Scaling of the energy for thin martensitic films

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Abstract

We study the scaling behaviour of thin martensitic films. Specifically we consider an elastic energy with two $SO(3)$ invariant wells which are strongly incompatible in the sense of Matos and Šverák, but whose two-dimensional projections are compatible. We show that in a thin film of thickness the energy per unit height scales like h. This scaling lies in between the classical membrane theory (where the energy per unit height is of order 1) and the Kirchhoff bending theory, which corresponds to a scaling of h^2.

1 Introduction

1.1 Main result

We study the scaling of the elastic energy for a thin film made of a multiphase material. Specifically we consider the cylindrical domain

$$\Omega_h := S \times \left(-\frac{h}{2}, \frac{h}{2}\right) \subset \mathbb{R}^3 \tag{1}$$

an elastic deformation

$$v : \Omega_h \to \mathbb{R}^3 \tag{2}$$

and its energy (per unit height)

$$E^h(v) := \frac{1}{h}\int_{\Omega_h} W(\nabla v(x))dx. \tag{3}$$

We suppose that stored energy density W, which is defined on the space $\mathbb{M}^{3\times 3}$ of 3×3 matrices is nonnegative and vanishes exactly on the set

$$K := SO(3) \cup SO(3)\,H, \quad \det H > 0 \tag{4}$$

which consists of two copies of the group $SO(3) = \{F \in \mathbb{M}^{3\times 3} : F^T F = Id_3, \det F = 1\}$ of rotations, corresponding to two preferred crystalline configurations or phases (see (1.7)-(1.9) and (1.10) below for the full list of assumptions on W). We are interested in low-energy deformations and these are characterized by the fact that ∇v is close to K, except possibly on a set of small measure.

Bhattacharya and James [5] made the crucial observation that for a number of interesting materials the low-energy states are very different in three-dimensional (bulk) samples and in the thin film limit. If Id represents the austenite (high-temperature) phase and H represents one of the martensitic phases then these are usually incompatible in bulk, in particular there are no nontrivial zero energy states. By contrast, the limiting thin film membrane energy

$$I_{\text{membrane}} := \int_S W_{\text{membrane}}(\nabla' v)\,dx', \quad \text{where } x' = (x_1, x_2), \quad \nabla' := (\partial_1, \partial_2), \tag{5}$$

which, roughly speaking, is the Γ-limit of E^h (see section 1.2 for a more detailed discussion), admits many nontrivial zero energy states, including lamellar arrangements of the two phases, as well as more complicated, e.g. tent-like, structures, see [5]. This drastic difference in behaviour stems from the fact that in three-dimensional compatibility requires the existence of an invariant plane (i.e. Id and RH have to agree on a plane, for some $R \in SO(3)$), while two-dimensional compatibility requires only an invariant line (in the film plane), i.e. one needs a vector v in the film plane such that $v = RHv$. Suppose such a vector v exists. Then in a film of small, but finite, thickness h the juxtaposition of the deformation gradients Id and RH (along a line in direction v) leads to a mismatch of the deformations of order h. Separating the two regions by a strip of width proportional to h one sees easily that there exist three-dimensional deformations $v^{(h)}$ which have a nontrivial thin-film limit and whose energy is bounded from above by Ch.

Our main result that this scaling is optimal. To state the result precisely it is convenient to introduce the rescaling $y(x) := v(x_1, x_2, hx_3)$ and the notation $\Omega := \Omega_1 = S \times (-1/2, 1/2)$. Then $y : \Omega \to \mathbb{R}^3$. As above we write $\nabla' y = (y_{,1}, y_{,2}) = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the gradient in the plane and $\nabla_h y = (y_{,1}, y_{,2}, \frac{1}{h} y_{,3})$. Thus the elastic energy per unit height is given by

$$I^h(y) := E^h(v) = \int_\Omega W(\nabla_h y) dx \,. \tag{6}$$

We assume that W is Borel measurable and satisfies

$$W \text{ is } C^2 \text{ in a neighbourhood of } K = SO(3) \cup SO(3)\, H, \tag{7}$$

$$W \text{ is frame indifferent: } W(F) = W(RF) \text{ for all } R \in K\,, \tag{8}$$

$$W(F) \geq C \mathrm{dist}^2(F\,, K),\ C > 0 \text{ and } W(F) = 0 \text{ if } F \in K\,. \tag{9}$$

We suppose that the two wells are strongly incompatible in the sense of Matos [21] and Šverák [31]. By polar decomposition and an orthogonal change of variables we may assume that H is diagonal, $H = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. The incompatibility condition then reads

$$\sum_{i=1}^{3} (1 - \lambda_i)(1 - \det H/\lambda_i) > 0. \tag{10}$$

By $O(2,3)$ we denote the set of linear isometries from $\mathbb{R}^2$ to $\mathbb{R}^3$, i.e. the set of all 3×2 matrices F with $F^T F = Id_2$. It is easy to see that the convex hull conv $O(2,3)$ of $O(2,3)$ is given by matrices with $F^T F \leq Id_2$ (in the sense of symmetric matrices), i.e. by all linear maps with Lipschitz constant less than or equal to 1. Let $\tilde{H} := \mathrm{diag}(\lambda_1, \lambda_2)$ be the two dimensional projection of H.

Theorem 1 *Suppose W satisfies conditions* (1.7)-(1.9) *and* (1.10). *Consider a sequence $y^{(h)}$ which satisfies*

$$\frac{1}{h} I^h(y^{(h)}) \leq C, \quad \textit{for all } 0 < h < h_0. \tag{11}$$

Then, as $h \to 0$, there exists a subsequence (not relabeled) such that $y^{(h)} \rightharpoonup y$ $W^{1,2}(\Omega, \mathbb{R}^3)$ (weakly) and y is independent of x_3.

Moreover $\nabla' y \in$ conv $O(2,3)$ or $\nabla' y\, \tilde{H}^{-1} \in$ conv $O(2,3)$ a.e. in S. In other words, $\mathcal{L}^2(S \setminus E_1 \cup E_2) = 0$, where

$$\begin{aligned} E_1 &:= \{x' \in S : (\nabla' y(x'))^T (\nabla' y(x')) \leq Id_2\} \\ E_2 &:= \{x' \in S : \tilde{H}^{-T} (\nabla' y(x'))^T (\nabla' y(x')) \tilde{H}^{-1} \leq Id_2\}. \end{aligned}$$

In addition we have

$$\liminf_{h\to 0} \frac{1}{h} I^h(y^{(h)})$$
$$\geq \; C \inf\{\mathcal{H}^1(\partial^* E) : E \textit{ has finite perimeter } E \subset E_1, S \setminus E \subset E_2\}. \quad (12)$$

Remark 2 *1. Note that the sets E_1 and E_2 need not be disjoint.*
2. The situation is complicated by the fact that in the thin film limit microstructure (i.e. fine scale oscillation of the deformation gradient) can arise from two different sources: phase mixtures and loss of compactness in thin films due to crumbling under compression, which already occurs in single phase materials (see (1.17) *below). It is due to crumbling that one can only assure that the limiting deformation gradient is in the convex hull of $O(2,3)$ (or $O(2,3)H$) rather than $O(2,3)$ itself. Even for single phase materials crumbling can only be excluded if one has the much stronger energy bound $I^h(y^{(h)}) \leq Ch^2$ (see [12], Section 5).*
3. Estimate (1.12) *assures that the scaling proportional to h is optimal, if the limit involves a nontrivial phase mixture. Indeed if $(1/h)I^h(y^{(h)}) \to 0$ then one can choose either $E_1 = S$ or $E_2 = S$ (up to a nullset), i.e. the corresponding limiting y can already be reached a single phase material (see next subsection).*
4. The scaling $I^h \sim h$ is an unconventional one in terms of classcial membrane and plate theories; it lies in between the scaling for membranes ($I^h \sim 1$) and Kirchhoff plates ($I^h \sim h^2$), see the next subsection for details.
5. The Γ-limit of the scaled functionals $(1/h)I^h$ is not known. One difficulty is that, in contrast to many other situations, one cannot expect the Γ-limit to be independent of the boundary conditions. Thus many of the usual cut-and-paste arguments do not apply, see the next subsection for further comments. For rods the corresponding Γ-limit is known, see [22].

1.2 Mathematical context

To put the result above in context, we very briefly review the theory of thin film limits for a single phase material, i.e. for energy functions W which satisfy the coercivity condition

$$W(F) \geq C \operatorname{dist}^2(F, SO(3)), \quad C > 0 \tag{13}$$

instead of (1.9) (for a more extended review with further references see [14]).

The derivation of lower dimensional theories of elasticity from the three-dimensional theory has a very long and distinguished history which dates back to the beginning of elasticity theory. Rigorous results, starting from *nonlinear* elasticity, however, have only been obtained since the early 90's beginning with the work of LeDret and Raoult [18, 19]. They showed, under an additional growth hypothesis on W from above, that the Γ-limit (with respect to strong L^2 convergence in $W^{1,2}(\Omega, \mathbb{R}^3)$) of the functionals I^h exist and is given by

$$I_{\text{membrane}} := \begin{cases} \int_S W_{\text{membrane}}(\nabla' v)\, dx' & \text{if } v_{,3} = 0, \\ \infty & \text{else} \end{cases} \tag{14}$$

The membrane energy can be computed in two steps. First one minimizes out the derivatives in the third component (corresponding to the third column of F) and defines the energy W_2 on 3×2 matrices by

$$W_2(G) = \min\{W(G + b \otimes e_3) : b \in \mathbb{R}^3\}. \tag{15}$$

Then W_{membrane} is given as the quasiconvex hull of W_2, i.e. by minimizing out over all possible fine-scale oscillations:

$$W_{\text{membrane}}(G) := \inf \left\{ \int_{(0,1)^2} W_2(G + \nabla'\varphi)\, dx' : \varphi \in C_0^\infty((0,1)^2; \mathbb{R}^3) \right\}. \tag{16}$$

For a single well material (i.e. if W vanishes on $SO(3)$ and satisfies (1.13)) the reduced energy W_2 vanishes on $O(2,3)$ and

$$W_{\text{membrane}}(G) = 0 \quad \Leftrightarrow \quad G \in \operatorname{conv} O(2,3) \tag{17}$$

(the convex, quasiconvex and rank-one convex hull of $O(2,3)$ agree). Thus the membrane energy is fully degenerate for compressions, which agrees with the physical intuition that a membrane can only withstand tension but not compression. Based on this intuition a so-called tension field theory for membranes has been used in the engineering literature for a long time [33, 28, 29]. Pipkin [25, 26] has shown that tension-field theory arises naturally as a consequence of relaxation and quasiconvexification.

We leave for a moment the case of single phase materials to mention that the limit considered by Bhattacharya and James [5] is slightly different from that studied by of LeDret and Raoult. They add a regularising higher gradient perturbation $\kappa^2|\nabla^2 v|^2$ to the integrand in (1.3) and (after the usual rescaling $y(x_1, x_2, x_3) = v(x_1, x_2, hx_3)$) pass to the limit $h \to 0$ at fixed $\kappa > 0$. They thus obtain a limiting two-dimensional energy which involves $W_2(\nabla' y)$ (plus a higher gradient contribution) rather then $W_{\text{membrane}}(\nabla' y)$. If one now takes the limit $\kappa \to 0$ the Bhattacharya-James limiting energy then one recovers (1.14). Shu [30] has shown that one also obtains (1.14) if one considers a $\kappa(h)$ with $\lim_{h\to 0} \kappa(h) = 0$, in fact he gives a detailed analysis of a variety of multiparameter limits related to the scale of material heterogeneities in the tangential and normal directions (corresponding, e.g. to polycrystals or multilayers).

Let us now return to single phase materials, i.e. those for which W vanishes on $SO(3)$ and satisfies (1.13). For those materials it has recently become possible to study also the Γ-limit of the rescaled functionals $h^{-\beta} I^h$ and to derive a full hierarchy of plate theories. For $\beta = 2$ one obtains Kirchhoff's geometrically nonlinear bending theory [11, 12, 23, 24]. This theory imposes the isometry constraint $\nabla' y \in O(2,3)$ and the energy is given by a quadratic expression in the curvature, more precisely in the second fundamental form $A = -(\nabla')^2 y \cdot \nu$, where $\nu := y_{,1} \wedge y_{,2}$ is the normal to the deformed surface. For $\beta = 4$ one obtains the von Kárman plate theory [13], in fact the full range of exponents $\beta \geq 2$ is now understood [14].

By contrast relatively little is known in the range $0 < \beta < 2$. Conti [8] has recently shown that for $0 < \beta < 1$ the Γ-limit of

$$\frac{1}{h^\beta} \left[I^h(y) - \int_\Omega h^\beta f(x') \cdot y(x) \, dx \right] \tag{18}$$

is given by

$$J(y) = \begin{cases} -\int_S f \cdot y \, dx' & \text{if } y_{,3} = 0, \quad \nabla' y \in \operatorname{conv} O(2,3), \\ \infty & \text{else.} \end{cases} \tag{19}$$

The range $1 \leq \beta < 2$ is largely unexplored in terms of rigorous analysis. In the context of delamination and blistering of thin films [15] one is lead to the study of compressive *Dirichlet* boundary conditions such as $y^{(h)}(x', x_3) = (\lambda x', h x_3)$ on $\partial S \times I$, with $0 \leq \lambda < 1$. If this boundary condition is imposed one can show that $ch \leq \inf I^h(y^{(h)}) \leq Ch$, with $c > 0$, see [4] (as well as [17, 3] for related work). For the extension to anisotropic boundary compression see [8]. The Γ-limit of $h^{-1} I^h$ is not known.

If instead of Dirichlet boundary conditions we only assume that $y^{(h)} \rightharpoonup (\lambda x', 0)$ in $W^{1,2}$ then much less is known. Constructions of S. Venkataramani and of Conti and Maggi [9] suggest that one has an upper bound $Ch^{5/3}$. No general lower bound is known, except for the (almost) trivial one $\liminf_{h \to 0} h^{-2} I^h(y^{(h)}) = \infty$. The scaling exponent $h^{5/3}$ has been suggested in the physics literature on crumbling as a natural exponent based on a formal scaling argument and an assumed equipartition of bending and stretching energy [20, 10] (see also [2, 6]; for complex folding patterns at free boundaries and their potential relevance for certain growth models in biology see as well as [27, 1]). For a single ridge with well-defined boundary conditions Venkataramini recently showed that the energy scales indeeed like $h^{5/3}$ [32].

2 Proof of the lower bound

The key ingredient in the proof is the following rigidity result.

Theorem 3 ([7], Thm. 2) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$ and $K := SO(n) \cup SO(n)\, H$, where $H = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$, $\lambda_i > 0$ such that $\sum_{i=1}^n (1-\lambda_i)\,(1-\det H/\lambda_i) > 0$. There exists a positive constant $C(\Omega,\, H)$ with the following property. For each $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there is an associated $R := R(u, \Omega) \in K$ such that*

$$\|\nabla u - R\|_{L^2(\Omega)} \leq C(\Omega,\, H)\, \|\operatorname{dist}(\nabla u\,, K\,)\|_{L^2(\Omega)}\ . \tag{20}$$

We note that the inequality (2.20) is invariant under uniform scaling and translation of the domain; e.g., the same value of C serves for $\lambda\Omega + c$, and the rescaled function $\lambda v((x-c)/\lambda)$ may be associated with the same choice of $R \in K$.

Proof of Theorem 1. Suppose first that $S = (0,1)^2$ and $h = 1/N$, $N \in \mathbb{N}$. Divide S into squares $S_{a,h}$ of side h with centre at a., i.e.,

$$S_{a,h} := a + \left(-\frac{h}{2}, \frac{h}{2}\right)^2, \quad a \in \mathbb{Z}_h^2, \qquad \mathbb{Z}_h := \left\{\frac{h}{2}, \frac{3h}{2}, \cdots, 1 - \frac{h}{2}\right\}. \tag{21}$$

Then $S = \cup S_{a,h}$, up to a set of measure zero. Let us define, $v^{(h)} : \Omega_h \to \mathbb{R}^3$ by $v^{(h)}(x_1, x_2, hx_3) := y^{(h)}(x_1, x_2, x_3)$, for all $x = (x_1, x_2, x_3) \in \Omega := \Omega_1$. Now we apply the rigidity Theorem 2.1 for $v^{(h)}$ for the domain $S_{a,h} \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ to obtain a constant $C > 0$, (independent of a and h) and $R_a^h \in K = SO(3) \cup SO(3)\, H$, such that

$$\int_{S_{a,h} \times \left(-\frac{h}{2}, \frac{h}{2}\right)} |\nabla v^{(h)}(z) - R_a^h|^2 dz \leq C \int_{S_{a,h} \times \left(-\frac{h}{2}, \frac{h}{2}\right)} \operatorname{dist}^2(\nabla v^{(h)}(z), K) dz. \tag{22}$$

This yields

$$\int_{S_{a,h} \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\nabla_h y^{(h)}(x) - R_a^h|^2 dx \leq C \int_{S_{a,h} \times \left(-\frac{1}{2}, \frac{1}{2}\right)} \operatorname{dist}^2(\nabla_h y^{(h)}, K) dx, \tag{23}$$

where, as before, $\nabla_h y^{(h)} := \left(\nabla' y^{(h)}, \frac{1}{h} y_{,3}\right)$.

Define the piecewise constant map $R^h : S \to K$ by $R^h := R_a^h$ in $S_{a,h}$. Summing (2.23) over all $S_{a,h}$ and using (1.9) and (1.11) we obtain

$$\begin{aligned}\int_{S \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\nabla_h y^{(h)}(x) - R^h(x')|^2 dx &\leq C \int_{S \times \left(-\frac{1}{2}, \frac{1}{2}\right)} \operatorname{dist}^2(\nabla_h y^{(h)}(x), K) dx \\ &\leq C\, h\,.\end{aligned} \tag{24}$$

Thus

$$\begin{cases} \nabla_h y^{(h)} - R^h \to 0 \quad \text{strongly in } L^2(\Omega, \mathbb{M}^{3\times 3}) \\ \nabla_h y^{(h)} \rightharpoonup (\nabla' y, b) \quad \text{weakly in } L^2(\Omega, \mathbb{M}^{3\times 3}) \\ R^h \rightharpoonup \overline{R} \quad \text{weakly in } L^2(S, \mathbb{M}^{3\times 3}). \end{cases} \tag{25}$$

From (2.25) we have $\overline{R} = (\nabla' y, b)$, and hence y is independent of x_3.

Let $\epsilon > 0$ be sufficiently small. We divide the family of squares $S_{a,h}$ into three different groups $\mathcal{A}_i$, $i = 0$, 1, 2 in the following manner.

$$a \in \mathcal{A}_0 \quad \text{if and only if } \int_{S_{a,h}\times\left(-\frac{1}{2},\frac{1}{2}\right)} W(\nabla_h y^{(h)}(x))dx \geq \epsilon\, h^2 \tag{26}$$

If $a \notin \mathcal{A}_0$, Theorem 3 implies that there exists $R^h_a \in K$ such that

$$\frac{1}{h^2}\int_{S_{a,h}\times I} |\nabla_h y^{(h)}(x) - R^h_a|^2 dx \leq \frac{C}{h^2}\int_{S_{a,h}\times I} W(\nabla_h y^{(h)}(x))dx \leq C\epsilon\,, \tag{27}$$

where $I := \left(-\frac{1}{2}, \frac{1}{2}\right)$. Now define

$$a \in \mathcal{A}_1 \quad \text{if and only if (2.27) holds for } R^h_a \in SO(3)\,, \tag{28}$$

$$a \in \mathcal{A}_2 \quad \text{if and only if (2.27) holds for } R^h_a \in SO(3)\, H\,. \tag{29}$$

Clearly $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, if ϵ is sufficiently small. Thus the sets

$$\Omega^h_i := \mathrm{int}\left(\cup_{a\in\mathcal{A}_i}\ \bar{S}_{a,h}\right)\,, \tag{30}$$

$i = 0$, 1, 2 are disjoint and cover S.

Note also that the area of the set Ω^h_0 is bounded by Ch in view of (1.11). We now would like to estimate the length of the boundary $\partial\Omega^h_1$. Clearly this boundary consists of a union of vertical on horizontal segments of lengths h. The main observation is that each such boundary segment must also be in the boundary of one of the squares in Ω^h_0 (see Lemma 4 below). Then a simple counting argument yields that the length of $\partial\Omega^h_1$ is bounded by a constant independent of h.

To state the argument precisely we introduce the following notation. Let $e_\pm := (0, \pm 1)$, and $\tilde{e}_\pm = (\pm 1, 0)$. Then $\partial S_{a,h}$ consists of four segments, namely the top and bottom boundaries $\Gamma^h_{a,e_\pm} := a + \left(-\frac{h}{2}, \frac{h}{2}\right) \times \left\{\pm\frac{h}{2}\right\}$ and the left and right boundaries $\Gamma^h_{a,\tilde{e}_\pm} := a + \left\{\pm\frac{h}{2}\right\} \times \left(-\frac{h}{2}, \frac{h}{2}\right)$. Thus the boundary of Ω_1^h is the union of the line segments $\Gamma_{a,e}^h \subset \partial S_{a,h}$ for $S_{a,h} \subset \Omega_1^h$, such that $\Gamma_{a,e}^h \cap \Omega_1^h = \emptyset$. In other words,

$$\partial\Omega_1^h = \bigcup_{\substack{a\in\mathcal{A}_1 \\ \Gamma_{a,e}^h\cap\Omega_1^h=\emptyset}} \Gamma_{a,e}^h \,. \tag{31}$$

Lemma 4 *Let $a \in \mathcal{A}_1$, and $\Gamma_{a,e}^h \subset \partial\Omega_1^h \setminus \partial S$ for some $e \in \{(\pm 1, 0),\, (0, \pm 1)\}$. Then $S_{b,h} \subset \Omega_0^h$, for $b := a + h\, e$.*

Proof of Lemma 4. Since $\Gamma_{a,e}^h \subset \partial\Omega_1^h \setminus \partial S$, the square $S_{b,h}$ belongs to S. Suppose $b \notin \mathcal{A}_0$ and apply Theorem 3 to the domain $(S_{a,h} \cup S_{b,h}) \times \left(-\frac{h}{2}, \frac{h}{2}\right)$. Thus there exists a constant $C > 0$ (independent of h, a and b) and $R_{ab}^h \in K$, such that

$$\begin{aligned}\frac{1}{h^2}\int_{\left(S_{a,h}\cup S_{b,h}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)} |\nabla_h y^{(h)} - R_{ab}^h|^2 dx &\le \frac{C}{h^2}\int_{\left(S_{a,h}\cup S_{b,h}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)} W(\nabla_h y^{(h)})\, dx \\ &\le 2C\,\epsilon\,. \end{aligned} \tag{32}$$

Since $S_{a,h} \subset \Omega_1^h$, there also exist $R_a^h \in SO(3)$ such that

$$\frac{1}{h^2}\int_{S_{a,h}\times\left(-\frac{1}{2},\frac{1}{2}\right)} |\nabla_h y^{(h)} - R_a^h|^2 dx \le C\,\epsilon\,. \tag{33}$$

Therefore (2.32) and (2.33) yield $|R_{ab}^h - R_a^h| \le 4C\epsilon$. Similarly there exist $R_b^h \in SO(3)H$ such that $|R_{ab}^h - R_b^h| \le 4C\epsilon$. We thus obtain a contradiction if ϵ is chosen sufficiently small, and the proof of the lemma is finished. $\Box$

Proof of Theorem 1 (ctd.). Let $a \in \mathcal{A}_1$, and $\Gamma_{a,e}^h \subset \partial\Omega_1^h \setminus \partial S$, for some e. Then by the Lemma 4, the square adjacent to the side $\Gamma_{a,e}^h$ is in Ω_0^h. There can be at most four edges $\Gamma_{a,e}^h \subset \partial\Omega_1^h \setminus \partial S$, touching a single square in Ω_0^h. Thus from (2.31), (2.26) and (1.11) we obtain

$$\begin{aligned}\mathcal{H}^1(\partial\Omega_1^h \setminus \partial S) &= \sum_{\substack{a\in\mathcal{A}_1 \\ \Gamma_{a,e}^h\subset\partial\Omega_1^h\setminus\partial S}} \mathcal{H}^1(\Gamma_{a,e}^h) \\ &= h\,\mathrm{card}\left\{a \in \mathcal{A}_1 \,:\, \Gamma_{a,e}^h \subset \partial\Omega_1^h \setminus \partial S\right\} \\ &\le 4h\,\mathrm{card}\mathcal{A}_0 \\ &\le 4h\,\frac{1}{h^2\epsilon}\, I^h(y^{(h)}) \\ &\le C\,, \end{aligned} \tag{34}$$

where 'card' stands for the cardinality of a set. Hence $\chi_{\Omega_1^h}$ is bounded in $BV(S)$, functions of bounded variation on S, (see example 1.4 in [16]) and passing to a subsequence, we get $\chi_{\Omega_1^h} \stackrel{*}{\rightharpoonup} \chi_E$ in $BV(S)$. Therefore by the lower semicontinuity and compactness theorems for BV functions (Theorem 1.9 and Theorem 1.19 in [16], respectively) we obtain

$$\mathrm{Per}(E) = \int_S |\nabla \chi_E| \le \liminf_{h\to 0} \int_S |\nabla \chi_{\Omega_1^h}| = \liminf_{h\to 0} \mathcal{H}^1(\partial\Omega_1^h \backslash \partial S) \le C \liminf_{h\to 0} \frac{1}{h} I^h(y^{(h)}),$$

and $\chi_{\Omega_1^h} \to \chi_E$ strongly in $L^1(S)$.

Furthermore it follows from (1.9) that $\mathrm{dist}^2(F, SO(3)) \le C(W(F)+1)$. Using (1.11) we deduce that

$$\int_{N_h} \mathrm{dist}^2(\nabla_h y^{(h)}, SO(3))\, dx \to 0, \quad \text{whenever } \mathcal{L}^3(N_h) \to 0. \tag{35}$$

Since the map $X \mapsto \mathrm{dist}(X, M)$ is convex, whenever M is a convex set, by standard convexity and lower semicontinuity arguments yield (with $I := (-1/2, 1/2)$)

$$\begin{aligned}
\int_\Omega \chi_{E\times I} \,\mathrm{dist}^2(\nabla' y, \mathrm{conv}\, O(2,3)) dx &\le \liminf_{h\to 0} \int_\Omega \chi_{E\times I} \,\mathrm{dist}^2(\nabla' y^{(h)}, \mathrm{conv}\, O(2,3)) dx \\
&\le \liminf_{h\to 0} \int_\Omega \chi_{E\times I} \,\mathrm{dist}^2(\nabla_h y^{(h)}, SO(3)) dx \\
&\le \liminf_{h\to 0} \int_\Omega \chi_{\Omega_1^h\times I} \,\mathrm{dist}^2(\nabla_h y^{(h)}, SO(3)) dx \\
&\le \liminf_{h\to 0} \sum_{a\in\mathcal{A}_1} \int_{S_{a,h}\times I} |\nabla_h y^{(h)} - R_a^h|^2 dx \\
&\le C \liminf_{h\to 0} \sum_{a\in\mathcal{A}_1} \int_{S_{a,h}\times I} W(\nabla_h y^{(h)}) dx \\
&\le C \liminf_{h\to 0} I^h(y^{(h)}) \\
&= 0,
\end{aligned} \tag{36}$$

where we used (2.35) to obtain the third inequality. Hence $\nabla' y \in \mathrm{conv}\, O(2,3)$ a.e. in E. Since $\mathcal{L}^2(\Omega_0^h) = h^2\, \mathrm{card}\mathcal{A}_0 \le \frac{1}{\epsilon} I^h(y^{(h)}) \to 0$ as $h \to 0$ we have

$$\chi_{\Omega_2^h} = 1 - \chi_{\Omega_0^h} - \chi_{\Omega_1^h} \to (1-\chi_E) \ \text{strongly in } L^1(S).$$

Applying the above arguments with $\nabla' y$ replaced by $(\nabla' y)\tilde{H}^{-1}$ and $\chi_{E\times I}$ replaced by $1-\chi_{E\times I}$, we conclude similarly that $(\nabla' y)\tilde{H}^{-1} \in \mathrm{conv}\, O(2,3)$

a.e. in $S \setminus E$. Thus $E \subset E_1$, $S \setminus E \subset E_2$ and

$$\mathrm{Per}(E) \le C \liminf \frac{1}{h} I^h(y^{(h)}) \, .$$

This finishes the proof for S being the unit square and $1/h \in \mathbb{N}$. For a general bounded open set S the assertion follows similarly by first considering the subset S^h which consists of the union all squares $S_{a,h}$ that are contained in S. $\square$

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