A remark on Serrin's Theorem

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Let $\Omega \subset \mathbb{R}^n$ be an open set. A Borel function $f = f(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ is called an *integrand* if, for every $(x, s) \in \Omega \times \mathbb{R}$,

$$f(x, s, \cdot) \quad \text{is convex on } \mathbb{R}^n .$$

Given an integrand f we consider the functional $I[f] : W^{1,1}_{\rm loc}(\Omega) \to [0, \infty]$ defined as

$$I[f](u) := \int_\Omega f(x, u(x), \nabla u(x))dx.$$

A classic problem in the Calculus of Variation is (see [20]) to find conditions on f ensuring that $I[f]$ is $L^1_{\rm loc}$ lower semicontinuous, that is

$$I[f](u) \le \liminf_{h\to\infty} I[f](u_h), \quad \forall u_h \to u \text{ in } L^1_{\rm loc}(\Omega).$$

Aronszajn provided a counterexample (see [19], [6] and for further modifications [16] and [15]) showing that lower semicontinuity in general fails even if f is continuous. On the other hand Serrin in [20] proved the following theorem:

Theorem 1 (Serrin's Theorem). *Assume f is a continuous integrand and that one of the following conditions holds:*

(i) $\lim_{|\xi|\to\infty} f(x, s, \xi) = \infty$*;*

(ii) $f(x, s, \cdot)$ *is strictly convex;*

(iii) the classical derivatives f_x, f_ξ *and* $f_{\xi x}$ *exist and are continuous functions.*

Then $I[f]$ *is* $L^1_{\rm loc}$ *lower semicontinuous on* $W^{1,1}_{\rm loc}$*.*

If we want to find out a necessary and sufficient condition to L^1_{loc} lower semicontinuity we are naturally lead to ask the question: what kind of relation, if any, does exist between the conditions (i), (ii) and (iii)? The aim of this note is to provide an answer to this question. We start by reviewing some research done in the last years on the subject.

First of all let us remark that the crux of the matter in Theorem 1 is the dependence of f from the x variable: indeed De Giorgi, Buttazzo and Dal Maso have proved in [9] that if $f = f(s, \xi)$ then the L^1 lower semicontinuity holds on $W^{1,1}$ under very mild hypotheses on f.

More recently a renewed interest in the problem was raised by the results in [16] where the condition (iii) was significantly weakened by replacing it with the assumption that, for every bounded set $B \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists $L = L(B) > 0$ such that

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \le L|x_1 - x_2|, \quad \forall (x_i, s, \xi) \in B, i = 1, 2. \tag{1}$$

Condition (1) is a Lipschtiz type of differentiability condition and in particular implies continuity in x. This condition can be further relaxed up to consider mere measurability in the x variable together with a Sobolev type of differentiability condition, as showed in [15]. Further generalizations (to bounded variation type of differentiability) are possible and have been explored in [12], [7] and [13]. Some kind of differentiability is necessary: for example the L^1 lower semicontinuity on $W^{1,1}$ is false if we replace (1) with an arbitrary Hölder condition, as showed by a counterexample in [15].

So far about condition (iii). Let us now spend a few words on the research carried over about conditions (i) and (ii). A natural question is: can we see them as particular cases of a more general condition of geometric type still sufficient to lower semicontinuity? The answer has been given in [14] and the condition is the following (this condition was independently introduced in [3] in the study of non coercive Dirichlet minimization problems and in [4] in connection with the regularity of convex envelopes).

Definition 2. *A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said demicoercive if there exist $c > 0$, $b \ge 0$ and $v \in \mathbb{R}^n$ such that*

$$c|\xi| \le \Psi(\xi) + v \cdot \xi + b, \quad \forall \xi \in \mathbb{R}^n .$$

For the basic properties of demicoercive functions we refer to [3]. Notice that a convex function Ψ such that $\lim\limits_{|\xi| \to \infty} \Psi(\xi) = \infty$ is demicoercive. Also strictly convex functions are demicoercive. There are demicoercive functions that are neither coercive or strictly convex, for example $\Psi(\xi) = \sum_{i=1}^n \max\{\xi_i, 0\}$. The following theorem holds.

Theorem 3 ([14], [18]). *If f is a lower semicontinuous integrand such that, for every $(x, s) \in \Omega \times \mathbb{R}$, $f(x, s, \cdot)$ is convex and demicoercive (or constantly equal to $\inf f$), then $I[f]$ is L^1_{loc} lower semicontinuous on $W^{1,1}_{\mathrm{loc}}$.*

The question is now if we can further unify the sufficient conditions found in Theorems 3 and 1. The main result of this note is to show that actually an integrand f of the type considered in Theorem 3 can be approximated by integrands $\{f_m\}_{m\in\mathbb{N}}$ of the type considered in (iii) in such a way that the lower semicontinuity of the $I[f_m]$ implies the lower semicontinuity of $I[f]$.

Theorem 4. *Let Σ be an open set in $\mathbb{R}^d$ and $f : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ be a non negative and lower semicontinuous function with $f(t, \cdot)$ convex and demicoercive. Then there exists a sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ of class $C^\infty(\Sigma \times \mathbb{R}^n, [0, \infty))$ such that*

$$f = \sup_{m\in\mathbb{N}} f_m.$$

Remark 5. As a consequence of this approximation result and of Proposition 1 below we have the quite surprising fact that the lower semicontinuous result under the assumption (i) or (ii) of Serrin's Theorem1 follows indeed from part (iii) of the same theorem! Also the more general Theorem 3 can be immediately deduced from part (iii) of Theorem 1 via Theorem 4. Moreover, the above theorem suggests that the search for a necessary and sufficient condition to L^1_{loc} lower semicontinuity has to move into the direction of finding conditions of the type introduced in [16].

The following proposition can be proved by a more or less standard argument based on a well known lemma concerning the integration of the supremum of nonnegative Borel functions (see [2, Lemma 2.35]).

Proposition 6. *Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of integrands such that $I[f_m]$ is $L^1_{\rm loc}$ lower semicontinuous on $W^{1,1}_{\rm loc}$. If $f:\Omega\times\mathbb{R}\times\mathbb{R}^n\to[0,\infty]$ is a function such that*

$$f(x,s,\xi)=\sup_{m\in\mathbb{N}}f_m(x,s,\xi)\quad\forall(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n,$$

then also $I[f]$ is $L^1_{\rm loc}$ lower semicontinuous on $W^{1,1}_{\rm loc}$.

The proof of Theorem 4 is based on an approximation theorem for demicoercive functions that was used to prove Theorem 3 in [14] (see Theorem 7 below), on a stability property for families of demicoercive functions (that has been used in [18] to treat the relaxation problem without coercivity assumptions, see Lemma 8, part (a)) and on an approximation result for integrands satisfying (3), an hypothesis on f, sufficient to $L^1_{\rm loc}$ lower semicontinuity, that was introduced by Fonseca and Leoni [11] (see Lemma 8, part (c)).

Theorem 7. *Let Σ be an open set in $\mathbb{R}^d$ and $f:\Sigma\times\mathbb{R}^n\to\mathbb{R}$ be a non negative and lower semicontinuous function with $f(t,\cdot)$ convex and demicoercive. Then there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ of lower semicontinuous functions with $f_k(t,\cdot)$ convex and demicoercive and a sequence of vectors $\{\xi_k\}_{k\in\mathbb{N}}\subset\mathbb{R}^n$ such that*

$$\xi\mapsto f_k(t,\xi+\xi_k)+1$$

is non negative and positively homogeneous of degree one and

$$f=\sup_{k\in\mathbb{N}}f_k.$$

Proof. See Section 2 in [14]. ☐

Lemma 8. *Let Σ be an open set in $\mathbb{R}^d$ and $f:\Sigma\times\mathbb{R}^n\to\mathbb{R}$ be a non negative and lower semicontinuous function with $f(t,\cdot)$ convex.*

(a) If $f(t,\cdot)$ is demicoercive and positively homogeneous of degree one then, for every $t_0\in\Sigma$, there exist a $\delta>0$ and non negative, convex and demicoercive function $\Phi:\mathbb{R}^n\to\mathbb{R}$ such that

$$\Phi(\xi)\le f(t,\xi),\quad\forall(t,\xi)\in B_\delta(t_0)\times\mathbb{R}^n\,.$$

(b) If $f(t,\cdot)$ is positively homogeneous of degree one and there exists $c_0>0$ such that, for every $(t,\xi)\in\Sigma\times\mathbb{R}^n$,

$$c_0|\xi|\le f(t,\xi),\tag{2}$$

then, for every $\varepsilon>0$ and $t_0\in\Sigma$, there exists $\delta>0$ such that

$$f(t_0,\xi)\le f(t,\xi)+\varepsilon(1+f(t,\xi)),\quad\forall(t,\xi)\in B_\delta(t_0)\times\mathbb{R}^n\,.\tag{3}$$

(c) If f satisfies (3) then there exist two sequences $\{a_k\}_{k\in\mathbb{N}}\subseteq C^\infty_c(\Sigma,[0,\infty))$ and $\{\Psi_k\}_{k\in\mathbb{N}}\subseteq C^\infty(\mathbb{R}^n,[0,\infty))$ with $\Psi_k:\mathbb{R}^n\to[0,\infty)$ convex such that

$$f(t,\xi)=\sup_{k\in\mathbb{N}}a_k(t)\Psi_k(\xi)\quad\forall(t,\xi)\in\Sigma\times\mathbb{R}^n\,.$$

Proof. Part (a) is proved in [18], Lemma 1.

Let us prove part (b). Let us fix $\varepsilon>0$ and $t_0\in\Sigma$: using the lower semicontinuity of f we that there exists $\delta>0$ such that, for every $(t,\xi)\in B_\delta(t_0)\times S^{n-1}$,

$$f(t_0,\xi)\le f(t,\xi)+\varepsilon c_0.$$

Then, by the positive homogeneity of f and (2), for every $(t,\xi) \in B_\delta(t_0) \times \mathbb{R}^n$,

$$f(t_0,\xi) \le f(t,\xi) + \varepsilon c_0|\xi| \le f(t,\xi) + \varepsilon f(t,\xi) \le f(t,\xi) + \varepsilon(1 + f(t,\xi)).$$

To prove (c) let us denote by ϱ a symmetric mollifier in $\mathbb{R}^n$, i.e. a nonnegative function from $C_c^\infty(\mathbb{R}^n,[0,\infty))$, such that $\varrho(-\xi) = \varrho(\xi)$ for all ξ and $\int \varrho(\xi)d\xi = 1$. Then, let us fix $t_0 \in \Sigma$, $\varepsilon > 0$, and set for all $\xi \in \mathbb{R}^n$

$$h_{\varepsilon,t_0}(\xi) = \frac{1}{1+\varepsilon}\int_{\mathbb{R}^n} \varrho(\zeta)[f(t_0,\xi+\varepsilon\zeta) - \varepsilon]\, d\zeta.$$

Clearly, $h_{\varepsilon,t_0} \to f(t_0,\cdot)$, as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}^n$ and, by the convexity of $f(t_0,\cdot)$ and the symmetry of ϱ, $h_{\varepsilon,t_0}(\xi) \le [f(t_0,\xi) - \varepsilon]/(1+\varepsilon)$ for all ξ. Still denoting by ϱ a symmetric mollifier in $\mathbb{R}^n$, let us set, for any $s \in \mathbb{R}$ and any $i \in \mathbb{N}$,

$$\sigma_i(s) = \int_{\mathbb{R}} \varrho(r)\Big(s + \frac{r}{i}\Big)^+ dr \tag{4}$$

and

$$g_{\varepsilon,t_0,i}(t,\xi) = \alpha_{\varepsilon,t_0}(t)\sigma_i(h_{\varepsilon,t_0}(\xi)), \quad \forall (t,\xi) \in \Sigma \times \mathbb{R}^n,$$

where $\alpha_{\varepsilon,t_0} \in C_c^\infty(B_\delta(t_0),[0,\infty))$, $0 \le \alpha_{\varepsilon,t_0} \le 1$, $\alpha_{\varepsilon,t_0}(t_0) = 1$, and δ is the positive number provided by (3). By construction and by (3), we then get that, for all $(t,\xi) \in \Sigma \times \mathbb{R}^n$

$$f(t,\xi) \ge g_{\varepsilon,t_0,i}(t,\xi) \tag{5}$$

and

$$f(t,\xi) = \sup_{\varepsilon\in(0,\infty)} \sup_{t_0\in\Sigma} \sup_{i\in\mathbb{N}} g_{\varepsilon,t_0,i}(t,\xi). \tag{6}$$

The result then follows by observing that, since all functions $g_{\varepsilon,t_0,i}$ are continuous, from (5) and (6) it follows easily that there exists a sequence $\gamma_k \in (0,\infty) \times \Sigma \times \mathbb{N}$ such that $f = \sup_{k\in\mathbb{N}} g_{\gamma_k}$ (see, for instance [11, Lemma 9.2]). □

We are ready to prove our result.

Proof of Theorem 4. We apply Theorem 7 to f to find a sequence of lower semicontinuous functions $\{f_k\}_{k\in\mathbb{N}}$ and a sequence of vectors $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that

$$h_k(t,\xi) := 1 + f_k(t,\xi+\xi_k) : \Sigma \times \mathbb{R}^n \to [0,\infty),$$

with

$$h_k(t,\cdot) \quad \text{positively homogeneous of degree one, convex and demicoervice.}$$

We apply Lemma 8 part (a) to each h_k and to each $t_0 \in \Sigma$ in order to find a non negative, convex and demicoercive function $\Phi[k,t_0] : \mathbb{R}^n \to [0,\infty)$ and a number $\delta[k,t_0]$ such that

$$\Phi[k,t_0](\xi) \le h_k(t,\xi) \quad \forall (t,\xi) \in B_{\delta[k,t_0]}(t_0) \times \mathbb{R}^n\,.$$

Furthermore we know there exists a vector $v[k,t_0] \in \mathbb{R}^n$ and a constant $c[k,t_0] > 0$ such that

$$c[k,t_0]|\xi| \le \Phi[k,t_0](\xi) + v[k,t_0]\cdot\xi, \quad \forall \xi \in \mathbb{R}^n\,.$$

By Lemma 8 part (b) and part (c) there are sequences of functions

$$\{a_j[k,t_0]\}_{j\in\mathbb{N}} \quad \text{and} \quad \{\Psi_j[k,t_0]\}_{j\in\mathbb{N}}$$

with $a_j[k,t_0] \in C_c^\infty(\Sigma,[0,\infty))$, $\mathrm{spt}(a_j[k,t_0]) \subset\subset B_{\delta[k,t_0]}(t_0)$ and $\Psi_j[k,t_0] : \mathbb{R}^n \to [0,\infty)$ convex and of class $C^\infty(\mathbb{R}^n,[0,\infty))$ such that, for every $(t,\xi) \in B_{\delta[k,t_0]}(t_0) \times \mathbb{R}^n$,

$$h_k(t,\xi) + v[k,t_0]\cdot\xi = \sup_{j\in\mathbb{N}} \{a_j[k,t_0](t)\Psi_j[k,t_0](\xi)\}\,.$$

Let us consider now a sequence of open sets $\Sigma_h \uparrow \Sigma$ with $\Sigma_h \subset\subset \Sigma$. For every fixed k we have that the family of balls

$$\left\{B_{\delta[k,t_0]}(t_0) : t_0 \in \Sigma_h\right\}$$

is an open covering of $\overline{\Sigma}_h$ so that for each $k, h \in \mathbb{N}$ there exist $N(k,h) \in \mathbb{N}$ and $\{(t_{k,h,i})\}_{i=1}^{N(k,h)} \subseteq \Sigma_h$ such that

$$\overline{\Sigma}_h \subseteq \bigcup_{i=1}^{N(k,h)} B_{\delta[k,t_{k,h,i}]}(t_{k,h,i}) =: \bigcup_{i=1}^{N(k,h)} B_{k,h,i}.$$

For each $k, h \in \mathbb{N}$, $i \in I(k,h) := \{1, \dots, N(k,h)\}$, we can find a sequence $\{\phi_{k,h,i,l}\}_{l\in\mathbb{N}}$ with

$$\sup_{l\in\mathbb{N}} \phi_{k,h,i,l}(t) = 1 \quad \forall t \in B_{k,h,i},$$

and such that $\phi_{k,h,i,l} \in C_c^\infty(\Sigma, [0,1])$ with $\operatorname{spt}(\phi_{k,h,i,l}) \subset\subset B_{k,h,i}$ and thus

$$\sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \phi_{k,h,i,l}(t) = 1 \quad \forall t \in \Sigma, k \in \mathbb{N}.$$

If we define

$$a_j[k, t_{k,h,i}] =: a_{k,h,i,j}, \quad \Psi_j[k, t_{k,h,i}] =: \Psi_{k,h,i,j}, \quad v[k, t_{k,h,i}] =: v_{k,h,i},$$

we have:

$$\begin{aligned}
f(t,\xi) &= \sup_{k\in\mathbb{N}} \left\{f_k(t,\xi)^+\right\} = \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \left\{\phi_{k,h,i,l}(t) f_k(t,\xi)^+\right\} \\
&= \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \left\{\phi_{k,h,i,l}(t) (h_k(t, \xi - \xi_k) - 1)^+\right\} \\
&= \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \left\{\phi_{k,h,i,l}(t) \sup_{j\in\mathbb{N}} (a_{k,h,i,j}(t)\Psi_{k,h,i,j}(\xi - \xi_k) - v_{k,h,i} \cdot (\xi - \xi_k) - 1)^+\right\} \\
&= \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \sup_{j\in\mathbb{N}} \left\{\phi_{k,h,i,l}(t) \left(a_{k,h,i,j}(t)\Psi_{k,h,i,j}(\xi - \xi_k) - v_{k,h,i} \cdot (\xi - \xi_k) - 1\right)^+\right\} \\
&= \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(k,h)} \sup_{l\in\mathbb{N}} \sup_{j\in\mathbb{N}} \sup_{m\in\mathbb{N}} \left\{\phi_{k,h,i,l}(t) \sigma_m \left(a_{k,h,i,j}(t)\Psi_{k,h,i,j}(\xi - \xi_k) - v_{k,h,i} \cdot (\xi - \xi_k) - 1\right)\right\},
\end{aligned}$$

where σ_m is defined as in (4). The conclusion then follows by a suitable re-labeling of the indexes. $\square$

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