A symmetry result for a general class of divergence form PDEs in fibered media

Milena Chermisi and Enrico Valdinoci

MC: *Department of Mathematical Sciences, Carnegie Mellon University, 5000 Forbes Avenue, 15213 Pittsburgh, PA*

EV: *Dipartimento di Matematica, Università di Roma Tor Vergata, I-00133 Roma Italy*

chermisi@andrew.cmu.edu

enrico.valdinoci@uniroma2.it

March 11, 2009

Keywords: possibly degenerate and singular elliptic PDEs, geometric analysis, rigidity and symmetry results.

2000 Mathematics Subject Classification: 35J70.

Abstract. In $\mathbb{R}^m \times \mathbb{R}^{n-m}$, endowed with coordinates $X = (x, y)$, we consider the PDE

$$-\mathrm{div}\,\big(a\big(x, |\nabla u|(X)\big)\nabla u(X)\big) = f(x, u(X)),$$

for which we prove a symmetry result.

EV is supported by FIRB project "Analysis and Beyond" and GNAMPA project "Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni".

1 Introduction

We consider the following PDE:

$$-\mathrm{div}\,\big(a\big(x, |\nabla u|(X)\big)\nabla u(X)\big) = f(x, u(X)) \qquad \text{in } \Omega, \tag{1.1}$$

under the following structural assumptions:

- $f = f(x, u) \in L^\infty(\mathbb{R}^m \times [-R, R])$ is differentiable in u with $f_u \in L^\infty(\mathbb{R}^m \times [-R, R])$, for any $R > 0$,
- $a = a(x, t)$ belongs to $L^\infty(\mathbb{R} \times [\beta_-, \beta_+])$, for any $\beta_+ > \beta_- > 0$,
- for any fixed $t \in (0, +\infty)$, $\inf_{x\in\mathbb{R}^m} a(x, t) > 0$,
- for any fixed $x \in \mathbb{R}^m$, the map $t \mapsto a(x, t)$ belongs to $C^1((0, +\infty))$,

- Ω is an open subset of $\mathbb{R}^n$.

Here, $u = u(X)$, with $X = (x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

The physical motivation for (1.1) comes from "fibered", or "stratified" media: namely, the medium, say $\Omega \subseteq \mathbb{R}^n$ is nonhomogeneous, but this nonhomogeneity only occurs in lower dimensional slices (here, the medium is homogeneous with respect to $y \in \mathbb{R}^{n-m}$ and nonhomogeneous with respect to $x \in \mathbb{R}^m$).

We remark that the assumptions we take here are very general: for instance, our model comprises, as particular cases, $p(x)$-Laplace and mean curvature operators, which may be therefore treated in a unified way (see Appendix A).

As customary, a weak solution of (1.1) is a function u satisfying

$$\int_\Omega a\big(x, |\nabla u|\big)\nabla u \cdot \nabla \xi \, dX = \int_\Omega f(x, u)\xi \, dX \tag{1.2}$$

for any $\xi \in C_0^\infty(\Omega)$.

In what follows, we always assume that

$$u \in C^1(\Omega) \cap C^2(\Omega \cap \{\nabla u \neq 0\}) \cap L^\infty(\Omega) \text{ and that } \nabla u \in L^\infty(\Omega, \mathbb{R}^n) \cap W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n). \tag{1.3}$$

We recall that these regularity assumptions are very mild, and automatically fulfilled in many cases of interest (see, for instance, [5, 17, 4] and the discussion after Theorem 1.1 in [7]).

In the sequel, we consider the map $\mathcal{B} : \mathbb{R}^m \times \big(\mathbb{R}^n \setminus \{0\}\big) \to \mathrm{Mat}(n \times n)$ given by

$$\mathcal{B}(x, \eta)_{ij} := a\big(x, |\eta|\big)\delta_{ij} + a_t\big(x, |\eta|\big)\frac{\eta_i \eta_j}{|\eta|} \tag{1.4}$$

for any $1 \leqslant i, j \leqslant n$, where $\mathrm{Mat}(n \times n)$ denotes the space of square $(n \times n)$-matrices.

We observe that

$$\mathcal{B} \text{ is a symmetric matrix.} \tag{1.5}$$

In what follows, we will suppose that

$$\text{the map } (x, y) \mapsto \mathcal{B}\big(x, \nabla u(x, y)\big) \text{ belongs to } L^\infty(\{\nabla u \neq 0\} \cap B_R) \tag{1.6}$$

for any $R > 0$.

A direct computation gives

$$\frac{d}{d\varepsilon}\Big[a\big(x, |\nabla u + \varepsilon \nabla \varphi|\big)(\nabla u + \varepsilon \nabla \varphi) \cdot \nabla \varphi\Big]_{\varepsilon = 0} = < \mathcal{B}(x, \nabla u)\nabla \varphi, \nabla \varphi > \tag{1.7}$$

for any smooth test function φ, where $<,>$ denotes the standard scalar product in $\mathbb{R}^n$.

We say that u is stable if

$$\int_\Omega < \mathcal{B}(x, \nabla u)\nabla \xi, \nabla \xi > - f_u(x, u)\xi^2 \, dX \;\geqslant\; 0 \tag{1.8}$$

for any $\xi \in C_0^\infty(\Omega)$.

We remark that the above integral is finite, thanks to (1.6).

The notion of stability given in (1.8) appears naturally in the calculus of variations setting and it is usually related to minimization and monotonicity properties. In particular, (1.7) and (1.8) state that the (formal) second variation of the energy functional associated to the equation has a sign (see also [11, 9, 1, 7, 3] for further stability results).

Moreover, a natural geometric condition that implies stability is monotonicity in one direction (for details, see Lemma B.1).

The main result of this paper is the following one:

Theorem 1.1. *Let u be a stable weak solution of* (1.1) *in whole $\mathbb{R}^n$.*
Assume that $\mathcal{B}(x, \nabla u(X))$ is positive definite at almost any $X = (x, y) \in \mathbb{R}^n$ and let $\mathcal{B}_\star(X)$ be its largest eigenvalue.
Suppose that there exists $C_o \geqslant 1$ in such a way that

$$\int_{B_R} \mathcal{B}_\star(X)\, |\nabla u(X)|^2 \, dX \leqslant C_o \, R^2, \tag{1.9}$$

for any $R \geqslant C_o$.
Then, there exist $\omega : \mathbb{R}^m \to \mathrm{S}^{n-m-1}$ and $u_o : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = u_o(x, \omega(x) \cdot y)$$

for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.
Also $\omega(x)$ is constant[1] *in any connected component of $\{\nabla_y u \neq 0\}$.*

Theorem 1.1 strengthens the results of [3], where we focused on the case of the $p(x)$-Laplace operators (in fact, only the case $p(x) \geqslant 2$ was considered in [3], while we can consider here also the case $p(x) > 1$, see Appendix A).

The first symmetry results for fibered nonlinearity were given in [12] (see also [13, 14] for a relation with fractional operators).

For explicit conditions that imply the energy bound in (1.9), we refer to Appendix B in [3].

Theorem 1.1 is proven in Section 4, after we obtain a general geometric formula in Section 2 and we perform some geometric analysis in Section 3. The paper ends with an Appendix, which contains the applications to $p(x)$-Laplace and mean curvature operators and recalls the relation between monotone and stable solutions.

2 A general geometric inequality

The goal of this section is to provide our framework with a general weighted Poincaré inequality (see the forthcoming Theorem 2.1), in which the L^2-norm of any test functions is bounded by the L^2-norm of its gradient, where the norms are appropriately weighted.

Remarkably, such weights have nice geometric meanings, which make such an inequality feasible for the application.

We recall that [15, 16] introduced a similar weighted Poincaré inequality in the classical uniformly elliptic semilinear framework.

The idea of making use of Poincaré-type inequalities on level sets to deduce suitable symmetries for the solutions was already in [6] and it has been also used in [2, 7, 13, 14]. For related Sobolev-Poincaré inequalities, see also [8].

Now, we state the following notation.

Fixed $x \in \mathbb{R}^m$ and $c \in \mathbb{R}$, we look at the level set

$$S := \{y \in \mathbb{R}^{n-m} \, : \, u(x, y) = c\}.$$

We will consider the regular points of S, that is, we define

$$L := \{y \in S \, : \, \nabla_y u(y, x) \neq 0\}.$$

[1] We stress that in the particular case of solutions u for which $\{\nabla_y u = 0\} = \emptyset$, Theorem 1.1 states that $\omega(x) = \omega$ is constant and so $u(x, y) = u_o(x, \omega \cdot y)$ for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

That is, u depends only on $(m + 1)$ Euclidean variables, namely it depends on $x \in \mathbb{R}^m$ and on the variable $t := \omega \cdot y$.

Note that L depends on the $x \in \mathbb{R}^m$ that we have fixed at the beginning, though we do not keep explicit track of this in the notation. In the same way, S has to be thought as the level set of u on the slice selected by the fixed x.

Let ∇_L to be the tangential gradient along L, that is, for any $y_o \in L$ and any $G : \mathbb{R}^{n-m} \to \mathbb{R}$ smooth in the vicinity of y_o, we set

$$\nabla_L G(y_o) := \nabla_y G(y_o) - \left(\nabla_y G(y_o) \cdot \frac{\nabla_y u(x, y_o)}{|\nabla_y u(x, y_o)|} \right) \frac{\nabla_y u(x, y_o)}{|\nabla_y u(x, y_o)|}. \tag{2.1}$$

Since L is a smooth $(n - m - 1)$-manifold, in virtue of the Implicit Function Theorem and (1.3), we can define the principal curvatures on it, denoted by

$$\kappa_1(x, y), \dots, \kappa_{n-m-1}(x, y),$$

for any $y \in L$.

We will then define the "total curvature" (also named "length of the second fundamental form" in the differential geometry textbooks) as

$$\mathcal{K}(x, y) := \sqrt{\sum_{j=1}^{n-m-1} \big(\kappa_j(x, y)\big)^2}.$$

Then, the geometric inequality which fits our scope is the following one:

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set.*
Assume that u is a stable weak solution of (1.1) in Ω.
Then,

$$\begin{aligned}
&\int_{\mathcal{R}} a\big(x, |\nabla u|\big) \big(\mathcal{S} + \mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2\big) \phi^2 + \frac{a_t\big(x, |\nabla u|\big)}{|\nabla u|} \mathcal{T} \phi^2 \\
&= \int_{\mathcal{R}} \Big[\sum_{j=1}^{n-m} < \mathcal{B}(x, \nabla u)\, \nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}(x, \nabla u)\, \nabla |\nabla_y u|, \nabla |\nabla_y u| > \Big] \phi^2 \\
&\qquad \leqslant \int_{\Omega} |\nabla_y u|^2 < \mathcal{B}(x, \nabla u) \nabla \phi, \nabla \phi >
\end{aligned} \tag{2.2}$$

for any $\phi \in C_0^\infty(\Omega)$, where

$$\mathcal{R} := \{(x, y) \in \Omega \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m} \,:\, \nabla_y u(x, y) \neq 0\}, \tag{2.3}$$

$$\mathcal{S} := \sum_{i=1}^{m} \sum_{j=1}^{n-m} (u_{x_i y_j})^2 - |\nabla_x |\nabla_y u||^2 \qquad \textit{and} \tag{2.4}$$

$$\mathcal{T} := \sum_{j=1}^{n-m} (\nabla u \cdot \nabla u_{y_j})^2 - (\nabla u \cdot \nabla |\nabla_y u|)^2. \tag{2.5}$$

Also

$$\mathcal{S},\, \mathcal{T} \geqslant 0 \textit{ on } \mathcal{R} \tag{2.6}$$

and

$$\begin{aligned}
&\mathcal{S}(X) = 0 \textit{ at some } X \in \mathbb{R}^n \\
&\textit{if and only if } \nabla_y u_{x_i}(X) \textit{ is parallel to } \nabla_y u(X) \\
&\textit{for any } i = 1, \dots, m.
\end{aligned} \tag{2.7}$$

Proof. By (1.4), we have that

$$\begin{aligned}&\int_\Omega a\big(x,|\nabla u|\big)\nabla u\cdot\Psi_{y_j}\\&=-\int_\Omega\left(a\big(x,|\nabla u|\big)\nabla u_{y_j}\cdot\Psi+a_t\big(x,|\nabla u|\big)\frac{\nabla u\cdot\nabla u_{y_j}}{|\nabla u|}\nabla u\cdot\Psi\right)\\&=-\int_\Omega<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\Psi>.\end{aligned}\tag{2.8}$$

for any $j=1,\ldots,n-m$ and any $\Psi\in C_0^\infty(\Omega,\mathbb{R}^{n-m})$.
The use of (1.2) and (2.8) with $\Psi:=\nabla\psi$ yields

$$\begin{aligned}\int_\Omega f_u(x,u)u_{y_j}\psi&=\int_\Omega(f(x,u))_{y_j}\psi=-\int_\Omega f(x,u)\psi_{y_j}=-\int_\Omega a\big(x,|\nabla u|\big)\nabla u\cdot\nabla\psi_{y_j}\\&=\int_\Omega<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\nabla\psi>\end{aligned}\tag{2.9}$$

for any $j=1,\ldots,n-m$ and any $\psi\in C_0^\infty(\Omega)$.
Actually, using (1.3) and (1.6), a standard density argument (see, e.g., formulas (2.2)–(2.8) in [3] for details) gives that

$$(2.9)\text{ holds for any }\psi\in W_0^{1,2}(\Omega)\tag{2.10}$$

and

$$(1.8)\text{ holds for any }\xi\in W_0^{1,2}(\Omega).\tag{2.11}$$

Thus, from (1.3) and (2.10), we may take $\psi:=u_{y_j}\phi^2$ in (2.9), where $\phi\in C_0^\infty(\Omega)$: we obtain that

$$\begin{aligned}&\int_\Omega\big[<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\nabla u_{y_j}>\phi^2+<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\nabla\phi^2>u_{y_j}\big]\\&\qquad=\int_\Omega f_u(x,u)u_{y_j}^2\phi^2.\end{aligned}\tag{2.12}$$

Now, we notice that, by (1.3) and Stampacchia's Theorem (see, e.g., Theorem 6.19 in [10]),

$$\begin{aligned}&\nabla|\nabla_y u|=0=\nabla u_{y_j}\\&\text{for a.e. }x\in\mathbb{R}^m\text{ and a.e. }y\in\mathbb{R}^{n-m}\text{ such that }\nabla_y u(x,y)=0.\end{aligned}\tag{2.13}$$

By (2.3), (2.12) and (2.13), we obtain

$$\int_{\mathcal{R}}\big[<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\nabla u_{y_j}>\phi^2+<\mathcal{B}(x,\nabla u)\nabla u_{y_j},\nabla\phi^2>u_{y_j}\big]=\int_\Omega f_u(x,u)u_{y_j}^2\phi^2.$$

We now sum over $j=1,...,n-m$ (dropping, for short, the dependences of $\mathcal{B}$) and we obtain

$$\int_{\mathcal{R}}\left[\sum_{j=1}^{n-m}<\mathcal{B}\nabla u_{y_j},\nabla u_{y_j}>\phi^2+\frac{1}{2}<\mathcal{B}\nabla|\nabla_y u|^2,\nabla\phi^2>\right]=\int_\Omega f_u(x,u)|\nabla_y u|^2\phi^2.\tag{2.14}$$

Now, we recall (2.11) and we choose $\xi:=|\nabla_y u|\phi$ in (1.8), obtaining

$$\begin{aligned}0\leqslant\int_{\mathcal{R}}\Big[&<\mathcal{B}\nabla|\nabla_y u|,\nabla|\nabla_y u|>\phi^2+<\mathcal{B}\nabla\phi,\nabla\phi>|\nabla_y u|^2\\&+2<\mathcal{B}\nabla|\nabla_y u|,\nabla\phi>|\nabla_y u|\phi\Big]-\int_\Omega f_u(x,u)|\nabla_y u|\phi^2,\end{aligned}$$

where (2.13) has been used once more.
This and (2.14) imply that

$$0 \leqslant \int_{\mathcal{R}} \Big[< \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \phi^2 + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 - \sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 \Big]. \tag{2.15}$$

Furthermore, by using (1.4) and (2.15), we are lead to the following equality:

$$\begin{aligned} &\sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \\ &\qquad = a\big(x, |\nabla u|\big)\mathcal{U} + \frac{a_t\big(x, |\nabla u|\big)}{|\nabla u|}\mathcal{T}, \end{aligned} \tag{2.16}$$

where $\mathcal{T}$ is as in (2.5) and

$$\mathcal{U} := \sum_{j=1}^{n-m} |\nabla u_{y_j}|^2 - \big|\nabla|\nabla_y u|\big|^2.$$

We also let $\mathcal{S}$ be as in (2.4): then, making use of formula (2.1) of [15], we have that, on $\mathcal{R}$,

$$\mathcal{U} - \mathcal{S} = \sum_{i,j=1}^{n-m} (u_{y_i y_j})^2 - \big|\nabla_y|\nabla_y u|\big|^2 = \mathcal{K}^2|\nabla_y u|^2 + |\nabla_L|\nabla_y u||^2. \tag{2.17}$$

Accordingly, plugging (2.17) into (2.16) yields

$$\begin{aligned} &\sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \\ &\quad = a\big(x, |\nabla u|\big)\left(\mathcal{S} + \mathcal{K}^2|\nabla_y u|^2 + |\nabla_L|\nabla_y u||^2\right) + \frac{a_t\big(x, |\nabla u|\big)}{|\nabla u|}\mathcal{T} \qquad \text{on } \mathcal{R}. \end{aligned} \tag{2.18}$$

Hence, formula (2.2) follows from (2.15) and (2.18).
Furthermore, if we set

$$\zeta_j := \nabla u \cdot \nabla u_{y_j} \qquad \text{for } j = 1, \ldots, n-m,$$

and

$$\zeta := (\zeta_1, \ldots, \zeta_{n-m}) \in \mathbb{R}^{n-m},$$

we have that, on $\mathcal{R}$,

$$\begin{aligned} -\mathcal{T} &= \left(\sum_{\ell=1}^{n} \partial_\ell u \partial_\ell |\nabla_y u|\right)^2 - |\xi|^2 = \left(\sum_{\ell=1}^{n} \partial_\ell u \frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y \partial_\ell u\right)^2 - |\xi|^2 \\ &= \left(\frac{\nabla_y u}{|\nabla_y u|} \cdot \xi\right)^2 - |\xi|^2 \leqslant 0, \end{aligned} \tag{2.19}$$

thanks to Cauchy-Schwarz inequality.
Analogously, for any $i = 1, \ldots, m$, on $\mathcal{R}$,

$$\big|\partial_{x_i}|\nabla_y u|\big| = \left|\frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y u_{x_i}\right| \leqslant |\nabla_y u_{x_i}| = \sqrt{\sum_{j=1}^{n-m} (u_{x_i y_j})^2}, \tag{2.20}$$

and

$$\text{equality holds in (2.20) if and only if } \nabla_y u_{x_i} \text{ is parallel to } \nabla_y u. \tag{2.21}$$

Therefore, from (2.20),

$$\begin{aligned}-\mathcal{S} &= |\nabla_x|\nabla_y u||^2 - \sum_{i=1}^{m}\sum_{j=1}^{n-m}(u_{x_i y_j})^2 \\ &= \sum_{i=1}^{m}\big(\partial_{x_i}|\nabla_y u|\big)^2 - \sum_{i=1}^{m}\sum_{j=1}^{n-m}(u_{x_i y_j})^2 \leqslant 0.\end{aligned}$$

This, (2.19) and (2.21) give (2.6) and (2.7), thus completing the proof of Theorem 2.1. ∎

3 Geometric Lemmata

The material of this section is closely related to the geometric analysis performed in [14] in the framework of fractional operators.

If $\mathcal{B}(x,\nabla u(X))$ is positive definite at some point $X=(x,y)\in\mathbb{R}^n\cap\{\nabla u\neq 0\}$, then given $\zeta\in\mathbb{R}^{n+1}$, we can define

$$\|\zeta\|_{\mathcal{B}(X)} := \sqrt{<\mathcal{B}(x,\nabla u(X))\zeta,\zeta>}.$$

The positive definiteness of $\mathcal{B}$ gives that the above definition is well-posed and, in view of (1.5), it is a norm.

Lemma 3.1. *If $\mathcal{B}(x,\nabla u(X))$ is positive definite at some point $X=(x,y)\in\mathbb{R}^n$, then*

$$\sum_{j=1}^{n-m} <\mathcal{B}(x,\nabla u(X))\nabla u_{y_j}(X),\nabla u_{y_j}(X)> - <\mathcal{B}(x,\nabla u(X))\nabla|\nabla_y u(X)|,\nabla|\nabla_y u(X)|> \geqslant 0. \tag{3.1}$$

Moreover, if equality holds in (3.1) *and $X\in\mathbb{R}^n\cap\{\nabla_y u\neq 0\}$, then there exist $v\in\mathrm{S}^{n-1}$, $c\in\mathbb{R}$, $c_1,\dots,c_n\geqslant 0$ such that*

$$\begin{aligned} &u_{y_j}(X)\nabla u_{y_j}(X) = c_j v \\ \textit{and}\qquad &\|\nabla u_{y_j}(X)\|_{\mathcal{B}(X)} = c|u_{y_j}(X)|, \qquad \textit{for } j=1,\dots,n-m. \end{aligned} \tag{3.2}$$

Proof. By Cauchy-Schwarz inequality, if $z=(z_1,\dots,z_{n-m}) := \nabla_y u$, we have that, at the point X,

$$\begin{aligned}\sqrt{<\mathcal{B}\nabla|\nabla_y u|,\nabla|\nabla_y u|>} &= \|\,\nabla|z|\,\|_{\mathcal{B}} = \Big\|\sum_{j=1}^{n-m}\frac{z_j}{|z|}\nabla z_j\Big\|_{\mathcal{B}} \\ &\leqslant \sum_{j=1}^{n-m}\Big\|\frac{z_j}{|z|}\nabla z_j\Big\|_{\mathcal{B}} = \frac{1}{|z|}\sum_{j=1}^{n-m}|z_j|\Big\|\nabla z_j\Big\|_{\mathcal{B}} \\ &\leqslant \frac{1}{|z|}\sqrt{\sum_{j=1}^{n-m}|z_j|^2}\sqrt{\sum_{j=1}^{n-m}\Big\|\nabla z_j\Big\|_{\mathcal{B}}^2} = \sqrt{\sum_{j=1}^{n-m}<\mathcal{B}\nabla u_{y_j},\nabla u_{y_j}>}.\end{aligned} \tag{3.3}$$

This gives (3.1).

Also, if equality holds in (3.1) (and therefore in (3.3)), the above computation says that all the vectors $(z_j/|z_j|)\nabla z_j$ are parallel and in the same direction (hence $z_j\nabla z_j$ are all parallel and in the same direction; thus, the first equality in (3.2) holds) and that $d\|\nabla z_j\|_{\mathcal{B}}+b|z_j|=0$ for some d, $b\in\mathbb{R}$, not both zero.

Since $\nabla_y u\neq 0$, with no loss of generality we may take $|z_1|=|u_{y_1}|\neq 0$, that implies $d\neq 0$. This gives the second inequality in (3.2). ∎

Corollary 3.2. *Let $\Omega \subseteq \mathbb{R}^n \cap \{\nabla_y u \neq 0\}$ be an open set for which $\mathcal{B}(x, \nabla u(X))$ is positive definite for almost any $X = (x, y)$ in Ω.*
Suppose that

$$\sum_{j=1}^{n-m} < \mathcal{B}(x, \nabla u(X))\nabla u_{y_j}(X), \nabla u_{y_j}(X) > - < \mathcal{B}(x, \nabla u(X))\nabla|\nabla_y u(X)|, \nabla|\nabla_y u(X)| >= 0 \qquad (3.4)$$

for almost any $X = (x, y)$ in Ω.
Then, for any level set L of u and any $x \in \Omega \cap L$, we have that

$$\mathcal{K} = 0 = |\nabla_L |\nabla_y u|| \qquad (3.5)$$

and

$$\nabla_y u_{x_i} \text{ is parallel to } \nabla_y u \text{ for any } i = 1, \dots, m \;. \qquad (3.6)$$

Proof. We make use of Lemma 3.1: accordingly, by (3.4) and (3.2), for almost any $X \in \Omega$, we can write

$$\begin{aligned} & u_{y_j}(X)\nabla u_{y_j}(X) = c_j(X)v(X) \\ \text{and} \qquad & \|\nabla u_{y_j}(X)\|_{\mathcal{B}(X)} = c(X)|u_{y_j}(X)|, \qquad \text{for } j = 1, \dots, n, \end{aligned} \qquad (3.7)$$

for suitable $v(X) \in \mathrm{S}^{n-1}$, $c(X) \in \mathbb{R}$, $c_1(X), \dots, c_n(X) \geqslant 0$.
In particular,

$$c_j(X)\|v(X)\|_{\mathcal{B}(X)} = \|u_{y_j}(X)\nabla u_{y_j}(X)\|_{\mathcal{B}(X)} = |c(X)|\, u^2_{y_j}(X)$$

and so, since $v(X) \neq 0$,

$$c_j(X) = \lambda(X)u^2_{y_j}(X), \qquad (3.8)$$

for a suitable $\lambda(X) \geqslant 0$.
Now, we define

$$\bar{c}_j(X) := \begin{cases} c_j(X)/u_{y_j}(X) & \text{if } u_{y_j}(X) \neq 0, \\ 0 & \text{if } u_{y_j}(X) = 0. \end{cases}$$

By Stampacchia Theorem (see, e.g., Theorem 6.19 in [10]), $\nabla u_{y_j} = 0$ almost everywhere in $\Omega \cap \{u_{y_j} = 0\}$, hence (3.7) says that

$$\nabla u_{y_j}(X) = \bar{c}_j(X)v(X) \qquad (3.9)$$

for almost any $x \in \Omega$.
Notice also that

$$\bar{c}_j(X) = \lambda(X)u_{y_j}(X), \qquad (3.10)$$

because of (3.8).
As a consequence, writing

$$v(X) = \big(v_1(X), \dots, v_m(X), v'(X)\big) \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \qquad (3.11)$$

we deduce from (3.9), (3.11) and (3.10) that

$$
\begin{aligned}
&\sum_{j=1}^{n-m} |\nabla_y u_{y_j}(X)|^2 - |\nabla_y |\nabla_y u(X)||^2 \\
&= \sum_{j=1}^{n-m} |\nabla_y u_{y_j}(X)|^2 - \Big| \sum_{j=1}^{n-m} \frac{u_{y_j}(X)}{|\nabla_y u(X)|} \nabla_y u_{y_j}(X) \Big|^2 \\
&= \sum_{j=1}^{n-m} |\bar{c}_j(X) v'(X)|^2 - \Big| \sum_{j=1}^{n-m} \frac{u_{y_j}(X)}{|\nabla_y u(X)|} \bar{c}_j(X) v'(X) \Big|^2 \\
&= |v'(X)|^2 \left(\sum_{j=1}^{n-m} \bar{c}_j^2(X) - |\nabla_y u(X)|^{-2} \Big| \sum_{j=1}^{n-m} u_{y_j}(X) \bar{c}_j(X) \Big|^2 \right) \\
&= |v'(X)|^2 \left(\sum_{j=1}^{n-m} \lambda^2(X) u_{y_j}^2(X) - |\nabla_y u(X)|^{-2} \Big| \sum_{j=1}^{n-m} \lambda(X) u_{y_j}^2(X) \Big|^2 \right) \\
&= |v'(X)|^2 \lambda^2(X) \Big(|\nabla_y u(X)|^2 - |\nabla_y u(X)|^{-2} |\nabla_y u(X)|^4 \Big) \;=\; 0.
\end{aligned}
$$

This and (2.17) imply that

$$\mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2 = 0$$

almost everywhere in Ω, and in fact everywhere in Ω by continuity, thus proving (3.5). What is more, exploiting (3.9) and (3.10), we see that for every $i = 1, \ldots, m$

$$u_{x_i y_j}(X) = \bar{c}_j(X) v_i(X) = \lambda(X) v_i(X) u_{y_j}(X),$$

and so

$$\nabla_y u_{x_i}(X) = \lambda(X) v_i(X) \nabla_y u(X),$$

almost everywhere in Ω, and in fact everywhere in Ω by continuity, which proves (3.6). ∎

4 Proof of Theorem 1.1

Given $\rho_1 \leqslant \rho_2$, we define

$$\mathcal{A}_{\rho_1,\rho_2} := \{ X \in \mathbb{R}^n \,:\, |X| \in [\rho_1, \rho_2] \}. \tag{4.1}$$

From (1.9) and Lemma A.2 of [3], applied here with

$$h(X) := \mathcal{B}_\star(X) |\nabla_y u(X)|^2,$$

we obtain

$$\int_{\mathcal{A}_{\sqrt{R},R}} \frac{\mathcal{B}_\star(X) |\nabla_y u(X)|^2}{|X|^2} \leqslant C_1 \log R \tag{4.2}$$

for a suitable $C_1 > 0$, if R is big.
Now we define

$$\phi_R(X) := \begin{cases} \log R & \text{if } |X| \leqslant \sqrt{R}, \\ 2 \log \big(R/|X| \big) & \text{if } \sqrt{R} < |X| < R, \\ 0 & \text{if } |X| \geqslant R \end{cases}$$

and we observe that

$$|\nabla \phi_R| \leqslant \frac{C_2\, \chi_{\mathcal{A}_{\sqrt{R},R}}}{|X|}, \tag{4.3}$$

for a suitable $C_2 > 0$.
Thus, plugging ϕ_R in (2.2) and recalling (3.1) and (4.3),

$$\begin{aligned} 0 &\leqslant (\log R)^2 \int_{\mathcal{R}\cap B_{\sqrt{R}}} \Big[\sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \Big] \\ &\leqslant \int_{\mathcal{R}} \Big[\sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \Big] \phi_R^2 \\ &\leqslant \int_{\mathcal{R}} < \mathcal{B}\nabla\phi_R, \nabla\phi_R > |\nabla_y u|^2 \\ &\leqslant \int_{\mathcal{R}\cap\mathcal{A}_{\sqrt{R},R}} \mathcal{B}_\star \, |\nabla\phi_R|^2 \, |\nabla_y u|^2 \\ &\leqslant C_2^2 \int_{\mathcal{A}_{\sqrt{R},R}} \frac{\mathcal{B}_\star \, |\nabla_y u|^2}{|X|^2}. \end{aligned} \tag{4.4}$$

Now, we divide (4.4) by $(\log R)^2$, we use (4.2) and we send $R \to +\infty$. In this way, we get

$$0 = \sum_{j=1}^{n-m} < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > - < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \qquad \text{for any } X = (x,y) \in \mathcal{R}. \tag{4.5}$$

Hence, from (3.5) and (4.5), we see that $\mathcal{K}$ and $\big|\nabla_L |\nabla_y u|\big|$ vanish identically on $\mathcal{R}$.
Then, by Lemma 2.11 of [7] (applied to the function $y \mapsto u(x,y)$, for any fixed $x \in \mathbb{R}^m$), we obtain that there exist $\omega : \mathbb{R}^m \to \mathrm{S}^{n-m-1}$ and $u_o : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that $u(x,y) = u_o(x, \omega(x)\cdot y)$ for any $(x,y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.
Also, from (3.6) and (4.5), we obtain that $\nabla_y u_{x_i}$ and $\nabla_y u$ are parallel.
This and Lemma A.1 of [3] imply that $\omega(x)$ is constant in any connected component of $\{\nabla_y u \neq 0\}$.
This completes the proof of Theorem 1.1. ■

Appendices

A Applications to the mean curvature and $p(x)$-Laplace operators

We observe that the assumptions of Theorem 1.1 are fulfilled by mean curvature operators, i.e., when the diffusion coefficient $a(.,.)$ has a product structure given by

$$a(x,t) = \alpha(x)\mathcal{A}_m(t), \qquad \mathcal{A}_m(t) = \frac{1}{\sqrt{1+t^2}},$$

and $p(x)$-Laplace operators, i.e., when

$$a(x,t) = \alpha(x)\mathcal{A}_p(t), \qquad \mathcal{A}_{p(x)}(t) = t^{p(x)-2},$$

where the function α is positive and bounded.
Indeed, in the mean curvature operator,

$$\begin{aligned} < \mathcal{B}v, v > &= \frac{\alpha}{(1+|\nabla u|^2)^{3/2}} \big[(1+|\nabla u|^2)|v|^2 + (\nabla u \cdot v)^2 \big] \\ &\geqslant \frac{\alpha |v|^2}{(1+|\nabla u|^2)^{3/2}} \end{aligned}$$

for any $v \in \mathbb{R}^{n+1}$, thence the positiveness of $\mathcal{B}$.
Similarly, for $p(x)$-Laplace operators, we have that

$$\begin{aligned} < \mathcal{B}v, v > &= \alpha |\nabla u|^{p-4} \left[|\nabla u|^2 |v|^2 + (p-2)(\nabla u \cdot v)^2 \right] \\ &\geqslant c_p \alpha |\nabla u|^{p-2} |v|^2, \end{aligned}$$

with

$$c_{p(x)} := \begin{cases} 1 & \text{when } p(x) \geqslant 2, \\ p(x) - 1 & \text{when } 1 < p(x) < 2, \end{cases}$$

thence the desired positivity of $\mathcal{B}$.
Also, condition (1.6) is satisfied for both the mean curvature operator and the $p(x)$-Laplace operator when $p(x) \geqslant 2$, and even for $p(x) > 1$ as long as $\{\nabla u = 0\} = \emptyset$.

B Monotonicity implies stability

The monotonicity in one direction implies stability:

Lemma B.1. *Assume that the symmetric matrix $\mathcal{B}$ defined in* (1.4) *is positive definite at almost any $X = (x, y) \in \mathbb{R}^n$. Let u be a weak solution of* (1.1) *in Ω and suppose that $\partial_{y_1} u > 0$ in Ω.*
Then, u is stable, that is (1.8) *holds.*

Proof. Fix $\xi \in C_0^\infty(\Omega)$. In view of (2.10), we may use (2.9) for $j = 1$ and $\psi := \dfrac{\xi^2}{u_{y_1}} \in W_0^{1,2}(\Omega)$.
This yields that

$$\begin{aligned} & \int_\Omega f_u(x,u) \xi^2 \, dX \\ = & \int_\Omega f_u(x,u) u_{y_1} \psi \, dX \\ = & \int_\Omega \left[\frac{2\xi}{u_{y_1}} < \mathcal{B}(x, \nabla u) \nabla u_{y_1}, \nabla \xi > - \frac{\xi^2}{(u_{y_1})^2} < \mathcal{B}(x, \nabla u) \nabla u_{y_1}, \nabla u_{y_1} > \right] dX \\ \leqslant & \int_\Omega < \mathcal{B}(x, \nabla u) \nabla \xi, \nabla \xi > \, dX, \end{aligned}$$

where the Cauchy-Schwarz-type inequality

$$2 < \mathcal{B}(x, \nabla u) v, w > \; \leqslant \; < \mathcal{B}(x, \nabla u) v, v > + < \mathcal{B}(x, \nabla u) w, w >, \qquad \forall v, w \in \mathbb{R}^n$$

was used in the last estimate. ■

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