QUASISTATIC EVOLUTION FOR CAM-CLAY PLASTICITY: EXAMPLES OF SPATIALLY HOMOGENEOUS SOLUTIONS

G. DAL MASO AND A. DESIMONE

Abstract. We study a quasistatic evolution problem for Cam-Clay plasticity under a special loading program which leads to spatially homogeneous solutions. Under some initial conditions, the solutions exhibit a softening behaviour and time discontinuities. The behavior of the solutions at the jump times is studied by a viscous approximation.

Keywords: Cam-Clay plasticity, softening behaviour, pressure-sensitive yield criteria, nonassociative plasticity, quasistatic evolution, rate independent processes, viscosity approximation.
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Contents

1. Introduction

The modified Cam-Clay model has been introduced in the engineering literature on soil mechanics as a conceptual tool to understand the irreversible deformations experienced by

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fine grained soils (clays) upon loading [10, 9, 8, 11]. One of the interesting features of this model is that, depending on the loading conditions, the stress-strain response may exhibit a hardening or a softening behaviour. Furthermore, it is an important example of nonassociative plasticity, for which a satisfactory mathematical theory is only partially developed [7].

We restrict our attention to the spatially homogeneous case in dimension n, with no volume forces. The system is driven by a time-dependent affine boundary condition $w(t,x)$, whose symmetrized spatial gradient $Ew(t,x)$ is independent of the space variable x and is denoted by $\xi(t)$. In this situation, the displacement $u(t,x)$ coincides with $w(t,x)$ and the unknowns are the elastic part $e(t)$ and the plastic part $p(t)$ appearing in the additive decomposition of the strain $Eu(t,x) = e(t) + p(t)$, as well as a scalar internal variable $z(t)$, which describes the time evolving yield surface. The stress $\sigma(t)$ is determined by the elastic part of the strain through the usual relation $\sigma(t) = \mathbb{C}e(t)$, where $\mathbb{C}$ is the tensor of elastic moduli.

One ingredient of the model is a closed convex cone $K \subset \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$, where $\mathbb{M}^{n\times n}_{sym}$ is the space of symmetric $n\times n$ matrices. It is assumed that K contains the half-line $\{0\}\times[0,+\infty)$. The stress is constrained by the inclusion $\sigma(t) \in K(z(t))$, where for every $\zeta \in [0,+\infty)$ we define $K(\zeta) := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : (\sigma,\zeta) \in K\}$. The interior of $K(\zeta)$ is the elastic domain corresponding to the value ζ of the internal variable, while its boundary $\partial K(\zeta)$ is the yield surface. In the typical applications, $\partial K(\zeta)$ is a suitable ellipsoid in the space $\mathbb{M}^{n\times n}_{sym}$.

The other ingredients of the model are the evolution laws for $p(t)$ and $z(t)$, resulting in the system

$$\begin{cases} e(t) + p(t) = \xi(t)\,, \qquad \sigma(t) = \mathbb{C}e(t) \in K(z(t))\,, \\ \dot p(t) \in N_{K(z(t))}(\sigma(t))\,, \\ \dot z(t) = \mathrm{tr}(\sigma(t))\,\mathrm{tr}(\dot p(t))\,, \end{cases} \tag{1.1}$$

where $N_{K(\zeta)}(\sigma)$ denotes the normal cone to $K(\zeta)$ at σ. The nonassociative nature of the problem is due to the fact that the equation for $\dot z$ in (1.1) does not depend on K. In view of the hypotheses on K, we have the monotonicity condition $\zeta_1 < \zeta_2 \Rightarrow K(\zeta_1) \subset K(\zeta_2)$. Therefore, if $\dot z(t) > 0$, the set $K(z(t))$ expands leading to a hardening response. On the contrary, if $\dot z(t) < 0$, the set $K(z(t)$ shrinks leading to a softening response. In the usual applications we have $\mathrm{tr}(\sigma) \le 0$ for every $\sigma \in K(\zeta)$, which reflects the compressive conditions typical of soil mechanics. Therefore, by the third line in (1.1), the hardening or softening behaviour is determined only by the sign of $\mathrm{tr}(\dot p)$. An energetic approach to a class of rate-independent plasticity problems which present only a softening behaviour has been proposed in [3].

To deal with the instabilities of the softening regime, we propose a viscosity approximation to (1.1), [4, 2]. Denoting the minimal distance projection of σ onto $K(\zeta)$ by $\pi_{K(\zeta)}(\sigma)$, for every $\varepsilon > 0$ we consider the unconstrained system

$$\begin{cases} e_\varepsilon(t) + p_\varepsilon(t) = \xi(t)\,, \qquad \sigma_\varepsilon(t) = \mathbb{C}e_\varepsilon(t)\,, \\ \dot p_\varepsilon(t) = N^\varepsilon_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))\,, \\ \dot z_\varepsilon(t) = \mathrm{tr}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))\,\mathrm{tr}(\dot p_\varepsilon(t))\,, \end{cases} \tag{1.2}$$

where $N^\varepsilon_{K(\zeta)}(\sigma) := \frac{1}{\varepsilon}(\sigma - \pi_{K(\zeta)}(\sigma))$ is the usual approximation of the normal to $K(\zeta)$. A viscosity solution $(e(t), p(t), \sigma(t), z(t))$ to (1.1) is defined as a left continuous map which, for almost every time t, is the pointwise limit of a sequence $(e_\varepsilon(t), p_\varepsilon(t), \sigma_\varepsilon(t), z_\varepsilon(t))$ of solutions of (1.2).

In this paper we study in detail the case where $\mathbb{C}e = e$ for every $e \in \mathbb{M}^{n\times n}_{sym}$, so that $\sigma(t) = e(t)$ and $\sigma_\varepsilon(t) = e_\varepsilon(t)$. Moreover, we assume that $K(\zeta)$ is the closed ball centered at $-\frac{1}{n}\zeta I$ with radius $\frac{1}{\sqrt{n}}\zeta$, namely,

$$K(\zeta) = \{\sigma \in \mathbb{M}^{n\times n}_{sym} : |\sigma + \tfrac{1}{n}\zeta I| \leq \tfrac{1}{\sqrt{n}}\zeta\} \tag{1.3}$$

where I is the identity matrix in $\mathbb{M}^{n\times n}_{sym}$. The fact that all the elements in the interior of $K(\zeta)$ are negative definite reflects the fact that the material can only sustain compressive stresses.

Given a constant $a_0 > 0$, and a matrix $e_0 \in \mathbb{M}^{n\times n}_{sym}$ with $\mathrm{tr}(e_0) = 0$ and $|e_0| = 1$, we consider the special loading path

$$\xi(t) = -a_0 \tfrac{1}{n} I + t \tfrac{1}{\sqrt{n}} e_0\,, \tag{1.4}$$

and the initial conditions $e_\varepsilon(0) = e(0) = -a_0\frac{1}{n}I$ and $z_\varepsilon(0) = z(0) = z_0$. Then $e_\varepsilon(t)$ and $e(t)$ have the form

$$e_\varepsilon(t) = -\tfrac{1}{n} x_\varepsilon(t) I + \tfrac{1}{\sqrt{n}} y_\varepsilon(t) e_0 \quad \text{and} \quad e(t) = -\tfrac{1}{n} x(t) I + \tfrac{1}{\sqrt{n}} y(t) e_0\,,$$

for suitable scalar function $x_\varepsilon(t), y_\varepsilon(t), x(t), y(t)$ satisfying $x_\varepsilon(0) = x(0) = a_0$ and $y_\varepsilon(0) = y(0) = 0$, while the constraint $\sigma(t) \in K(z(t))$ becomes

$$\sqrt{(x(t) - z(t))^2 + y(t)^2} \leq z(t)\,.$$

Since the initial condition must satisfy this constraint, we assume that $0 \leq a_0 \leq 2z_0$. Then the solution is given by

$$x_\varepsilon(t) = x(t) = a_0\,, \qquad y_\varepsilon(t) = y(t) = t\,, \qquad z_\varepsilon(t) = z(t) = z_0 \tag{1.5}$$

in the interval $[0, t_0]$, where t_0 satisfies $\sqrt{(a_0 - z_0)^2 + t_0^2} = z_0$. This corresponds to the elastic regime (see Fig. 1.1).

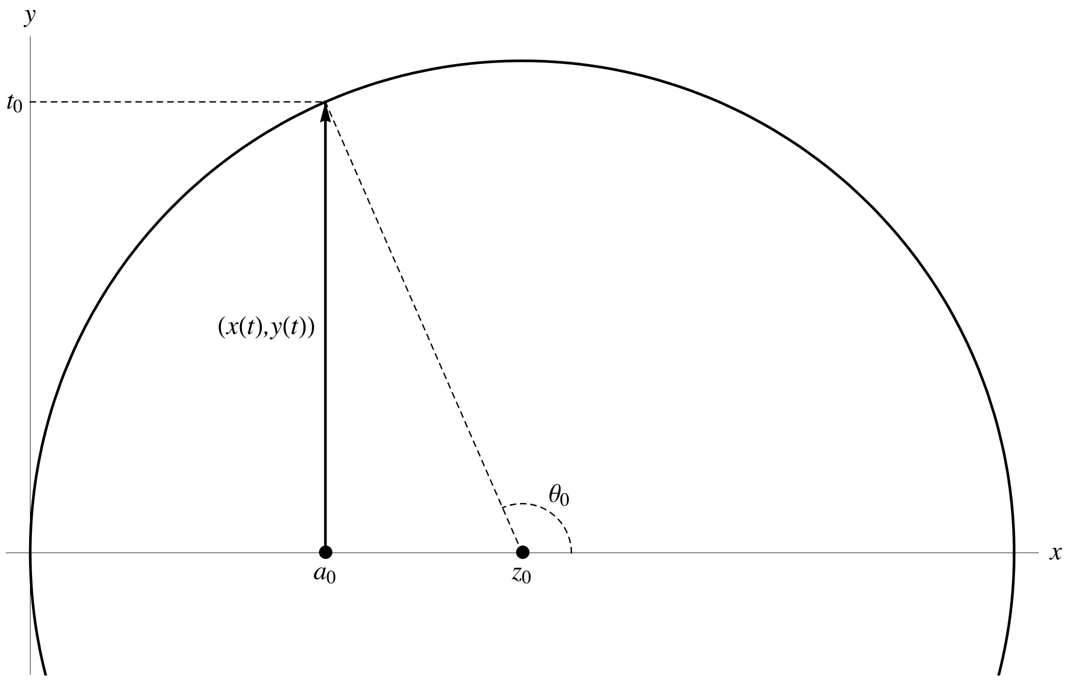

FIGURE 1.1. The elastic regime. The thick line segment is the trajectory of $(x(t), y(t))$ in the time interval $[0, t_0]$. The circle represents the yield surface in the (x, y) plane, which remains constant in this time interval.

After time t_0 the solution exhibits a plastic behaviour. To study the solution for $t > t_0$ we introduce polar coordinates

$$\begin{cases} x_\varepsilon(t) - z_\varepsilon(t) = \rho_\varepsilon(t)\cos\theta_\varepsilon(t)\,, \\ y_\varepsilon(t) = \rho_\varepsilon(t)\sin\theta_\varepsilon(t)\,, \end{cases} \qquad \begin{cases} x(t) - z(t) = \rho(t)\cos\theta(t)\,, \\ y(t) = \rho(t)\sin\theta(t)\,, \end{cases} \tag{1.6}$$

with $\rho_\varepsilon(t) > 0$ and $\rho(t) > 0$ and we consider the angle $\theta_0 \in [0,\pi]$ (see Fig. 1.1) such that

$$a_0 = z_0 + z_0\cos\theta_0 \quad \text{and} \quad t_0 = z_0\sin\theta_0\,. \tag{1.7}$$

To study the instabilities due to softening, it is convenient to introduce a *fast time* $s := \frac{1}{\varepsilon}t$. By contrast, the standard time t will be called *slow time*. In certain time intervals the problem has no singularities and the evolution can be studied using the slow time. The limit system in this case is called the system of the slow dynamics and is studied in Section 3. It is used to describe the limit behaviour in the hardening regime (Subsection 5.1) and in some cases of softening (Subsections 5.2 and 5.3).

In the softening regime, singular behavior may occur, which requires the use of the fast time s. The corresponding limit system is called the system of the fast dynamics and is studied in Section 6. It is formally obtained by rescaling time in (1.2) according to $s = \frac{t}{\varepsilon}$ and is used to determine the *transfer map* at a jump point $t_1 \geq t_0$, defined as the map

$$(\rho(t_1-), \theta(t_1-), z(t_1-)) \mapsto (\rho(t_1+), \theta(t_1+), z(t_1+))$$

where $+$ and $-$ refer to left and right limit, respectively (see Fig. 1.2). More precisely, the right limit $(\rho(t_1+), \theta(t_1+), z(t_1+))$ is given by the asymptotic value for $s \to +\infty$ of the solution $(\rho^f(s), \theta^f(s), z^f(s))$ of the system of the fast dynamics (6.30) whose limit as $s \to -\infty$ is given by $(\rho(t_1-), \theta(t_1-), z(t_1-))$.

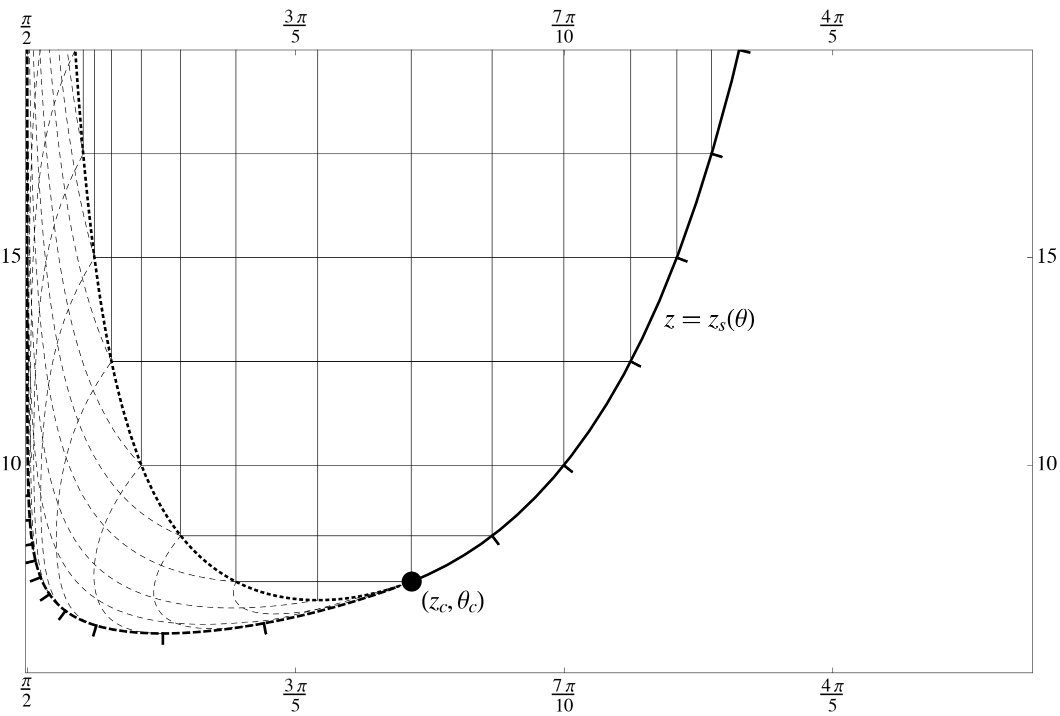

FIGURE 1.2. Transfer map in the (θ,ρ) plane. The solid rectilinear grid is transformed into the dotted curvilinear grid, the solid thick line is transformed into the dashed thick line, and the dotted line remains fixed.

The behaviour of the system in the plastic regime depends on the initial condition (θ_0, z_0) at time t_0 given in (1.7). If $0 \leq \theta_0 < \frac{\pi}{2}$, then we are in the hardening regime. The viscosity solution $(\rho(t), \theta(t), z(t))$ is continuous in time, is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on compact sets, and satisfies

$$\begin{gathered} \rho(t) = z(t) \quad \text{for} \quad t \in [t_0, +\infty)\,, \\ \dot\rho(t) > 0 \quad \text{and} \quad \dot\theta(t) > 0 \quad \text{for} \quad t \in [t_0, +\infty)\,, \\ \lim_{t\to+\infty} \rho(t) < +\infty \quad \text{and} \quad \lim_{t\to+\infty} \theta(t) = \tfrac{\pi}{2}\,. \end{gathered}$$

If $\frac{\pi}{2} < \theta_0 \leq \pi$, then we are in the softening regime and the viscosity solution $(\rho(t), \theta(t), z(t))$ may be discontinuous at a time $t_1 \geq t_0$ depending on the initial conditions (θ_0, z_0). The jump at the discontinuity time is determined by the transfer map considered above and satisfies the inequalities $0 < \rho(t_1+) = z(t_1+) < \rho(t_1-) = z(t_1-)$ and $\frac{\pi}{2} < \theta(t_1+) < \theta(t_1-)$.

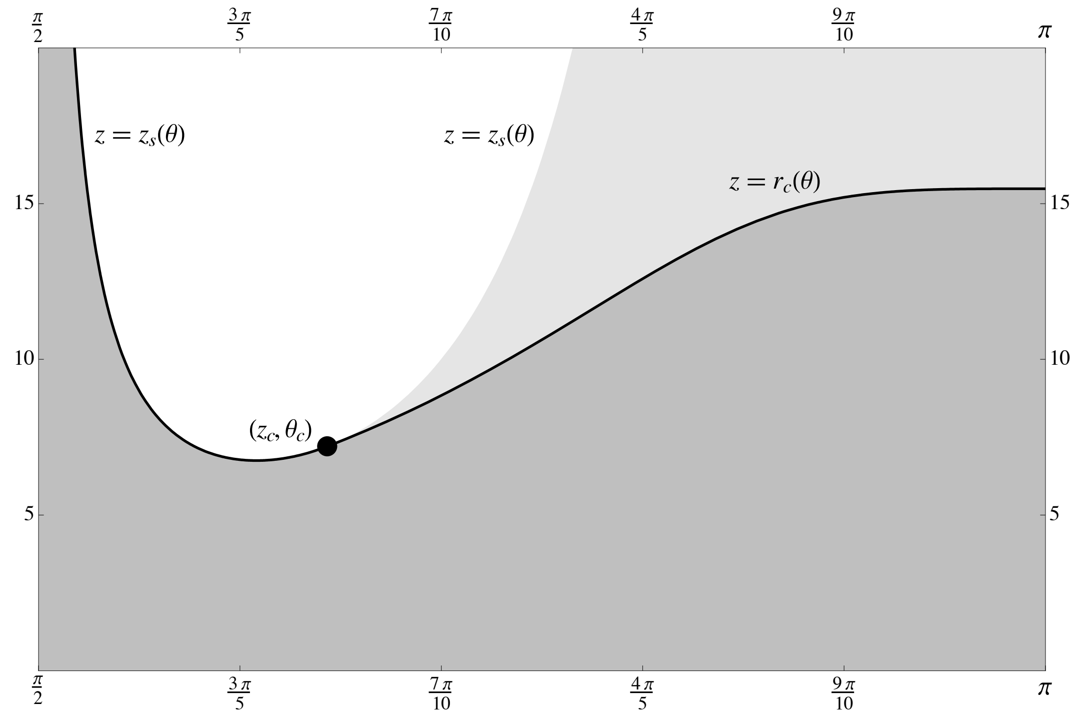

FIGURE 1.3. Phase diagram in the (θ, ρ) plane. Dark grey region (including the thick line): initial data (θ_0, z_0) of the plastic regime with continuous evolution. Light grey region: initial data with discontinuity time $t_1 > t_0$. White region: initial data with discontinuity time $t_1 = t_0$.

Three possible behaviours occur, according to the phase diagram illustrated in Fig. 1.3. A crucial role is played by the separation line $z = z_s(\theta)$, whose explicit formula is given by (3.4), by the critical line $z = r_c(\theta)$, described in (3.6), and by the critical point (z_c, θ_c) where the two lines meet, given explicitly in (3.3) and (3.5).

(a) If either $\frac{\pi}{2} < \theta_0 \leq \theta_c$ and $z_0 \leq z_s(\theta_0)$, or $\theta_c \leq \theta_0 \leq \pi$ and $z_0 \leq r_c(\theta_0)$, then the viscosity solution $(\rho(t), \theta(t), z(t))$ is continuous in time (see Fig. 1.4-1.6), is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on every compact subset of $[t_0, +\infty)$, and satisfies

$$\rho(t) = z(t) \quad \text{for} \quad t \in [t_0, +\infty)\,,$$
$$\dot\rho(t) < 0 \quad \text{and} \quad \dot\theta(t) < 0 \quad \text{for} \quad t \in [t_0, +\infty)\,,$$
$$\lim_{t\to+\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t\to+\infty} \theta(t) = \tfrac{\pi}{2}\,.$$

(b) If either $\frac{\pi}{2} < \theta_0 \le \theta_c$ and $z_0 > z_s(\theta_0)$, or $\theta_c < \theta_0 < \pi$ and $z_0 \ge z_s(\theta_0)$, then the viscosity solution $(\rho(t), \theta(t), z(t))$ is discontinuous at $t = t_0$. Moreover the solution $(\rho(t), \theta(t), z(t))$ is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on every compact subset of $(t_0, +\infty)$. It satisfies

$$\rho(t) = z(t) \quad \text{for} \quad t \in (t_0, +\infty)\,,$$
$$\dot\rho(t) < 0 \quad \text{and} \quad \dot\theta(t) < 0 \quad \text{for} \quad t \in (t_0, +\infty)\,,$$
$$\lim_{t\to+\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t\to+\infty} \theta(t) = \tfrac{\pi}{2}\,.$$

Finally, the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ are uniformly close to a rescaled version of $(\rho^f(s), \theta^f(s), z^f(s))$ in a suitable right neighbourhood of t_0.

(c) If $\theta_c < \theta_0 \le \pi$ and $r_c(\theta_0) < z_0 < z_s(\theta_0)$, then the viscosity solution $(\rho(t), \theta(t), z(t))$ is discontinuous at a time $t_1 > t_0$ (see Fig. 1.4-1.6). Moreover the solution $(\rho(t), \theta(t), z(t))$ is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on every compact subset of $[t_0, t_1) \cup (t_1, +\infty)$. It satisfies

$$\rho(t) = z(t) \quad \text{for} \quad t \in [t_0, t_1) \cup (t_1, +\infty)\,,$$
$$\dot\rho(t) < 0 \quad \text{and} \quad \dot\theta(t) < 0 \quad \text{for} \quad t \in [t_0, t_1) \cup (t_1, +\infty)\,,$$
$$\lim_{t\to+\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t\to+\infty} \theta(t) = \tfrac{\pi}{2}\,.$$

Finally, the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ are uniformly close to a rescaled version of $(\rho^f(s), \theta^f(s), z^f(s))$ in a suitable right neighbourhood of t_1.

Further details on the mechanical interpretation of the behaviour of the solutions are given in Section 8 using Cartesian coordinates $(x(t), y(t)$, see Fig. 8.1-8.3.

Extensions to general K and general loading conditions in the spatially uniform case, and extensions to non spatially uniform solutions will be considered in other forthcoming papers.

2. Formulation of the problem and general results

Let K be a closed convex cone in $\mathbb{M}^{n\times n}_{sym} \times [0, +\infty)$. For every $\zeta \in [0, +\infty)$ we define

$$K(\zeta) := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : (\sigma, \zeta) \in K\}\,.$$

Each set $K(\zeta)$ is closed and convex, and we have

$$K(\zeta) = \zeta\, K(1) \qquad \text{for every } \zeta \in [0, +\infty)\,. \tag{2.1}$$

We assume that $K(1)$ is bounded and that $0 \in K(1)$, hence

$$0 \in K(\zeta) \qquad \text{for every } \zeta \in [0, +\infty)\,, \tag{2.2}$$

and

$$|\sigma| \le M_K \zeta \qquad \text{for every } (\sigma, \zeta) \in K \tag{2.3}$$

for a suitable constant $M_K < +\infty$.

For every closed convex set $C \subset \mathbb{M}^{n\times n}_{sym}$ let $\pi_C \colon \mathbb{M}^{n\times n}_{sym} \to C$ be the minimal distance projection onto C. It follows from (2.1) that

$$\pi_{K(\zeta)}(\sigma) = \zeta \pi_{K(1)}(\tfrac{1}{\zeta}\sigma) \tag{2.4}$$

for every $\zeta > 0$ and every $\sigma \in \mathbb{M}^{n\times n}_{sym}$.

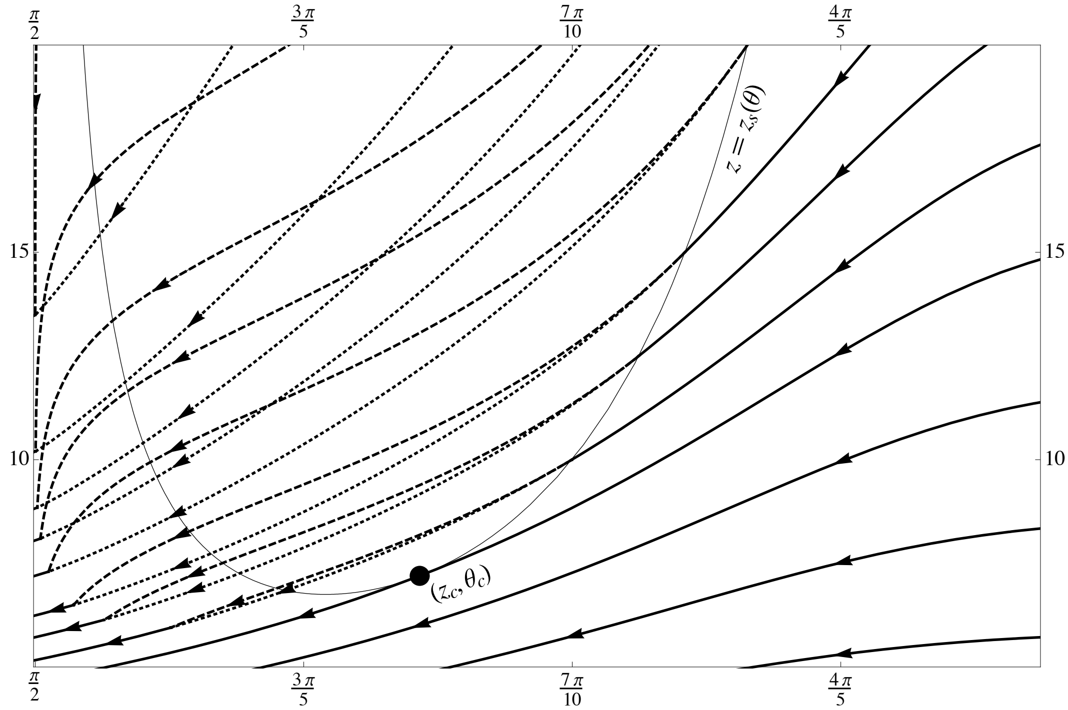

FIGURE 1.4. Trajectories in the (θ,ρ) plane for $\theta_0 = \frac{9}{10}\pi$ and 12 different values of $z_0 < z_s(\theta_0)$. Solid lines: trajectories of $(\theta(t),\rho(t)) = (\theta(t), z(t))$ (slow dynamics). Dashed lines: trajectories of $(\theta^f(s), \rho^f(s))$ (fast dynamics). Dotted lines: trajectories of $(\theta^f(s), z^f(s))$ (fast dynamics).

The following result will be used to prove the existence of a solution to the system (1.2) governing the viscous approximation of the original problem (1.1).

Lemma 2.1. *The map* $(\sigma,\zeta) \mapsto \pi_{K(\zeta)}(\sigma)$ *from* $\mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ *into* $\mathbb{M}^{n\times n}_{sym}$ *satisfies the Lipschitz estimate*

$$|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| \le |\sigma_2 - \sigma_1| + 2M_K|\zeta_2 - \zeta_1| \tag{2.5}$$

for every (σ_1,ζ_1), $(\sigma_2,\zeta_2) \in \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$.

Proof. It is enough to prove the estimate for (σ_1,ζ_1), $(\sigma_2,\zeta_2) \in \mathbb{M}^{n\times n}_{sym}\times[0,+\infty)$ with $0 < \zeta_1 \le \zeta_2$. Since $\pi_{K(\zeta_2)}$ has Lipschitz constant 1 on $\mathbb{M}^{n\times n}_{sym}$, from (2.3) and (2.4) we obtain

$$\begin{aligned}|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| &\le |\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_2)}(\sigma_1)| + |\pi_{K(\zeta_2)}(\sigma_1) - \pi_{K(\zeta_1)}(\sigma_1)| \le \\ &\le |\sigma_2-\sigma_1| + \big|\zeta_2\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \zeta_1\pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big| \le \\ &\le |\sigma_2-\sigma_1| + M_K|\zeta_2-\zeta_1| + \zeta_1\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big|\,.\end{aligned}$$

To prove (2.5) it is enough to show that

$$\zeta_1\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big| \le M_K|\zeta_2-\zeta_1|\,. \tag{2.6}$$

As $0 < \zeta_1 \le \zeta_2$, we have

$$\pi_{K(1)}\Big(\tfrac{1}{\zeta_1}\sigma_1 - \tfrac{\zeta_2-\zeta_1}{\zeta_1}\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\Big) = \pi_{K(1)}\Big(\tfrac{1}{\zeta_2}\sigma_1 + \tfrac{\zeta_2-\zeta_1}{\zeta_1}\big(\frac{1}{\zeta_2}\sigma_1 - \pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\big)\Big) = \pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\,.$$

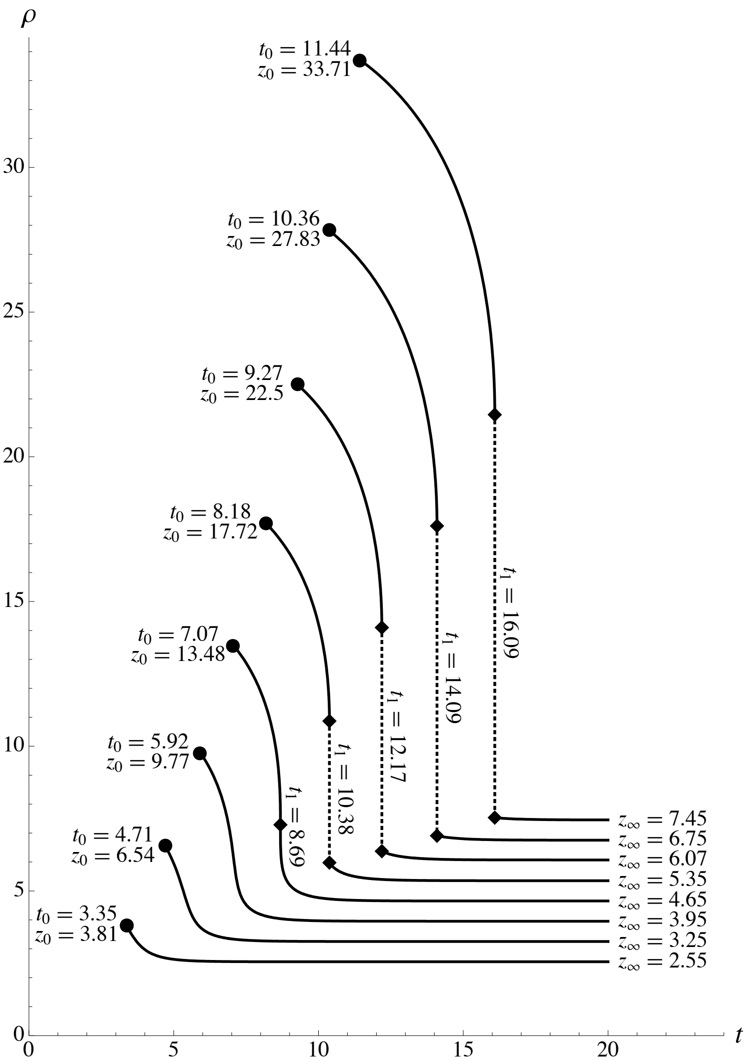

FIGURE 1.5. Graph of $\rho(t)$ in the plastic regime $t > t_0$ for $a_0 = 2$ and 8 different values of t_0 and z_0.

Since $\pi_{K(1)}$ has Lipschitz constant 1 on $\mathbb{M}^{n\times n}_{sym}$, we obtain

$$\left|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\frac{1}{\zeta_1}\sigma_1)\right| \le \tfrac{\zeta_2-\zeta_1}{\zeta_1}\left|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\right| \le M_K\tfrac{\zeta_2-\zeta_1}{\zeta_1}\,,$$

which gives (2.6). □

Let us fix $\xi \in W^{1,1}_{loc}([0,+\infty);\mathbb{M}^{n\times n}_{sym})$. For every $\varepsilon > 0$ system (1.2) is equivalent to

$$\begin{cases}\varepsilon\dot e_\varepsilon(t) = \varepsilon\dot\xi(t) - \mathbb{C}e_\varepsilon(t) + \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))\,,\\ \varepsilon\dot z_\varepsilon(t) = \mathrm{tr}(\pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t)))\,\mathrm{tr}(\mathbb{C}e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t)))\,.\end{cases} \tag{2.7}$$

Lemma 2.2. *For every $\varepsilon > 0$ and for every initial condition $e_\varepsilon(0) = e_0$ and $z_\varepsilon(0) = z_0 \ge 0$ system (2.7) has a unique solution defined for every $t \in [0,+\infty)$.*

Proof. As the right-hand sides are locally Lipschitz with respect to e and z by Lemma 2.1, it is enough to prove that for every $T > 0$ there is a constant $M_T > 0$ such that $|e_\varepsilon(t)| \le M_T$

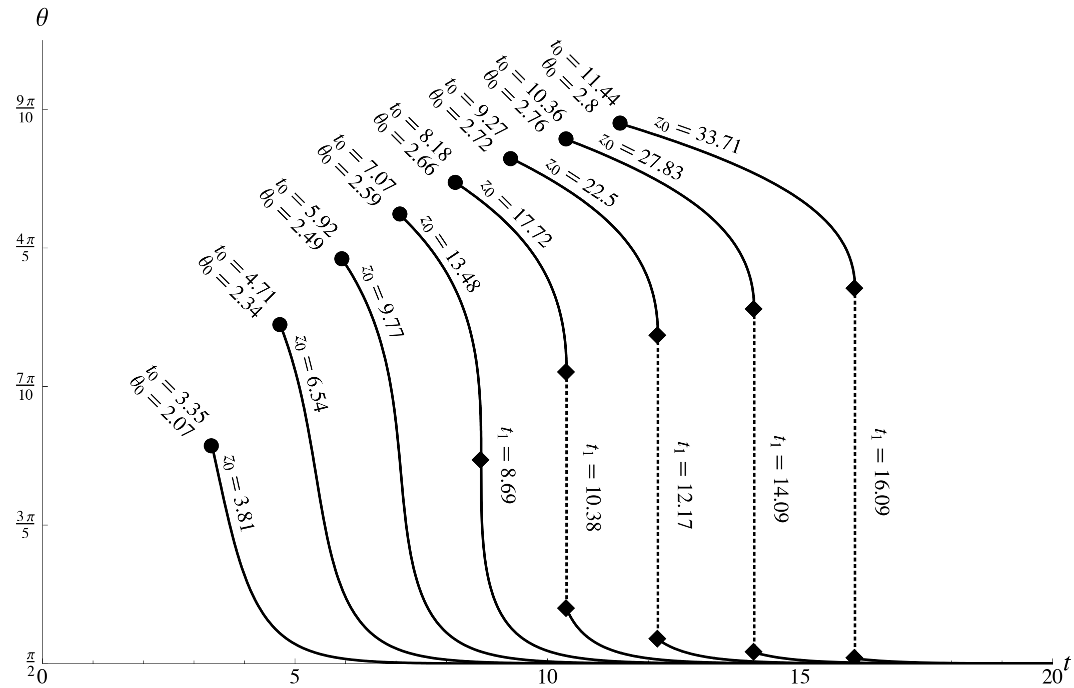

FIGURE 1.6. Graph of $\theta(t)$ in the plastic regime $t > t_0$ for $a_0 = 2$ and 8 different values of t_0 and z_0.

and $|z_\varepsilon(t)| \le M_T$ for every $t \in [0,T]$. Since $0 \in K(\zeta)$ for every $\zeta \in R$ by (2.2), we have $|\mathbb{C}e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))| \le |\mathbb{C}e_\varepsilon(t)| \le \beta_{\mathbb{C}}|e_\varepsilon(t)|$ and $|\pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))| \le |\mathbb{C}e_\varepsilon(t)| \le \beta_{\mathbb{C}}|e_\varepsilon(t)|$ for every $t \in [0,+\infty)$. Therefore, given $T > 0$, from the first equation in (2.7) we have

$$|e_\varepsilon(t)| \le A_T + \frac{\beta_{\mathbb{C}}}{\varepsilon}\int_0^t |e_\varepsilon(s)|\,ds \qquad \text{for every } t \in [0,T]\,.$$

with $A_T := |e_0| + \int_0^T |\dot\xi(s)|\,ds$. It follows from the Gronwall inequality that

$$|e_\varepsilon(t)| \le A_T \exp(T\,\beta_{\mathbb{C}}/\varepsilon) \qquad \text{for every } t \in [0,T]\,.$$

Then the second equation in (2.7) allows easily to obtain a constant $M_T > 0$ such that $|z_\varepsilon(t)| \le M_T$ for every $t \in [0,T]$. □

Lemma 2.3. *For every $\varepsilon > 0$, $e_0 \in \mathbb{M}^{n\times n}_{sym}$, and $z_0 > 0$ the solution $(e_\varepsilon, z_\varepsilon)$ of (2.7) with initial condition $e_\varepsilon(0) = e_0$ and $z_\varepsilon(0) = z_0$ satisfies $z_\varepsilon(t) > 0$ for every $t \in [0,+\infty)$.*

Proof. Suppose by contradiction that there exists $t_0 \in (0,+\infty)$ such that $z_\varepsilon(t_0) = 0$. Let e^*_ε be the solution of the Cauchy problem

$$\begin{cases} \varepsilon\dot e^*_\varepsilon(t) = \varepsilon\dot\xi(t) - \mathbb{C}e^*_\varepsilon(t)\,, \\ e^*_\varepsilon(t_0) = e_\varepsilon(t_0)\,, \end{cases} \tag{2.8}$$

and let $z^*_\varepsilon := 0$. Then $(e^*_\varepsilon, z^*_\varepsilon)$ would be a solution to (2.7) with $e^*_\varepsilon(t_0) = e_\varepsilon(t_0)$ and $z^*_\varepsilon(t_0) = z_\varepsilon(t_0)$. Since the right-hand side of (2.7) is locally Lipschitz with respect to e and z by Lemma 2.1, by uniqueness we would have $z_\varepsilon(t) = z^*_\varepsilon(t) = 0$ for every $t \in [0,+\infty)$, which contradicts the assumption $z_\varepsilon(0) = z_0 > 0$. □

For the rest of the paper we assume that $\mathbb{C}\xi = \xi$ for every $\xi \in \mathbb{M}^{n\times n}_{sym}$ and that $K(\zeta)$ is the closed ball centered at $-\frac{1}{n}\zeta I$ with radius $\frac{1}{\sqrt{n}}\zeta$, namely,

$$K(\zeta) = \{\sigma \in \mathbb{M}^{n\times n}_{sym} : |\sigma + \tfrac{1}{n}\zeta I| \le \tfrac{1}{\sqrt{n}}\zeta\}\,, \tag{2.9}$$

where I is the identity matrix in $\mathbb{M}^{n\times n}_{sym}$. In this case $\sigma_\varepsilon(t) = e_\varepsilon(t)$ and equation (2.7) simplifies to

$$\begin{cases} \varepsilon \dot e_\varepsilon(t) = \varepsilon\dot\xi(t) - e_\varepsilon(t) + \pi_{K(z_\varepsilon(t))}(e_\varepsilon(t))\,, \\ \varepsilon \dot z_\varepsilon(t) = \operatorname{tr}(\pi_{K(z_\varepsilon(t))}(e_\varepsilon(t)))\operatorname{tr}(e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(e_\varepsilon(t)))\,. \end{cases} \tag{2.10}$$

Moreover the projection onto $K(\zeta)$ is explicitly given by

$$\pi_{K(\zeta)}(\sigma) = -\tfrac{1}{n}\zeta I + \frac{\sigma + \frac{1}{n}\zeta I}{\max\{|\sigma + \frac{1}{n}\zeta I|, \frac{1}{\sqrt{n}}\zeta\}}\tfrac{1}{\sqrt{n}}\zeta\,.$$

Let us fix $e_0 \in \mathbb{M}^{n\times n}_{sym}$ with $\operatorname{tr}(e_0) = 0$ with $|e_0| = 1$. In the rest of the paper we consider $\xi(t)$ of the form

$$\xi(t) = -\tfrac{1}{n}a(t)I + \tfrac{1}{\sqrt{n}}b(t)e_0\,, \tag{2.11}$$

with a and b in $W^{1,1}_{loc}([0,\infty))$. In this case $\sigma_\varepsilon(t)$ and $e_\varepsilon(t)$ take the form

$$\sigma_\varepsilon(t) = e_\varepsilon(t) = -\tfrac{1}{n}x_\varepsilon(t)I + \tfrac{1}{\sqrt{n}}y_\varepsilon(t)e_0\,, \tag{2.12}$$

where the absolute values of the scalars $\frac{1}{\sqrt{n}}x_\varepsilon(t)$ and $\frac{1}{\sqrt{n}}y_\varepsilon(t)$ represent the norms of the spherical and deviatoric components of the stress, respectively. Moreover (2.10) is equivalent to the system

$$\begin{cases} \varepsilon \dot x_\varepsilon(t) = \varepsilon \dot a(t) - (x_\varepsilon(t) - z_\varepsilon(t)) + \dfrac{z_\varepsilon(t)\,(x_\varepsilon(t) - z_\varepsilon(t))}{u_\varepsilon(t)}\,, \\ \varepsilon \dot y_\varepsilon(t) = \varepsilon \dot b(t) - y_\varepsilon(t) + \dfrac{z_\varepsilon(t)\,y_\varepsilon(t)}{u_\varepsilon(t)}\,, \\ \varepsilon \dot z_\varepsilon(t) = \Big(z_\varepsilon(t) + \dfrac{z_\varepsilon(t)\,(x_\varepsilon(t) - z_\varepsilon(t))}{u_\varepsilon(t)}\Big)\Big(x_\varepsilon(t) - z_\varepsilon(t) - \dfrac{z_\varepsilon(t)\,(x_\varepsilon(t) - z_\varepsilon(t))}{u_\varepsilon(t)}\Big)\,, \end{cases} \tag{2.13}$$

where

$$u_\varepsilon(t) := \max\{z_\varepsilon(t), \sqrt{(x_\varepsilon(t) - z_\varepsilon(t))^2 + y_\varepsilon(t)^2}\}\,.$$

The corresponding viscosity solution $(e(t), p(t), \sigma(t), z(t))$ will be given by

$$\sigma(t) = e(t) = -\tfrac{1}{n}x(t)I + \tfrac{1}{\sqrt{n}}y(t)e_0 \quad \text{and} \quad p(t) = \tfrac{1}{n}(a(t) - x(t))I + \tfrac{1}{\sqrt{n}}(b(t) - y(t))e_0\,,$$

where $x(t)$, $y(t)$, and $z(t)$ are left continuous with respect to t and $x_\varepsilon(t) \to x(t)$, $y_\varepsilon(t) \to y(t)$, and $z_\varepsilon(t) \to z(t)$ for a.e. $t \in [0,+\infty)$.

Passing to polar coordinates through (1.6), system (2.13) becomes

$$\begin{cases} \varepsilon\,\dot\rho_\varepsilon(t) = \varepsilon\,(\dot a(t)\cos\theta_\varepsilon(t) + \dot b(t)\sin\theta_\varepsilon(t)) - \\ \qquad\qquad - (\rho_\varepsilon(t) - z_\varepsilon(t))^+\big(z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t))\cos^2\theta_\varepsilon(t) + 1\big)\,, \\ \varepsilon\,\rho_\varepsilon(t)\,\dot\theta_\varepsilon(t) = -\varepsilon\,(\dot a(t)\sin\theta_\varepsilon(t) - \dot b(t)\cos\theta_\varepsilon(t)) + \\ \qquad\qquad + (\rho_\varepsilon(t) - z_\varepsilon(t))^+ z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\sin\theta_\varepsilon(t)\,, \\ \varepsilon\,\dot z_\varepsilon(t) = (\rho_\varepsilon(t) - z_\varepsilon(t))^+ z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\,, \end{cases} \tag{2.14}$$

where $(\cdot)^+$ denotes the positive part. The polar coordinates of a viscosity solution are denoted by $(\rho(t), \theta(t), z(t))$. They are continuous from the left and $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t)) \to (\rho(t), \theta(t), z(t))$ for a.e. $t \in [0,+\infty)$.

Let us fix a_0 and z_0, with $0 \le a_0 \le 2\,z_0$ and $z_0 > 0$. In the rest of the paper we study the special (strain controlled) loading path

$$a(t) := a_0 \qquad \text{and} \qquad b(t) := t\,, \tag{2.15}$$

and the initial conditions

$$x_\varepsilon(0) = a_0\,, \qquad y_\varepsilon(0) = 0\,, \qquad z_\varepsilon(0) = z_0\,. \tag{2.16}$$

2.1. **The elastic regime.** The solution of (2.13) with loading path (2.15) and initial conditions (2.16) remains in the elastic regime in an interval $[0, t_0]$, where $t_0 := \sqrt{z_0^2 - (a_0 - z_0)^2}$ is the only positive number such that

$$\sqrt{(a_0 - z_0)^2 + t_0^2} = z_0\,. \tag{2.17}$$

More precisely we have

$$x_\varepsilon(t) = a_0\,, \qquad y_\varepsilon(t) = t\,, \qquad z_\varepsilon(t) = z_0 \tag{2.18}$$

for every $t \in [0, t_0]$. Indeed in this interval the functions defined by (2.18) satisfy the inequality $\sqrt{(x_\varepsilon - z_\varepsilon)^2 + y_\varepsilon^2} \le z_\varepsilon$, so that the system reduces to

$$\begin{cases} \varepsilon \dot{x}_\varepsilon(t) = \varepsilon \dot{a}(t)\,, \\ \varepsilon \dot{y}_\varepsilon(t) = \varepsilon \dot{b}(t)\,, \\ \varepsilon \dot{z}_\varepsilon(t) = 0\,, \end{cases} \tag{2.19}$$

which is trivially satisfied by (2.15) and (2.18). Therefore the viscosity solution satisfies

$$x(t) = a_0\,, \qquad y(t) = t\,, \qquad z(t) = z_0 \tag{2.20}$$

for every $t \in [0, t_0]$.

2.2. **The inelastic regime.** After time t_0 the solution exhibits a plastic behaviour. To study the solution for $t > t_0$ we use (2.14), which in case (2.15) becomes

$$\begin{cases} \varepsilon\, \dot{\rho}_\varepsilon(t) = \varepsilon \sin\theta_\varepsilon(t) - (\rho_\varepsilon(t) - z_\varepsilon(t))^+ \big(z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t)) \cos^2\theta_\varepsilon(t) + 1\big)\,, \\ \varepsilon\, \rho_\varepsilon(t)\, \dot{\theta}_\varepsilon(t) = \varepsilon \cos\theta_\varepsilon(t) + (\rho_\varepsilon(t) - z_\varepsilon(t))^+ z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t)) \cos\theta_\varepsilon(t) \sin\theta_\varepsilon(t)\,, \\ \varepsilon\, \dot{z}_\varepsilon(t) = (\rho_\varepsilon(t) - z_\varepsilon(t))^+ z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t)) \cos\theta_\varepsilon(t)\,. \end{cases} \tag{2.21}$$

By (2.17), there exists a unique $\theta_0 \in (0, \pi)$ such that

$$z_0 \cos\theta_0 = a_0 - z_0\,, \qquad z_0 \sin\theta_0 = t_0\,. \tag{2.22}$$

By elementary geometric considerations we have

$$0 \le a_0 < z_0 \implies \tfrac{\pi}{2} < \theta_0 \le \pi \qquad \text{and} \qquad z_0 < a_0 \le 2\, z_0 \implies 0 \le \theta_0 < \tfrac{\pi}{2}\,. \tag{2.23}$$

By (2.17), (2.18), and (2.22) we have

$$\rho_\varepsilon(t_0) = z_0\,, \qquad \theta_\varepsilon(t_0) = \theta_0\,, \qquad z_\varepsilon(t_0) = z_0\,. \tag{2.24}$$

Subtracting the third equation from the first one in (2.21) we obtain the following differential equation for the difference $\rho_\varepsilon(t) - z_\varepsilon(t)$:

$$\varepsilon\, (\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t)) = \varepsilon \sin\theta_\varepsilon(t) - (\rho_\varepsilon(t) - z_\varepsilon(t))^+ w_\varepsilon(t)\,, \tag{2.25}$$

where

$$w_\varepsilon(t) := z_\varepsilon(t)\,(1 + \cos\theta_\varepsilon(t))^2 \cos\theta_\varepsilon(t) + 1\,. \tag{2.26}$$

From (2.21) for every $t \in [t_0, +\infty)$ we obtain

$$\begin{aligned} \varepsilon\, \rho_\varepsilon(t)\, \dot{w}_\varepsilon(t) = {} & -\varepsilon\, z_\varepsilon(t)(1 + \cos\theta_\varepsilon(t))\,(1 + 3\cos\theta_\varepsilon(t)) \cos\theta_\varepsilon(t) \sin\theta_\varepsilon(t) - \\ & - (\rho_\varepsilon(t) - z_\varepsilon(t))\, z_\varepsilon(t)(1 + \cos\theta_\varepsilon(t))^3 \cos\theta_\varepsilon(t)\, v_\varepsilon(t)\,, \end{aligned} \tag{2.27}$$

where

$$v_\varepsilon(t) := z_\varepsilon(t)(1 + \cos\theta_\varepsilon(t) - 3\cos^2\theta_\varepsilon(t)) - (\rho_\varepsilon(t) - z_\varepsilon(t)) \cos\theta_\varepsilon(t)\,.$$

If $a_0 = z_0$, then $\theta_0 = \frac{\pi}{2}$ and, in this case,

$$\rho_\varepsilon(t) := z_0 + \varepsilon\big(1 - \exp(-\tfrac{t - t_0}{\varepsilon})\big)\,, \qquad \theta_\varepsilon(t) := \tfrac{\pi}{2}\,, \qquad z_\varepsilon(t) := z_0 \tag{2.28}$$

is the explicit solution of (2.21) with initial conditions (2.24). Then the viscosity solution obtained by taking the limit as $\varepsilon \to 0$ satisfies

$$\rho(t) = z_0 \,, \qquad \theta(t) = \tfrac{\pi}{2} \,, \qquad z(t) = z_0 \qquad \text{for every } t \in [t_0, +\infty) \,. \tag{2.29}$$

Lemma 2.4. *If $\theta_0 \neq \frac{\pi}{2}$, then $\theta_\varepsilon(t) \neq \frac{\pi}{2}$ for every $t \in [t_0, +\infty)$.*

Proof. Suppose $\theta_0 \neq \frac{\pi}{2}$ and suppose that there exists $\tau \in [t_0, +\infty)$ such that $\theta_\varepsilon(\tau) = \frac{\pi}{2}$. Let ρ_ε^τ be the solution of the Cauchy problem

$$\begin{cases} \varepsilon\, \dot\rho_\varepsilon^\tau(t) = \varepsilon - (\rho_\varepsilon^\tau(t) - z_\varepsilon(\tau))^+ \,, \\ \rho_\varepsilon^\tau(\tau) = \rho_\varepsilon(\tau) \,. \end{cases}$$

Then the triple

$$\rho_\varepsilon^\tau(t) \,, \qquad \theta_\varepsilon^\tau(t) := \tfrac{\pi}{2} \,, \qquad z_\varepsilon^\tau(t) := z_\varepsilon(\tau)$$

would be a solution of (2.21) which satisfies the Cauchy condition

$$\rho_\varepsilon^\tau(\tau) := \rho_\varepsilon(\tau) \,, \qquad \theta_\varepsilon^\tau(\tau) := \theta_\varepsilon(\tau) \,, \qquad z_\varepsilon^\tau(\tau) := z_\varepsilon(\tau) \,.$$

By uniqueness we must have $\theta_\varepsilon(t) = \theta_\varepsilon^\tau(t) = \frac{\pi}{2}$ for every t, which contradicts the fact that $\theta_\varepsilon(t_0) = \theta_0 \neq \frac{\pi}{2}$. This concludes the proof of (2.4). □

Lemma 2.5. *If $0 \le \theta_0 < \frac{\pi}{2}$, then $0 < \theta_\varepsilon(t) < \frac{\pi}{2}$ for every $t \in (t_0, +\infty)$. If $\frac{\pi}{2} < \theta_0 \le \pi$, then $\frac{\pi}{2} < \theta_\varepsilon(t) < \pi$ for every $t \in (t_0, +\infty)$.*

Proof. Assume $0 \le \theta_0 < \frac{\pi}{2}$. From the second equation in (5.12) it follows that $\dot\theta_\varepsilon(t_0) > 0$. Therefore the inequalities $0 < \theta_\varepsilon(t) < \frac{\pi}{2}$ are satisfied in a right neighbourhood of t_0. If they do not hold for every $t \in [t_0, +\infty)$, by Lemma 2.4 we can consider the first $\tau \in (t_0, +\infty)$ such that $\theta_\varepsilon(\tau) = 0$. Then $\dot\theta_\varepsilon(\tau) \le 0$. As $0 < \theta_\varepsilon(t) < \frac{\pi}{2}$ for every $t \in [t_0, \tau)$ by Lemma 2.4, from the second equation in (2.21) we obtain $\rho_\varepsilon(t)\, \dot\theta_\varepsilon(t) \ge \cos\theta_\varepsilon(t) > 0$ for every $t \in [t_0, \tau]$ and $\rho_\varepsilon(\tau)\, \dot\theta_\varepsilon(\tau) = 1$. As $\rho_\varepsilon(t_0) = z_0 > 0$, by continuity we have $\dot\theta_\varepsilon(t) > 0$ for every $t \in [t_0, \tau]$. This contradicts the inequality $\dot\theta_\varepsilon(\tau) \le 0$, and concludes the proof of the first implication. The second one is proved in the same way. □

Lemma 2.6. *We have $\rho_\varepsilon(t) > z_\varepsilon(t)$ for every $t \in (t_0, +\infty)$.*

Proof. We deduce from (2.25) that, if $\rho_\varepsilon(t) = z_\varepsilon(t)$ for some $t \in [t_0, +\infty)$, then $\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t) = \sin\theta_\varepsilon(t) > 0$, where the inequality follows from (2.28) and Lemma 2.5. Since $\rho_\varepsilon(t_0) = z_\varepsilon(t_0)$, we conclude that $\rho_\varepsilon(t) > z_\varepsilon(t)$ for every $t \in (t_0, +\infty)$. □

Lemma 2.7. *For every $t \in (t_0, +\infty)$ the following properties hold:*

$$\rho_\varepsilon(t) > 0 \,, \tag{2.30}$$

$$0 \le \theta_0 < \tfrac{\pi}{2} \quad \Longrightarrow \quad \dot\theta_\varepsilon(t) > 0 \quad \textit{and} \quad 0 < \theta_0 < \theta_\varepsilon(t) < \tfrac{\pi}{2} \,, \tag{2.31}$$

$$\tfrac{\pi}{2} < \theta_0 \le \pi \quad \Longrightarrow \quad \dot\theta_\varepsilon(t) < 0 \quad \textit{and} \quad \tfrac{\pi}{2} < \theta_\varepsilon(t) < \theta_0 < \pi, \tag{2.32}$$

$$0 \le \theta_0 < \tfrac{\pi}{2} \quad \Longrightarrow \quad \dot z_\varepsilon(t) > 0 \quad \textit{and} \quad z_\varepsilon(t) > z_0 \,, \tag{2.33}$$

$$\tfrac{\pi}{2} < \theta_0 \le \pi \quad \Longrightarrow \quad \dot z_\varepsilon(t) < 0 \quad \textit{and} \quad 0 < z_\varepsilon(t) < z_0 \,. \tag{2.34}$$

Proof. By Lemma 2.5 from the second equation in (2.21) and from (2.24) we obtain (2.31), (2.32), and (2.30). Implications (2.33) and (2.34) can be obtained from Lemmas 2.3, 2.5, and 2.6, using the third equation in (2.21). □

Lemma 2.8. *Assume $\frac{\pi}{2} < \theta_0 < \pi$. Then then $\rho_\varepsilon(t) \le \rho_\varepsilon(s) + \varepsilon$ whenever $t_0 \le s \le t$.*

Proof. Let us fix $s \ge t_0$ and $\eta > 0$. If the inequality

$$\rho_\varepsilon(t) \le \rho_\varepsilon(s) + (1 + \eta)\, \varepsilon \tag{2.35}$$

is not satisfied for every $t \ge s$, let τ be the first time after s with $\rho_\varepsilon(\tau) = \rho_\varepsilon(s) + (1 + \eta)\varepsilon$. Then $\dot\rho_\varepsilon(\tau) \ge 0$. From the first equation in (2.21) we obtain $\varepsilon\, \dot\rho_\varepsilon(\tau) \le \varepsilon - (\rho_\varepsilon(\tau) - z_\varepsilon(\tau))$.

By (2.34) and by the definition of τ we have $\varepsilon\,\dot\rho_\varepsilon(\tau)\le\varepsilon-(\rho_\varepsilon(s)+(1+\eta)\,\varepsilon-z_\varepsilon(s))$, so that Lemma 2.6 gives $\varepsilon\,\dot\rho_\varepsilon(\tau)\le-\eta\,\varepsilon$, which contradicts the inequality $\dot\rho_\varepsilon(\tau)\ge0$. This proves that (2.35) holds for every $t\ge s$. The conclusion can be obtained by taking the limit as $\eta\to0$. □

3. The slow dynamics

In this section we study in detail the behaviour of the solutions to the system of the slow dynamics.

3.1. **The trajectory of the slow dynamics.** In this subsection we study the equation

$$r'(\theta)=r(\theta)\frac{r(\theta)\,(1+\cos\theta)\,\sin\theta}{r(\theta)\,(1+\cos\theta)^2+1}\,, \tag{3.1}$$

that describes the trajectories followed along the slow dynamics.

Lemma 3.1. *Every solution of* (3.1) *with* $r(\theta^*)>0$ *for some* $\theta^*\in[0,\pi]$ *is defined for every* $\theta\in[0,\pi]$ *and satisfies* $r(\theta)>0$ *for every* $\theta\in[0,\pi]$ *and* $r'(\theta)>0$ *for every* $\theta\in(0,\pi)$.

Proof. Since the null function is a solution of the equation, if $r(\theta)$ is a solution of (3.1) and $r(\theta^*)>0$ for some θ^*, then $r(\theta)>0$ for every θ by uniqueness. Therefore, the right-hand side of (3.1) is positive for $\theta\in(0,\pi)$, which implies that $r'(\theta)>0$ on this interval.

To prove the global existence in the whole interval $[0,\pi]$, it is not restrictive to assume $\theta^*\in(0,\pi)$. The positivity and monotonicity of $r(\theta)$ imply that $[0,\theta^*]$ is contained in the maximal domain of existence of $r(\theta)$. To study the problem for $\theta>\theta^*$ we consider the inequalities

$$0<\rho\,\frac{\rho\,(1+\cos\theta)\,\sin\theta}{\rho\,(1+\cos\theta)^2+1}<\frac{\rho\,\sin\theta}{1+\cos\theta}$$

for every $\rho>0$ and every $\theta\in(0,\pi)$. Using an elementary comparison argument we deduce that the maximal domain of existence of $r(\theta)$ contains $[\theta^*,\pi)$ and

$$r(\theta)\le r(\theta^*)\frac{1+\cos\theta^*}{1+\cos\theta}\quad\text{for every }\theta^*\le\theta<\pi.$$

By (3.1) this inequality yields

$$r'(\theta)\le r(\theta)\,r(\theta^*)(1+\cos\theta^*)\quad\text{for every }\theta\in[\theta^*,\pi),$$

and this implies that π belongs to the maximal domain of existence of $r(\theta)$. □

Let λ_c be the unique negative solution of the equation $1+\lambda-3\lambda^2=0$, i.e.,

$$\lambda_c:=-\frac{1}{6}(\sqrt{13}-1)\simeq-0.43425\ldots\ , \tag{3.2}$$

and let

$$\theta_c:=\arccos\lambda_c\simeq2.0200\ldots\ . \tag{3.3}$$

We consider the function $z_s\colon[\frac{\pi}{2},\pi]\to[\frac{27}{4},+\infty]$ defined by

$$z_s(\theta):=-\frac{1}{(1+\cos\theta)^2\cos\theta}\quad\text{for }\theta\in(\tfrac{\pi}{2},\pi)\,,\qquad z_s(\tfrac{\pi}{2}):=z_s(\pi):=+\infty\,, \tag{3.4}$$

and we define

$$z_c:=z_s(\theta_c)=\tfrac{61}{18}+\tfrac{19}{18}\sqrt{13}\simeq7.1947\ldots\ . \tag{3.5}$$

We shall see that the graph of z_s plays the role of separation line between initial data leading to the slow dynamics and those leading to the fast dynamics.

Finally, let

$$r_c(\theta)\text{ be the solution of (3.1) with Cauchy condition }r_c(\theta_c)=z_c\,. \tag{3.6}$$

Lemma 3.2. *We have* $r_c(\theta_c)=z_s(\theta_c)$ *and* $r_c(\theta)<z_s(\theta)$ *for every* $\theta\in[\frac{\pi}{2},\theta_c)\cup(\theta_c,\pi]$.

Proof. By direct computation for every $\theta \in (\frac{\pi}{2}, \pi)$ we obtain

$$z_s'(\theta) = -\frac{(1+3\cos\theta)\sin\theta}{(1+\cos\theta)^3\cos^2\theta}\,,$$

$$z_s(\theta)\frac{z_s(\theta)\,(1+\cos\theta)\,\sin\theta}{z_s(\theta)\,(1+\cos\theta)^2+1} = -\frac{\sin\theta}{(1+\cos\theta)^3\cos\theta(1-\cos\theta)}\,,$$

so that in the interval $(\frac{\pi}{2}, \pi)$ the inequality

$$z_s'(\theta) > z_s(\theta)\frac{z_s(\theta)\,(1+\cos\theta)\,\sin\theta}{z_s(\theta)\,(1+\cos\theta)^2+1} = -\frac{\sin\theta}{(1+\cos\theta)^3\cos\theta(1-\cos\theta)} \tag{3.7}$$

is equivalent to

$$1+\cos\theta-3\cos^2\theta > 0\,.$$

Therefore (3.7) is satisfied $\theta < \theta_c$, and the opposite inequality holds for $\theta > \theta_c$. Since $r_c(\theta_c) = z_s(\theta_c)$ by (3.6), the inequality $r_c(\theta) < z_s(\theta)$ for $\theta \neq \theta_c$ follows from a comparison argument. ☐

Lemma 3.3. *Assume that*

$$\theta_c < \theta_0 \leq \pi \quad \textit{and} \quad r_c(\theta_0) \leq z_0 < z_s(\theta_0)\,. \tag{3.8}$$

Let $r_0(\theta)$ be the solution of (3.1) *with Cauchy condition $r_0(\theta_0) = z_0$. Then there exists $\theta_1 \in [\theta_c, \theta_0)$ such that*

$$r_0(\theta_1) = z_s(\theta_1) \qquad \textit{and} \quad r_0(\theta) < z_s(\theta) \quad \textit{for} \quad \theta \in (\theta_1, \theta_0]\,. \tag{3.9}$$

If $z_0 > r_c(\theta_0)$, then $\theta_1 > \theta_c$; if $z_0 = r_c(\theta_0)$, then $\theta_1 = \theta_c$.

Proof. Since $r_0(\theta_0) = z_0 \geq r_c(\theta_0)$, by comparison we have $r_0(\theta) \geq r_c(\theta)$ for every $\theta \in (0, \pi)$. In particular $r_0(\theta_c) \geq r_c(\theta_c) = z_s(\theta_c)$ and $r_0(\theta_0) = z_0 < z_s(\theta_0)$. Then (3.9) is satisfied by the greatest point θ_1 of $[\theta_c, \theta_0)$ such that $r_0(\theta_1) = z_s(\theta_1)$. If $z_0 > r_c(\theta_0)$, then $r_0(\theta) > r_c(\theta)$ by comparison, and $\theta_1 > \theta_c$ by Lemma 3.2. If $z_0 = r_c(\theta_0)$, then $r_0(\theta) = r_c(\theta)$ by uniqueness, and $\theta_1 = \theta_c$ by Lemma 3.2. ☐

Lemma 3.4. *Assume one of the following conditions:*

$$\tfrac{\pi}{2} < \theta_2 \leq \theta_c \quad \textit{and} \quad z_2 \leq z_s(\theta_2)\,, \tag{3.10}$$

$$\theta_c < \theta_2 < \pi \quad \textit{and} \quad z_2 < r_c(\theta_2)\,. \tag{3.11}$$

Let $r_2(\theta)$ be the solution of (3.1) *with Cauchy condition $r_2(\theta_2) = z_2$. Then*

$$r_2(\theta) < z_s(\theta) \quad \textit{for} \quad \theta \in (\tfrac{\pi}{2}, \theta_2)\,. \tag{3.12}$$

Proof. Assume (3.10). Then (3.7) holds for every $\theta \in (\frac{\pi}{2}, \theta_2)$ and $r_2(\theta_2) = z_2 \leq z_s(\theta_2)$, so that (3.12) follows from a comparison argument.

Assume (3.11). Since $r_2(\theta_2) = z_2 < r_c(\theta_2)$, by uniqueness we have $r(\theta) < r_c(\theta)$ for every $\theta \in \mathbb{R}$. In particular we have $r_2(\theta) < r_c(\theta) \leq z_s(\theta)$ for every $\theta \in (\frac{\pi}{2}, \theta_2)$. ☐

3.2. **The system of the slow dynamics.** In this subsection we study the system

$$\begin{cases} \dot\rho^{sl}(t) = \dfrac{\rho^{sl}(t)\,(1+\cos\theta^{sl}(t))\,\cos\theta^{sl}(t)\,\sin\theta^{sl}(t)}{\rho^{sl}(t)\,(1+\cos\theta^{sl}(t))^2\cos\theta^{sl}(t)+1}\,, \\[2ex] \dot\theta^{sl}(t) = \dfrac{\rho^{sl}(t)\,(1+\cos\theta^{sl}(t))^2\cos\theta^{sl}(t)+\cos\theta^{sl}(t)}{\rho^{sl}(t)\,\big(\rho^{sl}(t)\,(1+\cos\theta^{sl}(t))^2\cos\theta^{sl}(t)+1\big)}\,, \end{cases} \tag{3.13}$$

that will be satisfied during the slow dynamics. Let θ_c and z_c be the constants defined in (3.3) and (3.6), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6).

Lemma 3.5. *Assume* $0 \le \theta_0 < \frac{\pi}{2}$ *and let* $(\rho_0^{sl}, \theta_0^{sl})$ *be the solution of* (3.13) *with Cauchy conditions*

$$\rho_0^{sl}(t_0) = z_0 \quad \text{and} \quad \theta_0^{sl}(t_0) = \theta_0 \,. \tag{3.14}$$

Then $(\rho_0^{sl}, \theta_0^{sl})$ *is defined on* $[t_0, +\infty)$ *and*

$$\dot\rho_0^{sl}(t) > 0 \quad \text{and} \quad \dot\theta_0^{sl}(t) > 0 \quad \text{for} \quad t \in (t_0, +\infty)\,, \tag{3.15}$$

$$\lim_{t\to+\infty} \rho_0^{sl}(t) < +\infty \quad \text{and} \quad \lim_{t\to+\infty} \theta_0^{sl}(t) = \tfrac{\pi}{2}\,. \tag{3.16}$$

Proof. Let $r_0(\theta)$ be the solution of (3.1) with Cauchy condition $r_0(\theta_0) = z_0$, which is defined for every $\theta \in [0, \pi)$ by Lemma 3.1. Let us consider the solution $\theta_\flat(t)$ of the autonomous equation

$$\dot\theta_\flat(t) = \frac{r_0(\theta_\flat(t))\,(1+\cos\theta_\flat(t))^2\cos\theta_\flat(t) + \cos\theta_\flat(t)}{r_0(\theta_\flat(t))\big(r_0(\theta_\flat(t))\,(1+\cos\theta_\flat(t))^2\cos\theta_\flat(t) + 1\big)} \tag{3.17}$$

with Cauchy condition $\theta_\flat(t_0) = \theta_0$. We observe that the right-hand side of (3.17) is positive on $[\theta_0, \frac{\pi}{2})$ and vanishes for $\theta = \frac{\pi}{2}$. Then the theory of autonomous equations guarantees that $\theta_\flat(t)$ is defined for every $t \in [t_0, +\infty)$, $\dot\theta_\flat(t) > 0$ for every $t \in [t_0, +\infty)$, and $\theta_\flat(t) \to \frac{\pi}{2}$ as $t \to +\infty$.

Let $\rho_\flat(t) := r_0(\theta_\flat(t))$ for every $t \in [t_0, +\infty)$. Then $(\rho_\flat(t), \theta_\flat(t))$ is a solution of (3.13) defined on $[t_0, +\infty)$. Since it satisfies the Cauchy conditions (3.14), by uniqueness we have $(\rho_0^{sl}(t), \theta_0^{sl}(t)) = (\rho_\flat(t), \theta_\flat(t))$ for every $t \in [t_0, +\infty)$. This implies that $\dot\theta_0^{sl}(t) > 0$ for every $t \in [t_0, +\infty)$, and that $\theta_0^{sl}(t) \to \frac{\pi}{2}$ and $\rho_0^{sl}(t) \to r_0(\frac{\pi}{2}) < +\infty$ as $t \to +\infty$. Since $r_0'(\theta) > 0$ for every $\theta \in (0, \pi)$ by Lemma 3.1, we obtain $\dot\rho_0^{sl}(t) = r_0'(\theta_\flat(t))\,\dot\theta_\flat(t) > 0$ for every $t \in (t_0, +\infty)$. □

Lemma 3.6. *Assume* (3.8) *and let* $(\rho_0^{sl}, \theta_0^{sl})$ *be the solution of* (3.13) *with Cauchy conditions* (3.14). *Then there exist* $t_1 \in (t_0, +\infty)$, $z_1 \in (0, z_0)$, *and* $\theta_1 \in [\theta_c, \theta_0)$, *such that* $(\rho_0^{sl}, \theta_0^{sl})$ *is defined on* $[t_0, t_1)$ *and*

$$\lim_{t\to t_1} \rho_0^{sl}(t) = z_1\,, \qquad \lim_{t\to t_1} \theta_0^{sl}(t) = \theta_1\,, \qquad z_1 = z_s(\theta_1)\,, \tag{3.18}$$

$$\lim_{t\to t_1} \dot\rho_0^{sl}(t) = -\infty\,, \qquad \lim_{t\to t_1} \dot\theta_0^{sl}(t) = -\infty\,, \tag{3.19}$$

$$\dot\rho_0^{sl}(t) < 0 \quad \text{and} \quad \dot\theta_2^{sl}(t) < 0 \quad \text{for} \quad t \in [t_0, t_1)\,, \tag{3.20}$$

$$\rho_0^{sl}(t) < z_s(\theta_0^{sl}(t)) \quad \text{for every} \quad t \in [t_0, t_1)\,. \tag{3.21}$$

If $z_0 > r_c(\theta_0)$, *then* $\theta_1 > \theta_c$; *if* $z_0 = r_c(\theta_0)$, *then* $\theta_1 = \theta_c$ *and* $z_1 = z_c$.

Proof. Let $r_0(\theta)$ and θ_1 be as in Lemma 3.3, and let $z_1 := z_s(\theta_1)$. Let us consider the solution $\theta_\flat(t)$ of the autonomous equation (3.17) with Cauchy condition $\theta_\flat(t_0) = \theta_0$. By (3.9) the right-hand side of (3.17) is negative on (θ_1, θ_0) and tends to $-\infty$ for $\theta \to \theta_1$. Then the theory of autonomous equations guarantees that there exists $t_1 > t_0$ such that $\theta_\flat(t)$ is defined for every $t \in [t_0, t_1)$, $\dot\theta_\flat(t) < 0$ for every $t \in [t_0, t_1)$, and $\theta_\flat(t) \to \theta_1$ as $t \to t_1$.

Let $\rho_\flat(t) := r_0(\theta_\flat(t))$ for every $t \in [t_0, t_1)$. Then $(\rho_\flat(t), \theta_\flat(t))$ is a solution of (3.13) defined on $[t_0, t_1)$. Since it satisfies the Cauchy conditions (3.14), by uniqueness we have $(\rho_0^{sl}(t), \theta_0^{sl}(t)) = (\rho_\flat(t), \theta_\flat(t))$ for every $t \in [t_0, t_1)$. This implies that $\dot\theta_0^{sl}(t) > 0$ for every $t \in [t_0, t_1)$, and that $\theta_0^{sl}(t) \to \theta_1$ and $\rho_0^{sl}(t) \to r_0(\theta_1) = z_1$ as $t \to t_1$, where the last equality follows from (3.9) and from the definition of z_1. Since $r_0'(\theta) > 0$ for every $\theta \in (0, \pi)$ by Lemma 3.1, we obtain $\dot\rho_0^{sl}(t) = r_0'(\theta_\flat(t))\,\dot\theta_\flat(t) < 0$ for every $t \in (t_0, t_1)$. Inequality (3.21) follows from (3.9).

Finally, Lemma 3.3 guarantees that, if $z_0 > r_c(\theta_0)$, then $\theta_1 > \theta_c$, and if $z_0 = r_c(\theta_0)$, then $\theta_1 = \theta_c$, and hence hence $z_1 = z_s(\theta_c) = z_c$. □

Lemma 3.7. *Assume* (3.10) *or* (3.11)*, let* $t_1 \geq t_0$*, and let* $t_1^k \to t_1$*. Then there exists a unique solution* $(\rho_2^{sl}, \theta_2^{sl})$ *of* (3.13) *defined on* $(t_1, +\infty)$ *such that*

$$\rho_2^{sl}(t_1^k) \to z_2 \quad \textit{and} \quad \theta_2^{sl}(t_1^k) \to \theta_2 \,. \tag{3.22}$$

Moreover

$$\lim_{t \to t_1} \rho_2^{sl}(t) = z_2 \quad \textit{and} \quad \lim_{t \to t_1} \theta_2^{sl}(t) = \theta_2 \,, \tag{3.23}$$

$$\dot{\rho}_2^{sl}(t) < 0 \quad \textit{and} \quad \dot{\theta}_2^{sl}(t) < 0 \quad \textit{for} \quad t \in (t_1, +\infty) \,, \tag{3.24}$$

$$\lim_{t \to +\infty} \rho_2^{sl}(t) > 0 \quad \textit{and} \quad \lim_{t \to +\infty} \theta_2^{sl}(t) = \tfrac{\pi}{2} \,, \tag{3.25}$$

$$\rho_2^{sl}(t) < z_s(\theta_2^{sl}(t)) \quad \textit{for every} \quad t \in (t_1, +\infty) \,. \tag{3.26}$$

Proof. Let $r_2(\theta)$ be as in Lemma 3.4. By (3.12) we have

$$r_2(\theta)\,(1 + \cos\theta)^2 \cos\theta + 1 > 0 \qquad \text{for every } \theta \in [\tfrac{\pi}{2}, \theta_2) \,. \tag{3.27}$$

Let us consider the autonomous equation

$$\dot{\theta}_\sharp(t) = \frac{r_2(\theta_\sharp(t))\,(1 + \cos\theta_\sharp(t))^2 \cos\theta_\sharp(t) + \cos\theta_\sharp(t)}{r_2(\theta_\sharp(t))\,\big(r_2(\theta_\sharp(t))\,(1 + \cos\theta_\sharp(t))^2 \cos\theta_\sharp(t) + 1\big)} \,. \tag{3.28}$$

Since the right-hand side of this equation is negative on $(\frac{\pi}{2}, \pi)$ and vanishes at $\frac{\pi}{2}$, the theory of autonomous equations guarantees that there exists a unique solution $\theta_\sharp(t)$ of (3.28) defined for every $t \in (t_1, +\infty)$ and such that $\theta_\sharp(t) \to \theta_2$ as $t \to t_1$. Moreover $\theta_\sharp(t)$ is defined for every $t \in (t_1, +\infty)$, $\dot{\theta}_\sharp(t) < 0$, $\frac{\pi}{2} < \theta_\sharp(t) < \pi$, and $\theta_\sharp(t) \to \frac{\pi}{2}$ as $t \to +\infty$. Let $\rho_\sharp(t) := r_2(\theta_\sharp(t))$ for every $t \in (t_1, +\infty)$. From (3.1) and (3.28) it follows that $(\rho_\sharp(t), \theta_\sharp(t))$ is a solution of (3.13) defined on $(t_1, +\infty)$ and satisfies (3.23). Moreover $\dot{\rho}_\sharp(t), 0$ for every $t \in (t_1, +\infty)$ and $\rho_\sharp(t) = r_2(\theta_\sharp(t)) \to r_2(\frac{\pi}{2}) > 0$ as $t \to +\infty$. Since $\frac{\pi}{2} < \theta_\sharp(t) < \theta_2$ and $\rho_\sharp(t) := r_2(\theta_\sharp(t))$, by (3.27) we have $\rho_\sharp(t)\,\big(1 + \cos\theta_\sharp(t)\big)^2 \cos\theta_\sharp(t) + 1 > 0$ for every $t > t_1$, which proves (3.26).

To prove the uniqueness, let $(\rho^{sl}(t), \theta^{sl}(t))$ be a solution of (3.13) satisfying (3.22). By uniqueness we have $\theta^{sl}(t) \neq \frac{\pi}{2}$ for every t. As $\rho^{sl}(t)\,(1 + \cos\theta^{sl}(t))^2 \cos\theta^{sl}(t) + 1 > 0$ and $\cos\theta^{sl}(t) < 0$ for t near t_1, we deduce from the second equation in (3.13) that $\frac{\pi}{2} < \theta^{sl}(t) < \theta_2$ and $\dot{\theta}^{sl}(t) < 0$ for every $t \in (t_1, +\infty)$. It follows that there exists $r(\theta)$ such that $\rho^{sl}(t) = r(\theta^{sl}(t))$ for every $t \in (t_1, +\infty)$ and that $r(\theta)$ satisfies (3.1). Since $r(\theta^{sl}(t_1^k)) \to z_2$ by (3.22), we conclude that $r(\theta) = r_2(\theta)$ in a left neighbourhood of θ_2. This implies that $\theta^{sl}(t)$ satisfies (3.28). By (3.22) θ^{sl} and $\theta_\sharp$ satisfy the same Cauchy condition at t_1, therefore $\theta^{sl} = \theta_\sharp$ in a right neighbourhood of t_1. Since $\rho^{sl}(t) = r(\theta^{sl}(t))$, $r(\theta) = r_2(\theta)$, and $\rho_\sharp(t) := r_2(\theta_\sharp(t))$, we conclude that $\rho^{sl}(t) = \rho_\sharp(t)$ in a right neighbourhood of t_1. The equality is extended to all $t \in (t_1, +\infty)$ by uniqueness. □

4. Behaviour near the separation line

In this section we prove two technical lemmas which describe the behaviour of the solutions of the system near the points $(z_s(\theta), \theta, z_s(\theta))$, $\frac{\pi}{2} < \theta \leq \theta_c$, which correspond to the separation line $z = z_s(\theta)$ defined by (3.4).

4.1. **Behaviour near the critical point.**

In this subsection we study the behaviour of the system (2.21) near the point (z_c, θ_c, z_c), where θ_c and z_c are the constants defined in (3.3) and (3.5). Let $w_\varepsilon(t)$ be the function defined in (2.26).

Lemma 4.1. *Let* $\kappa \geq 1$*, let* $t_1 \in [t_0, +\infty)$*, and let* τ_δ *be a sequence in* $[t_0, +\infty)$*. Assume that*

$$|\tau_\delta - t_1| \leq \delta \,, \tag{4.1}$$

$$|\rho_\varepsilon(\tau_\delta) - z_c| + |\theta_\varepsilon(\tau_\delta) - \theta_c| + |z_\varepsilon(\tau_\delta) - z_c| \leq \delta \,, \tag{4.2}$$

for ε small enough. Then there exist three constants $\beta_1 > 0$, $\beta_2 > 0$, and $\delta_0 \in (0,1)$, a sequence ε_δ in $(0,+\infty)$, defined for $\delta \in (0,\delta_0)$, and a double sequence τ^δ_ε in $[t_0,+\infty)$, defined for $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,\varepsilon_\delta)$, such that

$$t_1 - \delta \le \tau_\delta \le \tau^\delta_\varepsilon \le t_1 + \beta_1\,\delta\,, \tag{4.3}$$

$$w_\varepsilon(\tau^\delta_\varepsilon) \ge 0\,, \tag{4.4}$$

$$\theta_\varepsilon(\tau^\delta_\varepsilon) \le \theta_c - \kappa\,\delta\,, \tag{4.5}$$

$$\sup_{\tau_\delta \le t \le \tau^\delta_\varepsilon} \big(|\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c|\big) \le \beta_2\sqrt{\delta}\,, \tag{4.6}$$

for every $\delta \in (0,\delta_0)$ and every $\varepsilon \in (0,\varepsilon_\delta)$.

Proof. We begin by observing that $1+\cos\theta_c - 3\cos^2\theta_c = 0$ and $1+3\cos\theta_c < 0$. Let us fix four constants a_0, b_0, c_0, d_0 such that

$$\begin{gathered}
0 < a_0 < (1+\cos\theta_c)(1+3\cos\theta_c)\cos\theta_c\sin\theta_c < 1\,,\\
0 < b_0 < -(1+\cos\theta_c)\cos\theta_c\sin\theta_c < 1\,,\\
0 < c_0 < (1+\cos\theta_c)^3\cos^2\theta_c < 1\,,\\
0 < d_0 < -z_c(1+\cos\theta_c)^3\cos\theta_c\,.
\end{gathered}$$

By continuity there exists $\eta > 0$ such that

$$\begin{gathered}
\eta < \tfrac12\,(\theta_c - \tfrac{\pi}{2}) < \tfrac12 < \tfrac12\,z_c\,,\\
-\rho < z^2(1+\cos\theta)^3\ \cos\theta\,(1+\cos\theta - 3\cos^2\theta) < \rho\,,\\
a_0\,\rho < z(1+\cos\theta)(1+3\cos\theta)\cos\theta\sin\theta < \rho\,,\\
b_0\,\rho < -z(1+\cos\theta)\cos\theta\sin\theta < \rho\,,\\
c_0\,\rho < z(1+\cos\theta)^3\cos^2\theta < \rho.\\
d_0\,\rho < -z^2(1+\cos\theta)^3\cos\theta\,,
\end{gathered} \tag{4.7}$$

for $|\theta-\theta_c| \le \eta$, $|\rho - z_c| \le \eta$, and $|z - z_c| \le \eta$.

Since the result has to be proved only for sufficiently small δ, we may also assume that

$$\delta < \tfrac18\,, \qquad \delta < \eta\,, \qquad 2\,\delta < \kappa\,. \tag{4.8}$$

We define

$$t^\delta_\varepsilon := \inf\{t \in (\tau_\delta,+\infty) : \theta_\varepsilon(t) < \theta_c - \kappa\,\delta\}\,, \tag{4.9}$$

$$\alpha^{\delta,\eta}_\varepsilon := \inf\{t \in (\tau_\delta,+\infty) : |\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c| > \eta\}\,, \tag{4.10}$$

$$s^{\delta,\eta}_\varepsilon := \min\{t^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon\}\,, \tag{4.11}$$

From (4.7) we obtain that

$$-\rho_\varepsilon(t) < z_\varepsilon(t)^2(1{+}\cos\theta_\varepsilon(t))^3\cos\theta_\varepsilon(t)(1{+}\cos\theta_\varepsilon(t){-}3\cos^2\theta_\varepsilon(t)) < \rho_\varepsilon(t)\,, \tag{4.12}$$

$$a_0\,\rho_\varepsilon(t) < z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t)\,(1+3\cos\theta_\varepsilon(t))\,\cos\theta_\varepsilon(t)\,\sin\theta_\varepsilon(t) < \rho_\varepsilon(t)\,, \tag{4.13}$$

$$b_0\,\rho_\varepsilon(t) < -z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\sin\theta_\varepsilon(t) < \rho_\varepsilon(t)\,, \tag{4.14}$$

$$c_0\,\rho_\varepsilon(t) < z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))^3\cos^2\theta_\varepsilon(t) < \rho_\varepsilon(t)\,, \tag{4.15}$$

$$d_0\,\rho_\varepsilon(t) < -z_\varepsilon(t)^2(1+\cos\theta_\varepsilon(t))^3\cos\theta_\varepsilon(t) \tag{4.16}$$

for every $t \in [\tau_\delta, \alpha^{\delta,\eta}_\varepsilon]$. Therefore (2.27) and (4.7) give

$$\varepsilon\dot w_\varepsilon(t) \le -\varepsilon a_0 + (\rho_\varepsilon(t){-}z_\varepsilon(t)) + (\rho_\varepsilon(t){-}z_\varepsilon(t))^2 \le -\varepsilon a_0 + 2(\rho_\varepsilon(t){-}z_\varepsilon(t))\,, \tag{4.17}$$

$$\varepsilon\dot w_\varepsilon(t) \ge -\varepsilon - (\rho_\varepsilon(t){-}z_\varepsilon(t)) + c_0(\rho_\varepsilon(t){-}z_\varepsilon(t))^2 \ge -\varepsilon - (\rho_\varepsilon(t){-}z_\varepsilon(t))\,, \tag{4.18}$$

for every $t \in [\tau_\delta, \alpha^{\delta,\eta}_\varepsilon]$.

Using the second equation in (2.21) we deduce from (4.14) that

$$\varepsilon\,\dot\theta_\varepsilon(t) \le -b_0\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau_\delta, \alpha^{\delta,\eta}_\varepsilon]\,.$$

From (4.17) and (4.18) we obtain

$$-1+\tfrac{1}{b_0}\dot\theta_\varepsilon(t)\le \dot w_\varepsilon(t)\le -a_0-\tfrac{2}{b_0}\dot\theta_\varepsilon(t) \quad \text{for every } t\in[\tau_\delta,\alpha_\varepsilon^{\delta,\eta}].$$

Integrating we get

$$\begin{aligned} w_\varepsilon(t)-w_\varepsilon(\tau_\delta)&\ge -(t-\tau_\delta)+\tfrac{1}{b_0}(\theta_\varepsilon(t)-\theta_\varepsilon(\tau_\delta))\,,\\ w_\varepsilon(t)-w_\varepsilon(\tau_\delta)&\le -a_0(t-\tau_\delta)-\tfrac{2}{b_0}(\theta_\varepsilon(t)-\theta_\varepsilon(\tau_\delta))\,, \end{aligned} \tag{4.19}$$

for every $t\in[\tau_\delta,\alpha_\varepsilon^{\delta,\eta}]$.

Since $z_c\,(1+\cos\theta_c)^2\cos\theta_c+1=0$, an elementary estimate of the first derivatives leads to the inequality $|z\,(1+\cos\theta)^2\cos\theta+1|\le|z-z_c|+8\,|\theta-\theta_c|$ for $|z-z_c|<\frac12$, so that (4.2) and (4.8) give

$$|w_\varepsilon(\tau_\delta)|\le 8\,\delta \tag{4.20}$$

for ε small enough. By (2.32), (4.2), and (4.9) we have

$$\theta_c-\kappa\,\delta\le\theta_\varepsilon(t)\le\theta_\varepsilon(\tau_\delta)\le\theta_c+\delta\le\theta_c+\kappa\,\delta \tag{4.21}$$

for every $t\in[\tau_\delta,t_\varepsilon^\delta]$, so that (4.19) gives

$$w_\varepsilon(t)\ge -8\,\delta-(t-\tau_\delta)-\tfrac{2\,\kappa}{b_0}\,\delta\,, \tag{4.22}$$

$$w_\varepsilon(t)\le 8\,\delta-a_0(t-\tau_\delta)+\tfrac{4\,\kappa}{b_0}\,\delta\,, \tag{4.23}$$

for every $t\in[\tau_\delta,s_\varepsilon^{\delta,\eta}]$.

Let

$$\hat\tau_\delta:=\tau_\delta+\kappa_1\,\delta\,,\qquad \text{where}\qquad \kappa_1:=\tfrac{9}{a_0}+\tfrac{4\,\kappa}{a_0b_0}\,. \tag{4.24}$$

Let us show, that

$$s_\varepsilon^{\delta,\eta}\le\hat\tau_\delta+2\delta. \tag{4.25}$$

Suppose, by contradiction, that $\hat\tau_\delta+2\delta<s_\varepsilon^{\delta,\eta}$. Then by (4.23) we have $w_\varepsilon(t)\le-\delta$ for every $t\in[\hat\tau_\delta,s_\varepsilon^{\delta,\eta}]$. Hence, (2.25) and (2.31) imply

$$\varepsilon(\dot\rho_\varepsilon(t)-\dot z_\varepsilon(t))\ge\varepsilon\sin\theta_0+\delta(\rho_\varepsilon(t)-z_\varepsilon(t))\quad\text{for every } t\in[\hat\tau_\delta,s_\varepsilon^{\delta,\eta}].$$

By comparison with the solution of the equation we obtain

$$\rho_\varepsilon(t)-z_\varepsilon(t)\ge\tfrac{\varepsilon}{\delta}\sin\theta_0\big(\exp(\tfrac{\delta}{\varepsilon}(t-\hat\tau_\delta))-1\big)\quad\text{for every } t\in[\hat\tau_\delta,s_\varepsilon^{\delta,\eta}]. \tag{4.26}$$

In particular, we have

$$\rho_\varepsilon(t)-z_\varepsilon(t)\ge\tfrac{\varepsilon}{\delta}\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\quad\text{for every } t\in[\hat\tau_\delta+\delta,s_\varepsilon^{\delta,\eta}],$$

so that (4.18) gives

$$\dot w_\varepsilon(t)\ge-1+\tfrac{1}{\delta}\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\big[-1+c_0\tfrac{\varepsilon}{\delta}\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\big]\quad\text{for every } t\in[\hat\tau_\delta+\delta,s_\varepsilon^{\delta,\eta}].$$

For ε small enough we have $-1+c_0\frac{\varepsilon}{\delta}\sin\theta_0\big(\exp(\frac{\delta^2}{\varepsilon})-1\big)>1$, hence

$$\dot w_\varepsilon(t)\ge-1+\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\quad\text{for every } t\in[\hat\tau_\delta+\delta,s_\varepsilon^{\delta,\eta}].$$

Integrating, we obtain

$$w_\varepsilon(t)\ge w_\varepsilon(\hat\tau_\delta+\delta)-(t-\hat\tau^\delta-\delta)+\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)(t-\hat\tau_\delta-\delta)\,, \tag{4.27}$$

for every $t\in[\hat\tau_\delta+\delta,s_\varepsilon^{\delta,\eta}]$. By using (4.22) we get $w_\varepsilon(\hat\tau_\delta+\delta)\ge-\kappa_2\,\delta$, with $\kappa_2:=9+\kappa_1+\frac{2\kappa}{b_0}$, so that (4.27) gives

$$w_\varepsilon(t)\ge-\kappa_2\,\delta+\big[-1+\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\big](t-\hat\tau_\delta-\delta)$$

for every $t\in[\hat\tau_\delta+\delta,s_\varepsilon^{\delta,\eta}]$. Using (4.23) for $t=\hat\tau_\delta+2\delta$, we obtain

$$\sin\theta_0\big(\exp(\tfrac{\delta^2}{\varepsilon})-1\big)\le 9+\kappa_2+\tfrac{4\,\kappa}{b_0}\,,$$

which leads to a contradiction for ε small enough. This concludes the proof of (4.25), which, together with (4.24), gives

$$s_\varepsilon^{\delta,\eta} \le \tau_\delta + (\kappa_1 + 2)\,\delta\,. \tag{4.28}$$

From (4.21) we have

$$|\theta_\varepsilon(t) - \theta_c| \le \kappa\,\delta \quad \text{for every } t \in [\tau_\delta, s_\varepsilon^{\delta,\eta}]\,. \tag{4.29}$$

From (4.22), (4.23), (4.28) if follows that

$$|w_\varepsilon(t)| \le \kappa_3\,\delta \quad \text{for every } t \in [\tau_\delta, s_\varepsilon^{\delta,\eta}]\,, \tag{4.30}$$

where $\kappa_3 := \kappa_1 + 3 + \frac{4\,\kappa}{b_0}$. Since the function

$$(\omega, \theta) \mapsto \frac{\omega - 1}{(1+\cos\theta)^2\cos\theta}$$

is Lipschitz continuous in $\mathbb{R}\times[\theta_c - \eta, \theta_c + \eta]$ and takes the value z_c at $(0, \theta_c)$ by the very definition of z_c (see (3.6)), there exists a constant $L \ge 1$ such that

$$|z_\varepsilon(t) - z_c| \le L(|w_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta_c|) \qquad \text{for every } t \in [\tau_\delta, \alpha_\varepsilon^{\delta,\eta}]\,. \tag{4.31}$$

By (4.29), (4.30), and (4.31) we have

$$|z_\varepsilon(t) - z_c| \le \kappa_4\,\delta \quad \text{for every } t \in [\tau_\delta, s_\varepsilon^{\delta,\eta}]\,. \tag{4.32}$$

where $\kappa_4 := L\,(\kappa_3 + \kappa)$.

By Lemmas 2.6 and 2.8 we have

$$z_\varepsilon(t) \le \rho_\varepsilon(t) \le \rho_\varepsilon(\tau_\delta) + \varepsilon \quad \text{for every } t \in [\tau_\delta, +\infty)\,,$$

so that for ε small enough (4.2) and (4.32) give

$$z_c - \kappa_4\,\delta \le \rho_\varepsilon(t) \le z_c + \delta + \varepsilon \le z_c + 2\delta \quad \text{for every } t \in [\tau_\delta, s_\varepsilon^{\delta,\eta}]\,,$$

which implies

$$|\rho_\varepsilon(t) - z_c| \le \kappa_4\,\delta \quad \text{for every } t \in [\tau_\delta, s_\varepsilon^{\delta,\eta}]\,, \tag{4.33}$$

Taking into account (4.10) and (4.11), if

$$\kappa_4\,\delta < \eta\,, \tag{4.34}$$

from (4.29), (4.32), and (4.33) we obtain $s_\varepsilon^{\delta,\eta} < \alpha_\varepsilon^{\delta,\eta}$, hence

$$s_\varepsilon^{\delta,\eta} = t_\varepsilon^\delta\,. \tag{4.35}$$

Therefore (4.28) yields

$$t_\varepsilon^\delta \le \tau_\delta + (\kappa_1 + 2)\,\delta\,, \tag{4.36}$$

which implies

$$\theta_\varepsilon(t_\varepsilon^\delta) \le \theta_c - \kappa\,\delta\,. \tag{4.37}$$

By (2.32) and (2.32)

we have

$$\tfrac{\pi}{2} < \theta_\varepsilon(t) \le \theta_c - \kappa\,\delta \qquad \text{for every } t \in [t_\varepsilon^\delta, +\infty)\,. \tag{4.38}$$

Since the function $\theta \mapsto 1 + \cos\theta - 3\cos^2\theta$ is concave on $[\frac{\pi}{2}, \theta_c]$, vanishes at θ_c, and takes the value 1 at $\frac{\pi}{2}$, using the inequality $\theta_c - \frac{\pi}{2} < \frac{1}{2}$ we obtain we have

$$1 + \cos\theta - 3\cos^2\theta \ge 2\,(\theta_c - \theta) \qquad \text{for every } \theta \in [\tfrac{\pi}{2}, \theta_c]\,. \tag{4.39}$$

It follows from (4.38) that

$$1 + \cos\theta_\varepsilon(t) - 3\cos^2\theta_\varepsilon(t) \ge 2\,(\theta_c - \theta_\varepsilon(t)) \ge 2\,\kappa\,\delta \quad \text{for every } t \in [t_\varepsilon^\delta, +\infty)\,. \tag{4.40}$$

Let us define

$$\tau_\varepsilon^\delta := \inf\{t \in (t_\varepsilon^\delta, +\infty) : w_\varepsilon(t) > 0\}\,, \tag{4.41}$$

$$\sigma_\varepsilon^{\delta,\eta} := \min\{\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta,\eta}\}\,. \tag{4.42}$$

From (2.27), (4.13), and (4.16) we obtain

$$\varepsilon\,\dot w_\varepsilon(t) \ge -\varepsilon + 2\,d_0\,\kappa\,\delta\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,. \tag{4.43}$$

Since $w_\varepsilon(t) \le 0$ for every $t \in [t^\delta_\varepsilon, \tau^\delta_\varepsilon]$, using (2.25) and (2.32) we get

$$\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t) \ge \sin\theta_0 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

hence

$$\rho_\varepsilon(t) - z_\varepsilon(t) \ge \rho_\varepsilon(t^\delta_\varepsilon) - z_\varepsilon(t^\delta_\varepsilon) + \sin\theta_0\,(t - t^\delta_\varepsilon) \ge \sin\theta_0\,(t - t^\delta_\varepsilon) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

where the last inequality follows from Lemma 2.6. Using (4.43) we obtain

$$\varepsilon\,\dot w_\varepsilon(t) \ge -\varepsilon + 2\,d_0\,\kappa\,\sin\theta_0\,\delta\,(t - t^\delta_\varepsilon) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

which gives

$$w_\varepsilon(t) \ge w_\varepsilon(t^\delta_\varepsilon) - (t - t^\delta_\varepsilon) + \kappa_5\,\tfrac{\delta}{\varepsilon}\,(t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

with $\kappa_5 := d_0\,\kappa\,\sin\theta_0$. Using (4.30) we obtain

$$w_\varepsilon(t) \ge -\kappa_3\,\delta - (t - t^\delta_\varepsilon) + \kappa_5\,\tfrac{\delta}{\varepsilon}\,(t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

hence

$$w_\varepsilon(t) \ge -\kappa_3\,\delta - \tfrac{2}{\kappa_5}\,\tfrac{\varepsilon}{\delta} + \tfrac{\kappa_5}{2}\,\tfrac{\delta}{\varepsilon}\,(t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,. \tag{4.44}$$

Since $w_\varepsilon(t) \le 0$ for every $t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]$, this implies

$$(\sigma^{\delta,\eta}_\varepsilon - t^\delta_\varepsilon)^2 \le \tfrac{2\,\kappa_3}{\kappa_5}\,\varepsilon + \tfrac{4}{\kappa_5^2\,\delta^2}\,\varepsilon^2\,,$$

so that

$$\sigma^{\delta,\eta}_\varepsilon - t^\delta_\varepsilon \le \delta \tag{4.45}$$

for ε small enough.

Using the second equation in (2.21) we deduce from (4.7) and (4.14) that

$$\varepsilon\,\dot\theta_\varepsilon(t) \ge -\tfrac{2}{z_c}\,\varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau_\delta, \alpha^{\delta,\eta}_\varepsilon]\,.$$

From (2.27), (4.13), (4.16), and (4.40) we obtain

$$\varepsilon\,\dot w_\varepsilon(t) \ge -\varepsilon + 2\,d_0\,(\theta_c - \theta_\varepsilon(t))\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [t^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]\,.$$

As $|\theta_c - \theta_\varepsilon(t)| < \eta$ for every $t \in [\tau_\delta, \alpha^{\delta,\eta}_\varepsilon]$, from the last two inequalities we obtain

$$\dot w_\varepsilon(t) \ge -1 - 2\,d_0\,(\theta_c - \theta_\varepsilon(t))\,\dot\theta_\varepsilon(t) - \tfrac{2\,d_0}{z_c}\,\eta \quad \text{for every } t \in [\,t^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]\,.$$

Let $\varphi_\varepsilon(t) := (\theta_c - \theta_\varepsilon(t))^2$. The previous inequality gives

$$\dot w_\varepsilon(t) \ge -a_1 + d_0\,\dot\varphi_\varepsilon(t) - \tfrac{2\,d_0}{z_c}\,\eta \quad \text{for every } t \in [\,t^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]\,,$$

so that

$$w_\varepsilon(t) - w_\varepsilon(t^\delta_\varepsilon) \ge -a_1(t - t^\delta_\varepsilon) + d_0\,(\varphi_\varepsilon(t) - \varphi_\varepsilon(t^\delta_\varepsilon)) - \tfrac{2\,d_0}{z_c}\,\eta\,(t - t^\delta_\varepsilon) \text{ for every } t \in [\,t^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]\,.$$

Since $w_\varepsilon(t) \le 0$ for every $t \in [t^\delta_\varepsilon, \tau^\delta_\varepsilon]$, the previous inequality, together with (4.30), (4.35), (4.45), and (4.37) gives

$$\varphi_\varepsilon(t) \le \kappa_6\,\delta \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,,$$

with $\kappa_6^2 := \kappa^2 + \frac{1}{d_0}\,(\kappa_3 + a_1 + \frac{2\,d_0}{z_c}\,\eta)$. It follows that

$$|\theta_\varepsilon(t) - \theta_c| \le \kappa_6\,\sqrt{\delta} \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,. \tag{4.46}$$

Since $w_\varepsilon(t) \le 0$ for every $t \in [t^\delta_\varepsilon, \tau^\delta_\varepsilon]$, for ε small enough we obtain from (4.44)

$$|w_\varepsilon(t)| \le (\kappa_3 + 1)\,\delta \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,. \tag{4.47}$$

These inequalities, together with (4.8) and (4.31), imply that

$$|z_\varepsilon(t) - z_c| \le \kappa_7\sqrt{\delta} \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta,\eta}_\varepsilon]\,. \tag{4.48}$$

where $\kappa_7 := L\,(\kappa_3 + \kappa_6 + 1)$.

By Lemmas 2.6 and 2.8 we have

$$z_\varepsilon(t) \le \rho_\varepsilon(t) \le \rho_\varepsilon(\tau_\varepsilon^\delta) + \varepsilon \quad \text{for every } t \in [\tau_\varepsilon^\delta, +\infty)\,,$$

so that for ε small enough (4.54) and (4.48) give

$$z_c - \kappa_7\sqrt{\delta} \le \rho_\varepsilon(t) \le z_c + \delta + \varepsilon \le z_c + 2\delta \quad \text{for every } t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]\,,$$

which implies

$$|\rho_\varepsilon(t) - z_c| \le \kappa_7\sqrt{\delta} \quad \text{for every } t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]\,. \tag{4.49}$$

There exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ inequalities (4.8) and (4.34) are satisfied and

$$\kappa_7\sqrt{\delta} < \eta\,.$$

It follows from (4.46), (4.48), and (4.49) that $\sigma_\varepsilon^{\delta,\eta} < \alpha_\varepsilon^{\delta,\eta}$ for ε small enough, hence

$$\sigma_\varepsilon^{\delta,\eta} = \tau_\varepsilon^\delta\,, \tag{4.50}$$

which implies $w_\varepsilon(\tau_\varepsilon^\delta) \ge 0$. This proves (4.4) for ε small enough.

Inequality (4.5) follows from (4.38). If $\delta \in (0, \delta_0)$ we have (4.35) and (4.50) for ε small enough, so that (4.3) follows from (4.28) and (4.45), with $\beta_1 := \kappa_1 + 3$, while (4.6) follows from (4.29), (4.32), (4.33), (4.46), (4.48), and (4.49), with $\beta_2 := 3\,\kappa_4 + 3\,\kappa_7$. ☐

4.2. **Behaviour near the left branch of the separation line.** The following lemma will be used to study the behaviour of the system when $\frac{\pi}{2} < \theta_0 \le \theta_c$ and $z_0 = z_s(\theta_0)$, where $z_s(\theta)$ is the function defined in (3.4). Note that (4.53) is always satisfied when $\theta_1 < \theta_c$ and δ is small.

Lemma 4.2. *Let $t_1 \ge t_0$, $\frac{\pi}{2} < \theta_1 \le \theta_c$, $z_1 = z_s(\theta_1)$, $\kappa_1 > 0$, and $\delta_0 \in (0,1)$. For every $\delta \in (0, \delta_0)$ let $\varepsilon_\delta \in (0, +\infty)$, and for every $\varepsilon \in (0, \varepsilon_\delta)$ let $\tau_\varepsilon^\delta \in [t_0, +\infty)$. Assume that for every $\delta \in (0, \delta_0)$ and every $\varepsilon \in (0, \varepsilon_\delta)$*

$$|\tau_\varepsilon^\delta - t_1| \le \delta\,, \tag{4.51}$$

$$w_\varepsilon(\tau_\varepsilon^\delta) \ge 0\,, \tag{4.52}$$

$$\theta_\varepsilon(\tau_\varepsilon^\delta) \le \theta_c - \kappa_1\,\delta\,, \tag{4.53}$$

$$|\rho_\varepsilon(\tau_\varepsilon^\delta) - z_1| + |\theta_\varepsilon(\tau_\varepsilon^\delta) - \theta_1| + |z_\varepsilon(\tau_\varepsilon^\delta) - z_1| \le \sqrt{\delta}\,. \tag{4.54}$$

Then there exist $\delta_1 > 0$, a sequence $\hat\varepsilon_\delta \in (0, +\infty)$, defined for $\delta \in (0, \delta_1)$, a double sequence $t_\varepsilon^\delta \in [t_0, +\infty)$, defined for $\delta \in (0, \delta_1)$ and $\varepsilon \in (0, \hat\varepsilon_\delta)$, and two constants $\gamma_1 > 0$ and $\gamma_2 > 0$, such that

$$t_1 - \delta \le \tau_\varepsilon^\delta \le t_\varepsilon^\delta \le t_1 + 2\,\delta\,, \tag{4.55}$$

$$w_\varepsilon(t_\varepsilon^\delta) \ge \delta^2\,, \tag{4.56}$$

$$|\rho_\varepsilon(t_\varepsilon^\delta) - z_\varepsilon(t_\varepsilon^\delta)| \le \gamma_1\,\tfrac{1}{\delta^2}\,\varepsilon\,, \tag{4.57}$$

$$\sup_{\tau_\varepsilon^\delta \le t \le t_\varepsilon^\delta} \big(|\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1|\big) \le \gamma_2\sqrt{\delta}\,, \tag{4.58}$$

for every $\delta \in (0, \delta_1)$ and every $\varepsilon \in (0, \hat\varepsilon_\delta)$.

Proof. Since $z_1 = z_s(\theta_1)$, we have $z_1\,(1 + \cos\theta_1)^2\cos\theta_1 + 1 = 0$. An elementary estimate of the first derivatives leads to the inequality $|z\,(1+\cos\theta)^2\cos\theta + 1| \le |z - z_1| + 8\,|\theta - \theta_1|$ for $|z - z_1| < \frac{1}{2}$, so that (4.54) gives

$$|w_\varepsilon(\tau_\varepsilon^\delta)| \le 8\sqrt{\delta} \tag{4.59}$$

for ε small enough. By (2.32), (2.32) , and (4.53) we have

$$\tfrac{\pi}{2} < \theta_\varepsilon(t) \le \theta_c - \kappa_1\,\delta \qquad \text{for every } t \in [\tau_\varepsilon^\delta, +\infty)\,. \tag{4.60}$$

It follows from (4.46) that

$$1 + \cos\theta_\varepsilon(t) - 3\cos^2\theta_\varepsilon(t) \ge 2\,(\theta_c - \theta_\varepsilon(t)) \ge 2\,\kappa_1\,\delta \quad \text{for every } t \in [\tau_\varepsilon^\delta, +\infty)\,. \tag{4.61}$$

Let us define

$$d_0 := -\tfrac{1}{2} z_1 (1+\cos\theta_1)^3 \cos\theta_1 \,. \tag{4.62}$$

Since

$$\begin{gathered} z_1\,(1+\cos\theta_1)^3\,\cos\theta_1\,(1+\cos\theta_1 - 3\cos^2\theta_1) < z_1\,, \\ -(1+\cos\theta_1)\cos\theta_1\sin\theta_1 < 1\,, \\ 0 < d_0 < -z_1(1+\cos\theta_1)^3\cos\theta_1\,, \end{gathered}$$

by continuity there exists $\eta > 0$ such that

$$2\,\eta < z_1 \quad \text{and} \quad 2\,\eta < \theta_1 - \tfrac{\pi}{2}\,, \tag{4.63}$$

$$z\,(1+\cos\theta)^3\,\cos\theta\,\big[z\,(1+\cos\theta-3\cos^2\theta) - (\rho - z)\cos\theta\big] < z_1\,\rho \tag{4.64}$$

$$-z(1+\cos\theta)\cos\theta\sin\theta < \rho\,, \tag{4.65}$$

$$d_0\,\rho < -z^2(1+\cos\theta)^3\cos\theta\,, \tag{4.66}$$

for $|\theta-\theta_1| \le \eta$, $|\rho - z_1| \le \eta$, and $|z - z_1| \le \eta$. Moreover,

$$-2\rho \le z\,(1+\cos\theta)\,(1+3\cos\theta)\cos\theta\sin\theta \le 2\rho \tag{4.67}$$

for every $0 < z \le \rho$ and every $\theta \in [\frac{\pi}{2}, \pi]$.

We set

$$\lambda := 3 + 4z_1 \qquad \text{and} \qquad \gamma_1 := \tfrac{6}{d_0\kappa_1}\,. \tag{4.68}$$

Since the result has to be proved only for sufficiently small δ, we may also assume that

$$\delta < \tfrac{1}{\lambda} < \tfrac{1}{3}\,, \qquad \delta < \eta < \tfrac{1}{2} z_1\,, \qquad \delta < \gamma_1\,, \qquad \delta < \tfrac{d_0\kappa_1\sin\theta_0}{2}\,. \tag{4.69}$$

For every $\varepsilon > 0$ and $\delta > 0$ we define

$$\tilde\tau^\delta_\varepsilon := \inf\{t \in (\tau^\delta_\varepsilon, +\infty) : w_\varepsilon(t) > \delta^2\}\,, \tag{4.70}$$

$$\tilde t^\delta_\varepsilon := \inf\{t \in (\tilde\tau^\delta_\varepsilon, +\infty) : w_\varepsilon(t) > w_\varepsilon(\tau^\delta_\varepsilon) + \lambda\,\delta^2\}\,, \tag{4.71}$$

$$t^\delta_\varepsilon := \inf\{t \in (\tilde\tau^\delta_\varepsilon, +\infty) : \rho_\varepsilon(t) - z_\varepsilon(t) < \gamma_1 \tfrac{1}{\delta^2}\,\varepsilon\}\,, \tag{4.72}$$

$$\alpha^{\delta,\eta}_\varepsilon := \inf\{t \in (\tau^\delta_\varepsilon, +\infty) : |\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1| > \eta\}\,, \tag{4.73}$$

$$\tilde\sigma^{\delta,\eta}_\varepsilon := \min\{\tilde\tau^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon\}\,, \qquad \tilde s^\delta_\varepsilon := \min\{\tilde t^\delta_\varepsilon, t^\delta_\varepsilon, \tau^\delta_\varepsilon + \delta^2\}\,, \qquad \tilde s^{\delta,\eta}_\varepsilon := \min\{\tilde s^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon\}\,. \tag{4.74}$$

Since $z_\varepsilon(t) \le \rho_\varepsilon(t)$ for every $t \in [t_0, +\infty)$ by Lemma 2.6, from (4.67) we obtain that

$$-2\,\rho_\varepsilon(t) \le z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t)\,(1+3\cos\theta_\varepsilon(t))\,\cos\theta_\varepsilon(t)\,\sin\theta_\varepsilon(t) \le 2\,\rho_\varepsilon(t) \tag{4.75}$$

for every $t \in [t_0, +\infty)$. By (4.65) and (4.66) we have

$$-z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\sin\theta_\varepsilon(t) < \rho_\varepsilon(t)\,, \tag{4.76}$$

$$d_0\,\rho_\varepsilon(t) < -z_\varepsilon(t)^2(1+\cos\theta_\varepsilon(t))^3\cos\theta_\varepsilon(t)\,, \tag{4.77}$$

for every $t \in [\tau^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]$. From (2.27). (4.61), (4.75) and (4.77) we obtain

$$\varepsilon\dot w_\varepsilon(t) \ge -2\varepsilon + 2d_0\,(\theta_c - \theta_\varepsilon(t))\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \ge -2\varepsilon + 2d_0\kappa_1\delta\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \tag{4.78}$$

for every $t \in [\tau^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]$.

Since $w_\varepsilon(t) \le \delta^2$ for every $t \in [\tau^\delta_\varepsilon, \tilde\tau^\delta_\varepsilon)$ by (4.70), from (2.25) and (2.32) we obtain that

$$\varepsilon\,(\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t)) \ge \varepsilon\sin\theta_0 - \delta^2(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde\tau^\delta_\varepsilon)\,.$$

By comparison we have

$$\rho_\varepsilon(t) - z_\varepsilon(t) \ge \varepsilon\,\tfrac{\sin\theta_0}{\delta^2}\big[1 - \exp\big(-\tfrac{\delta^2}{\varepsilon}(t - \tau^\delta_\varepsilon)\big)\big] \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde\tau^\delta_\varepsilon)\,,$$

so that (4.78) gives

$$\dot w_\varepsilon(t) \ge -2 + \tfrac{2d_0\kappa_1\sin\theta_0}{\delta}\,\big[1 - \exp\big(-\tfrac{\delta^2}{\varepsilon}(t - \tau^\delta_\varepsilon)\big)\big] \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde\sigma^{\delta,\eta}_\varepsilon)\,. \tag{4.79}$$

By (4.69) we have $-2+\frac{2d_0\kappa_1\sin\theta_1}{\delta}>2$. Integrating (4.79) and using the definition of $\tilde\tau^\delta_\varepsilon$ and (4.52) we obtain

$$\delta^2\geq w_\varepsilon(t)\geq 2(t-\tau^\delta_\varepsilon)-\tfrac{2d_0\,\kappa_1\sin\theta_0}{\delta^3}\,\varepsilon\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde\sigma^{\delta,\eta}_\varepsilon)\,. \tag{4.80}$$

This inequality implies

$$\tilde\sigma^{\delta,\eta}_\varepsilon-\tau^\delta_\varepsilon\leq\tfrac12\delta^2+\tfrac{d_0\kappa_1\sin\theta_0}{\delta^3}\,\varepsilon\leq\tfrac23\delta^2\,, \tag{4.81}$$

for ε small enough.

Using the second equation in (2.21) we deduce from (4.69) and (4.76) that

$$\varepsilon\,\dot\theta_\varepsilon(t)\geq-\tfrac{2}{z_1}\,\varepsilon-(\rho_\varepsilon(t)-z_\varepsilon(t))\quad\text{for every } t\in[\tau^\delta_\varepsilon,\alpha^{\delta,\eta}_\varepsilon].$$

As $|\theta_c-\theta_\varepsilon(t)|<\frac{\pi}{2}$ for every $t\in[t_0,+\infty)$ by (2.32) , from (4.78) we obtain

$$\dot w_\varepsilon(t)\geq-2-2\,d_0\,(\theta_c-\theta_\varepsilon(t))\,\dot\theta_\varepsilon(t)-\tfrac{2\,d_0\,\pi}{z_1}\geq-2-2\,d_0\,(\theta_1-\theta_\varepsilon(t))\,\dot\theta_\varepsilon(t)-\tfrac{2\,d_0\,\pi}{z_1}$$

for every $t\in[\tau^\delta_\varepsilon,\alpha^{\delta,\eta}_\varepsilon]$, where the last inequality follows from the inequalities $\dot\theta_\varepsilon(t)<0$ and $\theta_c\geq\theta_1$. Let $\varphi_\varepsilon(t):=(\theta_1-\theta_\varepsilon(t))^2$. The previous inequality gives

$$\dot w_\varepsilon(t)\geq-2+d_0\dot\varphi_\varepsilon(t)-\tfrac{2\,d_0\,\pi}{z_1}\quad\text{for every } t\in[\tau^\delta_\varepsilon,\alpha^{\delta,\eta}_\varepsilon]\,,$$

so that

$$w_\varepsilon(t)-w_\varepsilon(\tau^\delta_\varepsilon)\geq-2(t-\tau^\delta_\varepsilon)+d_0(\varphi_\varepsilon(t)-\varphi_\varepsilon(\tau^\delta_\varepsilon))-\tfrac{2\,d_0\,\pi}{z_1}(t-\tau^\delta_\varepsilon)$$

for every $t\in[\tau^\delta_\varepsilon,\alpha^{\delta,\eta}_\varepsilon]$. By (4.71) and (4.74) we have $w_\varepsilon(t)-w_\varepsilon(\tau^\delta_\varepsilon)\leq\lambda\,\delta^2$ for every $t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]$ and $\tilde s^{\delta,\eta}_\varepsilon-\tau^\delta_\varepsilon\leq\delta^2$. Therefore the previous inequality, together with (4.54), gives

$$\varphi_\varepsilon(t)\leq\kappa_2^2\,\delta^2\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]\,,$$

with $\kappa_2^2:=1+(\frac{\lambda+2}{d_0}+\frac{2\pi}{z_1})$. It follows that

$$|\theta_\varepsilon(t)-\theta_1|\leq\kappa_2\sqrt{\delta}\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]\,. \tag{4.82}$$

Since the function

$$(\omega,\theta)\mapsto\frac{\omega-1}{(1+\cos\theta)^2\cos\theta}$$

is Lipschitz continuous in $\mathbb{R}\times[\theta_1-\eta,\theta_1+\eta]$ and takes the value z_1 at $(0,\theta_1)$ by the hypothesis $z_1=z_s(\theta_1)$, there exists a constant $L\geq1$ such that

$$|z_\varepsilon(t)-z_1|\leq L(|w_\varepsilon(t)|+|\theta_\varepsilon(t)-\theta_1|)\qquad\text{for every } t\in[\tau^\delta_\varepsilon,\alpha^{\delta,\eta}_\varepsilon]\,. \tag{4.83}$$

Since for ε small enough

$$|w_\varepsilon(t)|\leq9\sqrt{\delta}\qquad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon] \tag{4.84}$$

by (4.59), (4.69), (4.73), and (4.74), from (4.82) we obtain

$$|z_\varepsilon(t)-z_1|\leq L\,(9+\kappa_2)\sqrt{\delta}\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]\,. \tag{4.85}$$

By Lemmas 2.6 and 2.8 we have

$$z_\varepsilon(t)\leq\rho_\varepsilon(t)\leq\rho_\varepsilon(\tau^\delta_\varepsilon)+\varepsilon\quad\text{for every } t\in[\tau^\delta_\varepsilon,+\infty)\,,$$

so that for ε small enough (4.54) and (4.85) give

$$z_1-L\,(9+\kappa_2)\sqrt{\delta}\leq\rho_\varepsilon(t)\leq z_1+\delta+\varepsilon\leq z_1+2\delta\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]\,,$$

which implies

$$|\rho_\varepsilon(t)-z_1|\leq L\,(9+\kappa_2)\sqrt{\delta}\quad\text{for every } t\in[\tau^\delta_\varepsilon,\tilde s^{\delta,\eta}_\varepsilon]\,. \tag{4.86}$$

Let $\gamma_2:=\kappa_2+18\,L+2\,L\,\kappa_2$. We choose $\delta_1\in(0,\delta_0)$ such that for every $\delta\in(0,\delta_1)$ inequalities (4.69) are satisfied and $\gamma_2\sqrt{\delta}<\eta$. Taking into account (4.73), from (4.83), (4.85), and (4.86) for every $\delta\in(0,\delta_1)$ we obtain $\tilde s^{\delta,\eta}_\varepsilon<\alpha^{\delta,\eta}_\varepsilon$ for ε small enough, hence

$$\tilde s^{\delta,\eta}_\varepsilon=\tilde s^\delta_\varepsilon<\alpha^{\delta,\eta}_\varepsilon\,. \tag{4.87}$$

By (4.74) and (4.81) we have $\tilde\sigma_\varepsilon^{\delta,\eta} \le \tilde s_\varepsilon^\delta < \alpha_\varepsilon^{\delta,\eta}$, hence $\tilde\sigma_\varepsilon^{\delta,\eta} = \tilde\tau_\varepsilon^\delta$ and

$$\tilde\tau_\varepsilon^\delta \le \tilde s_\varepsilon^\delta \le \tau_\varepsilon^\delta + \tfrac{2}{3}\delta^2 \,. \tag{4.88}$$

By the definition of $\tilde\tau_\varepsilon^\delta$ this implies that

$$w_\varepsilon(\tilde\tau_\varepsilon^\delta) \ge \delta^2 \quad \text{and} \quad w_\varepsilon(\tilde\tau_\varepsilon^\delta) \ge w_\varepsilon(\tau_\varepsilon^\delta)\,. \tag{4.89}$$

By (4.69), (4.73), and (4.78) we have $\dot w_\varepsilon(t) \ge \frac{9}{\delta}$ for every $t \in [\tilde\tau_\varepsilon^\delta, \tilde s_\varepsilon^\delta]$, so that by (4.89)

$$\lambda\,\delta^2 \ge w_\varepsilon(t) - w_\varepsilon(\tau_\varepsilon^\delta) \ge w_\varepsilon(t) - w_\varepsilon(\tilde\tau_\varepsilon^\delta) \ge \tfrac{9}{\delta}(t - \tilde\tau_\varepsilon^\delta) \quad \text{for every } t \in [\tilde\tau_\varepsilon^\delta, \tilde s_\varepsilon^\delta]\,. \tag{4.90}$$

By (4.69) this implies $\tilde s_\varepsilon^\delta - \tilde\tau_\varepsilon^\delta \le \frac{1}{3}\delta^2$, which, together with (4.88), gives

$$\tilde s_\varepsilon^\delta \le \tau_\varepsilon^\delta + \delta^2\,, \tag{4.91}$$

Let us prove that

$$t_\varepsilon^\delta < \tilde s_\varepsilon^\delta \tag{4.92}$$

for ε small enough. We argue by contradiction. If $t_\varepsilon^\delta \ge \tilde s_\varepsilon^\delta$, then (4.91) and the definition of $\tilde s_\varepsilon^\delta$ imply that $\tilde s_\varepsilon^\delta = \tilde t_\varepsilon^\delta$ and $\tilde t_\varepsilon^\delta \le \tilde\tau_\varepsilon^\delta + \delta^2$. Recalling (4.53) we obtain

$$w_\varepsilon(\tilde t_\varepsilon^\delta) = w_\varepsilon(\tau_\varepsilon^\delta) + \lambda\,\delta^2 \ge \lambda\,\delta^2\,.$$

Let $\sigma_\varepsilon^\delta$ be the last time in $[\tau_\varepsilon^\delta, \tilde t_\varepsilon^\delta]$ such that $w_\varepsilon(\sigma_\varepsilon^\delta) = \omega_\varepsilon^\delta := w_\varepsilon(\tau_\varepsilon^\delta) + \delta^2$ and let $\hat\sigma_\varepsilon^\delta$ be the first time in $[\sigma_\varepsilon^\delta, t_\delta^\eta]$ such that $w_\varepsilon(\hat\sigma_\varepsilon^\delta) = \hat\omega_\varepsilon^\delta := w_\varepsilon(\tau_\varepsilon^\delta) + 2\delta^2$. Let us prove that for ε small enough there exists $\hat t_\varepsilon^\delta \in [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]$ such that

$$\rho_\varepsilon(\hat t_\varepsilon^\delta) - z_\varepsilon(\hat t_\varepsilon^\delta) < \sqrt{\varepsilon}\,. \tag{4.93}$$

We argue by contradiction. If $\rho_\varepsilon(t) - z_\varepsilon(t) \ge \sqrt{\varepsilon}$ for every $t \in [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]$, for ε small enough from (4.78) we obtain

$$\dot w_\varepsilon(t) \ge -2 + 2d_0\kappa_1\delta\,\tfrac{1}{\sqrt{\varepsilon}} \ge d_0\kappa_1\delta\,\tfrac{1}{\sqrt{\varepsilon}} \quad \text{for every } t \in [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]\,, \tag{4.94}$$

so that w_ε is increasing on $[\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]$. Therefore there exists a function $u_\varepsilon\colon [\omega_\varepsilon^\delta, \hat\omega_\varepsilon^\delta] \to [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]$ such that

$$\rho_\varepsilon(t) - z_\varepsilon(t) = u_\varepsilon(w_\varepsilon(t)) \quad \text{for every } t \in [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]\,. \tag{4.95}$$

By (2.25) we have

$$\varepsilon(\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t)) \le \varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t))\,w_\varepsilon(t) \quad \text{for every } t \in [\sigma_\varepsilon^\delta, \tilde t_\varepsilon^\delta]\,. \tag{4.96}$$

From (4.94) and (4.96) we obtain

$$u_\varepsilon'(\omega) \le \tfrac{1}{d_0\kappa_1\delta}\sqrt{\varepsilon} - \tfrac{1}{d_0\kappa_1}\tfrac{1}{\sqrt{\varepsilon}}\,u_\varepsilon(\omega)\,\omega \quad \text{for every } \omega \in [\omega_\varepsilon^\delta, \hat\omega_\varepsilon^\delta]$$

By comparison with the solution of the equation we obtain

$$\begin{aligned} u_\varepsilon(\omega) \le{}& (\rho_\varepsilon(\sigma_\varepsilon^\delta) - z_\varepsilon(\sigma_\varepsilon^\delta))\exp\big(-\tfrac{1}{2d_0\kappa_1}\tfrac{1}{\sqrt{\varepsilon}}(\omega - \omega_\varepsilon^\delta)^2\big) + \\ &+ \tfrac{1}{d_0\kappa_1\delta}\sqrt{\varepsilon}\int_0^{\omega-\omega_\varepsilon^\delta}\exp\big(-\tfrac{1}{2d_0\kappa_1}\tfrac{1}{\sqrt{\varepsilon}}((\omega - \omega_\varepsilon^\delta)^2 - s^2)\big)\,ds \end{aligned}$$

for every $\omega \in [\omega_\varepsilon^\delta, \hat\omega_\varepsilon^\delta]$. For $\omega = \hat\omega_\varepsilon^\delta$ we obtain from (4.58) and (4.95)

$$\rho_\varepsilon(\hat\sigma_\varepsilon^\delta) - z_\varepsilon(\hat\sigma_\varepsilon^\delta) = u_\varepsilon(\hat\omega_\varepsilon^\delta) \le \gamma_3\delta\exp\big(-\tfrac{\delta^2}{2d_0\kappa_1}\tfrac{1}{\sqrt{\varepsilon}}\big) + \tfrac{1}{d_0\kappa_1}\sqrt{\varepsilon}\,\mu_\varepsilon\,,$$

with $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Since the right-hand side of this inequality is less than $\sqrt{\varepsilon}$ for ε small enough, we have contradicted the assumption $\rho_\varepsilon(t) - z_\varepsilon(t) \ge \sqrt{\varepsilon}$ for every $t \in [\sigma_\varepsilon^\delta, \hat\sigma_\varepsilon^\delta]$. This concludes the proof of (4.93).

As $w_\varepsilon(t) \ge \delta^2$ for every $t \in [\sigma_\varepsilon^\delta, \tilde t_\varepsilon^\delta]$, from (2.25) we obtain

$$\varepsilon(\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t)) \le \varepsilon - \delta^2(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\sigma_\varepsilon^\delta, \tilde t_\varepsilon^\delta]\,.$$

By comparison with the solution of the equation we get

$$\rho_\varepsilon(t) - z_\varepsilon(t) \le \tfrac{\varepsilon}{\delta^2} + (\rho_\varepsilon(\hat t_\varepsilon^\delta) - z_\varepsilon(\hat t_\varepsilon^\delta))\exp\big(-\tfrac{\delta^2}{\varepsilon}(t - \hat t_\varepsilon^\delta)\big) \quad \text{for every } t \in [\hat t_\varepsilon^\delta, \tilde t_\varepsilon^\delta]\,.$$

By (4.93) we have $\rho_\varepsilon(\hat t^\delta_\varepsilon) - z_\varepsilon(\hat t^\delta_\varepsilon) < \sqrt{\varepsilon}$, so that

$$\rho_\varepsilon(t) - z_\varepsilon(t) < \tfrac{\varepsilon}{\delta^2} + \sqrt{\varepsilon}\exp\big(-\tfrac{\delta^2}{\varepsilon}(t-\hat t^\delta_\varepsilon)\big) \quad \text{for every } t \in [\hat t^\delta_\varepsilon, \tilde t^\delta_\varepsilon]\,. \tag{4.97}$$

By (4.64) we have

$$z_\varepsilon(t)(1+\cos\theta_\varepsilon(t))^3\cos\theta_\varepsilon(t)\big[z_\varepsilon(t)(1+\cos\theta_\varepsilon(t)-3\cos^2\theta_\varepsilon(t))-(\rho_\varepsilon(t)-z_\varepsilon(t))\cos\theta_\varepsilon(t)\big] < z_1\rho_\varepsilon(t)\,,$$

for every $t \in [\tau^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]$, so that (2.27) and (4.75) imply

$$\varepsilon\,\dot w_\varepsilon(t) \le 2\varepsilon + z_1(\rho_\varepsilon(t) - z_\varepsilon(t)) \qquad \text{for every } t \in [\tau^\delta_\varepsilon, \alpha^{\delta,\eta}_\varepsilon]\,.$$

By (4.97) this yields

$$\varepsilon\,\dot w_\varepsilon(t) \le 2\,\varepsilon + z_1\tfrac{\varepsilon}{\delta^2} + z_1\sqrt{\varepsilon}\exp\big(-\tfrac{\delta^2}{\varepsilon}(t-\hat t^\delta_\varepsilon)\big) \quad \text{for every } t \in [\hat t^\delta_\varepsilon, \tilde t^\delta_\varepsilon]\,.$$

Since $w_\varepsilon(\hat t^\delta_\varepsilon) \le w_\varepsilon(\tau^\delta_\varepsilon) + 2\delta^2$, integrating we obtain

$$w_\varepsilon(t) - w_\varepsilon(\tau^\delta_\varepsilon) - 2\delta^2 \le (2+\tfrac{z_1}{\delta^2})\,(t-\hat t^\delta_\varepsilon) + \tfrac{z_1}{\delta^2}\sqrt{\varepsilon}\,\big[1-\exp\big(-\tfrac{\delta^2}{\varepsilon}(t-\hat t^\delta_\varepsilon)\big)\big] \quad \text{for every } t \in [\hat t^\delta_\varepsilon, \tilde t^\delta_\varepsilon]\,.$$

Taking $t = \tilde t^\delta_\varepsilon$ we find that

$$(\lambda-2)\delta^2 \le (2+\tfrac{z_1}{\delta^2})\,(\tilde t^\delta_\varepsilon - \hat t^\delta_\varepsilon) + \tfrac{z_1}{\delta^2}\sqrt{\varepsilon} \le (2+\tfrac{z_1}{\delta^2})\,(\tilde t^\delta_\varepsilon - \hat t^\delta_\varepsilon) + \delta^2$$

for ε small enough. Since $2\,\delta^2 < z_1$ by (4.69), the previous inequality gives $(\lambda-3)\,\delta^2 \le \frac{2\,z_1}{\delta^2}\,(\tilde t^\delta_\varepsilon - \hat t^\delta_\varepsilon)$ for ε small enough, hence $\tilde t^\delta_\varepsilon - \hat t^\delta_\varepsilon \ge \frac{\lambda-3}{2\,z_1}\,\delta^4$. If we apply (4.97) with $t = \tilde t^\delta_\varepsilon$ we obtain

$$\rho_\varepsilon(\tilde t^\delta_\varepsilon) - z_\varepsilon(\tilde t^\delta_\varepsilon) < \tfrac{\varepsilon}{\delta} + \sqrt{\varepsilon}\exp(-\tfrac{2\,\delta^6}{\varepsilon})\,,$$

which gives $\rho_\varepsilon(\tilde t^\delta_\varepsilon) - z_\varepsilon(\tilde t^\delta_\varepsilon) < \frac{\varepsilon}{\delta}$ for ε small enough. By the definition of t^δ_ε this implies $t^\delta_\varepsilon < \tilde t^\delta_\varepsilon$, which violates our hypothesis $t^\delta_\varepsilon \ge \tilde t^\delta_\varepsilon$ and concludes the proof of (4.92).

From (4.89), (4.90) and (4.92) we obtain $w_\varepsilon(t^\delta_\varepsilon) \ge \delta^2$, which proves (4.56). Inequalities (4.55) follow from (4.51), (4.69), (4.91), and (4.92). Inequality (4.57) follows from the definition of t^ε_δ, and (4.58) follows from (4.83), (4.85), and (4.86). ☐

5. Continuous evolution

In this section we consider two cases where the viscosity solution (ρ,θ,z) is continuous. In the first case $0 \le \theta_0 < \frac{\pi}{2}$ and the system exhibits a hardening behaviour by (2.33). In the second case $\frac{\pi}{2} < \theta_0 \le \pi$, so that we have a softening behaviour by (2.34), and we consider an additional condition on z_0 which implies that the viscosity solution (ρ,θ,z) is continuous. We begin by stating the result in the case of hardening, that will be proved in the next subsection.

Theorem 5.1. *Assume that $0 \le \theta_0 < \frac{\pi}{2}$ and let $(\rho^{sl}_0, \theta^{sl}_0)$ be defined as in Lemma 3.5. Then*

$$\rho(t) = z(t) = \rho^{sl}_0(t) \qquad \textit{and} \qquad \theta(t) = \theta^{sl}_0(t) \qquad\qquad \textit{for every } t \in [t_0,+\infty)\,. \tag{5.1}$$

Moreover

$$\sup_{t_0\le t\le\tau}\big(|\rho_\varepsilon(t)-\rho(t)| + |\theta_\varepsilon(t)-\theta(t)| + |z_\varepsilon(t)-z(t)|\big) \to 0 \tag{5.2}$$

for every $\tau \in (t_0,+\infty)$.

We now state the result in the case of softening with continuous evolution, that will be proved in Subsection 5.3. Let θ_c be the constant defined in (3.3), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6).

Theorem 5.2. *Assume one of the following conditions:*

$$\tfrac{\pi}{2} < \theta_0 \le \theta_c \quad \textit{and} \quad z_0 \le z_s(\theta_0)\,, \tag{5.3}$$

$$\theta_c < \theta_0 < \pi \quad \textit{and} \quad z_0 < r_c(\theta_0)\,. \tag{5.4}$$

Let $(\rho_2^{sl},\theta_2^{sl})$ *be defined as in Lemma 3.7 with* $t_1=t_0$, $\theta_2=\theta_0$, *and* $z_2=z_0$. *Then*

$$\rho(t)=z(t)=\rho_2^{sl}(t) \qquad \textit{and} \qquad \theta(t)=\theta_2^{sl}(t) \qquad\qquad \textit{for every } t\in[t_0,+\infty)\,. \tag{5.5}$$

Assume that

$$\theta_c<\theta_0<\pi \quad \textit{and} \quad z_0=r_c(\theta_0)\,. \tag{5.6}$$

Let $(\rho_0^{sl},\theta_0^{sl})$ *and* t_1 *be defined as in Lemma 3.6, and let* $(\rho_2^{sl},\theta_2^{sl})$ *be defined as in Lemma 3.7 with* $\theta_2=\theta_c$ *and* $z_2=z_c$. *Then*

$$\rho(t)=z(t)=\begin{cases}\rho_0^{sl}(t) & \textit{if } t\in[t_0,t_1)\,,\\ z_c & \textit{if } t=t_1\,,\\ \rho_2^{sl}(t) & \textit{if } t\in(t_1,+\infty)\,,\end{cases} \qquad \theta(t)=\begin{cases}\theta_0^{sl}(t) & \textit{if } t\in[t_0,t_1)\,,\\ \theta_c & \textit{if } t=t_1\,,\\ \theta_2^{sl}(t) & \textit{if } t\in(t_1,+\infty)\,.\end{cases} \tag{5.7}$$

In both cases we have

$$\sup_{t_0\le t\le\tau}\big(|\rho_\varepsilon(t)-\rho(t)|+|\theta_\varepsilon(t)-\theta(t)|+|z_\varepsilon(t)-z(t)|\big)\to 0 \tag{5.8}$$

for every $\tau\in(t_0,+\infty)$.

In the proof we shall use the following general result on continuous dependence on a parameter, whose proof can be found in [6] and [5] (see also [1]).

Theorem 5.3. *Let* f_ε *and* f_0 *be Carathéodory functions defined on* $[a,b]\times\mathbb{R}^m$ *with values in* $\mathbb{R}^m$, *let* t_ε, $t_0\in[a,b]$, *and let* x_ε, $x_0\in\mathbb{R}^m$. *Assume that there exist two constants* $L>0$ *and* $M>0$ *such that*

$$|f_\varepsilon(t,x_2)-f_\varepsilon(t,x_1)|\le L\,|x_2-x_1|\,,$$
$$|f_\varepsilon(t,x)|\le M\,,$$

for every $\varepsilon>0$, *every* $t\in[a,b]$, *and every* x, x_1, $x_2\in\mathbb{R}^m$. *Let* $y_\varepsilon(t)$ *and* $y_0(t)$ *be the solutions of the Cauchy problems*

$$\begin{cases}\dot y_\varepsilon(t)=f_\varepsilon(t,y(t))\,,\\ y_\varepsilon(t_\varepsilon)=x_\varepsilon\,,\end{cases} \qquad\qquad \begin{cases}\dot y_0(t)=f_\varepsilon(t,y(t))\,,\\ y_\varepsilon(t_0)=x_0\,.\end{cases}$$

If $t_\varepsilon\to t_0$, $x_\varepsilon\to x_0$, *and for every* $x\in\mathbb{R}^m$

$$\int_a^t f_\varepsilon(s,x)\,ds\to\int_a^t f(s,x)\,ds \qquad \textit{uniformy for } t\in[a,b]\,,$$

then $y_\varepsilon(t)\to y_0(t)$ *uniformly for* $t\in[a,b]$.

5.1. **Hardening.** In this subsection we prove Theorem 5.1 about the hardening regime.

Proof. By Lemma 2.5 we deduce from (2.25) that

$$\varepsilon(\dot\rho_\varepsilon(t)-\dot z_\varepsilon(t))\le\varepsilon-(\rho_\varepsilon(t)-z_\varepsilon(t))^+ \qquad \text{for every } t\in[t_0,+\infty)\,.$$

As $\rho_\varepsilon(t_0)-z_\varepsilon(t_0)=0$, by comparison we obtain that

$$\rho_\varepsilon(t)-z_\varepsilon(t)\le\varepsilon(1-e^{-\frac{1}{\varepsilon}(t-t_0)})\le\varepsilon \qquad \text{for every } t\in[t_0,+\infty)\,. \tag{5.9}$$

Let us define

$$\psi_\varepsilon(t):=\tfrac{1}{\varepsilon}(\rho_\varepsilon(t)-z_\varepsilon(t))\,. \tag{5.10}$$

By Lemma 2.6 and (5.9) we have

$$0\le\psi_\varepsilon(t)\le 1 \qquad \text{for every } t\in[t_0,+\infty)\,. \tag{5.11}$$

Passing to a subsequence, we may assume that $\psi_\varepsilon\rightharpoonup\psi$ weakly* in $L^\infty([t_0,+\infty))$, with $0\le\psi\le 1$ a.e. on $[t_0,+\infty)$.

From (2.21) we obtain

$$\begin{cases} \dot\rho_\varepsilon(t) = \sin\theta_\varepsilon(t) - \psi_\varepsilon(t)\,\big(z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))\cos^2\theta_\varepsilon(t) + 1\big)\,, \\ \rho_\varepsilon(t)\,\dot\theta_\varepsilon(t) = \cos\theta_\varepsilon(t) + \psi_\varepsilon(t)\,z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\sin\theta_\varepsilon(t)\,, \\ \dot z_\varepsilon(t) = \psi_\varepsilon(t)\,z_\varepsilon(t)\,(1+\cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\,. \end{cases} \tag{5.12}$$

By Lemma 2.6 and (2.33) we have $\rho_\varepsilon(t) \geq z_\varepsilon(t) \geq z_0$ for every $[t_0,+\infty)$. Therefore we can apply Theorem 5.3 and we obtain that $\rho_\varepsilon \to \rho$, $\theta_\varepsilon \to \theta$, and $z_\varepsilon \to z$ uniformly on compact subsets of $[t_0,+\infty)$, where (ρ,θ,z) is the solution of the Cauchy problem

$$\begin{cases} \dot\rho(t) = \sin\theta(t) - \psi(t)\,\big(z(t)\,(1+\cos\theta(t))\,\cos^2\theta(t) + 1\big)\,, \\ \rho(t)\,\dot\theta(t) = \cos\theta(t) + \psi(t)\,z(t)\,(1+\cos\theta(t))\cos\theta(t)\sin\theta(t)\,, \\ \dot z(t) = \psi(t)\,z(t)\,(1+\cos\theta(t))\cos\theta(t)\,, \\ \rho(t_0) = z_0\,, \qquad \theta(t_0) = \theta_0\,, \qquad z(t_0) = z_0\,. \end{cases} \tag{5.13}$$

By (2.31), passing to the limit we obtain we have hence

$$0 < \theta_0 \leq \theta(t) \leq \tfrac{\pi}{2} \qquad \text{for every } t \in [t_0,+\infty)\,. \tag{5.14}$$

By Lemma 2.6 and (5.9) $\rho_\varepsilon - z_\varepsilon \to 0$ strongly in $L^\infty([t_0,+\infty))$, hence $\rho(t) = z(t)$ for every $t \in [t_0,+\infty)$. From the first and third equations in (5.13) we obtain

$$\sin\theta(t) = \psi(t)\,\big(z(t)\,(1+\cos\theta(t))^2\cos\theta(t) + 1\big) \qquad \text{for a.e. } t \in [t_0,+\infty)\,.$$

By (5.14) we have $\sin\theta(t) > 0$ for every $t \in [t_0,+\infty)$, hence

$$\psi(t) = \frac{\sin\theta(t)}{\rho(t)\,(1+\cos\theta(t))^2\cos\theta(t) + 1} \qquad \text{for a.e. } t \in [t_0,+\infty)\,. \tag{5.15}$$

It follows that (ρ,θ) satisfies the system of the slow dynamics (3.13) in $[t_0,+\infty)$ with initial conditions (3.14), therefore $(\rho(t),\theta(t)) = (\rho_0^{sl}(t),\theta_0^{sl}(t))$ for every $t \in [t_0,+\infty)$. Since the limit does not depend on the subsequence, we obtain (5.2). ☐

5.2. **Convergence to the slow dynamics.** In this subsection we prove a general result on the convergence of the solutions of (2.21) to the solutions of the system of the slow dynamics. Let $z_s(\theta)$ be the function defined in (3.4).

Lemma 5.4. *Assume that*

$$t_0 \leq t_* < \tau < +\infty\,, \quad \tfrac{\pi}{2} < \theta_* \leq \pi\,, \quad 0 < z_* < z_s(\theta_*)\,. \tag{5.16}$$

Let $(\rho_*^{sl},\theta_*^{sl})$ *be the solution of* (3.13) *with Cauchy conditions*

$$\rho_*^{sl}(t_*) = z_* \quad \textit{and} \quad \theta_*^{sl}(t_*) = \theta_*\,, \tag{5.17}$$

and let t_ε^* *be a sequence in* $[t_0,+\infty)$. *Assume that*

$$\rho_*^{sl}(t) < z_s(\theta_*^{sl}(t)) \quad \textit{for every } t \in [t_*,\tau]\,, \tag{5.18}$$

$$t_\varepsilon^* \to t_*\,, \qquad \rho_\varepsilon(t_\varepsilon^*) \to z_*\,, \qquad \theta_\varepsilon(t_\varepsilon^*) \to \theta_*\,, \qquad z_\varepsilon(t_\varepsilon^*) \to z_*\,, \tag{5.19}$$

$$0 \leq \rho_\varepsilon(t_\varepsilon^*) - z_\varepsilon(t_\varepsilon^*) \leq \kappa\,\varepsilon\,, \tag{5.20}$$

for some $\kappa \geq 0$ *independent of* ε. *Then*

$$\sup_{t_\varepsilon^* \leq t \leq \tau} \big(|\rho_\varepsilon(t) - \rho_*^{sl}(t)| + |\theta_\varepsilon(t) - \theta_*^{sl}(t)| + |z_\varepsilon(t) - \rho_*^{sl}(t)|\big) \to 0\,. \tag{5.21}$$

Proof. For every $\alpha \in \mathbb{R}$ and $\eta > 0$ we define $q_\alpha^\eta \colon \mathbb{R} \to \mathbb{R}$ as the minimum distance projection into the interval $[\alpha-\eta,\alpha+\eta]$, i.e.,

$$q_\alpha^\eta(\beta) := \begin{cases} \alpha - \eta\,, & \text{if } \beta < \alpha - \eta\,, \\ \beta\,, & \text{if } \alpha - \eta \leq \beta \leq \alpha + \eta\,, \\ \alpha + \eta\,, & \text{if } \beta > \alpha + \eta\,. \end{cases} \tag{5.22}$$

Since the inequality in (5.18) is strict, from (3.4) we obtain

$$\rho_*^{sl}(t)\,(1+\cos\theta_*^{sl}(t))^2\cos\theta_*^{sl}(t)+1>0 \quad \text{for every } t\in[t_*,\tau]\,.$$

By continuity there exists $\eta>0$ such that

$$q^\eta_{\rho_*^{sl}(t)}(\rho)\,(1+\cos q^\eta_{\theta_*^{sl}(t)}(\theta))^2\cos q^\eta_{\theta_*^{sl}(t)}(\theta)+1\geq\eta \tag{5.23}$$

for every $t\in[t_*,\tau]$, $\theta\in\mathbb{R}$, $\rho\in\mathbb{R}$. Since $(z_*,\frac{\pi}{2})$ is a constant solution of (3.13), we have $\frac{\pi}{2}<\theta_*^{sl}(t)<\frac{3}{2}\pi$ for every $t\in[t_*,\tau]$. Therefore the second equation in (3.13) implies that $\dot\theta_*^{sl}(t)<0$, hence $\frac{\pi}{2}<\theta_*^{sl}(t)\leq\theta_*<\pi$ for every $t\in[t_*,\tau]$. We deduce that, if η is small enough, we have

$$\sin q^\eta_{\theta_*^{sl}(t)}(\theta)\geq\eta \qquad \text{for every } t\in[t_*,\tau] \text{and every } \theta\in\mathbb{R}\,. \tag{5.24}$$

Since $\rho_*^{sl}(t)>0$ for every $t\in[t_*,\tau]$, we may assume that

$$\rho_*^{sl}(t)\geq 2\eta \qquad \text{for every } t\in[t_*,\tau]\,. \tag{5.25}$$

Finally, we may also assume that

$$\kappa\,\eta<1\,, \tag{5.26}$$

where κ is the constant in (5.20).

Let us fix η satisfying (5.23)-(5.26), and let $(\rho_\varepsilon^\eta(t),\theta_\varepsilon^\eta(t),z_\varepsilon^\eta(t))$, $t\in[t_\varepsilon^*,\tau]$, be the solutions of the systems

$$\begin{cases}\varepsilon\,\dot\rho_\varepsilon^\eta(t)=\varepsilon\sin\tilde\theta_\varepsilon^\eta(t)-\\ \qquad -(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))^+\big(\tilde z_\varepsilon^\eta(t)\,(1+\cos\tilde\theta_\varepsilon^\eta(t))\cos^2\tilde\theta_\varepsilon^\eta(t)+1\big)\,,\\ \varepsilon\,\max\{\rho_\varepsilon^\eta(t),\eta\}\,\dot\theta_\varepsilon^\eta(t)=\varepsilon\cos\tilde\theta_\varepsilon^\eta(t)+\\ \qquad +(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))^+\tilde z_\varepsilon^\eta(t)\,(1+\cos\tilde\theta_\varepsilon^\eta(t))\cos\tilde\theta_\varepsilon^\eta(t)\sin\tilde\theta_\varepsilon^\eta(t)\,,\\ \varepsilon\,\dot z_\varepsilon^\eta(t)=(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))^+\tilde z_\varepsilon^\eta(t)\,(1+\cos\tilde\theta_\varepsilon^\eta(t))\cos\tilde\theta_\varepsilon^\eta(t)\,,\end{cases} \tag{5.27}$$

with Cauchy conditions

$$\rho_\varepsilon^\eta(t_\varepsilon^*)=\rho_\varepsilon(t_\varepsilon^*)\,,\qquad \theta_\varepsilon^\eta(t_\varepsilon^*)=\theta_\varepsilon(t_\varepsilon^*)\,,\qquad z_\varepsilon^\eta(t_\varepsilon^*)=z_\varepsilon(t_\varepsilon^*)\,, \tag{5.28}$$

where $\tilde\theta_\varepsilon^\eta(t):=q^\eta_{\theta_*^{sl}(t)}(\theta_\varepsilon^\eta(t))$ and $\tilde z_\varepsilon^\eta(t):=q^\eta_{\rho_*^{sl}(t)}(z_\varepsilon^\eta(t))$. By subtracting the third equation from the first one in (5.27) we get

$$\begin{aligned}\varepsilon\,(\dot\rho_\varepsilon^\eta(t)-\dot z_\varepsilon^\eta(t))&=\varepsilon\sin\tilde\theta_\varepsilon^\eta(t)-\\ &\quad-(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))^+\big(\tilde z_\varepsilon^\eta(t)\,(1+\cos\tilde\theta_\varepsilon^\eta(t))^2\cos\tilde\theta_\varepsilon^\eta(t)+1\big)\,.\end{aligned} \tag{5.29}$$

Therefore we deduce from (5.23) that

$$\varepsilon(\dot\rho_\varepsilon^\eta(t)-\dot z_\varepsilon^\eta(t))\leq\varepsilon-\eta(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))^+ \qquad \text{for every } t\in[t_*,\tau]\,.$$

As $0\leq\rho_\varepsilon^\eta(t_\varepsilon^*)-z_\varepsilon^\eta(t_\varepsilon^*)\leq\kappa\,\varepsilon$ by (5.20) and (5.28), by comparison we obtain that

$$\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t)\leq(\kappa\,\varepsilon-\tfrac{\varepsilon}{\eta})\exp\big(-\tfrac{\eta}{\varepsilon}(t-t_\varepsilon^*)\big)+\tfrac{\varepsilon}{\eta}\leq\tfrac{\varepsilon}{\eta} \qquad \text{for every } t\in[t_\varepsilon^*,\tau]\,, \tag{5.30}$$

where the last inequality follows from (5.26). Let us prove that

$$\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t)>0 \qquad \text{for every } t\in[t_*,\tau]\,. \tag{5.31}$$

If not, let τ be the first time in $(t_*,\tau]$ such that $\rho_\varepsilon^\eta(\tau)-z_\varepsilon^\eta(\tau)=0$. Clearly we have $\dot\rho_\varepsilon^\eta(\tau)-\dot z_\varepsilon^\eta(\tau)\leq 0$. By (5.24) and (5.29) we have $\dot\rho_\varepsilon^\eta(\tau)-\dot z_\varepsilon^\eta(\tau)=\sin\tilde\theta_\varepsilon^\eta(\tau)>0$, which contradicts the inequality $\dot\rho_\varepsilon^\eta(\tau)-\dot z_\varepsilon^\eta(\tau)\leq 0$ and concludes the proof of (5.31).

Let us define

$$\psi_\varepsilon^\eta(t):=\tfrac{1}{\varepsilon}(\rho_\varepsilon^\eta(t)-z_\varepsilon^\eta(t))\,. \tag{5.32}$$

By (5.30) and (5.31) we have

$$0\leq\psi_\varepsilon^\eta(t)\leq\tfrac{1}{\eta} \qquad \text{for every } t\in[t_*,\tau]\,. \tag{5.33}$$

Passing to a subsequence, we may assume that $\psi^\eta_\varepsilon \rightharpoonup \psi^\eta$ weakly* in $L^\infty([t_*,\tau])$ as $\varepsilon\to 0$, with $0\le \psi^\eta\le \frac{1}{\eta}$ a.e. on $[t_*,\tau]$. From (5.27) we obtain

$$\begin{cases} \dot\rho^\eta_\varepsilon(t) = \sin\tilde\theta^\eta_\varepsilon(t) - \psi^\eta_\varepsilon(t)\left(\tilde z^\eta_\varepsilon(t)\,(1+\cos\tilde\theta^\eta_\varepsilon(t))\cos^2\tilde\theta^\eta_\varepsilon(t)+1\right),\\ \max\{\rho^\eta_\varepsilon(t),\eta\}\,\dot\theta^\eta_\varepsilon(t) = \cos\tilde\theta^\eta_\varepsilon(t) + \psi^\eta_\varepsilon(t)\,\tilde z^\eta_\varepsilon(t)\,(1+\cos\tilde\theta^\eta_\varepsilon(t))\cos\tilde\theta^\eta_\varepsilon(t)\sin\tilde\theta^\eta_\varepsilon(t)\,,\\ \dot z^\eta_\varepsilon(t) = \psi^\eta_\varepsilon(t)\,\tilde z^\eta_\varepsilon(t)\,(1+\cos\tilde\theta^\eta_\varepsilon(t))\cos\tilde\theta^\eta_\varepsilon(t)\,. \end{cases} \tag{5.34}$$

We can regard (5.34) as a sequence of systems whose right-hand sides are given by

$$\begin{aligned} &F^\eta_\varepsilon(t,\rho,\theta,z) := \sin q^\eta_{\theta^{sl}_*(t)}(\theta) - \psi^\eta_\varepsilon(t)\left(q^\eta_{\rho^{sl}_*(t)}(z)\,(1+\cos q^\eta_{\theta^{sl}_*(t)}(\theta))\,\cos^2 q^\eta_{\theta^{sl}_*(t)}(\theta)+1\right),\\ &G^\eta_\varepsilon(t,\rho,\theta,z) := \cos q^\eta_{\theta^{sl}_*(t)}(\theta) + \psi^\eta_\varepsilon(t)\,q^\eta_{\rho^{sl}_*(t)}(z)(1+\cos q^\eta_{\theta^{sl}_*(t)}(\theta))\cos q^\eta_{\theta^{sl}_*(t)}(\theta)\sin q^\eta_{\theta^{sl}_*(t)}(\theta)\,,\\ &H^\eta_\varepsilon(t,\rho,\theta,z) := \psi^\eta_\varepsilon(t)\,q^\eta_{\rho^{sl}_*(t)}(z)\,(1+\cos q^\eta_{\theta^{sl}_*(t)}(\theta))\,\cos q^\eta_{\theta^{sl}_*(t)}(\theta)\,. \end{aligned}$$

By Theorem 5.3 we have $\rho^\eta_\varepsilon\to\rho^\eta$, $\theta^\eta_\varepsilon\to\theta^\eta$, and $z^\eta_\varepsilon\to z^\eta$ uniformly on $[t_*,\tau]$, where $(\rho^\eta,\theta^\eta,z^\eta)$ is the solution of the system

$$\begin{cases} \dot\rho^\eta(t) = \sin\tilde\theta^\eta(t) - \psi^\eta(t)\left(\tilde z^\eta(t)\,(1+\cos\tilde\theta^\eta(t))\cos^2\tilde\theta^\eta(t)+1\right),\\ \max\{\rho^\eta(t),\eta\}\,\dot\theta^\eta(t) = \cos\tilde\theta^\eta(t) + \psi^\eta(t)\,\tilde z^\eta(t)\,(1+\cos\tilde\theta^\eta(t))\cos\tilde\theta^\eta(t)\sin\tilde\theta^\eta(t)\,,\\ \dot z^\eta(t) = \psi^\eta(t)\,\tilde z^\eta(t)\,(1+\cos\tilde\theta^\eta(t))\cos\tilde\theta^\eta(t)\,, \end{cases} \tag{5.35}$$

with $\tilde\theta^\eta(t) := q^\eta_{\theta^{sl}_*(t)}(\theta^\eta(t))$ and $\tilde z^\eta(t) := q^\eta_{\rho^{sl}_*(t)}(z^\eta(t))$. Moreover

$$\rho^\eta(t_*) = z_*\,,\qquad \theta^\eta(t_*) = \theta_*\,,\qquad z^\eta(t_*) = z_*\,.$$

By (5.30) and (5.31) $\rho^\eta_\varepsilon - z^\eta_\varepsilon\to 0$ strongly in $L^\infty([t_*,\tau])$ as $\varepsilon\to 0$, hence $\rho^\eta(t) = z^\eta(t)$ for every $t\in[t_*,\tau]$. From the first and third equations in (5.35) we obtain

$$\sin\tilde\theta^\eta(t) = \psi^\eta(t)\left(\tilde\rho^\eta(t)\,(1+\cos\tilde\theta^\eta(t))^2\cos\tilde\theta^\eta(t)+1\right)\qquad \text{for a.e. } t\in[t_*,\tau]\,,$$

hence

$$\psi^\eta(t) = \frac{\sin\tilde\theta^\eta(t)}{\tilde\rho^\eta(t)\,(1+\cos\tilde\theta^\eta(t))^2\cos\tilde\theta^\eta(t)+1}\qquad \text{for a.e. } t\in[t_*,\tau]\,.$$

It follows that (ρ^η,θ^η) satisfies the system

$$\begin{cases} \dot\rho^\eta(t) = \dfrac{\tilde\rho^\eta(t)\left(1+\cos\tilde\theta^\eta(t)\right)\,\cos\tilde\theta^\eta(t)\,\sin\tilde\theta^\eta(t)}{\tilde\rho^\eta(t)\left(1+\cos\tilde\theta^\eta(t)\right)^2\cos\tilde\theta^\eta(t)+1}\,,\\[2ex] \max\{\rho^\eta(t),\eta\}\,\dot\theta^\eta(t) = \dfrac{\tilde\rho^\eta(t)\,(1+\cos\tilde\theta^\eta(t))^2\cos\tilde\theta^\eta(t)+\cos\tilde\theta^\eta(t)}{\tilde\rho^\eta(t)\,(1+\cos\tilde\theta^\eta(t))^2\cos\tilde\theta^\eta(t)+1}\,, \end{cases} \tag{5.36}$$

with Cauchy conditions

$$\rho^\eta(t_*) = z_*\quad\text{and}\quad \theta^\eta(t_*) = \theta_*\,. \tag{5.37}$$

By (5.25) we have $\max\{\rho^{sl}_*(t),\eta\} = \rho^{sl}_*(t)$ in a neighbourhood of $[t_*,\tau]$. Moreover, by (5.22) we have $q^\eta_{\rho^{sl}_*(t)}(\rho^{sl}_*(t)) = \rho^{sl}_*(t)$ and $q^\eta_{\theta^{sl}_*(t)}(\theta^{sl}_*(t)) = \theta^{sl}_*(t)$ in a neighbourhood of $[t_*,\tau]$. Since $(\rho^{sl}_*,\theta^{sl}_*)$ is the solution of (3.13) with Cauchy conditions (5.17), it satisfies also (5.36) with Cauchy conditions (5.37). By uniqueness we have $\rho^\eta(t) = \rho^{sl}_*(t)$ and $\theta^\eta(t) = \theta^{sl}_*(t)$ in a neighbourhood of $[t_*,\tau]$.

Since the limit does not depend on the subsequence, we conclude that $\rho^\eta_\varepsilon\to\rho^{sl}_*$, $\theta^\eta_\varepsilon\to\theta^{sl}_*$, and $z^\eta_\varepsilon\to\rho^{sl}_*$ uniformly in a neighbourhood of $[t_*,\tau]$ as $\varepsilon\to 0$. Then for ε small enough we have $\tilde\theta^\eta_\varepsilon(t) := q^\eta_{\theta^{sl}_*(t)}(\theta^\eta_\varepsilon(t)) = \theta^\eta_\varepsilon(t)$ and $\tilde z^\eta_\varepsilon(t) := q^\eta_{\rho^{sl}_*(t)}(z^\eta_\varepsilon(t)) = z^\eta_\varepsilon(t)$, and, recalling (5.26), $\max\{\rho^\eta_\varepsilon(t),\eta\} = \rho^\eta_\varepsilon(t)$ in a neighbourhood of $[t_*,\tau]$. From (5.27) we deduce that $(\rho^\eta_\varepsilon,\theta^\eta_\varepsilon,z^\eta_\varepsilon)$ satisfies (2.21) in a neighbourhood of $[t_*,\tau]$ for ε small enough. Since, by (5.28), $(\rho^\eta_\varepsilon,\theta^\eta_\varepsilon,z^\eta_\varepsilon)$ and $(\rho_\varepsilon,\theta_\varepsilon,z_\varepsilon)$ satisfy the same Cauchy condition at t^*_ε, by uniqueness we have that $(\rho^\eta_\varepsilon,\theta^\eta_\varepsilon,z^\eta_\varepsilon) = (\rho_\varepsilon,\theta_\varepsilon,z_\varepsilon)$ on $[t_*,\tau]$ for ε small enough. It follows that $\rho_\varepsilon\to\rho^{sl}_*$,

$\theta_\varepsilon \to \theta_*^{sl}$, and $z_\varepsilon \to \rho_*^{sl}$ uniformly in a neighbourhood of $[t_*, \tau]$ as $\varepsilon \to 0$. As $t_\varepsilon^* \to t_*$, this concludes the proof of (5.21). $\square$

5.3. **Softening with continuous evolution.** In this subsection we prove Theorem 5.2 describing the softening regime with a continuous evolution.

Proof. Let us fix $\tau \in (t_0, +\infty)$. Assume either $\frac{\pi}{2} < \theta_0 \le \theta_c$ and $z_0 < z_s(\theta_0)$, or $\theta_c < \theta_0 < \pi$ and $z_0 < r_c(\theta_0)$. Then we can apply Lemma 5.4 with $t_* = t_0$, $\theta_* = \theta_0$, $z_* = z_0$, $t_\varepsilon^* = t_0$, and $\kappa = 0$, since (5.18) is a consequence of (3.26). Therefore (5.5) and (5.8) follow from (5.21).

Assume $\frac{\pi}{2} < \theta_0 < \theta_c$ and $z_0 = z_s(\theta_0)$. To deal with the behaviour of the solutions near t_0 we apply Lemma 4.2 with $t_1 = t_0$, $\theta_1 = \theta_0$, $z_1 = z_0 = z_s(\theta_0)$, $\kappa_1 = 1$, $\tau_\varepsilon^\delta = t_0$, and $0 < \delta_0 < \theta_c - \theta_0$. Let δ_1, γ_1, γ_2, and t_ε^δ be the constants and the double sequence given by Lemma 4.2, and let δ_k be a decreasing sequence in $(0, \delta_1)$ converging to 0. For every k we have

$$|t_\varepsilon^{\delta_k} - t_0| \le 2\,\delta_k\,, \tag{5.38}$$

$$w_\varepsilon(t_\varepsilon^{\delta_k}) \ge \delta_k^2 \tag{5.39}$$

$$|\rho_\varepsilon(t_\varepsilon^{\delta_k}) - z_\varepsilon(t_\varepsilon^{\delta_k})| \le \gamma_1 \tfrac{1}{\delta_k^2}\,\varepsilon\,, \tag{5.40}$$

$$\sup_{t_0 \le t \le t_\varepsilon^\delta} \big(|\rho_\varepsilon(t) - z_0| + |\theta_\varepsilon(t) - \theta_0| + |z_\varepsilon(t) - z_0|\big) \le \gamma_2 \sqrt{\delta_k}\,, \tag{5.41}$$

for ε small enough. Using a diagonal argument and (5.40), we may assume that for every k there exist three constants $t_0^{\delta_k}$, $\theta_0^{\delta_k}$, and $z_0^{\delta_k}$ such that

$$t_\varepsilon^{\delta_k} \to t_0^{\delta_k}\,, \quad \rho_\varepsilon(t_\varepsilon^{\delta_k}) \to z_0^{\delta_k}\,, \quad \theta_\varepsilon(t_\varepsilon^{\delta_k}) \to \theta_0^{\delta_k}\,, \quad z_\varepsilon(t_\varepsilon^{\delta_k}) \to z_0^{\delta_k}\,, \tag{5.42}$$

as $\varepsilon \to 0$ along a suitable sequence independent of k. By (5.38), (5.39), and (5.41) for every k we have

$$|t_0^{\delta_k} - t_0| \le 2\,\delta_k\,, \tag{5.43}$$

$$z_0^{\delta_k}\,(1 + \cos\theta_0^{\delta_k})^2 \cos\theta_0^{\delta_k} + 1 \ge \delta_k^2\,, \tag{5.44}$$

$$|\theta_0^{\delta_k} - \theta_0| + |z_0^{\delta_k} - z_0| \le \gamma_2 \sqrt{\delta_k}\,. \tag{5.45}$$

Inequality (5.44) implies that $z_0^{\delta_k} < z_s(\theta_0^{\delta_k})$.

Let $(\rho_{\delta_k}^{sl}, \theta_{\delta_k}^{sl})$ be the solution of (3.13) with Cauchy conditions

$$\rho_{\delta_k}^{sl}(t_0^{\delta_k}) = z_0^{\delta_k} \quad \text{and} \quad \theta_{\delta_k}^{sl}(t_0^{\delta_k}) = \theta_0^{\delta_k}\,. \tag{5.46}$$

By (3.26) and (5.44) we have

$$\rho_{\delta_k}^{sl}(t) < z_s(\theta_{\delta_k}^{sl}(t)) \quad \text{for every } t \in [t_0^{\delta_k}, \tau]\,. \tag{5.47}$$

We can apply Lemma 5.4 with $t_* = t_0^{\delta_k}$, $z_* = z_0^{\delta_k}$, $\theta_* = \theta_0^{\delta_k}$, $(\rho_*^{sl}, \theta_*^{sl}) = (\rho_{\delta_k}^{sl}, \theta_{\delta_k}^{sl})$, $t_\varepsilon^* = t_\varepsilon^{\delta_k}$, and $\kappa = \gamma_1 \frac{1}{\delta_k^2}$. Indeed, (5.17) follows from (5.46), (5.18) from (5.47), (5.19) from (5.42), and (5.20) from (5.40). We conlude that for every k

$$\sup_{t_0^{\delta_k} \le t \le \tau} \big(|\rho_\varepsilon(t) - \rho_{\delta_k}^{sl}(t)| + |\theta_\varepsilon(t) - \theta_{\delta_k}^{sl}(t)| + |z_\varepsilon(t) - \rho_{\delta_k}^{sl}(t)|\big) \to 0\,. \tag{5.48}$$

as $\varepsilon \to 0$ along a sequence satisfying (5.42).

We deduce from (5.48) that $\rho_{\delta_k}^{sl}(t) = \rho_{\delta_h}^{sl}(t)$ and $\theta_{\delta_k}^{sl}(t) = \theta_{\delta_h}^{sl}(t)$ for every $t \in [t_0^{\delta_k}, \tau] \cap [t_0^{\delta_h}, \tau]$. Let $\tau_0 := \inf_k t_0^{\delta_k}$. Then there exists a solution (ρ^{sl}, θ^{sl}) of (3.13) in $(\tau_0, \tau]$ such that $\rho^{sl}(t) = \rho_{\delta_k}^{sl}(t)$ and $\theta^{sl}(t) = \theta_{\delta_k}^{sl}(t)$ for every $t \in [t_0^{\delta_k}, \tau]$. Since $t_0^{\delta_k} \to t_0$ as $k \to \infty$ by (5.43), while $\rho^{sl}(t_0^{\delta_k}) \to z_0$ and $\theta^{sl}(t_0^{\delta_k}) \to \theta_0$ by (5.45) and (5.46), the uniqueness result proved in Lemma 3.7 implies that $(\rho^{sl}, \theta^{sl}) = (\rho_0^{sl}, \theta_0^{sl})$ on $(t_0, \tau]$, hence $\rho_{\delta_k}^{sl}(t) = \rho_0^{sl}(t)$ and $\theta_{\delta_k}^{sl}(t) = \theta_0^{sl}(t)$ for every $t \in [t_0^{\delta_k}, \tau]$. As the limit does not depend on the sequence satisfying (5.42), the limit in (5.48) holds as $\varepsilon \to 0$.

Since

$$|\rho_\varepsilon(t) - \rho_0^{sl}(t)| + |\theta_\varepsilon(t) - \theta_0^{sl}(t)| + |z_\varepsilon(t) - \rho_0^{sl}(t)| \le$$
$$\le |\rho_\varepsilon(t) - z_0| + |\theta_\varepsilon(t) - \theta_0| + |z_\varepsilon(t) - z_0| + 2\,|z_0 - \rho_0^{sl}(t)| + |\theta_0 - \theta_0^{sl}(t)|\,,$$

it follows from (3.23) and (5.41) that there exists a sequence $\omega_k \to 0$ such that

$$\sup_{t_0<t\le t_0^{\delta_k}} \big(|\rho_\varepsilon(t) - \rho_0^{sl}(t)| + |\theta_\varepsilon(t) - \theta_0^{sl}(t)| + |z_\varepsilon(t) - \rho_0^{sl}(t)|\big) \le \omega_k\,.$$

By (5.48) we have

$$\limsup_{\varepsilon\to 0} \sup_{t_0<t\le \tau} \big(|\rho_\varepsilon(t) - \rho_0^{sl}(t)| + |\theta_\varepsilon(t) - \theta_0^{sl}(t)| + |z_\varepsilon(t) - \rho_0^{sl}(t)|\big) \le \omega_k\,,$$

which gives (5.8) as $k \to \infty$.

Assume that $\theta_c < \theta_0 < \pi$ and $z_0 = r_c(\theta_0)$, and let $(\rho_0^{sl}, \theta_0^{sl})$ and t_1 be defined as in Lemma 3.6. Let us fix a decreasing sequence $\delta_k \to 0$. Since $\rho_0^{sl}(t) \to z_c$ and $\theta_0^{sl}(t) \to \theta_c$ as $t \to t_1$ by Lemma 3.6, there exists a sequence τ^{δ_k} such that

$$t_1 - \delta_k < \tau^{\delta_k} < t_1\,, \qquad |\rho_0^{sl}(\tau^{\delta_k}) - z_c| < \tfrac{1}{6}\delta_k\,, \qquad |\theta_0^{sl}(\tau^{\delta_k}) - \theta_c| < \tfrac{1}{6}\delta_k\,. \tag{5.49}$$

We can apply Lemma 5.4 with $t_* = t_0$, $\theta_* = \theta_0$, $z_* = z_0$, $\tau = \tau^{\delta_k}$, $t_\varepsilon^* = t_0$, and $\kappa = 0$. Indeed, $z_0 = r_c(\theta_0) < z_s(\theta_0)$ by Lemma 3.2, and (5.18) follows from (3.20). By (5.21) we have

$$\sup_{t_0\le t\le \tau^{\delta_k}} \big(|\rho_\varepsilon(t) - \rho_0^{sl}(t)| + |\theta_\varepsilon(t) - \theta_0^{sl}(t)| + |z_\varepsilon(t) - \rho_0^{sl}(t)|\big) \le \tfrac{1}{2}\delta_k \tag{5.50}$$

for ε small enough. By (5.49) and (5.50) we have also

$$|\rho_\varepsilon(\tau^{\delta_k}) - z_c| + |\theta_\varepsilon(\tau^{\delta_k}) - \theta_c| + |z_\varepsilon(\tau^{\delta_k}) - z_c| \le \delta_k$$

for ε small enough. Then we can apply Lemma 4.1 with $\kappa = 1$, and we obtain a constant $\beta \ge 1$ and, for every k, a sequence $\tau_\varepsilon^{\delta_k}$ in $[t_0, +\infty)$, such that

$$t_1 - \delta_k \le \tau_{\delta_k} \le \tau_\varepsilon^{\delta_k} \le t_1 + \beta\,\delta_k\,, \tag{5.51}$$
$$w_\varepsilon(\tau_\varepsilon^{\delta_k}) \ge 0\,, \tag{5.52}$$
$$\theta_\varepsilon(\tau_\varepsilon^{\delta_k}) \le \theta_c - \delta_k\,, \tag{5.53}$$
$$\sup_{\tau^{\delta_k}\le t\le \tau_\varepsilon^{\delta_k}} \big(|\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c|\big) \le \sqrt{\beta}\sqrt{\delta_k}\,, \tag{5.54}$$

for ε small enough.

We now apply Lemma 4.2 with $\kappa_1 = \frac{1}{\beta}$ and obtain two constants $\gamma_1 > 0$ and $\gamma_2 > 0$, and, for every k, a new sequence $t_\varepsilon^{\delta_k}$ in $[t_0, +\infty)$, such that

$$t_1 - \delta_k \le \tau^{\delta_k} \le t_\varepsilon^{\delta_k} \le t_1 + 2\,\beta\,\delta_k\,, \tag{5.55}$$
$$w_\varepsilon(t_\varepsilon^{\delta_k}) \ge \tfrac{1}{\beta^2}\,\delta_k^2\,, \tag{5.56}$$
$$|\rho_\varepsilon(t_\varepsilon^{\delta_k}) - z_\varepsilon(t_\varepsilon^{\delta_k})| \le \tfrac{\gamma_1}{\beta^2}\tfrac{1}{\delta_k^2}\,\varepsilon\,, \tag{5.57}$$
$$\sup_{\tau_\varepsilon^{\delta_k}\le t\le t_\varepsilon^{\delta_k}} \big(|\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c|\big) \le \gamma_2\sqrt{\beta}\sqrt{\delta_k}\,, \tag{5.58}$$

for ε small enough.

Using a diagonal argument and (5.40), we may assume that for every k there exist three constants $t_1^{\delta_k}$, $\theta_1^{\delta_k}$, and $z_1^{\delta_k}$ such that

$$t_\varepsilon^{\delta_k} \to t_1^{\delta_k}\,, \quad \rho_\varepsilon(t_\varepsilon^{\delta_k}) \to z_1^{\delta_k}\,, \quad \theta_\varepsilon(t_\varepsilon^{\delta_k}) \to \theta_1^{\delta_k}\,, \quad z_\varepsilon(t_\varepsilon^{\delta_k}) \to z_1^{\delta_k}\,, \tag{5.59}$$

as $\varepsilon \to 0$ along a suitable sequence independent of k. By (5.55), (5.57), and (5.58) for every k we have

$$|t_1^{\delta_k} - t_1| \leq 2\,\beta\,\delta_k\,, \tag{5.60}$$

$$z_1^{\delta_k}\,(1+\cos\theta_1^{\delta_k})^2\cos\theta_1^{\delta_k} + 1 \geq \tfrac{1}{\beta^2}\,\delta_k^2\,, \tag{5.61}$$

$$|\theta_1^{\delta_k} - \theta_c| + |z_1^{\delta_k} - z_c| \leq \gamma_2\sqrt{\beta}\sqrt{\delta_k}\,. \tag{5.62}$$

Inequality (5.44) implies that $z_0^{\delta_k} < z_s(\theta_0^{\delta_k})$.

Let $(\rho^{sl}_{\delta_k}, \theta^{sl}_{\delta_k})$ be the solution of (3.13) with Cauchy conditions

$$\rho^{sl}_{\delta_k}(t_1^{\delta_k}) = z_1^{\delta_k} \quad \text{and} \quad \theta^{sl}_{\delta_k}(t_1^{\delta_k}) = \theta_1^{\delta_k}\,. \tag{5.63}$$

By (3.26) and (5.44) we have

$$\rho^{sl}_{\delta_k}(t) < z_s(\theta^{sl}_{\delta_k}(t)) \quad \text{for every } t \in [t_1^{\delta_k}, \tau]\,. \tag{5.64}$$

We can apply Lemma 5.4 with $t_* = t_1^{\delta_k}$, $z_* = z_1^{\delta_k}$, $\theta_* = \theta_1^{\delta_k}$, $t_*^\varepsilon = t_\varepsilon^{\delta_k}$, $(\rho_*^{sl}, \theta_*^{sl}) = (\rho^{sl}_{\delta_k}, \theta^{sl}_{\delta_k})$, and $\kappa = \frac{\gamma_1}{\beta^2}\frac{1}{\delta_k^2}$. Indeed, (5.17) follows from (5.63), (5.18) from (5.64), (5.19) from (5.59), and (5.20) from (5.57). We conclude that for every k

$$\sup_{t_\varepsilon^{\delta_k} \leq t \leq \tau} \big(|\rho_\varepsilon(t) - \rho^{sl}_{\delta_k}(t)| + |\theta_\varepsilon(t) - \theta^{sl}_{\delta_k}(t)| + |z_\varepsilon(t) - \rho^{sl}_{\delta_k}(t)|\big) \to 0\,. \tag{5.65}$$

as $\varepsilon \to 0$ along a sequence satisfying (5.59)

We deduce from (5.65) that $\rho^{sl}_{\delta_k}(t) = \rho^{sl}_{\delta_h}(t)$ and $\theta^{sl}_{\delta_k}(t) = \theta^{sl}_{\delta_h}(t)$ for every $t \in [t_1^{\delta_k}, \tau] \cap [t_1^{\delta_h}, \tau]$. Let $\tau_1 := \inf_k t_1^{\delta_k}$. Then there exists a solution (ρ^{sl}, θ^{sl}) of (3.13) in $(\tau_1, \tau]$ such that $\rho^{sl}(t) = \rho^{sl}_{\delta_k}(t)$ and $\theta^{sl}(t) = \theta^{sl}_{\delta_k}(t)$ for every $t \in [t_1^{\delta_k}, \tau]$. Let $(\rho_2^{sl}, \theta_2^{sl})$ be defined as in Lemma 3.7 with $\theta_2 = \theta_c$ and $z_2 = z_c$. Since $t_1^{\delta_k} \to t_1$ as $k \to \infty$ by (5.55), while $\rho^{sl}(t_1^{\delta_k}) \to z_c$ and $\theta^{sl}(t_1^{\delta_k}) \to \theta_c$ by (5.62) and (5.63), the uniqueness result proved in Lemma 3.7 implies that $(\rho^{sl}, \theta^{sl}) = (\rho_2^{sl}, \theta_2^{sl})$ on $(t_1, \tau]$, hence $\rho^{sl}_{\delta_k}(t) = \rho_2^{sl}(t)$ and $\theta^{sl}_{\delta_k}(t) = \theta_2^{sl}(t)$ for every $t \in [t_1^{\delta_k}, \tau]$. As the limit does not depend on the sequence satisfying (5.59), the limit in (5.65) holds as $\varepsilon \to 0$.

From (5.50), (5.51), and (5.65) we obtain (5.7), except for $t \in (t_1 - \delta_k, t_1 + 2\,\beta\,\delta_k)$. As $k \to \infty$ we obtain (5.7) on $[t_0, +\infty)$.

Since

$$\begin{aligned}|\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)| \leq \\ \leq |\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c| + 2\,|z_c - \rho(t)| + |\theta_c - \theta(t)|\,,\end{aligned}$$

it follows from (3.18), (3.23), (5.54), and (5.58) that there exists a sequence $\omega_k \to 0$ such that

$$\sup_{\tau^{\delta_k} < t \leq t_\varepsilon^{\delta_k}} \big(|\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)|\big) \leq \omega_k\,.$$

By (5.50), and (5.65) we have

$$\limsup_{\varepsilon \to 0} \sup_{t_0 \leq t \leq \tau} \big(|\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)|\big) \leq \omega_k\,,$$

which gives (5.8) as $k \to \infty$.

Assume that $\theta = \theta_c$ and $z_0 = z_c$, and let $(\rho_2^{sl}, \theta_2^{sl})$ be defined as in Lemma 3.7 with $t_1 = t_0$, $\theta_2 = \theta_c$, and $z_2 = z_c$. then we can apply Lemma (4.1) with $\kappa = 1$, $t_1 = \tau^\delta = t_0$. Given a decreasing sequence $\delta_k \to 0$, we obtain a constant $\beta \geq 1$ and, for every k, a sequence $\tau_\varepsilon^{\delta_k}$ in $[t_0, +\infty)$ which satisfies (5.51)-(5.54) for ε small enough. Then the proof can be concluded as in the previous case, replacing t_1 by t_0. □

6. The fast dynamics

In this section we study in detail the behaviour of the solutions to the system of the fast dynamics.

6.1. **The trajectory of the fast dynamics.**

In this subsection we study the system

$$\begin{cases} \varrho'(z) = -\cos\vartheta(z) - \dfrac{1}{z(1+\cos\vartheta(z))\cos\vartheta(z)}\,, \\ \vartheta'(z) = \dfrac{\sin\vartheta(z)}{\varrho(z)}\,, \end{cases} \tag{6.1}$$

that describes the trajectories followed along the fast dynamics. Using Cartesian coordinates, we consider the functions

$$\mathtt{x}(z) := z + \varrho(z)\,\cos\vartheta(z) \qquad \text{and} \qquad \mathtt{y}(z) := \varrho(z)\,\sin\vartheta(z)\,, \tag{6.2}$$

and (6.1) is equivalent to

$$\begin{cases} \mathtt{x}'(z) = -\dfrac{1}{z(1+\cos\vartheta(z))}\,, \\ \mathtt{y}'(z) = -\dfrac{\tan\vartheta(z)}{z(1+\cos\vartheta(z))}\,, \end{cases} \tag{6.3}$$

where

$$\cos\vartheta(z) = \frac{\mathtt{x}(z)-z}{\sqrt{(\mathtt{x}(z)-z)^2+\mathtt{y}(z)^2}} \qquad \text{and} \qquad \tan\vartheta(z) = \frac{\mathtt{y}(z)}{\mathtt{x}(z)-z}\,.$$

Let us fix $z_1>0$ and $\frac{\pi}{2}<\theta_1<\pi$ and consider the Cauchy conditions

$$\varrho(z_1) = z_1 \qquad \text{and} \qquad \vartheta(z_1) = \theta_1\,, \tag{6.4}$$

that in Cartesian coordinates become

$$\mathtt{x}(z_1) = z_1\,(1+\cos\theta_1) \qquad \text{and} \qquad \mathtt{y}(z_1) = z_1\sin\theta_1\,. \tag{6.5}$$

Let θ_c be the constant defined in (3.3), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6).

Lemma 6.1. *Let $z_1>0$ and $\frac{\pi}{2}<\theta_1<\pi$. Assume one of the following two conditions:*

$$z_1 > z_s(\theta_1)\,, \tag{6.6}$$

$$z_1 = z_s(\theta_1) \quad \textit{and} \quad \theta_1 > \theta_c\,. \tag{6.7}$$

Then there exists $z_2\in(0,z_1)$ such that (6.1) *with Cauchy condition* (6.4) *has a solution (ϱ,ϑ) defined in $[z_2,z_1]$ such that*

$$\varrho(z_2) = z_2 \qquad \textit{and} \qquad \varrho(z) > z \qquad \textit{for } z\in(z_2,z_1)\,. \tag{6.8}$$

Let $\theta_2 := \vartheta(z_2)$. Then we have

$$\tfrac{\pi}{2} < \theta_2 < \vartheta(z) < \theta_1 < \pi \qquad \textit{for } z\in(z_2,z_1)\,, \tag{6.9}$$

$$\varrho'(z) > 0 \quad \textit{and} \quad \vartheta'(z) > 0 \qquad \textit{for } z\in(z_2,z_1)\,, \tag{6.10}$$

$$z_2 \geq \tfrac{27}{4} \qquad \Longrightarrow \qquad \theta_2 < \theta_c\,, \tag{6.11}$$

$$\varrho'(z_2) > 1 \quad \textit{and} \quad z_2 < z_s(\theta_2)\,. \tag{6.12}$$

Proof. Let us consider the solution (ϱ,ϑ) of (6.1) with Cauchy condition (6.4) on its maximal left interval of existence $(z_e,z_1]$. By the singularities of the right-hand side we have that z, $\varrho(z)$, $\cos\vartheta(z)$, and $1+\cos\vartheta(z)$ cannot vanish for $z\in(z_e,z_1]$ so that $z_e\geq 0$ and $\frac{\pi}{2}<\vartheta(z)<\pi$ for every $z\in(z_e,z_1]$. Then (6.1) implies (6.10), which gives (6.9).

Let us define

$$\theta_e := \lim_{z\to z_e}\vartheta(z) = \inf_{z>z_e}\vartheta(z) \geq \tfrac{\pi}{2}\,,$$

$$\rho_e := \lim_{z\to z_e}\varrho(z) = \inf_{z>z_e}\varrho(z) \geq 0\,.$$

If $z_e = 0$, from the first equation in (6.1) we would have $\varrho'(z) \geq \frac{1}{2z}$ for every $z \in (0, z_1]$, and this contradicts the fact that the limit ρ_e is finite. Therefore $z_e > 0$.

We now show that $\rho_e < z_e$. If not, we would have $\rho_e \geq z_e > 0$, and hence $\theta_e > \frac{\pi}{2}$, otherwise the solution could be continued by solving a new Cauchy problem at z_e. By the second equation in (6.1) we have $\vartheta'(z) \to \frac{1}{\rho_e}$ as $z \to z_e$. Thus the first equation in (6.1) gives

$$\varrho'(z) \geq \frac{1}{2z_1 |\cos\vartheta(z)|} \geq \frac{\rho_e}{4z_1|z - z_e|},$$

for z near z_e, which contradicts again the finiteness of ρ_e. This proves that $\rho_e < z_e$.

It is convenient to introduce the function $\omega\colon [z_e, z_1] \to \mathbb{R}$ defined by

$$\omega(z) := z\,(1 + \cos\vartheta(z))^2 \cos\vartheta(z) + 1\,. \tag{6.13}$$

It follows from (6.1) that

$$\varrho'(z) - 1 = -\frac{\omega(z)}{z\,(1 + \cos\vartheta(z))\cos\vartheta(z)}\,. \tag{6.14}$$

Using (6.1) we obtain

$$\begin{aligned} \omega'(z)\,\varrho(z)\,(1+\cos\vartheta(z))^{-2} &= \varrho(z)\,\cos\vartheta(z) - z\,(1-\cos\vartheta(z))(1+3\cos\vartheta(z)) = \\ &= \varrho(z)\,\cos\vartheta(z) - z\,(1 + 2\cos\vartheta(z) - 3\cos^2\vartheta(z))\,. \end{aligned} \tag{6.15}$$

If (6.6) holds, then $\varrho'(z_1) < 1$, so that $\varrho(z) > z$ for all $z < z_1$ close to z_1. If, instead, (6.7) holds, then $\omega'(z_1)$ has the same sign as $-1 - \cos\theta_1 + 3\cos^2\theta_1$, which is positive by (6.7). Therefore $\omega'(z_1) > 0$ and $\omega(z_1) = 0$, hence $\omega(z) < 0$ for all $z < z_1$ close to z_1. From (6.14) we deduce that $\varrho'(z) < 1$, and hence $\varrho(z) > z$ for all $z < z_1$ close to z_1.

On the other hand the inequality $\rho_e < z_e$ gives $\varrho(z) < z$ for all $z > z_e$ close to z_e. Therefore there exists the greatest point z_2 in (z_e, z_1) such that $\varrho(z_2) = z_2$. Condition (6.8) is clearly satisfied, and implies

$$\varrho'(z_2) \geq 1\,. \tag{6.16}$$

By (6.6), (6.7), (6.16), and (6.14) we have

$$\omega(z_1) \leq 0 \qquad \text{and} \qquad \omega(z_2) \geq 0\,. \tag{6.17}$$

Since $\cos\theta_2 > \cos\theta_1$ by (6.9) and (6.10), if $\cos\theta_1 \geq \lambda_c$ we have also $\cos\theta_2 > \lambda_c$, where λ_c is the constant defined in (3.2). Therefore to prove (6.11) we may assume

$$\cos\theta_1 \leq \lambda_c \qquad \text{and} \qquad z_2 \geq \tfrac{27}{4}\,, \tag{6.18}$$

and we want to prove that $\cos\theta_2 > \lambda_c$. We argue by contradiction, assuming (6.18) and

$$\cos\theta_2 \leq \lambda_c < -\tfrac{1}{3}\,, \tag{6.19}$$

Since $\omega'(z_2)\,(1+\cos\theta_2)^{-2} = -1 - \cos\theta_2 + 3\cos^2\theta_2$ by (6.15), inequality (6.19) gives

$$\omega'(z_2) \geq 0\,. \tag{6.20}$$

As $\omega'(z_1)\,(1+\cos\theta_1)^{-2} = -1 - \cos\theta_1 + 3\cos^2\theta_1$ by (6.15), we have also

$$\omega'(z_1) \geq 0\,. \tag{6.21}$$

By (6.17) there exists a minimum point z_m of ω in $(z_2, z_1]$ and a maximum point z_M in $[z_2, z_m)$. By (6.21) we have

$$\omega'(z_m) = 0\,, \tag{6.22}$$

and by (6.20) we have

$$\omega'(z_M) = 0 \qquad \text{and} \qquad \omega''(z_M) \leq 0\,. \tag{6.23}$$

We want to prove that

$$\cos\vartheta(z_m) > -\tfrac{9}{10}\,. \tag{6.24}$$

As ϑ is increasing by (6.10), this inequality is trivial if $\cos\theta_1 > -\frac{9}{10}$, so we may assume that

$$-1 < \cos\theta_1 \le -\tfrac{9}{10}\,. \tag{6.25}$$

To prove (6.24) we argue by contradiction and assume that $\cos\vartheta(z_m) \le -\frac{9}{10}$. Let $\eta := 1+\cos\theta_1$, so that $0<\eta\le\frac{1}{10}$ and $\sin\theta_1 = \sqrt{2-\eta}\sqrt{\eta}$. By (6.3) and (6.9) we have $\mathtt{x}'(z)\le 0$ and $\mathtt{y}'(z)\ge 0$, so that, by (6.2),

$$\varrho(z)\cos\vartheta(z) \ge z_1\eta - z \quad \text{and} \quad \varrho(z)\,\sin\vartheta(z) \le z_1\sqrt{2-\eta}\sqrt{\eta} \quad \text{for every } z\in[z_2,z_1]\,. \tag{6.26}$$

This implies $z_2 > z_1\eta$ and $\varrho(z)^2 \le (z-z_1\eta)^2 + z_1^2(2-\eta)\eta \le z^2 + 2z_1^2\eta$, hence

$$\frac{\varrho(z)^2}{z^2} \le 1 + 2\frac{z_1^2}{z^2}\eta \qquad \text{for every } z\in[z_2,z_1]\,. \tag{6.27}$$

Since $\cos\vartheta(z_m)\le -\frac{9}{10}$, by (6.15) and (6.22) the polynomial $P_m(\lambda) := \varrho(z_m)\lambda - z_m(1+2\lambda-3\lambda^2)$ has a zero in the interval $(-1,-\frac{9}{10}]$. As $P_m(0) = -z_m < 0$, this implies $P_m(-\frac{9}{10})\le 0$, hence $\varrho(z_m) > \frac{323}{90}z_m > 3z_m$. By (6.27) we obtain $z_m \le \frac12 z_1\sqrt{\eta}$. As $-1+\eta = \cos\theta_1 \le \cos\vartheta(z) \le \cos\vartheta(z_m) \le -\frac{9}{10}$ for every $z\in[z_m,z_1]$, we have $0<\sin\vartheta(z)\le\frac12$ for every $z\in[z_m,z_1]$. Since the function $\lambda\mapsto -\lambda(1+\lambda)$ is increasing in $[-1,-\frac12]$, from (6.6) we obtain that $1\le z_1\eta^2(1-\eta)\le z_1\eta(1+\cos\vartheta(z))$ and $1\le z_1\eta^2(1-\eta)\le -z_1\eta(1+\cos\vartheta(z))\cos\vartheta(z)$ for every $z\in[z_m,z_1]$. By (6.3) we have

$$\mathtt{x}'(z) \ge -\frac{z_1}{z}\eta \qquad \text{and} \qquad \mathtt{y}'(z) \le \frac12\,\frac{z_1}{z}\,\eta \qquad \text{for every } z\in[z_m,z_1]\,.$$

Integrating we obtain

$$\mathtt{x}(z_1) - \mathtt{x}(z_1\sqrt{\eta}) \ge z_1\eta\log\sqrt{\eta} \quad \text{and} \quad \mathtt{y}(z_1)-\mathtt{y}(z_1\sqrt{\eta}) \le -\tfrac12 z_1\eta\log\sqrt{\eta}\,.$$

As $\mathtt{x}(z_1) = z_1\eta$ and $\mathtt{y}(z_1) = z_1\sqrt{2-\eta}\sqrt{\eta}$, we deduce from (6.2) that

$$\varrho(z_1\sqrt{\eta})\,\cos\vartheta(z_1\sqrt{\eta}) \le -z_1\big(\sqrt{\eta}-\eta+\eta\log\sqrt{\eta}\big)\,,$$
$$\varrho(z_1\sqrt{\eta})\,\sin\vartheta(z_1\sqrt{\eta}) \ge z_1\big(\sqrt{2-\eta}\sqrt{\eta}+\tfrac12\eta\log\sqrt{\eta}\big)\,.$$

As $\sqrt{\eta}\log\sqrt{\eta} \ge \frac{1}{\sqrt{10}}\log\frac{1}{\sqrt{10}} \ge -\frac{5}{12}$, we have $\sqrt{\eta}-\eta+\eta\log\sqrt{\eta} > 0$ and $\sqrt{2-\eta}\sqrt{\eta} + \frac12\eta\log\sqrt{\eta} > 0$ for every $\eta\in(0,\frac{1}{10}]$, so that

$$\begin{aligned}
\varrho(z_1\sqrt{\eta})^2 &\ge z_1^2\Big(\big(\sqrt{\eta}-\eta+\eta\log\sqrt{\eta}\big)^2 + \big(\sqrt{2-\eta}\sqrt{\eta}+\tfrac12\eta\log\sqrt{\eta}\big)^2\Big) \ge \\
&\ge z_1^2\eta\Big((1-\sqrt{\eta})^2 + 2\sqrt{\eta}\log\sqrt{\eta} + 2-\eta+\sqrt{2}\sqrt{\eta}\log\sqrt{\eta}\Big) \ge \\
&\ge z_1^2\eta\,(3-2\sqrt{\eta}+\tfrac72\sqrt{\eta}\log\sqrt{\eta}) \ge z_1^2\eta\,(\tfrac{37}{24}-2\sqrt{\eta}) \ge \\
&\ge z_1^2\eta\,(\tfrac{110}{81} - 2\tfrac{100}{81}\sqrt{\eta}) \ge \tfrac{100}{81}\,z_1^2\eta\,(1-\sqrt{\eta})^2 = \tfrac{100}{81}\,z_1^2\,(\sqrt{\eta}-\eta)^2
\end{aligned}$$

for every $\eta\in(0,\frac{1}{10}]$. Therefore (6.26) implies that $\cos^2\vartheta(z_1\sqrt{\eta}) \le \frac{81}{100}$. As $z_m < z_1\sqrt{\eta}$, we have $\cos\vartheta(z_m) > \cos\vartheta(z_1\sqrt{\eta}) \ge -\frac{9}{10}$, which contradicts our assumption $\cos\vartheta(z_m)\le -\frac{9}{10}$, and concludes the proof of (6.24).

Let $\lambda_M := \cos\vartheta(x_M)$. As $z_2\le z_M < z_m$ we have

$$-\tfrac{9}{10} < \lambda_M < -\tfrac13\,. \tag{6.28}$$

Since $\omega'(z_M) = 0$ by (6.23), using (6.1) and the equality

$$\frac{z_M}{\rho(z_M)} = \frac{\lambda_M}{(1-\lambda_M)(1+3\lambda_M)}\,,$$

that can be deduced from (6.15), we obtain

$$\omega''(z_M)\,\varrho(z_M)\,\frac{1+3\lambda_M}{1-\lambda_M} = -\frac{1}{z_M}\,(1+3\lambda_M) - (1+\lambda_M)(2+6\lambda_M+7\lambda_M^2-3\lambda_M^3)\,.$$

By (6.18), (6.23), and (6.28) we have

$$(1+\lambda_M)(2+6\lambda_M+7\lambda_M^2-3\lambda_M^3)+\tfrac{4}{27}\,(1+3\lambda_M)\leq 0\,. \tag{6.29}$$

Let us considerer the polynomial $P(\lambda):=(1+\lambda)(2+6\lambda+7\lambda^2-3\lambda^3)+\frac{4}{27}\,(1+3\lambda)=\frac{58}{27}+\frac{76}{9}\lambda+13\lambda^2+4\lambda^3-3\lambda^4$ and its derivative $P'(\lambda)=\frac{76}{9}+26\lambda+12\lambda^2-12\lambda^3=\frac{2}{9}(2+3\lambda)(19+30\lambda-18\lambda^2)$. Since $P'(\lambda)$ vanishes at $-\frac{2}{3}$, $-\frac{1}{6}(3\sqrt{7}-5)$, and $\frac{1}{6}(3\sqrt{7}+5)$, we deduce that $P(\lambda)$ has two local minima on $[-\frac{1}{10},-\frac{1}{3}]$ at the points $-\frac{1}{10}$ and $-\frac{1}{6}(3\sqrt{7}-5)$. By direct computation we see that $P(-\frac{1}{6}(3\sqrt{7}-5))=\frac{3053}{108}-\frac{21}{2}\sqrt{7}>\frac{52339}{270000}=P(-\frac{1}{10})$, so that $P(\lambda)>0$ for every $\lambda\in[-\frac{1}{10},-\frac{1}{3}]$. This contradicts (6.29) and concludes the proof of the implication (6.11).

Let us prove that (6.12). By (6.14) it is enough to prove that $\varrho'(z_2)>1$. We argue by contradiction, taking (6.16) into account. If $\varrho'(z_2)=1$, by (6.14) we have $\omega(z_2)=0$, hence

$$z_2=-\frac{1}{(1+\cos\theta_2)^2\cos\theta_2}\geq\tfrac{27}{4}\,.$$

Since, by (6.15), $\omega'(z_2)\,(1+\cos\theta_2)^{-2}=-1-\cos\theta_2+3\cos^2\theta_2$, by (6.11) we have $\omega'(z_2)<0$. As $\omega(z_2)=0$, this implies $\omega(z)<0$ for every $z>z_2$ close to z_2, so that by (6.14) $\varrho'(z)<1$, hence $\varrho(z)<z$ for every $z>z_2$ close to z_2, which contradicts (6.8). □

6.2. **The system of the fast dynamics.** In this subsection we study the solutions $(\rho^f(s),\theta^f(s),z^f(s))$ of the system of the fast dynamics

$$\begin{cases}\dot\rho^f(s)=-(\rho^f(s)-z^f(s))\left(z^f(s)\,(1+\cos\theta^f(s))\cos^2\theta^f(s)+1\right),\\ \rho^f(s)\,\dot\theta^f(s)=(\rho^f(s)-z^f(s))\,z^f(s)\,(1+\cos\theta^f(s))\cos\theta^f(s)\sin\theta^f(s)\,,\\ \dot z^f(s)=(\rho^f(s)-z^f(s))\,z^f(s)\,(1+\cos\theta^f(s))\,\cos\theta^f(s)\,,\end{cases} \tag{6.30}$$

under the additional condition $\rho^f(s)>z^f(s)>0$. In Cartesian coordinates this system is written as

$$\begin{cases}\dot x^f(s)=-(x^f(s)-z^f(s))\Big(1-\dfrac{z^f(s)}{\rho^f(s)}\Big)\\ \dot y^f(s)=-y^f(s)\Big(1-\dfrac{z^f(s)}{\rho^f(s)}\Big)\\ \dot z^f(s)=\Big(z^f(s)+\dfrac{(x^f(s)-z^f(s))\,z^f(s)}{\rho^f(s)}\Big)(x^f(s)-z^f(s))\Big(1-\dfrac{z^f(s)}{\rho^f(s)}\Big)\,,\end{cases} \tag{6.31}$$

where $\rho^f(s):=\sqrt{(x^f(s)-z^f(s))^2+y^f(s)^2}$.

Let θ_c be the constant defined in (3.3), and let $z_s(\theta)$ be the function defined in (3.4).

Lemma 6.2. *Let $z_1>0$ and $\frac{\pi}{2}<\theta_1<\pi$. Assume (6.6) or (6.7). Then there exists a solution of (6.30) such that*

$$\lim_{s\to-\infty}\rho^f(s)=z_1\,,\qquad\lim_{s\to-\infty}\theta^f(s)=\theta_1\,,\qquad\lim_{s\to-\infty}z^f(s)=z_1\,, \tag{6.32}$$

$$\rho^f(s)>z^f(s)\qquad\textit{for every } s\in\mathbb{R}\,. \tag{6.33}$$

The solution satisfying (6.32) and (6.33) is unique up to time translations, i.e., all such solutions have the form $(\rho^f(s-s_0),\theta^f(s-s_0),z^f(s-s_0))$ for some $s_0\in\mathbb{R}$. Moreover $\rho^f(s)=\varrho(z^f(s))$ and $\theta^f(s)=\vartheta(z^f(s))$ for every $s\in\mathbb{R}$, where (ϱ,ϑ) is the solution of (6.1) with Cauchy conditions (6.4). Finally,

$$\dot\rho^f(s)<0\,,\quad\dot\theta^f(s)<0\,,\quad\dot z^f(s)<0\quad\textit{for every } s\in\mathbb{R}\,, \tag{6.34}$$

$$\lim_{s\to+\infty}\rho^f(s)=z_2\,,\qquad\lim_{s\to+\infty}\theta^f(s)=\theta_2\,,\qquad\lim_{s\to+\infty}z^f(s)=z_2\,, \tag{6.35}$$

where z_2 and θ_2 are defined as in Lemma 6.1.

Proof. Let (ϱ,ϑ) be the solution of (6.1) with Cauchy conditions (6.4), and let $z^f(s)$ be a solution of the autonomous equation

$$\dot z^f(s)=(\varrho(z^f(s))-z^f(s))\,z^f(s)\,(1+\cos\vartheta(z^f(s)))\cos\vartheta(z^f(s))\,, \tag{6.36}$$

with $z_2<z^f(s)<z_1$ for some s. By Lemma 6.1 we have $(\varrho(z)-z)\,z\,(1+\cos\vartheta(z)\le 0$ for every $z\in[z_2,z_1]$, with equality only at $z=z_2$ and $z=z_1$. Then the theory of autonomous equations implies that $z^f(s)$ is defined for every $s\in\mathbb{R}$ and satisfies

$$\lim_{s\to-\infty} z^f(s)=z_1\,,\qquad \lim_{s\to+\infty} z^f(s)=z_2\,,\qquad \dot z^f(s)<0\quad\text{for every } s\in\mathbb{R}\,. \tag{6.37}$$

Let us define

$$\rho^f(s):=\varrho(z^f(s))\quad\text{and}\quad \theta^f(s):=\vartheta(z^f(s))\,. \tag{6.38}$$

By (6.1) and (6.36) $(\rho^f(s),\theta^f(s),z^f(s))$ is a solution of (6.30). Since $\varrho(z_1)=z_1$ and $\vartheta(z_1)=\theta_1$ by (6.4), condition (6.32) follows from (6.37). Since $z_2<z^f(s)<z_1$ for every $s\in\mathbb{R}$, inequality (6.33) follows from (6.8). Finally, (6.34) and (6.35) follow from (6.10), (6.37), and (6.38).

Suppose that $(\rho_*(s),\theta_*(s),z_*(s))$ is another solution of (6.30) satisfying (6.32) and (6.33). By uniqueness it is easy to see that $\theta_*(s)\neq\frac{\pi}{2}$ and $\theta_*(s)\neq\pi$ for every $s\in\mathbb{R}$. Recalling (6.32), we deduce that $\frac{\pi}{2}<\theta_*(s)<\pi$, so that (6.33) and the third equation in (6.30) imply that $\dot\theta_*(s)<0$ for every $s\in\mathbb{R}$. Then $z_*(s)\to z_*^\infty<z_1$ as $s\to+\infty$. Since $\theta_*(s)$ is decreasing, there exist two functions ϱ_* and ϑ_*, defined on (z_*^∞,z_1), such that

$$\rho_*(s):=\varrho_*(z_*(s))\quad\text{and}\quad \theta_*(s):=\vartheta_*(z_*(s))\,. \tag{6.39}$$

It follows from (6.30) that (ϱ_*,ϑ_*) satisfy (6.1) on (z_*^∞,z_1), and we deduce from (6.32) that $\varrho_*(z)\to z_1$ and $\vartheta_*(z)\to\theta_1$ as $z\to z_1$. By (6.4) (ϱ,ϑ) satisfies the same Cauchy conditions at z_0. By uniqueness we have $(\varrho_*,\vartheta_*)=(\varrho,\vartheta)$ on $(\max\{z_2,z_*^\infty\},z_1)$. Therefore (6.30) and (6.39) imply that $z_*(s)$ is a solution of (6.36) and $z_2<z_*(s)<z_1$ for s large enough (recall (6.32) and the monotonicity of $z_*(s)$). Then the theory of autonomous equations ensures that there exists $s_0\in\mathbb{R}$ such that $z_*(s)=z^f(s-s_0)$ for s large enough. Since $(\varrho_*,\vartheta_*)=(\varrho,\vartheta)$ near z_1, by (6.39) we have $\rho_*(s)=\rho^f(s-s_0)$ and $\theta_*(s)=\theta^f(s-s_0)$ for s large enough. These equalities are extended to every $s\in\mathbb{R}$ by the uniqueness of the solutions of a Cauchy problem for (6.30). ☐

7. Discontinuous evolution

In this subsection we consider the case where $\frac{\pi}{2}<\theta_0<\pi$ and the viscosity solution (ρ,θ,z) has a discontinuity at a time $t_1\ge t_0$ determined by the initial conditions. This solution follows the slow dynamics in $(t_0,t_1]$, has a jump at time t_1, governed by the system of the fast dynamics, and finally follows again the slow dynamics $(t_1,+\infty)$ with initial conditions at t_1 determined by the end point of the trajectory of the fast dynamics.

Let θ_c be the constant defined in (3.3), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6). The first theorem deals with the case $t_1>t_0$.

Theorem 7.1. *Assume*

$$\theta_c<\theta_0\le\pi\quad\textit{and}\quad r_c(\theta_0)<z_0<z_s(\theta_0)\,, \tag{7.1}$$

and let $(\rho_0^{sl},\theta_0^{sl})$ and t_1, z_1, and θ_1 be defined as in Lemma 3.6. Let (ρ^f,θ^f,z^f), z_2, and θ_2 be defined as in Lemma 6.2, and let $(\rho_2^{sl},\theta_2^{sl})$ be defined as in Lemma 3.7. Then

$$\rho(t)=z(t)=\begin{cases}\rho_0^{sl}(t) & \textit{if } t\in[t_0,t_1]\,,\\ \rho_2^{sl}(t) & \textit{if } t\in(t_1,+\infty)\,,\end{cases}\qquad \theta(t)=\begin{cases}\theta_0^{sl}(t) & \textit{if } t\in[t_0,t_1]\,,\\ \theta_2^{sl}(t) & \textit{if } t\in(t_1,+\infty)\,.\end{cases} \tag{7.2}$$

Moreover there exist three sequences of real numbers t^1_ε, τ^1_ε, *and* s_ε *such that for every* $\tau > t_1$ *we have*

$$t_0 < t^1_\varepsilon < \tau^1_\varepsilon \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} t^1_\varepsilon = \lim_{\varepsilon\to 0} \tau^1_\varepsilon = t_1 \,, \tag{7.3}$$

$$\sup_{t_0 \le t \le t^1_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^{sl}_0(t)| + |\theta_\varepsilon(t) - \theta^{sl}_0(t)| + |z_\varepsilon(t) - \rho^{sl}_0(t)|\big) \to 0 \,, \tag{7.4}$$

$$\sup_{t^1_\varepsilon \le t \le \tau^1_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^{f}_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta^{f}_\varepsilon(t)| + |z_\varepsilon(t) - z^{f}_\varepsilon(t)|\big) \to 0 \,, \tag{7.5}$$

$$\sup_{\tau^1_\varepsilon \le t \le \tau} \big(|\rho_\varepsilon(t) - \rho^{sl}_2(t)| + |\theta_\varepsilon(t) - \theta^{sl}_2(t)| + |z_\varepsilon(t) - \rho^{sl}_2(t)|\big) \to 0 \,, \tag{7.6}$$

where

$$\rho^f_\varepsilon(t) := \rho^f(\tfrac{1}{\varepsilon}t - s_\varepsilon) \,, \quad \theta^f_\varepsilon(t) := \theta^f(\tfrac{1}{\varepsilon}t - s_\varepsilon) \,, \quad z^f_\varepsilon(t) := z^f(\tfrac{1}{\varepsilon}t - s_\varepsilon) \,. \tag{7.7}$$

We now consider the case in which the discontinuity time is t_0.

Theorem 7.2. *Let* $z_0 > 0$ *and* $\frac{\pi}{2} < \theta_0 < \pi$. *Assume one of the following two conditions:*

$$z_0 > z_s(\theta_0) \,, \tag{7.8}$$

$$z_0 = z_s(\theta_0) \quad \textit{and} \quad \theta_0 > \theta_c \,. \tag{7.9}$$

Let (ρ^f, θ^f, z^f), z_2, θ_2 *be defined as in Lemma 6.2 with* $z_1 = z_0$ *and* $\theta_1 = \theta_0$, *and let* $(\rho^{sl}_2, \theta^{sl}_2)$ *be defined as in Lemma 3.7 with* $t_1 = t_0$ *Then*

$$\rho(t) = z(t) = \rho^{sl}_2(t) \quad \textit{and} \quad \theta(t) = \theta^{sl}_2(t) \quad \textit{for every } t \in (t_0, +\infty) \,. \tag{7.10}$$

Moreover there exist two sequences of real numbers τ^1_ε *and* s_ε *such that for every* $\tau > t_0$ *we have*

$$t_0 < \tau^1_\varepsilon \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} \tau^1_\varepsilon = t_0 \,, \tag{7.11}$$

$$\sup_{t_0 \le t \le \tau^1_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^{f}_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta^{f}_\varepsilon(t)| + |z_\varepsilon(t) - z^{f}_\varepsilon(t)|\big) \to 0 \,, \tag{7.12}$$

$$\sup_{\tau^1_\varepsilon \le t \le \tau} \big(|\rho_\varepsilon(t) - \rho^{sl}_2(t)| + |\theta_\varepsilon(t) - \theta^{sl}_2(t)| + |z_\varepsilon(t) - \rho^{sl}_2(t)|\big) \to 0 \,, \tag{7.13}$$

where ρ^f_ε, θ^f_ε, *and* z^f_ε *are defined in* (7.7).

The proof of both theorems will be given in Subsection 7.4.

7.1. **Transition to the fast dynamics.** We now describe the behaviour of the system in a small time interval $[\tau_\varepsilon, t^1_\varepsilon]$ where $\rho_\varepsilon(t) - z_\varepsilon(t)$ passes from a size of order ε to a size of order $\varepsilon^{1-\alpha}$ with $\alpha \in (0, \frac{1}{2})$. After t^1_ε the system will be governed by the fast dynamics. Let θ_c be the constant defined in (3.3), and let $z_s(\theta)$ and $w_\varepsilon(t)$ be the functions defined in (3.4) and (2.26).

Lemma 7.3. *Let* $z_1 > 0$ *and* $\frac{\pi}{2} < \theta_1 < \pi$. *Assume* (6.6) *or* (6.7). *Let* $t_1 \in [t_0, +\infty)$, *let* $\alpha \in (0, \frac{1}{2})$, *and let* τ_ε *be a sequence in* $[t_0, +\infty)$ *such that*

$$\tau_\varepsilon \to t_1 \,, \quad \rho_\varepsilon(\tau_\varepsilon) \to z_1 \,, \quad \theta_\varepsilon(\tau_\varepsilon) \to \theta_1 \,, \quad z_\varepsilon(\tau_\varepsilon) \to z_1 \,. \tag{7.14}$$

Then there exists a sequence t^1_ε *in* $[t_0, +\infty)$ *such that*

$$\tau_\varepsilon < t^1_\varepsilon \quad \textit{and} \quad t^1_\varepsilon \to t_1 \,, \tag{7.15}$$

$$w_\varepsilon(t^1_\varepsilon) \le -\varepsilon \,, \tag{7.16}$$

$$\rho_\varepsilon(t^1_\varepsilon) - z_\varepsilon(t^1_\varepsilon) \ge \varepsilon^{1-\alpha} \,, \tag{7.17}$$

$$\sup_{\tau_\varepsilon \le t \le t^1_\varepsilon} \big(|\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1|\big) \to 0 \,. \tag{7.18}$$

Proof. As $z_1 \geq z_s(\theta_1)$, we have $z_1(1+\cos\theta_1)^2\cos\theta_1 + 1 \leq 0$, so that (7.14) gives

$$\limsup_{\varepsilon\to 0} w_\varepsilon(\tau_\varepsilon) \leq 0\,. \tag{7.19}$$

Under the assumption (6.6) we have $z_1(1+\cos\theta_1)^2\cos\theta_1 + 1 < 0$. Therefore there exists $\eta > 0$ such that

$$z(1+\cos\theta)^2\cos\theta + 1 \leq -\eta \qquad \text{for } |\theta-\theta_1| \leq 2\eta \text{ and } |z-z_1| \leq 2\eta\,. \tag{7.20}$$

If, instead, (6.7) holds, then we have $\cos\theta_1 < \cos\theta_c = \lambda_c < -\frac{1}{3}$ by (3.2) and (3.3). This implies that $(1+\cos\theta_1)(1+3\cos\theta_1)\cos\theta_1\sin\theta_1 > 0$ and $(1+\cos\theta_1)^3\cos\theta_1(1+\cos\theta_1 - 3\cos^2\theta_1) > 0$. Therefore there exists $0 < \eta < \frac{1}{4}z_1$ such that

$$\begin{gathered} z\,(1+\cos\theta)\,(1+3\cos\theta)\cos\theta\sin\theta > \eta\,\rho\,, \\ z\,(1+\cos\theta)^3\,\cos\theta\,\big[z\,(1+\cos\theta-3\cos^2\theta) - (\rho - z)\cos\theta\big] > \eta\,\rho\,, \end{gathered} \tag{7.21}$$

for $|\rho - z_1| \leq 2\eta$, $|\theta-\theta_1| \leq 2\eta$, and $|z-z_1| \leq 2\eta$.

In both cases (6.6) and (6.7) we define

$$\tilde t^1_\varepsilon := \inf\{t \in (\tau_\varepsilon, +\infty) : w_\varepsilon(t) < -\varepsilon\}\,, \tag{7.22}$$

$$t^1_\varepsilon := \inf\{t \in (\tilde t^1_\varepsilon, +\infty) : \rho_\varepsilon(t) - z_\varepsilon(t) > \varepsilon^{1-\alpha}\}\,, \tag{7.23}$$

$$\alpha^\eta_\varepsilon := \inf\{t \in (\tau_\varepsilon, +\infty) : |\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) < z_1| < 2\eta\} \tag{7.24}$$

$$\tilde s^\eta_\varepsilon := \min\{\tilde t^1_\varepsilon, \alpha^\eta_\varepsilon\}\,, \qquad s^\eta_\varepsilon := \min\{t^1_\varepsilon, \alpha^\eta_\varepsilon\}\,. \tag{7.25}$$

By (7.14) for ε small enough we have

$$|\rho_\varepsilon(\tau_\varepsilon) - z_1| < \eta\,, \quad |\theta_\varepsilon(\tau_\varepsilon) - \theta_1| < \eta\,, \quad |z_\varepsilon(\tau_\varepsilon) - z_1| < \eta\,. \tag{7.26}$$

If (6.7) holds, by (7.21) for ε small enough we have

$$\begin{gathered} z_\varepsilon(t)(1+\cos\theta_\varepsilon(t)(1+3\cos\theta_\varepsilon(t))\cos\theta_\varepsilon(t)\sin\theta_\varepsilon(t) > \eta\rho_\varepsilon(t)\,, \\ z_\varepsilon(t)(1+\cos\theta_\varepsilon(t))^3\cos\theta_\varepsilon(t)\big[z_\varepsilon(t)(1+\cos\theta_\varepsilon(t) - 3\cos^2\theta_\varepsilon(t)) - \\ -(\rho_\varepsilon(t) - z_\varepsilon(t))\cos\theta_\varepsilon(t)\big] > \eta\rho_\varepsilon(t) \end{gathered}$$

for every $t \in [\tau_\varepsilon, \alpha^\eta_\varepsilon]$. Therefore, using Lemma 2.6 and (2.27), for ε small enough we obtain

$$\varepsilon\,\dot w_\varepsilon(t) < -\varepsilon\,\eta - \eta\,(\rho_\varepsilon(t) - z_\varepsilon(t)) \leq -\eta\,\varepsilon \qquad \text{for every } t \in [\tau_\varepsilon, \alpha^\eta_\varepsilon]\,. \tag{7.27}$$

This implies

$$w_\varepsilon(t) \leq w_\varepsilon(\tau_\varepsilon) - \eta(t-\tau_\varepsilon) \quad \text{for every} \quad t \in [\tau_\varepsilon, \alpha^\eta_\varepsilon]\,, \tag{7.28}$$

which gives

$$0 \leq \tilde s^\eta_\varepsilon - \tau_\varepsilon \leq \tfrac{1}{\eta}\max\{w_\varepsilon(\tau_\varepsilon), 0\} \tag{7.29}$$

for ε small enough. Recalling (7.27), we have

$$w_\varepsilon(t) < -\varepsilon \qquad \text{for every } t \in (\tilde s^\eta_\varepsilon, \alpha^\eta_\varepsilon]\,. \tag{7.30}$$

If (6.6) holds, for ε small enough we have $w_\varepsilon(\tau_\varepsilon) < -\varepsilon$ by (7.20) and (7.26), so that $\tilde s^\eta_\varepsilon = \tilde\tau^\eta_\varepsilon = \tau_\varepsilon$. In this case (7.30) follows directly from (7.20) and (7.24) for $\varepsilon < \eta$, while (7.29) is trivial.

In both cases (6.6) and (6.7), from (2.25), (2.32), and (7.30) we obtain

$$\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t) \geq \sin\theta_0 \quad \text{for every } t \in (\tilde s^\eta_\varepsilon, \alpha^\eta_\varepsilon]\,. \tag{7.31}$$

Integrating this inequality we obtain $\rho_\varepsilon(t) - z_\varepsilon(t) \geq (t - \tilde s^\eta_\varepsilon)\sin\theta_0$ for every $t \in (\tilde s^\eta_\varepsilon, \alpha^\eta_\varepsilon]$. As $\rho_\varepsilon(t) - z_\varepsilon(t) \leq \varepsilon^{1-\alpha}$ for every $t \in (\tilde t^1_\varepsilon, t^1_\varepsilon]$ by (7.23), from (7.25) we obtain

$$s^\eta_\varepsilon - \tilde s^\eta_\varepsilon \leq \tfrac{1}{\sin\theta_0}\varepsilon^{1-\alpha} \tag{7.32}$$

for ε small enough. From (7.14), (7.19), (7.29), and (7.32) it follows that

$$s^\eta_\varepsilon \to t_1 \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{7.33}$$

As $0 < z_\varepsilon(t) \le z_0$ for every $t \in [t_0, +\infty)$ by (2.34), using the third equation in (2.21) we obtain

$$\varepsilon\, \dot z_\varepsilon(t) \ge -2\, z_0\, (\rho_\varepsilon(t) - z_\varepsilon(t)) \qquad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^\eta]\,, \tag{7.34}$$

so that (7.27) gives for ε small enough

$$\dot w_\varepsilon(t) \le \tfrac{\eta}{2z_0} \dot z_\varepsilon(t) \qquad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^\eta]\,.$$

Since $w_\varepsilon(t) \ge -\varepsilon$ for every $t \in (\tau_\varepsilon, \tilde s_\varepsilon^\eta]$ by (7.22) and (7.25), we deduce that

$$-\varepsilon - w_\varepsilon(\tau_\varepsilon) \le w_\varepsilon(t) - w_\varepsilon(\tau_\varepsilon) \le \tfrac{\eta}{2z_0}(z_\varepsilon(t) - z_\varepsilon(\tau_\varepsilon)) \qquad \text{for every } t \in (\tau_\varepsilon, \tilde s_\varepsilon^\eta]\,,$$

so that for ε small enough

$$z_\varepsilon(t) \ge z_\varepsilon(\tau_\varepsilon) - \tfrac{2z_0}{\eta} \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon \qquad \text{for every } t \in [\tau_\varepsilon, \tilde s_\varepsilon^\eta]\,. \tag{7.35}$$

Since $\rho_\varepsilon(t) - z_\varepsilon(t) \le \varepsilon^{1-\alpha}$ for every $t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]$ by (7.23) and (7.25), from (7.34) we get

$$\dot z_\varepsilon(t) \ge -2\, z_0\, \varepsilon^{-\alpha} \qquad \text{for every } t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]\,.$$

Integrating and using (7.32) we obtain

$$z_\varepsilon(t) - z_\varepsilon(\tilde s_\varepsilon^\eta) \ge -2\, z_0\, \varepsilon^{-\alpha}(t - \tilde s_\varepsilon^\eta) \ge -\tfrac{2\, z_0}{\sin\theta_0}\, \varepsilon^{1-2\alpha} \qquad \text{for every } t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]\,,$$

which, together with (2.34) and (7.35), gives for ε small enough

$$z_\varepsilon(\tau_\varepsilon) - \tfrac{2z_0}{\eta} \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon - \tfrac{2\, z_0}{\sin\theta_0}\, \varepsilon^{1-2\alpha} \le z_\varepsilon(t) \le z_\varepsilon(\tau_\varepsilon) \qquad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta]\,.$$

By (7.14) and (7.19) this implies

$$\sup_{\tau_\varepsilon \le t \le s_\varepsilon^\eta} |z_\varepsilon(t) - z_1| \to 0 \qquad \text{as } \varepsilon \to 0\,. \tag{7.36}$$

By Lemmas 2.6 and 2.8 we have $z_\varepsilon(t) \le \rho_\varepsilon(t) \le \rho_\varepsilon(\tau_\varepsilon) + \varepsilon$ for every $t \in [\tau_\varepsilon, +\infty)$. Therefore (7.14) and (7.36) give

$$\sup_{\tau_\varepsilon \le t \le s_\varepsilon^\eta} |\rho_\varepsilon(t) - z_1| \to 0 \qquad \text{as } \varepsilon \to 0\,, \tag{7.37}$$

so that for ε small enough we obtain $\rho_\varepsilon(t) \ge \frac{1}{2} z_1$ for every $t \in [\tau_\varepsilon, s_\varepsilon^\eta]$.

Since, by (7.36), $z_\varepsilon(t) \le 2\, z_1$ for ε small enough, from the second equation in (2.21) we obtain

$$\varepsilon\, \dot\theta_\varepsilon(t) \ge -\tfrac{2}{z_1}\varepsilon - 4\, (\rho_\varepsilon(t) - z_\varepsilon(t)) \qquad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta]\,, \tag{7.38}$$

so that (7.27) gives

$$\dot w_\varepsilon(t) \le \tfrac{\eta}{4}\, \dot\theta_\varepsilon(t) + \tfrac{\eta}{2z_1} \qquad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta]\,.$$

Since $w_\varepsilon(t) \ge -\varepsilon$ for every $t \in (\tau_\varepsilon, \tilde s_\varepsilon^\eta]$ by (7.22) and (7.25), we deduce that

$$-\varepsilon - w_\varepsilon(\tau_\varepsilon) \le w_\varepsilon(t) - w_\varepsilon(\tau_\varepsilon) \le \tfrac{\eta}{4}(\theta_\varepsilon(t) - \theta_\varepsilon(\tau_\varepsilon)) + \tfrac{\eta}{2z_1}(t - \tau_\varepsilon) \quad \text{for every } t \in (\tau_\varepsilon, \tilde s_\varepsilon^\eta]\,,$$

so that by (7.29) for ε small enough we have

$$\theta_\varepsilon(t) \ge \theta_\varepsilon(\tau_\varepsilon) - (\tfrac{4}{\eta} + \tfrac{2}{\eta z_1}) \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon \quad \text{for every } t \in [\tau_\varepsilon, \tilde s_\varepsilon^\eta]\,. \tag{7.39}$$

Since $\rho_\varepsilon(t) - z_\varepsilon(t) \le \varepsilon^{1-\alpha}$ for every $t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]$ by (7.23) and (7.25), from (7.38) we get

$$\dot\theta_\varepsilon(t) \ge -\tfrac{2}{z_1} - 4\, \varepsilon^{-\alpha} \qquad \text{for every } t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]\,.$$

Integrating and using (7.32) we obtain

$$\theta_\varepsilon(t) - \theta_\varepsilon(\tilde s_\varepsilon^\eta) \ge -(\tfrac{2}{z_1} + 4\, \varepsilon^{-\alpha})\, (t - \tilde s_\varepsilon^\eta) \ge -\tfrac{2\varepsilon^\alpha + 4z_1}{z_1 \sin\theta_0}\, \varepsilon^{1-2\alpha} \quad \text{for every } t \in (\tilde s_\varepsilon^\eta, s_\varepsilon^\eta]\,,$$

which, together with (2.32) and (7.39), gives for ε small enough

$$\theta_\varepsilon(\tau_\varepsilon) - (\tfrac{4}{\eta} + \tfrac{2}{\eta z_1}) \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon - \tfrac{2\varepsilon^\alpha + 4z_1}{z_1 \sin\theta_0}\, \varepsilon^{1-2\alpha} \le \theta_\varepsilon(t) \le \theta_\varepsilon(\tau_\varepsilon)$$

for every $t \in [\tau_\varepsilon, s_\varepsilon^\eta]$. By (7.14) and (7.19) this implies

$$\sup_{\tau_\varepsilon \le t \le s_\varepsilon^\eta} |\theta_\varepsilon(t) - \theta_1| \to 0\,. \tag{7.40}$$

From (7.36), (7.37), and (7.40) we deduce that $s^\eta_\varepsilon < \alpha^\eta_\varepsilon$ for ε small enough. By (7.25) this implies $\tilde{s}^\eta_\varepsilon = \tilde{t}^1_\varepsilon$ and $s^\eta_\varepsilon = t^1_\varepsilon$. Therefore (7.33) gives (7.15), and (7.18) follows from (7.36), (7.37), and (7.40). Since $\tilde{t}^1_\varepsilon < +\infty$, by (7.22) we have $w_\varepsilon(\tilde{t}^1_\varepsilon) \le -\varepsilon$, so that (7.16) follows from (7.27). Since $t^1_\varepsilon < +\infty$, inequality (7.17) follows from the definition of t^1_ε given in (7.23). □

7.2. **Convergence to the fast dynamics.** Assume $\frac{\pi}{2} < \theta_0 \le \pi$, and let

$$z^\infty_\varepsilon := \lim_{t\to+\infty} z_\varepsilon(t) \ge 0\,.$$

By (2.34) there exist functions ϱ_ε and ϑ_ε defined on $(z^\infty_\varepsilon, z_0]$ such that

$$\rho_\varepsilon(t) = \varrho_\varepsilon(z_\varepsilon(t)) \quad \text{and} \quad \theta_\varepsilon(t) = \vartheta_\varepsilon(z_\varepsilon(t)) \quad \text{for every } t \in [t_0, +\infty). \tag{7.41}$$

From Lemma 2.6 it follows that

$$\varrho_\varepsilon(z) > z \quad \text{for every } z \in (z^\infty_\varepsilon, z_0), \tag{7.42}$$

and from (2.32) it follows that

$$\tfrac{\pi}{2} < \vartheta_\varepsilon(z) < \theta_0 < \pi \quad \text{for every } z \in (z^\infty_\varepsilon, z_0). \tag{7.43}$$

By (2.32) and (2.34) we have

$$\vartheta'_\varepsilon(z) > 0 \quad \text{for every } z \in (z^\infty_\varepsilon, z_0).$$

By (2.21) on the intervals $(z^\varepsilon_\infty, z^0_\varepsilon)$ the functions $(\varrho_\varepsilon, \vartheta_\varepsilon)$ are solutions to the system

$$\begin{cases} \varrho'_\varepsilon(z) = -\cos\vartheta_\varepsilon(z) - \dfrac{1}{z(1+\cos\vartheta_\varepsilon(z))\cos\vartheta_\varepsilon(z)} + \varepsilon\dfrac{\sin\vartheta_\varepsilon(z)}{F(z,\varrho_\varepsilon(z),\vartheta_\varepsilon(z))}\,, \\[2ex] \vartheta'_\varepsilon(z) = \dfrac{\sin\vartheta_\varepsilon(z)}{\varrho_\varepsilon(z)} + \varepsilon\dfrac{\cos\vartheta_\varepsilon(z)}{\varrho_\varepsilon(z)\,F(z,\varrho_\varepsilon(z),\vartheta_\varepsilon(z))}\,, \end{cases} \tag{7.44}$$

where

$$F(z,\rho,\theta) := (\rho - z)\, z\,(1+\cos\theta)\cos\theta\,.$$

Let $z_1 > 0$ and $\frac{\pi}{2} < \theta_1 < \pi$. Assume (6.6) or (6.7), and let (ϱ, ϑ), z_2, θ_2 be defined as in Lemma 6.1. Let us fix $\eta > 0$ such that $z_2(1+\cos\theta_1)|\cos\theta_2| > \eta$. This implies that

$$z(1+\cos\vartheta(z))|\cos\vartheta(z)| > \eta \qquad \text{for every } z \in [z_2, z_1]\,. \tag{7.45}$$

Given $\alpha \in (0, \frac12)$, we consider the auxiliary systems

$$\begin{cases} (\varrho^\eta_\varepsilon)'(z) = -\cos\vartheta^\eta_\varepsilon(z) + \dfrac{1}{G^\eta(z,\vartheta^\eta_\varepsilon(z))} - \varepsilon\,\dfrac{|\sin\vartheta^\eta_\varepsilon(z)|}{F^\eta_\varepsilon(z,\varrho^\eta_\varepsilon(z),\vartheta^\eta_\varepsilon(z))}\,, \\[2ex] (\vartheta^\eta_\varepsilon)'(z) = \dfrac{\sin\vartheta^\eta_\varepsilon(z)}{\max\{\varrho^\eta_\varepsilon(z),\eta\}} + \varepsilon\dfrac{|\cos\vartheta^\eta_\varepsilon(z)|}{F^\eta_\varepsilon(z,\varrho^\eta_\varepsilon(z),\vartheta^\eta_\varepsilon(z))}\,, \end{cases} \tag{7.46}$$

where

$$G^\eta(z,\theta) := \max\{z(1+\cos\theta)|\cos\theta|),\eta\}\,,$$
$$F^\eta_\varepsilon(z,\rho,\theta) := G^\eta(z,\theta)\,\max\{\rho - z, \varepsilon^{1-\alpha}\}\,.$$

Note that all solutions of (7.46) are defined for every $z \in \mathbb{R}$.

Lemma 7.4. *Let $z_1 > 0$ and $\frac{\pi}{2} < \theta_1 < \pi$. Assume* (6.6) *or* (6.7)*. Let $\alpha \in (0, \frac12)$, let ρ^1_ε, θ^1_ε, and z^1_ε be three sequences such that*

$$\rho^1_\varepsilon \to z_1\,, \quad \theta^1_\varepsilon \to \theta_1\,, \quad z^1_\varepsilon \to z_1\,, \tag{7.47}$$

$$z^1_\varepsilon\,(1+\cos\theta^1_\varepsilon)^2\cos\theta^1_\varepsilon + 1 < 0\,, \quad \rho^1_\varepsilon \ge z^1_\varepsilon + \varepsilon^{1-\alpha}\,, \tag{7.48}$$

and let $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ be the solution of (7.46) *with Cauchy conditions*

$$\varrho^\eta_\varepsilon(z^1_\varepsilon) = \rho^1_\varepsilon \quad \text{and} \quad \vartheta^\eta_\varepsilon(z^1_\varepsilon) = \theta^1_\varepsilon\,. \tag{7.49}$$

Then there exists $z_1^* \in (0, z_1)$, *depending on* η *in* (7.45), *but not on* ε, *such that for* ε *small enough we have*

$$\varrho_\varepsilon^\eta(z) > z + \varepsilon^{1-\alpha} \qquad \textit{for every } z \in [z_1^*, z_\varepsilon^1)\,, \tag{7.50}$$

Proof. Let us fix $\delta > 0$. From the second equation in (7.46) we have $-\frac{1}{\eta} \le (\vartheta_\varepsilon^\eta)'(z) \le \frac{1}{\eta} + \frac{\varepsilon^\alpha}{\eta}$ for every $z \in \mathbb{R}$, so that for ε small enough we have $|(\vartheta_\varepsilon^\eta)'(z)| \le 1 + \frac{1}{\eta}$. Recalling (7.49), by integrating we get $|\vartheta_\varepsilon^\eta(z) - \theta_\varepsilon^1| < (1 + \frac{1}{\eta})(z_\varepsilon^1 - z_1 + \delta)$ for every $z \in [z_1^*, z_\varepsilon^1]$. Using (7.47) for ε small enough we obtain

$$|\vartheta_\varepsilon^\eta(z) - \theta_1| < (1 + \tfrac{1}{\eta})\, 2\,\delta \qquad \text{for every } z \in [z_1^*, z_\varepsilon^1]. \tag{7.51}$$

Suppose that $z_1 > z_s(\theta_1)$, so that

$$-\cos\theta_1 - \frac{1}{z_1(1+\cos\theta_1)\cos\theta_1} < 1\,.$$

By continuity there exist $\delta_1 > 0$ such that

$$-\cos\theta + \frac{1}{G^\eta(z,\theta)} = -\cos\theta + \frac{1}{\max\{z(1+\cos\theta)|\cos\theta|), \eta\}} < 1 \tag{7.52}$$

for $|\theta - \theta_1| \le \delta_1$ and $|z - z_1| \le \delta_1$. Let us fix $\delta > 0$ with

$$(1 + \tfrac{1}{\eta})\, 2\,\delta < \delta_1\,. \tag{7.53}$$

Since $z_\varepsilon^1 < z_1 + \delta_1$ for ε small enough by (7.47), using (7.51), (7.52), (7.53), and the first equation in (7.46) we deduce that $(\varrho_\varepsilon^\eta)'(z) < 1$ for every $z \in [z_1^*, z_\varepsilon^1]$. As $\varrho_\varepsilon^\eta(z_\varepsilon^1) \ge z_\varepsilon^1 + \varepsilon^{1-\alpha}$ by (7.48), after integration we obtain (7.50).

Suppose now that $\theta_c < \theta_1 < \pi$ and $z_1 = z_s(\theta_1)$. Let us consider the function

$$\omega_\varepsilon^\eta(z) := z\,(1 + \cos\vartheta_\varepsilon^\eta(z))^2 \cos\vartheta_\varepsilon^\eta(z) + 1\,. \tag{7.54}$$

From (7.46) we obtain

$$\begin{aligned}(\omega_\varepsilon^\eta)'(z) = \alpha_\varepsilon^\eta(z)\Big(&\big(\max\{\varrho_\varepsilon^\eta(z), \eta\} - z\big)\cos\vartheta_\varepsilon^\eta(z) - z\,\big(1 + \cos\vartheta_\varepsilon^\eta(z) - 3\cos^2\vartheta_\varepsilon^\eta(z)\big)\Big) - \\ &- \beta_\varepsilon^\eta(z)\, z\,\big(1 + 3\cos\vartheta_\varepsilon^\eta(z)\big)\,\sin\vartheta_\varepsilon^\eta(z)\,,\end{aligned} \tag{7.55}$$

where $\alpha_\varepsilon^\eta(z) \ge 0$ and $\beta_\varepsilon^\eta(z) \ge 0$ for every $z \in \mathbb{R}$. Since $\theta_c < \theta_1 < \pi$, by (3.2) and (3.3) we have $\cos\theta_1 < \cos\theta_c = \lambda_c < -\frac{1}{3}$. This implies that $1 + \cos\theta_1 - 3\cos^2\theta_1 < 0$ and $z_1(1 + 3\cos\theta_1)\sin\theta_1 < 0$. As $z_1 > \eta$ by (7.45), by continuity there exists $\delta_1 > 0$ such that

$$\begin{gathered}\big(\max\{\rho, \eta\} - z\big)\cos\theta - z\,(1 + \cos\theta - 3\cos^2\theta) > 0\,, \\ z\,(1 + 3\cos\theta)\sin\theta < 0\,,\end{gathered} \tag{7.56}$$

for $|\rho - z_1| \le \delta_1$, $|\theta - \theta_1| \le \delta_1$, and $|z - z_1| \le \delta_1$. Let us fix $\delta > 0$ satisfying (7.53).

From the first equation in (7.46) we have $-1 - \frac{\varepsilon^\alpha}{\eta} \le (\varrho_\varepsilon^\eta)'(z) \le 1 + \frac{1}{\eta}$ for every $z \in \mathbb{R}$, so that for ε small enough we have $|(\varrho_\varepsilon^\eta)'(z)| \le 1 + \frac{1}{\eta}$. Recalling (7.49), by integrating we get $|\varrho_\varepsilon^\eta(z) - \rho_\varepsilon^1| < (1 + \frac{1}{\eta})(z_\varepsilon^0 - z_1 + \delta)$ for every $z \in [z_1^*, z_\varepsilon^1]$. Using (7.47) for ε small enough we obtain $|\varrho_\varepsilon^\eta(z) - z_1| < \delta_1$ for every $z \in [z_1^*, z_\varepsilon^1]$.

Since $z_\varepsilon^1 < z_1 + \delta_1$ for ε small enough by (7.47), using (7.51), (7.53), (7.55), and (7.56) we obtain $(\omega_\varepsilon^\eta)'(z) \ge 0$ for every $z \in [z_1^*, z_\varepsilon^1]$. By (7.48), (7.49), and (7.54) we have $\omega_\varepsilon^\eta(z_\varepsilon^1) < 0$. It follows that $\omega_\varepsilon^\eta(z) < 0$ for every $z \in [z_1^*, z_\varepsilon^1]$. This implies

$$-\cos\vartheta_\varepsilon^\eta(z) - \frac{1}{z(1 + \cos\vartheta_\varepsilon^\eta(z))\cos\vartheta_\varepsilon^\eta(z)} < 1 \qquad \text{for every } z \in [z_1^*, z_\varepsilon^1]\,,$$

and hence

$$-\cos\vartheta_\varepsilon^\eta(z) + \frac{1}{G^\eta(z, \cos\vartheta_\varepsilon^\eta(z))} < 1 \qquad \text{for every } z \in [z_1^*, z_\varepsilon^1]\,.$$

Therefore the first equation in (7.46) gives $(\varrho_\varepsilon^\eta)'(z) < 1$ for every $z \in [z_1^*, z_\varepsilon^1]$. As $\varrho_\varepsilon^\eta(z_\varepsilon^1) \ge z_\varepsilon^1 + \varepsilon^{1-\alpha}$ by (7.48), after integration we obtain (7.50). □

Lemma 7.5. *Let $z_1 > 0$ and $\frac{\pi}{2} < \theta_1 < \pi$. Assume (6.6) or (6.7). Let $t_1 \in [t_0, +\infty)$, let $\alpha \in (0, \frac{1}{2})$, and let t^1_ε be a sequence in $[t_0, +\infty)$ such that*

$$t^1_\varepsilon \to t_1\,, \quad \rho_\varepsilon(t^1_\varepsilon) \to z_1\,, \quad \theta_\varepsilon(t^1_\varepsilon) \to \theta_1\,, \quad z_\varepsilon(t^1_\varepsilon) \to z_1\,, \tag{7.57}$$

$$w_\varepsilon(t^1_\varepsilon) < 0\,, \quad \rho_\varepsilon(t^1_\varepsilon) - z_\varepsilon(t^1_\varepsilon) \geq \varepsilon^{1-\alpha}\,. \tag{7.58}$$

Let $z^1_\varepsilon := z_\varepsilon(t^1_\varepsilon)$, let $(\varrho_\varepsilon, \vartheta_\varepsilon)$ be the functions defined in (7.41), and let (ϱ, ϑ) be the solution of (6.1) with Cauchy condition (6.4). Then for ε small enough there exists $z^2_\varepsilon \in (z^\infty_\varepsilon, z^1_\varepsilon)$ such that

$$\varrho_\varepsilon(z^2_\varepsilon) = z^2_\varepsilon + \varepsilon^{1-\alpha} \quad \textit{and} \quad \varrho_\varepsilon(z) > z + \varepsilon^{1-\alpha} \quad \textit{for } z \in (z^2_\varepsilon, z^1_\varepsilon)\,. \tag{7.59}$$

Let $\theta^2_\varepsilon := \vartheta_\varepsilon(z^2_\varepsilon)$. Then

$$z^2_\varepsilon \to z_2 \quad \textit{and} \quad \theta^2_\varepsilon \to \theta_2\,,$$

where z_2 and θ_2 are defined as in Lemma 6.1. Moreover

$$\sup_{z^2_\varepsilon \leq z \leq z^1_\varepsilon} \big(|\varrho_\varepsilon(z) - \varrho(z)| + |\vartheta_\varepsilon(z) - \vartheta(z)|\big) \to 0 \qquad \textit{as } \varepsilon \to 0\,. \tag{7.60}$$

Proof. Let us define

$$\rho^1_\varepsilon := \rho_\varepsilon(t^1_\varepsilon)\,, \quad \theta^1_\varepsilon := \theta_\varepsilon(t^1_\varepsilon)\,, \quad z^1_\varepsilon := z_\varepsilon(t^1_\varepsilon)\,. \tag{7.61}$$

We consider the auxiliary system

$$\begin{cases} (\varrho^\eta)'(z) = -\cos\vartheta^\eta(z) + \dfrac{1}{\max\{z(1+\cos\vartheta^\eta(z))|\cos\vartheta^\eta(z)|, \eta\}}\,, \\ (\vartheta^\eta)'(z) = \dfrac{\sin\vartheta^\eta(z)}{\max\{\varrho^\eta(z), \eta\}}\,, \end{cases} \tag{7.62}$$

whose solutions are defined for every $z \in \mathbb{R}$. Since $1/F^\eta_\varepsilon(z, \rho, \theta) \leq \varepsilon^\alpha/\eta$ for every $(z, \rho, \theta) \in \mathbb{R}^3$, by (7.58) and (7.61) the solutions $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ considered in Lemma 7.4 converge uniformly on compact subsets of $\mathbb{R}$ to the solution $(\varrho^\eta, \vartheta^\eta)$ of (7.62) with Cauchy conditions

$$\varrho^\eta(z_1) = z_1 \quad \text{and} \quad \vartheta^\eta(z_1) = \theta_1\,. \tag{7.63}$$

As $\varrho'(z_2) > 0$ by (6.12), there exists $z^*_2 \in (0, z_2)$ such that (ϱ, ϑ) is defined on $[z^*_2, z_1]$ and

$$\varrho(z) < z \qquad \text{for every } z \in [z^*_2, z_2)\,. \tag{7.64}$$

By (7.45) we may also suppose that

$$z\,(1 + \cos\vartheta(z))|\cos\vartheta(z)| > \eta \qquad \text{for every } z \in [z^*_2, z_1]\,. \tag{7.65}$$

Since (ϱ, ϑ) is a solution of (6.1), inequality (7.65) implies that (ϱ, ϑ) is a solution of (7.62). Since $(\varrho^\eta, \vartheta^\eta)$ and (ϱ, ϑ) satisfy the same Cauchy conditions at z_1 by (6.4) and (7.63), we conclude that $(\varrho^\eta, \vartheta^\eta) = (\varrho, \vartheta)$ on $[z^*_2, z_1]$. Therefore $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ converges to (ϱ, ϑ) uniformly on $[z^*_2, z_1]$. By (7.64) and (7.65) for ε small enough we have $\varrho^\eta_\varepsilon(z^*_2) < z^*_2$ and

$$z\,(1 + \cos\vartheta^\eta_\varepsilon(z))|\cos\vartheta^\eta_\varepsilon(z)| > \eta \qquad \text{for every } z \in [z^*_2, z_1]\,. \tag{7.66}$$

Let z^*_1 be the constant introduced in Lemma 7.4. Since $\varrho^\eta_\varepsilon(z) > z + \varepsilon^{1-\alpha}$ for every $z \in [z^*_1, z^1_\varepsilon)$, we can consider the greatest point z^2_ε of $[z^*_2, z^*_1]$ such that $\varrho^\eta_\varepsilon(z^2_\varepsilon) = z^2_\varepsilon + \varepsilon^{1-\alpha}$, and we have

$$\varrho^\eta_\varepsilon(z^2_\varepsilon) = z^2_\varepsilon + \varepsilon^{1-\alpha} \qquad \text{and} \qquad \varrho^\eta_\varepsilon(z) > z + \varepsilon^{1-\alpha} \qquad \text{for } z \in (z^2_\varepsilon, z^1_\varepsilon)\,. \tag{7.67}$$

The uniform convergence of $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ to (ϱ, ϑ) on $[z^*_2, z^*_1]$ implies that $z^2_\varepsilon \to z_2$ and $\vartheta^\eta_\varepsilon(z^2_\varepsilon) \to \vartheta(z_2) = \theta_2$.

From (7.66) and (7.67) we deduce that $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ satisfies (7.44) in the interval $[z^2_\varepsilon, z^1_\varepsilon]$. Since $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)$ and $(\varrho_\varepsilon, \vartheta_\varepsilon)$ satisfy the same Cauchy conditions at z^1_ε by (7.41), (7.49), and (7.61), we conclude that $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon) = (\varrho_\varepsilon, \vartheta_\varepsilon)$ on $[z^2_\varepsilon, z^1_\varepsilon]$. This implies

$$\sup_{z^2_\varepsilon \leq z \leq z^1_\varepsilon} \big(|\varrho_\varepsilon(z) - \varrho(z)| + |\vartheta_\varepsilon(z) - \vartheta(z)|\big) \to 0$$

and the convergence $\theta^2_\varepsilon := \vartheta_\varepsilon(z^2_\varepsilon) = \vartheta^\eta_\varepsilon(z^2_\varepsilon) \to \theta_2$. $\square$

Lemma 7.6. *Under the assumptions of Lemma 7.5, let* τ^1_ε *be the time such that*

$$z_\varepsilon(\tau^1_\varepsilon) = z^2_\varepsilon\,, \tag{7.68}$$

and let η *be the constant in* (7.45)*. Then*

$$0 < \tau^1_\varepsilon - t^1_\varepsilon < \tfrac{1}{\eta}\,\varepsilon^\alpha$$

for ε *small enough*

Proof. By Lemma 2.6, (2.21), and (7.41) the function $z_\varepsilon(t)$ is a solution of the autonomous equation

$$\varepsilon\dot z_\varepsilon(t) = (\varrho_\varepsilon(z_\varepsilon(t)) - z_\varepsilon(t))\, z_\varepsilon(t)\,(1+\cos\vartheta_\varepsilon(z_\varepsilon(t)))\cos\vartheta_\varepsilon(z_\varepsilon(t)) \tag{7.69}$$

in the interval $[t_1, +\infty)$. Let $z^1_\varepsilon := z_\varepsilon(t^1_\varepsilon)$. As $z_\varepsilon(\tau^1_\varepsilon) = z^2_\varepsilon$ by (7.68), equation (7.69) gives

$$\tau^1_\varepsilon - t^1_\varepsilon = \varepsilon\int_{z^1_\varepsilon}^{z^2_\varepsilon} \frac{dz}{(\varrho_\varepsilon(z) - z)\, z\,(1+\cos\vartheta_\varepsilon(z))\cos\vartheta_\varepsilon(z)}\,,$$

so that the conclusion follows from (7.45) and (7.59). $\square$

Lemma 7.7. *Under the assumptions of Lemma 7.5, let* (ρ^f, θ^f, z^f) *be defined as in Lemma 6.2, and let* τ^1_ε *be the time introduced in Lemma 7.6. Then for every* $\varepsilon > 0$ *there exists* $s_\varepsilon \in \mathbb{R}$ *such that*

$$\sup_{t^1_\varepsilon \le t \le \tau^1_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^f_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta^f_\varepsilon(t)| + |z_\varepsilon(t) - z^f_\varepsilon(t)|\big) \to 0\,, \tag{7.70}$$

where ρ^f_ε, θ^f_ε, *and* z^f_ε *are defined in* (7.7)*.*

Proof. Let ρ^1_ε, θ^1_ε, and z^1_ε be defined as in (7.61). By (7.41) we have $\rho_\varepsilon(t) = \varrho_\varepsilon(z_\varepsilon(t))$ and $\theta_\varepsilon(t) = \vartheta_\varepsilon(z_\varepsilon(t))$ for every $t \in [t_0, +\infty)$, where $(\varrho_\varepsilon, \vartheta_\varepsilon)$ satisfies (7.44) and the Cauchy condition

$$\varrho_\varepsilon(z^1_\varepsilon) = \rho^1_\varepsilon \quad\text{and}\quad \vartheta_\varepsilon(z^1_\varepsilon) = \theta^1_\varepsilon\,. \tag{7.71}$$

Moreover z_ε satisfies (7.69), so that the function $\zeta_\varepsilon(s) := z_\varepsilon(\varepsilon s)$ satisfies the equation

$$\dot\zeta_\varepsilon(s) = (\varrho_\varepsilon(\zeta_\varepsilon(s)) - \zeta_\varepsilon(s))\,\zeta_\varepsilon(s)\,(1+\cos\vartheta_\varepsilon(\zeta_\varepsilon(s)))\,\cos\vartheta_\varepsilon(\zeta_\varepsilon(s)) \tag{7.72}$$

for $s \in [\frac{1}{\varepsilon}t_0, +\infty)$. As explained at the end of the proof of Lemma 7.5 we have $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon) = (\varrho_\varepsilon, \vartheta_\varepsilon)$ on $[z^2_\varepsilon, z^1_\varepsilon]$. By (7.68) the function ζ_ε satisfies also the equation

$$\dot\zeta_\varepsilon(s) = (\varrho^\eta_\varepsilon(\zeta_\varepsilon(s)) - \zeta_\varepsilon(s))\,\zeta_\varepsilon(s)\,(1+\cos\vartheta^\eta_\varepsilon(\zeta_\varepsilon(s)))\,\cos\vartheta^\eta_\varepsilon(\zeta_\varepsilon(s)) \tag{7.73}$$

for $s \in [\frac{1}{\varepsilon}t^1_\varepsilon, \frac{1}{\varepsilon}\tau^1_\varepsilon]$.

By (7.57) and (7.61) we have $z^1_\varepsilon \to z_1$, while $z^2_\varepsilon \to z_2$ by Lemma 7.5. Since $z_2 < z^f(0) < z_1$ by the monotonicity of z^f and by (6.32) and (6.35), we have that $z^2_\varepsilon < z^f(0) < z^1_\varepsilon$ for ε small enough. By (7.61) and (7.68) we have $\zeta_\varepsilon(\frac{1}{\varepsilon}t^1_\varepsilon) = z^1_\varepsilon$ and $\zeta_\varepsilon(\frac{1}{\varepsilon}\tau^1_\varepsilon) = z^2_\varepsilon$. Since ζ_ε is decreasing by (2.34), there exists a unique $s_\varepsilon \in (\frac{1}{\varepsilon}t^1_\varepsilon, \frac{1}{\varepsilon}\tau^1_\varepsilon)$ such that $\zeta_\varepsilon(s_\varepsilon) = z^f(0)$.

Let ζ^η_ε be the maximal solution of (7.73) with Cauchy condition $\zeta^\eta_\varepsilon(0) = z^f(0)$ and let $\zeta^\ominus_\varepsilon$ be the solution of (7.72) on $(-\infty, 0]$ with Cauchy condition $\zeta^\ominus_\varepsilon(0) = z^f(0)$. The theory of autonomous systems implies that ζ^η_ε is defined in a neighborhood of the interval $[0, +\infty)$, is decreasing, and satisfies $\zeta^\eta_\varepsilon(s) \to z_1$ as $s \to +\infty$. Taking into account (7.42) and (7.43), the theory of autonomous systems guarantees that $\zeta^\ominus_\varepsilon$ is defined on the whole interval $(-\infty, 0]$, is decreasing, and satisfies $\zeta^\ominus_\varepsilon(s) \to z_1$ as $s \to -\infty$. By uniqueness we have

$$\zeta_\varepsilon(s) = \zeta^\eta_\varepsilon(s - s_\varepsilon) \qquad \text{for every } s \in [\tfrac{1}{\varepsilon}t^1_\varepsilon, \tfrac{1}{\varepsilon}\tau^1_\varepsilon]\,, \tag{7.74}$$

$$\zeta_\varepsilon(s) = \zeta^\ominus_\varepsilon(s - s_\varepsilon) \qquad \text{for every } s \in [\tfrac{1}{\varepsilon}t_0, \tfrac{1}{\varepsilon}\tau^1_\varepsilon]\,. \tag{7.75}$$

In the proof of Lemma 7.5 we have seen that $(\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon) \to (\varrho, \vartheta)$ as $\varepsilon \to 0$ uniformly on $[z^*_2, z_0]$, where (ϱ, ϑ) is the solution of (6.1) with Cauchy conditions (6.4), and $\delta_2 > 0$

satisfies (7.65). Continuing as in the same proof we can construct z_ε^η, with $z_\varepsilon^\eta \to z_2$ as $\varepsilon \to 0$, such that

$$\varrho_\varepsilon^\eta(z_\varepsilon^\eta) = z_\varepsilon^\eta \quad \text{and} \quad \varrho_\varepsilon^\eta(z) > z \quad \text{for } z \in (z_\varepsilon^\eta, z_\varepsilon^1).$$

Let us prove that ζ_ε^η converges to z^f uniformly on $[s_0, +\infty)$ for every $s_0 < 0$. Let us fix $\lambda > 0$. By (6.35) we can find $s_2 \in (0, +\infty)$ such that $|z^f(s) - z_2| < \lambda$ for any $s \in [s_2, +\infty)$. Since $(\varrho_\varepsilon^\eta, \vartheta_\varepsilon^\eta) \to (\varrho, \vartheta)$ as $\varepsilon \to 0$ uniformly on $[z_2^*, z_0]$, and z^f satisfies (6.36), we have $\zeta_\varepsilon^\eta \to z^f$ uniformly in $[s_0, s_2]$ for every $s_0 < 0$. For $s \geq s_2$ the monotonicity of ζ_ε^η gives

$$\begin{aligned} |\zeta_\varepsilon^\eta(s) - z^f(s)| &\leq |\zeta_\varepsilon^\eta(s) - z_\varepsilon^\eta| + |z_\varepsilon^\eta - z_2| + |z_2 - z^f(s)| \leq \\ &\leq \zeta_\varepsilon^\eta(s_2) - z_\varepsilon^\eta + |z_\varepsilon^\eta - z_2| + \lambda \leq |\zeta_\varepsilon^\eta(s_2) - z^f(s_2)| + 2|z_\varepsilon^\eta - z_2| + 2\lambda, \end{aligned}$$

so that

$$\sup_{s_0 \leq s} |\zeta_\varepsilon^\eta(s) - z^f(s)| \leq \sup_{s_0 \leq s \leq s_2} |\zeta_\varepsilon^\eta(s) - z^f(s)| + 2|z_\varepsilon^\eta - z_2| + 2\lambda.$$

Since z_ε^η tends to z_2 and λ is arbitrary, the uniform convergence of ζ_ε^η to z^f on compact subsets of $\mathbb{R}$ implies the uniform convergence on all of $[s_0, +\infty)$.

Let us prove now that $\zeta_\varepsilon^\ominus$ converges to z^f uniformly on $(\infty, 0]$. We first observe that for every $s_0 < 0$ there exists $\varepsilon_0 > 0$ such that $\zeta_\varepsilon^\eta(s) = \zeta_\varepsilon^\ominus(s)$ for every $s \in [s_0, 0]$ and every $\varepsilon \in (0, \varepsilon_0)$. Indeed, by the uniform convergence of ζ_ε^η to z^f and the properties of z^f listed Lemma 6.2 there exists ε_0 such that $z_\varepsilon^2 < \zeta_\varepsilon^\eta(s) < z_\varepsilon^1$ for every $s \in [s_0, 0]$ and every $\varepsilon \in (0, \varepsilon_0)$. As observed in the proof of Lemma 7.5 we have $(\varrho_\varepsilon^\eta, \vartheta_\varepsilon^\eta) = (\varrho_\varepsilon, \vartheta_\varepsilon)$ on $[z_\varepsilon^2, z_\varepsilon^1]$. This implies that on the interval $[s_0, 0]$ the function ζ_ε^η is in fact solution of (7.72). Since it satisfies the same Cauchy conditions as $\zeta_\varepsilon^\ominus$, by uniqueness we have $\zeta_\varepsilon^\eta(s) = \zeta_\varepsilon^\ominus(s)$ for every $s \in [s_0, 0]$.

Let us fix $\lambda > 0$. By (6.32) we can find $s_0 \in (\infty, 0)$ such that $|\rho^f(s) - z_1| < \lambda$ for any $s \in (-\infty, s_0]$. Since $\zeta_\varepsilon^\ominus = \zeta_\varepsilon^\eta$ on $[s_0, 0]$ for ε small enough, we have that $\zeta_\varepsilon^\ominus \to z^f$ uniformly on $[s_0, 0]$. For $s \leq s_0$ the monotonicity of $\zeta_\varepsilon^\ominus$ gives

$$\begin{aligned} |\zeta_\varepsilon^\ominus(s) - z^f(s)| &\leq |\zeta_\varepsilon^\ominus(s) - z_1| + |z_1 - z^f(s)| \leq \\ &\leq z_1 - \zeta_\varepsilon^\ominus(s_0) + \lambda \leq |\zeta_\varepsilon^\ominus(s_0) - z^f(s_0)| + 2\lambda, \end{aligned}$$

so that

$$\sup_{s \leq 0} |\zeta_\varepsilon^\ominus(s) - z^f(s)| \leq \sup_{s_0 \leq s \leq 0} |\zeta_\varepsilon^\ominus(s) - z^f(s)| + 2\lambda.$$

Since λ is arbitrary, the uniform convergence of $\zeta_\varepsilon^\ominus$ to z^f on compact subsets of $(-\infty, 0]$ implies the uniform convergence on all of $(-\infty, 0]$.

By (7.74) and (7.75), the uniform convergence of ζ_ε^η and $\zeta_\varepsilon^\ominus$ to z^f gives

$$\sup_{\frac{1}{\varepsilon} t_\varepsilon^1 \leq s \leq \frac{1}{\varepsilon} \tau_\varepsilon^1} |\zeta_\varepsilon(s) - z^f(s - s_\varepsilon)| \to 0.$$

Since $z_\varepsilon(t) = \zeta_\varepsilon(\frac{t}{\varepsilon})$, this implies

$$\sup_{t_\varepsilon^1 \leq t \leq \tau_\varepsilon^1} |z_\varepsilon(t) - z_\varepsilon^f(t)| \to 0. \tag{7.76}$$

By Lemma 6.2 we have $\rho^f(s) = \varrho(z^f(s))$ and $\theta^f(s) = \vartheta(z^f(s))$ for every $s \in \mathbb{R}$. By (2.34) and (7.68) we have $z_\varepsilon^2 \leq z_\varepsilon(t) \leq z_\varepsilon^1$ for every $t \in [t_\varepsilon^1, \tau_\varepsilon^1]$. It follows from (7.60) that

$$\sup_{t_\varepsilon^1 \leq t \leq \tau_\varepsilon^1} \big(|\rho_\varepsilon(t) - \varrho(z_\varepsilon(t))| + |\theta_\varepsilon(t) - \vartheta(z_\varepsilon(t))|\big) \to 0.$$

Since $\rho^f_\varepsilon(t) = \varrho(z^f_\varepsilon(t))$ and $\theta^f_\varepsilon(t) = \vartheta(z^f_\varepsilon(t))$, using (7.76) and the uniform continuity of ϱ and ϑ we obtain

$$\begin{aligned}\sup_{t^1_\varepsilon \le t \le \tau^1_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^f_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta^f_\varepsilon(t)|\big) \le \\ \le \sup_{t^1_\varepsilon \le t \le \tau^1_\varepsilon} \big(|\rho_\varepsilon(t) - \varrho(z_\varepsilon(t))| + |\theta_\varepsilon(t) - \vartheta(z_\varepsilon(t))|\big) + \\ + \sup_{t^1_\varepsilon \le t \le \tau^1_\varepsilon} \big(|\varrho(z_\varepsilon(t)) - \varrho(z^f_\varepsilon(t))| + |\vartheta(z_\varepsilon(t)) - \vartheta(z^f_\varepsilon(t)))|\big) \to 0\,,\end{aligned}$$

which, together with (7.76), gives (7.70). □

7.3. **Transition to the slow dynamics.** We now describe the behaviour of the system in a small time interval $[\tau^1_\varepsilon, t_\varepsilon]$ after which the system is governed by the slow dynamics. During this transition $\rho_\varepsilon(t) - z_\varepsilon(t)$ decreases from the value $\varepsilon^{1-\alpha}$, attained at $t = \tau^1_\varepsilon$, to a value of order ε, attained at $t = t_\varepsilon$.

Lemma 7.8. *Let $\frac{\pi}{2} < \theta_2 \le \pi$, let $0 < z_2 < z_s(\theta_2)$, let $t_1 \in [t_0, +\infty)$, let $\alpha \in (0, \frac12)$, and let τ^1_ε be a sequence in $[t_0, +\infty)$ such that*

$$\tau^1_\varepsilon \to t_1\,, \quad \rho_\varepsilon(\tau^1_\varepsilon) \to z_2\,, \quad \theta_\varepsilon(\tau^1_\varepsilon) \to \theta_2\,, \quad z_\varepsilon(\tau^1_\varepsilon) \to z_2\,, \tag{7.77}$$

$$\rho_\varepsilon(\tau^1_\varepsilon) - z_\varepsilon(\tau^1_\varepsilon) = \varepsilon^{1-\alpha}\,. \tag{7.78}$$

Then there exist a sequence t_ε in $[t_0, +\infty)$ and a constants $\beta_1 > 0$ such that

$$\tau^1_\varepsilon < t_\varepsilon \quad \textit{and} \quad t_\varepsilon \to t_1 \quad \textit{as } \varepsilon \to 0\,, \tag{7.79}$$

$$\rho_\varepsilon(t_\varepsilon) - z_\varepsilon(t_\varepsilon) \le \kappa\,\varepsilon \quad \textit{for } \varepsilon \textit{ small enough}\,, \tag{7.80}$$

$$\sup_{\tau^1_\varepsilon \le t \le t_\varepsilon} \big(|\rho_\varepsilon(t) - z_2| + |\theta_\varepsilon(t) - \theta_2| + |z_\varepsilon(t) - z_2)|\big) \to 0 \quad \textit{as } \varepsilon \to 0\,. \tag{7.81}$$

Proof. As $z_2 < z_s(\theta_2)$, we have $z_2(1 + \cos\theta_2)^2 \cos\theta_2 + 1 > 0$. Let $\kappa > 0$ be such that $z_2(1 + \cos\theta_2)^2 \cos\theta_2 + 1 > \frac{2}{\kappa}$. Under our hypotheses, by continuity there exists $\eta > 0$, with $\eta < \frac{z_2}{2}$, such that

$$z(1 + \cos\theta)^2 \cos\theta + 1 \ge \tfrac{2}{\kappa} \quad \text{for } |\theta - \theta_2| < \eta \text{ and } |z - z_2| < \eta\,. \tag{7.82}$$

We define

$$t_\varepsilon := \inf\{t \in (\tau^1_\varepsilon, +\infty) : \rho_\varepsilon(t) - z_\varepsilon(t) \ge \kappa\varepsilon\}\,, \tag{7.83}$$

$$\alpha^\eta_\varepsilon := \inf\{t \in (\tau^1_\varepsilon, +\infty) : |\theta_\varepsilon(t) - \theta_2| + |z_\varepsilon(t) - z_2| > \eta\}\,, \tag{7.84}$$

$$s^\eta_\varepsilon := \min\{t_\varepsilon, \alpha^\eta_\varepsilon\}\,. \tag{7.85}$$

Since $\rho_\varepsilon(t) - z_\varepsilon(t) \ge \kappa\,\varepsilon$ for every $t \in [\tau^1_\varepsilon, t_\varepsilon]$, from (7.82) we obtain

$$(z_\varepsilon(t)(1 + \cos\theta_\varepsilon(t))^2 \cos\theta_\varepsilon(t) + 1)(\rho_\varepsilon(t) - z_\varepsilon(t)) \ge 2\varepsilon \quad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,.$$

Therefore (2.25) gives

$$\dot\rho_\varepsilon(t) - \dot z_\varepsilon(t) \le -\varepsilon \quad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,, \tag{7.86}$$

which, after integration, yields

$$s^\eta_\varepsilon - \tau^1_\varepsilon \le \varepsilon^{1-\alpha}\,. \tag{7.87}$$

By Lemma 2.6, (7.78), and (7.86) we have $0 < \rho_\varepsilon(t) - z_\varepsilon(t) \le \rho_\varepsilon(\tau^1_\varepsilon) - z_\varepsilon(\tau^1_\varepsilon) = \varepsilon^{1-\alpha}$ for every $t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]$. Since $0 < z_\varepsilon(t) \le z_0$ by (2.34), from the third equation in (2.21) we have

$$\dot z_\varepsilon(t) \ge -2\, z_\varepsilon(t)\, \varepsilon^{-\alpha} \ge -2\, z_0\, \varepsilon^{-\alpha} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,,$$

which, together with (2.34) and (7.87), implies

$$z_\varepsilon(\tau^1_\varepsilon) \ge z_\varepsilon(t) \ge z_\varepsilon(\tau^1_\varepsilon) - 2\, z_0\, \varepsilon^{-\alpha}(t - \tau^1_\varepsilon) \ge z_\varepsilon(\tau^1_\varepsilon) - 2\, z_0\, \varepsilon^{1-2\alpha} \quad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,,$$

which gives

$$|z_\varepsilon(t) - z_2| \le |z_\varepsilon(\tau^1_\varepsilon) - z_2| + 2\, z_0\, \varepsilon^{1-2\alpha} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,. \tag{7.88}$$

From the second equation in (2.21) we have

$$\rho_\varepsilon(t)\dot\theta_\varepsilon(t) \le -1 - 2z_0\varepsilon^{-\alpha} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,.$$

Moreover, by Lemma 2.6 we have

$$\rho_\varepsilon(t) > z_\varepsilon(t) \ge z_2 - \eta > \tfrac{z_2}{2} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,. \tag{7.89}$$

Thus, recalling (2.32),

$$0 \ge \dot\theta_\varepsilon(t) \ge -\tfrac{2}{z_2} - 4\,\varepsilon^{-\alpha} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,, \tag{7.90}$$

and, integrating and using (7.87), we obtain

$$\theta_\varepsilon(\tau^1_\varepsilon) \ge \theta_\varepsilon(t) \ge \theta_\varepsilon(\tau^1_\varepsilon) - (\tfrac{2}{z_2} + 4\,\varepsilon^{-\alpha})(t - \tau^1_\varepsilon) \ge \theta_\varepsilon(\tau^1_\varepsilon) - (\tfrac{2}{z_2} + 4)\,\varepsilon^{1-2\alpha}$$

for every $t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]$, which gives

$$|\theta_\varepsilon(t) - \theta_2| \le |\theta_\varepsilon(\tau^1_\varepsilon) - \theta_2| + (\tfrac{2}{z_2} + 4)\,\varepsilon^{1-2\alpha} \qquad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon]\,. \tag{7.91}$$

By (7.77) we have $|z_\varepsilon(\tau^1_\varepsilon) - z_2| + \varepsilon^{1-2\alpha} + |\theta_\varepsilon(\tau^1_\varepsilon) - \theta_2| + (\tfrac{2}{z_2} + 4)\,\varepsilon^{1-2\alpha} < \eta$ for ε small enough. Therefore (7.88) and (7.91) give $s^\eta_\varepsilon < \alpha^\eta_\varepsilon$ for ε small enough. By (7.85) this implies $s^\eta_\varepsilon = t_\varepsilon$, so that (7.87) gives $t_\varepsilon - \tau^1_\varepsilon \le \varepsilon^{1-\alpha}$, which concludes the proof of (7.79). Since $t_\varepsilon < +\infty$, inequality (7.80) follows from the definition of t_ε given in (7.83), while (7.81) follows from (7.88) and (7.91). ☐

7.4. **Softening with discontinuity.** In this subsection we prove Theorems 7.1 and 7.2 describing the softening regime with a discontinuity.

Proof of Theorem 7.1. Let us prove that there exists a sequence τ_ε in $[t_0, +\infty)$ such that

$$\tau_\varepsilon \to t_1\,, \quad \rho_\varepsilon(\tau_\varepsilon) \to z_1, \quad \theta_\varepsilon(\tau_\varepsilon) \to \theta_1, \quad z_\varepsilon(\tau_\varepsilon) \to z_1\,, \tag{7.92}$$

$$\sup_{t_0 \le t \le \tau_\varepsilon} \big(|\rho_\varepsilon(t) - \rho^{sl}_0(t)| + |\theta_\varepsilon(t) - \theta^{sl}_0(t)| + |z_\varepsilon(t) - \rho^{sl}_0(t)|\big) \to 0\,. \tag{7.93}$$

Let us fix an integer $k > 0$. We can apply Lemma 5.4 with $t_* = t_0$, $\tau = t_1 - \frac{1}{k}$, $\theta_* = \theta_0$, $z_* = z_0$, and $t^*_\varepsilon = t_0$. Indeed, (5.18) follows from (3.21). By (5.21) we have

$$\sup_{t_0 \le t \le t_1 - \frac{1}{k}} \big(|\rho_\varepsilon(t) - \rho^{sl}_0(t)| + |\theta_\varepsilon(t) - \theta^{sl}_0(t)| + |z_\varepsilon(t) - \rho^{sl}_0(t)|\big) \to 0\,.$$

Let us fix a decreasing sequence $a_k \to 0$. There exists a decreasing sequence $\varepsilon_k \to 0$ such that for every $\varepsilon \in (0, \varepsilon_k]$ we have

$$\sup_{t_0 \le t \le t_1 - \frac{1}{k}} \big(|\rho_\varepsilon(t) - \rho^{sl}_0(t)| + |\theta_\varepsilon(t) - \theta^{sl}_0(t)| + |z_\varepsilon(t) - \rho^{sl}_0(t)|\big) \le a_k\,. \tag{7.94}$$

We now define $\tau_\varepsilon := t_1 - \frac{1}{k}$ for every $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$. Then $\tau_\varepsilon \to t_1$ as $\varepsilon \to 0$, and (7.93) follows from (7.94). From (7.93) we obtain, in particular,

$$|\rho_\varepsilon(\tau_\varepsilon) - \rho^{sl}_0(\tau_\varepsilon)| + |\theta_\varepsilon(\tau_\varepsilon) - \theta^{sl}_0(\tau_\varepsilon)| + |z_\varepsilon(\tau_\varepsilon) - \rho^{sl}_0(\tau_\varepsilon)| \to 0\,.$$

Since $\tau_\varepsilon \to t^1$, this implies (7.92) thanks to (3.18).

Let us fix $\alpha \in (0, \frac{1}{2})$. By Lemma 7.3 there exists a sequence t^1_ε in $[t_0, +\infty)$ which satisfies (7.15)-(7.18). By (3.18) we have

$$\sup_{\tau_\varepsilon \le t \le t^1_\varepsilon} \big(|\rho^{sl}_0(t) - z_1| + |\theta^{sl}_0(t) - \theta_1|\big) \to 0\,.$$

Together with (7.18) and (7.93), this proves (7.4).

By Lemmas 7.6 and 7.7 there exists $\tau^1_\varepsilon > t^1_\varepsilon$ such that (7.70) holds and $\tau^1_\varepsilon \to t_1$ as $\varepsilon \to 0$. This proves (7.5) and concludes the proof of (7.3).

By Lemma 7.8 there exists a sequence t_ε in $[t_0, +\infty)$ which satisfies (7.79)-(7.81). By (3.23) we have

$$\sup_{\tau^1_\varepsilon \le t \le t_\varepsilon} \big(|\rho^{sl}_2(t) - z_2| + |\theta^{sl}_2(t) - \theta_2|\big) \to 0\,,$$

which, together with (7.81), gives

$$\sup_{\tau_\varepsilon^1\le t\le t_\varepsilon}\big(|\rho_\varepsilon(t)-\rho_2^{sl}(t)|+|\theta_\varepsilon(t)-\theta_2^{sl}(t)|+|z_\varepsilon(t)-\rho_2^{sl}(t)|\big)\to 0\,. \tag{7.95}$$

We can apply now Lemma 5.4 with $t_*=t_1$, $\theta_*=\theta_2$, $z_*=z_2$, and $t_\varepsilon^*=\tau_\varepsilon^1$. Indeed, hypothesis (5.18) follows from (3.26), while (5.19) and (5.20) are satisfied thanks to (7.79)-(7.81). We deduce that (5.21) holds with $\rho_*^{sl}=\rho_2^{sl}$. Together with (7.95), this proves (7.6). Equalities (7.2) follow from (7.3), (7.4), and (7.6). □

Proof of Theorem 7.2. Let us fix $\alpha\in(0,\frac12)$. We apply Lemma 7.3 with $z_1=z_0$, $\theta_1=\theta_0$, $t_1=\tau_\varepsilon=t_0$, and we find a sequence t_ε^1 in $[t_0,+\infty)$ which satisfies (7.15)-(7.18) with $\tau_\varepsilon=t_0$. In particular we have

$$|t_\varepsilon^1-t_1|+|\rho_\varepsilon(t_\varepsilon^1)-z_1|+|\theta_\varepsilon(t_\varepsilon^1)-\theta_1|+|z_\varepsilon(t_\varepsilon^1)-z_1|\to 0\,. \tag{7.96}$$

By Lemmas 7.6 and 7.7 there exists $\tau_\varepsilon^1>t_\varepsilon^1$ which satisfies (7.11) and (7.70). In particular we have

$$|\rho_\varepsilon(t_\varepsilon^1)-\rho_\varepsilon^f(t_\varepsilon^1)|+|\theta_\varepsilon(t_\varepsilon^1)-\theta_\varepsilon^f(t_\varepsilon^1)|+|z_\varepsilon(t_\varepsilon^1)-z_\varepsilon^f(t_\varepsilon^1)|\to 0\,,$$

which, together with (7.96), gives

$$|\rho_\varepsilon^f(t_\varepsilon^1)-z_1|+|\theta_\varepsilon^f(t_\varepsilon^1)-\theta_1|+|z_\varepsilon^f(t_\varepsilon^1)-z_1|\to 0\,, \tag{7.97}$$

By (6.32), (6.34), and (7.7) we have

$$|\rho_\varepsilon^f(t)-z_1|+|\theta_\varepsilon^f(t)-\theta_1|+|\rho_\varepsilon^f(t)-z_1|\le|\rho_\varepsilon^f(t_\varepsilon^1)-z_1|+|\theta_\varepsilon^f(t_\varepsilon^1)-\theta_1|+|\rho_\varepsilon^f(t_\varepsilon^1)-z_1|$$

for every $t\in(-\infty,t_\varepsilon^1]$, so that (7.97) gives

$$\sup_{t_0\le t\le t_\varepsilon^1}\big(|\rho_\varepsilon^f(t)-z_1|+|\theta_\varepsilon^f(t)-\theta_1|+|\rho_\varepsilon^f(t)-z_1|\big)\to 0\,.$$

Together with (7.18) and (7.70) this proves (7.12).

We apply Lemma 7.8 with $t_1=t_0$ and find a sequence t_ε converging to t_0 which satisfies (7.79)-(7.81). As in the proof of Theorem 7.1 we obtain (7.95). We can apply now Lemma 5.4 with $t_*=t_0$, $\theta_*=\theta_2$, $z_*=z_2$, and $t_\varepsilon^*=t_\varepsilon$. Indeed, hypothesis (5.18) follows from (3.26), while (5.19) and (5.20) are satisfied thanks to (7.79)-(7.81). We deduce that (5.21) holds with $\rho_*^{sl}=\rho_2^{sl}$. Together with (7.95), this proves (7.13). Equalities (7.10) follow from (7.11)-(7.13). □

8. Mechanical interpretation of the results

We conclude the paper with some comments on the mechanical interpretation of our results. We first recall that the scalar variables x and y are related to the stress by the formula

$$\sigma(t)=e(t)=-\tfrac1n x(t)I+\tfrac{1}{\sqrt n}y(t)e_0\,,$$

where $e_0\in\mathbb{M}^{n\times n}_{sym}$ is a fixed traceless matrix with unit norm. It follows that $-x(t)$ is the trace of the stress, so that, with the usual sign conventions, $\frac{x(t)}{n}$ is the pressure. The scalar $\frac{1}{\sqrt n}|y(t)|$ is the norm of the deviatoric part of the stress, usually denoted by q in soil mechanics. For simplicity, in what follows we will call x and y the pressure coefficient and the deviatoric stress coefficient.

By (1.6) we have

$$x(t)=z(t)+\rho(t)\cos\theta(t)\qquad\text{and}\qquad y(t)=\rho(t)\sin\theta(t)\,.$$

Since $\rho(t)=z(t)$ for every viscosity solution (Theorems 5.1, 5.2, 7.1, and 7.2), we conclude that

$$x(t)=\rho(t)\,(1+\cos\theta(t))\quad\text{and}\quad y(t)=\rho(t)\sin\theta(t)\quad\text{for every } t\in[t_0,+\infty)\,.$$

From the above mentioned theorems and from Lemmas 3.5-3.7 it follows that $\rho(t)>0$ and $\frac{\pi}{2}\le\theta(t)\le\pi$ for every $t\in[t_0,+\infty)$. Moreover, $\theta(t)=\pi$ only at the initial time

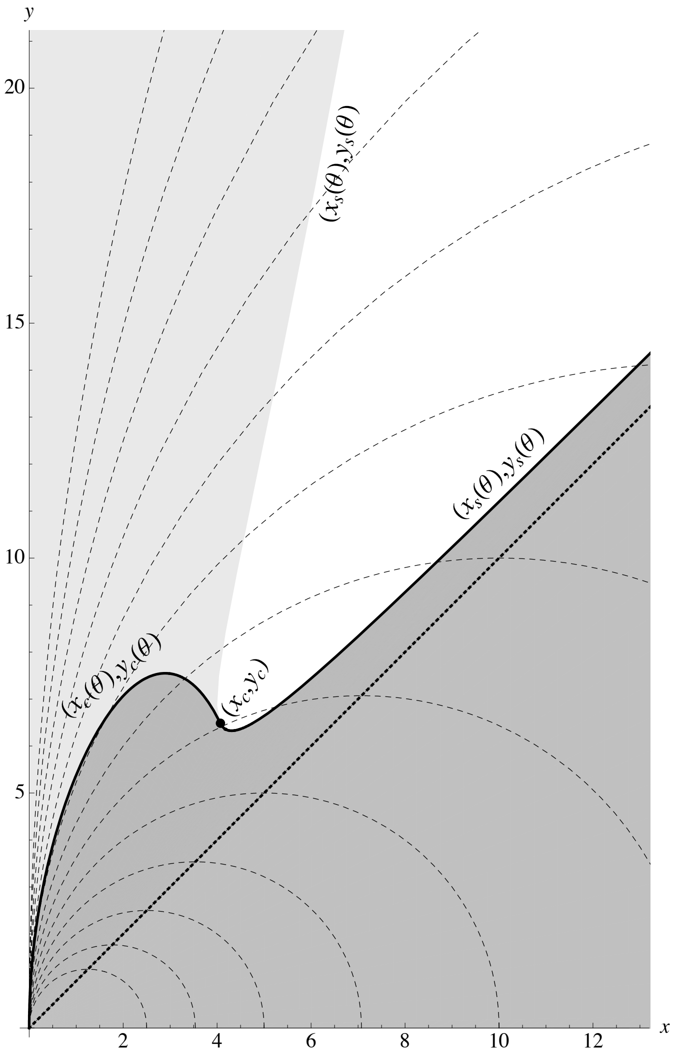

FIGURE 8.1. Phase diagram in the (x, y) plane. Dark grey region (including the thick line): initial data (x_0, y_0) of the plastic regime producing a continuous evolution. Light grey region: initial data producing a discontinuity at time $t_1 > t_0$. White region: initial data producing a discontinuity at time $t_1 = t_0$. The dotted line is composed of fixed points and separates softening behaviour (above the line) from hardening behaviour (below the line).

$t = t_0$ for the special loading program corresponding to $a_0 = 0$ (i.e., in the absence of a preconsolidation pressure, see (1.4)) so that $t_0 = 0$. Using also (2.20) we deduce that

$$x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0 \quad \text{for every } t \in [0, +\infty)\,.$$

and that $x(t) = 0$ if and only if $t = t_0 = 0$ and $a_0 = 0$, while $y(t) = 0$ if and only if $t = 0$.

Plastic behaviour starts at $t = t_0$. The initial data for the plastic regime are given by

$$x_0 := x(t_0) = z_0\,(1 + \cos\theta_0) = a_0 \quad \text{and} \quad y_0 := y(t_0) = z_0 \sin\theta_0 = t_0\,,$$

respectively. In Cartesian coordinates the separation line $\rho = z_s(\theta)$ and the critical line $\rho = r_c(\theta)$ of the (ρ, θ) plane introduced in (3.4) and (3.6) become the parametric curves

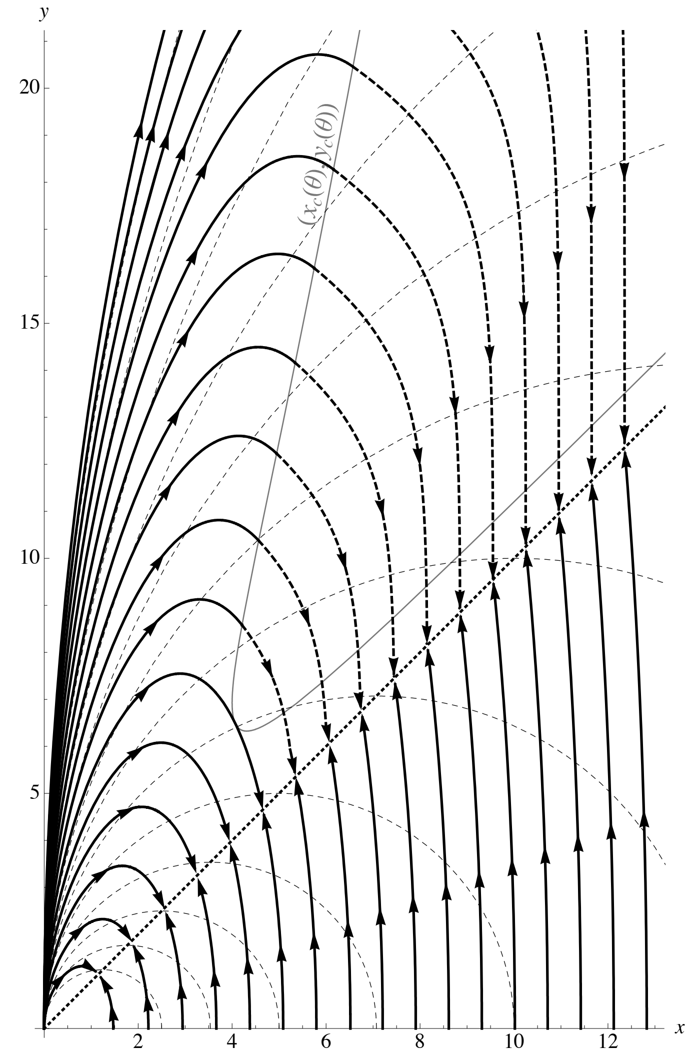

FIGURE 8.2. Trajectories of $(x(t), y(t))$ in the plastic regime for several values of the initial data (x_0, y_0). The evolution for $t > t_0$ is obtained following the trajectory through (x_0, y_0) in the sense of the arrow. Solid lines: slow dynamics. Dashed lines: fast dynamics. Dotted line: fixed points.

defined by

$$x_s(\theta) := z_s(\theta)\,(1+\cos\theta) \quad \text{and} \quad y_s(\theta) := z_s(\theta)\sin\theta \quad \text{for } \theta \in (\tfrac{\pi}{2}, \pi)\,,$$
$$x_c(\theta) := r_c(\theta)\,(1+\cos\theta) \quad \text{and} \quad y_c(\theta) := r_c(\theta)\sin\theta \quad \text{for } \theta \in [\tfrac{\pi}{2}, \pi]\,.$$

The critical point (θ_c, z_c) becomes

$$x_c := z_c\,(1+\cos\theta_c) \quad \text{and} \quad y_c := z_c\sin\theta_c\,.$$

The phase diagram in the (x, y) plane is obtained from Fig. 1.3 by a change of variables and is shown in Fig. 8.1.

The trajectories of $(x(t), y(t))$ are shown in Fig. 8.2, while Fig. 8.3 illustrates the behaviour of $x(t)$ and $y(t)$ as functions of t. Note that, by our choice of the loading program (1.4), t is proportional to the norm of the imposed deviatoric strain. Moreover, we note

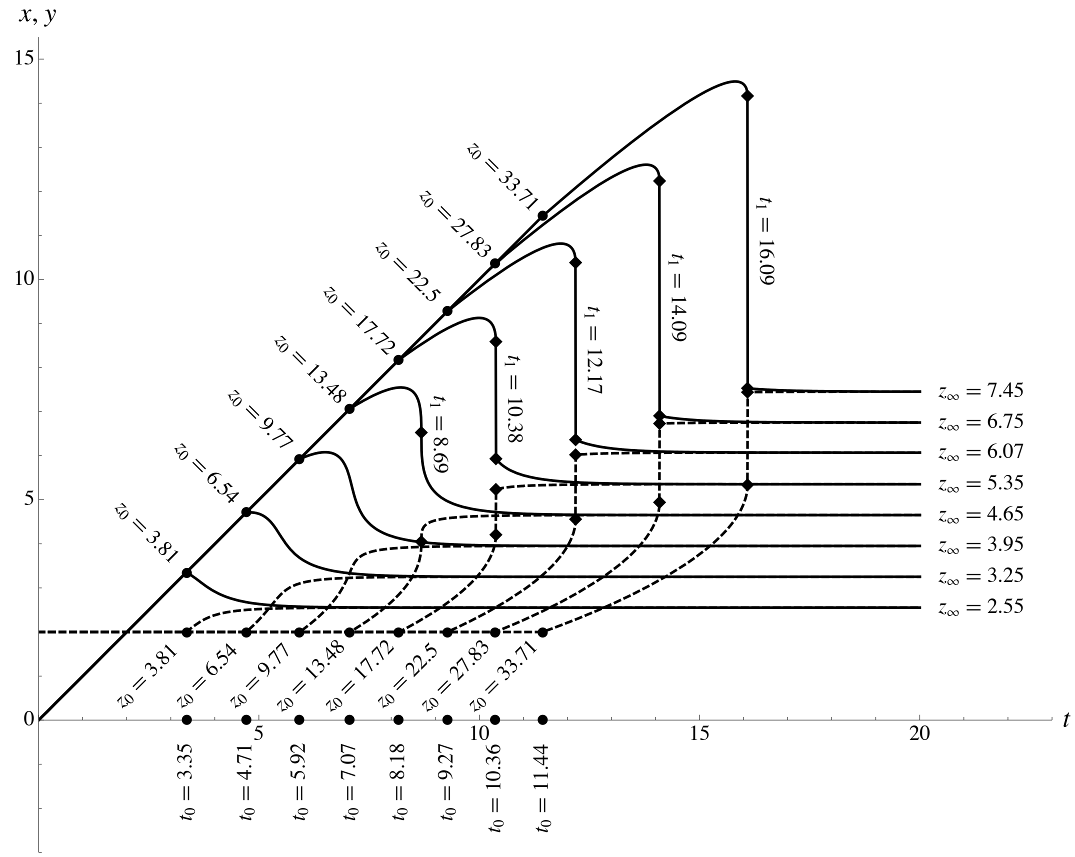

FIGURE 8.3. Deviatoric stress coefficient $y(t)$ and pressure coefficient $x(t)$ as functions of the imposed deviatoric strain t for $a_0 = 2$ and 8 different values of $z_0 > 2$, leading to a softening behaviour. Solid lines: the functions $y(t)$. Dashed lines: the functions $x(t)$.

that the straight line $x = y$ (critical state line) is composed of fixed points. Each trajectory $x(t), y(t)$ tends to a fixed point as $t \to \infty$. The region below the critical state line is invariant, and all solutions therein display a hardening behaviour, namely, $z(t)$ is increasing. Moreover, $y(t)$ is increasing. Both of these properties follow from (3.15) and Theorem 5.1.

In the region above the critical state line the trajectories exhibit softening, namely, $z(t)$ is decreasing. Some trajectories are continuous and follow the system of the slow dynamics. Other trajectories in this region exhibit a discontinuity, which may occur either at $t = t_0$ or at $t > t_0$. They follow the system of the slow dynamics in the intervals of continuity, and their trajectories follow instantaneously the system of the fast dynamics at the jump time. These different behaviours are described in Figure 8.1. The monotonicity of $z(t)$ in the intervals of slow dynamic follows from (3.20), (3.24), and Theorem 5.2. For what concerns the jump governed by the fast dynamics, we observe that under the assumptions of Lemma 6.1, the solution $(\mathtt{x}, \mathtt{y})$ of (6.3) with Cauchy condition (6.5) satisfies $\mathtt{x}'(z) < 0$ and $\mathtt{y}'(z) > 0$ for every $z \in (z_2, z_1)$ in view of (6.9). Finally in the intervals of fast dynamics described by the solution (x^f, y^f, z^f) of (6.31), $\dot{x}^f(s) > 0$, $\dot{y}^f(s) < 0$, and $\dot{z}^f(s) < 0$ for every $s \in \mathbb{R}$ in view of (6.34).

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