Models of defects in atomistic systems

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1 Introduction

According to the Weak-Membrane Model by Blake and Zisserman [8], a simple way to model free-discontinuity energies in a finite-difference scheme is by considering truncated quadratic

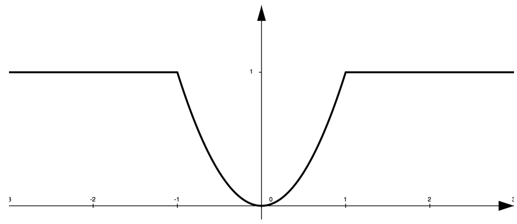

Figure 1: A truncated quadratic potential

energy densities (Fig. 1). The energy of such a (n-dimensional) scheme can then be written as

$$E(u) = \sum_{i,j} (u_i - u_j)^2 \wedge 1,$$

where u_i is a real parameter (the vertical displacement of the 'discrete membrane'), and the sum is performed over nearest neighbors in a cubic grid parameterized by $\mathbb{Z}^n$.

Thanks to a scaling argument due to Chambolle [21], which leads to the energies

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^n \Big(\Big(\frac{u_i - u_j}{\varepsilon}\Big)^2 \wedge \frac{1}{\varepsilon}\Big),$$

this discrete model can be approximated by a continuous energy defined on special functions with bounded variation. In fact, if we limit the interactions in the sum to the nearest neighbors in the portion of $\varepsilon\mathbb{Z}^n$ contained in some fixed Ω, and we interpret the values u_i as the discretization of a function defined in Ω, then these energies can be studied using the methods of Γ-convergence, and their limit is then given by a *fracture energy*

$$F(u) = \int_\Omega |\nabla u|^2\, dx + \int_{\Omega\cap S(u)} \|\nu\|_1 d\mathcal{H}^{n-1}$$

(see [21, 22, 16]), where $S(u)$ is the fracture site, ν is its normal and u is the macroscopic displacement outside the fracture site. The correct functional setting for these kinds of energies is the space $GSBV(\Omega)$ of (generalized) special functions of bounded variation in Ω introduced by Ambrosio and De Giorgi (see [11, 3]). From an atomistic standpoint, the energy $(u_i-u_j)^2\wedge 1$ can be interpreted as that of a 'defected' quadratic spring, which breaks after reaching a critical elongation; the collective behavior of such a system gives rise to the possibility of fracture. The critical scaling in E_ε is precisely the one that allows this behavior but forbids the accumulation of 'broken springs' on sets of dimension larger than $n-1$ while keeping the energy bounded. Note that the truncated quadratic potentials are a prototypical example to which the study of more general convex-concave atomistic potentials can be often reduced such as for Lennard Jones ones (see [18, 20])

If not all springs are 'defected', but a portion of them are simple quadratic linear springs, with corresponding energy $(u_i-u_j)^2$ (for which the Γ-limit is simply the Dirichlet integral and no discontinuity is allowed for the limit u), then the problem is more complex, and a continuous description must take into account the location and 'micro-geometry' of the two types of springs. In a probabilistic setting the location of the defected springs can be modeled in terms of realizations of i.i.d. random variables. In dimension two an analysis by Braides and Piatnitski [19] shows that the Γ-limit is deterministic and depends almost surely on the probability p of the weak springs. Its form is of 'fracture type' if p is above the *percolation threshold*, while it coincides with the Dirichlet integral for all values of p below that threshold.

A deterministic study leads necessarily to a more complex statement. In this case we look at possible Γ-limits of energies of the form

$$E_\varepsilon(u)=\sum_{i,j}\varepsilon^n f^\varepsilon_{ij}\Big(\frac{u_i-u_j}{\varepsilon}\Big),$$

where, for each ε, $f^\varepsilon_{ij}(z)$ may be chosen arbitrarily to be either z^2 or $z^2\wedge(1/\varepsilon)$.

It must be noted beforehand that, whatever the limit percentage of weak interaction is, we can obtain in the limit both the Dirichlet integral, and the Weak-Membrane Energy above; i.e., that even if we prescribe that for every subdomain $A\subset\Omega$ we have

$$\lim_{\varepsilon\to 0}\frac{\#\{(i,j)\in A\cap\varepsilon\mathbb{Z}^n: f^\varepsilon_{ij}(z)=z^2\wedge(1/\varepsilon)\}}{\#\{(i,j)\in A\cap\varepsilon\mathbb{Z}^n\}}=\theta$$

for any $\theta\in[0,1]$, we may obtain both such energies as Γ-limits for suitable choices of f^ε_{ij} (see [19] and Section 3.4 below). This is in contrast with formally similar problems where damaged springs are modeled as still quadratic with an energy density αz^2 with a constant $\alpha<1$ (for this 'discrete G-closure' problem see Braides and Francfort [15], and Braides and Gloria [17]). This observation leads to conjecturing that indeed the possible limit energies F are (independent of the limit density and) characterized by the two inequalities deriving from the comparison with the extreme cases; i.e.,

$$F(u)\le\int_\Omega|\nabla u|^2\,dx\qquad\text{if } u\in H^1(\Omega),$$

$$F(u)\ge\int_\Omega|\nabla u|^2\,dx+\int_{S(u)}\|\nu\|_1 d\mathcal{H}^{n-1}\qquad\text{if } u\in GSBV(\Omega).$$

The two inequalities imply that indeed $F(u)=\int_\Omega|\nabla u|^2\,dx$ if $u\in H^1(\Omega)$, and suggest the conjecture that we may obtain as limits all lower-semicontinuous energies of the form

$$F(u)=\int_\Omega|\nabla u|^2\,dx+\int_{S(u)}\varphi(x,u^+-u^-,\nu)d\mathcal{H}^{n-1}\text{ if } u\in GSBV(\Omega),$$

($u^\pm$ denote the traces of u on both sides of $S(u)$), where

- $\nu\mapsto\varphi(x,z,\nu)$ is even and $\varphi(x,z,\nu)\ge\|\nu\|_1$
- $z\mapsto\varphi(x,z,\nu)$ is even, and is increasing for positive z.

A complete proof of such a conjecture is not within the possibilities of the present knowledge of free-discontinuity functionals, even in the homogeneous case, i.e., with $\varphi(x,z,\nu)=\varphi(z,\nu)$.

Indeed, for such energy densities the condition for lower semicontinuity is *BV-ellipticity* (see Ambrosio and Braides [2]), which is the analog for interfacial energies of the condition of *quasiconvexity* for integral functionals (see Morrey [25]), and turns out to be necessary and sufficient if φ satisfies an inequality from above $\varphi(z,\nu) \le C|z|$. This last growth condition is not in general satisfied by our energies, and without this assumption neither we can apply known representation results (as those by Braides and Chiadò Piat [14] or Bouchitté et al. [9]), nor we can characterize the energy density (indeed, the problem of removing growth conditions is one of the main issues also in the theory of vector energies; see Ball and Murat [7]). But even when growth assumptions from above are satisfied and the function φ is BV-elliptic this information is of little help since explicit constructions of BV-elliptic energy densities (e.g., in the spirit of the construction of quasiconvex functions by relaxation as that by Šverák [28]) or their variational approximation by simpler energies (e.g., in the spirit of appoximation of quasiconvex energies by homogenization of polyconvex functionals as by Braides [10]) are not available in general, as are not available for arbitrary quasiconvex functions.

We will then restrict our analysis to classes of simpler energy densities, proving a number of results, each of its particular interest (summarized in Theorem 2.2)

1) $\varphi = \varphi(\nu)$ even. In this case the condition of BV-ellipticity is equivalent to the convexity of (the one-homogneous extension of) φ. We will prove that all such energy densities can be obtained in the limit;

2) $\varphi = \varphi(z)$. The form of the energies E_ε implies that φ is even and $z \mapsto \varphi(z)$ is increasing on $(0,+\infty)$. Moreover the growth condition gives $\varphi(z) \ge \sup_\nu \|\nu\|_1 = \sqrt{n}$. In this case the condition of BV-ellipticity is equivalent to the *subadditivity* of φ; i.e. that $\varphi(z+z') \le \varphi(z)+\varphi(z')$ for all z, z'. This condition is rather complex, and is implied by the concavity of φ on $(0,+\infty)$. We will prove the approximation result for this restricted but important class of energy densities;

3) $\varphi = \varphi(x)$ lower semicontinuous. In this case the only condition for approximation is $\varphi(x) \ge \sqrt{n}$.

Moreover we can obtain $\varphi(x,z,\nu) = \varphi_1(\nu)\varphi_2(z)\varphi_3(x)$ by combining the approximation constructions above.

We note that other types of energies can be obtained as Γ-limits; for example, those of the form

$$F(u) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)} \varphi(x, u^+ - u^-)d\mathcal{H}^{n-1} \text{ if } u \in SBV(\Omega),$$

with the constraint that $S(u) \subset K$ where K is a fixed $n-1$-dimensional surface. Indeed, such types of energies will be the building blocks of our approximation strategy. In fact, for case (1) above we will first use this construction with K a network of planar surfaces and φ suitable constants on each surface of the network, and then use an approximation procedure similar to the one by Ansini and Iosifescu [6] to obtain an arbitrary convex φ. Note that in particular we may obtain as φ any constant not smaller than $\sqrt{n}$, so that case (3) can be derived by localizing such a construction. To obtain case (2), we first treat the case of K a single hyperplane and $\varphi(x,z) = c_1 + c_2 z^2$. This can be obtained following arguments similar to those by Ansini [4] to approximate the energy density $c(u^+ - u^-)^2$ on a surface (Neumann sieve) coupled with the description of the effect of pinning sites at the critical scaling by Sigalotti [26, 27]. Note that the computation of the interfacial energy gives the same constant as in the continuous case for $n = 2$, while it highlights a more complex behavior for $n \ge 3$, where a fraction of the total contribution is actually given by the strong springs at the interface, which sums up to the contribution distributed away from the interface and summarized in a capacitary formula. By repeating this argument on more parallel surfaces concentrating to the same hyperplane we can recover an arbitrary concave function by approximation with subadditive envelopes of families of functions as above (this is the only argument where concavity is used). Finally the use of a network of hyperplanes as above allows for a radially symmetric target φ.

The paper is organized as follows. In Section 2 we introduce the necessary notation to state the main result of the paper (Theorem 2.2). In Section 3 we treat discrete energies with defects on coordinate hyperplanes. Its main result is Theorem 3.1, where we describe the effect of a (small) percentage of strong springs distributed on a planar interface, and which exhibits an interesting separation of scales effect. Another result of independent interest is Theorem 3.11, which treats the case when weak springs in that interface are substituted by voids (in other words, we consider two quadratic discrete media weakly connected through a hypersurface). In Section 4 we prove a number of Γ-convergence results for functionals defined on $GSBV$ starting from the energies obtained in Theorem 3.1, eventually proving Theorem 2.2 by successive constructions. Finally, Section 5 is an appendix devoted to a short proof of a discrete Poincaré inequality in our simplified context.

2 Setting of the problem. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. For fixed $\varepsilon > 0$ we consider the lattice $\varepsilon\mathbb{Z}^n \cap \Omega =: \Omega_\varepsilon$ and we denote by $\mathcal{A}_\varepsilon(\Omega)$ the set of functions

$$\mathcal{A}_\varepsilon(\Omega) = \{u : \varepsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}\}.$$

We define the set of all *nearest neighbors* (using a terminology borrowed from Mechanics we will also call such sets the *springs* in Ω).

$$M_\varepsilon(\Omega) = \{\{a, b\} : a, b \in \varepsilon\mathbb{Z}^n \cap \Omega \text{ and } |a - b| = \varepsilon\}. \tag{1}$$

We will simply write M_ε if Ω is fixed and no confusion is possible. In order not to count the interactions twice, nearest neighbors are defined as sets containing two points, and not as pairs in $(\varepsilon\mathbb{Z}^n \cap \Omega) \times (\varepsilon\mathbb{Z}^n \cap \Omega)$. We can equivalently state our results in the latter notation, in which case we must take care in considering symmetric subsets of $(\varepsilon\mathbb{Z}^n \cap \Omega) \times (\varepsilon\mathbb{Z}^n \cap \Omega)$ only.

With fixed a subset $W_\varepsilon \subseteq M_\varepsilon$, we define the functional $F^{W_\varepsilon} : \mathcal{A}_\varepsilon(\Omega) \to [0, +\infty)$ as

$$\begin{aligned} F^{W_\varepsilon}(u) &= \sum_{\{a,b\}\in M_\varepsilon\setminus W_\varepsilon} \varepsilon^n \Big(\frac{u(a) - u(b)}{\varepsilon}\Big)^2 + \sum_{\{a,b\}\in W_\varepsilon} \varepsilon^n \Big(\Big(\frac{u(a) - u(b)}{\varepsilon}\Big)^2 \wedge \frac{1}{\varepsilon}\Big), \\ &= \sum_{\{a,b\}\in M_\varepsilon} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), \end{aligned} \tag{2}$$

where

$$f^\varepsilon_{a,b}(z) = \begin{cases} \Big(\dfrac{z}{\varepsilon}\Big)^2 & \text{if } \{a, b\} \in M_\varepsilon \setminus W_\varepsilon \\ \Big(\dfrac{z}{\varepsilon}\Big)^2 \wedge \dfrac{1}{\varepsilon} & \text{if } \{a, b\} \in W_\varepsilon. \end{cases}$$

Remark 2.1 A function $u \in \mathcal{A}_\varepsilon(\Omega)$ will be identified with the piecewise-constant measurable function given by $u(x) = u(z^\varepsilon_x)$, where z^ε_x is the closest point to x in $\varepsilon\mathbb{Z}^n$ (which is uniquely defined up to a set of zero measure). In this definition, we set $u(z) = 0$ if $z \in \varepsilon\mathbb{Z}^n \setminus \Omega$. In this way $A_\varepsilon(\Omega)$ will be regarded as a subset of $L^1(\Omega)$.

With the identification above, and a slight abuse of notation, we can extend F^{W_ε} to a functional $F^{W_\varepsilon} : L^1(\Omega) \to [0, +\infty]$ as

$$F^{W_\varepsilon}(u) = \begin{cases} \displaystyle\sum_{\{a,b\}\in M_\varepsilon} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), & \text{if } u \in \mathcal{A}_\varepsilon(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

The notation of (3) will be "localized" to subsets A of Ω by setting

$$F^{W_\varepsilon}(u; A) = \begin{cases} \displaystyle\sum_{\{a,b\}\in M_\varepsilon(A)} \varepsilon^n f^\varepsilon_{a,b}(u(a) - u(b)), & \text{if } u \in \mathcal{A}_\varepsilon(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{4}$$

(and accordingly for other functionals).

We will study the Γ-convergence of families of such functionals with varying W_ε with respect to the L^1-convergence.

Given an arbitrary distribution of weak springs $W_j = W_{\varepsilon_j}$, we may define the limit density in Ω of the weak springs W_j. This can be done after identifying each weak spring with a scaled Dirac delta; i.e., when W_j is identified with the measure

$$\lambda^{W_j} = \frac{\varepsilon_j^n}{n} \sum_{\{a,b\}\in W_j} \delta_{(a+b)/2}.$$

Upon passing to a subsequence, λ^{W_j} has a weak* limit λ in the sense of measures. Furthermore, since this limit is simply the Lebesgue measure if $W_j = M_{\varepsilon_j}(\mathbb{R}^n)$, then λ is absolutely continuous with respect to $\mathcal{L}^n$, so that we can write $\lambda = \theta\,\mathcal{L}^n$, with $0 \le \theta \le 1$. We will then simply write

$$W_j \to \theta\,. \tag{5}$$

We will show that all results that we obtain can also be obtained by prescribing θ

As an important preliminary step in Section 3 we will consider the case of W_ε concentrating on coordinate hyperplanes, and in Section 4 we will use that result to obtain a wide class of limit energies. Before stating the main result of that section we introduce the necessary function setting.

2.1 Special functions of bounded variations

Our limit energies will be defined on the Ambrosio-De Giorgi space of *generalized special functions with bounded variation* $GSBV(\Omega)$ (for all the definitions in this section see e.g. [3, 11]).

We recall that the space $SBV(\Omega)$ is defined as the set of functions u in $BV(\Omega)$ such that their measure distributional derivative Du admits the representation $Du = \nabla u\,\mathcal{L}^n + (u^+ - u^-)\nu(u)\mathcal{H}^{n-1} \llcorner S(u)$, where

- ∇u is the *approximate differential* of u
- $S(u)$ is the set of essential discontinuity points or *jump set* of u
- $\nu(u)$ is the measure theoretical *normal* to $S(u)$, which is defined $\mathcal{H}^{n-1}$ on $S(u)$
- $u^\pm$ are the *traces* of u on both sides of $S(u)$.

$\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure in $\mathbb{R}^n$ and the $n-1$-dimensional Hausdorff measure, respectively. $\lambda \llcorner B$ denotes the restriction of the measure λ to B; i.e., $(\lambda \llcorner B)(A) = \lambda(A \cap B)$. A function u belongs to $GSBV(\Omega)$ if for all $T > 0$ its truncations $u_T := (u \wedge T) \vee (-T)$ belong to $SBV(\Omega)$.

2.2 Statement of the main result

The results of the final section are (partly) summarized in the following theorem, which is the main result in the paper.

Theorem 2.2 *Let $\varphi : \mathbb{R}^n \to [0,+\infty)$ be any convex, even and positively homogeneous function of degree one with*

$$\varphi(w) \ge \|w\|_1 := \sum_{j=1}^n |w_j|,$$

$\psi : (0,+\infty) \to [1,+\infty)$ be any concave function, $a : \Omega \to [1,+\infty)$ be any lower semicontinuous function, and let $F : L^1(\Omega) \to [0,+\infty]$ be given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap S(u)} a(x)\varphi(\nu(u))\psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1} \\ \qquad\qquad \text{if } u \in GSBV(\Omega) \text{ and } \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{6}$$

Then there exists a family W_ε such that functionals F^{W_ε} given by (2) *Γ-converge to F above in the L^1-topology. Furthermore, for any $\theta \in L^\infty(\Omega)$ with $0 \le \theta \le 1$ we can additionally choose W_ε with $W_\varepsilon \to \theta$ in the sense of* (5)

2.3 Preliminary results

The case when $W_\varepsilon = M_\varepsilon$ in (2) is described by the following result by Chambolle [22] (see also Braides and Gelli [16])

Theorem 2.3 (Blake-Zisserman weak membrane) *The functionals defined by*

$$F_\varepsilon(u) = \sum_{\{a,b\}\in M_\varepsilon(\Omega)} \varepsilon^n \Big(\Big(\frac{u(a)-u(b)}{\varepsilon}\Big)^2 \wedge \frac{1}{\varepsilon}\Big) \tag{7}$$

on $\mathcal{A}_\varepsilon(\Omega)$ and extended to $L^1(\Omega)$ by $+\infty$ as in (3) *Γ-converge with respect to the $L^1(\Omega)$-convergence to the functional defined by*

$$F(u) = \int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap S(u)} \|\nu(u)\|_1\,d\mathcal{H}^{n-1}$$

if $u \in GSBV(\Omega)$ and $\mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty$, and $+\infty$ otherwise. Furthermore, if (u_ε) is a bounded sequence in $L^\infty(\Omega)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$ then, up to extraction of a subsequence, it converges to a function in $SBV(\Omega)$.

Note that F is an anisotropic version of the Mumford-Shah functional, and enjoys all the coerciveness and lower-semicontinuity properties of that functional (see [11]).

Remark 2.4 (1) Since for general W_ε we have $F_\varepsilon \le F^{W_\varepsilon}$ (F_ε as in (7)), the previous result provides a lower bound for all our Γ-limits, and in particular it implies that their domain will always be contained in $\{u \in GSBV(\Omega) : \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty\}$.

(2) Since all energies are decreasing by truncation (i.e., $F^{W_\varepsilon}(u_T) \le F^{W_\varepsilon}(u)$), it will suffice to characterize Γ-limits on $\{u \in SBV(\Omega)\cap L^\infty(\Omega) : \mathcal{H}^{n-1}(S(u)\cap\Omega) < +\infty\}$.

(3) If $W_\varepsilon = \emptyset$ (all springs are strong), then the Γ-limit is simply the Dirichlet integral.

(4) For general W_ε, by comparison with the cases above, the Γ-limit always exists and is equal to the Dirichlet integral on functions $u \in H^1(\Omega)$.

We will use well-known results on $GSBV$-functions, referring to [3, 11] when needed. We only recall the following approximation result, since it will be crucial to understand our strategy. To that end we introduce the following set of "piecewise-Lipschitz functions".

Definition 2.5 *We denote by $\mathrm{PC}(\Omega)$ the set of all functions $u \in SBV(\Omega)$ such that $S(u)$ is a finite union of $(n-1)$-dimensional simplices with disjoint closures, and $u \in W^{1,\infty}(\Omega\setminus\overline{S(u)})$.*

The set $\mathrm{PC}(\Omega)$ is "strongly dense" in $SBV(\Omega)$ as implied by the following theorem.

Theorem 2.6 (Cortesani-Toader [23]) *For all $u \in SBV(\Omega)\cap L^\infty(\Omega)$ there exists a sequence (u_j) in $\mathrm{PC}(\Omega)$ such that*

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx &+ \int_{\Omega\cap S(u)} f(\nu(u), u^+ - u^-)\,d\mathcal{H}^{n-1} \\ &= \lim_j \Big(\int_\Omega |\nabla u_j|^2\,dx + \int_{\Omega\cap S(u_j)} f(\nu(u_j), u_j^+ - u_j^-)\,d\mathcal{H}^{n-1}\Big)\end{aligned}$$

for all continuous f. Moreover we can take $u_j \in C^\infty(\Omega\setminus\overline{S(u_j)})\cap W^{k,\infty}(\Omega\setminus\overline{S(u_j)})$ for all k.

Remark 2.7 For functions $u \in \mathrm{PC}(\Omega)$ it is easily seen that the Γ-limit in Theorem 2.3 is actually a pointwise limit (see [22, 16]).

The approximation result above will guarantee that it is sufficient to prove the Γ-limsup inequality for functions in $SBV(\Omega)\cap L^\infty(\Omega)$, whose jump set is a finite union of $n-1$-dimensional simplices and are smooth outside that jump set. In particular, for those functions the jump set is contained in a finite union of hyperplanes. It will be crucial then first to construct limit energies whose domain implies the constraint that the jump set be (a union of simplices) contained in a given finite union of hyperplanes, and then remove that constraint through a homogenization procedure by considering an "invading" family of hyperplanes.

3 Discrete energies with defects on coordinate hyperplanes

This section will be the cornerstone of our approximation procedure. We will analyze the case when the 'defected springs' W_ε are located across a coordinate hyperplane, that we can assume being $x_n = 0$. More precisely, pairs in W_ε have their middle point on the set $\mathbb{Z}^{n-1} \times \varepsilon/2$, contained in the hyperplane $\{x_n = \varepsilon/2\}$ (essentially, the hyperplane $\{x_n = 0\}$).

Notation. In this section we will use a notation suitable to the discrete setting. The subscript ε will indicate the intersection with $\varepsilon\mathbb{Z}^n$, so that in particular $\Omega_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^n$.

The closed cube centered in x and with side length $2L$ will be denoted by $Q(L;x) = x + [-L,L]^n$. If $x = 0$ then $Q_L = Q(L,0)$. Accordingly, we will write $Q_\varepsilon(L;x) = \Omega_\varepsilon \cap \big(x+[-L,L]^n\big)$. Intersections of boundary of cubes with $\varepsilon\mathbb{Z}^n$ will be denoted by $\mathcal{L}(L;x) = \Omega_\varepsilon \cap \partial(x+[-L,L]^n)$, and we write $\mathcal{L}(L) = \mathcal{L}(L,0)$ when $x = 0$.

A subset A of $\varepsilon\mathbb{Z}^n$ is identified with the measurable set in $\mathbb{R}^n$ obtained as the union of the ε-cubes centered in A. We will highlight this identification with boldface cases: $\mathbf{A} = \bigcup_{a\in A}\mathcal{C}(a)$, where $\mathcal{C}(a) = \mathcal{C}_\varepsilon(a) = Q(\varepsilon/2;a) = a + \big[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\big]^n$. Finally, $[t]$ stands for the integer part of t.

Statement of the result. The following theorem describes the situation when all springs (parameterized by $\varepsilon_j\mathbb{Z}^{n-1}$) across the coordinate hyperplane are defected except those on a lattice $\delta_j\mathbb{Z}^{n-1}$ with $\delta_j >> \varepsilon_j$.

We denote by K the set $K = \Omega \cap \{x \in \mathbb{R}^n :\ x_n = 0\}$. If necessary, a vector $a \in \mathbb{R}^n$ will be written as $a = (a', a_n)$, with $a' = (a_1,\dots,a_{n-1})$.

Theorem 3.1 *Let (ε_j) be a positive sequence such that $\varepsilon_j \to 0$ as $j \to +\infty$. Let (δ_j) be a positive infinitesimal sequence such that $\delta_j/\varepsilon_j \in \mathbb{N}$ and $\lim_j \delta_j/\varepsilon_j = +\infty$. We assume that (ε_j) and (δ_j) are such that*

$$\varepsilon_j = \begin{cases} e^{-\beta(1+o(1))/\delta_j} \text{ as } j \to +\infty & \text{if } n = 2 \\ \beta^{2-n}\delta_j^{(n-1)/(n-2)}(1+o(1)) \text{ as } j \to +\infty & \text{if } n > 2 \end{cases} \tag{8}$$

where β is a positive constant. For all $j \in \mathbb{N}$ we set

$$W_{\varepsilon_j} = \{\{a,b\} \in M_j :\ a' = b', a_n = 0, b_n = \varepsilon_j, a' \in \varepsilon_j\mathbb{Z}^{n-1} \setminus \delta_j\mathbb{Z}^{n-1}\}. \tag{9}$$

Let C_n be defined as follows:

$$C_n = \frac{\pi}{2} \text{ if } n = 2, \qquad C_n = \frac{l_n}{4 + l_n} \text{ if } n > 2, \tag{10}$$

where

$$l_n = \lim_{T\to+\infty} \min\Big\{ \sum_{\{a,b\}\in M_1(Q_T)} (v(a) - v(b))^2 :\ v = 1 \text{ on } \partial Q_T,\ v(0) = v(e_n) = 0\Big\}$$

(recall that $M_1(\Omega)$ denotes the set of nearest neighbors of $\mathbb{Z}^n$ inside Ω) is a positive constant for all $n > 2$. Let $F : L^1(\Omega) \to [0,+\infty]$ be given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_{S(u)} \Big(1 + \frac{C_n}{\beta}|u^+ - u^-|^2\Big) d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega),\ S(u) \subseteq K \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have

(i) (coerciveness) *for any sequence* $(u_j)_j$ *bounded in* $L^1(\Omega)$ *such that* $\sup_j F^{W_{\varepsilon_j}}(u_j) < +\infty$ *there exist a subsequence* $(u_{j_h})_h$ *and a function* $u \in SBV(\Omega)$ *with* $S(u) \subseteq K$ *such that* $u_{j_h} \to u$ *as* $h \to +\infty$ *in* $L^1(\Omega)$*;*

(ii) (lower bound) *for all* $u \in L^1(\Omega)$ *and* $u_j \to u$ *in* $L^1(\Omega)$ *we have* $F(u) \leq \liminf_j F^{W_{\varepsilon_j}}(u_j)$*;*

(iii) (upper bound) *for all* $u \in \mathrm{PC}(\Omega)$ *there exists* $u_j \to u$ *in* $L^1(\Omega)$ *such that* $F(u) = \lim_j F^{W_{\varepsilon_j}}(u_j)$.

Note that the coerciveness is an immediate consequence of Remark 2.4 (1) and (3) (the latter applied on all open sets not intersecting K). The rest of the theorem will be proven throughout this section, separately proving the upper and lower bounds.

Remark 3.2 1) Note that, since we have the constraint $S(u) \subset K$ for the jump set, the domain of F is actually contained in $H^1(\Omega \setminus K)$;

2) It is worth noting the two different definitions of the constant C_n in the cases $n = 2$, which are connected to capacitary issues due to the presence of a portion of strong springs on the interface. In particular,

• in the case $n = 2$ the constant is the same as the one for the Neumann sieve for continuous problems. This highlights that capacitary potentials in dimension 2 are logarithmic, and their contribution is at a scale much larger than the lattice;

• in the case $n \geq 3$ a scaling argument leads to a *discrete capacitary problem* involving the capacity l_n of a *discrete dipole* in $\mathbb{Z}^n$;

3) Hypothesis (8) can be restated in terms of the percentage $p_j = (\varepsilon_j/\delta_j)^{n-1}$ of strong springs at the interface, which now reads

$$p_j = \begin{cases} \dfrac{\varepsilon_j |\log \varepsilon_j|}{\beta}(1 + o(1)) & \text{if } n = 2 \\ \dfrac{\varepsilon_j}{\beta}(1 + o(1)) & \text{if } n > 2. \end{cases}$$

Remark 3.3 The constraint in the theorem can be generalized to $S(u) \subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero, where K is the closure of a relatively open subset A of a coordinate hyperplane with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, or more in general $S(u) \subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero, where K is the closure of a relatively open subset A of a union of coordinate hyperplanes. with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$. The proof is exactly the same, upon noticing that the constraint $S(u) \subset K$ up to sets of $\mathcal{H}^{n-1}$-measure zero is closed, thus compatible with the lower bound, and that the proof of the upper bound only involves a local argument.

For notational simplicity we will often write F_j in place of $F^{W_{\varepsilon_j}}$.

3.1 Two technical lemmas

In this section we state and prove the discrete analogs of the key propositions that allow the treatment of perforated domains as envisaged by Ansini and Braides [5] and restated for transmission problems across an interface by Ansini [4]. These results will be used to reduce to the case when the competing functions are constant on the upper and lower parts of the boundary of suitable squares centered in the strong springs of the interface. This reduction will then allow to estimate the contribution close to the strong springs at the interface with suitable discrete capacitary problems.

The set Z_j will be defined as

$$Z_j = \{(z', 0) : z' \in \delta_j \mathbb{Z}^{n-1} : Q(\delta_j/2; (z', 0)) \subset \Omega\}. \tag{11}$$

Recall that $\delta_j/\varepsilon_j \in \mathbb{N}$, so that $\delta_j \mathbb{Z}^{n-1} \subset \varepsilon_j \mathbb{Z}^{n-1}$.

Lemma 3.4 *Let $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in SBV(\Omega)$ with $S(u) \subset K$. We assume that $u_j \to u$ in $L^1(\Omega)$ and $\sup_j F_j(u_j) < +\infty$. With fixed $\alpha < 1/2$, let $\rho_j = \alpha\delta_j$. Let $k \in \mathbb{N}$ be fixed. Then, for all $l \in Z_j$ there exists $k_l \in \{0, 1, \dots, k-1\}$ such that, having set*

$$C_j^l = Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{k_l}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{k_l+1}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big), \qquad C_j^{l\pm} = C_j^l \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\} \quad (12)$$

$$u_j^{l\pm} = \frac{1}{\sharp C_j^{l\pm}} \sum_{a \in C_j^{l\pm}} u_j(a), \qquad \rho_j^l = \Big[\frac{3}{4}\frac{1}{2^{k_l}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, \quad (13)$$

there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ such that $w_j \to u$ in $L^1(\Omega)$ satisfying the following conditions:

$$w_j = u_j \text{ on } \Omega_j \setminus \bigcup_{l \in Z_j} C_j^l, \qquad w_j = u_j^{l\pm} \text{ on } \mathcal{L}_{\varepsilon_j}(\rho_j^l, l) \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\}, \quad (14)$$

$$\Big|\sum_{l \in Z_j}\Big(F_j(u_j; C_j^{l+}) + F_j(u_j; C_j^{l-}) - \big(F_j(w_j; C_j^{l+}) + F_j(w_j; C_j^{l-})\big)\Big)\Big| \leq \frac{c}{k}. \quad (15)$$

Proof. For all $h \in \{0, 1, \dots, k-1\}$ we define

$$C_{j,h}^l = Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^h}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{1}{2^{h+1}}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big), \quad C_{j,h}^{l\pm} = C_{j,h}^l \cap \{\pm(x_n - \varepsilon_j/2) \geq 0\},$$

$$u_{j,h}^{l\pm} = \frac{1}{\sharp C_{j,h}^{l\pm}} \sum_{a \in C_{j,h}^{l\pm}} u_j(a), \qquad \rho_{j,h}^l = \Big[\frac{3}{4}\frac{1}{2^h}\frac{\rho_j}{\varepsilon_j}\Big]\varepsilon_j.$$

For fixed $h \in \{0, 1, \dots, k-1\}$ we consider a function $\phi = \phi_{j,h}^l \in C_c^\infty(\mathbf{C}_{j,h}^l)$ such that $\phi = 1$ on $\partial Q(\rho_{j,h}^l, l)$ and $\|\nabla\phi\|_\infty \leq c/\delta_j$. We define a sequence $w_{l,h}^j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ as follows:

$$w_{l,h}^j(a) = \begin{cases} \phi(a)u_{j,h}^{l+} + (1 - \phi(a))u_j(a) & \text{if } a_n \geq \varepsilon_j \\ \phi(a)u_{j,h}^{l-} + (1 - \phi(a))u_j(a) & \text{if } a_n \leq 0. \end{cases} \quad (16)$$

We focus our attention on $C_{j,h}^{l+}$ and notice that for all $\{a, b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})$ we have:

$$w_{l,h}^j(a) - w_{l,h}^j(b) = (1 - \phi(b))(u_j(a) - u_j(b)) + (\phi(a) - \phi(b))(u_{j,h}^{l+} - u_j(a)).$$

Moreover, by Jensen's inequality and the assumptions on ϕ we get that $|\phi(a) - \phi(b)|^2 \leq \varepsilon_j^2\|\nabla\phi\|_\infty^2 \leq c\varepsilon_j^2\delta_j^{-2}$ for all $\{a, b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})$. There follows that

$$\begin{aligned}
&\sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |w_{l,h}^j(a) - w_{l,h}^j(b)|^2 \varepsilon_j^{n-2} \\
&\leq c \sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} \big(|1 - \phi(b)|^2 |u_j(a) - u_j(b)|^2 + |\phi(a) - \phi(b)|^2 |u_{j,h}^{l+} - u_j(a)|^2\big)\varepsilon_j^{n-2} \\
&\leq c \sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} + \frac{c}{\delta_j^2}\varepsilon_j^n \sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |u_{j,h}^{l+} - u_j(a)|^2.
\end{aligned}$$

By Lemma 5.2 we deduce that the last term above can be estimated as follows:

$$\frac{c}{\delta_j^2}\varepsilon_j^n \sum_{a \in C_{j,h}^{l+}} |u_{j,h}^{l+} - u_j(a)|^2 \leq c \sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2};$$

hence, $\displaystyle\sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |w_{l,h}^j(a) - w_{l,h}^j(b)|^2 \varepsilon_j^{n-2} \leq c \sum_{\{a,b\} \in M_{\varepsilon_j}(C_{j,h}^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2}.$

Note that the constant c in the right-hand side can be chosen independent of $h \in \{0,\dots,k-1\}$, thanks to the fact that the sets $C^{l+}_{j,h}$ are obtained from homothetic sets. Arguing similarly on $C^{l-}_{j,h}$ we deduce that

$$\sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l-}_{j,h})} |w^j_{l,h}(a)-w^j_{l,h}(b)|^2\varepsilon_j^{n-2} \le c \sum_{\{a,b\}\in M_{\varepsilon_j}(C^{l-}_{j,h})} |u_j(a)-u_j(b)|^2\varepsilon_j^{n-2}.$$

Therefore there exists $c>0$ (independent of h) such that $F_j(w^j_{l,h};C^{l\pm}_{j,h})+F_j(u_j;C^{l\pm}_{j,h}) \le c\,F_j(u_j;C^{l\pm}_{j,h})$. Summing up over $h\in\{0,1,\dots,k-1\}$ we get

$$\sum_{h=0}^{k-1}\big(F_j(w^j_{l,h};C^{l+}_{j,h})+F_j(u_j;C^{l+}_{j,h})+F_j(w^j_{l,h};C^{l-}_{j,h})+F_j(u_j;C^{l-}_{j,h})\big)\le cF_j(u_j;Q_{\varepsilon_j}(\rho_j,l)).$$

Hence there exists $k_l\in\{0,1,\dots,k-1\}$ such that

$$F_j(w^j_{l,h};C^{l+}_{j,k_l})+F_j(u_j;C^{l+}_{j,k_l})+F_j(w^j_{l,h};C^{l-}_{j,k_l})+F_j(u_j;C^{l-}_{j,k_l})\le \frac{c}{k}F_j(u_j;Q_{\varepsilon_j}(\rho_j,l)).$$

Now, we set $C^l_j=C^l_{j,k_l}$, $C^{l\pm}_j=C^{l\pm}_{j,k_l}$, $u^{l\pm}_j=u^{l\pm}_{j,k_l}$, $\rho^l_j=\rho^l_{j,k_l}$, and we define $w_j:\Omega_j\to\mathbb{R}$ as follows:

$$w_j(a)=\begin{cases} w^j_{l,k_l}(a) & \text{if } a\in C^l_j, \quad l\in Z_j\\ u_j(a) & \text{if } a\in\Omega_j\setminus\bigcup_{l\in Z_j}C^l_j.\end{cases} \tag{17}$$

The sequence (w_j) defined in (17) satisfies all the required conditions. In fact, $w_j\equiv u_j$ on $\Omega_j\setminus\bigcup_{l\in Z_j}C^l_j$, $w_j=u^{l\pm}_j$ on $\partial Q\big(\rho^l_j,l\big)\cap\{\pm(x_n-\varepsilon_j/2)\ge 0\}$ and

$$\begin{aligned}
&\Big|\sum_{l\in Z_j}\Big(F_j(u_j;C^{l+}_j)+F_j(u_j;C^{l-}_j)-\big(F_j(w_j;C^{l+}_j)+F_j(w_j;C^{l-}_j)\big)\Big)\Big|\\
&\quad\le\sum_{l\in Z_j}\big(F_j(u_j;C^{l+}_j)+F_j(u_j;C^{l-}_j)+F_j(w_j;C^{l+}_j)+F_j(w_j;C^{l-}_j)\big)\\
&\quad\le\frac{c}{k}\sum_{l\in Z_j}F_j(u_j;Q_{\varepsilon_j}(\rho_j,l))\le\frac{c}{k}F_j(u_j)\le\frac{c}{k}.
\end{aligned}$$

Moreover, we show that $w_j\to u$ in $L^1(\Omega)$:

$$\begin{aligned}
\int_\Omega|w_j-u|dx &\le \int_{\Omega\setminus\bigcup_{l\in Z_j}\mathbf{C}^l_j}|w_j-u|dx+\sum_{l\in Z_j}\int_{\mathbf{C}^l_j}|w_j-u|dx\\
&\le \int_\Omega|u_j-u|dx+\sum_{l\in Z_j}\Big(\sum_{a\in C^{l+}_j}|u_j(a)-u^{l+}_j|\varepsilon_j^n+\sum_{a\in C^{l-}_j}|u_j(a)-u^{l-}_j|\varepsilon_j^n\Big).
\end{aligned}$$

By Hölder's inequality and Lemma 5.2 we have

$$\begin{aligned}
\sum_{l\in Z_j}\sum_{a\in\mathbf{C}^{l\pm}_j}|u_j(a)-u^{l\pm}_j|\varepsilon_j^n &= \sum_{l\in Z_j}\sum_{a\in C^{l\pm}_j}|u_j(a)-u^{l\pm}_j|\varepsilon_j^n\\
\le\sum_{l\in Z_j}\varepsilon_j^{n/2}\Big(\sum_{a\in C^{l\pm}_j}|u_j(a)-u^{l\pm}_j|^2\varepsilon_j^n\Big)^{1/2}\big(\sharp C^{l\pm}_j\big)^{1/2} &\le c\delta_j^{n/2}\delta_j(\sharp Z_j)^{1/2}\big(F_j(u_j)\big)^{1/2}\le c\delta_j^{3/2}.
\end{aligned}$$

In conclusion $\limsup_j\int_\Omega|w_j-u|dx\le\limsup_j\Big(\int_\Omega|u_j-u|dx+c\delta_j^{3/2}\Big)=0$ as desired. □

Proposition 3.5 *Let (u_j) be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u_j \to u$ in $L^1(\Omega)$ for some $u \in SBV(\Omega)$ with $S(u) \subset K$. Assume that (u_j) is bounded in $L^\infty(\Omega)$. We fix $k \in \mathbb{N}$ and consider a positive infinitesimal sequence $\rho_j = \alpha\delta_j$, with $\alpha < 1/2$. Following the notation of Lemma 3.4, we fix (arbitrarily) $k_l \in \{0, 1 \ldots, k-1\}$ and we denote by $u_j^{l\pm}$ the discrete average of u_j on $C^{l\pm}_{j,k_l}$. Then we have*

$$\lim_j \sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \delta_j^{n-1} = \int_{\Omega\cap S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1}. \tag{18}$$

Proof. For all $l \in Z_j$ we define $I_j^l = l + [-\delta_j/2, \delta_j/2)^{n-1} \subseteq K$. Let $\psi_j : \Omega \cap K \to [0, +\infty)$ be given by

$$\psi_j(x', 0) = \sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \chi_{I_j^l}(x') = \begin{cases} |u_j^{l+} - u_j^{l-}|^2 & \text{for } x' \in I_j^l, \quad l \in Z_j, \\ 0 & \text{otherwise.} \end{cases}$$

We have to prove that $\lim_j \int_{\Omega\cap K} \psi_j(x', 0)dx' = \int_{\Omega\cap S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1}$.

Since (u_j) is bounded in $L^\infty(\Omega)$, we have

$$\begin{aligned} &\Big| \int_{\Omega\cap K} \Big(\sum_{l\in Z_j} |u_j^{l+} - u_j^{l-}|^2 \chi_{I_j^l}(x') - |u^+(x', 0) - u^-(x', 0)|^2 \Big) dx' \Big| \\ &\qquad \le c \sum_{l\in Z_j} \int_{I_j^l} \big(|u_j^{l+} - u^+(x', 0)| + |u_j^{l-} - u^-(x', 0)| \big) dx'. \end{aligned}$$

We want to prove that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l\pm} - u^\pm(x', 0)| dx' = 0. \tag{19}$$

We notice that

$$\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l\pm} - u^\pm(x', 0)| dx' \le \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l\pm} - u_j(x', \varepsilon_j)| dx' + \int_{I_j^l} |u^\pm(x', 0) - u_j(x', \varepsilon_j)| dx'.$$

We focus our attention on $\Omega^+ = \Omega \cap \{x_n > 0\}$ and we prove that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)| dx' = 0. \tag{20}$$

By Hölder's and Jensen's inequalities we get

$$\begin{aligned} \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)| dx' &\le c \sum_{l\in Z_j} \delta_j^{(n-1)/2} \Big(\int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)|^2 dx' \Big)^{1/2} \\ \le \quad c\big(\sharp Z_j\big)^{1/2} \delta_j^{(n-1)/2} \Big(\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)|^2 dx' \Big)^{1/2} &\le c \Big(\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)|^2 dx' \Big)^{1/2}. \end{aligned}$$

By construction $\int_{I_j^l} |u_j^{l+} - u_j(x', \varepsilon_j)|^2 dx' = \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a', \varepsilon_j)|^2 \varepsilon_j^{n-1}$. We claim that the following inequality holds:

$$\begin{aligned} &\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a', \varepsilon_j)|^2 \varepsilon_j^{n-1} \\ &\qquad \le c \Big(\frac{1}{\delta_j} \sum_{a\in R_j^{l+}} |u_j^{l+} - u_j(a)|^2 \varepsilon_j^n + \delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} \Big), \end{aligned} \tag{21}$$

where $R_j^{l+} = \Omega_j \cap \{I_j^l \times [\varepsilon_j, \delta_j/2]\}$ and the constant c is independent of j and l. By applying Lemma 5.2 to the first term in (21) we deduce that

$$\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \le c\delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2}.$$

By summing over $l \in Z_j$ we get

$$\begin{aligned}\sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u_j(x',\varepsilon_j)|^2 dx' &\le c\delta_j \sum_{l\in Z_j} \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2} \\ &\le c\delta_j \sup_j F_j(u_j) \le c\delta_j\end{aligned}$$

and this implies that (20) holds. Moreover, by the definition of the trace of a function in $SBV(\Omega)$ (actually in $H^1(\Omega\setminus K)$), we deduce that

$$\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u^+(x',0) - u_j(x',\varepsilon_j)| dx' = 0. \tag{22}$$

By (20) and (22) we deduce that $\limsup_j \sum_{l\in Z_j} \int_{I_j^l} |u_j^{l+} - u^+(x',0)| dx' = 0$, as desired. By arguing similarly on Ω^-, we can conclude that (19) holds. It remains to show that inequality (21) holds. We have:

$$\begin{aligned}&\sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\ &\le c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j^{l+} - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\ &\le c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} \big(|u_j^{l+} - u_j(b)|^2 + |u_j(b) - u_j(a',\varepsilon_j)|^2\big) \varepsilon_j^{n-1} \\ &\le c\frac{\varepsilon_j}{\delta_j} \sum_{b\in R_j^{l+}} |u_j^{l+} - u_j(b)|^2 \varepsilon_j^{n-1} + c\frac{\varepsilon_j^n}{\delta_j^n} \sum_{b\in R_j^{l+}} \sum_{(a',\varepsilon_j)\in\Omega_j:\ a'\in I_j^l} |u_j(b) - u_j(a',\varepsilon_j)|^2 \varepsilon_j^{n-1} \\ &\le \frac{c}{\delta_j} \sum_{b\in R_j^{l+}} |u_j^{l+} - u_j(b)|^2 \varepsilon_j^n + c\delta_j \sum_{\{a,b\}\in M_{\varepsilon_j}(R_j^{l+})} |u_j(a) - u_j(b)|^2 \varepsilon_j^{n-2}\end{aligned}$$

as desired. □

3.2 Lower bound

In this section we prove the lower bound for the sequence F_j by combining a scale-separation and a capacitary argument. The energy of a sequence $F_j(u_j)$, with $u_j \to u$, can be decomposed in

- a bulk energy away from the interface;
- an interfacial term due to the presence of the weak springs at the interface. This term corresponds to the surface term of the Blake-Zisserman weak membrane;
- an additional interfacial term decoupled from the previous one due to the presence of a (small) percentage of strong springs at the interface. This is a quadratic term on the interface depending on the discontinuity $|u^+ - u^-|^2$.

The key argument is the separation of scales at the interface. By using Lemma 3.4 we can separately examine the energy contributions on cubes of side length $\alpha\delta_j$ (with α small) and centered on the strong springs, and the energy elsewhere. Outside those cubes a lower bound is given by the Blake-Zisserman weak membrane (with an error vanishing with α). Again by Lemma 3.4 it is not restrictive to suppose that on the (upper/lower) boundary of the cubes the value of the functions u_j is exactly $u^{\pm}$, and a capacitary argument then allows to compare the contribution on each cube by a term $\delta_j^{n-1} c_n |u^+ - u^-|^2$, which gives a Riemann sum converging to the correct interfacial energy (the precise statement uses Proposition 3.5).

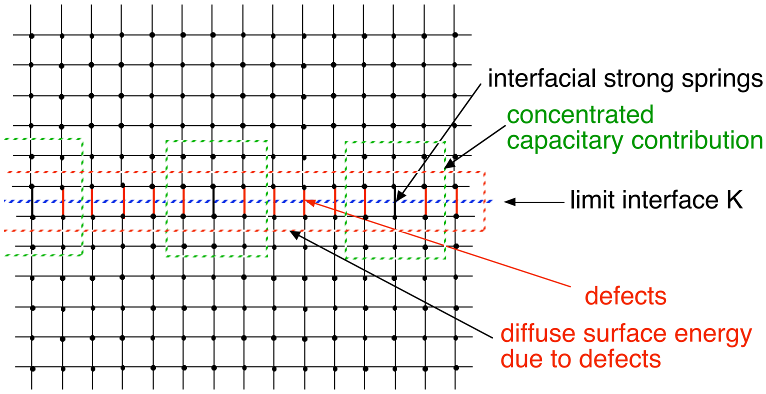

Figure 2: Scale and concentration effects on the interface

Proposition 3.6 (lower bound) *Let $u_j \to u$ in $L^1(\Omega)$, with $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in SBV(\Omega) \cap H^1(\Omega \setminus K)$. Then $\liminf_j F_j(u_j) \geq F(u)$.*

Remark 3.7 In our computation of the lower bound, we have found it convenient to deal with the contribution due to the quadratic strong springs (both on the interface and elsewhere) separately from that of the weak springs. To that end we introduce the energies (in "localized form" on subsets A of Ω)

$$G_j(v; A) = F_j(v; A) - \sum_{a \in \Omega_j \cap K,\ a \notin Z_j} \varepsilon_j^{n-2} ((v(a) - v(a', \varepsilon_j))^2 \wedge \varepsilon_j) \tag{23}$$

for any $v \in \mathcal{A}_{\varepsilon_j}(A)$, and $G_j(v) = G_j(v, \Omega)$. The Γ-limit of G_j is of interest in itself (see Section 3.5).

Proof. Let $k \in \mathbb{N}$ and let $\alpha < 1/2$. By applying Lemma 3.4 to (u_j) we build a sequence $w_j \to u$ in $L^1(\Omega)$ satisfying conditions (14)–(15). We define the set $E_j \subset \Omega_j$ as $E_j = \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l; l)$. Since $\liminf_j F_j(u_j) \geq \liminf_j F_j(u_j; E_j) + \liminf_j F_j(u_j; \Omega_j \setminus E_j)$, we will estimate the contributions of u_j on E_j and $\Omega_j \setminus E_j$ separately (step **A** and **B** respectively).

A. We want to prove that

$$\liminf_j F_j(u_j; E_j) \geq \frac{C_n}{\beta} \int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} - \frac{c}{k}. \tag{24}$$

Lemma 3.4 implies that

$$\begin{aligned}\liminf_j F_j(u_j;E_j) &\geq \liminf_j \sum_{l\in Z_j}\Big(F_j(u_j;Q_{\varepsilon_j}(\rho^l_j,l)\cap\{x_n\geq\varepsilon_j\})\\
&\quad +F_j(u_j;Q_{\varepsilon_j}(\rho^l_j,l)\cap\{x_n\leq 0\})+(u_j(l',\varepsilon_j)-u_j(l',0))^2\varepsilon_j^{n-2}\Big)-\frac{c}{k}\\
&\geq \liminf_j \sum_{l\in Z_j}\Big(F_j(w_j;Q_{\varepsilon_j}(\rho^l_j,l)\cap\{x_n\geq\varepsilon_j\})\\
&\quad +F_j(w_j;Q_{\varepsilon_j}(\rho^l_j,l)\cap\{x_n\leq 0\})+(u_j(l',\varepsilon_j)-u_j(l',0))^2\varepsilon_j^{n-2}\Big)-\frac{c}{k}\\
&= \liminf_j \sum_{l\in Z_j} G_j(w_j;Q_{\varepsilon_j}(\rho^l_j,l))-\frac{c}{k},\end{aligned}\tag{25}$$

where G_j in defined in (23).

Having fixed $l\in Z_j$, we look for an estimate from below for $G_j(w_j;Q_{\varepsilon_j}(\rho^l_j,l))$. Let $\widetilde{w}_j\in\mathcal{A}_{\varepsilon_j}(Q_{\varepsilon_j}(\rho_j,l))$ be defined as

$$\widetilde{w}_j(a)=\begin{cases}u^{l+}_j & \text{if } a\in(Q_{\varepsilon_j}(\rho_j,l)\setminus Q_{\varepsilon_j}(\rho^l_j,l))\cap\{x_n\geq\varepsilon_j\}\\ w_j(a) & \text{if } a\in Q_{\varepsilon_j}(\rho^l_j,l)\\ u^{l-}_j & \text{if } a\in(Q_{\varepsilon_j}(\rho_j,l)\setminus Q_{\varepsilon_j}(\rho^l_j,l))\cap\{x_n\leq 0\}.\end{cases}$$

By construction $G_j(w_j;Q_{\varepsilon_j}(\rho^l_j,l))=G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l))$ and, minimizing over all v subject to the boundary conditions satisfied by $\widetilde{w}_j$,

$$G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l))\geq\inf\big\{G_j(v,Q_{\varepsilon_j}(\rho_j,l)):\ v=u^{l\pm}_j \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j,l)\big\}.$$

After writing $\widetilde{w}_j=\frac{(u^{l+}_j+u^{l-}_j)}{2}+\frac{(u^{l+}_j-u^{l-}_j)}{2}v_j$, by a translation and a scaling argument we get

$$G_j(\widetilde{w}_j;Q_{\varepsilon_j}(\rho_j,l))\geq\frac{(u^{l+}_j-u^{l-}_j)^2}{4}\inf\big\{G_j(v,Q_{\varepsilon_j}(\rho_j)):v=\pm1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j)\big\}.\tag{26}$$

We denote by m_j the rescaled infimum

$$m_j=\varepsilon_j^{2-n}\inf\big\{G_j(v;Q_{\varepsilon_j}(\rho_j)):v=\pm1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}(\rho_j)\big\}.$$

We want to study the asymptotic behavior of m_j by comparing it with the infimum

$$\mu_j=\inf\Big\{\sum_{\{a,b\}\in M_{\varepsilon_j}(Q(\rho_j))}(v(a)-v(b))^2:\ v=1 \text{ on } \mathcal{L}_{\varepsilon_j}(\rho_j),\ v(0)=v(\varepsilon_j e_n)=0\Big\}.$$

The limit behavior of μ_j has been studied in details in [27]; to our purposes we recall that

$$\lim_{j\to+\infty}\mu_j=l_n\in(0,+\infty) \text{ if } n\geq 3,\qquad \lim_{j\to+\infty}\beta\frac{\mu_j}{\delta_j}=l_2\in(0,+\infty) \text{ if } n=2,\tag{27}$$

where

$$l_n=\lim_{T\to+\infty}\min\Big\{\sum_{\{a,b\}\in M_1(Q_T)}(v(a)-v(b))^2:\ v=1 \text{ on } \partial Q_T,\ v(0)=v(e_n)=0\Big\}>0\tag{28}$$

for $n\geq 3$, and

$$l_2=\lim_{T\to+\infty}\log T\,\min\Big\{\sum_{\{a,b\}\in M_1(Q_T)}(v(a)-v(b))^2:\ v=1 \text{ on } \partial Q_T,\ v(0)=v(e_n)=0\Big\}=2\pi\tag{29}$$

(T is understood to be integer).

We now focus our attention on m_j. It is not restrictive to substitute the cube $Q_{\varepsilon_j}(\rho_j)$ with a cube with center in $(0', \varepsilon/2)$ (for which we use the same symbol, with abuse of notation), so that we may use a symmetry argument. First, we note that if $z_j \in \mathcal{A}_{\varepsilon_j}(Q_{\varepsilon_j}(\rho_j))$ is a minimizer for m_j, then it satisfies the following condition:

$$z_j(x', x_n) = -z_j(x', -x_n + \varepsilon_j). \tag{30}$$

In fact, let $\overline{z}_j$ be the function $\overline{z}_j(x', x_n) := -z_j(x_1, \ldots, -x_n + \varepsilon_j)$. Then $\overline{z}_j$ is a minimizer for m_j by construction. The function $(z_j + \overline{z}_j)/2$ is a test function for m_j; by the strict convexity of G_j we get that if $z_j \neq \overline{z}_j$ then

$$G_j\Big(\frac{z_j + \overline{z}_j}{2}; Q_{\varepsilon_j}(\rho_j)\Big) < \frac{1}{2}G_j(z_j; Q_{\varepsilon_j}(\rho_j)) + \frac{1}{2}G_j(\overline{z}_j; Q_{\varepsilon_j}(\rho_j)) = m_j.$$

There follows that $z_j = \overline{z}_j$; i.e., condition (30) holds.

Now, let

$$\gamma = z_j(0, \ldots, 0, \varepsilon_j) \tag{31}$$

denote the 'half-elongation of the strong spring at the interface' (hence, $z_j(0) = -\gamma$). We note that

$$\begin{aligned}
&G_j(z_j; Q_{\varepsilon_j}(\rho_j) \cap \{x_n \geq \varepsilon_j\}) \\
&\quad = \min\big\{G_j(v; Q_{\varepsilon_j}(\rho_j) \cap \{x_n \geq \varepsilon_j\}) : \ v = 1 \text{ on } \mathcal{L}^+_{\varepsilon_j}(\rho_j), v(0, \ldots, 0, \varepsilon_j) = \gamma\big\} \\
&\quad = (1-\gamma)^2 \min\big\{G_j(v; Q_{\varepsilon_j}(\rho_j) \cap \{x_n \geq \varepsilon_j\}) : \ v = 1 \text{ on } \mathcal{L}^+_{\varepsilon_j}(\rho_j), v(0, \ldots, 0, \varepsilon_j) = 0\big\} \\
&\quad = \frac{1}{2}\mu_j(1-\gamma)^2.
\end{aligned}$$

By (30) we deduce that $G_j(z_j; Q_{\varepsilon_j}(\rho_j)) = 2 \times \frac{1}{2}\mu_j(1-\gamma)^2 + 4\gamma^2$, which attains its minimum for $\gamma = \mu_j/(\mu_j + 4)$, hence

$$m_j = \frac{4\mu_j}{\mu_j + 4}. \tag{32}$$

The asymptotic behavior of m_j can be specified as follows:

$$\lim_{j\to+\infty} m_j = \frac{4l_n}{l_n + 4} =: C_n \text{ if } n \geq 3, \qquad \lim_{j\to+\infty} \beta\,\frac{m_j}{\delta_j} = l_2 =: C_2 \text{ if } n = 2. \tag{33}$$

By (25) and (26) we get

$$\liminf_j F_j(u_j; E_j) \geq \liminf_j \sum_{l\in Z_j} \frac{(u_j^{l+} - u_j^{l-})^2}{4}\varepsilon_j^{n-2} m_j - \frac{c}{k}.$$

Note that by (33) if $n \geq 3$ we have

$$\varepsilon_j^{n-2} m_j = \frac{1}{\beta}\delta_j^{n-1} m_j(1 + o(1)) = \frac{1}{\beta}\delta_j^{n-1} C_n(1 + o(1)), \tag{34}$$

while if $n = 2$ we have

$$\varepsilon_j^{n-2} m_j = \delta_j \cdot \frac{m_j}{\delta_j} = \frac{\delta_j}{\beta} C_2(1 + o(1)). \tag{35}$$

In both cases, taking into account Proposition 3.5, we then deduce that

$$\liminf_j F_j(u_j; E_j) \geq \frac{C_n}{\beta}\int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} - \frac{c}{k}.$$

B. We want to prove that

$$\liminf_j F_j(u_j; \Omega_j \setminus E_j) \geq \int_\Omega |\nabla u|^2 dx + (1-\alpha)\mathcal{H}^{n-1}(S(u)). \tag{36}$$

Having fixed a parameter $s > 0$, we consider the "s-neighborhood of K" defined by

$$P_j^s = \{a \in \Omega_j \setminus E_j : \ |a_n| \leq s\}$$

and its complement $R_j^s = (\Omega_j \setminus E_j) \setminus P_j^s = \{a \in \Omega_j \setminus E_j : \ |a_n| > s\}$. Since

$$\liminf_j F_j(u_j; \Omega_j \setminus E_j) \geq \liminf_j F_j(u_j; R_j^s) + \liminf_j F_j(u_j; P_j^s),$$

we can estimate the contribution of u_j separately near the (hyperplane containing the) jump set and far from it (steps **B.1** and **B.2** below, respectively). By letting $s \to 0^+$, we will finally get the desired inequality (36).

B.1 First, we focus our attention on P_j^s and we prove that

$$\liminf_j F_j(u_j; P_j^s) \geq (1-\alpha)\mathcal{H}^{n-1}(S(u)). \tag{37}$$

The proof of (37) will be performed through the blow-up technique. For all $A \in \mathcal{B}(\Omega)$, we set

$$\lambda_j(A) = F_j(u_j; P_j^s \cap A),$$

which defines a family of measures. The family (λ_j) is equi-bounded; i.e., $\sup_j |\lambda_j|(\Omega) \leq \sup_j F_j(u_j) < +\infty$. Hence, there exists $\mu \in \mathcal{M}^+(\Omega)$ such that λ_j converges weakly* to λ up to subsequences. We consider the Radon-Nykodim decomposition of λ with respect to $\mathcal{H}^{n-1} \llcorner S(u)$: there exists a non-negative function $g \in L^1(\Omega)$ such that $\lambda = g\mathcal{H}^{n-1} \llcorner S(u) + \lambda^s$, where $\lambda^s \in \mathcal{M}^+(\Omega)$ is such that $\lambda^s \perp (\mathcal{H}^{n-1} \llcorner S(u))$. We want to prove that

$$g(x_0) \geq (1-\alpha) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u). \tag{38}$$

To this end we follow an argument by contradiction: we assume that

$$g(x_0) < (1-\alpha) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u).$$

Let $x_0 \in S(u)$. We denote by Q the open cube $Q = (-1/2, 1/2)^n$. We can assume that

$$\lim_{\rho \to 0^+} \frac{\lambda(x_0 + \rho Q)}{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))} = g(x_0), \qquad \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))}{\rho^{n-1}} = 1, \tag{39}$$

$$\lim_{\rho \to 0^\pm} \frac{1}{\rho^n} \int_{x_0 + \rho Q^\pm} |u(x) - u^\pm(x_0)| dx = 0, \tag{40}$$

since these properties are satisfied up to a set of zero-$\mathcal{H}^{n-1}$ measure. Moreover, up to a countable set of values of ρ, we can assume that

$$\lambda(\partial(x_0 + \rho Q)) = 0. \tag{41}$$

By (39)and (41) we get

$$\begin{aligned} g(x_0) &= \lim_{\rho \to 0^+} \lim_{j \to +\infty} \frac{\lambda_j(x_0 + \rho Q)}{\mathcal{H}^{n-1}(S(u) \cap (x_0 + \rho Q))} = \lim_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \lim_{j \to +\infty} \lambda_j(x_0 + \rho Q) \\ &= \lim_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \lim_{j \to +\infty} F_j(u_j; P_j^s \cap (x_0 + \rho Q)). \end{aligned}$$

By a diagonal argument we can find a sequence $\rho_j \to 0$ such that

$$g(x_0) = \lim_{j \to +\infty} \frac{1}{\rho_j^{n-1}} F_j(u_j; P_j^s \cap (x_0 + \rho_j Q)).$$

We now rescale the space variable by defining

$$A = \frac{\varepsilon_j}{\rho_j}\Big[\frac{a - x_0}{\varepsilon_j}\Big], \quad \text{for all } a \in (x_0 + \rho_j Q) \cap P_j^s,$$

and we set

$$v_j(A) = u_j(\rho_j A + x_0) = u_j(a), \quad \text{for all } a \in (x_0 + \rho_j Q) \cap P_j^s.$$

Up to a further diagonalization we can find a subsequence v_j (not relabelled) such that

$$v_j \to u_0 \text{ in } L^1(Q) \;\text{ and }\; g(x_0) = \lim_j \frac{1}{\rho_j^{n-1}} G_j(v_j; Q),$$

where

$$u_0(x) = \begin{cases} u^+(x_0) & \text{for } x > 0 \\ u^-(x_0) & \text{for } x \le 0 \end{cases}$$

and

$$G_j(v_j; Q) = F_j(u_j; P_j^s \cap (x_0 + \rho_j Q)).$$

By the modification of De Giorgi's method for matching boundary conditions adapted to the discrete setting (see e.g. [1]), we can build a sequence $\widetilde{v}_j \in \mathcal{A}_{\varepsilon_j/\rho_j}(Q)$ such that $v_j \to u_0$ in $L^1(Q)$,

$$\frac{1}{\rho_j^{n-1}} G_j(\widetilde{v}_j; Q) \le \frac{1}{\rho_j^{n-1}} G_j(v_j; Q) + o(1)$$

and

$$\widetilde{v}_j(a) = u^\pm(x_0) \text{ for } a \in \frac{\varepsilon_j}{\rho_j}\mathbb{Z}^n \cap Q: \; a_n \in \pm[1 - \varepsilon_j/\rho_j, 1]. \tag{42}$$

Let I_j be the set

$$I_j = \Big\{A \in Q \cap \Big[\frac{P_j^s - x_0}{\varepsilon_j}\Big]\frac{\varepsilon_j}{\rho_j} : A_n = 0, \; (\widetilde{v}_j(A) - \widetilde{v}_j(A', \varepsilon_j/\rho_j))^2 < \varepsilon_j\Big\}$$

of springs (both weak and strong) where the elongation is below the 'fracture threshold'. We claim that the cardinality of I_j satisfies the following condition:

$$\frac{\sharp I_j}{\rho_j^{n-1}\varepsilon_j^{1-n}(1 - \alpha^{n-1})} \ge \beta \qquad \text{for } j \ge j_0, \tag{43}$$

for some constant $\beta \in (0, 1]$ and $j_0 \in \mathbb{N}$. This can be proved through an argument by contradiction: we assume that for all $\beta > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$\sharp I_j < \beta \rho_j^{n-1}\varepsilon_j^{1-n}(1 - \alpha^{n-1}) \;\text{ for } j \ge j_0.$$

Having set

$$I_j^c = \Big\{A \in Q \cap \Big[\frac{P_j^s - x_0}{\varepsilon_j}\Big]\frac{\varepsilon_j}{\rho_j} : A_n = 0, \; (\widetilde{v}_j(A) - \widetilde{v}_j(A', \varepsilon_j/\rho_j))^2 \ge \varepsilon_j\Big\},$$

there follows that

$$\sharp I_j^c \ge (1 - \beta)\rho_j^{n-1}\varepsilon_j^{1-n}(1 - \alpha^{n-1}) \;\text{ for } j \ge j_0.$$

Hence for all $j \ge j_0$ we have

$$\frac{1}{\rho_j^{n-1}} G_j(\widetilde{v}_j; Q) \ge \frac{1}{\rho_j^{n-1}} \sharp I_j^c \varepsilon_j^{n-1} \ge \frac{1}{\rho_j^{n-1}}(1 - \beta)\rho_j^{n-1}\varepsilon_j^{1-n}(1 - \alpha^{n-1})\varepsilon_j^{n-1}.$$

Since $\rho_j^{1-n} G_j(\widetilde{v}_j; Q) < 1 - \alpha$ by assumption (for j large enough), we get

$$(1 - \beta)(1 - \alpha^{n-1}) < 1 - \alpha.$$

By letting $\beta \to 0^+$ we get $1-\alpha^{n-1} < 1-\alpha$, which is in contrast with the assumption $\alpha < 1/2$.

Let $A \in I_j$. Note that by Hölder's inequality

$$\begin{aligned}(u^+(x_0)-u^-(x_0))^2 &\le \Big(\sum_{B\in Q:\ B'=A'} (\widetilde{v}_j(B)-\widetilde{v}_j(B',B_n+\varepsilon_j/\rho_j))\Big)^2 \\ &\le \sharp\{B\in Q:\ B'=A'\} \sum_{B\in Q:\ B'=A'} (\widetilde{v}_j(B)-\widetilde{v}_j(B',B_n+\varepsilon_j/\rho_j))^2 \\ &\le \frac{\rho_j}{\varepsilon_j} \sum_{B\in Q:\ B'=A'} (\widetilde{v}_j(B)-\widetilde{v}_j(B',B_n+\varepsilon_j/\rho_j))^2.\end{aligned}$$

By summing up over $A \in I_j$ we get

$$\begin{aligned}\sum_{A\in I_j}(u^+(x_0)-u^-(x_0))^2 &\le \frac{\rho_j}{\varepsilon_j}\sum_{A\in I_j}\sum_{B\in Q:\ B'=A'} (\widetilde{v}_j(B)-\widetilde{v}_j(B',B_n+\varepsilon_j/\rho_j))^2 \\ &\le \frac{\rho_j}{\varepsilon_j^{n-1}}\sum_{A\in I_j}\sum_{B\in Q:\ B'=A'} (\widetilde{v}_j(B)-\widetilde{v}_j(B',B_n+\varepsilon_j/\rho_j))^2\varepsilon_j^{n-2}\end{aligned}$$

hence (for $j \ge j_0$)

$$\begin{aligned}(u^+(x_0)-u^-(x_0))^2 &\le \frac{1}{\sharp I_j}\frac{\rho_j}{\varepsilon_j^{n-1}}G_j(\widetilde{v}_j;Q) \\ &\le \frac{1}{\beta\rho_j^{n-1}\varepsilon_j^{1-n}(1-\alpha^{n-1})}\frac{\rho_j}{\varepsilon_j^{n-1}}G_j(\widetilde{v}_j;Q) \le c\rho_j.\end{aligned}$$

By letting $j \to +\infty$, we get $u^+(x_0) = u^-(x_0)$, which is in contradiction with the assumption $x_0 \in S(u)$. In conclusion, our arguments imply that

$$g(x_0) \ge (1-\alpha) \quad \text{for} \quad \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S(u).$$

Finally, by (38) we deduce that

$$\lambda(\Omega) \ge \int_\Omega g\, d(\mathcal{H}^{n-1} \llcorner S(u)) \ge (1-\alpha)\mathcal{H}^{n-1}(S(u)),$$

which implies the desired inequality (37):

$$\liminf_j F_j(u_j;P_j^s) = \liminf_j \lambda_j(\Omega) \ge (1-\alpha)\mathcal{H}^{n-1}(S(u)).$$

B.2. To estimate the contribution of u_j on $R_j^s = (\Omega\setminus E_j)\cap\{x\in\mathbb{R}^n:\ |x_n|>s\}$ is suffices to recall that the weak-membrane functional is always a lower bound, from which we obtain

$$\begin{aligned}\liminf_j F_j(u_j;R_j^s) &\ge \int_{\Omega\cap\{|x_n|>s\}} |\nabla u|^2dx + \mathcal{H}^{n-1}(S(u)\cap\{x\in\Omega:\ |x_n|>s\}) \\ &= \int_{\Omega\cap\{|x_n|>s\}} |\nabla u|^2dx \qquad (44)\end{aligned}$$

since $\mathcal{H}^{n-1}(S(u)\cap\{x\in\Omega:\ |x_n|>s\}) = 0$.

Taking into account (37) and (44) and letting $s\to 0^+$, we deduce that the contribution of u_j outside E_j can be estimated as follows:

$$\liminf_j F_j(u_j;\Omega_j\setminus E_j) \ge (1-\alpha)\mathcal{H}^{n-1}(S(u)) + \int_\Omega |\nabla u|^2dx,$$

as desired. □

Remark 3.8 From the characterization of m_j in (32), and the limit behavior of μ_j (described in (27) we deduce that the 'elongation of the strong springs' (scaled by ε_j) at the interface $2\gamma_j$ (defined in (31)) asymptotically vanishes in the case $n=2$, while it is finite and given by (33) if $n \ge 3$.

3.3 Upper bound

We now prove the upper bound for our energies.

Proposition 3.9 (upper bound) *For all $u \in \mathrm{PC}(\Omega)$ such that $S(u) \subseteq K$, there exists a sequence (v_j) such that $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$, $v_j \to u$ in $L^1(\Omega)$ and*

$$\limsup_j F_j(v_j) \le F(u). \tag{45}$$

Proof. For all $j \in \mathbb{N}$ we denote by u_j the function $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ defined as the discretization of u on the lattice $\Omega_j = \varepsilon_j \mathbb{Z}^n \cap \Omega$:

$$u_j(a) = u(a) \text{ for } a \in \Omega_j.$$

(If $a \in \Omega_j \cap S(u)$ we set $u_j(a) = u^-(x_0)$). Let $k \in \mathbb{N}$ and $\alpha > 0$ be such that $2^{k+1}\alpha < 1/2$. Let (ρ_j) be a positive infinitesimal sequence of the form $\rho_j = 2^{k+1}\alpha\delta_j$. By applying Lemma 3.4 to the sequence $u_j \to u$, we get a new sequence $w_j \to u$ satisfying conditions (14)–(15). We want to modify the functions w_j on the cubes $Q_{\varepsilon_j}(\rho_j^l, l)$, $l \in Z_j$, in order to get a recovery sequence for u. Following the notation of the lemma, we note that

$$\rho_j^l \ge \Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j \qquad \text{for all } l \in Z_j.$$

For fixed $l \in Z_j$, we denote by $z_j^l \in \mathcal{A}_{\varepsilon_j}(\mathbb{R}^n)$ the minimizer of the following minimum problem:

$$\min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \ v = u_j^{l\pm} \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\}, \tag{46}$$

where G_j is defined by (23). We define the sequence $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ as follows:

$$v_j(a) = \begin{cases} z_j^l(a-l) & \text{if } a \in Q_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j, l\Big), \ l \in Z_j \\ u_j^{l\pm} & \text{if } a \in \Omega^{\pm} \cap \Big(Q_{\varepsilon_j}(\rho_j^l, l) \setminus Q_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big), l\Big), \ l \in Z_j \\ w_j(a) & \text{if } \Omega_j \setminus \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l, l). \end{cases}$$

We want to prove that (v_j) is a recovery sequence for u. By construction

$$F_j(v_j) \quad \le \quad \sum_{l \in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) + F_j\Big(v_j; \Omega_j \setminus \bigcup_{l \in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)\Big).$$

Having fixed $l \in Z_j$, we focus our attention on $F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l))$. By construction:

$$\begin{aligned} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) &\le G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + \varepsilon_j^{n-1}\sharp\big(K \cap Q_{\varepsilon_j}(\rho_j^l, l)\big) \\ &\le G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + \varepsilon_j^{n-1}(2^{k+1}\alpha)^{n-1}\frac{\delta_j^{n-1}}{\varepsilon_j^{n-1}}. \end{aligned}$$

By summing over $l \in Z_j$ we get

$$\begin{aligned} &\sum_{l \in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) \le \sum_{l \in Z_j} G_j(z_j; Q_{\varepsilon_j}(\alpha\delta_j)) + c(2^{k+1}\alpha)^{n-1} \\ &= \sum_{l \in Z_j} \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \ v = u_j^{l\pm} \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\} \\ &\qquad + c(2^{k+1}\alpha)^{n-1} \\ &= \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) : \ v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\} \sum_{l \in Z_j} \frac{(u_j^{l+} - u_j^{l-})^2}{4} + c(2^{k+1}\alpha)^{n-1}. \end{aligned}$$

Having defined the scaled minimum problems

$$m_j = \varepsilon_j^{2-n} \min\Big\{G_j(v; Q_{\varepsilon_j}(\alpha\delta_j)) :\ v = \pm 1 \text{ on } \mathcal{L}^{\pm}_{\varepsilon_j}\Big(\Big[\frac{\alpha\delta_j}{\varepsilon_j}\Big]\varepsilon_j\Big)\Big\},$$

their asymptotic behavior is given by (34) and (35). By Proposition 3.5 we then get

$$\limsup_j \sum_{l\in Z_j} F_j(v_j; Q_{\varepsilon_j}(\rho_j^l, l)) \le \frac{C_n}{\beta}\int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} + c(2^{k+1}\alpha)^{n-1}. \tag{47}$$

Finally, we estimate the contribution of v_j on $\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)$. By Lemma 3.4 we have

$$\begin{aligned}
&F_j(v_j; \Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)) \\
&\quad = F_j\Big(w_j; (\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)) \cap \Omega^+\Big) + F_j\Big(w_j; (\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)) \cap \Omega^-\Big) \\
&\quad + \sum_{a\in K\cap(\Omega_j\setminus\bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l,l))} \varepsilon_j^{n-2}(w_j(a) - w_j(a', \varepsilon_j))^2 \wedge \varepsilon_j^{n-1} \\
&\quad \le F_j\Big(u_j; (\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)) \cap \Omega^+\Big) + F_j\Big(u_j; (\Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)) \cap \Omega^-\Big) \\
&\quad + \sum_{a\in K\cap(\Omega_j\setminus\bigcup_{l\in Z_j} Q_{\varepsilon_j}(\alpha 2^{k+1}\delta_j,l))} \varepsilon_j^{n-2}(u_j(a) - u_j(a', \varepsilon_j))^2 \wedge \varepsilon_j^{n-1} + c(\alpha 2^{k+1})^{n-1} + \frac{c}{k} \\
&\quad \le F_j^{V_j}(u_j) + c(\alpha 2^{k+1})^{n-1} + \frac{c}{k}.
\end{aligned}$$

where $V_j := \{\{a, b\} \in M_j :\ a \in K \cap \Omega_j,\ b = a + \varepsilon_j e_n\} \subseteq M_j$. Taking Remark 2.7 into account we get

$$\limsup_j F^{V_j}(u_j) \le \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S(u)).$$

There follows that

$$\limsup_j F_j\Big(v_j; \Omega_j \setminus \bigcup_{l\in Z_j} Q_{\varepsilon_j}(\rho_j^l, l)\Big) \le \frac{c}{k} + \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S(u)). \tag{48}$$

By (47) and (48) we deduce that

$$\begin{aligned}
\liminf_j F_j(v_j) \quad &\le \quad \frac{C_n}{\beta}\int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} \\
&\qquad + \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(\Omega) + \frac{c}{k} + c(2^{k+1}\alpha)^{n-1}.
\end{aligned}$$

By letting first $\alpha \to 0^+$ and then $k \to +\infty$, we get (45). □

3.4 Limits with prescribed density of weak springs

We will show that all the constructions throughout the paper can be repeated also with *any* prescribed limit density $\theta \in L^\infty(\Omega; [0, 1])$.

Note that in the construction considered in this section, the weak springs are until now concentrated on a $(n-1)$-dimensional hyperplane, so that θ is identically 0 a.e. We now show that for all given θ our construction can be repeated with a different choice of W_j such that (5) holds.

Proposition 3.10 (Prescribed density of weak springs) *Let $F : SBV(\Omega) \to [0, +\infty]$ be the functional*

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_{S(u)} \Big(\frac{C_n}{\beta}|u^+ - u^-|^2 + 1\Big) d\mathcal{H}^{n-1} & \text{if } S(u) \subseteq K \\ +\infty & \text{otherwise.} \end{cases} \tag{49}$$

For all $\theta \in L^\infty(\Omega; [0,1])$ there exists a sequence of arrangements (W_j) such that $W_j \to \theta$, $F \leq \Gamma\text{-}\liminf_j F_j$, and $\Gamma\text{-}\lim_j F_j = F$ on $\mathrm{PC}(\Omega)$

Proof. It is sufficient to prove the thesis with a choice of W_j with limit density 1. The result will then follow by comparison by taking $W'_j \subset W_j$, W'_j containing the weak springs in Theorem 3.1 and $W'_j \to \theta$. Note that, by the compactness of Γ-convergence ([12]), we can always suppose that Γ-limits exist when needed (even though we characterize them only on $\mathrm{PC}(\Omega)$).

We fix $\eta > 0$ and $N \in \mathbb{N}$, and define $W_j = W^{\eta,N}_{\varepsilon_j} = W^1_j \cup W^2_j$ as follows:

- W^1_j defined as the set W_{ε_j} in (9) (weak springs at the interface)
- W^2_j defined by (here we use the notation $\widehat{a}_j = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$)

$$W^2_j = \{\{a,b\} \in M_1 : |a_n| \geq \eta \text{ and } |b_n| \geq \eta\} \setminus \{\{a,b\} : \widehat{a}_j = \widehat{b}_j \in \varepsilon N \mathbb{Z}^{n-1}\} \tag{50}$$

is the set of all springs outside the η-tubular neighborhood of the interface and not lying between any two neighboring points of the lattice $\varepsilon N \mathbb{Z}^n$. In other words, the strong springs outside the η-tubular neighborhood of the interface are exactly those on straight segments nearest neighbors of the lattice $\varepsilon N \mathbb{Z}^n$.

Note that for this choice of W_j the limit density of weak springs is $\theta_{\eta,N}$ given by

$$\theta_{\eta,N}(x) = \begin{cases} 0 & \text{if } |x_n| \leq \eta \\ \frac{1}{N^{n-1}} & \text{if } |x_n| > \eta. \end{cases} \tag{51}$$

We will denote by $F^{\eta,N}_j$ the functional with W_j as set of weak springs, and we will use the usual notation for its localized version. To prove that the Γ-limit is given by F it suffices to show the lower bound, since the upper bound follows by Theorem 3.1 by comparison since the set W_j contains the one in that theorem.

We consider now a sequence $u_j \to u$ such that $\liminf_j F^{\eta,N}_j(u_j) < +\infty$. It is not restrictive to suppose that indeed $\sup_j F^{\eta,N}_j(u_j) < +\infty$. Note that we can apply Theorem 3.1 to $F^{\eta,N}_j(\cdot; \Omega_\eta)$, where $\Omega_\eta = \Omega \cap \{|x_n| < \eta\}$, since $F^{\eta,N}_j$ coincides with the energy therein on Ω_η. In particular we deduce that the Γ-limit is finite only on functions $u \in H^1(\Omega_\eta \setminus K) \cap SBV(\Omega_\eta)$, and we have

$$\liminf_j F^{\eta,N}_j(u_j; \Omega_\eta) \geq \int_{\Omega_\eta} |\nabla u|^2 \, dx + \int_{S(u)} \Big(\frac{C_n}{\beta}|u^+ - u^-|^2 + 1\Big) d\mathcal{H}^{n-1} \tag{52}$$

We now focus our attention on $\Omega \cap \{x_n > \eta/2\}$ and we define the sequence $v_j : (\Omega \cap \{x_n > \eta/2\}) \cap \varepsilon_j N \mathbb{Z}^n \to \mathbb{R}$ as $v_j(a) = u_j(a)$ for $a \in (\Omega \cap \{x_n > \eta\}) \cap \varepsilon_j N \mathbb{Z}^n$. By construction (v_j) satisfies

$$\sum_{|a-b| = N\varepsilon_j} \Big(\frac{v_j(a) - v_j(b)}{N\varepsilon_j}\Big)^2 (\varepsilon_j N)^n \leq c N^{n-2} F^{\eta,N}_j(u_j).$$

By Remark 2.4(3) (applied to $\varepsilon N \mathbb{Z}^n$ in place of $\varepsilon \mathbb{Z}^n$) up to subsequences, $v_j \to v \in H^1(\Omega \cap \{x_n > \eta/2\})$. We now denote by χ_j the characteristic function of the set

$$\bigcup_{k \in \varepsilon_j N \mathbb{Z}^n} (k + (-\varepsilon_j/2, \varepsilon_j/2)^n),$$

which converge weakly* in $L^\infty(\Omega)$ to the constant N^{-n}. This implies that $u_j \chi_j \to N^{-n} u$ in $L^1(\Omega \cap \{x_n > \eta/2\})$ and $v_j \chi_j \to N^{-n} v$ in $L^1(\Omega \cap \{x_n > \eta/2\})$. After noticing that $\chi_j u_j \equiv \chi_j v_j$, we conclude that u coincides on $\Omega \cap \{x_n > \eta\}$ with a function $v \in H^1(\Omega \cap \{x_n > \eta/2\})$. By a similar argument on $\Omega \cap \{x_n < -\eta/2\}$ we conclude that $u \in H^1(\Omega \cap \{|x_n| > \eta/2\})$.

As a result, we have $u \in H^1(\Omega \setminus K)$. By Remark 2.4(4), applied to $\Omega \cap \{|x_n| > \eta\}$ we have

$$\liminf_j F_j^{\eta,N}(u_j; \Omega \cap \{|x_n| > \eta\}) \geq \int_{\Omega \cap \{|x_n| > \eta\}} |\nabla u|^2 \, dx. \tag{53}$$

Taking into account (52) and (53) we conclude that

$$\liminf_j F_j^{\eta,N}(u_j) \geq \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \Big(\frac{C_n}{\beta} |u^+ - u^-|^2 + 1 \Big) d\mathcal{H}^{n-1}$$

as desired.

Since $\theta_{\eta,N}$ to 1 as $\eta \to 0$ and $N \to +\infty$, we can choose $\eta_j \to 0$ and $N_j \to +\infty$ such that, having redefined $W_j = W_{\varepsilon_j}^{\eta_j, N_j}$ the corresponding Γ-limit still satisfied the thesis, thus obtaining the desired result. Note that this last argument uses a diagonalization procedure, which is possible by the metrizability properties of Γ-convergence (see [24] Theorem 10.22, which requires a common lower bound for all functionals with a coercive energy. In our case that energy is the Mumford-Shah functional, after identification of the functions in $\mathcal{A}(\Omega)$ with suitable functions in $SBV(\Omega)$ – see e.g. [22]). □

3.5 The discrete Neumann sieve problem

We consider the energies

$$G_j(u) = F_j(u) - \sum_{a \in \Omega_j \cap K,\ a \notin Z_j} \varepsilon_j^{n-2} ((u(a) - u(a', \varepsilon_j))^2 \wedge \varepsilon_j) \tag{54}$$

for any $v \in \mathcal{A}_{\varepsilon_j}(A)$, introduced (in a local form) in (23) and used in the proof of Theorem 3.1

The energies G_j do not take the weak springs into account, which are replaced by 'voids', and are the discrete analog of the energy of a "Neumann sieve" [4], where the interface is now free (i.e., we have Neumann boundary conditions at the interface) except for the strong springs (see Fig. 3). The Γ-limit of G_j consists of the quadratic part of the limit of F_j and is described

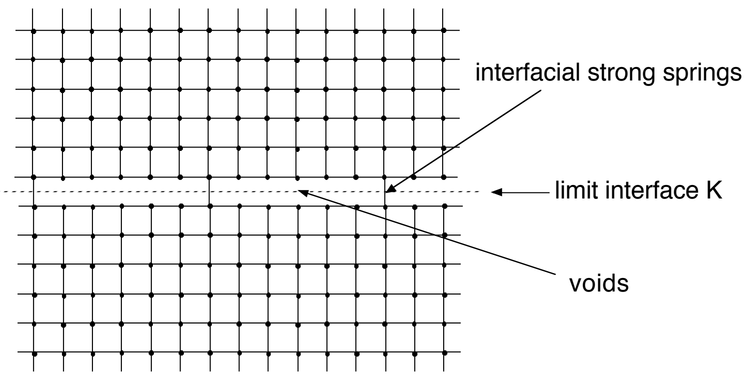

Figure 3: The discrete Neumann sieve

as follows.

Theorem 3.11 *The functionals G_j defined by (54) Γ-converge, with respect to the strong convergence in $L^1(\Omega)$, to the functional $G : L^1(\Omega) \to [0, +\infty]$ given by*

$$G(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \frac{C_n}{\beta} \int_{S(u)} |u^+ - u^-|^2 d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega),\ S(u) \subseteq K \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is contained in that of Theorem 3.1, whose separation of scale argument is precisely to consider quadratic and non-quadratic interactions separately. Note that in this case we can prove the Γ-limsup inequality for all functions in $H^1(\Omega\setminus K)\cap SBV(\Omega)$, since a mollification argument easily shows the density in energy of the set $\mathrm{PC}(\Omega)$. □

4 Closure results for a class of free-discontinuity fracture energies

In the previous section we have obtained as limits of discrete energies (in the sense of Theorem 3.1) functionals of the form

$$F_{K,b}(u,\Omega)=\int_\Omega |\nabla u|^2\,dx+\int_{S(u)\cap\Omega}(1+b|u^+-u^-|^2)\,d\mathcal{H}^{n-1}, \tag{55}$$

with the constraint that $S(u)\subset K$ up to $\mathcal{H}^{n-1}$-negligible sets, where K is the closure of an open set A of the union of coordinate hyperplanes with $\mathcal{H}^{n-1}(K)=\mathcal{H}^{n-1}(A)$, and $b\geq 0$ is any positive constant (by the arbitrariness of β in Theorem 3.1).

Scope of this section is to describe a wide class of $GSBV$ energies obtained as Γ-limits of energies of the form (55) with varying K and b. In order not to overburden notation, in the definition of the energies u is understood to be in $GSBV(\Omega)$, and the energies are extended to $+\infty$ where not explicitly defined.

Even though applying some new arguments, in this section we will use well-known techniques in Geometric Measure Theory, so we will feel free to drop some details in order to lighten the presentation.

Notation. We now deal with energies on the continuum, for which we find it convenient to change the notation used in a discrete setting. In particular note that, with a slight abuse of notation, in this section cubes will be open and not closed.

4.1 Energies with the constraint $S(u)\subset K$

In this section we consider varying K_j converging to some given K, and examine the class of energy densities that can be obtained in this way. The limit energies will be of the form

$$F_{K,f}(u,\Omega)=\int_\Omega |\nabla u|^2\,dx+\int_{S(u)\cap\Omega} f(x,|u^+-u^-|,\nu)\,d\mathcal{H}^{n-1}, \tag{56}$$

with the constraint that $S(u)\subset K$ up to $\mathcal{H}^{n-1}$-negligible sets.

We will construct approximating F_j with the domain characterized by the corresponding constraint that $S(u)\subset K_j$ up to $\mathcal{H}^{n-1}$-negligible sets starting from (55), such that the following theorem holds.

Theorem 4.1 *If $F_{K,f}$ is given by* (56)*. Then*
(i) *for all $u\in SBV(\Omega)$ and $u_j\to u$ we have $\liminf_j F_j(u_j)\geq F_{K,f}(u)$;*
(i) *for all $u\in \mathrm{PC}(\Omega)$ there exist $u_j\in\mathrm{PC}(\Omega)$ with $u_j\to u$ and $\limsup_j F_j(u_j)\leq F_{K,f}(u)$.*

The construction of K_j and F_j, as well as the proof of the corresponding version of Theorem 4.1 will depend on the target K and $F_{K,f}$, and are described below in the various cases.

1. K is the closure of an open set A of a union of coordinate hyperplanes with $\mathcal{H}^{n-1}(K)=\mathcal{H}^{n-1}(A)$, and $f(x,z,\nu)=a+b|z|^2$, $a\geq 1$ and $b\geq 0$. In this case we denote $F_{K,a,b}=F_{K,f}$.

We define the approximating energies as $F_j(u)=F_{K_j,b/a}(u,\Omega)$ ($F_{K,b}$ as in (55)), where K_j are *oscillating fracture sites* defined as follows. For the sake of simplicity it is not restrictive to suppose that $K \subset \{x_n = 0\}$. For all $i \in \mathbb{Z}^{n-1}$ we consider the coordinate open cube $Q^{n-1}_{1/2}(i)$ of centre i and side length $1/2$ in $\mathbb{R}^{n-1}$ and correspondingly the coordinate parallelepiped

$$R^a_i = \overline{Q^{n-1}_{1/2}(i)} \times \Big[0, \frac{a-1}{(n-1)\vee 2}\Big]$$

in $\mathbb{R}^n$. We then set

$$\begin{aligned} K_j &= \bigcup\Big\{\Big(K \cup \frac{1}{j}\partial R^a_i\Big) \setminus \Big(K \cap \frac{1}{j}\partial R^a_i\Big) : \frac{1}{j}Q^{n-1}_{1/2}(i) \subset K\Big\} \\ &\cup \bigcup\Big\{\partial R^a_i : \frac{1}{j}Q^{n-1}_{1/2}(i) \cap K \neq \emptyset,\ \frac{1}{j}Q^{n-1}_{1/2}(i) \not\subset K\Big\} \end{aligned} \tag{57}$$

(see Fig. 4). Note in particular that we have $\mathcal{H}^{n-1} \llcorner K_j \rightharpoonup^* a\mathcal{H}^{n-1} \llcorner K$.

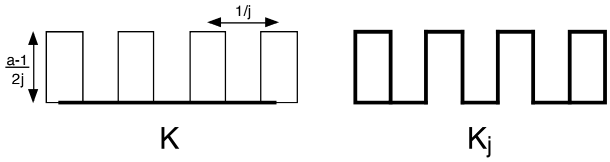

Figure 4: Construction of oscillating fracture sites (two-dimensional picture)

Proof of Theorem 4.1–*Case 1.* Let $u_j \to u$ in $L^1(\Omega)$ with $\sup_j F_j(u_j) < +\infty$. Then we have $u_j \to u$ in $SBV(\Omega)$ and $u_j \rightharpoonup u$ in $H^1_{\rm loc}(\Omega \setminus \overline{K})$. Hence, $u \in SBV(\Omega) \cap H^1(\Omega\setminus\overline{K})$; i.e., $S(u) \subset K$ $\mathcal{H}^{n-1}$-a.e., and u belongs to the domain of K.

For $\mathcal{H}^{n-1}$-a.e. $x_0 \in S(u)$, by a blow up argument around x_0 we can find (up to a relabeling of the indices j and possible extraction of subsequences) a sequence v_j converging in $L^1(Q_1(0))$ (we use the notation $Q_1(0) = Q^n_1(0)$) to the function

$$u^{x_0}(x) = \begin{cases} u^+(x_0) & \text{if } x_n > 0 \\ u^-(x_0) & \text{if } x_n < 0, \end{cases} \tag{58}$$

and $\sup_j F_{K_j,a/b}(v_j, Q_1(0)) < +\infty$. Note that we then have $v_j \rightharpoonup u^{x_0}$ in $H^1_{\rm loc}(Q_1(0) \setminus \overline{K})$, and that $v_j - u^{x_0} \to 0$ in $L^2(Q_1(0) \cap \{x_n = t\})$ for a.a. $-1/2 < t < 1/2$. Note that by the blow up argument around x_0 we can suppose that $Q^{n-1}_1(0) \subset K$.

We first prove that $\liminf_j \mathcal{H}^{n-1}(S(v_j) \cap Q_1(0)) \geq a$. Suppose otherwise that $\mathcal{H}^{n-1}(S(v_j) \cap Q_1(0)) < a$; i.e., that

$$\mathcal{H}^{n-1}(K_j \setminus S(v_j)) \geq c > 0 \tag{59}$$

for j sufficiently large. We can find disjoint smooth one-dimensional paths γ^j_y in $Q_1(0)$ indexed by $y \in Q_1(0) \cap K_j$ with the two endpoints in $Q_1(0) \cap \{x_n = \pm 1/2\}$, respectively, such that $Q_1(0) = \bigcup\Big\{\gamma^j_y : y \in K_j\Big\}$ and $\mathcal{H}^1(\gamma^j_y) = 1 + o(1)$ as $j \to +\infty$. For $\mathcal{H}^{n-1}$-a.a. $y \in K_j \setminus S(v_j)$ the functions v_j belong to $H^1(\gamma^j_y)$. For fixed $\delta > 0$ such that

$$v_j - u^{x_0} \to 0 \text{ in } L^2(Q_1(0) \cap \{x_n = \pm\delta\}) \tag{60}$$

we set $x^{\delta,\pm}_{y,j} = \gamma^j_y \cap \{x_n = \pm\delta\}$, and estimate

$$|v_j(x^{\delta,+}_{y,j}) - v_j(x^{\delta,+}_{y,j})| \leq \int_{\gamma^j_y \cap \{-\delta < x_n < \delta\}} |\nabla v_j| d\mathcal{H}^{n-1} \leq c\sqrt{\delta}\Big(\int_{\gamma^j_y} |\nabla v_j|^2 d\mathcal{H}^{n-1}\Big)^{1/2}.$$

Integrating this inequality for $y \in K_j \setminus S(v_j)$ by (59), (60) and (58) we then obtain

$$|u^+(x_0) - u^-(x_0)|^2 \leq c \liminf_j \delta \int_{Q_1(0)} |\nabla v_j|^2\, dx \leq c\delta,$$

contradicting that $x_0 \in S(u)$ by the arbitrariness of δ. The same type of argument, used by comparing v_j on $\{x_n = \pm\delta\}$ and on K_j shows that

$$\liminf_j \int_{K_j} |v_j^\pm - u^\pm(x_0)|^2 \, d\mathcal{H}^{n-1} \le c\delta$$

so that indeed

$$\begin{aligned}\lim_j \int_{S(v_j)\cap Q_1(0)} |v_j^+ - v_j^-|^2 \, d\mathcal{H}^{n-1} &= \lim_j \int_{K_j\cap Q_1(0)} |u^+(x_0) - u^-(x_0)|^2 \, d\mathcal{H}^{n-1} \\ &= a|u^+(x_0) - u^-(x_0)|^2.\end{aligned}$$

The blow-up method of Fonseca and Müller allows then to conclude that

$$\liminf_j F_j(u) \ge \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)\cap\Omega} (a + b|u^+ - u^-|^2) \, d\mathcal{H}^{n-1}$$

since the inequality of the bulk part follows trivially from the lower semicontinuity of the Dirichlet integral.

Let $u \in \mathrm{PC}(\Omega)$ with $F_{K,a,b}(u) < +\infty$. To check the limsup inequality we simply extend the restriction of u to $\{x_n < 0\}$ by reflexion to a neighborhood of K and denote it by $\widetilde{u}$. The sequence u_j is simply given by

$$u_j(x) = \begin{cases} \widetilde{u}(x) & \text{if } x \in \bigcup\{\frac{1}{j}R_i^a : \frac{1}{j}Q_{1/2}^{n-1}(i) \cap K \ne \emptyset\} \\ u(x) & \text{otherwise,} \end{cases} \tag{61}$$

which satisfies the constraint $S(u_j) \subset K_j$, and for which the desidered inequality immediately follows. Note that $u \in PC(\Omega)$. □

2. K is the closure of a relatively open set A of a union of (not necessarily coordinate) hyperplanes $\Pi_m = \{\langle x - x_m, \nu_m\rangle = 0\}$, $m \in M$, with $\mathcal{H}^{n-1}(K) = \mathcal{H}^{n-1}(A)$, and $f(z,\nu) = (a + b|z|^2)\|\nu\|_1$, $a \ge 1$ and $b \ge 0$. We use the same notation of Case 1: $F_{K,a,b} = F_{K,f}$.

The approximating functionals will be of the same form $F_j = F_{K_j,a,b}$, with K_j subsets of coordinate hyperplanes as in Case 1. It is sufficient to consider the case of a single hyperplane $K \subset \Pi_0 = \{\langle x - x_0, \nu_0\rangle = 0\}$. We then consider the sets of indices

$$I_1^j = \Big\{ i \in \mathbb{Z}^n : \frac{1}{j}Q_1(i) \cap K \ne \emptyset, \overline{\frac{1}{j}Q_1(i)} \cap (\Pi_0 \setminus K) = \emptyset \Big\} \tag{62}$$

$$I_2^j = \Big\{ i \in \mathbb{Z}^n : \frac{1}{j}Q_1(i) \cap K \ne \emptyset, \overline{\frac{1}{j}Q_1(i)} \cap (\Pi_0 \setminus K) \ne \emptyset \Big\}, \tag{63}$$

and define (see Fig. 5)

$$K_j = \Big(\bigcup_{i\in I_2^j} \frac{1}{j}\partial Q_1(i) \Big) \cup \partial\Big(\{x : x - x_0, \nu_0\rangle > 0\} \cap \bigcup_{i\in I_1^j} \frac{1}{j}Q_1(i) \Big). \tag{64}$$

Proof of Theorem 4.1–*Case 2.* After noting that $\mathcal{H}^{n-1} \llcorner K_j \rightharpoonup^* \|\nu_0\|_1 \mathcal{H}^{n-1} \llcorner K$ the proof of the liminf inequality follows word for word that of Case 1, with the hyperplane $\{x_n = 0\}$ substituted by Π_0.

As for the limsup inequality, the sequence u_j is simply given by

$$u_j(x) = \begin{cases} \widetilde{u}(x) & \text{if } x \in \bigcup\{\frac{1}{j}Q_1(i) : \frac{1}{j}Q_1(i) \cap K \ne \emptyset\} \\ u(x) & \text{otherwise,} \end{cases} \tag{65}$$

where $\widetilde{u}$ is an extension by symmetry of the restriction of u to $\Omega \cap \{x : x - x_0, \nu_0\rangle < 0\}$. This sequence belongs to $\mathrm{PC}(\Omega)$, satisfies $S(u_j) \subset K_j$, and the desidered inequality immediately follows. Again, the general case is obtained by the usual localization arguments. □

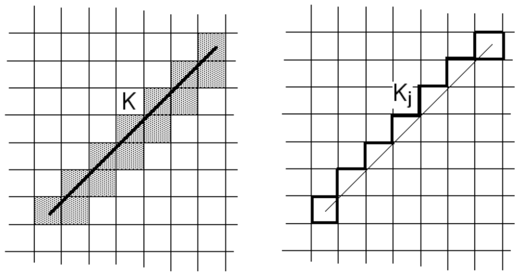

Figure 5: Construction of oscillating fracture sites for non-coordinate planes

Remark 4.2 (generalizations) (i) The construction works exactly in the same way when K is a subset of a smooth hypersurface, in which case ν stands for the normal to that surface;

(ii) Since the construction is independent on each Π_m we can choose $f(x,z,\nu) = (a_m + b_m|z|^2)\|\nu\|_1$ for $x \in \Pi_m$;

(iii) By localizing the construction we also may choose lower-semicontinuous piecewise constant a and b, and then by approximation $f(x,z,\nu) = (a(x) + b(x)|z|^2)\|\nu\|_1$, for any choice of lower semicontinuous coefficients a and b, with $a \geq 1$ and $b \geq 0$.

3. K as in Case 2 above, and $f(x,z,\nu) = \psi(|z|)\|\nu\|_1$, where $\psi : (0,+\infty) \to \mathbb{R}$ is of the form

$$\psi(z) = \min\Big\{\sum_{m\in J}(a_m + b_m z_m^2) : J \subset \{0,\dots,M\},\ J \neq \emptyset,\ \sum_{m\in J} z_m = z\Big\}, \tag{66}$$

where $M \in \mathbb{N}$ is fixed, and $a_0,\dots,a_M$ and $b_0,\dots,b_M$ are given numbers with $a_m \geq 1$ and $b_m \geq 0$. We use the notation: $F_{K,\psi} = F_{K,f}$.

By reasoning locally, it is not restrictive to suppose that K is a subset of a single hyperplane $\{\langle x - x_0, \nu_0\rangle = 0\}$. The approximating energies will be obtained by *piling up* $M+1$ copies of K, on which energies as in Remark 4.2(ii) are considered. More precisely, we define

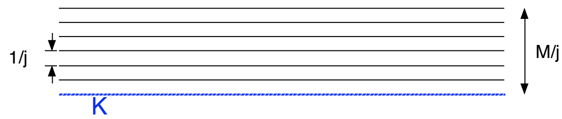

Figure 6: 'Micro-cracks' piling up to a 'macro-crack'

$$F_j(u) = \int_\Omega |\nabla u|^2\,dx + \sum_{m=0}^{M} \int_{S(u)\cap(K+\frac{m}{j}\nu_0)} (a_m + b_m|u^+ - u^-|^2)\|\nu_m\|_1\,d\mathcal{H}^{n-1}, \tag{67}$$

with the constraint that $S(u) \subset \bigcup_m (K + \frac{m}{j}\nu_0)$. For the sake of simplicity we may suppose that $K \subset \{x_n = 0\}$.

Proof of Theorem 4.1*–Case 3.* Let $u_j \to u$ in $L^1(\Omega)$ and in SBV with equibounded energy. It suffices to check the liminf inequality on the interfacial part. Note that $u_j^+\big(y, \frac{M}{j}\big) \to u^+(y,0)$, $u_j^-(y,0) \to u^-(y,0)$ and that by Poincaré's inequality and the equi-boundedness of the L^2 norms of ∇u_j, for a.a. $y \in K$ we have, upon extraction of subsequences, $u_j^+\big(y, \frac{m-1}{j}\big) - u_j^-\big(y, \frac{m}{j}\big) \to 0$ for all $m = 1,\dots,M$. For all $y \in K$ we set

$$J_j(y) = \Big\{m \in \{0,\dots,M\} : u^+\Big(y, \frac{m}{j}\Big) \neq u^-\Big(y, \frac{m}{j}\Big)\Big\}.$$

We then conclude by Fatou's Lemma:

$$\begin{aligned}
&\liminf_j \sum_{m=0}^{M} \int_{S(u)\cap(K+\frac{m}{j}e_n)} (a_m + b_m|u^+ - u^-|^2)\, d\mathcal{H}^{n-1} \\
&= \liminf_j \int_K \sum_{m\in J_j(y)} \Big(a_m + b_m\Big|u^+(y,\frac{m}{j}) - u^-(y,\frac{m}{j})\Big|^2\Big)\, d\mathcal{H}^{n-1} \\
&\geq \int_K \liminf_j \sum_{m\in J_j(y)} \Big(a_m + b_m\Big|u^+(y,\frac{m}{j}) - u^-(y,\frac{m}{j})\Big|^2\Big)\, d\mathcal{H}^{n-1} \\
&\geq \int_K \psi(|u^+ - u^-|)\, d\mathcal{H}^{n-1} \geq \int_{S(u)} \psi(|u^+ - u^-|)\, d\mathcal{H}^{n-1}.
\end{aligned}$$

As for the limsup inequality, we can perform the proof in the case $M = 1$, the general case following by induction. By the Lipschitz continuity of u outside $\overline{S(u)}$ and the continuity of the functions $z \mapsto a_m + b_m z^2$, for fixed $\eta > 0$ we can find a function $v_\eta : S(u) \to \mathbb{R}$ such that v is constant on each cube $\eta Q_1^{n-1}(i) \cap S(u)$ for all $i \in \mathbb{Z}^{n-1}$, and for almost all $y \in S(u)$

$$\begin{aligned}
&\chi_{\{v\neq u^-\}}(y)(a_0 + b_0|v(y) - u^-(y)|^2) + \chi_{\{v\neq u^+\}}(y)(a_1 + b_1|v(y) - u^+(y)|^2) \\
&\leq \psi(|u^+(y) - u^-(y)|) + r_\eta,
\end{aligned}$$

with $r_\eta \to 0$ as $\eta \to 0$. We then fix δ_j with $1 >> \delta_j^2 >> \frac{1}{j}$ and define functions $v_j^\eta \in W^{1,\infty}(\{x_n = 0\})$ as functions with minimal Lipschitz constant satisfying

$$\begin{aligned}
v_j^\eta(y) = v_\eta(y) \text{ if } y \in \eta Q_1^{n-1}(j) \cap S(u) \text{ and } \operatorname{dist}\Big(y, (\{x_n = 0\} \setminus S(u)) \cup \bigcup_{i\neq j} \eta Q_1^{n-1}(j)\Big) > \delta, \\
v_j^\eta(y) = u(y,0) \text{ if } y \in \{x_n = 0\} \setminus S(u).
\end{aligned}$$

By construction we have $|\nabla v_j^\eta| \leq \frac{C}{\delta_j}$. If we define u_j as

$$u_j = \begin{cases} u(x) & \text{if } x_n < 0 \\ v_j^\eta(y) & \text{if } x = (y,t) \text{ with } 0 \leq t \leq 1/j \\ u(x - \frac{1}{j}e_n) & \text{if } x_n > 1/j, \end{cases}$$

then we have $u_j \to u$ and

$$\limsup_j F_j(u) \leq \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \psi(|u^+ - u^-|)\, d\mathcal{H}^{n-1} + r_\eta \mathcal{H}^{n-1}(S(u)).$$

The thesis then follows by the arbitrariness of η. □

Remark 4.3 If K is the closure of a relatively open subset of a locally finite union of hyperplanes $\Pi_l = \{\langle x - x_l, \nu_l\rangle = 0\}$, $l \in L$, then we can localize the argument above, and obtain limits with $f(x,z,\nu) = \psi_l(|z|)\|\nu_l\|_1$ if $x \in \Pi_l$, where each ψ_l is of the form (66).

4.2 Homogenized energies

In this section we consider sequences of planar systems invading the space $\mathbb{R}^n$. As a consequence the constraint $S(u) \subset K$ will be lost in the limit, and will appear only through inequalities on the limit energy densities. Moreover, by a density argument of the functions in $\mathrm{PC}(\Omega)$ the Γ-limit will be characterized on the whole $GSBV(\Omega)$.

4. Limit energies of the type

$$F_\varphi(u,\Omega)=\int_\Omega |\nabla u|^2\,dx+\int_{S(u)\cap\Omega}\varphi(\nu)\,d\mathcal{H}^{n-1}, \tag{68}$$

where φ is any even convex function positively homogeneous of degree one with $\varphi(\nu)\geq\|\nu\|_1$. Note that in particular we can obtain the Mumford-Shah functionals corresponding to $\psi(\nu)=c\|\nu\|_2$ (the Euclidean norm), provided that $c\geq\sqrt{n}$.

To define the approximating functionals we consider a dense sequence $\{\nu_k : k\in\mathbb{N}\}$ in S^{n-1}, set $\Pi_k=\{x:\langle x,\nu_k\rangle=0\}$, and consider the family of hyperplanes (see Fig. 7)

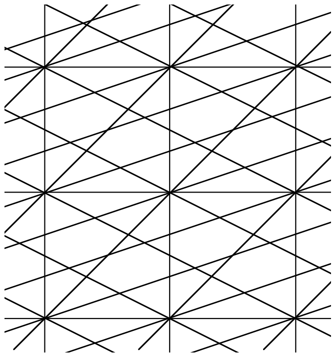

Figure 7: A system of hyperplanes

$$\Big\{\frac{1}{j}\mathbb{Z}^n+\Pi_k : k=1,\dots,j\Big\}=:\{\Pi^j_m : m\in M_j\}, \tag{69}$$

where M_j is a set of indices such that $\Pi^j_m\neq\Pi^j_{m'}$ if $m\neq m'$ (we can directly take $M_j=\mathbb{Z}^n\times\{1,\dots,j\}$ if all ν_k are irrational directions: i.e., if $t\nu_k\in\mathbb{Z}^n$ only if $t=0$).

We then take

$$K_j=\bigcup_{m\in M_j}\Pi^j_m \tag{70}$$

and F_j defined as in Remark 4.2(ii) by $F_{K_j,\{a^j_m\},\{b^j_m\}}(u,\Omega)$, where $a^j_m=\frac{1}{\|\nu^j_m\|_1}\varphi(\nu^j_m)$ and $b^j_m=0$; i.e.,

$$F_j(u)=\int_\Omega|\nabla u|^2\,dx+\sum_{m\in M_j}a^j_m\,\mathcal{H}^{n-1}(S(u)\cap\Pi^j_m\cap\Omega) \tag{71}$$

always with the constraint $S(u)\subset K_j$.

Theorem 4.4 *The functionals F_j in* (71) *Γ-converge to the functional F_φ in* (68)*. Moreover, recovery sequences can be constructed in* PC(Ω).

Proof. To prove the liminf inequality it suffices to remark that if $F_j(u_j)<+\infty$ then

$$F_j(u_j)=\int_\Omega|\nabla u_j|^2\,dx+\int_{S(u_j)\cap\Omega}\varphi(\nu(u_j))\,d\mathcal{H}^{n-1}=F_\varphi(u_j),$$

so that the desired inequality immediately follows from the lower semicontinuity of F_φ.

To prove the limsup inequality by approximation it suffices to treat the case when $u\in PC(\Omega)$; in particular $S(u)$ is a finite union of $n-1$-dimensional simplexes with disjoint closures.

If $S(u)=K_0$ is a single simplex in $\Pi_0=\{\langle x-x_0,\nu_0\rangle\}$ with $x_0\in K_0$, then we find a neighborhood U of $\overline{K}_0$, choose ν_{k_j} with $k_j\leq j$ converging to ν_0 and $x_j\in\frac{1}{j}\mathbb{Z}^n$ converging to x_0, and smooth invertible $\Phi_j\in\mathrm{Id}+C^\infty_0(U;U)$ with $\Phi_j\to\mathrm{Id}$ in $W^{1,\infty}(U;\mathbb{R}^m)$ and $\Phi_j(K_0)=x_j+$

$R_j(K_0+(x_j-x_0))$, where R_j is a rotation such that $R_j\nu_0=\nu_j$. Then, we set $u_j(x)=u(\Phi_j^{-1}x)$, so that $S(u_j)\subset\{\langle x-x_j,\nu_{k_j}\rangle\}$ (note that this hyperplane is of the form Π^j_m), and

$$\limsup_j F_j(u_j)=\int_\Omega|\nabla u|^2\,dx+\varphi(\nu_0)\mathcal{H}^{n-1}(K_0)=F(u).$$

If $S(u)$ is composed of more than one simplex then the same construction must be repeated locally, taking care of choosing disjoint neighborhoods. □

5. Limit energies of the type

$$F_\psi(u,\Omega)=\int_\Omega|\nabla u|^2\,dx+\int_{S(u)\cap\Omega}\psi(|u^+-u^-|)\,d\mathcal{H}^{n-1},\tag{72}$$

where ψ is any concave function on $(0,+\infty)$ with $\inf\psi\geq\sqrt{n}$. Note that this constraint is optimal, and derives from the inequality $\psi(z)\geq\|\nu\|_1$, which must hold for all $z>0$ and $\nu\in S^{n-1}$.

Since ψ is concave, we can find two sequences $\{a_j\}$ and $\{b_j\}$ such that

$$\psi(z)=\inf\{a_j+b_jz^2:j=0,1,\ldots\}$$

for all $z>0$ (see Fig. 8). Moreover, the convergence is uniform on bounded subsets of $(0,+\infty)$.

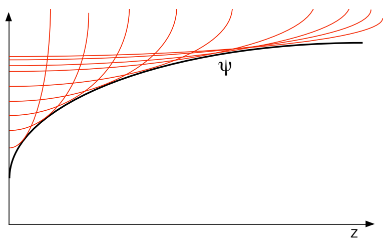

Figure 8: Approximation of a concave function

Note that $a_j\geq\sqrt{n}$ and $b_j\geq 0$ for all j. We define ψ_j as in (66) by

$$\psi_j(z)=\min\Big\{\sum_{l\in J}(a_l+b_lz_l^2):J\subset\{0,\ldots,j\},\ J\neq\emptyset,\ \sum_{l\in J}z_l=z\Big\}.\tag{73}$$

We have $\psi_j\geq\psi$ and $\psi_j\to\psi$ uniformly on bounded sets of $(0,+\infty)$.

We consider the hyperplanar networks $\{\Pi^j_m\}_{m\in M_j}$ as in (69), and their union as the corresponding K_j defined in (70). Denoted by ν^j_m the normal to Π^j_m,

$$\begin{aligned}\psi^j_m(z)&=\frac{1}{\|\nu^j_m\|_1}\psi_j(z)\qquad(74)\\&=\min\Big\{\sum_{l\in J}\Big(\frac{a_l}{\|\nu^j_m\|_1}+\frac{b_l}{\|\nu^j_m\|_1}z_l^2\Big):J\subset\{0,\ldots,j\},\ J\neq\emptyset,\ \sum_{l\in J}z_l=z\Big\}.\end{aligned}$$

Since $a_l\geq\sqrt{n}\geq\|\nu\|_1$ for all $\nu\in S^{n-1}$ the functions ψ^j_m satisfy the hypotheses of Remark 4.3 (with the system of planes $\{\Pi^j_m\}$ in place of $\{\Pi_l\}$). The functionals F_j are then defined by $F_{K_j,\{\psi^j_m\}}(u,\Omega)$ as in Remark 4.3; namely,

$$\begin{aligned}F_j(u)&=\int_\Omega|\nabla u|^2\,dx+\sum_{m\in M_j}\int_{S(u)\cap\Pi^j_m\cap\Omega}\psi^j_m(|u^+-u^-|)\|\nu^j_m\|_1\,d\mathcal{H}^{n-1}\\&=\int_\Omega|\nabla u|^2\,dx+\sum_{m\in M_j}\int_{S(u)\cap\Pi^j_m\cap\Omega}\psi_j(|u^+-u^-|)\,d\mathcal{H}^{n-1},\qquad(75)\end{aligned}$$

with the constraint $S(u)\subset K_j$.

Theorem 4.5 *The functionals F_j in* (75) Γ*-converge to the functional F_ψ in* (72)*. Moreover recovery sequences can be constructed in* $\mathrm{PC}(\Omega)$*.*

Proof. To prove the liminf inequality it suffices to remark that, since $\psi_j \geq \psi$, if $F_j(u_j) < +\infty$ then

$$F_j(u_j) \geq \int_\Omega |\nabla u_j|^2\,dx + \int_{S(u_j)\cap\Omega} \psi(|u_j^+ - u_j^-|)\,d\mathcal{H}^{n-1} = F_\psi(u_j),$$

so that the desired inequality immediately follows from the lower semicontinuity of F_ψ.

To prove the converse inequality, we can follow the same construction of Theorem 4.4. For the functions u_j defined therein we have

$$\begin{aligned}\limsup_j F_j(u_j) &\leq \lim_j \Big(\int_\Omega |\nabla u|^2\,dx + \int_{S(u)} \psi_j(|u^+ - u^-|)\,d\mathcal{H}^{n-1}\Big)\\ &= \int_\Omega |\nabla u|^2\,dx + \int_{S(u)} \psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1}\end{aligned}$$

by the uniform convergence of ψ_j to ψ on bounded sets of $(0, +\infty)$ (recall that we can always assume u in L^∞ by a truncation argument). □

Remark 4.6 (a wider class of surface energy densities) Theorem 4.5 is sharp on the set of concave target functions ψ. The same proof holds for a wider class, namely that of non-decreasing lower-semicontinuous subadditive functions that can be written as an infimum of functions ψ_j as in (66). It is not clear if there is a more transparent characterization of this class, which is strictly larger than the class of concave functions, containing for example

$$\psi_1(z) = \sqrt{n}\,\min\Big\{j + \frac{1}{j}z^2 : j = 1, 2, \dots\Big\}$$

(the subadditive envelope of $\sqrt{n}(1+z^2)$), and $\psi_2(z) = \sqrt{n}\,\min\{1+z^2, 2\}$, or if these are all the accessible energy densities. Note that not all subadditive non-decresing functions are in this class, as for example

$$\psi_3(z) = \begin{cases} \sqrt{n} & \text{if } z \leq 1\\ 2\sqrt{n} & \text{if } z > 1\end{cases}$$

(all functions ψ such that $\sup\psi \leq 2\inf\psi$ are subadditive), which does not seem to be an accessible target function.

Remark 4.7 (Locally inhomogeneous energies) We can reach all energies of the form

$$F(u) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)\cap\Omega} a(x)\psi(|u^+ - u^-|)\varphi(\nu(u))\,d\mathcal{H}^{n-1} \tag{76}$$

with a lower semicontinuous with $a \geq 1$, ψ concave with $\psi \geq 1$, and φ convex and $\varphi(\nu) \geq \|\nu\|_1$ on S^{n-1}. The approximating energies can be easily constructed by localizing the arguments in this section and the previous.

4.3 Proof of Theorem 2.2

We are eventually in the position to prove Theorem 2.2 using a diagonal procedure. Since all functionals we considered have the weak-membrane functional as a lower bound, we can use the metrizability of Γ-convergence ([24] Theorem 10.22), and a diagonal argument to deduce that there exists a sequence of sets K_i such that the energies $F_{K_i,b}$ defined as in (55) Γ-converge to the energy F in (76).

We recall that if $u \in \mathrm{PC}(\Omega)$ then the recovery sequences constructed in Section 4.1 again belong to $\mathrm{PC}(\Omega)$, while again we have used a density argument with that set in Section 4.2. As a consequence, also the functionals

$$H_{K_i,b}(u) = \begin{cases} F_{K_i,b} & \text{if } u \in \mathrm{PC}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge to the same F.

On the other hand, Theorem 3.1 ensures that for all i there exist a family $W^i_{\varepsilon_j}$ such that the Γ-limit F^i of $F^{W^i_{\varepsilon_j}}_{\varepsilon_j}$, which we can always suppose exists up to subsequences, satisfies $F_{K_i,b} \le F^i \le H_{K_i,b}$, so that also F^i Γ-converges to F. We then conclude the existence of $W_{\varepsilon_j} = W^{i_j}_{\varepsilon_j}$ satisfying the thesis of Theorem 2.2 by a diagonal argument. Finally, note that again by Theorem 3.1, for given θ we can always suppose that the limit density of $W^i_{\varepsilon_j}$ is θ for all i, and then that this holds also for W_{ε_j}. □

5 Appendix: discrete Poincarè's inequalities

We give a simple proof of the discrete version of Poincarè's inequality in the simplified situations of the paper.

Lemma 5.1 (A discrete version of Poincarè-Wirtinger's Lemma) *Let $\Omega \subset \mathbb{R}^n$ be a finite union of rectangles, and $p > 1$. There exists ε_0 and a constant $C = C(p, \Omega)$ such that for all $\varepsilon < \varepsilon_0$ and $u : \Omega_\varepsilon \to \mathbb{R}^m$, having set $\widetilde{u} = \frac{1}{\sharp\Omega_\varepsilon} \sum_{a\in\Omega_\varepsilon} u(a)$, we have*

$$\sum_{a\in\Omega_\varepsilon} |u(a) - \widetilde{u}|^p \varepsilon^n \le C \sum_{\{a,b\}\in M_\varepsilon} \Big|\frac{u(a) - u(b)}{\varepsilon}\Big|^p \varepsilon^n. \tag{77}$$

Proof. By construction we have

$$\begin{aligned} \sum_{a\in\Omega_\varepsilon} |u(a) - \widetilde{u}|^p \varepsilon^n &= \sum_{a\in\Omega_\varepsilon} \Big|u(a) - \frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} u(b)\Big|^p \varepsilon^n \\ &= \sum_{a\in\Omega_\varepsilon} \Big|\frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} (u(a) - u(b))\Big|^p \le \sum_{a\in\Omega_\varepsilon} \frac{1}{\sharp\Omega_\varepsilon} \sum_{b\in\Omega_\varepsilon} |u(a) - u(b)|^p \varepsilon^n. \end{aligned}$$

We want to estimate the term $\frac{1}{\sharp\Omega_\varepsilon} \sum_{a,b\in\Omega_\varepsilon} |u(a) - u(b)|^p$, by comparing it with the sum of all nearest-neighbor interactions.

Consider the case when Ω is a single rectangle of side-lengths $L_1, \dots, L_n$. Then we may consider a path connecting a and b composed of n segments in the directions $e_1, \dots, e_n$ in that order, and the points a_i on that path, so that (by Jensen's inequality) for ε small

$$|u(a) - u(b)|^p \le \frac{\max\{L_1, \dots, L_n\}^{p-1}}{\varepsilon^{p-1}} \sum_i |u(a_i) - u(a_{i-1})|^p.$$

Since each pair of nearest neighbors belongs at most to $\max\{L_1, \dots, L_n\}/\varepsilon$ such paths and $\sharp\Omega_\varepsilon$ is approximately $L_1 \cdots L_n/\varepsilon^n$, we obtain

$$\frac{1}{\sharp\Omega_\varepsilon} \sum_{a,b\in\Omega_\varepsilon} |u(a) - u(b)|^p \le c \frac{(\max\{L_1, \dots, L_n\})^p}{L_1 \cdots L_n} \sum_{\{a,b\}\in M_\varepsilon} \Big|\frac{u(a) - u(b)}{\varepsilon}\Big|^p \varepsilon^n$$

If Ω is a union of N rectangles of side-lenghts $L_1^j, \ldots, L_n^j$ then we obtain the thesis with a constant

$$C = c\frac{(\sum_j \max\{L_1^j, \ldots, L_n^j\})^p}{\sum_j L_1^j \cdots L_n^j} \tag{78}$$

by following the same reasoning, but joining points a, b in Ω by a path through possibly all the rectangles. □

Lemma 5.2 (Rescaled version of Lemma 5.1) *Let Ω, ε, p be as in Lemma 5.1 and let C be the constant in (77). We fix $\delta > 0$. We denote by Ω^δ the rescaled set $\Omega^\delta = \{x \in \mathbb{R}^n : x/\delta \in \Omega\}$ and by $\Omega_\varepsilon^\delta$ the lattice $\Omega_\varepsilon^\delta = \Omega^\delta \cap \varepsilon\mathbb{Z}^n$ (and accordingly, the set of nearest neighbors M_ε^δ). Then, for $\varepsilon < \varepsilon_0$ and for all $u : \Omega_\varepsilon^\delta \to \mathbb{R}^m$ we have:*

$$\sum_{a \in \Omega_\varepsilon^\delta} |u(a) - \widetilde{u}|\varepsilon^n \leq C\delta^p \sum_{\{a,b\} \in M_\varepsilon^\delta} \left|\frac{u(a) - u(b)}{\varepsilon}\right|^p \varepsilon^n, \tag{79}$$

where $\widetilde{u} = (\sharp\Omega_\varepsilon^\delta)^{-1} \sum_{a \in \Omega_\varepsilon^\delta} u(a)$.

Proof. By applying Lemma 5.1 to the function $v : \Omega_{\varepsilon/\delta} \to \mathbb{R}^m$ defined as $v(A) = u(\delta A)$ for $A \in \Omega_{\varepsilon/\delta}$, we get:

$$\sum_{A \in \Omega_{\varepsilon/\delta}} \left|v(A) - \frac{1}{\sharp(\Omega_{\varepsilon/\delta})} \sum_{B \in \Omega_{\varepsilon/\delta}} v(B)\right| \frac{\varepsilon^n}{\delta^n} \leq C \sum_{\{A,B\} \in M_{\varepsilon/\delta}} \left|\frac{v(A) - v(B)}{\varepsilon/\delta}\right|^p \frac{\varepsilon^n}{\delta^n}.$$

Scaling back the space variable we obtain

$$\sum_{a \in \Omega_\varepsilon^\delta} \left|u(a) - \frac{1}{\sharp(\Omega_\varepsilon^\delta)} \sum_{b \in \Omega_\varepsilon^\delta}\right| \varepsilon^n \leq C\delta^p \sum_{\{a,b\} \in M_\varepsilon^\delta} \left|\frac{u(a) - u(b)}{\varepsilon}\right|^p \varepsilon^n,$$

with C as in (78) as desired. □

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