holds.
Results related to Theorem 10 have been recently given in [FG08]. We remark that, differently from [FG08], we do not need to assume any analytic regularity in Theorem 10, $\partial\Omega$ does not need to be connected, and Ω does not need to be bounded.
Also, Theorem 10 gives a new proof of Theorem 4.2.5 on page 67 of [Hen06] (see also [HO03]): we devote Appendix D to this new proof.
When conditions (17), (21) or (22) are dropped, Theorem 10 may not hold and very interesting counterexamples have been constructed in [FGLP09].
We remark that the condition that $\partial\Omega$ is smooth cannot, in general, be dropped, see Appendix E.
Also, we observe that Theorem 10 is a partial counterpart to the classical result of [Ser71] to unbounded domains.
Moreover, it is worth noticing that the statements of Theorem 10 may not hold if the Dirichlet boundary condition in (16) is given only on Γ instead on the whole of $\partial\Omega$ (see Appendix F).
Other "nice" domains are the ones which possess a rotational symmetry. In this setting, the next result is an energy bound (namely, formula (26) below), valid for solutions of a semilinear PDE in a rotationally invariant domain, with a rotationally invariant boundary data, in dimension $n \le 11$:

Theorem 11. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz and $C^3(\mathbb{R} \setminus \{0\})$. Let $O := (0, \Psi(0))$. Let $f \in C^1(\mathbb{R})$. Let*

$$\Omega := \Big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \Psi\big(|x'|\big)\Big\} \tag{23}$$

and $\Omega_o := \{(s,t) \in \mathbb{R}^2 \text{ s.t. } t > \Psi(s)\}$.
Let u_o, $v_o : \partial\Omega_o \to \mathbb{R}$, and let ν be the exterior normal to $\partial\Omega$.

Let

$$\begin{cases} u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{O\}) \cap W^{1,\infty}(\Omega), \\ \Delta u(x) + f(u(x)) = 0 \text{ for any } x \in \Omega. \end{cases} \tag{24}$$

Suppose that $u(x', x_n) = u_o(|x'|, x_n)$ *for any* $(x', x_n) \in \partial\Omega$ *and that* $\partial_\nu u(x', x_n) = v_o(|x'|, x_n)$ *for any* $(x', x_n) \in \partial\Omega \setminus \{O\}$.
Then, $u(\cdot, x_n)$ *is rotationally symmetric, that is there exists* $u_\star : \Omega_o \to \mathbb{R}$ *for which*

$$u(x) = u_\star(|x'|, x_n) \text{ for any } x = (x', x_n) \in \Omega. \tag{25}$$

Also, let F *be a primitive of* f*, and define*

$$c_u := \sup_{r \in \left[\inf_{\mathbb{R}} u, \sup_{\mathbb{R}} u\right]} F(r).$$

For any $R \geq 0$*, let*

$$\mathcal{E}(R) := \int_{\Omega \cap B_R} \frac{|\nabla u(x)|^2}{2} - F(u(x)) + c_u \, dx.$$

Assume that $\partial_{x_n} u(x) > 0$ *for any* $x \in \Omega$.
Then, there exists $C \geq 1$*, only depending on* n*,* $\|u\|_{W^{1,\infty}(\mathbb{R}^n)}$ *and* $\|f\|_{C^1(\mathbb{R})}$ *such that, if* $n \leq 11$,

$$\mathcal{E}(R) \leq CR^{n-1}. \tag{26}$$

As a consequence of Theorem 11, we obtain the following onedimensional symmetry result, which establishes that overdetermined problems in rotationally symmetric domains in dimension $n \leq 3$ possess a solution only if the domain is a halfspace:

Corollary 12. *Let* $\Psi : \mathbb{R} \to \mathbb{R}$ *be globally Lipschitz and* $C^3(\mathbb{R} \setminus \{0\})$*,* $O := (0, \Psi(0))$*, and* $f \in C^1(\mathbb{R})$*. Let* Ω *be as in* (23) *and* u *as in* (24).
Assume that $\partial_{x_n} u(x) > 0$ *for any* $x \in \Omega$.
Suppose that

$$u \text{ and } |\nabla u| \text{ are constant on } (\partial\Omega) \setminus \{O\}. \tag{27}$$

Then, if $n \leq 3$*, there exists* $u_\sharp : \mathbb{R} \to \mathbb{R}$ *such that* $u(x', x_n) = u_\sharp(x_n)$ *for any* $(x', x_n) \in \Omega$. *Moreover,* Ψ *is constant and* Ω *is a halfspace.*

A particular case of Corollary 12 is when the domain is a cone. In this case, we have:

Corollary 13. *Let* $n \leq 3$*,* $c \in \mathbb{R}$*,* $f \in C^1(\mathbb{R})$*,* $\alpha \geq 0$ *and*

$$\Omega := \big\{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n > \alpha|x'|\big\}.$$

Let ν *be the exterior normal on* $(\partial\Omega) \setminus \{0\}$.
Suppose that there exists $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\}) \cap W^{1,\infty}(\Omega)$ *such that*

$$\begin{cases} \Delta u + f(u) = 0, \ u > 0 \text{ in } \Omega, \\ u = 0, \ \partial_\nu u = c \text{ on } (\partial\Omega) \setminus \{0\}. \end{cases} \tag{28}$$

Then, $\alpha = 0$.

We observe that Corollary 13 states that the only case in which a cone admits a solution of the overdetermined problem in (28) is when the cone is, in fact, a halfspace. Such result was also obtained in [FV10b] under the additional sign requirement that $f \geq 0$ (and $n = 3$): here such an assumption is dropped, and Corollary 13 is just a plain consequence of Corollary 12, since, if $\alpha > 0$, the domain is coercive and so $\partial_{x_n} u > 0$ (hence all the assumptions of Corollary 12 are fulfilled), see Theorem 1.3 of [BCN97].

1.4 Overdetermined eigenvalue problems on epigraphs

Now, we consider the particular case in which the overdetermined problem is set on a Lipschitz epigraph and the nonlinearity is, in fact, linear (i.e., the PDE boils down to an eigenvalue problem). This setting is more complicated than what it may look like at a first glance, and the classification results in [FV10a] and their counterparts given by the counterexample in [Sic10] may be seen as a bridge between the setting in [Ser71] and the one in [BCN97]. Indeed, the counterexample in [Sic10] is obtained by perturbing the symmetric cylinder $B \times \mathbb{R}$, where B is a ball. On the other hand, smooth, globally Lipschitz epigraphs could be seen as a perturbation of a halfspace (another symmetric domain). Hence, one might think to give an answer to the question posed in [BCN97] (as quoted in (2) here), in any dimension, by looking at the overdetermined eigenvalue problem for a smooth, globally Lipschitz epigraph.
But this case is ruled by the following observation:

Theorem 14. *Let $\lambda \in \mathbb{R}$. Let Ω be a globally Lipschitz epigraph of $\mathbb{R}^n$, with $n \geq 2$. Then, there exists no solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ of*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

In the forthcoming sections, we prove the results stated above.

2 Proof of Theorem 5

The proof is based on the Unique Continuation Principle. By possibly taking a smaller $\epsilon_o > 0$, we have that $B_{\epsilon_o}(p_o) \cap (\partial\Omega) \subset \Gamma$. We let $w(x) := u^{(1)}(x) - u^{(2)}(x)$. Then

$$\Delta w(x) + \alpha(x) w(x) + \beta(x) \cdot \nabla w(x) = 0 \tag{30}$$

in Ω, for suitable $\alpha \in L^\infty(\Omega \cap B_{\epsilon_o}(p_o))$, $\beta \in L^\infty(\Omega \cap B_{\epsilon_o}(p_o), \mathbb{R}^n)$. Now, we extend α, β and w to vanish identically in $B_{\epsilon_o}(p_o) \setminus \Omega$. By construction, α and β are bounded in $B_{\epsilon_o}(p_o)$, and $w \in C^1(B_{\epsilon_o}(p_o)) \cap W^{2,2}(B_{\epsilon_o}(p_o))$.

Moreover, for any $\psi \in C_0^\infty(B_{\epsilon_o}(p_o))$,

$$\begin{aligned}
\int_{B_{\epsilon_o}(p_o)} \nabla w \cdot \nabla \psi &= \int_{B_{\epsilon_o}(p_o)\cap\Omega} \nabla w \cdot \nabla \psi \\
&= \int_{\partial(B_{\epsilon_o}(p_o)\cap\Omega)} \psi \partial_\nu w - \int_{B_{\epsilon_o}(p_o)\cap\Omega} \psi \Delta w \\
&= 0 - \int_{B_{\epsilon_o}(p_o)\cap\Omega} \psi(\alpha w + \beta \cdot \nabla w) \\
&= - \int_{B_{\epsilon_o}(p_o)} \psi(\alpha w + \beta \cdot \nabla w),
\end{aligned}$$

that is w is a weak solution of (30) in $B_{\epsilon_o}(p_o)$.
Accordingly, by the Unique Continuation Principle (see, e.g., [Hör83a, Hör83b]), w vanishes identically in $B_{\epsilon_o}(p_o)$ and so in Ω. ∎

3 Proof of Theorem 6

The idea for the proof is that, when the Dirichlet and Neumann data agree on some piece of the boundary, this piece may be enlarged a little bit, thanks to Theorem 5, and this rules out the possibility that the two different natural boundaries $\mathcal{C}$ and $\mathcal{M}$ may bifurcate one from the other. The details of the proof go as follow. By possibly replacing Γ with $\Gamma \cap B_{\epsilon_o/2}(p_o)$, we may and do suppose that,

$$u \in W^{2,2}(\Omega \cap B_{\epsilon_o/2}(x)), \text{ for any } x \in \Gamma. \tag{31}$$

We define

$$\begin{aligned}
\mathcal{X} := \big\{ x \in \mathcal{C} \text{ s.t. there exists } \alpha(x) > 0 \text{ for which}& \\
\mathcal{C} \cap B_{\alpha(x)}(x) \text{ is a } C^1\text{-subset of } \partial\Omega,& \\
\mathcal{C} \cap B_{\alpha(x)}(x) = \mathcal{M} \cap B_{\alpha(x)}(x),& \\
u = \phi, \partial_\nu u = \psi \text{ on } \mathcal{C} \cap B_{\alpha(x)}(x),& \\
\text{and } u \in W^{2,2}\big(\Omega \cap B_{\alpha(x)}(x)\big) \big\}.&
\end{aligned} \tag{32}$$

We observe that, if $x \in \mathcal{X}$, then

$$\mathcal{C} \cap B_{\alpha(x)}(x) \subseteq \mathcal{X}. \tag{33}$$

Indeed, if $y \in \mathcal{C} \cap B_{\alpha(x)}(x)$, we take $\alpha(y) := \alpha(x) - |x-y| > 0$, we notice that $B_{\alpha(y)}(y) \subseteq B_{\alpha(x)}(x)$, hence $\mathcal{C} \cap B_{\alpha(y)}(y)$ is a C^1-subset of $\partial\Omega$, $\mathcal{C} \cap B_{\alpha(y)}(y) = \mathcal{M} \cap B_{\alpha(y)}(y)$, $u = \phi$ and $\partial_\nu u = \psi$ on $\mathcal{C} \cap B_{\alpha(y)}(y) \subseteq \mathcal{C} \cap B_{\alpha(x)}(x)$, and $u \in W^{2,2}\big(\Omega \cap B_{\alpha(x)}(x)\big) \subseteq W^{2,2}\big(\Omega \cap B_{\alpha(y)}(y)\big)$, and so (33) follows.
On the other hand, $\mathcal{C} \cap B_{\alpha(x)}(x) = \mathcal{M} \cap B_{\alpha(x)}(x)$ for all $x \in \mathcal{X}$. Therefore, we may write (33) as:

$$\text{if } x \in \mathcal{X} \text{ then } \mathcal{M} \cap B_{\alpha(x)}(x) = \mathcal{C} \cap B_{\alpha(x)}(x) \subseteq \mathcal{X}.$$

From this and (31), we obtain that

$$\Gamma \subseteq \mathcal{X} \text{ and } \mathcal{X} \text{ is open in the natural topologies of } \mathcal{C} \text{ and of } \mathcal{M}. \tag{34}$$

We claim that

$$\mathcal{X} \text{ is closed in the natural topologies of } \mathcal{C} \text{ and of } \mathcal{M}. \tag{35}$$

To check this, we take a sequence $P_k \in \mathcal{X}$ such that

$$\lim_{k\to+\infty} P_k = P. \tag{36}$$

The limit in (36) is in the topology of $\mathcal{C}$, and therefore in the topology of $\mathbb{R}^n$ too. In order to prove (35), we will show that $P \in \mathcal{X}$. If $P \in \Gamma$, we have that $P \in \mathcal{X}$, thanks to (34), therefore we may focus on the case in which

$$P \notin \Gamma. \tag{37}$$

Notice that

$$P \in \mathcal{C} \cap \mathcal{M}, \tag{38}$$

because $P_k \in \mathcal{X} \subseteq \mathcal{C} \cap \mathcal{M}$ and the latter set is closed in $\mathbb{R}^n$.
Also, since $u \in C^1(\tilde{\Omega})$, we have that

$$\begin{aligned} & u(P) = \lim_{k\to+\infty} u(P_k) = \phi(P) \\ \text{and} \qquad & \partial_\nu u(P) = \lim_{k\to+\infty} \partial_\nu u(P_k) = \psi(P). \end{aligned}$$

In particular,

$$\partial_\nu (u-\phi)(P) = (\psi - \partial_\nu \phi)(P) \neq 0,$$

due to (5). From this and the Implicit Function Theorem, we deduce that there exists $r \in (0, r(P))$ such that $\{u-\phi=0\} \cap B_r(P)$ is a connected C^1-graph, that we denote by $\mathcal{G}$. Up to a rotation, we may suppose that such a graph is in the vertical direction, say $\mathcal{G} := \{x_n = g(x')\}$; we let $\mathcal{G}^+ := \{x_n > g(x')\}$ and $\mathcal{G}^- := \{x_n < g(x')\}$. Here above, we used the notation in (7) for $r(P)$: we remark that $r(P)$ is defined because $P \in \mathcal{M} \setminus \Gamma$, due to (37) and (38), and, of course, the restriction $r < r(P)$ may be taken without loss of generality. Then, for small positive r,

$$\partial\Omega \cap B_r(P) \,=\, \mathcal{C} \cap B_r(P) \,\subseteq\, \{u-\phi=0\} \cap B_r(P) \,=\, \mathcal{G} \cap B_r(P). \tag{39}$$

By (36), we take k_o large enough so that

$$P_{k_o} \in B_{r/2}(P),$$

and we use (32) to see that

$$B_{\alpha_o}(P_{k_o}) \cap \mathcal{C} \subseteq B_r(P) \cap \mathcal{M}$$

and $u = \phi$, $\partial_\nu u = \psi$ on $\mathcal{C} \cap B_{\alpha_o}(P_{k_o})$, where

$$\alpha_o := \min\Big\{\alpha(P_{k_o}), \frac{r}{2}\Big\}.$$

The sets involved are sketched in Figure 2, in which the dashed horizontal straight line represents $\mathcal{M}$, the solid curve represents $\mathcal{C}$, and $\mathcal{M}$ and $\mathcal{C}$ agree on the dotted horizontal straight line near P_{k_o} (of course, this is just an "a-priori" picture, and we will show that these lines do coincide!).

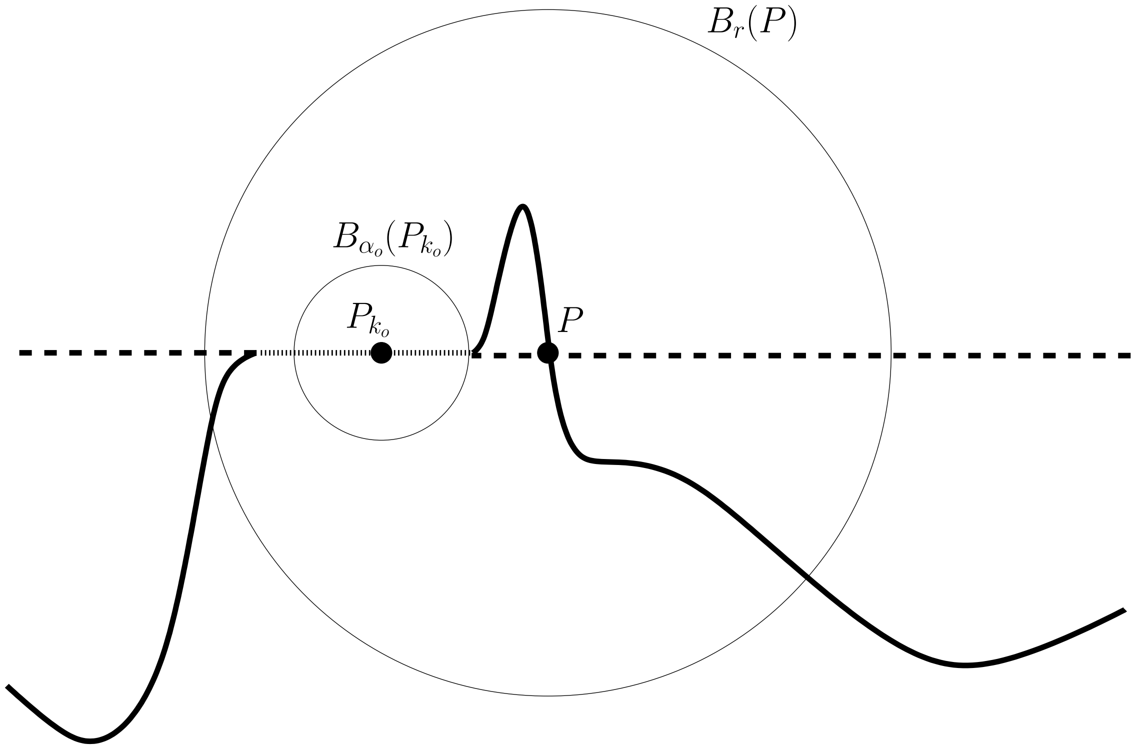

Figure 2: *Drawing the proof of* (35)

Now, we show that

$$\mathcal{C} \cap B_r(P) \text{ is a } C^1\text{-subset of } \partial\Omega. \tag{40}$$

To prove this, we use that $P_{k_o} \in \mathcal{X} \cap B_r(P)$ and we argue as follows. Since $\mathcal{C} \cap B_{\alpha(P_{k_o})}(P_{k_o})$ is a C^1-subset of $\partial\Omega$, we have that Ω lies on one side of it with respect to the vertical direction, by (4): that is to say, by possibly reverting the orientation of the vertical direction and taking $\alpha(P_{k_o})$ suitably small, we may suppose that

$$\mathcal{G}^+ \cap B_{\alpha(P_{k_o})}(P_{k_o}) \subseteq \Omega \tag{41}$$

and

$$\mathcal{G}^- \cap B_{\alpha(P_{k_o})}(P_{k_o}) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{42}$$

We claim that

$$\mathcal{G}^+ \cap B_r(P) \subseteq \Omega. \tag{43}$$

Suppose not: then there exists a point $Z \in \mathcal{G}^+ \cap B_r(P) \cap (\mathbb{R}^n \setminus \Omega)$. Take a curve $\vartheta : [0,1] \to \mathcal{G}^+ \cap B_r(P)$ such that $\vartheta(0) \in \Omega$ and $\vartheta(1) = Z$ (thanks to (41) such a curve exists), and let $t_\star := \sup\{t \in [0,1] \text{ s.t. } \vartheta(t) \in \Omega\}$. Then, $Z_\star := \vartheta(t_\star) \in (\partial\Omega) \cap \mathcal{G}^+ \cap B_r(P)$. In particular, by the Dirichlet boundary datum in (8), we have that

$$Z_\star \in \partial\Omega \subseteq \{u - \phi = 0\},$$

and so, by (39), we obtain that $Z_\star \in \mathcal{G}$. This is in contradiction with the fact that $Z_\star \in \mathcal{G}^+$, hence (43) is established.
In the same way (by reverting the roles of Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ and by using (42) instead of (41)), one obtains that

$$\mathcal{G}^- \cap B_r(P) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{44}$$

From (43) and (44), we conclude that $(\partial\Omega) \cap B_r(P) = \mathcal{G} \cap B_r(P)$. So, by (39),

$$\mathcal{C} \cap B_r(P) = (\partial\Omega) \cap B_r(P) = \mathcal{G} \cap B_r(P) \tag{45}$$

and this completes the proof of (40).
Now, we use (7), according to which u and $u^{(P)}$ satisfy the same PDE in $B_{r(P)}(P) \cap \Omega$, and they have the same Dirichlet and Neumann data on $\mathcal{C} \cap B_{\alpha_o}(P_{k_o})$, due to (32).
By (7) and Theorem 5 (up to taking a smaller $r(P)$), it follows that $u(x) = u^{(P)}(x)$ for any $x \in B_{r(P)}(P)$. In particular, since the latter is an open set that contains P, there exists $\rho(P) \in (0, r)$ for which $B_{\rho(P)}(P) \subseteq B_{r(P)}(P)$, and $\{u^{(P)} - \phi = 0\} \cap B_{\rho(P)}(P) = \mathcal{M} \cap B_{\rho(P)}(P)$.
So,

$$u(x) = u^{(P)}(x) \text{ for any } x \in B_{\rho(P)}(P) \cap \Omega \tag{46}$$

and

$$\mathcal{G} \cap B_{\rho(P)}(P) = \{u - \phi = 0\} \cap B_{\rho(P)}(P) = \{u^{(P)} - \phi = 0\} \cap B_{\rho(P)}(P) = \mathcal{M} \cap B_{\rho(P)}(P).$$

This and (45) imply that

$$\mathcal{C} \cap B_{\rho(P)}(P) = \mathcal{M} \cap B_{\rho(P)}(P). \tag{47}$$

Also, for any $x \in \mathcal{M} \cap B_{\rho(P)}(P)$, we deduce from (46), (47) and (7) that $u(x) = u^{(P)}(x) = \phi(x)$ and $\partial_\nu u(x) = \partial_\nu u^{(P)}(x) = \psi(x)$.
Moreover, by (6) and (46) that

$$u = u^{(P)} \in W^{2,2}\big(B_{\rho(P)}(P) \cap \Omega\big).$$

These observations, together with (47) and (40), says that $P \in \mathcal{X}$.
This completes the proof of (35).
Then, from (34) and (35), we have that

$$\mathcal{X} = \mathcal{C} = \mathcal{M}, \tag{48}$$

which implies (9) and (10). ∎

4 Proof of Corollary 8

We use Corollary 7, with $\mathcal{M} = \partial\Omega$ and

$$\Gamma := \{(x' x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n = \Psi(x') \text{ and } x' \in U\}.$$

We remark that condition (13) is fulfilled in this case due to the Cauchy-Kowaleskaya Theorem (see, e.g., [Eva98]), since Ψ and $u \mapsto u - u^3$ are analytic. Also, $\partial\Omega$ is connected.
Then, Corollary 7 gives that u and $\partial_\nu u$ are constant on the whole of $\partial\Omega$.
This and Theorem 1.7 of [FV10a] imply the desired result. ∎

5 Proof of Theorem 9

By (15), we observe that Γ is noncharacteristic for the PDE in (14), i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x, r, p)\nu_i(x)\nu_j(x) \neq 0 \quad \text{ for any } x \in \Gamma,\, r \in \mathbb{R},\, p \in \mathbb{R}^n.$$

Therefore, claim (i) follows from the Cauchy-Kowaleskaya Theorem (see, e.g., [Eva98]) and claims (ii) and (iii) follow from Theorem 5. ∎

6 Proof of Theorem 10

We deal with the case of (21), the case of (17) (or (22)) being very similar, just replacing portions of hyperplanes with portions of spheres (or cylinders), formula (19) with formula (18) (or (20)), and the 1D symmetry with the rotational (or cylindrical) one.
The idea for the proof is to use Theorem 6 to obtain that the overdetermined prescription occurs on the whole of $\{x_n = 0\}$. From this, one can use Theorem 5 to obtain that u is symmetric near $\{x_n = 0\}$, and then the Unique Continuation Principle to extend this symmetry everywhere.

Here are the details of the argument. First we extend u outside Ω, so that $u \in C^1(\mathbb{R}^n)$. From elliptic regularity (see, e.g., Lemma 6.18 in [GT01]), we conclude that $u \in C^{2,\alpha}(\Omega \cup \Gamma)$, and so, fixed $p_o \in \Gamma$, $u \in W^{2,2}(\Omega \cap B_{\epsilon_o}(p_o))$.
Now, we define $u_\star$ as in (19), and we set $u^\star(x) := u_\star(x_n)$. Furthermore, we may suppose, with no loss of generality, that

$$\tau \text{ gives the maximal interval of existence of the solution in (19).} \tag{49}$$

Now, we apply Theorem 6 with $\tilde{\Omega} := \mathbb{R}^n$, $\mathcal{M} := \{x_n = 0\}$ and $\Gamma := B_{r_o}(p_o) \cap \{x_n = 0\}$: for this, we observe that condition (7) is satisfied in this case thanks to the Cauchy Theorem for ODEs (see the footnote on page 9). Therefore, by Theorem 6, we conclude that

$$\{x_n = 0\} \subseteq \partial\Omega, \text{ and } u = 0 \text{ and } \partial_\nu u = c \text{ on } \{x_n = 0\}. \tag{50}$$

Therefore, by (19) and Theorem 5, we have that $u = u^\star$ in $\Omega \cap \{|x_n| < \tau\}$. Also $\{|x_n| < \tau\}$ contains Ω, by (49), so

$$u = u^\star \text{ in } \Omega. \tag{51}$$

This gives the desired symmetry for u. In particular, the connected component of $\partial\Omega$ which contains Γ has to agree with $\{x_n = 0\}$.
Now, we classify the domain. For this, first of all, we observe that

$$\text{the zeroes of } u^\star \text{ cannot accumulate,} \tag{52}$$

due to Cauchy's Uniqueness Theorem for ODEs.
We know from (51) that the level sets of u have planar symmetry (or rotational, or cylindrical symmetry, in the other cases), and so, by (52),

$$\partial\Omega \text{ is contained on non-accumulating, parallel hyperplanes} \tag{53}$$

(or concentric spheres, or cylinders, in the other cases). By (51), these hyperplanes are normal to the nth direction of the coordinate frame.
We claim that

$$\text{any connected component of } \partial\Omega \text{ is a hyperplane} \tag{54}$$

(or a sphere, or a cylinder in the other cases). To prove this, let C be a connected component of $\partial\Omega$. In particular, C is closed in $\partial\Omega$ and so in $\mathbb{R}^n$. By (53), we have that C is a C^0-subset of $\partial\Omega$ contained in some hyperplane Π, thus $C = \Pi$, and this proves (54).
From (54), the domain Ω may be classified as stated in Theorem 10. ∎

7 Proof of Theorem 11

Let $\mathcal{R}$ be a rotation of $\mathbb{R}^{n-1}$. For any $x = (x', x_n) \in \Omega$ let $v(x) := u(\mathcal{R}x', x_n)$. By construction, v satisfies (24) and $v(x) = u_o(|x'|, x_n) = u(x)$ when $x \in \partial\Omega$.
Moreover, if we consider the rotation of $\mathbb{R}^n$ given by

$$\mathcal{R}_\star := \begin{pmatrix} \mathcal{R} & 0 \\ 0 & 1 \end{pmatrix},$$

we deduce from the rotational symmetry of $\partial\Omega$ that if ν is the exterior normal at $x \in \partial\Omega$, then $\tilde\nu = (\tilde\nu', \tilde\nu_n) := \mathcal{R}_\star \nu = (\mathcal{R}\nu', \nu_n)$ is the exterior normal at $\tilde x = (\tilde x', \tilde x_n) := \mathcal{R}_\star x = (\mathcal{R}x', x_n) \in \partial\Omega$. As a consequence, for any $x \in \partial\Omega$,

$$\begin{aligned}\partial_\nu v(x) &= \nu \cdot \nabla\big(u(\mathcal{R}x', x_n)\big) = \nu \cdot \nabla\big(u(\mathcal{R}_\star x)\big) = \nu \cdot \big(\mathcal{R}_\star^T \nabla u(\mathcal{R}_\star x)\big)\\ &= (\mathcal{R}_\star \nu) \cdot (\nabla u(\tilde x)) = \tilde\nu \cdot (\nabla u(\tilde x)) = \partial_{\tilde\nu} u(\tilde x) = v_o(|\tilde x'|, \tilde x_n) = v_o(|x'|, x_n) = \partial_\nu u(x).\end{aligned}$$

Therefore, by Theorem 5, we have that v is identically equal to u (in order to apply Theorem 5, notice that u is C^2 up to $\partial\Omega \setminus \{O\}$ by Lemma 6.18 in [GT01]).
Since $\mathcal{R}$ is an arbitrary rotation of $\mathbb{R}^{n-1}$, we have proved (25).
Now, we prove (26). For this, we notice that, since u is monotone and bounded, we can define, for any $x' \in \mathbb{R}^{n-1}$,

$$\overline{u}(x') := \lim_{n\to+\infty} u(x', x_n).$$

By (25), we have that $\overline{u}$ is rotationally symmetric, i.e. we can write

$$\overline{u}(x') = \overline{u_\star}(|x'|), \tag{55}$$

for a suitable $\overline{u_\star} : \mathbb{R} \to \mathbb{R}$.
Moreover, by the monotonicity of u, we have that u is stable (see, e.g., [AAC01], or Section 7 in [FSV08]), that is

$$\int_\Omega |\nabla\varphi(x)|^2 - f'(u(x))\big(\varphi(x)\big)^2\, dx \geq 0 \tag{56}$$

for any $\varphi \in C_0^\infty(\Omega)$.
Thus, by passing to the limit (56), we have that $\overline{u}$ is stable as well. Using this, (55), Theorem 2.2(i) in [Vil07] (see also [CC04] for related results), and the fact that $n-1 \leq 10$, we deduce that $\overline{u}$ is constant.
Then, (26) follows from a classical, variational, energy argument of [AAC01] (see, e.g., Lemma 9.1 in [FV10a]). ∎

8 Proof of Corollary 12

The proof makes use of (26) and of some rigidity results in the literature. From Theorem 11 and the fact that $n \leq 3$, we know that $\mathcal{E}(R) \leq CR^2$. Hence, repeating verbatim the argument[4] in the proof of Corollary 9.4 of [FV10a], we see that $u(x) = u_\sharp(\omega \cdot x)$ for any $x \in \Omega$, for suitable $\omega = (\omega', \omega_n) \in \mathrm{S}^{n-1}$ and $u_\sharp : \mathbb{R} \to \mathbb{R}$.
So, recalling (25), we see that u and its level sets have both rotational and planar symmetry, which implies the thesis of Corollary 12. ∎

[4] We remark that the argument in the proof of Corollary 9.4 of [FV10a] needs condition (27), and this is why we require such condition in the statement of Corollary 12 here. See also the version of the Geometric Sternberg–Zumbrun–Poincaré Inequality for Lipschitz epigraphs as given in Theorem 1 of [FV10b].

9 Proof of Theorem 14

The idea for the proof is that, since u is bounded, $f(r)$ can be modified for large r without affecting the problem, and this trick makes some results of [BCN97] available for our purposes.
Here are the details. We argue by contradiction, supposing that such a solution u exists. Then, we have that

$$\lambda > 0. \tag{57}$$

To establish this, suppose, by contradiction, that $\lambda \le 0$. Then, $\Delta u \ge 0$ in Ω, so Lemma 2.1 in [BCN97] gives that $u \le 0$ in Ω, in contradiction with our assumptions.
We define

$$\begin{aligned} M &:= \sup_{\Omega} u, \\ s_0 &:= 1, \\ s_1 &:= M+1, \\ \mu &:= M+2, \\ f(r) &:= \begin{cases} \lambda r & \text{if } r \le s_1, \\ \lambda s_1(\mu - r) & \text{if } r > s_1. \end{cases} \end{aligned}$$

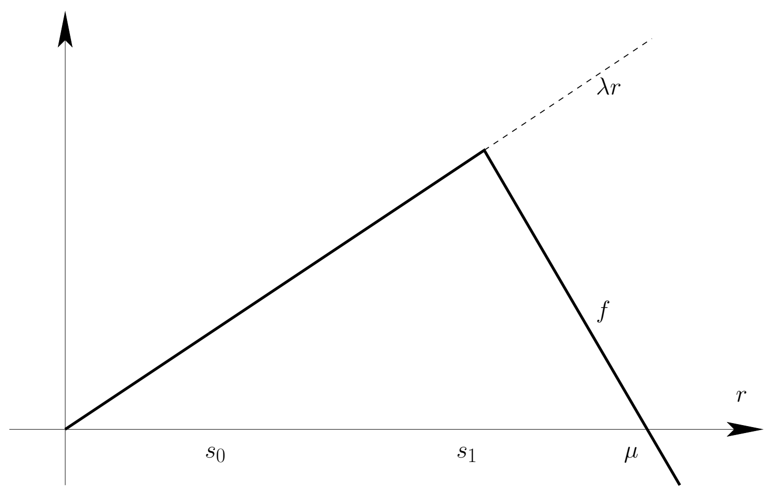

Figure 3: *The function f in the proof of Theorem 14*

As a consequence of (57) we have that f is globally Lipschitz continuous, $f > 0$ in $(0, \mu)$, and $f \le 0$ in $[\mu, +\infty)$, that is Condition (1) on page 1090 of [BCN97] is fulfilled; also, $f(r) \ge \lambda r$ when $r \in [0, s_0]$, and f is decreasing in (s_1, μ), therefore Conditions (2) and (3) on page 1090 of [BCN97] are satisfied.
Moreover, $\Delta u + f(u) = \Delta u + \lambda u = 0$ in Ω. Consequently, Theorem 1.2(b) on page 1091 of [BCN97] says that

$$\lim_{x_n \to +\infty} u(0, x_n) = \mu > M,$$

which is in contradiction with the definition of M. ∎

Appendix A

The purpose of this appendix is to show that the results of this paper are general enough to relate, for instance, with some of the results of [FG08].
For instance, with our methods, we obtain:

Theorem 15. *Let $c \in \mathbb{R} \setminus \{0\}$. Let Ω be an open, connected and bounded subset of $\mathbb{R}^n$.*
Suppose that $\partial\Omega$ is connected.
Let $\Gamma \subseteq \partial\Omega$ be nonempty and relatively open in $\partial\Omega$.
Assume that there exists an open set $\Omega' \subset \mathbb{R}^n$ with a connected, analytic boundary such that $\Gamma \subseteq \partial\Omega'$.
Assume also that f is an analytic function.
If there exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \Gamma, \end{cases} \tag{58}$$

then $\Omega = \Omega'$, Ω is a ball and u is radially symmetric.

Theorem 15 may be proved directly with the results of this paper, via the following argument. We observe that by the Cauchy-Kowaleskaya Theorem (see, e.g., [Eva98]), for any $P \in \partial\Omega'$, one can a find a solution $u^{(P)}$ of (7) on $B_{r(P)}(P)$. So, we apply Theorem 6 with $\phi := 0$, $\psi := c \neq 0$, $\mathcal{C} := \partial\Omega$, $\mathcal{M} := \partial\Omega'$ and

$$\tilde{\Omega} := \Omega \cup \left(\bigcup_{P \in \partial\Omega'} B_{r(P)}(P) \right).$$

We remark that here it is very important to have stated Theorem 6 in the way we did. Then, we can extend $u \in C^2(\tilde{\Omega})$ by the analytic continuation, by setting $u := u^{(P)}$ in $B_{r(P)}(P)$, and (5) and (7) are true by construction. Hence, we obtain from Theorem 6 that $\partial\Omega = \mathcal{C} = \mathcal{M} = \partial\Omega'$, and $u = 0$ and $\partial_\nu u = c$ on $\partial\Omega$.
With this, one can use the result of [Ser71], and obtain the desired claim. ∎

Appendix B

The purpose of this appendix is to point out that condition (5) cannot, in general, be dropped.

To check this, let $n \geq 2$,

$$\begin{aligned} & u(x', x_n) := x_n^2/2, \\ & f := -1, \\ & \Gamma := \{x = (x', 0) \in \mathbb{R}^n \text{ s.t. } |x'| < 1\}, \\ & \mathcal{C} := \{x = (x', 0) \in \mathbb{R}^n\}, \\ & \Omega := \{x \in \mathbb{R}^n \text{ s.t. } x_n \in (0, 1)\} \\ \text{and} \qquad & \tilde{\Omega} := \{x \in \mathbb{R}^n \text{ s.t. } x_n \in (-1, 2)\}. \end{aligned}$$

Let $\mathcal{M}$ be a smooth, compact hypersurface such that

$$\Gamma \subset \mathcal{M} \subset \{x \in \mathbb{R}^n \text{ s.t. } |x'| < 2\}.$$

Let also ν be the exterior normal of $\mathcal{M}$, and ϕ, $\psi \in C^\infty(\mathbb{R}^n)$ such that

$$\phi(x) = u(x) \text{ and } \psi(x) = \partial_\nu u(x) \text{ for any } x = (x', x_n) \in \mathbb{R}^n \text{ with } |x'| \leq 3,$$
$$\text{and } \phi(4, 0, \dots, 0) = \psi(4, 0, \dots, 0) = 1.$$

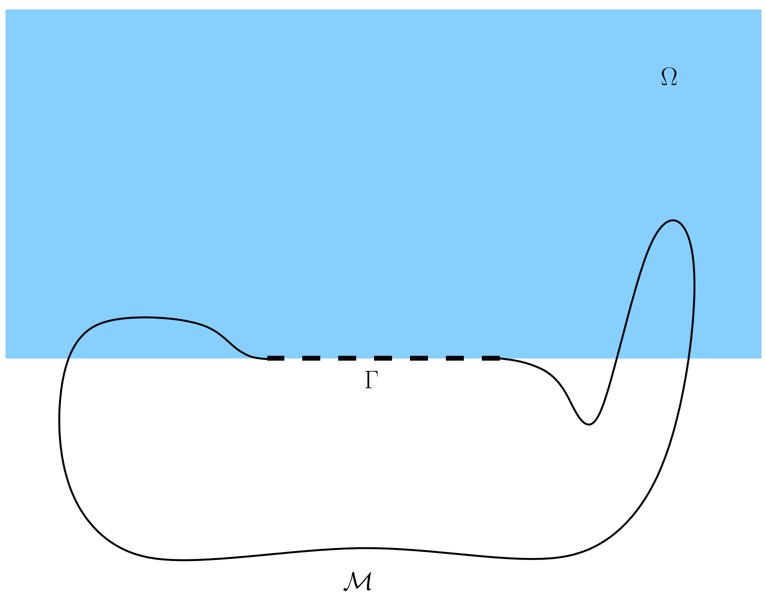

Figure 4: *The sets involved in Appendix B*

We observe that

$$(\mathcal{M} \setminus \Gamma) \cap \{x_n = 0\} \neq \varnothing.$$

Indeed, if not, we would have that $\mathcal{M} \cap \{x_n = 0\} = \Gamma$, but this cannot be since $\mathcal{M} \cap \{x_n = 0\}$ is a closed subset of $\mathbb{R}^n$, while Γ is not.

Then, all the assumptions of Theorem 6 are satisfied, with the only exception of condition (5), which is violated on $(\mathcal{M}\setminus\Gamma)\cap\{x_n = 0\}$. Notice, indeed, that (7) is satisfied by choosing $r(P) := 1$ and $u^{(P)} := u$.
The theses of Theorem 6 also do not hold in this case. Actually, (9) is obviously not true here; also $u(4, 0, \dots, 0) = 0 \neq 1 = \phi(4, 0, \dots, 0)$ and $|\nabla u(4, 0, \dots, 0)| = 0 \neq 1 = |\psi(4, 0, \dots, 0)|$, which shows that (10) does not hold as well. ■

Appendix C

Concerning the assumptions on Γ in Theorem 6, we now observe that the condition that Γ is a C^1-subset of $\partial\Omega$ cannot be relinquished.
Let $n \geq 2$, $\Omega := \{x_1 > 0\}$, $\tilde{\Omega} := \mathbb{R}^n$, $\mathcal{M} := \{x_n = 0\}$, $\Gamma := \{0\}$, $u(x) := \tanh(x_1/\sqrt{2})$, $\phi(x) := 0$, $\psi \in C^\infty(\mathbb{R}^n, [1/10, 2])$ with $\psi(0, \dots, 0, 0) := 1/\sqrt{2}$ and $\psi(0, \dots, 0, 1) = 1$. In this case, $\mathcal{C} = \partial\Omega = \{x_1 = 0\}$, and u is a solution of the Allen-Cahn equation (i.e., $f(u) = u - u^3$). Condition (5) is satisfied, and condition (7) holds since ψ never vanishes.
So, all the assumptions of Theorem 6 are met, with the exception that Γ is not open in $\partial\Omega$. Notice that in this case, the thesis in (9) of Theorem 6 fail.
Moreover, the thesis in (10) cannot be true, since $(0, \dots, 0, 1) \in \mathcal{C}$ but $|\nabla u(0, \dots, 0, 1)| = 1/\sqrt{2} \neq 1 = \psi(0, \dots, 0, 1)$. ■

Appendix D

Now, we show that Theorem 4.2.5 on page 67 of [Hen06] (see also [HO03]) about optimal shapes in elliptic problems may be easily obtained as a consequence of the results given here, thus obtaining a new proof of it.
Very sketchy, the setting in [Hen06] is the following. For any $k \in \mathbb{N}$, one considers the kth eigenvalue $\lambda_k(\Omega)$ of the operator $-\Delta$ with Dirichlet boundary datum on a convex domain Ω with prescribed Lebesgue measure.
For a given $k \in \mathbb{N}$, it is always possible to find a domain $\Omega_\star$ that attains the minimum value of $\lambda_k(\Omega)$ among all the possible choices of convex domains Ω with prescribed Lebesgue measure: see Theorem 2.4.1 on page 35 of [Hen06].
The case $k = 2$ and $n = 2$ turns out to be particularly ineteresting, since the optimal convex planar shape has to satisfy additional geometric properties: see page 64 of [Hen06]. In particular, dealing with a conjecture of Troesch, the following result turned out to be very important:

Theorem 16. *[[HO03] and [Hen06]] The stadium (i.e. the convex hull of two identical tangent discs) does not realize the minimum of λ_2 among plane convex domains of given area.*

We can give a new proof of such a result, via the following argument. Let us call S the stadium. We observe that the boundary of S is $C^{1,1}$. Also, $\partial S = C_1 \cup C_2 \cup L_1 \cup L_2$, where C_1 and C_2 are two (open) arcs of circumference, and L_1 and L_2 are two (closed) segments.
Let $u \in L^2(S)$ be the eigenfunction corresponding to $\lambda_2(S)$. By Theorem 9.15 of [GT01], we know that $u \in W^{2,2}(S)$ and so, by Sobolev embedding, $u \in C^{0,1/2}(\overline{S}) \subset L^p(S)$ for any $p > 1$.

Consequently, by applying again Theorem 9.15 of [GT01], we conclude that $u \in W^{2,p}(S)$ for any $p > 1$ and so $u \in C^1(\overline{S})$ (an argument of this type is also outlined on page 9 of [Hen06]). Furthermore, by formula (4.5) on page 64 of [Hen06], we have that $\partial_\nu u$ is constant along C_1 (and also along C_2). Then, we can apply Theorem 10 (with $\Omega := S$, $f(u) := \lambda_2(S)u$ and $\Gamma := C_1$). We obtain that both u and S possess rotational symmetry, in contradiction with the actual shape of S. ∎

Appendix E

In Theorem 10, one cannot drop the regularity assumption on $\partial\Omega$. For instance, if $\partial\Omega$ is not smooth, the solution may have the desired symmetry, but the domain may not. For example, take $u(x) := \sin x_n$, $\Gamma := \{x_n = 0\}$, $\Omega := \{x_n \in (0, 2\pi)\} \setminus \Sigma$, for *any* nonempty closed set Σ such that $\Sigma \subset \{x \in \mathbb{R}^n \text{ s.t. } |x'| > 1 \text{ and } x_n = \pi\}$: in this case u has planar symmetry but Ω does not. ∎

Appendix F

In this appendix, we remark that the statements of Theorem 10 may not hold if the Dirichlet boundary condition in (16) is given only on Γ instead on the whole of $\partial\Omega$. For instance, take f for which the solution $u_\star$ of the ODE in (18) (resp., (19) or (20)) is defined in the whole of $\mathbb{R}$ and let u to be $u_\star$ extended by symmetry. Let us consider a domain Ω whose boundary is given by two connected components, Γ and Λ, with $\Gamma := \partial B_1$ being a sphere (resp., $\Gamma := \{x_n = 0\}$ being a hyperplane, or $\Gamma := \partial B_1'$ being a cylinder), and Λ not a sphere (resp., not a hyperplane, not a cylinder). Then

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 \text{ on } \Gamma, \\ \partial_\nu u = c \text{ on } \Gamma \end{cases}$$

i.e., the Dirichlet datum in (16) is given only on Γ instead on the whole of $\partial\Omega$. But in this case, of course, it is not possible to obtain any classification of the shape of Ω.
This says that it is essential to prescribe the Dirichlet boundary condition in (16) on the whole of $\partial\Omega$. ∎

Acknowledgments

This work has been supported by *FIRB* project "Analysis and Beyond" and *GNAMPA* project "Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni".

References

[AAC01] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (2001), no. 1-3, 9–33, Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. MR 1843784 (2002f:35080)

[AC81] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. MR 618549 (83a:49011)

[Ale56] A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large. I*, Vestnik Leningrad. Univ. **11** (1956), no. 19, 5–17. MR 0086338 (19,167c)

[BCN97] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1089–1111. MR 1470317 (98k:35064)

[CC04] Xavier Cabré and Antonio Capella, *On the stability of radial solutions of semilinear elliptic equations in all of* $\mathbb{R}^n$, C. R. Math. Acad. Sci. Paris **338** (2004), no. 10, 769–774. MR 2059485 (2005a:35084)

[DG79] Ennio De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978) (Bologna), Pitagora, 1979, pp. 131–188. MR 533166 (80k:49010)

[DSJ09] Daniela De Silva and David Jerison, *A singular energy minimizing free boundary*, J. Reine Angew. Math. **635** (2009), 1–21. MR 2572253 (2010k:49003)

[DSV10] Daniela De Silva and Enrico Valdinoci, *A fully nonlinear problem with free boundary in the plane*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 1, 111–132. MR 2668875

[DZ01] M. C. Delfour and J.-P. Zolésio, *Shapes and geometries*, Advances in Design and Control, vol. 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, Analysis, differential calculus, and optimization. MR 1855817 (2002i:49002)

[Eva98] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR 1625845 (99e:35001)

[FG08] Ilaria Fragalà and Filippo Gazzola, *Partially overdetermined elliptic boundary value problems*, J. Differential Equations **245** (2008), no. 5, 1299–1322. MR 2436831

[FGLP09] Ilaria Fragalà, Filippo Gazzola, Jimmy Lamboley, and Michel Pierre, *Counterexamples to symmetry for partially overdetermined elliptic problems*, Analysis (Munich) **29** (2009), no. 1, 85–93. MR 2524107

[Fre77] Jens Frehse, *On Signorini's problem and variational problems with thin obstacles*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 2, 343–362. MR 0509085 (58 #22987)

[FSV08] Alberto Farina, Berardino Sciunzi, and Enrico Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 741–791. MR 2483642 (2009j:58020)

[FV09] Alberto Farina and Enrico Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*, Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 74–96. MR 2528756

[FV10a] ———, *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems*, Arch. Ration. Mech. Anal. **195** (2010), no. 3, 1025–1058. MR 2591980 (2011a:35092)

[FV10b] ———, *Overdetermined problems in unbounded domains with Lipschitz singularities*, Rev. Mat. Iberoam. **26** (2010), no. 3, 965–974.

[GNN79] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR 544879 (80h:35043)

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364 (2001k:35004)

[Hen06] Antoine Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006. MR 2251558 (2007h:35242)

[HO03] Antoine Henrot and Edouard Oudet, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Arch. Ration. Mech. Anal. **169** (2003), no. 1, 73–87. MR 1996269 (2004j:35215)

[Hör83a] Lars Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis. MR 717035 (85g:35002a)

[Hör83b] ———, *The analysis of linear partial differential operators. II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 257, Springer-Verlag, Berlin, 1983, Differential operators with constant coefficients. MR 705278 (85g:35002b)

[HP05] Antoine Henrot and Michel Pierre, *Variation et optimisation de formes*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 48, Springer, Berlin, 2005, Une analyse géométrique. [A geometric analysis]. MR 2512810 (2009m:49003)

[Ser71] James Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318. MR 0333220 (48 #11545)

[Sic10] Pieralberto Sicbaldi, *New extremal domains for the first eigenvalue of the Laplacian in flat tori*, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 329–344. MR 2592974

[Sok56] I. S. Sokolnikoff, *Mathematical theory of elasticity*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956, 2d ed. MR 0075755 (17,800b)

[Vil07] Salvador Villegas, *Asymptotic behavior of stable radial solutions of semilinear elliptic equations in $\mathbb{R}^N$*, J. Math. Pures Appl. (9) **88** (2007), no. 3, 241–250. MR 2355457 (2008g:34134)

[Vog92] Andrew L. Vogel, *Symmetry and regularity for general regions having a solution to certain overdetermined boundary value problems*, Atti Sem. Mat. Fis. Univ. Modena **40** (1992), no. 2, 443–484. MR 1200301 (94b:35187)

[WCG95] N. B. Willms, Marc Chamberland, and G. M. L. Gladwell, *A duality theorem for an overdetermined eigenvalue problem*, Z. Angew. Math. Phys. **46** (1995), no. 4, 623–629. MR MR1345815 (96e:35121)

[Wei71] H. F. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Rational Mech. Anal. **43** (1971), 319–320. MR 0333221 (48 #11546)