Short-time Heat Flow and Functions of Bounded Variation in $\mathbf{R}^N$

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Abstract

We prove a characterisation of sets with finite perimeter and BV functions in terms of the short time behaviour of the heat semigroup in $\mathbf{R}^N$. For sets with smooth boundary a more precise result is shown.

Résumé

On prouve une caractérisation des ensembles avec perimètre fini et des fonctions à variation bornée en termes du comportément du semi-groupe de la chaleur dans $\mathbf{R}^N$ au voisinage de $t = 0$. On prouve aussi un résultat plus précis pour les ensembles avec frontière assez régulière.

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1 Introduction

Sets with finite perimeter have been introduced by E. De Giorgi in the fifties (see [5], [6]), as a part of the theory of functions of bounded variation, in order to deal with geometric variational problems and have proved to be very useful in several contexts. The first researches of De Giorgi were connected with the investigations of R. Caccioppoli, and in fact sets with finite perimeter are also called *Caccioppoli sets*. Let us refer to [2] for the properties of sets with finite perimeter and BV functions. De Giorgi's original definition of the perimeter of a (measurable) set $E \subset \mathbf{R}^N$ was based on the heat semigroup $(T(t))_{t\geq 0}$ in $\mathbf{R}^N$, because of its regularising effects, and can be phrased as follows:

$$P(E) = \lim_{t\to 0} \|\nabla_x T(t)\chi_E\|_{L^1(\mathbf{R}^N)},$$

where χ_E denotes the characteristic function of E. We denote by $p_N : \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}_+ \to \mathbf{R}$ the Gauss-Weierstrass kernel, defined by

$$p_N(x, y, t) := \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}},$$

so that $T(t)u(x) = \int_{\mathbf{R}^N} p_N(x, y, t)u(y)dy$ for every $u \in L^1(\mathbf{R}^N)$.

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In [8] M. Ledoux investigated in a different perspective some connections between the heat semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbf{R}^N)$ and the isoperimetric inequality, observing that the L^2-inequality

$$\|T(t)\chi_E\|_{L^2(\mathbf{R}^N)} \leq \|T(t)\chi_B\|_{L^2(\mathbf{R}^N)} \qquad t \geq 0 \tag{1.1}$$

for all sets E with smooth boundary with the same volume $|E|$ as the ball B implies the isoperimetric inequality. By the self-adjointness of the operators $T(t)$ and

$$\|T(t)\chi_E\|^2_{L^2(\mathbf{R}^N)} = \langle T(t)\chi_E, T(t)\chi_E\rangle = \langle T(2t)\chi_E, \chi_E\rangle, \tag{1.2}$$

the behaviour of $\langle T(t)\chi_E, \chi_E\rangle$ is related to the L^2-norm of $T(t)\chi_E$, where we use the notation $\langle f, g\rangle = \int_{\mathbf{R}^N} fg dx$ whenever the integral is finite. Notice that (1.1) can be easily deduced from the Riesz-Sobolev inequality (see e.g. [9, Theorem 3.7])

$$\int_{\mathbf{R}^N\times\mathbf{R}^N} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbf{R}^N\times\mathbf{R}^N} f^*(x)g^*(x-y)h^*(y)dxdy, \tag{1.3}$$

where ϕ^* denotes the spherical symmetrisation of ϕ. Taking $f = h = \chi_E$ and $g = g^* = p_N(\cdot, \cdot, t)$ in (1.3), so that $f^* = h^* = \chi_B$, the inequality (1.1) follows immediately:

$$\begin{aligned}
\|T(t)\chi_E\|^2_{L^2(\mathbf{R}^N)} &= \langle T(2t)\chi_E, \chi_E\rangle \\
&= \int_{\mathbf{R}^N\times\mathbf{R}^N} \chi_E(x)\chi_E(y)p_N(x, y, 2t)dxdy \\
&\leq \int_{\mathbf{R}^N\times\mathbf{R}^N} \chi_B(x)\chi_B(y)p_N(x, y, 2t)dxdy \\
&= \langle T(2t)\chi_B, \chi_B\rangle = \|T(t)\chi_B\|^2_{L^2(\mathbf{R}^N)}
\end{aligned}$$

In [8] one important point has been the formula

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_B, \chi_{B^c}\rangle = P(B), \tag{1.4}$$

where B is a ball, and the inequality

$$\sqrt{\frac{\pi}{t}}\langle T(t)\chi_E, \chi_{E^c}\rangle \leq P(E) \qquad \text{for every } t \geq 0, \tag{1.5}$$

which has been generalised in [12] for all $E \subset \mathbf{R}^N$ such that either E or its complementary set E^c has a finite volume (otherwise, both terms are infinite). If E and B have the same volume, from the elementary relation

$$|E| = \langle T(t)\chi_E, \chi_E\rangle + \langle T(t)\chi_E, \chi_{E^c}\rangle \qquad \text{for every } t \geq 0$$

and (1.2) it follows that the L^2-inequality (1.1) is equivalent to

$$\langle T(t)\chi_E, \chi_{E^c}\rangle \geq \langle T(t)\chi_B, \chi_{B^c}\rangle \qquad \text{for every } t \geq 0,$$

and the semigroup inequality (1.1) implies the isoperimetric inequality for Caccioppoli sets in $\mathbf{R}^N$. In connection with these results, it seems to be interesting to pursue the investigation of the relationships between the perimeter of a set and the short-time behaviour of the heat semigroup. The asymptotic expansion of the heat semigroups on regular manifolds or submanifolds has been deeply investigated (see e.g. [7], and also [10]), and in fact a *localised* version of (1.4) can be proved for sets with smooth boundary (i.e., such that the unique projection property in a tubular neighbourhood of the boundary holds), see Theorem 2.1. This result is in fact stronger than (1.4) itself, see Theorem 2.4. In Section 3 we prove that equality (1.4) holds true not only for smooth sets, but for all Caccioppoli sets B. Our proof is based upon the measure-theoretic properties of the reduced boundary. We also show that the finiteness of the limit on the left hand side characterises sets of finite perimeter. Let us point out that the same characterisation of finite perimeter sets is also proved, following a different approach based on the study of the behaviour of the difference quotients of u, in the papers [3], [4], [11] (see also [1]), where convolution kernels more general than the Gauss-Weierstrass one are considered. Our approach is more geometric in spirit, and gives directly the optimal constants.

Section 4 is devoted to some remarks concerning the short-time behaviour of the heat semigroup for general BV functions. Recalling that $u \in BV(\mathbf{R}^N)$ if $u \in L^1(\mathbf{R}^N)$ and its distributional gradient is a ($\mathbf{R}^N$-valued) Radon measure with finite total variation given by

$$|Du|(\mathbf{R}^N) = \sup\Big\{\int_{\mathbf{R}^N} u \,\mathrm{div}\, g dx : \; g \in [C_c^1(\mathbf{R}^N)]^N, \; \|g\|_{L^\infty(\mathbf{R}^N)} \le 1\Big\},$$

we show that the equality

$$|Du|(\mathbf{R}^N) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x) - u(y)| p_N(x,y,t) dx dy$$

holds. Since $P(E) = |D\chi_E|(\mathbf{R}^N)$ if $u = \chi_E$, the above equality is equivalent to (1.4), and the finiteness of the right hand side characterises the BV functions. As a consequence, defining (as in [14]) the interpolation space

$$(L^1(\mathbf{R}^N), D(\Delta))_{1/2,\infty} = \Big\{u \in L^1(\mathbf{R}^N) : \; \sup_{0<t<1} \frac{1}{\sqrt{t}} \|T(t)u - u\|_{L^1(\mathbf{R}^N)} < +\infty\Big\}, \tag{1.6}$$

we have that $BV(\mathbf{R}^N) \subset (L^1(\mathbf{R}^N), D(\Delta))_{1/2,\infty}$ and, from a known characterisation of the above interpolation space, an embedding theorem for BV into a Besov space follows.

Notation We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure (which coincides with the classical measure for k-dimensional smooth submanifolds). For every measure μ and measurable set E, we denote by $\mu \llcorner E$ the restriction measure, i.e., $\mu \llcorner E(B) = \mu(E \cap B)$ for all measurable B. For E measurable and $t \in [0,1]$ we denote by E^t the set of points of density t, i.e., we set

$$x \in E^t \quad \Longleftrightarrow \quad \lim_{\varrho\to 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho|} = t. \tag{1.7}$$

Recall that the *essential boundary* of E is $\partial^* E = \mathbf{R}^N \setminus (E^0 \cup E^1)$, and the *reduced boundary* $\mathcal{F}E$ is defined as follows. For $E \subset \mathbf{R}^N$ such that $\chi_E \in BV_{loc}(\mathbf{R}^N)$, i.e., E has locally finite perimeter, $x \in \mathrm{supp}|D\chi_E|$ belongs to $\mathcal{F}E$ if the limit

$$\nu_E(x) := \lim_{\varrho\to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))}$$

exists in $\mathbf{R}^N$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : E \to \mathbf{S}^{N-1}$ is called the *generalised inner normal* to E, and, see e.g. [2, Theorem 3.59], for every $x \in \mathcal{F}E$ the hyperplane $\pi_x = \{y : y \cdot \nu_E(x) = 0\}$ is the *approximate tangent plane* to $\mathcal{F}E$ at x, i.e.,

$$\lim_{\varrho\to 0} \frac{1}{\varrho^{N-1}} \int_E \phi\Big(\frac{x-y}{\varrho}\Big) dy = \int_{\pi_x} \phi(y) d\mathcal{H}^{N-1}(y) \qquad \forall\ \phi \in C_c(\mathbf{R}^N). \tag{1.8}$$

Moreover, see e.g. [2, Theorem 3.78],

$$\mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0. \tag{1.9}$$

and, as a consequence, the distributional derivative of χ_F is given by the $\mathbf{R}^N$-valued measure

$$D\chi_F = \nu_F \mathcal{H}^{N-1} \llcorner \mathcal{F}F. \tag{1.10}$$

2 Diffusion for regular sets

In this section we study the short-time behaviour of $T(t)\chi_A$ for sets A with smooth boundary. By *smooth* we mean the minimal regularity ensuring the unique projection property in a tubular neighbourhood of the boundary. To this end, the Lipschitz continuity of the unit normal vector field is the natural requirement. We say that $A \subset \mathbf{R}^N$ is *uniformly $C^{1,1}$-regular* if there are $\varrho, L > 0$ such that for every $p \in \partial A$ the set $\partial A \cup B_\varrho(p)$ is the graph of a $C^{1,1}$ function ψ with $\|\nabla\psi\|_\infty \le L$. Setting

$$A^\delta := \{x \in A^c : \mathrm{dist}(x, \partial A) \le \delta\}, \qquad A_\varepsilon := \{x \in A : \mathrm{dist}(x, \partial A) \le \varepsilon\} \tag{2.1}$$

for $\delta, \varepsilon > 0$, we prove the following theorem.

Theorem 2.1 *Let $A \subset \mathbf{R}^N$ be uniformly $C^{1,1}$-regular. Let A_ε and A^δ be an inner and outer tubular neighborhood of ∂A defined in (2.1). Then for every continuous $\varphi : \mathbf{R}^N \to \mathbf{R}$ with compact support the equality*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_{A_\varepsilon}, \varphi\chi_{A^\delta}\rangle = \int_{\partial A} \varphi\, d\mathcal{H}^{N-1}$$

holds.

Equality (1.4) for uniformly $C^{1,1}$-regular sets follows easily from Theorem 2.1 (see Theorem 2.4). Our proof of Theorem 2.1 is based on a preliminary one-dimensional computation and then on a partition of unity argument, reflecting the physical insight that for short time the heat flow is approximately "transversal" to the boundary.

The first step in the proof of Theorem 2.1 is the one-dimensional computation below.

Lemma 2.2 *Fix $\delta, \varepsilon > 0$, and for $t > 0$ define the functions f_t by*

$$f_t(z) := \frac{1}{\sqrt{t}} \int_0^\delta \chi_{\{-\varepsilon \le r + \sqrt{2t}\, z \le 0\}}(r)\, dr, \qquad z \le 0. \tag{2.2}$$

Then $f_t \ge f_s$ for every $0 < t < s$ and

$$\lim_{t\to 0} f_t(z) = -\sqrt{2}\, z, \qquad z \le 0.$$

Proof. In order to compute the integral in (2.2), notice that the integrand is nonzero if and only if

$$r \in I_t = [0,\delta] \cap [-\varepsilon - \sqrt{2t}\, z, -\sqrt{2t}\, z],$$

and that the interval I_t is

$$I_t = \begin{cases} [-\varepsilon - \sqrt{2t}z, \delta] & \text{if } z \le \min\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\}\,, \\ [0,\delta] & \text{if } z \in [-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}]\,, \\ [-\varepsilon - \sqrt{2t}z, -\sqrt{2t}z] & \text{if } z \in [-\delta/\sqrt{2t}, -\varepsilon/\sqrt{2t}]\,, \\ [0, -\sqrt{2t}z] & \text{if } z \ge \max\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\}\,. \end{cases}$$

Notice that only one between the second and third possibility for I_t may occur, according to the relative size of ε and δ. Consequently, since

$$f_t(z) = \frac{1}{\sqrt{t}}|I_t|\,,$$

we have

$$f_t(z) = \begin{cases} (\delta+\varepsilon)/\sqrt{t} + \sqrt{2}z & -\infty < z \le \min\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\}\,, \\ \min\{\varepsilon,\delta\}/\sqrt{t} & \min\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\} \le z \le \max\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\}\,, \\ -\sqrt{2}\,z & \max\{-\varepsilon/\sqrt{2t}, -\delta/\sqrt{2t}\} \le z \le 0\,, \end{cases}$$

and the assertion follows immediately. □

In order to fix the notation to be used in the proof of Theorem 2.1, let us recall the geometric properties of smooth boundaries which we are going to use. We refer e.g. to [15, Section I.2] for a detailed discussion on the subject.

Proposition 2.3 *Let $A \subset \mathbf{R}^N$ be a uniformly $C^{1,1}$-regular set. Then, there are $\varepsilon, \delta > 0$ such that the maps*

$$\begin{array}{lll} i) & \partial A \times [0,\delta] \to A^\delta, & (p,d) \mapsto p + d\cdot\nu(p) \\ ii) & \partial A \times [0,\varepsilon] \to A_\varepsilon, & (p,d) \mapsto p - d\cdot\nu(p), \end{array}$$

where $\nu(p)$ is the outward unit normal to ∂A at p, are $C^{1,1}$-diffeomorphisms.

Moreover, for every $\eta > 0$ there is a locally finite covering $\mathcal{V} = (V_i)$ of ∂A and $C^{1,1}$ diffeomorphisms

$$\psi_i : D_i \to V_i\,, \quad D_i \text{ open subset of } \mathbf{R}^{N-1}$$

such that

$$\begin{array}{ll} |D\nu(p)| \le \eta & p \in V_i \\ |D\psi_i(\xi) - R_i| \le \eta & \xi \in D_i \end{array} \tag{2.3}$$

for suitable linear maps R_i. For every V_i and ψ_i define

$$\Psi_i(\xi,\varrho) := \psi_i(\xi) + \varrho\nu(\psi_i(\xi))\,, \qquad \xi \in D_i\,, \varrho \in (-\varepsilon,\delta).$$

Denoting by U_i the open set

$$U_i := \Psi_i\big(D_i \times (-\varepsilon,\delta)\big),$$

the family $\mathcal{U} = (U_i)$ turns out to be a covering of $A_\varepsilon \cup A^\delta$, and

$$|D\Psi_i - L_i| < \eta \tag{2.4}$$

for every i, where L_i are orthogonal maps such that $|L_i e_N - \nu| \le \eta$.

Proof of Theorem 2.1. Recall that we denote by p_N the Gauss-Weierstrass kernel. We divide the proof in three steps.
Step 1 - We first consider the case when $A = \{x \in \mathbf{R}^N : \ x_N < 0\}$, so that we have

$$A^\delta = \mathbf{R}^{N-1} \times (0, \delta)\,, \quad A_\varepsilon = \mathbf{R}^{N-1} \times (-\varepsilon, 0)$$

and φ independent of x_N. Denoting $x = (x', x_N)$ (where $x' = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}$) observe that $p_N(x, y, t) = p_{N-1}(x', y', t)p_1(x_N, y_N, t)$ and take $\varphi = \varphi(x')$ in $C_c(\mathbf{R}^{N-1})$. Then we have

$$\begin{aligned}
&\sqrt{\frac{\pi}{t}} \int_{A^\delta} \int_{A_\varepsilon} p_N(x, y, t)\varphi(x)\, dy\, dx = \\
&= \sqrt{\frac{\pi}{t}} \int_{A^\delta} \int_{A_\varepsilon} p_{N-1}(x', y', t)p_1(x_N, y_N, t)\varphi(x')\, dy\, dx \\
&= \sqrt{\frac{\pi}{t}} \int_{\mathbf{R}^{N-1}} \int_{\mathbf{R}^{N-1}} p_{N-1}(x', y', t)\varphi(x') \int_0^\delta \int_{-\varepsilon}^0 p_1(x_N, y_N, t)\, dy_N\, dx_N\, dy'\, dx' \\
&= \Big(\int_{\mathbf{R}^{N-1}} \varphi(x')dx' \Big) \sqrt{\frac{\pi}{t}} \int_0^\delta \int_{-\varepsilon}^0 p_1(x_N, y_N, t)\, dy_N\, dx_N.
\end{aligned}$$

Let us now consider only $\sqrt{\frac{\pi}{t}} \int_0^\delta dx_N \int_{-\varepsilon}^0 p_1(x_N, y_N, t)\, dy_N$ in the last line above: taking the new variable $z = (y_N - x_N)/\sqrt{2t}$ we obtain

$$\begin{aligned}
&\sqrt{\frac{\pi}{t}} \int_0^\delta \int_{-\varepsilon}^0 p_1(x_N, y_N, t)\, dy_N\, dx_N = \\
&\qquad = \sqrt{\frac{\pi}{t}} \int_0^\delta \int_{\frac{-\varepsilon - x_N}{\sqrt{2t}}}^{\frac{-x_N}{\sqrt{2t}}} \frac{e^{-z^2/2}}{2\sqrt{\pi t}} \sqrt{2t}\, dz\, dx_N \\
&\qquad = \frac{1}{\sqrt{2t}} \int_0^\delta \int_{\mathbf{R}} e^{-z^2/2} \chi_{(-\varepsilon,0)}(\sqrt{2t}z + x_N)\, dz\, dx_N \\
&\qquad = \frac{1}{\sqrt{2t}} \int_{-\infty}^0 e^{-z^2/2} \int_0^\delta \chi_{(-\varepsilon,0)}(\sqrt{2t}z + x_N)\, dx_N\, dz.
\end{aligned}$$

By Lemma 2.2,

$$\lim_{t\to 0} \frac{1}{\sqrt{2t}} \int_0^\delta \chi_{(-\varepsilon,0)}(\sqrt{2t}z + x_N)\, dx_N = -z,$$

and then, by the monotone convergence theorem we conclude that

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{A^\delta} \int_{A_\varepsilon} p_N(x, y, t)\varphi(x)\, dy\, dx = \int_{\mathbf{R}^{N-1}} \varphi(x')dx'. \tag{2.5}$$

Step 2 - Take now a function $\varphi \in C_c(\mathbf{R}^N)$ and denote by A the same set as in *Step* 1. Notice that

$$\omega(\tau) := \sup_{(x', x_N) \in \mathbf{R}^{N-1} \times [-\tau, \tau]} |\varphi(x', x_N) - \varphi(x', 0)| \to 0 \quad \text{as } \tau \to 0. \tag{2.6}$$

We can then write

$$\begin{aligned}
&\sqrt{\frac{\pi}{t}} \int_{A^\delta} \int_{A_\varepsilon} p_N(x, y, t)\varphi(x)\, dy\, dx = \\
&\qquad = \sqrt{\frac{\pi}{t}} \int_{A^\delta} \int_{A_\varepsilon} p_N(x, y, t)\big[\varphi(x', 0) + (\varphi(x) - \varphi(x', 0))\big]\, dy\, dx\,.
\end{aligned}$$

By *Step* 1 we have that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{A^\delta}\int_{A_\varepsilon} p_N(x,y,t)\varphi(x',0)\,dy\,dx = \int_{\mathbf{R}^{N-1}}\varphi(x',0)dx'\,.$$

It remains to prove that

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{A^\delta}\int_{A_\varepsilon} p_N(x,y,t)[\varphi(x)-\varphi(x',0)]\,dy\,dx = 0\,. \tag{2.7}$$

As done in *Step* 1 we obtain, denoting by K the projection of spt φ on $\{x\in\mathbf{R}^N\,|\,x_N=0\}$

$$\begin{aligned}
\Big|\sqrt{\frac{\pi}{t}}\int_{A^\delta}\int_{A_\varepsilon} & p_N(x,y,t)[\varphi(x)-\varphi(x',0)]\,dy\,dx\Big| \leq \\
&\leq \sqrt{\frac{\pi}{t}}\int_K\int_0^\delta\int_{-\varepsilon}^0|\varphi(x)-\varphi(x',0)|p_1(x_N,y_N,t)dy_N dx_N dx' \\
&= \sqrt{\frac{\pi}{t}}\int_K\int_0^\delta\int_{\mathbf{R}}|\varphi(x)-\varphi(x',0)|\chi_{(-\varepsilon,0)}(y_N)\frac{e^{-|x_N-y_N|^2/4t}}{(4\pi t)^{1/2}}dy_N dx_N dx' \\
&= \frac{1}{\sqrt{2t}}\int_K\int_0^\delta\int_{-\infty}^0|\varphi(x)-\varphi(x',0)|\chi_{(-\varepsilon,0)}(x_N+\sqrt{2t}z)e^{-z^2/2}dz dx_N dx'
\end{aligned}$$

where in the last equality we have set $z=(y_N-x_N)/\sqrt{2t}$.
Now fix $\sigma>0$: then we can find $z_0<0$ for which

$$\Big|\int_{-\infty}^{z_0} z e^{-z^2/2}dz\Big| < \sigma\,. \tag{2.8}$$

Going on estimating, from above we derive, for $-\sqrt{2t}z_0\leq\delta$,

$$\begin{aligned}
\Big|\sqrt{\frac{\pi}{t}}\int_{A^\delta}\int_{A_\varepsilon} & p_N(x,y,t)[\varphi(x)-\varphi(x',0)]\,dy\,dx\Big| \leq \\
&= \frac{1}{\sqrt{2t}}\int_K\int_{-\infty}^{z_0}\int_0^\delta|\varphi(x)-\varphi(x',0)|\chi_{(-\varepsilon,0)}(x_N+\sqrt{2t}z)e^{-z^2/2}dx_N dz dx' + \\
&\quad+\frac{1}{\sqrt{2t}}\int_K\int_{z_0}^0\int_0^\delta|\varphi(x)-\varphi(x',0)|\chi_{(-\varepsilon,0)}(x_N+\sqrt{2t}z)e^{-z^2/2}dx_N dz dx' \\
&\leq \omega(\delta)\int_K\int_{-\infty}^{z_0}\frac{f_t(z)}{\sqrt{2}}e^{-z^2/2}dz dx' + \int_K\int_{z_0}^0\omega(-\sqrt{2t}z_0)\frac{f_t(z)}{\sqrt{2}}e^{-z^2/2}dz dx'
\end{aligned}$$

where ω is defined in (2.6) and f_t is as in Lemma 2.2. By Lemma 2.2 we know that $f_t(z)\to-\sqrt{2}z$ and $f_t\leq f_s$ for $0<t<s$. Then we infer, also by (2.8),

$$\lim_{t\to 0^+}\omega(\delta)\int_K\int_{-\infty}^{z_0}\frac{f_t(z)}{\sqrt{2}}e^{-z^2/2}dz dx' = -\omega(\delta)|K|\int_{-\infty}^{z_0} z e^{-z^2/2}dz dx' < \omega(\delta)|K|\sigma\,.$$

Since $|\frac{f_t(z)}{\sqrt{2}}e^{-z^2/2}|\leq|ze^{-z^2/2}|$ and $\omega(-\sqrt{2t}z_0)$ converges uniformly to 0 as $t\to 0+$, we finally obtain (2.7).
Step 3 - Now we consider a uniformly $C^{1,1}$-regular set A, fix a function $\varphi\in C_c(\mathbf{R}^N)$ and

use the notation in Proposition 2.3. Assume at first that the support of φ is contained in a fixed U_i (this hypothesis can easily be removed by a partition of unity argument). Since for every $i \in \mathbf{N}$, $x \in U_i$ the function $p_N(x,y,t)/\sqrt{t}$ goes to 0 as $t \to 0$ for every $y \in A_\varepsilon \setminus U_i$, by dominated convergence we have

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{A^\delta} dx \int_{A_\varepsilon} p_N(x,y,t)\varphi(x)\, dy = \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{A^\delta \cap U_i} dx \int_{A_\varepsilon \cap U_i} p_N(x,y,t)\varphi(x)\, dy.$$

Since the index i is fixed, we may drop this index everywhere, and write the kernel $p_N(x,y,t)$, $x \in A^\delta \cap U$ and $y \in A_\varepsilon \cap U$, using the new variables in $D \times [0,\delta]$ and $D \times [-\varepsilon, 0]$, i.e., we may write

$$\begin{array}{ll} y = \Psi(z,r) & z \in D\,, r \in [0,\delta]\,, \\ x = \Psi(\xi,\varrho) & \xi \in D\,, \varrho \in [-\varepsilon,0]\,. \end{array}$$

so that

$$\begin{aligned} &\psi(z) = \psi(\xi) + D\psi(\xi)\cdot(z-\xi) + o(|z-\xi|)\,, \\ &\nu(\psi(z)) = \nu(\psi(\xi)) + D\nu(\psi(\xi))\cdot\big[D\psi(\xi)\cdot(z-\xi)\big] + o(|z-\xi|)\,. \end{aligned} \tag{2.9}$$

Then, using these equalities, we obtain

$$\begin{aligned} &\int_{A^\delta}\int_{A_\varepsilon} p_N(x,y,t)\varphi(x)\, dy\, dx = \\ &\int_{D\times(0,\delta)}\int_{D\times(-\varepsilon,0)} \tilde{p}_N\big((z,r),(\xi,\varrho),t\big)\varphi(\psi(z)+r\nu(\psi(z)))dz\, dr\, d\xi\, d\varrho \end{aligned}$$

where we have defined

$$\tilde{p}_N\big((z,r),(\xi,\varrho),t\big) = p_N\big(\Psi(z,r),\Psi(\xi,\varrho),t)|D\Psi(z,r)||D\Psi(\xi,\varrho)|\,.$$

By (2.4), we have

$$\begin{array}{ll} |D\Psi(\xi,\varrho)| = 1 + O(\eta) & \text{in } D\times(0,\delta)\,, \\ |D\Psi(z,r)| = 1 + O(\eta) & \text{in } D\times(-\varepsilon,0) \end{array} \tag{2.10}$$

and consequently

$$|D\Psi^{-1}(x)| = 1 + O(\eta) \qquad \text{and} \qquad |D\psi^{-1}(x)| = 1 + O(\eta) \qquad \text{in } U\,. \tag{2.11}$$

Then, writing

$$\Psi(z,r) = \Psi(\xi,\varrho) + [D\Psi - L]((z,r)-(\xi,\varrho)) + L((z,r)-(\xi,\varrho)) + o(|(z,r)-(\xi,\varrho)|)$$

for the orthogonal map L given by Proposition 2.3, we can compute, using (2.4),

$$\begin{aligned} \big|\Psi(z,r)-\Psi(\xi,\varrho)\big|^2 &= \big|[D\Psi - L]((z,r)-(\xi,\varrho)) + L((z,r)-(\xi,\varrho)) + o(|(z,r)-(\xi,\varrho)|)\big|^2 \\ &= |(z,r)-(\xi,\varrho)|^2\big(1+O(\eta)\big) \end{aligned}$$

and consequently $p_N\big(\Psi(z,r),\Psi(\xi,\varrho),t) = \frac{1}{(4\pi t)^{N/2}} \exp\Big\{-\dfrac{|(z-\xi,r-\varrho)|^2}{4t}\Big\}\big(1+O(\eta)\big)$ by which finally

$$\tilde{p}_N\big(\Psi(z,r),\Psi(\xi,\varrho),t) \;=\; \frac{1}{(4\pi t)^{N/2}} \exp\Big\{-\frac{|(z-\xi,r-\varrho)|^2}{4t}\Big\}\big(1+O(\eta)\big)\,.$$

Then from the above equality and (2.10) we derive

$$\sqrt{\frac{\pi}{t}}\int_{A^\delta}\int_{A_\varepsilon} p_N(x,y,t)\varphi(x)\,dy\,dx =$$
$$= \big(1+O(\eta)\big)\sqrt{\frac{\pi}{t}}\int_{D\times(0,\delta)}\int_{D\times(-\varepsilon,0)} p_N\big((z,r),(\xi,\varrho),t\big)\varphi(\Psi(\xi,\varrho))d\xi\,d\varrho\,dz\,dr\,.$$

Taking the limit as $t\to 0$, by *Step* 2 and (2.11) we obtain

$$\begin{aligned}\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{A^\delta}dx\int_{A_\varepsilon} p_N(x,y,t)\varphi(x)\,dy &= \big(1+O(\eta)\big)\int_D \varphi(\Psi(\xi,0))d\xi\\ &= \big(1+O(\eta)\big)\int_D \varphi(\psi(\xi))d\xi\\ &= \big(1+O(\eta)\big)\int_{\partial A\cap U}\varphi(x)|D\psi^{-1}(x)|d\mathcal{H}^{N-1}(x)\\ &= \big(1+O(\eta)\big)\int_{\partial A\cap U}\varphi(x)d\mathcal{H}^{N-1}(x)\end{aligned}$$

which concludes the proof. □

Of course, for sets with smooth boundary, equality (1.4) can be deduced from Theorem 2.1.

Theorem 2.4 *Let $A\subset\mathbf{R}^N$ be a compact subset with $C^{1,1}$-boundary ∂A. Then*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_A,\chi_{A^c}\rangle = P(A)$$

holds.

Proof. From the elementary relations

$$\begin{aligned}\langle T(t)\chi_{A_\varepsilon},\chi_{A^\delta}\rangle+\langle T(t)\chi_{A_\varepsilon},\chi_{A^c\setminus A^\delta}\rangle &= \langle T(t)\chi_{A_\varepsilon},\chi_{A^c}\rangle,\\ \langle T(t)\chi_{A_\varepsilon},\chi_{A^c}\rangle+\langle T(t)\chi_{A\setminus A_\varepsilon},\chi_{A^c}\rangle &= \langle T(t)\chi_A,\chi_{A^c}\rangle\end{aligned}$$

it follows that

$$\begin{aligned}\langle T(t)\chi_{A_\varepsilon},\chi_{A^\delta}\rangle &\le \langle T(t)\chi_{A_\varepsilon},\chi_{A^c}\rangle,\\ \langle T(t)\chi_{A_\varepsilon},\chi_{A^c}\rangle &\le \langle T(t)\chi_A,\chi_{A^c}\rangle\end{aligned}$$

and therefore, together with (1.5),

$$\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_{A_\varepsilon},\chi_{A^\delta}\rangle \le \sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_A,\chi_{A^c}\rangle \le P(A).$$

Then by this last equality and Theorem 2.1 it follows that

$$\begin{aligned}P(A) &= \lim_{t\to 0}\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_{A_\varepsilon},\chi_{A^\delta}\rangle \le \liminf_{t\to 0}\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_A,\chi_{A^c}\rangle\\ &\le \limsup_{t\to 0}\sqrt{\frac{\pi}{t}}\,\langle T(t)\chi_A,\chi_{A^c}\rangle \le P(A).\end{aligned}$$

□

Remark 2.5 Since for compact subsets $A \subset \mathbf{R}^N$ with smooth boundary ∂A the perimeter $P(A)$ and the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}(\partial A)$ coincide (cf [2, Proposition 3.62]), we also have

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}}\, \langle T(t)\chi_A, \chi_{A^c}\rangle = \mathcal{H}^{N-1}(\partial A).$$

3 Diffusion for Caccioppoli sets

In this section we show that a measurable set $E \subset \mathbf{R}^N$ has finite perimeter if and only if $\sqrt{\pi/t}\langle T(t)\chi_E, \chi_{E^c}\rangle$ is bounded. To begin with, let us study the short-time behaviour of $T(t)$ with respect to two arbitrary sets of finite perimeter. Recall that for E with finite perimeter ν_E denotes the generalised (or measure-theoretic) inner unit normal to the reduced boundary.

Theorem 3.1 *Let $E, F \subset \mathbf{R}^N$ be sets of finite perimeter. Then the following equality holds:*

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}}\langle \chi_E - T(t)\chi_E, \chi_F\rangle = \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x)\cdot \nu_F(x) d\mathcal{H}^{N-1}(x). \tag{3.1}$$

Proof. Since

$$T(t)\chi_E - \chi_E = \int_0^t \Delta T(s)\chi_E ds,$$

we have

$$\langle T(t)\chi_E - \chi_E, \chi_F\rangle = \int_0^t \langle \Delta T(s)\chi_E, \chi_F\rangle ds.$$

Moreover, by (1.10), integrating by parts we obtain

$$\begin{aligned}\langle \Delta T(s)\chi_E, \chi_F\rangle &= \int_{\mathbf{R}^N} \Delta T(s)\chi_E(x)\chi_F(x)dx = -\int_{\mathbf{R}^N} \nabla T(s)\chi_E(x)\cdot dD\chi_F(x)\\ &= -\int_{\mathcal{F}F} \nabla T(s)\chi_E(x)\cdot \nu_F(x)d\mathcal{H}^{N-1}(x).\end{aligned}$$

Notice that, if we define for every $x \in \mathcal{F}E$ and $s > 0$ the measures

$$d\mu_{s,x} = \mathcal{L}^N \llcorner \left(\frac{E-x}{\sqrt{s}}\right),$$

we have

$$\begin{aligned}\nabla T(s)\chi_E(x) &= \int_E \nabla_x \left(\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{N/2}}\right) dy = -\int_E \frac{(x-y)}{2s}\frac{e^{-\frac{|x-y|^2}{4s}}}{(4\pi s)^{N/2}}dy\\ &= \frac{1}{2\sqrt{s}}\int_{\frac{E-x}{\sqrt{s}}} \frac{e^{-|z|^2/4}}{(4\pi)^{N/2}} z dz.\\ &= \frac{1}{2\sqrt{s}}\int_{\mathbf{R}^N} \frac{e^{-|z|^2/4}}{(4\pi)^{N/2}} z d\mu_{s,x}(z).\end{aligned}$$

Moreover, setting, for every $x \in \mathcal{F}E$,

$$H_{\nu_E(x)} = \left\{z \in \mathbf{R}^N : z\cdot \nu_E(x) \geq 0\right\},$$

the existence of the approximate tangent plane for $x \in \mathcal{F}E$, see (1.8), implies that the measures $\mu_{s,x}$ are locally weakly* convergent as $s \to 0$ to the measure

$$d\mu_x = \mathcal{L}^N \llcorner H_{\nu_E(x)},$$

and also

$$\lim_{s\to 0} \int_{\mathbf{R}^N} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{N/2}} d\mu_{s,x}(z) = \int_{H_{\nu_E(x)}} z \cdot \nu_F(x) \frac{e^{-|z|^2/4}}{(4\pi)^{N/2}} dz.$$

Summing up, we can write

$$\sqrt{\frac{\pi}{t}} \langle \chi_E - T(t)\chi_E, \chi_F \rangle = \int_{\mathcal{F}F} g(x,t) d\mathcal{H}^{N-1}(x),$$

where $g : \mathcal{F}F \times (0, +\infty)$ is given by

$$g(x,t) = \sqrt{\frac{\pi}{t}} \int_0^t \frac{1}{2\sqrt{s}} \int_{\mathbf{R}^N} \frac{e^{-|z|^2/4}}{(4\pi)^{N/2}} z \cdot \nu_F(x) d\mu_{s,x}(z) ds,$$

and by (1.9) we have

$$\lim_{t\to 0^+} g(x,t) = \begin{cases} \dfrac{\sqrt{\pi}}{(4\pi)^{N/2}} \displaystyle\int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz & \text{for } x \in \mathcal{F}E \cap \mathcal{F}F \\ 0 & \text{for } x \in \left(E^0 \cup E^1\right) \cap \mathcal{F}F, \end{cases}$$

where E^0, E^1 are defined according to (1.7). Since

$$|g(x,t)| \le \frac{\sqrt{\pi}}{(4\pi)^{N/2}} \int_{\mathbf{R}^N} |z| e^{-|z|^2/4} dz = c_N,$$

we can apply Lebesgue dominated convergence theorem and obtain

$$\begin{aligned} &\exists \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \langle \chi_E - T(t)\chi_E, \chi_F \rangle = \frac{\sqrt{\pi}}{(4\pi)^{N/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} z \cdot \nu_F(x) e^{-|z|^2/4} dz d\mathcal{H}^{N-1}(x) \\ &= \frac{\sqrt{\pi}}{(4\pi)^{N/2}} \int_{\mathcal{F}E\cap\mathcal{F}F} \int_{H_{\nu_E(x)}} (\nu_E(x) \cdot \nu_F(x))(z \cdot \nu_E(x)) e^{-|z|^2/4} dz d\mathcal{H}^{N-1}(x) \\ &= \int_{\mathcal{F}E\cap\mathcal{F}F} \nu_E(x) \cdot \nu_F(x) d\mathcal{H}^{N-1}(x), \end{aligned}$$

because $\nu_F(x) = (\nu_E(x) \cdot \nu_F(x))\nu_E(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$ and

$$\frac{\sqrt{\pi}}{(4\pi)^{N/2}} \int_{H_{\nu_E(x)}} z \cdot \nu_E(x) e^{-|z|^2/4} dz = 2(4\pi)^{(N-1)/2} \qquad \forall\ x \in \mathcal{F}E.$$

□

Remark 3.2 Notice that if $|F \setminus E| = 0$ in the preceding statement, then $\nu_E(x) = \nu_F(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$, hence the equality

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \langle \chi_E - T(t)\chi_E, \chi_F \rangle = \mathcal{H}^{N-1}(\mathcal{F}E \cap \mathcal{F}F) \tag{3.2}$$

holds.

As a special case, we may take $E = F$ in the above theorem, and obtain the following result, which generalises formula (1.4).

Theorem 3.3 *Let $E \subset \mathbf{R}^N$ be a set of finite perimeter; then the following equality holds*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\langle T(t)\chi_E, \chi_{E^c}\rangle = P(E). \tag{3.3}$$

Proof. Since $\|T(t)\chi_E\|_{L^1(\mathbf{R}^N)} = |E|$ for all $t \geq 0$, inserting $F = E$ in (3.2) we obtain

$$\langle \chi_E - T(t)\chi_E, \chi_E\rangle = \langle \chi_E - T(t)\chi_E, 1 - \chi_{E^c}\rangle = \langle T(t)\chi_E, \chi_{E^c}\rangle$$

and the assertion follows. □

Let us now prove the reverse implication of Theorem 3.3.

Theorem 3.4 *Let $E \subset \mathbf{R}^N$ be a set such that either E or E^c has finite measure, and*

$$\liminf_{t\to 0^+}\frac{\langle T(t)\chi_E, \chi_{E^c}\rangle}{\sqrt{t}} < +\infty.$$

Then E has finite perimeter.

Proof. Assume that $|E| < +\infty$. We can write

$$\begin{aligned}
\frac{1}{\sqrt{t}}\langle T(t)\chi_E, \chi_{E^c}\rangle &= \frac{1}{(4\pi)^{N/2}\sqrt{t}}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\chi_{E^c}(x)\chi_E(x+\sqrt{t}y)e^{-|y|^2/4}dydx \\
&= \frac{1}{(4\pi)^{N/2}\sqrt{t}}\int_{\mathbf{R}^N}e^{-|y|^2/4}\int_{\mathbf{R}^N}\Big(\chi_{E-\sqrt{t}y}(x) - \chi_E(x)\chi_{E-\sqrt{t}y}(x)\Big)\,dxdy \\
&= \frac{1}{(4\pi)^{N/2}\sqrt{t}}\int_{\mathbf{R}^N}e^{-|y|^2/4}\Big(|E| - |E\cap(E-\sqrt{t}y)|\Big)\,dy \\
&= \frac{1}{2(4\pi)^{N/2}\sqrt{t}}\int_{\mathbf{R}^N}e^{-|y|^2/4}|E\triangle(E-\sqrt{t}y)|dy \\
&= \frac{1}{2(4\pi)^{N/2}}\int_{\mathbf{R}^N}|y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy,
\end{aligned}$$

where $E\triangle F = (E\cup F)\setminus(E\cap F)$. Then, if we define

$$|D_\nu\chi_E| = \liminf_{t\to 0^+}\frac{|E\triangle(E-t\nu)|}{t},$$

from the previous estimate we get that

$$\int_{\mathbf{R}^N}|y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy \leq \liminf_{t\to 0^+}\int_{\mathbf{R}^N}|y|e^{-|y|^2/4}\frac{|E\triangle(E-\sqrt{t}y)|}{\sqrt{t}|y|}dy < +\infty.$$

Noticing that

$$\int_{\mathbf{R}^N}|y|e^{-|y|^2/4}|D_{y/|y|}\chi_E|dy = C_N\int_{\mathbf{S}^{N-1}}|D_\nu\chi_E|d\nu,$$

we have proved that

$$\int_{\mathbf{S}^{N-1}}|D_\nu\chi_E|d\nu < +\infty.$$

This implies that the function $\nu \mapsto |D_\nu \chi_E|$ is finite for a.e. $\nu \in \mathbf{S}^{N-1}$; in particular, there exist $M > 0$ and an orthonormal system of coordinates $\nu_1, \ldots, \nu_N$ of Lebesgue points of $|D_\nu \chi_E|$ such that

$$|D_{\nu_i} \chi_E| \le M, \quad \forall i = 1, \ldots, N.$$

Without loss of generality, we can assume that $\nu_i = e_i$; now, if $\phi \in C_c^1(\mathbf{R}^N)$, the function

$$\phi_t(x) = \frac{\phi(x + te_i) - \phi(x)}{t}$$

is uniformly convergent to $\partial_i \phi(x)$. This implies that

$$\int_{\mathbf{R}^N} \chi_E(x) \partial_i \phi(x) dx = \lim_{t \to 0^+} \int_{\mathbf{R}^N} \chi_E(x) \phi_t(x) dx.$$

But

$$\int_{\mathbf{R}^N} \chi_E(x) \phi_t(x) dx = \int_{\mathbf{R}^N} \frac{\chi_E(x - te_i) - \chi_E(x)}{t} \phi(x) dx,$$

hence

$$\left| \int_{\mathbf{R}^N} \chi_E(x) \phi_t(x) dx \right| \le \|\phi\|_\infty \frac{|E \triangle (E + te_i)|}{t}.$$

From this it follows that

$$\begin{aligned} \left| \int_{\mathbf{R}^N} \chi_E(x) \partial_i \phi(x) dx \right| &\le \|\phi\|_\infty \liminf_{t \to 0^+} \frac{|E \triangle (E + te_i)|}{t} \\ &= \|\phi\|_\infty |D_i \chi_E| \le M \|\phi\|_\infty. \end{aligned}$$

In the end, we have proved that

$$\int_{\mathbf{R}^N} \chi_E(x) \operatorname{div} \phi(x) dx \le NM \|\phi\|_\infty, \quad \forall \phi \in C_c^1(\mathbf{R}^N),$$

and then $\chi_E \in BV(\mathbf{R}^N)$. □

Remark 3.5 Given E with finite perimeter and setting $f(x,t) = T(t)\chi_E(x) - \chi_E(x)$, we have $\int_{\mathbf{R}^N} f dx = 0$ for all $t \ge 0$, hence $\int_{\mathbf{R}^N} f^+ dx = \int_{\mathbf{R}^N} f^- dx$. Since $f^+ = (T(t)\chi_E - \chi_E)\chi_{E^c}$ we obtain

$$\|T(t)\chi_E - \chi_E\|_{L^1(\mathbf{R}^N)} = 2\langle T(t)\chi_E, \chi_{E^c} \rangle \tag{3.4}$$

and therefore from equality (3.3) we deduce

$$P(E) = \lim_{t \to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \|T(t)\chi_E - \chi_E\|_{L^1(\mathbf{R}^N)}. \tag{3.5}$$

Moreover, since by [12, Proposition 8] $\frac{1}{2}\sqrt{\pi/t}\|T(t)\chi_E - \chi_E\|_{L^1(\mathbf{R}^N)} \le P(E)$ for all $t > 0$, the limit in (3.5) is in fact a supremum.

4 Diffusion for BV functions

In this section we apply the results of the preceding one to general BV functions and prove a characterisation of BV functions which generalises Theorems 3.3 and 3.4. Let us recall the *coarea formula*, which is repeatedly used in the sequel (see e.g. [2, Theorem 3.40] for details). For every $u \in L^1(\mathbf{R}^N)$ the following equality holds:

$$|Du|(\mathbf{R}^N) = \int_{\mathbf{R}} P(\{u > \tau\})d\tau, \tag{4.1}$$

and both sides are finite if and only if $u \in BV(\mathbf{R}^N)$.

Theorem 4.1 *Let $u \in BV(\mathbf{R}^N)$; then the following equality holds:*

$$|Du|(\mathbf{R}^N) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x) - u(y)| p_N(x,y,t)dxdy.$$

Proof. From (3.4), (3.5) and the coarea formula (4.1), setting $E_\tau = \{u > \tau\}$ and noting that, for almost every $x, y \in \mathbf{R}^N$

$$\int_{\mathbf{R}} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)|d\tau = |u(x) - u(y)|, \tag{4.2}$$

we get by Remark 3.5

$$\begin{aligned}
|Du|(\mathbf{R}^N) &= \int_{\mathbf{R}} P(E_\tau)d\tau \geq \int_{\mathbf{R}} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}^N} |T(t)\chi_{E_\tau} - \chi_{E_\tau}|dxd\tau \\
&\geq \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |\chi_{E\tau}(y) - \chi_{E_\tau}(x)| p_N(x,y,t)dxdyd\tau \\
&= \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x) - u(y)| p_N(x,y,t)dxdy.
\end{aligned}$$

For the reverse inequality, from the Fatou Lemma and from the fact that

$$\chi_{E_\tau}(x)\chi_{E_\tau}(y) \neq 0 \quad \text{iff} \quad \tau < \min[u(x), u(y)],$$

we have that

$$\begin{aligned}
|Du|(\mathbf{R}^N) &= \int_{\mathbf{R}} P(E_\tau)d\tau = \int_{\mathbf{R}} \lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{t}} \int_{\mathbf{R}^N} T(t)\chi_{E_\tau}(x)\chi_{E_\tau^c}(x)dxd\tau \\
&\leq \liminf_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{t}} \int_{\mathbf{R}} \int_{\mathbf{R}^N\times\mathbf{R}^N} (\chi_{E\tau}(y) - \chi_{E_\tau}(y)\chi_{E_\tau}(x)) p_N(x,y,t)dxdyd\tau \\
&= \liminf_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} (u(y) - \min\{u(x), u(y)\}) p_N(x,y,t)dxdy.
\end{aligned}$$

But then the assertion follows, since $\min\{u(x), u(y)\} = \frac{1}{2}[u(x) + u(y) - |u(x) - u(y)|]$. □

The following result is the reverse implication of Theorem 4.1 and is indeed a corollary of Theorem 3.4.

Proposition 4.2 *Let $u \in L^1(\mathbf{R}^N)$ be such that*

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x)-u(y)|\, p_N(x,y,t)dxdy < +\infty\,; \tag{4.3}$$

then $u \in BV(\mathbf{R}^N)$.

Proof. Using (4.2) and setting again $E_\tau = \{u > \tau\}$, we can write

$$\int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x)-u(y)|\, p_N(x,y,t)dxdy = \int_{\mathbf{R}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|\, p_N(x,y,t)dxdyd\tau\,.$$

Since $\langle T(t)\chi_E, \chi_{E^c}\rangle \geq 0$ (see Remark 3.5) by the above equality and 4.3 we get

$$\begin{aligned} 0 &\leq \int_{\mathbf{R}} \liminf_{t\to 0} \frac{1}{\sqrt{t}} \langle T(t)\chi_{E_\tau}, \chi_{E^c_\tau}\rangle d\tau \leq \liminf_{t\to 0} \int_{\mathbf{R}} \frac{1}{\sqrt{t}} \langle T(t)\chi_{E_\tau}, \chi_{E^c_\tau}\rangle d\tau \\ &\leq \liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} \int_{\mathbf{R}} |\chi_{E_\tau}(x)-\chi_{E_\tau}(y)|\, p_N(x,y,t)dxdyd\tau < +\infty\,. \end{aligned}$$

In particular, by Theorem 3.4, for a.e. $\tau \in \mathbf{R}$ the set E_τ has finite perimeter and the limit $\lim_{t\to 0} \frac{1}{\sqrt{t}} \langle T(t)\chi_{E_\tau}, \chi_{E^c_\tau}\rangle$ exists. Then by the coarea formula and Theorem 3.3,

$$\begin{aligned} |Du|(\mathbf{R}^N) &= \int_{\mathbf{R}} P(E_\tau)d\tau = \int_{\mathbf{R}} \lim_{t\to 0} \sqrt{\frac{\pi}{t}} \langle T(t)\chi_{E_\tau}, \chi_{E^c_\tau}\rangle d\tau \\ &\leq \liminf_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\mathbf{R}^N\times\mathbf{R}^N} |u(x)-u(y)|\, p_N(x,y,t)dxdy < +\infty \end{aligned}$$

that is, $u \in BV(\mathbf{R}^N)$. □

Let us add a further result concerning BV functions, which in some sense complements Theorem 3.1. In order to state it, let us recall some properties of $u \in BV(\mathbf{R}^N)$. Setting

$$\begin{aligned} u^\vee(x) &= \inf\left\{ t \in [-\infty,+\infty] : \lim_{\varrho\to 0} \varrho^{-N} |\{u > t\} \cap B_\varrho(x)| = 0 \right\}, \\ u^\wedge(x) &= \sup\left\{ t \in [-\infty,+\infty] : \lim_{\varrho\to 0} \varrho^{-N} |\{u < t\} \cap B_\varrho(x)| = 0 \right\}. \end{aligned}$$

and $S_u = \{x \in \mathbf{R}^N : \ u^\wedge(x) < u^\vee(x)\}$, a Borel unit vector field ν_u is defined $\mathcal{H}^{N-1}$-a.e. in S_u in such a way that the measure Du, restricted to S_u, can be represented as $Du \llcorner S_u = (u^\vee - u^\wedge)\nu_u \mathcal{H}^{N-1} \llcorner S_u$, see e.g. [2, Theorem 3.78].

Theorem 4.3 *Let $u, v \in BV(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, or $u, v \in BV(\mathbf{R}^N)$, $v \in L^\infty(\mathbf{R}^N)$. Then*

$$\lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{t}} \langle u - T(t)u, v\rangle = \int_{S_u \cap S_v} (u^\vee - u^\wedge)(v^\vee - v^\wedge)\nu_u \cdot \nu_v d\mathcal{H}^{N-1},$$

where $E_\tau = \{u > \tau\}$ and $F_\sigma = \{v > \sigma\}$. The previous identity makes also sense for any $u, v \in BV_{\rm loc}(\mathbf{R}^N)$ without the assumption to be L^2, possibly having on both sides infinite value.

Proof. Notice that

$$x \in \mathcal{F}E_\tau \Rightarrow \tau \in [u^\wedge(x), u^\vee(x)].$$

Then if x is a Lebesgue point either for u or for v, we have either

$$\int_{\mathbf{R}} \chi_{\mathcal{F}E_\tau}(x) d\tau = 0 \qquad \text{or} \qquad \int_{\mathbf{R}} \chi_{\mathcal{F}F_\sigma}(x) d\sigma = 0.$$

Moreover, since for $\mathcal{H}^{N-1}$-a.e. $x \in S_v$ and $\sigma \in (v^\wedge(x), v^\vee(x))$ there holds $x \in \mathcal{F}F_\sigma$, we get

$$\begin{aligned}
\int_{\mathbf{R}\times\mathbf{R}} \int_{\mathcal{F}E_\tau \cap \mathcal{F}F_\sigma} \nu_{E_\tau} \cdot \nu_{F_\sigma} d\mathcal{H}^{N-1} d\tau d\sigma &= \int_{\mathbf{R}} \int_{\mathcal{F}E_\tau} \int_{\mathbf{R}} \nu_{E_\tau} \cdot \nu_{F_\sigma} \chi_{\mathcal{F}F_\sigma} d\sigma d\mathcal{H}^{N-1} d\tau \\
&= \int_{\mathbf{R}} \int_{\mathcal{F}E_\tau \cap S_v} (v^\vee - v^\wedge) \nu_{E_\tau} \cdot \nu_v d\mathcal{H}^{N-1} d\tau \\
&= \int_{S_v} \int_{\mathbf{R}} (v^\vee - v^\wedge) \nu_{E_\tau} \cdot \nu_v \chi_{\mathcal{F}E_\tau} d\tau d\mathcal{H}^{N-1} \\
&= \int_{S_u \cap S_v} (u^\vee - u^\wedge)(v^\vee - v^\wedge) \nu_u \cdot \nu_v d\mathcal{H}^{N-1}.
\end{aligned}$$

□

Remark 4.4 Immediate consequences of (1.2) and Theorem 4.3 is the following equality:

$$\lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{2t}} \left(\langle u, v\rangle - \langle T(t)u, T(t)v\rangle\right) = \int_{S_u \cap S_v} (u^\vee - u^\wedge)(v^\vee - v^\wedge) \nu_u \cdot \nu_v d\mathcal{H}^{N-1}.$$

In fact,

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{2t}} \left(\langle u, v\rangle - \langle T(t)u, T(t)v\rangle\right) &= \lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{2t}} \left(\langle u, v\rangle - \langle T(2t)u, v\rangle\right) \\
&= \lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{2t}} \left(\langle u - T(2t)u, v\rangle\right).
\end{aligned}$$

Remark 4.5 The interpolation space $(L^1(\mathbf{R}^N), D(\Delta))_{1/2,\infty}$ defined in (1.6) coincides with the Besov space $B^{1/2}_{1\infty}(\mathbf{R}^N)$ (see [13, Theorem 4*]). From Theorem 4.1 it follows that the BV-norm is stronger than the $B^{1/2}_{1\infty}(\mathbf{R}^N)$ norm, because clearly

$$\|T(t)u - u\|_{L^1(\mathbf{R}^N)} \le \int_{\mathbf{R}^N \times \mathbf{R}^N} |u(x) - u(y)| p_N(x, y, t) dx dy$$

for every $t > 0$. On the other hand, Remark 3.5 shows that in the above estimate the equality holds if $u = \chi_E$ and $P(E)$ is finite. Let us point out an example showing that $(L^1(\mathbf{R}^N), D(\Delta))_{1/2,\infty}$ is larger than $BV(\mathbf{R}^N)$. Taking $N = 1$ and using the following norm, which is equivalent to the natural norm in the above interpolation space (see [14, Theorem 1.13.6.1])

$$\|u\| = \sup_{0<t<1} \frac{\|(T_A(t) - I)^2 u\|_{L^1(\mathbf{R})}}{t} < \infty,$$

where in our case $Au = u'$ and $T_A(t)$ is the translation semigroup $T_A(t)u(x) = u(x + t)$, it is easily seen that the function $u(x) = \log|x| \cdot \chi_{(-1,1)}(x)$ belongs to $L^1(\mathbf{R})$ and to $(L^1(\mathbf{R}), D(\Delta))_{1/2,\infty}$, but u does not belong to $BV(\mathbf{R})$.

References

[1] L. Ambrosio: Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* **158** (2004), 227-260.

[2] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity problems,* Oxford U. P., 2000.

[3] H. Brézis: How to recognize constant functions. Connections with Sobolev spaces, *Russian Math. Surveys* **57** (2002), 693-708.

[4] J. Dávila: On an open question about functions of bounded variation, *Calc. Var.* **15** (2002), 519-527.

[5] E. De Giorgi: Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni, *Ann. Mat. Pura Appl.* **(4) 36** (1954), 191-213.

[6] E. De Giorgi: Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni, *Ric. di Mat.* **4** (1955), 95–113.

[7] P. Gilkey, M. van den Berg: Heat content asymptotics of a Riemannian manifold with boundary, *J. Funct. Anal.* **120** (1994), 48-71.

[8] M. Ledoux: Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space, *Bull. Sci. Math.* **118** (1994), 485-510.

[9] E.H. Lieb, M. Loss: *Analysis,* Second edition, 2001, Amer. Math. Soc.

[10] M. Miranda (Jr), D. Pallara, F. Paronetto, M. Preunkert: *Heat Semigroup and BV Functions on Riemannian Manifolds,* forthcoming.

[11] K. Pietruska-Paluba: Heat kernels on metric spaces and a characterisation of constant functions, *Manuscripta Math.* **115** (2004), 389–399.

[12] M. Preunkert: A semigroup version of the isoperimetric inequality, *Semigroup Forum* **68** (2004), 233-245.

[13] M. H. Taibleson: On the theory of Lipschitz spaces of distributions on Euclidean n-spaces I, *J. Math. Mech.* **13** (1964), 407-479.

[14] H. Triebel: *Interpolation theory, function spaces, differential operators,* North-Holland 1978.

[15] J. Wloka: *Partial Differential Equations,* Cambridge U. P., 1987.