APERIODIC FRACTIONAL OBSTACLE PROBLEMS

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Abstract. We determine the asymptotic behaviour of (bilateral) obstacle problems for fractional energies in rather general aperiodic settings via Γ-convergence arguments. As further developments we consider obstacles with random sizes and shapes located on points of standard lattices, and the case of random homothetics obstacles centered on random Delone sets of points.

Obstacle problems for non-local energies occur in several physical phenomenona, for which our results provide a description of the first order asympotitc behaviour.

1. Introduction

Non-local energies and operators have been actively investigated over recent years. They arise in problems from different fields, the most celebrated being Signorini's problem in contact mechanics: finding the equilibria of an elastic body partially laying on a surface and acted upon part of its boundary by unilateral shear forces (see [43], [31]). In the anti-plane setting the elastic energy can be then expressed in terms of the seminorm of a $H^{1/2}$ function, or equivalently as the boundary trace energy of a $W^{1,2}$ displacement.

As further examples we mention applications in elasticity, for instance in phase field theories for dislocations (see [34] and the references therein); in heat transfer for optimal control of temperature across a surface [33], [6]; in equilibrium statistical mechanics to model free energies of Ising spin systems with Kac potentials on lattices (see [2] and the references therein); in fluid dynamics to describe flows through semi-permeable membranes [30]; in financial mathematics in pricing models for American options [4]; and in probability in the theory of Markov processes (see [10], [11] and the references therein).

Many efforts have been done to extend the existing theories for (fully non-linear) second order elliptic equations to non-local equations. Regularity has been developed for integro-differential operators [10], [11], [22], and for obstacle problems for the fractional laplacian (see [20], [44] and the references therein). Periodic homogenization has been studied for a quite general class of non-linear, non-local uniformly elliptic equations [41] and for obstacle problems for the fractional laplacian [19], [32].

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In this paper we determine the homogenization limit for bilateral obstacle problems involving fractional type energies. In doing that we employ a variational approach by following De Giorgi's Γ-convergence theory. Many contributions in literature are related to the analogous problem for energies defined in ordinary Sobolev spaces or equivalently for local elliptic operators. The setting just mentioned will be referred to in the sequel as the *local case* in contrast to the non-local framework object of our analysis. Starting from the seminal papers by Marchenko and Khruslov [37], Rauch and Taylor [39], [40], and Cioranescu and Murat [24] there has been an outgrowing interest on this kind of problems with different approaches (see the books [7], [15], [16], [23], [28] and the references therein on this subject). We limit ourselves to stress that Γ-convergence theory was successfully applied to tackle the problem and to solve it in great generality (see [29], [26], [27]).

Let us briefly resume the contents of this paper in a model case (for all the details and the precise assumptions see section 3). With fixed a bounded set T, and a discrete and homogeneous distribution of points $\Lambda = \{\mathrm{x}^{\mathbf{i}}\}_{\mathbf{i}\in\mathbf{Z}^n}$ (see Definition 2.2), define for all $j \in \mathbf{N}$ the *obstacle set* $T_j \subseteq \mathbf{R}^n$ by $T_j = \cup_{\mathbf{i}\in\mathbf{Z}^n}\left(\varepsilon_j\,\mathrm{x}^{\mathbf{i}} + \varepsilon_j^{n/(n-sp)}T\right)$, where $(\varepsilon_j)_{j\in\mathbf{N}}$ is a positive infinitesimal sequence. Given a bounded, open and connected subset U of $\mathbf{R}^n$, $n \geq 2$, with Lipschitz regular boundary we consider the functionals $\mathcal{F}_j : L^p(U) \to [0, +\infty]$ given by

$$\mathcal{F}_j(u) := \int_{U\times U}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy \quad \text{ if } u \in W^{s,p}(U),\ \tilde{u} = 0 \text{ cap}_{s,p} \text{ q.e. on } T_j \cap U$$

and $+\infty$ otherwise. Here, $W^{s,p}(U)$ is the Sobolev-Slobodeckij space for $s \in (0,1)$, $p \in (1,+\infty)$ and $sp \in (1, n)$, $\text{cap}_{s,p}$ is the related variational (p,s)-capacity, and $\tilde{u}$ denotes the precise representative of $u \in W^{s,p}(U)$ which is defined except on a $\text{cap}_{s,p}$-negligible set (see subsections 2.4 and 2.5).

In Theorem 3.3 we show that the asymptotic behaviour of the sequence $(\mathcal{F}_j)_{j\in\mathbf{N}}$ is described in terms of $\Gamma(L^p(U))$-convergence by the functional $\mathcal{F} : L^p(U) \to [0,+\infty]$ defined by

$$\mathcal{F}(u) = \int_{U\times U}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy + \theta\,\text{cap}_{s,p}(T)\int_U |u(x)|^p\beta(x)\,dx$$

if $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$. The quantities θ and β represent respectively the limit density and the limit normalized distribution of the set of points $\varepsilon_j\Lambda$ and can be explicitely calculated in some cases (see (3.1) and (3.3) for the exact definitions, Remark 3.2 for further comments, and Examples 3.5-3.7 where some cases are discussed in details).

Adding zero boundary conditions, Γ-convergence then implies the convergence for minimizers and minimum values of $(\mathcal{F}_j)_{j\in\mathbf{N}}$ to the respective quantities of $\mathcal{F}$. More generally, we can study the asymptotic behaviour of anisotropic variations of the Gagliardo seminorm (see subsection 3.4).

We remark that it is not our aim to establish a general abstract theory as Dal Maso did in the local case [26],[27] (see [9] for related results in random settings), nor to consider the most general framework for homogenization as the one proposed by Nguetseng in the linear, local and non degenerate case [38]; but rather we aim at giving explicit constructive results for a sufficiently broad class of fractional energies and non-periodic obstacles.

The main novelty of the paper is that we extend the asymptotic analysis of obstacle probelms for local energies to non-local ones. In doing that we use intrinsic arguments and give a self-contained proof. We avoid extension techniques with which the original problem is transformed into a homogenization problem at the boundary by rewriting fractional seminorms as trace energies for local (but degenerate) functionals in one dimension higher (see [25], [19], [32]). This is the well known harmonic extension procedure for $H^{1/2}$ functions; more recently it was proved to hold true also for the fractional laplacian, corresponding to $p = 2$ and $s \in (0,1)$ above, by Caffarelli and Silvestre [21]. Because of our approach, we are able to deal with non-local energies which can not be included in the previous frameworks. Generalizations are likely to be done in several directions, the most immediate being the choice of the obstacle condition. For, we have confined ourselves to the basic bilateral zero obstacle condition in the scalar case only for the sake of simplicity, improvements to vector valued problems and unilateral obstacles seem to be at hand (see [5], [32]).

Our analysis do not need the usual periodicity or almost periodicity assumptions for the distribution of obstacles. We deal with aperiodic settings defined after *Delone set of points* (see subsection 2.3), the set Λ introduced above. No regularity or simmetry conditions are imposed on Λ, only two simple geometric properties are assumed: *discreteness* and *homogeneity*. They turn out to be physically reasonable conditions; as a matter of fact Delone sets have been introduced in n-dimensional mathematical cristallography to model many non-periodic structures such as quasicrystals (see [42]). Essentially, these assumptions guarantee that points in Λ can neither cluster nor be scattered away.

Furthermore, we show that with minor changes the same tools are suited also to deal with some random settings. In particular, we deal with obstacles with random sizes and shapes located on points of a standard lattice in $\mathbf{R}^n$, a setting introduced by Caffarelli and Mellet [18], [19]; and consider also homothetics random copies of a given obstacle located on random lattices following Blanc, Le Bris and Lions [13], [14] (for more details see section 4).

As a byproduct, our approach yields also an intrinsic proof of the results first obtained by [19] that avoids the extension techniques in [21]. A different proof using Γ-convergence methods, but still relying on those extension techniques, was given by the author in [32].

The key tools of our analysis are Lemmas 3.8 and 3.9 below. By means of these results we reduce the Γ-limit process to families of functions which are constants on suitable annuli surrounding the obstacle sets. Lemma 3.9 is the counterpart in the current non-local framework of the joining lemma in varying domains for gradient energies on standard Sobolev spaces proven by Ansini and Braides [5]. It is a variant of an idea by De Giorgi in the setting of varying domains, on the way of matching boundary conditions by increasing the energy only up to a small error. As in the local case the proofs of Lemmas 3.8 and 3.9 exploit De Giorgi's slicing/averaging principle and the fact that Poincaré-Wirtinger inequalities are qualitatively invariant under families of biLipschitz mappings with equibounded Lipschitz constants.

Despite this, the non-local behaviour of fractional energies introduces several additional difficulties into the problem: Lemmas 3.8 and 3.9 do not follow from routine modifications of the arguments used in the local case. New ideas have to be worked out mainly to control the long-range interaction terms. A major role in doing that is played by the counting arguments in Proposition 2.5, Hardy inequality (see Theorem 2.8), and the estimates on singular kernels in Lemma A.1.

The paper is organized as follows. In section 2 we list the necessary prerequisites on Γ-convergence, Sobolev-Slobodeckij spaces and variational capacities giving precise references for those not proved. Section 3 is devoted to the exact statement and the proof of the homogenization result for deterministic distribution of obstacles. To avoid unnecessary generality we deal with the model case of fractional seminorms. Generalizations to anisotropic kernels are postponed to subsection 3.4. The ideas of section 3 are then used in section 4 to deal with the two different random settings mentioned before. Finally, we give the proof of an elementary technical result, though instrumental for us, in Appendix A.

2. Preliminaries and Notations

2.1. **Basic Notations.** We use standard notations for Lebesgue and Hausdorff measures, and for Lebesgue and Sobolev function spaces.

The Euclidean norm in $\mathbf{R}^n$ is denoted by $|\cdot|$, the maximum one by $|\cdot|_\infty$. $B_r(x)$ stands for the Euclidean ball in $\mathbf{R}^n$ with centre x and radius $r>0$, and we write simply B_r in case $x=\underline{0}$. As usual $\omega_n := \mathcal{L}^n(B_1)$.

Given a set $E \subset \mathbf{R}^n$ its complement will be indifferently denoted by E^c or $\mathbf{R}^n \setminus E$. Its interior and closure are denoted by $\mathrm{int(E)}$ and $\overline{E}$, respectively. Given two sets $E \subset\subset F$ in $\mathbf{R}^n$, a *cut-off function between E and F* is any $\varphi \in \mathrm{Lip}(\mathbf{R}^n,[0,1])$ such that $\varphi|_{\overline{E}} \equiv 1$, $\varphi|_{\mathbf{R}^n\setminus F} \equiv 0$, and $\mathrm{Lip}(\varphi) \leq 1/\mathrm{dist}(E,\partial F)$.

Given an open set $A \subseteq \mathbf{R}^n$ the collections of its open, Borel subsets are denoted by $\mathcal{A}(A)$, $\mathcal{B}(A)$, respectively. The diagonal set in $\mathbf{R}^n \times \mathbf{R}^n$ is denoted by Δ, and for every $\delta > 0$ its open δ-neighborhood by $\Delta_\delta := \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : |x-y| < \delta\}$. Accordingly, for any set $E \subseteq \mathbf{R}^n$ and for any $\delta > 0$

$$E_\delta := \{x \in \mathbf{R}^n : \mathrm{dist}(x,E) < \delta\}, \qquad E_{-\delta} := \{x \in E : \mathrm{dist}(x,\partial E) > \delta\}. \tag{2.1}$$

In the following U will always be an open and connected subset of $\mathbf{R}^n$ whose boundary is Lipschitz regular.

In several computations below the letter c generically denotes a positive constant. We assume this convention since it is not essential to distinguish from one specific constant to another, leaving understood that the constant may change from line to line. The parameters on which each constant c depends will be explicitely highlighted.

2.2. **Γ-convergence.** We recall the notion of Γ-convergence introduced by De Giorgi in a generic metric space (X, d) endowed with the topology induced by d (see [28],[15]). A sequence of functionals $F_j : X \to [0, +\infty]$ Γ*-converges* to a functional $F : X \to [0, +\infty]$ in $u \in X$, in short $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if the following two conditions hold:

(i) (*liminf inequality*) $\forall\ (u_j)_{j\in\mathbf{N}}$ converging to u in X, we have $\liminf_j F_j(u_j) \geq F(u)$;

(ii) (*limsup inequality*) $\exists\ (u_j)_{j\in\mathbf{N}}$ converging to u in X such that $\limsup_j F_j(u_j) \leq F(u)$.

We say that F_j Γ*-converges* to F (or $F = \Gamma\text{-}\lim_j F_j$) if $F(u) = \Gamma\text{-}\lim_j F_j(u)\ \forall u \in X$. We may also define the *lower* and *upper* Γ*-limits* as

$$\Gamma\text{-}\limsup_j F_j(u) = \inf\{\limsup_j F_j(u_j) :\ u_j \to u\},$$

$$\Gamma\text{-}\liminf_j F_j(u) = \inf\{\liminf_j F_j(u_j) :\ u_j \to u\},$$

respectively, so that conditions (i) and (ii) are equivalent to $\Gamma\text{-limsup}_j F_j(u) = \Gamma\text{-liminf}_j F_j(u) = F(u)$. Moreover, the functions $\Gamma\text{-limsup}_j F_j$ and $\Gamma\text{-liminf}_j F_j$ are lower semicontinuous.

One of the main reasons for the introduction of this notion is explained by the following fundamental theorem (see [28, Theorem 7.8]).

Theorem 2.1. *Let $F = \Gamma\text{-}\lim_j F_j$, and assume there exists a compact set $K \subset X$ such that $\inf_X F_j = \inf_K F_j$ for all j. Then there exists $\min_X F = \lim_j \inf_X F_j$. Moreover, if $(u_j)_{j\in\mathbf{N}}$ is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.*

2.3. **Non-periodic tilings.** In the ensuing sections we will deal with a general framework extending the usual periodic setting. The partition of $\mathbf{R}^n$ we consider is obtained via the *Voronoï tessellation* related to a fixed *Delone set of points* Λ. We refer to the by now classical book of M. Senechal [42] for all the relevant results.

Definition 2.2. *A point set $\Lambda \subset \mathbf{R}^n$ is a* Delone (or Delaunay) set *if it satisfies*

(i) Discreteness*: there exists $r > 0$ such that for all $x, y \in \Lambda$, $x \neq y$, $|x - y| \geq 2r$;*

(ii) Homogeneity or Relative Density*: there exists $R > 0$ such that $\Lambda \cap B_R(x) \neq \emptyset$ for all $x \in \mathbf{R}^n$.*

It is then easy to show that Λ is countably infinite. Hence, from now on we use the notation $\Lambda = \{\mathrm{x}^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$. By the very definition the quantities

$$r_\Lambda := \frac{1}{2}\inf\{|x - y| :\ x, y \in \Lambda,\ x \neq y\}, \quad R_\Lambda := \inf\{R > 0 :\ \Lambda \cap B_R(x) \neq \emptyset\ \ \forall x \in \mathbf{R}^n\} \tag{2.2}$$

are finite and strictly positive; R_Λ is called the *covering radius* of Λ.

Definition 2.3. *Let $\Lambda \subset \mathbf{R}^n$ be a Delone set, the* Voronoï cell *of a point $\mathrm{x}^{\mathtt{i}} \in \Lambda$ is the set of points*

$$V^{\mathtt{i}} := \{y \in \mathbf{R}^n :\ |y - \mathrm{x}^{\mathtt{i}}| \leq |y - \mathrm{x}^{\mathtt{k}}|,\ \textit{for all}\ \mathtt{i} \neq \mathtt{k}\}.$$

The Voronoï tessellation *induced by Λ is the partition of $\mathbf{R}^n$ given by $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$.*

In the following proposition we collect several interesting properties of Voronoï tessellations (see [42, Propositions 2.7, 5.2]).

Proposition 2.4. *Let $\Lambda \subset \mathbf{R}^n$ be a Delone set and $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ its induced Voronoï tessellation, then*

(i) the $V^{\mathtt{i}}$'s are convex polytopes fitting together along whole faces, and have no interior points in common;

(ii) if $V^{\mathtt{i}}$ and $V^{\mathtt{k}}$ share a vertex z, then $\mathrm{x}^{\mathtt{i}}$ and $\mathrm{x}^{\mathtt{k}}$ lie on $\partial B_\rho(z)$ with $\Lambda \cap B_\rho(z) = \emptyset$, $\rho \le R_\Lambda$.

Hence, $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ is a tiling, *i.e. the $V^{\mathtt{i}}$'s are closed, have no interior points in common and $\cup_{\mathtt{i}} V^{\mathtt{i}} = \mathbf{R}^n$. More precisely,*

(iii) $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ is a normal *tiling: for each tile $V^{\mathtt{i}}$ we have $B_{r_\Lambda}(\mathrm{x}^{\mathtt{i}}) \subseteq V^{\mathtt{i}} \subseteq \overline{B}_{R_\Lambda}(\mathrm{x}^{\mathtt{i}})$;*

(iv) $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ is a locally finite *tiling: $\#(\Lambda \cap B_\rho(x)) < +\infty$ for all $x \in \mathbf{R}^n$, $\rho > 0$.*

Further properties that will be repeatedly used in our analysis are summarized below. We omit their proofs since they are justified by elementary counting arguments. For any $A \in \mathcal{A}(\mathbf{R}^n)$ we set

$$\mathcal{I}_\Lambda(A) := \{\mathtt{i} \in \mathbf{Z}^n : V^{\mathtt{i}} \subseteq A\}, \qquad \mathscr{I}_\Lambda(A) := \{\mathtt{i} \in \mathbf{Z}^n : V^{\mathtt{i}} \cap \partial A \neq \emptyset\}. \tag{2.3}$$

Proposition 2.5. *Let $\Lambda \subset \mathbf{R}^n$ be a Delone set and $\{V^{\mathtt{i}}\}_{\mathtt{i}\in\mathbf{Z}^n}$ its induced Voronoï tessellation. Then,*

$$\omega_n r_\Lambda^n \#(\mathcal{I}_\Lambda(A)) \le \mathcal{L}^n(A), \quad \omega_n r_\Lambda^n \#(\mathscr{I}_\Lambda(A)) \le (\partial A)_{R_\Lambda}, \quad \mathcal{L}^n\left(A \setminus \cup_{\mathcal{I}_\Lambda(A)} V^{\mathtt{i}}\right) \le (\partial A)_{R_\Lambda}. \tag{2.4}$$

In particular, there exists a constant $c = c(n) > 0$ such that for every $\mathtt{i} \in \mathbf{Z}^n$, $m \in \mathbf{N}$ it holds

$$\#\{\mathtt{k} \in \mathcal{I}_\Lambda(A) : m r_\Lambda < |\mathrm{x}^{\mathtt{i}} - \mathrm{x}^{\mathtt{k}}|_\infty \le (m+1) r_\Lambda\} \le c\, m^{n-1}. \tag{2.5}$$

2.4. **Sobolev-Slobodeckij spaces.** Let $A \subseteq \mathbf{R}^n$ be any bounded open Lipschitz set, $p \in (1, +\infty)$, $s \in (0,1)$ and $ps \in (1,n)$, by $W^{s,p}(A)$ we denote the usual Sobolev-Slobodeckij space, or Besov space $B^s_{p,p}(A)$. The space is Banach if equipped with the norm $\|u\|_{W^{s,p}(A)} = \|u\|_{L^p(A)} + |u|_{W^{s,p}(A)}$, where

$$|u|^p_{W^{s,p}(A)} := \int_{A\times A} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\,.$$

We will use several properties of fractional Sobolev spaces, giving precise references for those employed in the sequel in the respective places mainly referring to [1] and [45].

In the indicated ranges for the parameters p, s it turns out that $W^{s,p}$ is a reflexive space (see [45, Thm 4.8.2]), Sobolev embedding and Sobolev-Gagliardo-Nirenberg inequality hold (see [1, Chapter V]), and traces are well defined (see (2.9) below). We remark that the restriction on s are necessary, since otherwise $W^{s,p}(A)$ contains only constant functions if $s \ge 1$, while for $ps < 1$ traces are not well defined (see also (2.9) below). The exclusion of the other cases is related to capacitary issues (see subsection 2.5).

Poincarè-Wirtinger and Poincarè inequalities in fractional Sobolev spaces are instrumental tools in the sequel. Thus, we state explicitely those results in the form we need. Their proof clearly follows from the usual argument by contradiction once $W^{s,p}$ is reflexive and is endowed with a trace operator.

Theorem 2.6. *Let $n \geq 1$, $p \in (1, +\infty)$, and $s \in (0,1)$. Let $A \subset \mathbf{R}^n$ be a bounded, connected open set, and O any measurable subset of A with $\mathcal{L}^n(O) > 0$. Then for any function $u \in W^{s,p}(A)$,*

$$\|u - u_O\|^p_{L^p(A)} \leq c_{PW} |u|^p_{W^{s,p}(A)}, \tag{2.6}$$

for a constant $c_{PW} = c_{PW}(n,p,s,O,A)$. Moreover, for any $u \in W^{s,p}_0(U)$ we have

$$\|u\|^p_{L^p(A)} \leq c_P |u|^p_{W^{s,p}(A)}, \tag{2.7}$$

for a constant $c_P = c_P(n,p,s,A)$.

Remark 2.7. *Let $(\Phi_t)_{t \in \mathscr{T}}$ be a family of biLipschitz maps on A with $\sup_{\mathscr{T}}(\mathrm{Lip}(\Phi_t) + \mathrm{Lip}(\Phi_t^{-1})) < +\infty$, then a simple change of variables implies that the constants $c_{PW}(n,p,s,\Phi_t(O),\Phi_t(A))$ and $c_P(n,p,s,\Phi_t(A))$ are uniformly-bounded in t. In particular, a scaling argument and Hölder inequality yield for any $z \in \mathbf{R}^n$ and $r > 0$ and for some $c = c(n,p,s,O,A) > 0$*

$$\|u - u_{z+rO}\|^p_{L^p(z+rA)} \leq c\, r^{sp} |u|^p_{W^{s,p}(z+rA)}. \tag{2.8}$$

A similar conclusion holds for Poincaré inequality (2.7).

Next, we recall the fractional version of Hardy inequality (see [45, Theorem 4.3.2/1, Remark 2 pp. 319-320] and [46] for further comments). To this aim we introduce the space $\hat{W}^{s,p}(B_1) := \{u \in W^{s,p}(\mathbf{R}^n) : \mathrm{spt} u \subset \overline{B}_1\}$. It is clear that $C^\infty_0(B_1)$ is dense in $\hat{W}^{s,p}(B_1)$ for any $p \in (1, +\infty)$ and that $\hat{W}^{s,p}(B_1) \subseteq W^{s,p}_0(B_1)$. The latter inclusion is strict if $s - 1/p \in \mathbf{N}$, and more precisely it holds

$$W^{s,p}_0(B_1) = \begin{cases} \hat{W}^{s,p}(B_1) & \text{if } s > 1/p - 1,\ s - 1/p \notin \mathbf{N} \\ W^{s,p}(B_1) & \text{if } s \in (0, 1/p]. \end{cases} \tag{2.9}$$

Theorem 2.8. *There exists a constant $c = c(n,p,s)$ such that for every $u \in \hat{W}^{s,p}(B_1)$ we have*

$$\int_{B_1} \frac{|u(x)|^p}{\mathrm{dist}(x, \partial B_1)^{sp}}\, dx \leq c \left(|u|^p_{W^{s,p}(B_1)} + \|u\|^p_{L^p(B_1)} \right).$$

Remark 2.9. *The usual scaling argument, Poincaré inequality (2.7) and Theorem 2.8 then yield*

$$\int_{B_r} \frac{|u(x)|^p}{\mathrm{dist}(x, \partial B_r)^{sp}}\, dx \leq c\, |u|^p_{W^{s,p}(B_r)} \tag{2.10}$$

for every $r > 0$ and $u \in \hat{W}^{s,p}(B_r)$, for a constant $c = c(n,p,s)$ independent from r.

2.5. **Fractional capacities.** We recall the notion of variational capacity for fractional Sobolev spaces and prove some properties relevant in the developments below. Those properties, straightforward in the local case, require more work in the non-local one. Let $p \in (1, +\infty)$ and $s \in (0,1)$ be given as before, for any set $T \subseteq \mathbf{R}^n$ define

$$\mathrm{cap}_{s,p}(T) := \inf_{\{A \in \mathcal{A}(\mathbf{R}^n):\, A \supseteq T\}} \inf \left\{ |u|^p_{W^{s,p}(\mathbf{R}^n)} : u \in W^{s,p}(\mathbf{R}^n),\ u \geq 1\, \mathcal{L}^n \text{ a.e. on } A \right\}, \tag{2.11}$$

with the usual convention $\inf\emptyset=+\infty$. The set function in (2.11) turns out to be a Choquet capacity (see [1, Chapter V]). Recall that a property holds $\mathrm{cap}_{s,p}$ *quasi everywhere*, in short $\mathrm{cap}_{s,p}$ q.e. on A, if it holds up to a set of $\mathrm{cap}_{s,p}$ zero. In particular, any function u in $W^{s,p}(A)$, $A\in\mathcal{A}(\mathbf{R}^n)$, has a *precise representative* $\tilde u$ defined $\mathrm{cap}_{s,p}$ q.e. and the following formula holds (see [1, Proposition 5.3])

$$\mathrm{cap}_{s,p}(T):=\inf\left\{|w|^p_{W^{s,p}(\mathbf{R}^n)}:\,w\in W^{s,p}(\mathbf{R}^n),\,\tilde w\geq 1\text{ q.e. on }T\right\}. \tag{2.12}$$

Coercivity of the fractional norm is ensured only in the $L^{p^*}(\mathbf{R}^n)$ topology, $p^*:=np/(n-sp)$ is the Sobolev exponent relative to p and s (see [1, Chapter V]). Thus, a minimizer for the capacitary problem exists in the homogeneous space $\dot W^{s,p}(\mathbf{R}^n)=\{u\in L^{p^*}(\mathbf{R}^n):\,|u|_{W^{s,p}(\mathbf{R}^n)}<+\infty\}$. Uniqueness is guaranteed by the strict convexity of the fractional energy, thus the minimizer of (2.11), (2.12) will be denoted by u^T and called the *capacitary potential* for T.

Remark 2.10. *If $ps>n$ points have positive capacity and $W^{s,p}$ is embedded into C^0. In this case it is well known that the homogenized obstacle problem turns out to be trivial, this is the reason why we required $ps<n$. The borderline case $ps=n$ deserves an analysis similar to that we will perform but different in some details, so that its study is not dealt with in this paper (see for instance [24] and [17] in the local framework).*

Let us prove that set inclusion induces an ordering among capacitary potentials. As a byproduct we also show that admissible functions in the capacitary problem can be taken with values in $[0,1]$.

Lemma 2.11. *If $T\subseteq F$, then $0\leq u^T\leq u^F\leq 1$ $\mathcal{L}^n$ a.e. on $\mathbf{R}^n$.*

Proof. First, take note that for all $u,\,v\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$\begin{aligned}|(u\vee v)(x)-(u\vee v)(y)|&\leq|u(x)-u(y)|\vee|v(x)-v(y)|,\\ |(u\wedge v)(x)-(u\wedge v)(y)|&\leq|u(x)-u(y)|\vee|v(x)-v(y)|.\end{aligned} \tag{2.13}$$

In particular, uniqueness of the capacitary potential u^T and by choosing $v\equiv 0$ in $(2.13)_1$ and $v\equiv 1$ in $(2.13)_2$ imply that $0\leq u^T\leq 1$ $\mathcal{L}^n$ a.e. on $\mathbf{R}^n$ for any subset $T\subset\mathbf{R}^n$.

Moreover, (2.13) yields that $u^T\vee u^F$ and $u^T\wedge u^F\in\dot W^{s,p}(\mathbf{R}^n)$. Set $U_T=\{u^T\leq u^F\}$ and $U_F=\{u^F<u^T\}$, and assume by contradiction that $\mathcal{L}^n(U_F)>0$. By taking $u^T\vee u^F$ as test function in the minimum problem $\mathrm{cap}_{s,p}(F)$ and recalling the strict minimality of u^F, we infer $|u^F|^p_{W^{s,p}(\mathbf{R}^n)}<|u^T\vee u^F|^p_{W^{s,p}(\mathbf{R}^n)}$. An easy computation then leads to

$$|u^F|^p_{W^{s,p}(U_F)}+2\int_{U_T\times U_F}\frac{|u^F(x)-u^F(y)|}{|x-y|^{n+sp}}dxdy<|u^T|^p_{W^{s,p}(U_F)}+2\int_{U_T\times U_F}\frac{|u^T(x)-u^F(y)|}{|x-y|^{n+sp}}dxdy.$$

The latter inequality can be used to estimate the fractional norm of $u^T\wedge u^F$ as follows

$$\begin{aligned}|u^T\wedge u^F|^p_{W^{s,p}(\mathbf{R}^n)}&<|u^T|^p_{W^{s,p}(\mathbf{R}^n)}\\ &+2\int_{U_T\times U_F}\frac{|u^T(x)-u^F(y)|^p+|u^F(x)-u^T(y)|^p-|u^T(x)-u^T(y)|^p-|u^F(x)-u^F(y)|^p}{|x-y|^{n+sp}}dxdy.\end{aligned} \tag{2.14}$$

We claim that the second term on the rhs in (2.14) is non-positive, this would imply that the strict inequality sign holds above, which in turn would give a contradiction since $u^T \wedge u^F$ is a test function for the capacitary problem related to T.

To conclude consider the auxiliary function $\psi(s,t) := |s-u^T(y)|^p + |u^T(x)-t|^p - |s-t|^p - |u^T(y)-u^T(x)|^p$ and the set $H := \{(s,t) : s \geq u^T(x),\ t < u^T(y)\}$. Elementary calculations yield that $\max_H \psi \leq 0$. Take note that the numerator of the integrand of the second term in the rhs of (2.14) equals to $\psi(u^F(x), u^F(y))$ and $u^F(x) \geq u^T(x)$ $\mathcal{L}^n$ a.e. $x \in U_T$, and $u^F(y) < u^T(y)$ $\mathcal{L}^n$ a.e. $y \in U_F$. □

In the sequel we are interested into relative capacities, for which we introduce two different notions. The first one is useful in the Γ-liminf inequality, the second in the Γ-limsup inequality, respectively. For every $0 < r \leq R$ set

$$\mathrm{cap}_{s,p}(T, B_R; r) := \inf\left\{|w|^p_{W^{s,p}(B_R)} : w \in W^{s,p}(\mathbf{R}^n),\ w = 0 \text{ on } \mathbf{R}^n \setminus \overline{B}_r,\ \tilde{w} \geq 1 \text{ q.e. on } T\right\},$$

and

$$C_{s,p}(T, B_R) := \inf\left\{|w|^p_{W^{s,p}(\mathbf{R}^n)} : w \in W^{s,p}(\mathbf{R}^n),\ w = 0 \text{ on } \mathbf{R}^n \setminus \overline{B}_R,\ \tilde{w} \geq 1 \text{ q.e. on } T\right\}.$$

To prove the convergence of relative capacities to the global one we introduce some notation to simplify the calculations below: for any $\mathcal{L}^n$-measurable function w and any $\mathcal{L}^{n\times n}$-measurable set $E \subseteq U \times U$ consider the *locality defect* of the $W^{s,p}$ seminorm

$$\mathcal{D}_{s,p}(w, E) := \int_E \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}} dxdy\,.$$

The terminology is justified since given two disjoint subdomains $A, B \subseteq U$, it holds

$$|w|^p_{W^{s,p}(A\cup B)} = |w|^p_{W^{s,p}(A)} + |w|^p_{W^{s,p}(B)} + 2\mathcal{D}_{s,p}(w, A \times B). \tag{2.15}$$

In particular, $\mathcal{D}_{s,p}(w, A\times A) = |w|^p_{W^{s,p}(A)}$.

We are now in a position to prove the claimed converge result for relative capacities. Actually, we show uniform convergence for families of equi-bounded sets. A different argument yielding pointwise convergence will be exploited in the generalizations of subsection 3.4 (see Lemma 3.12). The latter is sufficient for the proof of Theorem 3.3; the advantage of the approach below is that it can be carried over straightforward to the case of obstacles with random sizes and shapes for which uniform convergence is necessary (see subsection 4.1).

Lemma 2.12. *For all $\rho > 0$ it holds*

$$\lim_{r\to+\infty} \sup_{T\subseteq B_\rho} |C_{s,p}(T, B_r) - \mathrm{cap}_{s,p}(T)| = 0. \tag{2.16}$$

Moreover, there exists a constant $c = c(n,s,p)$ such that for all $0 < \rho < r < R$

$$\sup_{T\subseteq B_\rho} \left(\mathrm{cap}_{s,p}(T) - \mathrm{cap}_{s,p}(T, B_R; r)\right) \leq \frac{c\, r^{sp}}{(R-r)^{sp}} C_{s,p}(B_\rho, B_r). \tag{2.17}$$

In addition, if $R(r)/r \to +\infty$ as $r \to +\infty$ we have

$$\lim_{r\to+\infty}\sup_{T\subseteq B_\rho}|\mathrm{cap}_{s,p}(T)-\mathrm{cap}_{s,p}(T,B_{R(r)};r)|=0. \tag{2.18}$$

Proof. It is clear from the very definitions that $(0,+\infty)\ni r\to C_{s,p}(T,B_r)$ is monotone decreasing, and moreover that $\mathrm{cap}_{s,p}(T,B_R;r)\le C_{s,p}(T,B_r)$, $\mathrm{cap}_{s,p}(T)\le C_{s,p}(T,B_r)$.

Let us first prove (2.16). To this aim take u^T, $u^{B_\rho}\in \dot{W}^{s,p}(\mathbf{R}^n)$ the capacitary potentials of T and B_ρ, respectively; by Lemma 2.11 we have $0\le u^T\le u^{B_\rho}\le 1$ $\mathcal{L}^n$ a.e. on $\mathbf{R}^n$. In addition, u^{B_ρ} is radially symmetric and decreasing (to zero) since the fractional norm is stricly convex, rotation invariant and decreasing under radial rearrangements (for the last result see [3, Section 9]).

With fixed $\delta>0$ consider the Lipschitz map $\psi_\delta(t):=\frac{t-\delta}{1-\delta}\vee 0$, $\mathrm{Lip}(\psi_\delta)\le(1-\delta)^{-1}$, and set $w_\delta(x):=\psi_\delta(u^T(x))$. Up to $\mathcal{L}^n$ negligible sets, $\{w_\delta>0\}=\{u^T>\delta\}\subseteq\{u^{B_\rho}>\delta\}\subseteq B_{R_\delta}$, for some $R_\delta\to+\infty$ as $\delta\to0^+$. Then $w_\delta\in W^{s,p}(\mathbf{R}^n)$ with

$$|w_\delta|^p_{W^{s,p}(\mathbf{R}^n)}\le\frac{1}{(1-\delta)^p}|u^T|^p_{W^{s,p}(\mathbf{R}^n)}=\frac{1}{(1-\delta)^p}\mathrm{cap}_{s,p}(T),\quad \|w_\delta\|_{L^p(\mathbf{R}^n)}\le\frac{1}{1-\delta}\|u^T\|_{L^p(B_{R_\delta})}.$$

Moreover, $\tilde{w}_\delta\ge1$ q.e. on T. Since $\mathrm{cap}_{s,p}(\cdot)$ is an increasing set function, we infer

$$0\le C_{s,p}(T,R_\delta)-\mathrm{cap}_{s,p}(T)\le\left(\frac{1}{(1-\delta)^p}-1\right)\mathrm{cap}_{s,p}(T)\le\left(\frac{1}{(1-\delta)^p}-1\right)\mathrm{cap}_{s,p}(B_\rho).$$

In conclusion, (2.16) follows from the monotonicity properties of $r\to C_{s,p}(T,B_r)$.

To prove (2.17) take note that any admissible function u for the minimum problem defining $\mathrm{cap}_{s,p}(T,B_R;r)$ is admissible for the one defining $\mathrm{cap}_{s,p}(T)$, too. Furthermore, for some constant $c=c(n,s,p)$ it holds

$$\begin{aligned}\mathrm{cap}_{s,p}(T)&\le|u|^p_{W^{s,p}(\mathbf{R}^n)}\overset{(2.15)}{=}|u|^p_{W^{s,p}(B_R)}+|u|^p_{W^{s,p}(B_R^c)}+2\mathcal{D}_{s,p}(u,B_R\times B_R^c)\\&\overset{u|_{B_r^c}=0}{=}|u|^p_{W^{s,p}(B_R)}+2\int_{B_r}dx\int_{B_R^c}\frac{|u(x)|^p}{|x-y|^{n+sp}}dy\overset{(ii)\ \text{Lemma A.1}}{\le}|u|^p_{W^{s,p}(B_R)}+c\int_{B_r}\frac{|u(x)|^p}{\mathrm{dist}^{sp}(x,\partial B_R)}dx\\&\le|u|^p_{W^{s,p}(B_R)}+\frac{c}{(R-r)^{sp}}\int_{B_r}|u(x)|^pdx\le|u|^p_{W^{s,p}(B_R)}+\frac{c\,r^{sp}}{(R-r)^{sp}}|u|^p_{W^{s,p}(B_r)}.\end{aligned}$$

In the last inequality we used the scaled version of Poincarè inequality (2.7) as follows from Remark 2.7. By passing to the infimum on the admissible test functions we infer

$$\mathrm{cap}_{s,p}(T)-\mathrm{cap}_{s,p}(T,B_R;r)\le\frac{c\,r^{sp}}{(R-r)^{sp}}\mathrm{cap}_{s,p}(T,B_R;r)\le\frac{c\,r^{sp}}{(R-r)^{sp}}C_{s,p}(T,B_r).$$

We deduce statement (2.17) since $C_{s,p}(\cdot,B_r)$ is a monotone increasing set function.

Eventually, (2.18) follows at once from (2.16), (2.17), and the fact that $\mathrm{cap}_{s,p}(T,B_R;r)\le C_{s,p}(T,B_r)$.

□

Remark 2.13. *Clearly estimate (2.17) blows up for $r=R$. In such a case by using Hardy inequality one can only prove that $\mathrm{cap}_{s,p}(T)\le(1+c)\mathrm{cap}_{s,p}(T,B_R,R)$ for some $c=c(n,s,p)>0$.*

Remark 2.14. *If ξ_r is a $(1/r)$-minimizer for $C_{s,p}(T,B_r)$ and $R(r)/r\to+\infty$ as $r\to+\infty$, then*

$$\lim_{r\to+\infty}\mathcal{D}_{s,p}(\xi_r,B_{R(r)}\times B^c_{R(r)})=0.$$

Indeed, being ξ_r admissible for the problem defining $\mathrm{cap}_{s,p}(T,B_R;r)$, for all $r<R$, with

$$\mathrm{cap}_{s,p}(T,B_R;r)\le|\xi_r|^p_{W^{s,p}(B_R)}\le|\xi_r|^p_{W^{s,p}(\mathbf{R}^n)}\le C_{s,p}(T,B_r)+\frac{1}{r},$$

from (2.18) we conclude.

3. Deterministic Setting

3.1. **Statement of the Main Result.** Consider Delone sets $\Lambda_j=\{\mathrm{x}^{\mathtt{i}}_j\}_{\mathtt{i}\in\mathbf{Z}^n}$, and let $R_j:=R_{\Lambda_j}$, $\mathcal{I}_j(A):=\mathcal{I}_{\Lambda_j}(A)$, $\mathscr{I}_j(A):=\mathscr{I}_{\Lambda_j}(A)$, for all $A\in\mathcal{A}(U)$ (see (2.3) for the definition of $\mathcal{I}_j$, $\mathscr{I}_j$). Fix $r_j\in(0,r_{\Lambda_j}]$, and assume that the r_j's and Λ_j's are such that

$$\lim_j r_j=0,\quad (1\le)\limsup_j(R_j/r_j)<+\infty, \tag{3.1}$$

$$\lim_j \#\mathcal{I}_j(U)\,r_j^n=\theta\in(0,+\infty), \tag{3.2}$$

$$\mu_j:=\frac{1}{\#\mathcal{I}_j(U)}\sum_{\mathtt{i}\in\mathcal{I}_j(U)}\delta_{\mathrm{x}^{\mathtt{i}}_j}\to\mu:=\beta\mathcal{L}^n\llcorner U\qquad w^*\text{-}C_b(U), \tag{3.3}$$

for some $\beta\in L^1(U,[0,+\infty])$ with $\|\beta\|_{L^1(U)}=1$.

Remark 3.1. *Condition (3.2) implies that $r_j\sim r_{\Lambda_j}$ since $\limsup_j\#\mathcal{I}_j(U)\,r^n_{\Lambda_j}<+\infty$ by $(2.4)_1$ and $\liminf_j\#\mathcal{I}_j(U)\,r^n_{\Lambda_j}>0$ by $(2.4)_2$ (see also Remark 3.4).*

Remark 3.2. *It is well known that the w^*-$C_b(U)$ convergence of $(\mu_j)_{j\in\mathbf{N}}$ to μ in (3.3) can be restated as*

$$\mu_j(A)\to\mu(A)\qquad\textit{for all } A\in\mathcal{A}(U)\textit{ with }\mu(U\cap\partial A)=0. \tag{3.4}$$

Proposition 2.5 and conditions (3.1), (3.2) imply that assumption (3.3) is always satisfied up to subsequences. First, let us show that any w^*-$C^0_0(U)$ cluster point of the probability measures $(\mu_j)_{j\in\mathbf{N}}$ is absolutely continuous w.r.to $\mathcal{L}^n\llcorner U$. For, let μ_j converge to μ w^*-$C^0_0(U)$, then since $B_{r_j}(\mathrm{x}^{\mathtt{i}}_j)\subseteq V^{\mathtt{i}}_j$, for every $\varphi\in C^0_0(U)$ and $\delta>0$ uniform continuity yields for j sufficiently big

$$\int_U\varphi\,d\mu_j=\frac{1}{\#\mathcal{I}_j(U)}\sum_{\mathcal{I}_j(U)}\varphi(\mathrm{x}^{\mathtt{i}}_j)\le\frac{1}{\#\mathcal{I}_j(U)}\sum_{\mathcal{I}_j(U)}\fint_{V^{\mathtt{i}}_j}\varphi\,dx+\delta\le\frac{1}{\omega_n r_j^n\#\mathcal{I}_j(U)}\int_U|\varphi|\,dx+\delta.$$

By taking as test functions $\pm\varphi$ and first letting $j\to+\infty$ and then $\delta\to0^+$ we infer

$$\left|\int_U\varphi\,d\mu\right|\le\frac{1}{\omega_n\theta}\int_U|\varphi|\,dx.$$

The latter inequality implies $\mu\ll\mathcal{L}^n\llcorner U$. Actually, since $V^{\mathtt{i}}_j\subseteq\overline{B}_{R_j}(\mathrm{x}^{\mathtt{i}}_j)$ arguing as above it follows $\beta\in L^\infty(U)$ with $(\liminf_j r_j/R_j)^n\le\omega_n\theta\beta(x)\le1$ for $\mathcal{L}^n$ a.e. $x\in U$.

Furthermore, take A', $A\in\mathcal{A}(U)$ such that $A'\subset\subset A\subset\subset U$, then for j sufficiently big we have

$$\mu_j(\overline{A}) = \frac{\#\{\mathtt{i} \in \mathcal{I}_j(U) : \mathrm{x}_j^{\mathtt{i}} \in \overline{A}\}}{\#\mathcal{I}_j(U)} \geq \frac{\#\mathcal{I}_j(A') + \#\mathscr{I}_j(A')}{\#\mathcal{I}_j(U)}$$
$$= 1 - \frac{\#\mathcal{I}_j(U \setminus \overline{A'})}{\#\mathcal{I}_j(U)} \overset{(2.4)_1,(2.4)_2}{\geq} 1 - \frac{\mathcal{L}^n(U \setminus \overline{A'})}{\left(\frac{r_{\Lambda_j}}{R_{\Lambda_j}}\right)^n \mathcal{L}^n(U) - (\partial U)_{R_{\Lambda_j}}}.$$

Hence, equi-tightness of $(\mu_j)_{j\in\mathbf{N}}$ follows from

$$\liminf_j \mu_j(\overline{A}) \geq 1 - \limsup_j \left(\frac{R_{\Lambda_j}}{r_{\Lambda_j}}\right)^n \frac{\mathcal{L}^n(U \setminus \overline{A'})}{\mathcal{L}^n(U)}.$$

Thus, Prokhorov theorem gives the w^*-$C_b(U)$ compactness in (3.3) up to subsequences.

With fixed a bounded set T, for all $j \in \mathbf{N}$ define the *obstacle set* $T_j \subseteq \mathbf{R}^n$ by $T_j = \cup_{\mathtt{i}\in\mathbf{Z}^n} T_j^{\mathtt{i}}$ where

$$T_j^{\mathtt{i}} := \mathrm{x}_j^{\mathtt{i}} + \lambda_j T, \quad \text{ and } \lambda_j := r_j^{n/(n-sp)}. \tag{3.5}$$

Take note that $T_j^{\mathtt{i}} \subseteq V_j^{\mathtt{i}}$ for all $\mathtt{i} \in \mathbf{Z}^n$ and $j \in \mathbf{N}$.

Consider the functionals $\mathcal{F}_j : L^p(U) \to [0, +\infty]$ defined by

$$\mathcal{F}_j(u) = \begin{cases} |u|^p_{W^{s,p}(U)} & \text{if } u \in W^{s,p}(U),\ \tilde{u} = 0 \text{ cap}_{s,p} \text{ q.e. on } T_j \cap U \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Theorem 3.3. *Let $U \in \mathcal{A}(\mathbf{R}^n)$ be bounded and connected with Lipschitz regular boundary; and assume that (3.1)-(3.3) are satisfied.*

The sequence $(\mathcal{F}_j)_{j\in\mathbf{N}}$ Γ-converges in the $L^p(U)$ topology to $\mathcal{F} : L^p(U) \to [0, +\infty]$ defined by

$$\mathcal{F}(u) = |u|^p_{W^{s,p}(U)} + \theta \,\mathrm{cap}_{s,p}(T) \int_U |u(x)|^p \beta(x)\, dx \tag{3.7}$$

if $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$.

Remark 3.4. *If $\theta \in \{0, +\infty\}$ simple comparison arguments show that the conclusions of Theorem 3.3 still hold true, though the Γ-limit is trivial in both cases.*

Let us show some examples of sets of points included in the framework above. In the sequel $(\varepsilon_j)_{j\in\mathbf{N}}$ will always denote a positive infinitesimal sequence.

Example 3.5. *Given a Delone set of points Λ in $\mathbf{R}^n$ let $\Lambda_j := \varepsilon_j \Lambda$, then $r_{\Lambda_j} = \varepsilon_j r_\Lambda$ and $R_{\Lambda_j} = \varepsilon_j R_\Lambda$. If $r_j \sim r_{\Lambda_j}$ assumptions (3.2) and (3.3) hold true up to the extraction of a subsequence according to Remark 3.2, respectively. Several ways of generating Delone sets of points are discussed in [42].*

More explicit examples can be obtained as follows (see Examples 4.9, 4.10 for the stochastic versions).

Example 3.6. *Let $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism satisfying*

$$\|\nabla\Phi\|_{L^\infty(\mathbf{R}^n)} \leq M, \quad \textit{and} \quad \inf_{\mathbf{R}^n} \det \nabla\Phi \geq \nu > 0.$$

Then the smallest eigenvalue of $\nabla^t\Phi\nabla\Phi$ is greater than νM^{1-n}, and thus for all $x, y \in \mathbf{R}^n$

$$\nu M^{1-n}|x-y| \leq |\Phi(x) - \Phi(y)| \leq M|x-y|.$$

Set $\Lambda_j = \{\Phi(\varepsilon_j \mathtt{i})\}_{\mathtt{i}\in\mathbf{Z}^n}$, then $(\nu M^{1-n}/2)\varepsilon_j \leq r_{\Lambda_j} \leq R_{\Lambda_j} \leq M\varepsilon_j$. An easy computation and (3.4) yield the w^*-$C_b(U)$ convergence of the measures μ_j in (3.3) to $\mu = \beta\mathcal{L}^n \llcorner U$ with

$$\beta(x) = \left(\int_U \det \nabla\Phi^{-1}(x)\,dx\right)^{-1} \det \nabla\Phi^{-1}(x).$$

Eventually, if $r_j \in (0, r_{\Lambda_j}]$ with $r_j/\varepsilon_j \to \gamma > 0$ we have $\theta = \gamma^n \int_U \det \nabla\Phi^{-1}(x)\,dx$.

Example 3.7. *Let Φ be a diffeomorphism as in the previous example, and set $\Lambda_j = \{\varepsilon_j\Phi(\mathtt{i})\}_{\mathtt{i}\in\mathbf{Z}^n}$. As before, we have $(\nu M^{1-d}/2)\varepsilon_j \leq r_{\Lambda_j} \leq R_{\Lambda_j} \leq M\varepsilon_j$, by using (3.4) it can be checked that if $(\det \nabla\Phi^{-1})(\cdot/\varepsilon_j)_{j\in\mathbf{N}}$ converge to g weakly $*$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$, the measures μ_j in (3.3) converge w^*-$C_b(U)$ to $\mu = \beta\mathcal{L}^n \llcorner U$ with*

$$\beta(x) = \left(\int_U g(x)\,dx\right)^{-1} g(x).$$

By choosing $r_j \in (0, r_{\Lambda_j}]$ with $r_j/\varepsilon_j \to \gamma > 0$, we have $\theta = \gamma^n \int_U g(x)\,dx$.

3.2. **Technical Lemmas.** To prove Theorem 3.3 we establish two technical results which are instrumental for our strategy. Roughly speaking we show that the Γ-limit can be computed on sequences of functions matching the values of their limit on suitable annuli surrounding the obstacle sets. To give a proof as much clear as possible we first work in an unscaled setting in Lemma 3.8, and then turn to the framework of interest in Lemma 3.9. The method of proof is elementary and based on a clever slicing and averaging argument, looking for those zones where the energy does not concentrate. The relevant property we prove is that the energetic error of the construction we perform is estimated by a local term: a measure.

Lemma 3.8. *Let Λ be a Delone set of points. For any $m \in \mathbf{N}$, $m \geq 2$, $\rho \in (0, r_\Lambda/2)$ and $\mathtt{i} \in \mathcal{I}_\Lambda(U)$ let $A'_\mathtt{i} = \mathrm{x}^\mathtt{i} + B_{\rho/m} \setminus \overline{B}_{\rho/m^2}$, $A_\mathtt{i} = \mathrm{x}^\mathtt{i} + B_\rho \setminus \overline{B}_{\rho/m^3}$, and $\varphi_\mathtt{i}(\cdot) = \varphi(\cdot - \mathrm{x}^\mathtt{i})$, where φ is a cut-off function between $B_{\rho/m} \setminus \overline{B}_{\rho/m^2}$ and $B_\rho \setminus \overline{B}_{\rho/m^3}$.*

Then there exists a constant $c = c(n, p, s) > 0$ such that for any function $u \in W^{s,p}(U)$, and any $\#\mathcal{I}_\Lambda(U)$-tuple of vectors $\{z_\mathtt{i}\}_{\mathtt{i}\in\mathcal{I}_\Lambda(U)}$, $z_\mathtt{i} \in \mathbf{R}^n$, the function

$$w(x) = \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \varphi_\mathtt{i}(x) z_\mathtt{i} + \left(1 - \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \varphi_\mathtt{i}(x)\right) u(x)$$

belongs to $W^{s,p}(U)$, $u = z_\mathtt{i}$ on $A'_\mathtt{i}$ and $w = u$ on $U \setminus \overline{A}$, with $A := \cup_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} A_\mathtt{i}$; in addition for every measurable set $E \subseteq U \times U$ it holds

$$|\mathcal{D}_{s,p}(w, E) - \mathcal{D}_{s,p}(u, E)| \leq c \left(\mathcal{D}_{s,p}(u, U \times A) + m^{2p}\rho^{-sp} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_\mathtt{i}} |u(y) - z_\mathtt{i}|^p\,dy\right). \quad (3.8)$$

Proof. By construction $w = u$ on $U \setminus \overline{A}$ and $u = z_{\mathtt{i}}$ on $A'_{\mathtt{i}}$ for each i.

To prove that $w \in W^{s,p}(U)$ and estimate (3.8) in case $E = U \times U$ we use (2.15) to get

$$|w|^p_{W^{s,p}(U)} = |u|^p_{W^{s,p}(U\setminus\overline{A})} + |w|^p_{W^{s,p}(A)} + 2\mathcal{D}_{s,p}(w, A \times (U \setminus \overline{A})). \tag{3.9}$$

In order to control the last two terms on the rhs above we will use two different splitting of the oscillation $w(x) - w(y)$ corresponding roughly to short range and long range interactions estimates.

First decompose further the seminorm of w on A in (3.9) as follows,

$$|w|^p_{W^{s,p}(A)} = \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_{\mathtt{i}}\times A_{\mathtt{i}}} \dots dxdy + \sum_{\{(\mathtt{i},\mathtt{k}):\, \mathtt{i}\neq\mathtt{k}\}} \int_{A_{\mathtt{i}}\times A_{\mathtt{k}}} \dots dxdy =: I_1 + I_2. \tag{3.10}$$

Next we deal with I_1: since $\varphi_{\mathtt{l}} \equiv 0$ on $A_{\mathtt{i}}$ for $\mathtt{l} \neq \mathtt{i}$ and $0 \leq \varphi_{\mathtt{i}} \leq 1$ we have

$$\begin{aligned} I_1 &= \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_{\mathtt{i}}\times A_{\mathtt{i}}} \frac{|(1-\varphi_{\mathtt{i}}(x))u(x) - (1-\varphi_{\mathtt{i}}(y))u(y) + (\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y))z_{\mathtt{i}}|^p}{|x-y|^{n+sp}} dxdy \\ &\overset{\pm(1-\varphi_{\mathtt{i}}(x))u(y)}{\leq} 2^{p-1} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \left(|u|^p_{W^{s,p}(A_{\mathtt{i}})} + \int_{A_{\mathtt{i}}\times A_{\mathtt{i}}} \frac{|(\varphi_{\mathtt{i}}(y) - \varphi_{\mathtt{i}}(x))(u(y) - z_{\mathtt{i}})|^p}{|x-y|^{n+sp}} dxdy \right). \end{aligned}$$

Take note that $\mathrm{Lip}(\varphi_{\mathtt{i}}) \leq 2m^2/\rho$ for all $\mathtt{i} \in \mathcal{I}_\Lambda(U)$, then Fubini theorem and (i) in Lemma A.1 applied with $\nu = n + (s-1)p$ and $O = A_{\mathtt{i}}$ imply

$$\begin{aligned} I_1 &\leq 2^{p-1} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \left(|u|^p_{W^{s,p}(A_{\mathtt{i}})} + \left(\frac{2m^2}{\rho}\right)^p \int_{A_{\mathtt{i}}\times A_{\mathtt{i}}} \frac{|u(y) - z_{\mathtt{i}}|^p}{|x-y|^{n+(s-1)p}} dxdy \right) \\ &\leq c \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \left(|u|^p_{W^{s,p}(A_{\mathtt{i}})} + m^{2p}\rho^{-sp} \int_{A_{\mathtt{i}}} |u(y) - z_{\mathtt{i}}|^p dy \right). \end{aligned} \tag{3.11}$$

To estimate I_2 we rewrite w as follows

$$w(x) - w(y) = u(x) - u(y) + \sum_{\mathtt{l}\in\mathcal{I}_\Lambda(U)} \varphi_{\mathtt{l}}(x)(z_{\mathtt{l}} - u(x)) + \sum_{\mathtt{l}\in\mathcal{I}_\Lambda(U)} \varphi_{\mathtt{l}}(y)(u(y) - z_{\mathtt{l}}). \tag{3.12}$$

With the help of (3.12) and since $\varphi_{\mathtt{l}} \equiv 0$ on $A \setminus A_{\mathtt{l}}$ we infer

$$I_2 = \sum_{\{(\mathtt{i},\mathtt{k}):\, \mathtt{i}\neq\mathtt{k}\}} \int_{A_{\mathtt{i}}\times A_{\mathtt{k}}} \frac{|u(x) - u(y) + \varphi_{\mathtt{i}}(x)(z_{\mathtt{i}} - u(x)) + \varphi_{\mathtt{k}}(y)(u(y) - z_{\mathtt{k}})|^p}{|x-y|^{n+sp}} dxdy.$$

Thus, we can bound each summand in I_2 as follows

$$\begin{aligned} \int_{A_{\mathtt{i}}\times A_{\mathtt{k}}} \frac{|w(x) - w(y)|^p}{|x-y|^{n+sp}} dxdy &\leq 3^{p-1}\mathcal{D}_{s,p}(u, A_{\mathtt{i}} \times A_{\mathtt{k}}) \\ +3^{p-1} \int_{A_{\mathtt{i}}} dx \int_{A_{\mathtt{k}}} \frac{|\varphi_{\mathtt{i}}(x)(u(x) - z_{\mathtt{i}})|^p}{|x-y|^{n+sp}} dy &+ 3^{p-1} \int_{A_{\mathtt{k}}} dy \int_{A_{\mathtt{i}}} \frac{|\varphi_{\mathtt{k}}(y)(u(y) - z_{\mathtt{k}})|^p}{|x-y|^{n+sp}} dx. \end{aligned} \tag{3.13}$$

Further, take note that for every fixed $(\mathtt{i},\mathtt{k})$, with $\mathtt{i} \neq \mathtt{k}$, if $x \in A_{\mathtt{i}}$ and $y \in A_{\mathtt{k}}$ we have $|x-y| \geq \mathrm{dist}(A_{\mathtt{i}}, A_{\mathtt{k}}) \geq |x^{\mathtt{i}} - x^{\mathtt{k}}| - r_\Lambda$, and then $|x-y| \geq |x^{\mathtt{i}} - x^{\mathtt{k}}|/2$ since $|x^{\mathtt{i}} - x^{\mathtt{k}}| \geq 2r_\Lambda$. Hence, being

$\mathcal{L}^n(A_{\mathtt{k}}) \leq \omega_n \rho^n$ for all $\mathtt{k} \in \mathbf{Z}^n$, from (2.5) and the choice $\rho \in (0, r_\Lambda/2)$ it follows for some constant $c = c(n) > 0$

$$\sum_{\{\mathtt{k}\colon \mathtt{k}\neq\mathtt{i}\}} \int_{A_{\mathtt{k}}} \frac{1}{|x-y|^{n+sp}} dy \leq c \sum_{\{\mathtt{k}\colon \mathtt{k}\neq\mathtt{i}\}} \frac{\rho^n}{|x^{\mathtt{i}} - x^{\mathtt{k}}|^{n+sp}} \tag{3.14}$$
$$\leq c \sum_{h\geq 1} \sum_{\{\mathtt{k}\neq\mathtt{i},\, hr_\Lambda < |x^{\mathtt{i}}-x^{\mathtt{k}}|_\infty \leq (h+1)r_\Lambda\}} \frac{\rho^n}{(hr_\Lambda)^{n+sp}} \leq \frac{c}{\rho^{sp}} \sum_{h\geq 1} \frac{1}{h^{1+sp}}.$$

By summing up on $(\mathtt{i},\mathtt{k})$, $\mathtt{i} \neq \mathtt{k}$, Fubini theorem, (3.13) and (3.14) imply for some $c = c(n,p,s) > 0$

$$I_2 \leq 3^{p-1} \mathcal{D}_{s,p}(u, \cup_{\mathtt{i}}(A_{\mathtt{i}} \times \cup_{\mathtt{k}\neq\mathtt{i}} A_{\mathtt{k}})) + \frac{c}{\rho^{sp}} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_{\mathtt{i}}} |u(x) - z_{\mathtt{i}}|^p dx, \tag{3.15}$$

We turn now to the locality defect term in (3.9). We use again equality (3.12) and notice that $\varphi_{\mathtt{i}} \equiv 0$ on $U \setminus \overline{A}_{\mathtt{i}}$ to infer

$$\mathcal{D}_{s,p}(w, A\times(U\setminus\overline{A})) \leq 2^{p-1}\mathcal{D}_{s,p}(u, A\times(U\setminus\overline{A})) + 2^{p-1} \int_{A\times(U\setminus\overline{A})} \frac{\left|\sum_{\mathcal{I}_\Lambda(U)} \varphi_{\mathtt{i}}(x)(u(x)-z_{\mathtt{i}})\right|^p}{|x-y|^{n+sp}} dxdy$$
$$= 2^{p-1}\mathcal{D}_{s,p}(u, A\times(U\setminus\overline{A})) + 2^{p-1} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_{\mathtt{i}}\times(U\setminus\overline{A})} \frac{|\varphi_{\mathtt{i}}(x)(u(x)-z_{\mathtt{i}})|^p}{|x-y|^{n+sp}} dxdy. \tag{3.16}$$

We fix $\mathtt{i} \in \mathbf{Z}^n$ and define $\Delta'_{\mathtt{i}} = (A_{\mathtt{i}} \times (U\setminus\overline{A})) \cap \Delta_\rho$, then by using $|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \leq 2m^2|x-y|/\rho$, with $\varphi_{\mathtt{i}}(y) = 0$ for $y \in U\setminus\overline{A}$, Fubini theorem and a direct integration yield

$$\int_{\Delta'_{\mathtt{i}}} \frac{|\varphi_{\mathtt{i}}(x)(u(x)-z_{\mathtt{i}})|^p}{|x-y|^{n+sp}} dxdy \leq (2m^2)^p\rho^{-p} \int_{\Delta'_{\mathtt{i}}} \frac{|u(x)-z_{\mathtt{i}}|^p}{|x-y|^{n+(s-1)p}} dxdy \tag{3.17}$$
$$\leq (2m^2)^p\rho^{-p} \int_{A_{\mathtt{i}}} dx \int_{B_\rho(x)} \frac{|u(x)-z_{\mathtt{i}}|^p}{|x-y|^{n+(s-1)p}} dy = \frac{n\omega_n(2m^2)^p}{p(1-s)} \rho^{-sp} \int_{A_{\mathtt{i}}} |u(x)-z_{\mathtt{i}}|^p dx.$$

Let now $\Delta''_{\mathtt{i}} = (A_{\mathtt{i}} \times (U\setminus\overline{A})) \setminus \Delta_\rho$, then we argue as above using $|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \leq 1$ to get again by a direct integration

$$\int_{\Delta''_{\mathtt{i}}} \frac{|\varphi_{\mathtt{i}}(x)(u(x)-z_{\mathtt{i}})|^p}{|x-y|^{n+sp}} dxdy \leq \int_{\Delta''_{\mathtt{i}}} \frac{|u(x)-z_{\mathtt{i}}|^p}{|x-y|^{n+sp}} dxdy$$
$$\leq \int_{A_{\mathtt{i}}} dx \int_{\mathbf{R}^n\setminus B_\rho(x)} \frac{|u(x)-z_{\mathtt{i}}|^p}{|x-y|^{n+sp}} dy = \frac{n\omega_n}{sp} \rho^{-sp} \int_{A_{\mathtt{i}}} |u(x)-z_{\mathtt{i}}|^p dx. \tag{3.18}$$

By taking into account (3.16)-(3.18) we deduce

$$\mathcal{D}_{s,p}(w, A\times(U\setminus\overline{A})) \leq 2^{p-1}\mathcal{D}_{s,p}(u, A\times(U\setminus\overline{A})) + c\, m^{2p}\rho^{-sp} \sum_{\mathtt{i}\in\mathcal{I}_\Lambda(U)} \int_{A_{\mathtt{i}}} |u(x)-z_{\mathtt{i}}|^p dx. \tag{3.19}$$

By collecting (3.9), (3.11), (3.15), and (3.19) we infer $w \in W^{s,p}(U)$ and estimate (3.8) for $U \times U$.

For any measurable subset E in $U \times U$, being $w = u$ on $U \setminus \overline{A}$, we have

$$|\mathcal{D}_{s,p}(w,E) - \mathcal{D}_{s,p}(u,E)| = |\mathcal{D}_{s,p}(w, E\cap(U\times A)) - \mathcal{D}_{s,p}(u, E\cap(U\times A))|$$
$$\leq |w|^p_{W^{s,p}(A)} + \mathcal{D}_{s,p}(w, (U\setminus A)\times A) + \mathcal{D}_{s,p}(u, U\times A).$$

Eventually, (3.8) follows at once by taking into account (3.10)-(3.19). □

By scaling Lemma 3.8 we establish a joining lemma for fractional type energies.

Before starting the proof we fix some notation: having fixed $m \in \mathbf{N}$ and set $\mathcal{I}_j := \mathcal{I}_{\Lambda_j}(U)$, for all $\mathtt{i} \in \mathcal{I}_j$ and $h \in \mathbf{N}$ let

$$B_j^{\mathtt{i},h} = \{x \in \mathbf{R}^n : |x - \mathrm{x}_j^{\mathtt{i}}| < m^{-3h} r_j\}, \quad C_j^{\mathtt{i},h} := \{x \in \mathbf{R}^n : m^{-3h-2} r_j < |x - \mathrm{x}_j^{\mathtt{i}}| < m^{-3h-1} r_j\}.$$

Clearly, we have $C_j^{\mathtt{i},h} \subset B_j^{\mathtt{i},h} \setminus \overline{B}_j^{i,h+1} \subset V_j^{\mathtt{i}}$.

Lemma 3.9. *Let $(u_j)_{j\in\mathbf{N}}$ be converging to u in $L^p(U)$ with $\sup_j |u_j|_{W^{s,p}(U)} < +\infty$. With fixed m, $N \in \mathbf{N}$, for every $j \in \mathbf{N}$ there exists $h_j \in \{1, \dots, N\}$ and a function $w_j \in W^{s,p}(U)$ such that*

$$w_j \equiv u_j \ \text{ on } U \setminus \cup_{\mathtt{i}\in\mathcal{I}_j} (\overline{B}_j^{i,h_j} \setminus B_j^{i,h_j+1}), \tag{3.20}$$

$$w_j(x) \equiv (u_j)_{C_j^{\mathtt{i},h_j}} \ \text{ on } C_j^{\mathtt{i},h_j}, \tag{3.21}$$

for some $c = c(n,p,s,m) > 0$ it holds for every measurable set E in $U \times U$

$$|\mathcal{D}_{s,p}(u_j, E) - \mathcal{D}_{s,p}(w_j, E)| \le \frac{c}{N} |u_j|^p_{W^{s,p}(U)}, \tag{3.22}$$

and the sequences $(w_j)_{j\in\mathbf{N}}$, $(\zeta_j)_{j\in\mathbf{N}}$, with $\zeta_j := \sum_{\mathtt{i}\in\mathcal{I}_j(U)} (u_j)_{C_j^{\mathtt{i},h_j}} \chi_{V_j^{\mathtt{i}}}$, converge to u in $L^p(U)$.

In addition, if $u_j \in L^\infty(U)$

$$\|w_j\|_{L^\infty(U)} \le \|u_j\|_{L^\infty(U)}. \tag{3.23}$$

Proof. Given m, $N \in \mathbf{N}$, then for every $j \in \mathbf{N}$ and $h \in \{1, \dots, N\}$ fixed, apply Lemma 3.8 with $(A')^h_{\mathtt{i}} := C_j^{\mathtt{i},h}$, $A^h_{\mathtt{i}} := B_j^{\mathtt{i},h} \setminus \overline{B}_j^{i,h+1}$, $z_{\mathtt{i}} = (u_j)_{C_j^{\mathtt{i},h}}$, $\mathtt{i} \in \mathcal{I}_j$. Take note that $\rho = m^{-3h} r_j$. If $w_j^{i,h}$ denotes the resulting function and $A^h = \cup_{\mathtt{i}\in\mathcal{I}_j} A^h_{\mathtt{i}}$, then for some constant $c = c(n,p,s)$ and for any measurable set E in $U \times U$ by (3.8) it holds

$$|\mathcal{D}_{s,p}(u_j, E) - \mathcal{D}_{s,p}(w_j, E)| \le c\,\mathcal{D}_{s,p}(u_j, U \times A^h) + c\, m^{2p} \left(\frac{m^{3h}}{r_j}\right)^{ps} \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{A^h_{\mathtt{i}}} |u_j - (u_j)_{C_j^{\mathtt{i},h}}|^p dx.$$

This estimate, together with the scaled Poincarè-Wirtinger inequality (2.8) with $r = m^{-3h} r_j$, gives

$$|\mathcal{D}_{s,p}(u_j, E) - \mathcal{D}_{s,p}(w_j, E)| \le c \left(\mathcal{D}_{s,p}(u_j, U \times A^h) + |u_j|^p_{W^{s,p}(A^h)}\right) \le c\,\mathcal{D}_{s,p}(u_j, U \times A^h), \tag{3.24}$$

fro some $c = c(n,p,s,m) > 0$. By summing up and averaging on h, being the A^h's disjoint, we find $h_j \in \{1, \dots, N\}$ such that

$$\mathcal{D}_{s,p}(u_j, U \times A^{h_j}) \le \frac{1}{N} \mathcal{D}_{s,p}(u_j, U \times \cup_h A^h). \tag{3.25}$$

Set $w_j := w_j^{i,h_j}$, then (3.20) and (3.21) are satisfied by construction, and moreover (3.24) and (3.25) imply (3.22).

To prove that $(w_j)_{j\in\mathbf{N}}$ converges to u in $L^p(U)$ we use (2.8), with $r = m^{-3h} r_j$, and the very definition of w_j as convex combination of u_j and the mean value $(u_j)_{C_j^{i,h_j}}$ on $B_j^{\mathtt{i},h_j} \setminus \overline{B}_j^{i,h_j+1}$ to get

$$\|u_j - w_j\|^p_{L^p(U)} = \|u_j - w_j\|^p_{L^p(A^{h_j})} = \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - w_j\|^p_{L^p(B_j^{\mathtt{i},h_j} \setminus B_j^{i,h_j+1})}$$

$$\leq \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{i,h_j}}\|^p_{L^p(B_j^{\mathtt{i},h_j}\setminus B_j^{i,h_j+1})} \leq c\left(\frac{r_j}{m^{3h_j}}\right)^{ps} \sum_{\mathtt{i}\in\mathcal{I}_j} |u_j|^p_{W^{s,p}(B_j^{\mathtt{i},h_j}\setminus B_j^{i,h_j+1})} \leq c r_j^{ps}|u_j|^p_{W^{s,p}(U)},$$

where $c = c(n,p,s,m) > 0$.

Eventually, let us show the convergence of $(\zeta_j)_{j\in\mathbf{N}}$ to u in $L^p(U)$. To this aim we prove that $(\zeta_j - u_j)_{j\in\mathbf{N}}$ is infinitesimal in $L^p(U)$. Fix any number M with $\sup_j R_j/r_j < M < +\infty$ (see (3.1)), we claim that for some constant $c = c(n,p,s,m,M,N) > 0$ we have

$$\sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} \leq c\, r_j^{sp}|u_j|^p_{W^{s,p}(U)}. \tag{3.26}$$

Given this for granted the conclusion is a straightforward consequence of the definition of ζ_j, of (3.26), of $(2.4)_3$ and of the equi-integrability of $(|u_j|^p)_{j\in\mathbf{N}}$, i.e.

$$\|\zeta_j - u_j\|^p_{L^p(U)} = \sum_{\mathtt{i}\in\mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} + \|u_j\|^p_{L^p(U\setminus\cup_{\mathcal{I}_j}V_j^{\mathtt{i}})}.$$

To prove (3.26) we use (2.8) and the fact that the balls $B_{Mr_j}(\mathrm{x}_j^{\mathtt{i}})$ have (uniformly) finite overlapping. More precisely, the inclusions $V_j^{\mathtt{i}} \subseteq \overline{B}_{R_j}(\mathrm{x}_j^{\mathtt{i}}) \subseteq B_{Mr_j}(\mathrm{x}_j^{\mathtt{i}})$ and (2.8) applied with $r = r_j$ give for some $c = c(n,p,s,m,M,N) > 0$

$$\|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(V_j^{\mathtt{i}})} \leq \|u_j - (u_j)_{C_j^{\mathtt{i},h_j}}\|^p_{L^p(B_{Mr_j}(\mathrm{x}_j^{\mathtt{i}}))} \leq c\, r_j^{ps}|u_j|^p_{W^{s,p}(B_{Mr_j}(\mathrm{x}_j^{\mathtt{i}}))}.$$

Moreover, since Λ_j is a Delone set and by definition $r_j \leq r_{\Lambda_j}$, an elementary counting argument implies $\sup_{\mathtt{i}\in\mathbf{Z}^n} \#\{\mathtt{k}\in\mathbf{Z}^n : B_{Mr_j}(\mathrm{x}_j^{\mathtt{i}}) \cap B_{Mr_j}(\mathrm{x}_j^{\mathtt{k}}) \neq \emptyset\} \leq (2M+1)^n$ for all $j \in \mathbf{N}$.

Finally, (3.23) follows by construction. □

3.3. **Proof of the Γ-convergence.** We establish the Γ-convergence result in Theorem 3.3. It will be a consequence of Propositions 3.10, 3.11, below. We start with the lower bound inequality.

Proposition 3.10. *For every $u_j \to u$ in $L^p(U)$ we have*

$$\liminf_j F_j(u_j) \geq \mathcal{F}(u).$$

Proof. Fix $N \in \mathbf{N}$, $\delta > 0$, and set $m = \lfloor 1/\delta \rfloor \in \mathbf{N}$, $\lfloor\cdot\rfloor$ denoting the integer part function. Consider the sequence $(w_j)_{j\in\mathbf{N}}$ provided by Lemma 3.9. We do not highlight its dependence on δ, N for the sake of notational convenience. We remark that whatever the choice of δ and N is, we have that $(w_j)_{j\in\mathbf{N}}$ converges to u in $L^p(U)$ and that for some $c = c(n,p,s,\delta)$ it holds

$$\left(1 + \frac{c}{N}\right) \liminf_j F_j(u_j) \geq \liminf_j F_j(w_j). \tag{3.27}$$

Furthermore, we note that for j sufficiently big $\cup_{\mathtt{i}\in\mathcal{I}_j}(V_j^{\mathtt{i}} \times V_j^{\mathtt{i}}) \subseteq \Delta_\delta$, and thus

$$\liminf_j F_j(w_j) \geq \liminf_j \left(\int_{U\times U\setminus\Delta_\delta} \frac{|w_j(x) - w_j(y)|^p}{|x-y|^{n+sp}} dx\, dy + \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} \right)$$

$$\geq \int_{U\times U\setminus\Delta_\delta} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy + \liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})}, \tag{3.28}$$

thanks to Fatou lemma. We claim that

$$\liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} \geq \theta\,(\mathrm{cap}_{s,p}(T)-\epsilon_\delta)\int_U |u(x)|^p\beta(x)dx, \tag{3.29}$$

with $\epsilon_\delta > 0$ infinitesimal as $\delta \to 0^+$. Given this for granted, by (3.27) inequality (3.28) rewrites as

$$\left(1+\frac{c}{N}\right)\liminf_j F_j(u_j) \geq \int_{U\times U\setminus\Delta_\delta} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy + \theta\,(\mathrm{cap}_{s,p}(T)-\epsilon_\delta)\int_U |u(x)|^p\beta(x)dx. \tag{3.30}$$

The thesis then follows by passing to the limit first as $N \to +\infty$ and then as $\delta \to 0^+$ in (3.30).

To conclude we are left with proving (3.29). We keep the notation of Lemma 3.9, and further set $B_j^{\mathtt{i}} := \{x \in \mathbf{R}^n : |x - \mathrm{x}_j^{\mathtt{i}}| < m^{-(3h_j+1)}r_j\}$, for all $\mathtt{i} \in \mathcal{I}_j$. Take note that $B_j^{\mathtt{i}} \subseteq V_j^{\mathtt{i}}$. We have

$$|w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} \geq \inf\left\{|w|^p_{W^{s,p}(B_j^{\mathtt{i}})} : w \in W^{s,p}(\mathbf{R}^n),\ w = (u_j)_{C_j^{\mathtt{i},h_j}} \text{ on } C_j^{\mathtt{i},h_j},\ \tilde{w} = 0 \text{ q.e. on } T_j^{\mathtt{i}}\right\} \tag{3.31}$$

$$= \inf\left\{|w|^p_{W^{s,p}(B_j^{\mathtt{i}})} : w \in W^{s,p}(\mathbf{R}^n),\ w = 0 \text{ on } C_j^{\mathtt{i},h_j},\ \tilde{w} = (u_j)_{C_j^{\mathtt{i},h_j}} \text{ q.e. on } T_j^{\mathtt{i}}\right\}$$

$$= |(u_j)_{C_j^{\mathtt{i},h_j}}|^p \mathrm{cap}_{s,p}\left(T_j^{\mathtt{i}}, B_j^{\mathtt{i}}; \frac{r_j}{m^{3h_j+2}}\right) = \lambda_j^{n-ps}|(u_j)_{C_j^{\mathtt{i},h_j}}|^p \mathrm{cap}_{s,p}\left(T, B_{\frac{r_j}{m^{3h_j+1}\lambda_j}}; \frac{r_j}{m^{3h_j+2}\lambda_j}\right).$$

The last equality is justified by an elementary translation and scaling argument. Thanks to (2.17) in Lemma 2.12, by recalling that $h_j \in \{1,\dots,N\}$, we get the following estimate

$$\mathrm{cap}_{s,p}\left(T, B_{\frac{r_j}{m^{3h_j+1}\lambda_j}}; \frac{r_j}{m^{3h_j+2}\lambda_j}\right) \geq \mathrm{cap}_{s,p}(T) - \frac{c}{(m-1)^{sp}} C_{s,p}\left(B_1, B_{\frac{r_j}{m^{3h_j+2}\lambda_j}}\right).$$

Hence, if $A \in \mathcal{A}(U)$ is such that $A \subset\subset U$, for j sufficiently big we infer

$$\sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} \geq \left(\mathrm{cap}_{s,p}(T) - \frac{c}{(m-1)^{sp}} C_{s,p}\left(B_1, B_{\frac{r_j}{m^{3h_j+2}\lambda_j}}\right)\right)\int_A |\zeta_j(x)|^p \Psi_j(x)dx,$$

where $\Psi_j(x) := \sum_{\mathtt{i}\in\mathcal{I}_j} \lambda_j^{n-sp}(\mathcal{L}^n(V_j^{\mathtt{i}}))^{-1}\chi_{V_j^{\mathtt{i}}}(x)$ and ζ_j is defined in Lemma 3.9. Take note that by $(2.4)_2$ we have

$$\left|\int_{A'} \Psi_j(x)dx - \lambda_j^{n-sp}\#(\mathcal{I}_j(A'))\right| \leq \lambda_j^{n-sp}\#(\mathscr{I}_j(A')) \leq \omega_n^{-1}\mathcal{L}^n((\partial A')_{R_j})$$

for any $A' \in \mathcal{A}(U)$. Testing the inequality above for all cubes in U with sides parallel to the coordinate axes, centers and vertices with rational coordinates yields $\Psi_j \to \theta\,\beta$ weak* $L^\infty(U)$ by (3.2) and (3.3).

By taking this into account, the thesis follows at once by the convergence of relative capacities to the global one proved in (2.16) of Lemma 2.12, the strong convergence of $(\zeta_j)_{j\in\mathbf{N}}$ to u in $L^p(U)$ established in Lemma 3.9, and eventually by letting A increase to U. □

In the next proposition we prove that the lower bound established in Proposition 3.10 is attained. In doing that the main difficulty is to show that the inequalities in (3.28) are optimal. Due to the insight provided by Proposition 3.10 we show that the capacitary contribution is concentrated along the diagonal set Δ and is due to short range interactions. Instead, long range interactions are responsible for the non-local term in the limit.

Proposition 3.11. *For every $u \in L^p(U)$ there exists a sequence $(u_j)_{j\in\mathbf{N}}$ such that $u_j \to u$ in $L^p(U)$ and*

$$\limsup_j F_j(u_j) \leq \mathcal{F}(u).$$

Proof. We may assume $u \in W^{1,\infty}(U)$ by a standard density argument and the lower semicontinuity of Γ- $\limsup \mathcal{F}_j$.

With fixed $N \in \mathbf{N}$, let $\xi_N \in W^{s,p}(\mathbf{R}^n)$ be such that $\xi_N = 0$ on $\mathbf{R}^n \setminus \overline{B}_N$, $\tilde{\xi}_N \geq 1$ $\mathrm{cap}_{s,p}$ q.e. on T and $|\xi_N|^p_{W^{s,p}(\mathbf{R}^n)} \leq C_{s,p}(T, B_N) + 1/N$, and let $\zeta \in C_0^\infty(B_N)$ be any function such that $\zeta \equiv 1$ on B_{r_Λ}, $(\mathrm{Lip}\zeta)^p \leq 2$ and $0 \leq \zeta \leq 1$.

Let $(w_j)_{j\in\mathbf{N}}$ be the sequence obtained from u by applying Lemma 3.9 with $m = 2$. We keep the notation introduced there and further set

$$\begin{aligned}
&B_j^{\mathtt{i}} := \{x \in \mathbf{R}^n : |x - \mathrm{x}_j^{\mathtt{i}}| < 2^{-3h_j} r_j\}, \;\; u_j^{\mathtt{i}} := u_{C_j^{\mathtt{i},h_j}} \quad \text{for every } \mathtt{i} \in \mathcal{I}_j, \\
&\hat{B}_j^{\mathtt{i}} := B_{N\lambda_j}(\mathrm{x}_j^{\mathtt{i}}) \cap U \quad \text{for every } \mathtt{i} \in \mathscr{I}_j, \; \mathscr{I}_j := \mathscr{I}_{\Lambda_j}(U), \\
&U_j := U \setminus \left(\left(\cup_{\mathcal{I}_j} B_j^{\mathtt{i}} \right) \cup \left(\cup_{\mathscr{I}_j} \hat{B}_j^{\mathtt{i}} \right) \right).
\end{aligned}$$

Then, recalling that $\lambda_j = r_j^{n/(n-sp)}$, define

$$u_j(x) := \begin{cases} w_j(x) & U_j \\ \left(1 - \xi_N\left(\frac{x - \mathrm{x}_j^{\mathtt{i}}}{\lambda_j}\right)\right) u_j^{\mathtt{i}} & B_j^{\mathtt{i}}, \, \mathtt{i} \in \mathcal{I}_j \\ \left(1 - \zeta\left(\frac{x - \mathrm{x}_j^{\mathtt{i}}}{\lambda_j}\right)\right) w_j(x) & \hat{B}_j^{\mathtt{i}}, \, \mathtt{i} \in \mathscr{I}_j. \end{cases} \tag{3.32}$$

For the sake of notational simplicity we have not highlighted the dependence of the sequence $(u_j)_{j\in\mathbf{N}}$ on the parameter $N \in \mathbf{N}$. Clearly, $(u_j)_{j\in\mathbf{N}}$ converges strongly to u in $L^p(U)$, and moreover it satisfies the obstacle condition by construction. The rest of the proof is devoted to show that $u_j \in W^{s,p}(U)$ with

$$\limsup_j \mathcal{F}_j(u_j) \leq \mathcal{F}(u) + \epsilon_\delta + \epsilon_N,$$

where $\epsilon_\delta \to 0^+$ as $\delta \to 0^+$ and $\epsilon_N \to 0^+$ as $N \to +\infty$.

A first reduction can be done by computing the energy of u_j only on a neighborhood of the diagonal Δ. Indeed, Lebesgue dominated convergence and the stated convergence of $(u_j)_{j\in\mathbf{N}}$ to u in $L^p(U)$ imply

$$\lim_j \mathcal{D}_{s,p}(u_j, (U \times U) \setminus \Delta_\delta) = \mathcal{D}_{s,p}(u, (U \times U) \setminus \Delta_\delta).$$

In addition, since $u_j \equiv w_j$ on U_j by (3.22) in Lemma 3.9 we have for some constant $c = c(n,p,s)$

$$\limsup_j \mathcal{D}_{s,p}(u_j,(U_j\times U_j)\cap\Delta_\delta) \leq \limsup_j \mathcal{D}_{s,p}(w_j,(U\times U)\cap\Delta_\delta) \leq \left(1+\frac{c}{N}\right)\mathcal{D}_{s,p}(u,(U\times U)\cap\Delta_\delta) = \epsilon_\delta. \tag{3.33}$$

The conclusion then follows provided we show that

$$\limsup_j \left(\mathcal{D}_{s,p}(u_j,(U\times(U\setminus\overline{U}_j))\cap\Delta_\delta) + \mathcal{D}_{s,p}(u_j,((U\setminus\overline{U}_j)\times U_j)\cap\Delta_\delta)\right)$$
$$\leq \theta\,\mathrm{cap}_{s,p}(T)\int_U |u(x)|^p\beta(x)dx + \epsilon_N + \epsilon_\delta. \tag{3.34}$$

In order to prove this we introduce the following splitting of the left hand side above:

$$\mathcal{D}_{s,p}(u_j,(U\times(U\setminus\overline{U}_j))\cap\Delta_\delta) \leq \sum_{\mathtt{i}\in\mathcal{I}_j}|u_j|^p_{W^{s,p}(B^{\mathtt{i}}_j)} + \sum_{\{(\mathtt{i},\mathtt{k})\in\mathcal{I}^2_j:\,0<|\mathrm{x}^{\mathtt{i}}_j-\mathrm{x}^{\mathtt{k}}_j|<\delta\}}\mathcal{D}_{s,p}(u_j,B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j)$$
$$+2\sum_{\mathtt{i}\in\mathcal{I}_j}\mathcal{D}_{s,p}(u_j,(B^{\mathtt{i}}_j\times U_j)\cap\Delta_\delta) + \sum_{(\mathtt{i},\mathtt{k})\in\mathscr{I}^2_j}\mathcal{D}_{s,p}(u_j,\hat{B}^{\mathtt{i}}_j\times\hat{B}^{\mathtt{k}}_j)$$
$$+2\sum_{\mathtt{i}\in\mathscr{I}_j}\mathcal{D}_{s,p}(u_j,(\hat{B}^{\mathtt{i}}_j\times U_j)\cap\Delta_\delta) + 2\sum_{(\mathtt{i},\mathtt{k})\in\mathcal{I}_j\times\mathscr{I}_j}\mathcal{D}_{s,p}\left(u_j,(B^{\mathtt{i}}_j\times\hat{B}^{\mathtt{k}}_j)\cap\Delta_\delta\right) =: I^1_j+\ldots+I^6_j.$$

Next we estimate separately each term I^h_j, $h\in\{1,\ldots,6\}$. Since the computations below are quite involved, we divide our argument into several steps to provide a proof as clear as possible. Take note that all the constants c appearing in the rest of the proof depend only on n, p, s, hence this dependence will no longer be indicated.

Step 1. Estimate of I^1_j:

$$\limsup_j I^1_j \leq \theta\,(\mathrm{cap}_{s,p}(T)+\epsilon_N)\int_U |u(x)|^p\beta(x)dx. \tag{3.35}$$

A change of variables yields

$$I^1_j = \lambda_j^{n-sp}\sum_{\mathtt{i}\in\mathcal{I}_j}|u^{\mathtt{i}}_j|^p|\xi_N|^p_{W^{s,p}(\lambda_j^{-1}(B^{\mathtt{i}}_j-\mathrm{x}^{\mathtt{i}}_j))}$$
$$\leq \left(C_{s,p}(T,B_N)+\frac{1}{N}\right)\sum_{\mathtt{i}\in\mathcal{I}_j}r^n_j|u^{\mathtt{i}}_j|^p = \left(C_{s,p}(T,B_N)+\frac{1}{N}\right)\int_U|\zeta_j(x)|^p\Psi_j(x)dx,$$

where $\Psi_j(x)=\sum_{\mathtt{i}\in\mathcal{I}_j}\lambda_j^{n-sp}(\mathcal{L}^n(V^{\mathtt{i}}_j))^{-1}\chi_{V^{\mathtt{i}}_j}(x)$ and ζ_j is defined in Lemma 3.9. Arguing as in Proposition 3.10 and by Lemma 2.12 we conclude (3.35).

Step 2. Estimate of I^2_j:

$$\limsup_j I^2_j \leq \epsilon_\delta. \tag{3.36}$$

Take note that by the very definition of u_j in (3.32) for any $(x,y)\in B^{\mathtt{i}}_j\times B^{\mathtt{k}}_j$, $\mathtt{i}\neq\mathtt{k}$ and $\mathtt{i},\mathtt{k}\in\mathcal{I}_j$, we get

$$u_j(x)-u_j(y) = \left(u^{\mathtt{i}}_j-u^{\mathtt{k}}_j\right) - \xi_N\left(\lambda_j^{-1}(x-\mathrm{x}^{\mathtt{i}}_j)\right)u^{\mathtt{i}}_j + \xi_N\left(\lambda_j^{-1}(y-\mathrm{x}^{\mathtt{k}}_j)\right)u^{\mathtt{k}}_j.$$

Hence, we can bound I_j^2 as follows

$$I_j^2 \le 3^{p-1} \sum_{\mathtt{i}\in\mathcal{I}_j} \sum_{\{\mathtt{k}\in\mathcal{I}_j:\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|<\delta\}} \int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|u_j^{\mathtt{i}}-u_j^{\mathtt{k}}|^p}{|x-y|^{n+sp}}dxdy$$

$$+3^p\|u\|^p_{L^\infty(U)} \sum_{\mathtt{i}\in\mathcal{I}_j} \sum_{\{\mathtt{k}\in\mathcal{I}_j:\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|<\delta\}} \int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|\xi_N(\lambda_j^{-1}(x-\mathrm{x}_j^{\mathtt{i}}))|^p}{|x-y|^{n+sp}}dxdy =: I_j^{2,1}+I_j^{2,2}.$$

Since $|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|/2 \le |x-y| \le 2|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|$ for any $(x,y)\in B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}$, $\mathtt{i},\mathtt{k}\in\mathcal{I}_j$ with $\mathtt{i}\neq\mathtt{k}$, we infer $|u_j^{\mathtt{i}}-u_j^{\mathtt{k}}| \le 2\mathrm{Lip}(u)|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|$ being $u\in\mathrm{Lip}(U)$, and so we deduce

$$\int_{B_j^{\mathtt{i}}\times B_j^{\mathtt{k}}} \frac{|u_j^{\mathtt{i}}-u_j^{\mathtt{k}}|^p}{|x-y|^{n+sp}}dxdy \le c\,\mathrm{Lip}^p(u)\frac{r_j^{2n}}{|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|^{n+(s-1)p}}. \tag{3.37}$$

To go on further we notice that for every fixed $\mathtt{i}\in\mathcal{I}_j$ we have

$$\{\mathtt{k}\in\mathcal{I}_j:\, 0<|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|_\infty<\delta\} \subseteq \cup_{h=2}^{\lfloor\delta/r_j\rfloor}\{\mathtt{k}\in\mathcal{I}_j:\, hr_j \le |\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|_\infty < (h+1)r_j\},$$

where $\lfloor t\rfloor$ denotes the integer part of t. The latter inclusion together with $(2.4)_1$, (2.5) and (3.37) entail

$$I_j^{2,1} \le c\,\mathrm{Lip}^p(u)\sum_{\mathtt{i}\in\mathcal{I}_j}\sum_{h=2}^{\lfloor\delta/r_j\rfloor}\sum_{\{\mathtt{k}\in\mathcal{I}_j:\, hr_j\le|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|_\infty<(h+1)r_j\}} \frac{r_j^{n-(s-1)p}}{h^{n+(s-1)p}}$$

$$\overset{(2.4)_1,(2.5)}{\le} c\,\mathrm{Lip}^p(u)\sum_{h=2}^{\lfloor\delta/r_j\rfloor}\frac{r_j^{-(s-1)p}}{h^{1+(s-1)p}} \le c\,\mathrm{Lip}^p(u)\delta^{(1-s)p}. \tag{3.38}$$

In the last inequality we used that $\sum_{h=2}^M h^{-(1+\gamma)} \le (M^{-\gamma})/(-\gamma)$, for any $\gamma<0$ and $M\in\mathbf{N}$.

To deal with $I_j^{2,2}$ we use a similar argument. Indeed, for every $\mathtt{i}\in\mathcal{I}_j$ we have

$$\sum_{\{\mathtt{k}\in\mathcal{I}_j:\,\mathtt{k}\neq\mathtt{i}\}}\int_{B_j^{\mathtt{k}}}\frac{1}{|x-y|^{n+sp}}dy \le c\sum_{\{\mathtt{k}\in\mathcal{I}_j:\,\mathtt{k}\neq\mathtt{i}\}}\frac{r_j^n}{|\mathrm{x}_j^{\mathtt{i}}-\mathrm{x}_j^{\mathtt{k}}|^{n+sp}} \overset{(2.5)}{\le} \frac{c}{r_j^{sp}}\sum_{h\ge1}\frac{1}{h^{1+sp}}.$$

Thus, being $\xi_N(\lambda_j^{-1}(\cdot-\mathrm{x}_j^{\mathtt{i}}))$ supported in $B_j^{\mathtt{i}}$, a change of variables yields

$$I_j^{2,2} \le c\,\|u\|^p_{L^\infty(U)}2^{Nn}\lambda_j^n r_j^{-n-sp}\|\xi_N\|^p_{L^p(B_N)} = c\,\|u\|^p_{L^\infty(U)}2^{Nn}r_j^{\frac{(sp)^2}{n-sp}}\|\xi_N\|^p_{L^p(B_N)}. \tag{3.39}$$

Clearly, (3.38) and (3.39) imply (3.36).

Step 3. Estimate of I_j^3*:*

$$\limsup_j I_j^3 \le \epsilon_\delta + \epsilon_N. \tag{3.40}$$

Being $u_j\equiv w_j$ on U_j and $\mathrm{spt}(\xi_N(\lambda_j^{-1}(\cdot-\mathrm{x}_j^{\mathtt{i}}))) \subseteq B_j^{\mathtt{i}}$, we find

$$I_j^3 \le c\,\|u\|^p_{L^\infty(U)}\sum_{\mathtt{i}\in\mathcal{I}_j}\mathcal{D}_{s,p}(\xi_N(\lambda_j^{-1}(\cdot-\mathrm{x}_j^{\mathtt{i}})), B_j^{\mathtt{i}}\times(U\setminus B_j^{\mathtt{i}}))$$

$$+c\sum_{\mathtt{i}\in\mathcal{I}_j}\int_{B_j^{\mathtt{i}}\times U_j}\frac{|w_j(x)-u_j^{\mathtt{i}}|^p}{|x-y|^{n+sp}}dxdy + c\sum_{\mathtt{i}\in\mathcal{I}_j}\int_{(B_j^{\mathtt{i}}\times U_j)\cap\Delta_\delta}\frac{|w_j(y)-w_j(x)|^p}{|x-y|^{n+sp}}dxdy$$

$$=: I_j^{3,1} + I_j^{3,2} + I_j^{3,3}.$$

Note that by a change of variables the integral $I_j^{3,1}$ rewrites as

$$I_j^{3,1} \le c\,\|u\|^p_{L^\infty(U)} \lambda_j^{n-sp} r_j^{-n} \mathcal{D}_{s,p}\left(\xi_N, B_{\frac{r_j}{8^{h_j}\lambda_j}} \times \left(\mathbf{R}^n \setminus \overline{B}_{\frac{r_j}{8^{h_j}\lambda_j}}\right)\right) = \epsilon_N, \tag{3.41}$$

by Remark 2.14. To deal with the term $I_j^{3,2}$ we first integrate out y and then use Hardy inequality:

$$I_j^{3,2} \le c \sum_{\mathtt{i}\in\mathcal{I}_j} \int_{B_j^{\mathtt{i}}} \frac{|w_j(x) - u_j^{\mathtt{i}}|^p}{\mathrm{dist}^{sp}(x, \partial B_j^{\mathtt{i}})} dx \le c \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(B_j^{\mathtt{i}})} \le c\,\mathcal{D}_{s,p}(w_j, (U\times U)\cap\Delta_\delta) \stackrel{(3.33)}{=} \epsilon_\delta. \tag{3.42}$$

Finally, for what $I_j^{3,3}$ is concerned we have

$$I_j^{3,3} \le c\,\mathcal{D}_{s,p}(w_j, (U\times U)\cap\Delta_\delta) \stackrel{(3.33)}{=} \epsilon_\delta. \tag{3.43}$$

By collecting (3.41)-(3.43) we infer (3.40).

Step 4. Estimate of I_j^4:

$$\limsup_j I_j^4 \le \epsilon_\delta. \tag{3.44}$$

The very definition of u_j in (3.32) gives for any $(x,y) \in \hat{B}_j^{\mathtt{i}} \times \hat{B}_j^{\mathtt{k}}$, $\mathtt{i},\mathtt{k} \in \mathscr{I}_j$

$$\begin{aligned} u_j(x) - u_j(y) &= \left(1 - \zeta\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right)\right) w_j(x) - \left(1 - \zeta\left(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{k}})\right)\right) w_j(y) \\ &= \left(1 - \zeta\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right)\right)(w_j(x) - w_j(y)) + \left(\zeta\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right) - \zeta\left(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{k}})\right)\right) w_j(y). \end{aligned}$$

Distinguishing the couples of the form $(\mathtt{i},\mathtt{i})$, $\mathtt{i} \in \mathscr{I}_j$, from the others we bound I_j^4 as follows

$$\begin{aligned} I_j^4 &\le c\,\mathcal{D}_{s,p}(w_j, (U\times U)\cap\Delta_\delta) + c\,\|u\|^p_{L^\infty(U)} \sum_{\mathtt{i}\in\mathscr{I}_j} \left|\zeta\left(\lambda_j^{-1}(\cdot - \mathrm{x}_j^{\mathtt{i}})\right)\right|^p_{W^{s,p}(\hat{B}_j^{\mathtt{i}})} \\ &+ c\,\|u\|^p_{L^\infty(U)} \sum_{\{(\mathtt{i},\mathtt{k})\in\mathscr{I}_j^2:\,\mathtt{i}\neq\mathtt{k}\}} \int_{\hat{B}_j^{\mathtt{i}}\times\hat{B}_j^{\mathtt{k}}} \frac{|\zeta(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})) - \zeta(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{k}}))|^p}{|x-y|^{n+sp}} dxdy := I_j^{4,1} + I_j^{4,2} + I_j^{4,3}. \end{aligned}$$

A change of variables and $(2.4)_2$ yield

$$I_j^{4,1} + I_j^{4,2} \le c\,\mathcal{D}_{s,p}(w_j, (U\times U)\cap\Delta_\delta) + c\,\mathcal{L}^n\left((\partial U)_{R_j}\right)|\zeta|^p_{W^{s,p}(B_N)} \stackrel{(3.33)}{\le} \epsilon_\delta + c\,\mathcal{L}^n\left((\partial U)_{R_j}\right). \tag{3.45}$$

To deal with the term $I_j^{4,3}$ first note that

$$I_j^{4,3} \le c\,\|u\|^p_{L^\infty(U)} \sum_{\mathtt{i}\in\mathscr{I}_j} \int_{\hat{B}_j^{\mathtt{i}}\times(U\setminus\hat{B}_j^{\mathtt{i}})} \frac{|\zeta(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}}))|^p}{|x-y|^{n+sp}} dxdy,$$

then integrate out y, scale back the x variable, and finally use Hardy inequality taking into account that $\zeta \in C_c^\infty(B_N)$:

$$\begin{aligned} I_j^{4,3} &\le c\,\|u\|^p_{L^\infty(U)} \sum_{\mathtt{i}\in\mathscr{I}_j} \int_{\hat{B}_j^{\mathtt{i}}} \frac{|\zeta(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}}))|^p}{\mathrm{dist}^{sp}(x, \partial\hat{B}_j^{\mathtt{i}})} dx = c\,\|u\|^p_{L^\infty(U)} \lambda_j^{n-sp} \sum_{\mathtt{i}\in\mathscr{I}_j} \int_{B_N} \frac{|\zeta(x)|^p}{\mathrm{dist}^{sp}(x, \partial B_N)} dx \\ &\le c\,\|u\|^p_{L^\infty(U)} \lambda_j^{n-sp} \#(\mathscr{I}_j) |\zeta|^p_{W^{s,p}(B_N)} \le c\|u\|^p_{L^\infty(U)} \mathcal{L}^n\left((\partial U)_{R_j}\right)|\zeta|^p_{W^{s,p}(B_N)} \end{aligned} \tag{3.46}$$

by $(2.4)_2$. In conclusion, (3.45) and (3.46) give (3.44).

Step 5. Estimate of I_j^5:

$$\limsup_j I_j^5 \le \epsilon_\delta. \tag{3.47}$$

The computations are similar to the previous step once one notices that for any $(x,y) \in \hat{B}_j^{\mathtt{i}} \times U_j$, $\mathtt{i} \in \mathscr{I}_j$, it holds

$$u_j(x) - u_j(y) = \left(1 - \zeta\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right)\right) w_j(x) - w_j(y) = (w_j(x) - w_j(y)) - \zeta\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{k}})\right) w_j(x).$$

Hence, we bound I_j^5 by the sum of two terms, the first analogous to $I_j^{4,1}$ and the second to $I_j^{4,3}$. Thus, (3.47) follows.

Step 6. Estimate of I_j^6:

$$\limsup_j I_j^6 \le \epsilon_\delta. \tag{3.48}$$

For $(x,y) \in B_j^{\mathtt{i}} \times \hat{B}_j^{\mathtt{k}}$, $\mathtt{i} \in \mathcal{I}_j$ and $\mathtt{k} \in \mathscr{I}_j$, we write

$$\begin{aligned} u_j(x) - u_j(y) = &\left(1 - \xi_N\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right)\right)(u_j^{\mathtt{i}} - w_j(x)) \\ &+ \left(1 - \xi_N\left(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}})\right)\right) w_j(x) - \left(1 - \zeta\left(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{i}})\right)\right) w_j(y). \end{aligned}$$

Thus, we infer

$$\begin{aligned} I_j^6 \le c \sum_{(\mathtt{i},\mathtt{k}) \in \mathcal{I}_j \times \mathscr{I}_j} \int_{B_j^{\mathtt{i}} \times \hat{B}_j^{\mathtt{k}}} \frac{|w_j(x) - u_j^{\mathtt{i}}|^p}{|x-y|^{n+sp}} dxdy + c\,\|u\|_{L^\infty(U)}^p \sum_{(\mathtt{i},\mathtt{k}) \in \mathcal{I}_j \times \mathscr{I}_j} \int_{B_j^{\mathtt{i}} \times \hat{B}_j^{\mathtt{k}}} \frac{|\zeta(\lambda_j^{-1}(y - \mathrm{x}_j^{\mathtt{k}}))|^p}{|x-y|^{n+sp}} dxdy \\ + c\,\|u\|_{L^\infty(U)}^p \sum_{(\mathtt{i},\mathtt{k}) \in \mathcal{I}_j \times \mathscr{I}_j :\, 0 < |\mathrm{x}_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{k}}| < \delta\}} \int_{B_j^{\mathtt{i}} \times \hat{B}_j^{\mathtt{k}}} \frac{|\xi_N(\lambda_j^{-1}(x - \mathrm{x}_j^{\mathtt{i}}))|^p}{|x-y|^{n+sp}} dxdy := I_j^{6,1} + I_j^{6,2} + I_j^{6,3}. \end{aligned}$$

Clearly, $I_j^{6,1}$ can be estimated as $I_j^{3,2}$, $I_j^{6,2}$ as $I_j^{4,3}$, and $I_j^{6,3}$ as $I_j^{2,2}$. In conclusion, (3.48) follows.

Step 7: Conclusion. By collecting Step 1 - Step 6 we infer

$$\limsup_j \mathcal{F}_j(u_j) \le \mathcal{F}(u) + \epsilon_\delta + \epsilon_N,$$

with the two terms on the rhs above infinitesimal as $\delta \to 0^+$ and as $N \to +\infty$, respectively. □

3.4. **Generalizations.** Anisotropic and homogeneous variations of the fractional semi-norm can be treated essentially in the same way. Consider a $\mathcal{L}^n$-measurable kernel $K : \mathbf{R}^n \setminus \{\underline{0}\} \to (0, +\infty)$ such that for all $z \in \mathbf{R}^n \setminus \{\underline{0}\}$ and for some constant $\alpha \ge 1$ it holds

$$K(tz) = t^{-(n+sp)} K(z) \quad t > 0, \qquad \alpha^{-1}|z|^{-(n+sp)} \le K(z) \le \alpha |z|^{-(n+sp)}. \tag{3.49}$$

Define $\mathcal{K} : W^{s,p}(\mathbf{R}^n) \times \mathcal{A}(\mathbf{R}^n) \to [0, +\infty)$ as

$$\mathcal{K}(u, A) := \int_{A \times A} K(x-y)|u(x) - u(y)|^p \, dxdy, \tag{3.50}$$

dropping the dependence on the set of integration in case $A = \mathbf{R}^n$. All the relevant quantities introduced in the preceding sections have analogous counterparts simply replacing the kernel $|\cdot|^{-(n+sp)}$ with K. For instance, the locality defect associated to the energy $\mathcal{K}$ is given by

$$\mathcal{D}_{\mathcal{K}}(u,E) := \int_E K(x-y)|u(x)-u(y)|^p dxdy,$$

for any $E \subseteq U \times U$ $\mathcal{L}^{n\times n}$-measurable. We point out that in general, $\mathcal{D}_{\mathcal{K}}(u, A\times B) \neq \mathcal{D}_{\mathcal{K}}(u, B\times A)$, A and B being measurable subsets of $\mathbf{R}^n$. Nevertheless, a splitting formula similar to (2.15) holds.

The only relevant changes are in the proof of Lemma 2.12 in which we exploited the invariance of the kernel $|\cdot|^{-(n+sp)}$ under rotations to establish (2.16). As already noticed before the statement of Lemma 2.12, and as it turns out from the proof of Propositions 3.10, 3.11, in the current deterministic setting it is sufficient to prove pointwise convergence of the relative capacities to infer the lower bound estimate. This is the content of the next lemma. The argument below does not give uniform convergence.

Lemma 3.12. *Let $r > 0$ and define for every $T \subset \mathbf{R}^n$*

$$C_{\mathcal{K}}(T, B_r) := \inf\left\{\mathcal{K}(w) :\ w \in W^{s,p}(\mathbf{R}^n),\ w = 0 \text{ on } \mathbf{R}^n \setminus B_r,\ \tilde{w} \geq 1 \text{ q.e. on } T\right\}.$$

Then

$$\lim_{r\to+\infty} C_{\mathcal{K}}(T, B_r) = \mathrm{cap}_{\mathcal{K}}(T). \tag{3.51}$$

Moreover, there exists a constant $c = c(n,s,p,\alpha)$ such that for all $0 < r < R$

$$\mathrm{cap}_{\mathcal{K}}(T) - \mathrm{cap}_{\mathcal{K}}(T, B_R; r) \leq \frac{c\, r^{sp}}{(R-r)^{sp}} C_{\mathcal{K}}(T, B_r), \tag{3.52}$$

where $\mathrm{cap}_{\mathcal{K}}(T, B_R; r) := \inf\{\mathcal{K}(w, B_R) :\ w \in W^{s,p}(\mathbf{R}^n),\ w = 0 \text{ on } \mathbf{R}^n \setminus B_r,\ \tilde{w} \geq 1 \text{ q.e. on } T\}$.

Proof. Estimate (3.52) can be derived as we did for (2.17) thanks to $(3.49)_2$.

It is clear that $\mathrm{cap}_{\mathcal{K}}(T) \leq C_{\mathcal{K}}(T, B_r) \leq C_{\mathcal{K}}(T, B_R)$ for all $0 < R < r$, so that

$$\lim_{r\to+\infty} C_{\mathcal{K}}(T, B_r) \geq \mathrm{cap}_{\mathcal{K}}(T).$$

Fix $r > 0$ such that $T \subseteq B_{r/2}$ and let $u \in W^{s,p}(\mathbf{R}^n)$ be such that $0 \leq u \leq 1$ $\mathcal{L}^n$ a.e. on $\mathbf{R}^n$, $\tilde{u} \geq 1$ q.e. on T. Given φ_r a cut-off function between $B_{r/2}$ and $B_{3r/4}$ set $w_r := \varphi_r u$. By construction $w_r = u$ on $B_{r/2}$, $w_r \equiv 0$ on $\mathbf{R}^n \setminus B_{3r/4}$ and $\tilde{w}_r \geq 1$ q.e. on T. We claim that $w_r \in W^{s,p}(\mathbf{R}^n)$ with

$$\begin{aligned}\mathcal{K}(w_r, B_r) &\leq \mathcal{K}(u, B_{r/2}) \\ &+ c(n,p,s,\alpha)\left(\mathcal{D}_{\mathcal{K}}(u, B_{r/2}\times(B_r\setminus \overline{B}_{r/2})) + \mathcal{D}_{\mathcal{K}}(u, (B_r\setminus \overline{B}_{r/2})\times B_{r/2}) + r^{-ps}\int_{B_r\setminus \overline{B}_{r/2}} |u|^p dx\right).\end{aligned} \tag{3.53}$$

To this aim we bound the energy of w_r on B_r by

$$\mathcal{K}(w_r, B_r) \leq \mathcal{K}(u, B_{r/2}) + \mathcal{D}_{\mathcal{K}}(w_r, B_{r/2}\times(B_r\setminus \overline{B}_{r/2})) + \mathcal{D}_{\mathcal{K}}(w_r, (B_r\setminus \overline{B}_{r/2})\times B_{r/2}). \tag{3.54}$$

We estimate the first of the two locality defect terms in (3.54), the argument for the second being analogous. In doing that we can follow the argument used in Lemma 3.8 for the locality defect term (see (3.17)-(3.19)). Then, by $(3.49)_2$ and since $0 \leq \varphi_r \leq 1$, we infer

$$\begin{aligned}
&\mathcal{D}_{\mathcal{K}}(w_r, B_{r/2} \times (B_r \setminus \overline{B}_{r/2})) \\
&\quad \overset{\pm\varphi_r(x)u(y)}{\leq} 2^p \left(\mathcal{D}_{\mathcal{K}}(u, B_{r/2} \times (B_r \setminus \overline{B}_{r/2}) + \alpha \int_{B_{r/2}\times(B_r\setminus\overline{B}_{r/2})} \frac{|(\varphi_r(x) - \varphi_r(y))u(y)|^p}{|x-y|^{n+sp}} dxdy \right) \\
&\quad \leq c(n,p,s,\alpha) \left(\mathcal{D}_{\mathcal{K}}(u, B_{r/2} \times (B_r \setminus \overline{B}_{r/2}) + r^{-ps} \int_{B_r\setminus\overline{B}_{r/2}} |u|^p dy \right).
\end{aligned}$$

Formula (3.53) then follows at once.

Furthermore, since $w_r = 0$ on $B^c_{3r/4}$, $(3.49)_2$ and (A.2) in Lemma A.1 yield

$$\mathcal{D}_{\mathcal{K}}(w_r, B_r \times (\mathbf{R}^n \setminus \overline{B}_r)) \leq c(n,p,s)\alpha\, r^{-ps} \int_{B_r} |u|^p dx,$$

and thus we conclude $w_r \in W^{s,p}(\mathbf{R}^n)$.

Eventually, since $\lim_r \mathcal{D}_{\mathcal{K}}(u, \mathbf{R}^n \times (\mathbf{R}^n \setminus \overline{B}_{r/2})) = \lim_r \mathcal{D}_{\mathcal{K}}(u, (\mathbf{R}^n \setminus \overline{B}_{r/2}) \times \mathbf{R}^n) = 0$ and $u \in L^p(\mathbf{R}^n)$, from (3.53) we conclude

$$\lim_r C_{\mathcal{K}}(T, B_r) \leq \limsup_r \mathcal{K}(w_r, B_r) \leq \mathcal{K}(u).$$

Taking the infimum on all admissible functions u we conclude. □

With fixed $U \in \mathcal{A}(\mathbf{R}^n)$, consider $\mathcal{K}_j : L^p(U) \to [0, +\infty]$ given by

$$\mathcal{K}_j(u) = \begin{cases} \mathcal{K}(u, U) & \text{if } u \in W^{s,p}(U),\ \tilde{u} = 0 \text{ cap}_{s,p} \text{ q.e. on } T_j \cap U \\ +\infty & \text{otherwise.} \end{cases} \tag{3.55}$$

Theorem 3.13. *Let $U \in \mathcal{A}(\mathbf{R}^n)$ be bounded and connected with Lipschitz regular boundary.*

Then the sequence $(\mathcal{K}_j)_{j\in\mathbf{N}}$ Γ-converges in the $L^p(U)$ topology to $\mathscr{K} : L^p(U) \to [0, +\infty]$ defined by

$$\mathscr{K}(u) = \mathcal{K}(u) + \theta\, \text{cap}_{\mathcal{K}}(T) \int_U |u(x)|^p \beta(x)\, dx \tag{3.56}$$

if $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$, where

$$\text{cap}_{\mathcal{K}}(T) := \inf \{\mathcal{K}(w) :\ w \in W^{s,p}(\mathbf{R}^n),\ \tilde{w} \geq 1 \text{ q.e. on } T\}.$$

Proof. The proof is the same of Theorem 3.3 a part from the necessary changes to the various technical lemmas preceding Theorem 3.3. We indicate how these preliminaries must be appropriately restated.

We have shown the (pointwise) convergence of relative capacities in Lemma 3.12. Changing the relevant quantities according to the substitution of the kernel $|\cdot|^{-(n+sp)}$ with K, Lemmas 3.8 and 3.9 have analogous statements since the splitting formula (2.15) does.

In the proof of Proposition 3.10 we use the homogeneity of the kernel K (see $(3.49)_1$) for the scaling argument in (3.31) leading to the analogue of formula (3.29). This is the reason why we ask for $(3.49)_1$.

Proposition 3.11 needs few changes: in the definition of u_j choose ξ_N to be a $(1/N)$-minimizer of $C_{\mathcal{K}}(T,B_N)$, the estimate of I^1_j follows straightforward. For what the terms I^h_j, $h\in\{2,\dots,6\}$ are concerned it suffices to take note that by condition $(3.49)_2$ the locality defect satisfies $\alpha^{-1}\mathcal{D}_{s,p}(u,E)\leq \mathcal{D}_{\mathcal{K}}(u,E)\leq \alpha\mathcal{D}_{s,p}(u,E)$. Hence, we can follow exactly Steps 2-6 to conclude. □

Remark 3.14. *Dropping the homogeneity assumption* $(3.49)_1$ *on the kernel* K *the* Γ*-limit might not exist. This fact had already been noticed in the local case (see [5]). Though, in such a framework abstract integral representation and compactness arguments for local functional imply* Γ*-convergence up to subsequences.*

4. Random Settings

In this section we extend our analysis to two different random settings. First, we deal with obstacles having random sizes and shapes located on points of a periodic lattice as introduced by Caffarelli and Mellet [18], [19] (see also [32]). In particular, we provide a self-contained proof of the results in [19], [32] avoiding extension techniques. Second, we consider random homothetics copies of a given obstacle set placed on random Delone sets of points following the approach by Blanc, Le Bris and Lions to define the energy of microscopic stochastic lattices [13] and to study some variants of the usual stochastic homogenization theory [14].

We have not been able to work out a unified approach for the two frameworks described above. The main issue for this being related to the interplay between the weighted version of the pointwise ergodic theorem in Theorem 4.2 below and stationarity for random Delone sets of points (see (4.16)).

In both cases we are given a probability space $(\Omega,\mathscr{P},\mathbb{P})$ such that the group $\mathbf{Z}^n$ acts on Ω via measure-preserving transformations $\tau_{\mathtt{i}}:\Omega\to\Omega$. The σ-subalgebra of $\mathscr{P}$ of the invariant sets of the $\tau_{\mathtt{i}}$'s, i.e. $O\subseteq\Omega$ such that $\tau_{\mathtt{i}}O=O$ for all $\mathtt{i}\in\mathbf{Z}^n$, is denoted by $\mathscr{I}$. Recall that $(\tau_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$ is said to be *ergodic* if $\mathscr{I}$ is trivial, i.e. $O\in\mathscr{I}$ satisfies either $\mathbb{P}(O)=0$ or $\mathbb{P}(O)=1$.

In the sequel we keep the notation introduced in Section 3 highlighting the dependence of relevant quantities on ω when needed.

4.1. **Obstacles with Random sizes and shapes.** In this subsection we deal with the case of obstacles with random sizes and shapes located on points of a lattice. We restrict to the standard cubic one only for the sake of simplicity (see Remark 4.3 below for extensions). More precisely, let $\Lambda_j=\varepsilon_j\mathbf{Z}^n$ with $(\varepsilon_j)_{j\in\mathbf{N}}$ a positive infinitesimal sequence, then we have $\mathrm{x}^{\mathtt{i}}_j=\varepsilon_j\mathtt{i}$, $V^{\mathtt{i}}_j=\varepsilon_j(\mathtt{i}+[-1/2,1/2]^n)$, $r_{\Lambda_j}=\varepsilon_j/2$ and $R_{\Lambda_j}=\sqrt{n}\varepsilon_j/2$. In addition, if $r_j=\varepsilon_j/2$ then $\theta=\mathcal{L}^n(U)/2^n$ and $\beta\equiv 1/\mathcal{L}^n(U)$. Thus, to simplify the presentation we will use the scaling parameter ε_j instead of r_j, and the more intuitive notation $Q^{\mathtt{i}}_j$ for $V^{\mathtt{i}}_j$.

Let us now fix the assumptions on the distribution of obstacles originally introduced by Caffarelli and Mellet [18], [19]. For all $\omega\in\Omega$ and $j\in\mathbf{N}$ the obstacle set $T_j(\omega)\subseteq\mathbf{R}^n$ is given by $T_j(\omega):=\cup_{\mathtt{i}\in\mathbf{Z}^n}T^{\mathtt{i}}_j(\omega)$, where the sets $T^{\mathtt{i}}_j(\omega)\subseteq Q^{\mathtt{i}}_j$, satisfy the following conditions:

(O1). *Capacitary Scaling:* There exists a process $\gamma : \mathbf{Z}^n \times \Omega \to [0, +\infty)$ such that for all $\mathtt{i} \in \mathbf{Z}^n$ and $\omega \in \Omega$

$$\mathrm{cap}_{s,p}(T_j^{\mathtt{i}}(\omega)) = \varepsilon_j^n \, \gamma(\mathtt{i}, \omega).$$

Moreover, for some $\gamma_0 > 0$ we have for all $\mathtt{i} \in \mathbf{Z}^n$ and $\mathbb{P}$ a.s. $\omega \in \Omega$

$$\gamma(\mathtt{i}, \omega) \leq \gamma_0. \tag{4.1}$$

(O2). *Stationarity of the Process:* The process $\gamma : \mathbf{Z}^n \times \Omega \to [0, +\infty)$ is stationary w.r.to the family $(\tau_{\mathtt{i}})_{\mathtt{i} \in \mathbf{Z}^n}$, i.e. for all $\mathtt{i}, \mathtt{k} \in \mathbf{Z}^n$ and $\omega \in \Omega$

$$\gamma(\mathtt{i} + \mathtt{k}, \omega) = \gamma(\mathtt{i}, \tau_{\mathtt{k}}\omega). \tag{4.2}$$

(O3). *Strong Separation:* There exists a positive infinitesimal sequence $(\delta_j)_{j \in \mathbf{N}}$, with $\delta_j = o(\varepsilon_j)$ and $\varepsilon_j^{1 + \frac{n}{n - sp}} = O(\delta_j)$, such that $T_j^{\mathtt{i}}(\omega) \subseteq \mathrm{x}_j^{\mathtt{i}} + \delta_j(Q_j^{\mathtt{i}} - \mathrm{x}_j^{\mathtt{i}})$ for all $\mathtt{i} \in \mathbf{Z}^n$, $\omega \in \Omega$.

Take note that by (O2) we have $\gamma(\mathtt{i}, \omega) = \gamma(\underline{0}, \tau_{-\mathtt{i}}\omega)$, hence the random variables $\gamma(\mathtt{i}, \omega)$ are identically distributed. The common value of their expectations is denoted by $\mathbb{E}[\gamma]$, i.e.

$$\mathbb{E}[\gamma] := \int_\Omega \gamma(\underline{0}, \omega) d\mathbb{P}.$$

Moreover, $\mathbb{E}[\gamma, \mathscr{I}]$ denotes the *conditional expectation* of the process, i.e. the unique $\mathscr{I}$-measurable function in $L^1(\Omega, \mathbb{P})$ such that for every set $O \in \mathscr{I}$

$$\int_O \gamma(\underline{0}, \omega) d\mathbb{P} = \int_O \mathbb{E}[\gamma, \mathscr{I}](\omega) d\mathbb{P}.$$

In the sequel we analyze the asymptotics of the energies $\mathcal{K}_j : L^p(U) \times \Omega \to [0, +\infty]$ given by

$$\mathcal{K}_j(u, \omega) = \begin{cases} \mathcal{K}(u) & \text{if } u \in W^{s,p}(U),\ \tilde{u} = 0 \ \mathrm{cap}_{s,p} \text{ q.e. on } T_j(\omega) \cap U \\ +\infty & \text{otherwise.} \end{cases} \tag{4.3}$$

where $\mathcal{K}$ is the functional in (3.50).

Theorem 4.1. *Let $U \in \mathcal{A}(\mathbf{R}^n)$ be bounded and connected with Lipschitz regular boundary, and assume that the kernel K satisfies (3.49) and is invariant under rotations.*

If (O1)-(O3) hold true, then there exists a set $\Omega' \subseteq \Omega$ of full probability such that for all $\omega \in \Omega'$ the sequence $(\mathcal{K}_j(\cdot, \omega))$ Γ-converges in the $L^p(U)$ topology to $\mathscr{K} : L^p(U) \times \Omega \to [0, +\infty]$ defined by

$$\mathscr{K}(u, \omega) = \mathcal{K}(u) + \mathbb{E}[\gamma, \mathscr{I}] \int_U |u(x)|^p \, dx \tag{4.4}$$

if $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$.

If in addition $(\tau_{\mathtt{i}})_{\mathtt{i} \in \mathbf{Z}^n}$ is ergodic then the Γ-limit is deterministic, i.e. $\mathbb{E}[\gamma, \mathscr{I}] = \mathbb{E}[\gamma]$ $\mathbb{P}$ a.s. in Ω.

The proof of Theorem 4.1 builds upon Theorem 3.13 and upon a weighted ergodic theorem established in [32, Theorem 4.1]. We give the proof of the latter result for the sake of convenience.

Theorem 4.2. *Let γ be satisfying (O2), then for every bounded open set $V \subset \mathbf{R}^n$ with $\mathcal{L}^n(\partial V) = 0$ we have $\mathbb{P}$ a.s. in Ω*

$$\lim_j \frac{1}{\#\mathcal{I}_j(V)} \sum_{\mathtt{i} \in \mathcal{I}_j(V)} \gamma(\mathtt{i}, \omega) = \mathbb{E}[\gamma, \mathscr{I}], \tag{4.5}$$

and

$$\Psi_j(x, \omega) := \sum_{\mathtt{i} \in \mathcal{I}_j(V)} \gamma(\mathtt{i}, \omega) \chi_{Q_j^{\mathtt{i}}}(x) \to \mathbb{E}[\gamma, \mathscr{I}] \quad \text{weak}^* \, L^\infty(V). \tag{4.6}$$

Proof. Define the operators $T_{\mathtt{i}} : L^1(\Omega, \mathbb{P}) \to L^1(\Omega, \mathbb{P})$ by $T_{\mathtt{i}}(f) := f \circ \tau_{\mathtt{i}}$ for every $\mathtt{i} \in \mathbf{Z}^n$. The group property of $(\tau_{\mathtt{i}})_{\mathtt{i} \in \mathbf{Z}^n}$ implies that $\mathscr{S} = \{T_{\mathtt{i}}\}_{\mathtt{i} \in \mathbf{Z}^n}$ is a multiparameter semigroup generated by the commuting isometries $T_{\mathtt{e}_r}$ for $r \in \{1, \ldots, n\}$, being $\{\mathtt{e}_1, \ldots, \mathtt{e}_n\}$ the canonical basis of $\mathbf{R}^n$.

We define a process F on bounded Borel sets V of $\mathbf{R}^n$ with values in $L^\infty(\Omega, \mathbb{P})$. Set $Q_1 := [-1/2, 1/2]^n$ and let F be as follows

$$F_V(\omega) := \sum_{\{\mathtt{i} \in \mathbf{Z}^n : \, \mathtt{i} + Q_1 \subseteq V\}} \gamma(\mathtt{i}, \omega),$$

with the convention that $F_V(\omega) := 0$ if the set of summation is empty. It is clear that F is *additive*, that is it satisfies

(i) F is stationary: $T_{\mathtt{i}} \circ F_V = F_{V + \mathtt{i}}$ for all $\mathtt{i} \in \mathbf{Z}^n$;
(ii) $F_{V_1 \cup V_2} = F_{V_1} + F_{V_2}$, for disjoint V_1, V_2;
(iii) the random variables F_V are integrable; and
(iv) the spatial constant of the process $\overline{\gamma}(F) := \inf\{j^{-n} \int_\Omega F_{jQ_1} d\mathbb{P}\} \in (0, \infty)$.

Indeed, (i) and (iv) follow by (O2) (actually $\overline{\gamma}(F) = \mathbb{E}[\gamma]$), (ii) by the very definition of F, and (iii) by the positivity of γ and (4.1). It then follows from [36, Theorem 2] and [36, Remark (b) p.294] that there exists $\overline{f} \in L^1(\Omega, \mathbb{P})$ such that for all $V \in \mathcal{B}(\mathbf{R}^n)$ bounded with $\mathcal{L}^n(\partial V) = 0$ and $\mathbb{P}$ a.s. in Ω

$$\lim_{j \to +\infty} j^{-n} F_{jV}(\omega) = \mathcal{L}^n(V) \overline{f}(\omega). \tag{4.7}$$

Stationarity and boundedness of γ together with (4.7) yield that for any $O \in \mathscr{I}$ we have

$$\int_O \overline{f}(\omega) \, d\mathbb{P} = \lim_{j \to +\infty} j^{-n} \int_O F_{jQ_1}(\omega) \, d\mathbb{P} = \int_O \gamma(\underline{0}, \omega) \, d\mathbb{P},$$

so that the limit $\overline{f}$ is actually given by $\mathbb{E}[\gamma, \mathscr{I}]$ (see for instance [35, Theorem 2.3 page 203]).

In particular, (4.7) still holds by substituting $(j)_{j \in \mathbf{N}}$ with any diverging sequence $(a_j)_{j \in \mathbf{N}} \subseteq \mathbf{N}$. Take $a_j = \lfloor 1/\varepsilon_j \rfloor$ ($\lfloor t \rfloor$ stands for the integer part of t), then notice that for every $\delta > 0$ and j sufficiently big we have

$$a_j^{-n} \left| \sum_{\mathtt{i} \in \mathcal{I}_j(V)} \gamma(\mathtt{i}, \omega) - F_{a_j V}(\omega) \right| \leq \gamma_0 a_j^{-n} \# \left(\mathcal{I}_j(V) \triangle \{\mathtt{i} \in \mathbf{Z}^n : \, a_j^{-1}(\mathtt{i} + Q_1) \subseteq V\} \right) \leq \gamma_0 \mathcal{L}^n((\partial V)_\delta),$$

here $\triangle$ denotes the symmetric difference between the relevant sets. Since $\varepsilon_j^n \#\mathcal{I}_j(V) \to \mathcal{L}^n(V)$ and $a_j \varepsilon_j \to 1$ we infer (4.5).

Eventually, in order to prove (4.6) consider the family $\mathscr{Q}$ of all open cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes, and with center and vertices having rational coordinates. To show the claimed weak* convergence it suffices to check that $\mathbb{P}$ a.s. in Ω it holds

$$\lim_j \int_\Omega \Psi_j(x,\omega)\chi_Q(x)\,d\mathcal{L}^n = \mathcal{L}^n(Q)\mathbb{E}[\gamma,\mathscr{I}]$$

for any $Q \in \mathscr{Q}$ with $Q \subseteq V$, V as in the statement above. We have

$$\left|\int_Q (\Psi_j(x,\omega) - \mathbb{E}[\gamma,\mathscr{I}])d\mathcal{L}^n\right| \leq \left|\varepsilon_j^n \sum_{\mathtt{i}\in\mathcal{I}_j(Q)} \gamma(\mathtt{i},\omega) - \mathcal{L}^n(Q)\mathbb{E}[\gamma,\mathscr{I}]\right| + 2\gamma_0\mathcal{L}^n\left(Q \setminus \cup_{\mathtt{i}\in\mathcal{I}_j(Q)} Q_j^{\mathtt{i}}\right),$$

and thus (4.5) and the denumerability of $\mathscr{Q}$ yield that the rhs above is infinitesimal $\mathbb{P}$ a.s. in Ω.

□

Proof (of Theorem 4.6). We define Ω' as any subset of full probability in Ω for which Theorem 4.2 holds true for U. Then, we follow the lines of the proof of Theorem 3.13 pointing out only the necessary changes.

First, we note that by assumption (O3), in particular condition $\delta_j = o(\varepsilon_j)$, we can still apply Lemma 3.9 in this framework. Hence, to get the lower bound inequality we argue as in Proposition 3.10, we need only to substitute (3.29) suitably. Formula (3.31) is replaced by

$$\begin{aligned}|w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} &\geq \varepsilon_j^n |(u_j)_{C_j^{\mathtt{i},h_j}}|^p \mathrm{cap}_{s,p}\left(\lambda_j^{-1}(T_j^{\mathtt{i}}(\omega) - \mathrm{x}_j^{\mathtt{i}}), B_{\frac{\varepsilon_j}{m^{3h_j+1}\lambda_j}}; \frac{\varepsilon_j}{m^{3h_j+2}\lambda_j}\right)\\ &\geq (\gamma(\mathtt{i},\omega) - \epsilon_m)\ \varepsilon_j^n\ |(u_j)_{C_j^{\mathtt{i},h_j}}|^p,\end{aligned}$$

where $\epsilon_m > 0$ is infinitesimal as $m \to +\infty$. To infer the last inequality we have taken into account (O1) and the uniform convergence of the relative capacities established in (2.17) of Lemma 2.12. This is guaranteed by condition $\varepsilon_j^{1+\frac{n}{n-sp}} = O(\delta_j)$ in (O3), which ensures that the rescaled obstacle sets $\lambda_j^{-1}(T_j^{\mathtt{i}}(\omega) - \mathrm{x}_j^{\mathtt{i}})$ are equi-bounded.

By summing up all the contributions, by Theorem 4.2 and by recalling that the sequence $(\zeta_j)_{j\in\mathbf{N}}$ defined in Lemma 3.9 converges strongly to u in $L^p(U)$ we infer

$$\liminf_j \sum_{\mathtt{i}\in\mathcal{I}_j} |w_j|^p_{W^{s,p}(V_j^{\mathtt{i}})} \geq \liminf_j \int_A \Psi_j(x,\omega)|\zeta_j|^p dx \geq (\mathbb{E}[\gamma,\mathscr{I}] - \epsilon_m)\int_A |u(x)|^p dx,$$

for all $A \in \mathcal{A}(U)$ with $A \subset\subset U$. By increasing A to U and by letting $m \to +\infty$ we conclude.

The upper bound inequality is established as in Proposition 3.11 by substituting ξ_N in the definition of u_j with $\xi_j^{\mathtt{i},N}$, a $(1/N)$-minimizer of $C_{\mathcal{K}}(\lambda_j^{-1}(T_j^{\mathtt{i}}(\omega) - \mathrm{x}_j^{\mathtt{i}}), B_N)$. The uniform convergence of those relative capacities is guaranteed by the analogue of (2.16) in Lemma 2.12. This is the reason why we suppose K to be invariant under rotations.

The estimate of I_j^1 then follows straightforward. For what the terms I_j^h, $h \in \{2, \dots, 6\}$ are concerned, take note that since $\alpha^{-1}\mathcal{D}_{s,p}(u,E) \le \mathcal{D}_{\mathcal{K}}(u,E) \le \alpha\mathcal{D}_{s,p}(u,E)$ (see $(3.49)_2$), we can follow exactly Steps 2-6 to conclude. □

Remark 4.3. *If Λ is a generic periodic n-dimensional lattice in $\mathbf{R}^n$ we can argue analogously. By definition the points of Λ belong to the orbit of a $\mathbf{Z}$-module generated by n linearly independent vectors, and the Voronoï cells turn out to be congruent polytopes (see* [42, Chapter 2]*). Clearly, the obstacle set $T_j(\omega)$ and stationarity for the process $\gamma(\cdot,\omega)$ have to be defined according to the group of translations associated to vectors in the mentioned $\mathbf{Z}$-module which leave Λ invariant.*

4.2. **Random Delone set of points.** According to Blanc, Le Bris and Lions (see [13], [14]) a *random Delone set* is a random variable $\Lambda : \Omega \to (\mathbf{R}^n)^{\mathbf{Z}^n}$ satisfying Definition 2.2 $\mathbb{P}$ a.s. in Ω. Then, both r_Λ and $R_\Lambda : \Omega \to \mathbf{R}$ turn out to be random variables: r_Λ because of its very definition $(2.2)_1$, and R_Λ since it can be characterized as $R_\Lambda(\cdot) = \sup_{\mathbf{Q}^n} \mathrm{dist}(x, \Lambda(\cdot))$.

We will deal with sequences $\Lambda_j : \Omega \to (\mathbf{R}^n)^{\mathbf{Z}^n}$ of random Delone sets fulfilling conditions analogous to (3.1)-(3.3) (see below for relevant examples), that is for $\mathbb{P}$ a.e. $\omega \in \Omega$ it holds

$$\lim_j \|r_j\|_{L^\infty(\Omega,\mathbb{P})} = 0, \quad (1 \le) \limsup_j \|R_j/r_j\|_{L^\infty(\Omega,\mathbb{P})} < +\infty, \tag{4.8}$$

$$\lim_j \#(\Lambda_j(\omega) \cap U) r_j^n(\omega) = \theta(\omega) \in (0, +\infty), \tag{4.9}$$

$$\mu_j(\cdot,\omega) := \frac{1}{\#(\Lambda_j(\omega)\cap U)} \sum_{\mathtt{i}\in\Lambda_j(\omega)\cap U} \delta_{\mathtt{x}_j^{\mathtt{i}}(\omega)}(\cdot) \to \mu(\cdot,\omega) := \beta(\cdot,\omega)\mathcal{L}^n \llcorner U \quad w^*\text{-}C_b(U), \tag{4.10}$$

for some $\mathcal{B}(U)\otimes\mathscr{P}$ measurable function β such that $\beta(\cdot,\omega) \in L^1(U,[0,+\infty])$ and $\|\beta(\cdot,\omega)\|_{L^1(U)} = 1$.

Remark 4.4. *In (4.10) the set of indices $\Lambda_j(\omega)\cap U$ susbsitutes $\mathcal{I}_j(U,\omega)$ to ensure the measurability of θ, β. For, the measurability of $\#(\Lambda(\omega)\cap U)$ follows easily from that of the random Delone set Λ.*

Despite this, since $0 \le \#(\Lambda_j(\omega)\cap U) - \#\mathcal{I}_j(U,\omega) \le \#\mathscr{I}_j(U,\omega)$, by (4.8) we have for $\mathbb{P}$ a.e. $\omega\in\Omega$

$$\mu_j(\cdot,\omega) - \frac{1}{\#\mathcal{I}_j(U,\omega)} \sum_{\mathtt{i}\in\mathcal{I}_j(U,\omega)} \delta_{\mathtt{x}_j^{\mathtt{i}}(\omega)}(\cdot) \to 0 \quad w^*\text{-}C_b(U), \quad \lim_j \#\mathcal{I}_j(U,\omega) r_j^n(\omega) = \theta(\omega). \tag{4.11}$$

Remark 4.5. *Contrary to the deterministic setting we do not know whether conditions (4.9), (4.10) are satisfied up to the subsequences or not. Nevertheless, in Examples 4.9, 4.10 we show some sets of points satisfying all the conditions listed above. The only piece of information we extract from (4.8) is that for a subsequence (not relabeled in what follows) we have the convergence*

$$\langle\mu_j(\cdot,\omega),\varphi\rangle_{C_b(U),\mathcal{P}(U)} \to \langle\mu(\cdot,\omega),\varphi\rangle_{C_b(U),\mathcal{P}(U)} \qquad w^*\text{-}L^\infty(\Omega,\mathbb{P}) \quad \text{for all } \varphi\in C_b(U). \tag{4.12}$$

Thus, condition (4.10) is equivalent to the strong convergence in $L^1(\Omega,\mathbb{P})$ for all $\varphi\in C_b(U)$ of the (sub)sequences $(\langle\mu_j(\cdot,\omega),\varphi\rangle_{C_b(U),\mathcal{P}(U)})$.

To establish (4.12) take note that μ_j is a Young Measure (see [8]*). More precisely, $\mu_j : (\Omega,\mathscr{P},\mathbb{P}) \to \mathcal{P}(U)$, being $\mathcal{P}(U)$ the space of probability measures on U, is a measurable map, i.e. $\mu_j(A,\omega)$ is* *measurable for all $A \in \mathcal{B}(U)$. By taking into account the uniform conditon* (4.8) *and by arguing as in Remark 3.2 we infer that the family $(\mu_j)_{j\in\mathbf{N}}$ is parametrized tight (see* [8, Definition 3.3]*): given $\delta > 0$ for a compact set $C_\delta \subset U$ it holds*

$$(\mathbb{P} \otimes \mu_j)(\Omega \times C_\delta) = \int_\Omega \mu_j(C_\delta, \omega) d\mathbb{P}(\omega) \geq 1 - \delta.$$

Thus, by parametrized Prohorov theorem (see [8, Theorem 4.8]*)* (4.12) *holds true up to a subsequence.*

Finally, in view of Remark 3.2 and (4.8) *we note that it is not restrictive to suppose that the limit measure in* (4.10) *is absolutely continuous w.r.to $\mathcal{L}^n \llcorner U$ $\mathbb{P}$ a.s. in Ω.*

We fix a bounded set $T \subset \mathbf{R}^n$ and define the obstacle set $T_j(\omega) := \cup_{\mathtt{i}\in\mathbf{Z}^n} T_j^{\mathtt{i}}(\omega)$ as the union of random rescaled and translated copies of T according to (3.5), i.e.

$$T_j^{\mathtt{i}}(\omega) := \mathrm{x}_j^{\mathtt{i}}(\omega) + \lambda_j(\omega) T, \qquad \text{where } \lambda_j(\omega) := r_j(\omega)^{n/(n-sp)}. \tag{4.13}$$

The asymptotics of the energies $\mathcal{K}_j : L^p(U) \times \Omega \to [0, +\infty]$ given by

$$\mathcal{K}_j(u,\omega) = \begin{cases} \mathcal{K}(u) & \text{if } u \in W^{s,p}(U),\ \tilde{u} = 0 \ \mathrm{cap}_{s,p} \text{ q.e. on } T_j(\omega) \cap U \\ +\infty & \text{otherwise.} \end{cases} \tag{4.14}$$

where $\mathcal{K}$ is the functional in (3.50), is a straightforward generalization of Theorem 3.3.

Theorem 4.6. *Let $U \in \mathcal{A}(\mathbf{R}^n)$ be bounded and connected with Lipschitz regular boundary.*

Assume $\Lambda_j : \Omega \to (\mathbf{R}^n)^{\mathbf{Z}^n}$ is a sequence of random Delone sets satisfying (4.8)-(4.10) *$\mathbb{P}$ a.s. in Ω. Then, $\mathbb{P}$ a.s. in Ω the sequence $(\mathcal{K}_j(\cdot,\omega))$ Γ-converges in the $L^p(U)$ topology to $\mathscr{K} : L^p(U) \times \Omega \to [0, +\infty]$ defined by*

$$\mathscr{K}(u,\omega) = \mathcal{K}(u) + \theta(\omega)\, \mathrm{cap}_{\mathcal{K}}(T) \int_U |u(x)|^p \beta(x,\omega)\, dx \tag{4.15}$$

if $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$.

To ensure that the Γ-limit is deterministic we introduce a sequential version of stationarity for random lattices as defined by Blanc, Le Bris and Lions [13], [14]: there exists a positive and infinitesimal sequence $(\delta_j)_{j\in\mathbf{N}}$ such that for all $\mathtt{i} \in \mathbf{Z}^n$ and $\mathbb{P}$ a.s. in Ω.

$$\Lambda_j(\tau_{\mathtt{i}}\omega) = \Lambda_j(\omega) - \mathtt{i}\delta_j. \tag{4.16}$$

Corollary 4.7. *If the assumptions of Theorem 4.6 hold true, if $(\Lambda_j)_{j\in\mathbf{N}}$ satisfies* (4.16)*, and if the family $(\tau_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$ is ergodic, then r_{Λ_j} is constant $\mathbb{P}$ a.s. in Ω for all $j \in \mathbf{N}$, and there exists $\hat{\beta} \in L^1(U)$ such that for $\mathbb{P}$ a.e. $\omega \in \Omega$ we have $\beta(\cdot,\omega) = \hat{\beta}$ $\mathcal{L}^n$ a.e. in U.*

In addition, if r_j is constant $\mathbb{P}$ a.s. in Ω for all $j \in \mathbf{N}$, then θ is constant $\mathbb{P}$ a.s. in Ω. In particular, the Γ-limit in (4.15) *is deterministic, i.e. $\mathbb{P}$ a.s. in Ω the functional in* (4.15) *takes the form*

$$\mathscr{K}(u,\omega) = \mathcal{K}(u) + \theta\, \mathrm{cap}_{\mathcal{K}}(T) \int_U |u(x)|^p \hat{\beta}(x)\, dx$$

for every $u \in W^{s,p}(U)$, $+\infty$ otherwise in $L^p(U)$.

Proof. We show that r_{Λ_j} is invariant under the action of $(\tau_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$, this suffices to conclude since ergodicity implies that the only invariant random variables are $\mathbb{P}$ a.s. equal to constants. We work with $\mathcal{I}_j(A,\omega)$ in place of $\Lambda_j(\omega)\cap A$, since $0\leq\#(\Lambda_j(\omega)\cap A)-\#\mathcal{I}_j(A,\omega)\leq\#\mathscr{I}_j(A,\omega)$ for all $A\in\mathcal{A}(U)$ as already pointed out for $A=U$ in Remark 4.4.

Given any $\mathtt{i}\in\mathbf{Z}^n$, (4.16) yields $V_j^{\mathtt{k}}(\tau_{\mathtt{i}}\omega)=V_j^{\mathtt{k}_{\mathtt{i}}}(\omega)-\mathtt{i}\delta_j$ for some $\mathtt{k}_{\mathtt{i}}\in\mathbf{Z}^n$, which implies in turn that $r_{\Lambda_j}(\tau_{\mathtt{i}}\omega)=r_{\Lambda_j}(\omega)$, and thus r_{Λ_j} is equal to a constant $\mathbb{P}$ a.s. in Ω for every $j\in\mathbf{N}$.

In addition, we have also that $\#\mathcal{I}_j(A,\tau_{\mathtt{i}}\omega)=\#\mathcal{I}_j(A+\mathtt{i}\delta_j,\omega)$ for every $A\in\mathcal{A}(U)$, then if $\mathcal{L}^n(\partial A)=0$ for any $\delta>0$ and j sufficiently big it holds $\#\mathcal{I}_j(A_{-\delta},\omega)\leq\#\mathcal{I}_j(A,\tau_{\mathtt{i}}\omega)\leq\#\mathcal{I}_j(A_\delta,\omega)$ (see (2.1) for the definition of A_δ and $A_{-\delta}$). In particular, we infer

$$|\#\mathcal{I}_j(A,\tau_{\mathtt{i}}\omega)-\#\mathcal{I}_j(A,\omega)|\leq\#(\mathcal{I}_j\cup\mathscr{I}_j)(A_\delta\setminus\overline{A},\omega)\vee\#(\mathcal{I}_j\cup\mathscr{I}_j)(A\setminus\overline{A_{-\delta}},\omega). \tag{4.17}$$

If $\omega\in\Omega$ is such that (4.10) holds, for any $x_0\in U$ and $r\in(0,\operatorname{dist}(x_0,\partial U))$ we have by (2.4), $(4.11)_1$ and (4.17)

$$\int_{B_r(x_0)}\beta(x,\tau_{\mathtt{i}}\omega)dx=\lim_j\frac{\#\mathcal{I}_j(B_r(x_0),\tau_{\mathtt{i}}\omega)}{\#\mathcal{I}_j(U,\tau_{\mathtt{i}}\omega)}=\lim_j\frac{\#\mathcal{I}_j(B_r(x_0),\omega)}{\#\mathcal{I}_j(U,\omega)}=\int_{B_r(x_0)}\beta(x,\omega)dx.$$

In turn, from the latter equality we infer that if

$$\hat\beta(x_0,\omega):=\limsup_{r\to0^+}\fint_{B_r(x_0)}\beta(x,\omega)dx,$$

then $\hat\beta(x_0,\omega)=\hat\beta(x_0,\tau_{\mathtt{i}}\omega)$ for all $\mathtt{i}\in\mathbf{Z}^n$, $x_0\in U$ and for $\mathbb{P}$ a.e. $\omega\in\Omega$. Thus, $\hat\beta(x_0,\omega)$ is $\mathbb{P}$ a.s. equal to a constant for every $x_0\in U$; Lebesgue-Besicovitch differentiation theorem yields the conclusion.

Eventually, suppose that r_j is constant, then (4.17), $\mathcal{L}^n(\partial U)=0$ and (4.8) imply that $\theta(\tau_{\mathtt{i}}\omega)=\theta(\omega)$ for all $\mathtt{i}\in\mathbf{Z}^n$. In conclusion, θ is equal to a constant $\mathbb{P}$ a.s. in Ω. □

We discuss some examples related to sets of points introduced in [13] and [14]. As before, $(\varepsilon_j)_{j\in\mathbf{N}}$ denotes a positive infinitesimal sequence.

Example 4.8. *Let us consider ensembles of points which are stationary perturbations of a standard periodic lattice. More precisely, given a random variable $X:(\Omega,\mathscr{P},\mathbb{P})\to\mathbf{R}^n$ define the family $(X_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$ by $X_{\mathtt{i}}(\omega):=X(\tau_{\mathtt{i}}(\omega))$. By construction $(X_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$ is stationary w.r.to $(\tau_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$. Let $\Lambda(\omega):=\{\mathrm{x}^{\mathtt{i}}(\omega)\}_{\mathtt{i}\in\mathbf{Z}^n}$ where $\mathrm{x}^{\mathtt{i}}(\omega):=\mathtt{i}+X_{\mathtt{i}}(\omega)$, and $\Lambda_j(\omega):=\varepsilon_j\Lambda(\omega)$. A simple computation shows that (4.16) is satisfied with $\delta_j=\varepsilon_j$. Moreover, if $M:=\sup_{\mathtt{i}}\|X_{\mathtt{i}}-X\|_{L^\infty(\Omega,\mathbb{P})}<1$, then Λ is a random Delone set with $1-M\leq r_\Lambda\leq R_\Lambda\leq1+M$ $\mathbb{P}$ a.s. in Ω. In conclusion, $(\Lambda_j)_{j\in\mathbf{N}}$ satisfies both (4.8) and (4.16). We do not know which additional conditions must be imposed on X to ensure (4.9) and (4.10).*

Example 4.9. *We consider a stochastic diffeomorphism as introduced by Blanc, Le Bris and Lions [13]; that is a field $\Phi:\mathbf{R}^n\times\Omega\to\mathbf{R}^n$ such that $\Phi(\cdot,\omega)$ is a diffeomorphism for $\mathbb{P}$ a.e. $\omega\in\Omega$ satisfying*

$$\operatorname{ess-sup}_{\mathbf{R}^n\times\Omega}\|\nabla\Phi(x,\omega)\|\leq M<+\infty,\quad\textit{and}\quad\operatorname{ess-inf}_{\mathbf{R}^n\times\Omega}\det\nabla\Phi(x,\omega)\geq\nu>0.$$

Recalling Example 3.6 (see also [13, Proposition 4.7]), let $\Lambda_j(\omega) = \{\Phi(\varepsilon_j \mathtt{i}, \omega)\}_{\mathtt{i}\in\mathbf{Z}^n}$*, then* $\mathbb{P}$ *a.s. in* Ω *it holds* $(\nu M^{1-n}/2)\varepsilon_j \leq r_{\Lambda_j}(\omega) \leq R_{\Lambda_j}(\omega) \leq M\varepsilon_j$ *and*

$$\beta(x,\omega) = \left(\int_U \det \nabla\Phi^{-1}(x,\omega)\, dx\right)^{-1} \det \nabla\Phi^{-1}(x,\omega).$$

By choosing r_j *such that* $r_j/\varepsilon_j \to \gamma > 0$ $\mathbb{P}$ *a.s. in* Ω*, we have* $\theta = \gamma^n \int_U \det \nabla\Phi^{-1}(x,\cdot)\, dx$ $\mathbb{P}$ *a.s. in* Ω*. In conclusion,* Λ_j *are random Delone sets satisfying (4.8)-(4.10).*

Example 4.10. *Consider as before a stochastic diffeomorphism, and assume in addition* Φ *to be stationary w.r.to* $(\tau_{\mathtt{i}})_{\mathtt{i}\in\mathbf{Z}^n}$*, that is for every* $\mathtt{i} \in \mathbf{Z}^n$*, for a.e.* $x \in \mathbf{R}^n$ *and for* $\mathbb{P}$ *a.s. in* Ω *it holds*

$$\Phi(x, \tau_{\mathtt{i}}(\omega)) = \Phi(x + \mathtt{i}, \omega).$$

Let $\Lambda_j(\omega) = \{\varepsilon_j \Phi(\mathtt{i}, \omega)\}_{\mathtt{i}\in\mathbf{Z}^n}$*, we have* $(\nu M^{1-n}/2)\varepsilon_j \leq r_{\Lambda_j}(\omega) \leq R_{\Lambda_j}(\omega) \leq M\varepsilon_j$*. Thus* Λ_j *are random Delone sets satisfying (4.8). For what (4.9) and (4.10) are concerned by [12, Lemmas 2.1, 2.2] and [13, Remark 1.9],* $\mathbb{P}$ *a.s. in* Ω *it holds*

$$\#(\Lambda_j(\omega) \cap U)\varepsilon_j^n \to \left(\det\left(\mathbb{E}\left[\int_{[0,1]^n} \nabla\Phi(x,\cdot)\, dx\right]\right)\right)^{-1} \mathcal{L}^n(U).$$

Then by using (3.4) it follows

$$\frac{1}{\#(\Lambda_j(\omega) \cap U)} \sum_{\mathtt{i}\in\Lambda_j(\omega)\cap U} \delta_{\mathrm{x}_j^{\mathtt{i}}(\omega)} \to \frac{1}{\mathcal{L}^n(U)} \mathcal{L}^n \llcorner U \quad w^*\text{-}C_b(U).$$

Take also note that Λ_j *satisifes (4.16) with* $\delta_j \equiv 0$*. Hence,* r_{Λ_j} *is constant by Corollary 4.7, and moreover if we choose* $r_j \in (0, \nu M^{1-n}\varepsilon_j/2)$ *such that* $r_j/\varepsilon_j \to \gamma > 0$*, the* Γ*-limit of the energies in (4.14) is given by the functional*

$$\mathscr{K}(u) = \mathcal{K}(u) + \frac{\gamma^n \operatorname{cap}_{\mathcal{K}}(T)}{\det\left(\mathbb{E}\left[\int_{[0,1]^n} \nabla\Phi(x,\cdot)\, dx\right]\right)} \int_U |u(x)|^p\, dx.$$

for $u \in W^{s,p}(U)$*,* $\mathscr{K}(u) \equiv +\infty$ *otherwise in* $L^p(U)$*.*

Eventually, let us remark that stationary diffeomorphisms according to Blanc, Le Bris and Lions [13] satisfy the weaker condition $\nabla\Phi(x, \tau_{\mathtt{i}}\omega) = \nabla\Phi(x + \mathtt{i}, \omega)$*. The sets of points generated by* Φ *are not stationary according to (4.16) in general.*

Appendix A.

We prove some elementary bounds on the singular kernels that were crucial in the computations of subsections 3.2 and 3.3.

Lemma A.1. *Let* $\nu > 0$*, then there exists a positive constant* $c(n,\nu)$ *such that for every measurable set* $O \subset \mathbf{R}^n$ *it holds* *(i) if* $\nu \in (0,n)$ *and* $\operatorname{dist}(z,O)=0$

$$\int_O \frac{1}{|x-z|^\nu}dx \le c(n,\nu)(\mathcal{L}^n(O))^{1-\nu/n}. \tag{A.1}$$

(ii) if $\nu \in (n,+\infty)$ *and* $\operatorname{dist}(z,O)>0$

$$\int_O \frac{1}{|x-z|^\nu}\,dx \le c(n,\nu)\,(\operatorname{dist}(z,O))^{n-\nu}\,, \tag{A.2}$$

Proof. The lemma is an easy application of Cavalieri formula.

Let us start with (i). Clearly, we may suppose $\mathcal{L}^n(O)<+\infty$ the inequality being trivial otherwise. Then, by setting $\overline{s}=(\mathcal{L}^n(O)/\omega_n)^{1/n}$ a direct integration yields[1]

$$\int_O |x-z|^{-\nu}dx = \int_0^{+\infty}\mathcal{L}^n(\{x\in O:\,|x-z|\le t^{-1/\nu}\})dt = \nu\int_0^{\operatorname{diam}(O)}\frac{\mathcal{L}^n(\{x\in O:\,|x-z|\le s\})}{s^{1+\nu}}ds$$

$$= \nu\left(\int_0^{\overline{s}}+\int_{\overline{s}}^{\operatorname{diam}(O)}\right)\ldots ds \le \nu\omega_n\int_0^{\overline{s}}s^{n-\nu-1}ds+\nu\mathcal{L}^n(O)\int_{\overline{s}}^{\operatorname{diam}(O)}s^{-\nu-1}ds$$

$$= \frac{\nu}{n-\nu}\omega_n^{\nu/n}(\mathcal{L}^n(O))^{1-\nu/n}+\mathcal{L}^n(O)\left(-(\operatorname{diam}(O))^{-\nu}+\left(\frac{\mathcal{L}^n(O)}{\omega_n}\right)^{-\nu/n}\right)\le c(n,\nu)(\mathcal{L}^n(O))^{1-\nu/n}.$$

Inequality (A.2) easily follows from a direct integration. More precisely, we have

$$\int_O |x-z|^{-\nu}\,dx \le \int_{\mathbf{R}^n\setminus\overline{B}_{\operatorname{dist}(z,O)}(z)}|x-z|^{-\nu}\,dx = \frac{n\omega_n}{\nu-n}\,(\operatorname{dist}(z,O))^{n-\nu}\,.$$

□

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