Optimal Urban Networks via Mass Transportation

G. Buttazzo

A. Pratelli

S. Solimini

E. Stepanov

(G. Buttazzo) Dipartimento di Matematica, Univ. of Pisa - Via Buonarroti 2, 56127 Pisa, ITALY
E-mail address: `buttazzo@dm.unipi.it`

(A. Pratelli) Scuola Normale Superiore - Piazza dei Cavalieri 7, 56126 Pisa, ITALY
E-mail address: `aldo.pratelli@unipv.it`

(S. Solimini) Dipartimento di Matematica, Univ. of Bari, Via E. Orabona 4 70125 Bari, ITALY
E-mail address: `solimini@dm.uniba.it`

(E. Stepanov) Dipartimento di Matematica, Univ. of Pisa - Via Buonarroti 2, 56127 Pisa, ITALY
E-mail address: `e.stepanov@sns.it`

2000 *Mathematics Subject Classification.* 49J45, 49Q10, 49Q15, 49Q20, 90B06, 90B10,90B20

Key words and phrases. Mass transportation, optimal networks, urban planning

Abstract. The monograph is dedicated to a class of models of optimization of transportation networks (urban traffic networks or networks of railroads and highways) in the given geographic area. One assumes that the data on distributions of population and of services/workplaces (i.e. sources and sinks of the network) as well as the costs of movement with and without the help of the network to be constructed, are known. Further, the models take into consideration both the cost of everyday movement of the population and the cost of construction and maintenance of the network, the latter being determined by a given function on the total length of the network. The above data suffice, if one considers optimization in long-term prospective, while for the short-term optimization one also needs to know the transport plan of everyday movements of the population (i.e. the information on "who goes where"). Similar models can also be adapted for the optimization of networks of different nature, like telecommunication, pipeline or drainage networks. In the monograph we study the most general problem settings, namely, when neither the shape nor even the topology of the network to be constructed is a priori known.

To be more precise, given a region $\Omega \subseteq \mathbb{R}^N$, we will model the transportation network to be constructed by an a priori generic Borel set $\Sigma \subseteq \Omega$. We consider then the mass transportation problem in which the paths inside and outside the network Σ are charged differently. The aim is to find the best location for Σ, in order to minimize a suitable cost functional $\mathfrak{F}(\Sigma)$, which is given by the sum of the cost of transportation of the population, and the penalization term depending on the length of the network, which represents the cost of construction and maintenance of the network. To study the problem of existence of optimal solutions, we present first a relaxed version of the optimization problem, where the network is represented by a Borel measure rather than a set, and we prove the existence of a relaxed solution. We will study then the properties of optimal relaxed solutions (measures) and prove that, under suitable assumptions, the relaxed solution solve the original problem, i.e. in fact they correspond to rectifiable sets, and therefore can be called classical solutions. However, it will be shown that in general the problem studied may have no classical solutions. We will also study some topological properties of optimal networks, like closedness and the number of connected components. In particular, we find rather sharp conditions on problem data, which ensure the existence of closed optimal networks and/or optimal networks having at most countably many connected components. Finally, we will prove a general regularity result on optimal networks. Namely, we will show that an optimal network is covered by a finite number of Lipschitz curves of uniformly bounded length, although it may have even uncountably many connected components.

Contents

Introduction

The present monograph treats one particular class of mathematical models arising in urban planning, namely, the models of optimization of transportation networks such as urban traffic networks, networks of tram or metro lines, railroads or highways. The optimization is performed so as to take into account the known data of the distributions of the population and of services/workplaces (or, more generally, sources and sinks of the network), the costs of the transportation with and without using the network to be constructed, and the budgetary restrictions on construction and maintenance of the network, as well as, in certain cases, the transportation plan of everyday movement of the population. The functional to be minimized corresponds to the overall cost of everyday transportation of population from their homes to the services together with the cost of construction and maintenance of the network. It is important to emphasize that the shape and even the topology of the network is considered a priori unknown.

From the most general point of view such models belong to the class of economical optimal resource planning problems which were first studied in [**33**]. In the simplest cases under additional restrictions on the network such problems reduce to problems of minimization of so called average distance functionals (see [**17**]), and are similar to the well-known discrete problems of optimization of service locations (so-called Fermat-Weber, or k-median problems) studied by many authors (see, e.g. [**5, 54, 55, 38**]). Similar as well as slightly different models have been proposed for telecommunication, pipeline and drainage networks in [**9, 30, 35**], and are recently subject to extensive study (see, for instance, [**6, 7, 8, 14, 22, 25, 36, 42, 43, 48, 52, 39, 56, 57**]. The common kernel of all such models is the general (i.e. not necessarily discrete) setting of the Monge-Kantorovich optimal transportation problem (see, e.g. [**31, 32, 53, 1, 27, 26, 46, 20, 29**]), which we recall below.

Given a metric space (X, d) and two finite Borel measures f^+ and f^- with the same total mass, the Monge transportation problem (see for instance [**1, 46**]) consists in determining, among all the *transport maps*, i.e. the Borel maps $T : X \to X$ such that $T_\# f^+ = f^-$ (where $T_\#$ stands for the push-forward operator), the ones which minimize the transportation cost

$$\int_X c\big(x, T(x)\big)\, df^+(x)\,,$$

where $c : X \times X \to \mathbb{R}^+$ is a given lower semicontinuous function ($c(x,y) = d(x,y)$ in the classical Monge problem). The existence of optimal transport maps has been widely investigated in many papers (we refer for instance to [**1, 2, 20, 27, 44**] for an overview on the problem). In many situations, for instance when the measure f^+ has singular parts, the existence of optimal transport maps may fail and the problem has to be considered in the relaxed form introduced by Kantorovich, where one looks for optimal transport plans, i.e. the nonnegative measures γ on $X \times X$ which minimize the transportation cost

$$\iint_{X\times X} c(x,y)\, d\gamma(x,y)$$

among all the transport plans. The *transport plans* (see Definition 1.4) are the measures γ on $X \times X$ with marginals $\pi_{1\#}\gamma$ and $\pi_{2\#}\gamma$ respectively equal to f^+ and f^-; the meaning of such a measure γ is to represent the way to transport f^+ onto f^- that, roughly speaking, moves a mass $\gamma\big(\{(x,y)\}\big)$ from the point x to the point y. More formally we should say that, for any couple of subsets C, D of X, γ moves a total amount $\gamma(C \times D)$ of mass from the set C to the set D.

In this monograph we consider a problem of urban planning, in which we take as ambient space a region $\Omega \subseteq \mathbb{R}^N$, with $N \geq 2$ since the one-dimensional case is in fact trivial; the measure f^+ represents the density of the population in the urban area Ω and the measure f^- represents the density of the services or workplaces. We also consider a Borel set $\Sigma \subseteq \Omega$ of finite $\mathscr{H}^1$ length, which represents the urban transportation network that has to be constructed to minimize the cost of transporting f^+ on f^- according to some suitable cost functional.

Once the set Σ is given, the cost $d_\Sigma(x,y)$ to be paid in order to connect any two points x and y of Ω is defined as the least "price" of moving along a Lipschitz curve connecting x and y given by the number

$$\delta_\Sigma(\theta) := A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big)\,.$$

The functions A and B are two given nondecreasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $A(0) = B(0) = 0$, A being continuous and B lower semicontinuous: $A(s)$ is the "cost" of covering a distance s by own means, that is a number including the expenses for the fuel, the fare of the highway, the fatigue of moving by feet, the time consumption and so on; on the other hand, $B(s)$ represents the cost of covering the distance s making use of the transportation network (i.e. the "cost of the ticket").

In this monograph, we assume the point of view of an "ideal city", where the only goal is to minimize the total expenses for the people; therefore, the number $B(s)$ should be regarded just as a tax that people pay to contribute to the cost of the network when they use it, and the case $B \equiv 0$, corresponding to a situation where everybody can use the public transportation for free, is the simplest (and most common in the literature) choice in this ideal setting. An opposite point of view, where the owner of the network aims to maximize his total income by choosing a suitable pricing policy B, has been studied in [**15**].

Having fixed the set Σ, the population will naturally try to minimize its expenses, that is, people choose to move following a transport plan γ minimizing

$$I_\Sigma(\gamma) := \iint_{X\times X} d_\Sigma(x,y)\,d\gamma(x,y)$$

among all admissible transport plans, and we denote by $MK(\Sigma)$ the respective minimum (or the infimum if the minimum is not achieved). We want to find a network Σ minimizing the total cost for the people. However, $MK(\Sigma)$ is not the only cost to be considered: otherwise, a network of infinite length covering the whole Ω would be clearly the optimal choice. We will then consider also a very general cost function $H\big(\mathscr{H}^1(\Sigma)\big)$ for the maintenance of the network, that will depend on the length $\mathscr{H}^1(\Sigma)$ of Σ and that diverges if the length goes to ∞. For instance, one can set

$$H(l) := \begin{cases} 0, & \text{if } l \le L\,, \\ +\infty, & \text{if } l > L\,, \end{cases}$$

which corresponds to a situation where one is allowed to build a network of total length not exceeding L. Our goal is then to find an optimal network Σ_{opt} which minimizes $\Sigma \mapsto MK(\Sigma) + H\big(\mathscr{H}^1(\Sigma)\big)$ among the admissible sets Σ.

The above problem can be considered as a long-term optimization model. In fact, in this case while choosing the optimal network Σ one is allowed to change freely the transportation plan γ (i.e. it is supposed that people may consider it more convenient to choose different destinations for their everyday movements, e.g. change the shops they usually use or even change their workplace, in view of the cost of transportation), which is only reasonable in a quite long-term prospective. On the contrary, the reasonable model for the short-term prospective is obtained by considering given the transport plan γ (i.e. the information on “who goes where” in the everyday movements) and thus minimizing $\Sigma \mapsto I_\Sigma(\gamma) + H\big(\mathscr{H}^1(\Sigma)\big)$ among the admissible sets Σ. However, it is easy to notice, similarly to [**15**], that the short-term optimization problem is in fact much simpler than the long-term one. Hence in this monograph we concentrate on studying the latter with all the results applying also to the former.

Plan of the monograph.

In Chapter 1 we define the general problem setting, and in Chapter 2 we briefly report some known facts in the case when Σ is a priori required to be connected.

In Chapter 3 we show that without extra assumptions on the functions A, B and H there may be no optimal networks. Therefore, we introduce a relaxed version of the problem, where the sets are replaced by Radon measures, and in particular each set Σ corresponds to the measure $\mathscr{H}^1 \llcorner \Sigma$. Then, we show the existence of optimal “relaxed networks”, and in particular we prove that optimal measures μ on Ω of the form $\mu = a(x)\mathscr{H}^1 \llcorner \Sigma$ for a one-dimensional rectifiable set Σ and a Borel function $a : \Sigma \to [0,1]$ always exist. Roughly speaking, this means that there is an optimal transportation network concentrated on a Borel set Σ, but it has a pointwise density in $[0,1]$: the density 1 corresponds to a standard railway, where covering a path of length l has a cost $B\big(l\big)$. In general, covering a path of length l on a network of density $0 \leq p \leq 1$ costs $A\big((1-p)l\big) + B\big(pl\big)$, as if one covers a length pl on the network, and the remaining $(1-p)l$ by own means. Moreover we show that, under suitable assumptions, there are also "classical solutions", that is, optimal networks which naturally correspond to sets (in other words, relaxed solutions with the coefficient $a(x)$ above taking only values 0 and 1). However, we give counterexamples showing that this does not always occur.

In Chapter 4 we consider two questions, namely whether or not there exists an optimal classical network which is closed, or which has only countably many connected components. We present counterexamples to show that this is not always the case, even when classical solutions exist. However, we are able to find conditions under which one has the existence of an optimal classical network that is closed or has countably many connected components.

Finally, in Chapter 5 we prove that, under suitable hypotheses, there is a classical optimal network that is covered by a finite number of Lipschitz curves of uniformly bounded length, even if it may still have infinitely many (even more than countably many) connected components.

CHAPTER 1

Problem setting

In this chapter we introduce the notation and the preliminaries to rigorously set the problem of optimal networks. The formulation in the sense of L. Kantorovich, by using *transport plans*, i.e. measures on the product space $\Omega\times\Omega$, will be presented together with a second equivalent formulation where the main tools are the so-called *transport path measures* that are measures on the family of curves in Ω. This seems to be a very natural formulation that has already been used in previous papers (see for instance [**19, 51, 4, 45**]) and that allows to obtain in a rather simple way existence results and necessary conditions of optimality.

1. Notation and preliminaries

In this monograph the ambient space Ω is assumed to be a bounded, closed, $N-$dimensional convex subset of $\mathbb{R}^N$, $N \geq 2$, equipped with the Euclidean distance; the convexity assumption is made here only for simplicity of presentation; in fact, all the results are still valid in the more general case of bounded Lipschitz domains. For any pair of Lipschitz paths θ_1, $\theta_2 : [0,1] \to \Omega$, we introduce the distance

$$
\begin{aligned}
d_\Theta(\theta_1,\theta_2) := \inf \Big\{ &\max_{t\in[0,1]} |\theta_1(t) - \theta_2(\varphi(t))| \,, \\
&\varphi : [0,1] \to [0,1] \text{ increasing and bijective} \Big\} ,
\end{aligned} \tag{1.1}
$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. We define then Θ as the set of the equivalence classes of Lipschitz paths in Ω parametrized over $[0,1]$, where two paths θ_1 and θ_2 are considered equivalent whenever $d_\Theta(\theta_1,\theta_2) = 0$: it is easily noticed that Θ is a metric space equipped with the distance d_Θ. Moreover, simple examples show that the infimum in (1.1) might not be attained. It will be often useful to remind that, given any sequence $\{\theta_n\}$ of paths in Θ with uniformly bounded Euclidean lengths, by Ascoli–Arzelà Theorem one can find a $\theta \in \Theta$ such that (possibly up to a subsequence) $\theta_n \xrightarrow{d_\Theta} \theta$. This implies, in particular, that the corresponding curves $\theta_n([0,1])$ converge in the Hausdorff distance to $\theta([0,1])$, while the converse implication is not true. Notice that

$$
\theta_n \xrightarrow{d_\Theta} \theta \quad \Longrightarrow \quad \mathscr{H}^1(\theta) \leq \liminf_{n\to\infty} \mathscr{H}^1(\theta_n) \,,
$$

where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure.

In the sequel, for the sake of brevity we will abuse the notation calling θ also the set $\theta([0,1]) \subseteq \Omega$, when not misleading. We call *endpoints* of the path θ the points $\theta(0)$ and $\theta(1)$, and, given two paths $\theta_1, \theta_2 \in \Theta$ such that $\theta_1(1) = \theta_2(0)$, the *composition* $\theta_1 \cdot \theta_2$ is defined by the formula

$$\theta_1 \cdot \theta_2(t) := \begin{cases} \theta_1(2t) & \text{for } 0 \leq t \leq 1/2\,, \\ \theta_2(2t-1) & \text{for } 1/2 \leq t \leq 1\,. \end{cases}$$

As already introduced in Chapter , we let now A, $B : \mathbb{R}^+ \to \mathbb{R}^+$ be the costs of moving by own means and by using the network, i.e. $A(s)$ (resp. $B(s)$) is the cost corresponding to a part of the itinerary of length s covered by own means (resp. with the use of the network). This means that, if the urban network is a Borel set $\Sigma \subseteq \Omega$ of finite length, the total cost of covering a path $\theta \in \Theta$ is given by

$$\delta_\Sigma(\theta) := A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big)\,, \tag{1.2}$$

since the length $\mathscr{H}^1(\theta \setminus \Sigma)$ is covered by own means and the length $\mathscr{H}^1(\theta \cap \Sigma)$ is covered by the use of the network. Concerning the functions A and B, we make from now on the following assumptions:

$$A \text{ is nondecreasing, continuous and } A(0) = 0\,; \tag{1.3}$$

$$B \text{ is nondecreasing, l.s.c. and } B(0) = 0\,. \tag{1.4}$$

Note that these hypotheses follow the intuition: the meaning of the assumptions $A(0) = 0$, $B(0) = 0$ and of the monotonicity are obvious, while the continuity of the function A means that a slightly longer path cannot have a much higher cost, and it is a natural assumption once one moves by own means. On the contrary, a continuity assumption on the function B would rule out some of the most common pricing policies which occur in many real life urban transportation networks: for instance, often such a pricing policy is given by a fixed price (the price of a single ticket) for any positive distance, or is a piecewise constant function.

We define now a "distance" on Ω which depends on Σ and is given by the least cost of the paths connecting two points: in short,

$$d_\Sigma(x,y) := \inf\left\{\delta_\Sigma(\theta) :\ \theta \in \Theta,\, \theta(0) = x,\, \theta(1) = y\right\}\,. \tag{1.5}$$

The infimum in the above definition is not always attained, as we will see in Example 1.8. Moreover, it has to be pointed out that in general the function d_Σ is not a distance; for instance, with $A(s) = B(s) = s^2$ it is easy to see that the triangle inequality does not hold. However, when A and B are subadditive functions, i.e.

$$A(s_1 + s_2) \leq A(s_1) + A(s_2) \text{ for all } s_1,\ s_2\ \in \mathbb{R}^+\,,$$
$$B(s_1 + s_2) \leq B(s_1) + B(s_2) \text{ for all } s_1,\ s_2\ \in \mathbb{R}^+\,,$$

and they are strictly positive on $(0, +\infty)$, then an easy computation shows that d_Σ is in fact a distance (the strict positivity is needed to ensure that

$d_\Sigma(x,y)=0$ implies $x=y$). Nevertheless, with an abuse of notation, we will call d_Σ a distance in any case.

LEMMA 1.1. *For any $\theta\in\Theta$ and any $\varepsilon>0$, there is a path $\theta_\varepsilon\in\Theta$ such that*

$$\theta_\varepsilon(0)=\theta(0)\,,\qquad \theta_\varepsilon(1)=\theta(1)\,,\qquad d_\Theta(\theta,\theta_\varepsilon)<\varepsilon\,,$$
$$\mathscr{H}^1(\theta_\varepsilon)<\mathscr{H}^1(\theta)+\varepsilon\,,\qquad \mathscr{H}^1(\theta_\varepsilon\cap\Sigma)=0\,.$$

PROOF. Since $\Omega\subseteq\mathbb{R}^N$ and $N\geq 2$, we can take a more than countable family $\{\theta_i\}_{i\in I}$ of elements of Θ such that

- $\theta_i(0)=\theta(0)$ and $\theta_i(1)=\theta(1)$ for each $i\in I$;
- $d_\Theta(\theta,\,\theta_i)<\varepsilon$ and $\mathscr{H}^1(\theta_\varepsilon)<\mathscr{H}^1(\theta)+\varepsilon$ for each $i\in I$;
- for all $i,\,j\in I$ with $i\neq j$, $\theta_i\cap\theta_j$ consists of finitely many points.

The proof of this assertion is trivial if the curve θ is given by a finite union of segments, as Figure 1 shows. The general case is now easily achieved approximating any path θ by a finite union of segments as needed.

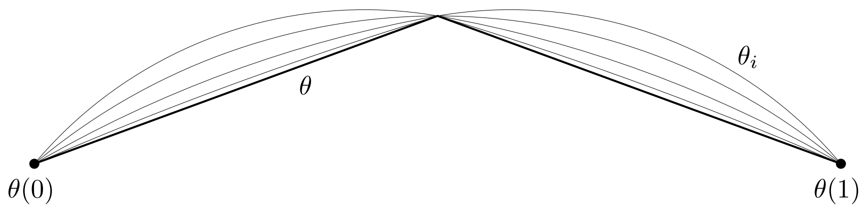

FIGURE 1. The path θ and some paths θ_i

The thesis can be then proved making use of the paths θ_i: since $\mathscr{H}^1(\Sigma)<\infty$, the condition $\mathscr{H}^1(\theta_i\cap\Sigma)>0$ may occur at most for a countable set of indices $i\in I$; one then concludes just by taking one of the remaining paths θ_i. □

COROLLARY 1.2. *For any $\theta\in\Theta$, $\varepsilon>0$ and $l\leq\mathscr{H}^1(\theta\cap\Sigma)$, there is a path $\theta_{l,\varepsilon}\in\Theta$ such that*

$$\theta_{l,\varepsilon}(0)=\theta(0)\,,\qquad \theta_{l,\varepsilon}(1)=\theta(1)\,,\qquad d_\Theta(\theta,\theta_{l,\varepsilon})<\varepsilon\,,$$
$$\mathscr{H}^1(\theta_{l,\varepsilon})<\mathscr{H}^1(\theta)+\varepsilon\,,\qquad \mathscr{H}^1(\theta_{l,\varepsilon}\cap\Sigma)=l\,.$$

PROOF. This follows easily by Lemma 1.1: let $t\in[0,1]$ be such that

$$\mathscr{H}^1\big(\theta([0,t])\big)=l\,,$$

and define θ_1 to be the restriction of θ to $[0,t]$ and θ_2 to be the restriction of θ to $[t,1]$, so that

$$\theta=\theta_1\cdot\theta_2\,,\qquad \mathscr{H}^1(\theta_1\cap\Sigma)=l\,.$$

It suffices then to apply Lemma 1.1 to θ_2 and to compose θ_1 with the resulting path. □

Proposition 1.3. *The function $d_\Sigma : \Omega \times \Omega \to \mathbb{R}^+$ is continuous.*

Proof. This is a consequence of (1.3): take $(x, y) \in \Omega \times \Omega$ and a path θ between x and y with

$$\delta_\Sigma(\theta) < d_\Sigma(x, y) + \varepsilon .$$

Then, given any pair $(\tilde{x}, \tilde{y}) \in \Omega \times \Omega$, we can define a path between $\tilde{x}$ and $\tilde{y}$ by setting $\tilde{\theta} := \alpha \cdot \theta \cdot \beta$ for any choice of paths α and β connecting $\tilde{x}$ to x and y to $\tilde{y}$ respectively. Thanks to Lemma 1.1, we may choose α and β having $\mathscr{H}^1$-negligible intersection with Σ and length less than $|x - \tilde{x}| + \varepsilon$ and $|y - \tilde{y}| + \varepsilon$ respectively. We infer thus

$$\begin{aligned} d_\Sigma(\tilde{x}, \tilde{y}) \le \delta_\Sigma(\tilde{\theta}) &\le A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x - \tilde{x}| + |y - \tilde{y}| + 2\varepsilon\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big) \\ &= \delta_\Sigma(\theta) + A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x - \tilde{x}| + |y - \tilde{y}| + 2\varepsilon\big) - A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) \\ &\le d_\Sigma(x, y) + \varepsilon + A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x - \tilde{x}| + |y - \tilde{y}| + 2\varepsilon\big) \\ &\qquad\qquad - A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) , \end{aligned}$$

and the upper semicontinuity of d_Σ follows since $\varepsilon > 0$ is arbitrary and A is continuous.

Concerning the lower semicontinuity of d_Σ, suppose that $x_n \to x$, $y_n \to y$ and that $d_\Sigma(x_n, y_n) \to d$ as $n \to \infty$. This means that there exist paths θ_n connecting x_n to y_n and satisfying $\delta_\Sigma(\theta_n) \to d$. Composing as before θ_n with short paths α_n and β_n connecting x to x_n and y_n to y respectively, and having

$$\mathscr{H}^1(\alpha_n \cap \Sigma) = \mathscr{H}^1(\beta_n \cap \Sigma) = 0 ,$$

we find the paths $\tilde{\theta}_n$ between x and y satisfying

$$\delta_\Sigma(\tilde{\theta}_n) = \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta_n \setminus \Sigma) + \mathscr{H}^1(\alpha_n) + \mathscr{H}^1(\beta_n)\big) - A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big) .$$

Since $\delta_\Sigma(\theta_n) \to d$ and since

$$\mathscr{H}^1(\alpha_n) + \mathscr{H}^1(\beta_n) \to 0 ,$$

the conclusion follows if $\mathscr{H}^1(\theta_n \setminus \Sigma)$ is uniformly bounded, because A is continuous hence uniformly continuous on compact sets. At last, if $\mathscr{H}^1(\theta_n \setminus \Sigma)$ is not uniformly bounded, then

$$\mathscr{H}^1(\theta_n \setminus \Sigma) > |x - y| + 1$$

for n arbitrarily large; in this case, we could directly take a path θ close to the segment connecting x to y and having negligible intersection with Σ, so that

$$\delta_\Sigma(\theta) = A\big(\mathscr{H}^1(\theta)\big) \le A\big(|x - y| + 1\big) \le A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big) \le \delta_\Sigma(\theta_n) ,$$

and hence, the thesis follows in this case too. $\square$

The problem we want to study is to find the best transportation network Σ to move the population from their “homes” to their “workplaces”. To set the problem, we consider two probability measures f^+, $f^- \in \mathcal{M}^+(\Omega)$

describing the distributions of homes and workplaces respectively. The following notion is often used in transportation theory; throughout the monograph, $\pi_i : \Omega \times \Omega \to \Omega$, $i = 1, 2$, stands for the i−th projection, and for a Borel map $g : X \to Y$ the push-forward $g_\# : \mathcal{M}^+(X) \to \mathcal{M}^+(Y)$ is defined by

$$g_\# \mu(A) := \mu\big(g^{-1}(A)\big) \qquad\qquad \text{for any Borel set } A \subseteq Y\,.$$

Definition 1.4. *A* transport plan *is a positive measure* $\gamma \in \mathcal{M}^+(\Omega \times \Omega)$*, the marginals of which are* f^+ *and* f^-*, i.e.*

$$\pi_{1\#}\gamma = f^+\,, \qquad\qquad \pi_{2\#}\gamma = f^-\,.$$

One can intuitively think that $\gamma(x, y)$ is the number of people moving from x to y, or, more precisely, that $\gamma(C \times D)$ is the number of people living in $C \subseteq \Omega$ and working in $D \subseteq \Omega$. To each transport plan γ we associate the total cost of transportation according to the formula

$$I_\Sigma(\gamma) := \iint_{\Omega\times\Omega} d_\Sigma(x, y)\, d\gamma(x, y)\,. \tag{1.6}$$

The Monge-Kantorovich optimal transport problem consists in finding a transport plan $\bar\gamma \in \mathcal{M}^+(\Omega \times \Omega)$ (which is usually called *optimal transport plan*) minimizing I_Σ.

It is important to notice that the transport plan γ gives no precise information on *how* the mass is moving (i.e. which trajectories are chosen for transportation). To be able to recover such an information we will make use of the following definition, already used in [**45**] (a quite similar idea was already used elsewhere, for instance in [**19, 51, 4**]).

Definition 1.5. *A* transport path measure *(shortly t.p.m. in the sequel) is a measure* $\eta \in \mathcal{M}^+(\Theta)$ *with the property that its first and last projections are* f^+ *and* f^-*, i.e.*

$$p_{0\#}\eta = f^+ \qquad\qquad p_{1\#}\eta = f^-\,, \tag{1.7}$$

where for $t = 0, 1$ *we denote by* $p_t : \Theta \to \Omega$ *the function* $p_t(\theta) := \theta(t)$*.*

It is important to understand the meaning of the above definition: roughly speaking, if η is a t.p.m., then $\eta(\theta)$ indicates the amount of mass to be moved along the path θ; more precisely, $\eta(E)$ is the mass following the paths contained in $E \subseteq \Theta$. The meaning of condition (1.7) is then clear, since $p_{0\#}\eta$ and $p_{1\#}\eta$ are respectively the measure from which η starts and the measure to which it is transported.

We are now able to define the total cost of transportation associated to any t.p.m. by the formula

$$C_\Sigma(\eta) := \int_\Theta \delta_\Sigma(\theta)\, d\eta(\theta)\,. \tag{1.8}$$

Finally, we denote by $MK(\Sigma)$ the infimum of the above costs, namely,

$$MK(\Sigma) := \inf \{ C_\Sigma(\eta) \,:\, \eta \text{ is a t.p.m.} \} \,. \tag{1.9}$$

The purpose of this monograph is to study the problem of finding the best possible network Σ: in other words, we want to find a set Σ having minimal total cost of usage (defined below). To do that, as already discussed in Chapter , we consider a function $H : \mathbb{R}^+ \to \overline{R}^+$, where $H(l)$ represents the maintenance cost of a network Σ of length $\mathscr{H}^1(\Sigma) = l$. We assume on H the natural conditions

$$H \text{ is nondecreasing, l.s.c., } H(0) = 0 \text{ and } H(l) \to \infty \text{ as } l \to \infty \,. \tag{1.10}$$

Finally, the *total cost* of usage of Σ is defined by the formula

$$\mathfrak{F}(\Sigma) := MK(\Sigma) + H\big(\mathscr{H}^1(\Sigma)\big) \,. \tag{1.11}$$

Our goal is to study the problem of minimizing the functional $\mathfrak{F}$.

2. Properties of optimal paths and relaxed costs

In (1.5) we defined a distance in Ω as the infimum of the costs of the paths connecting two given points. We show now the possibility to choose a Borel selection of paths which have almost minimal costs in the sense of proposition below.

PROPOSITION 1.6. *For any $\varepsilon > 0$ there is a Borel function $q_\varepsilon : \Omega \times \Omega \to \Theta$ such that $q_\varepsilon(x, y)$ is a path connecting x to y with*

$$\delta_\Sigma\big(q_\varepsilon(x, y)\big) < d_\Sigma(x, y) + \varepsilon \,. \tag{1.12}$$

PROOF. Fix a $\rho > 0$ and let $\{x_i\}$ be a finite set of points in Ω such that

$$\bigcup B(x_i, \rho) \supseteq \Omega \,.$$

Let then $C_{ij} \subseteq \Omega \times \Omega$ be pairwise disjoint Borel sets covering $\Omega \times \Omega$, each contained in $B\big((x_i, x_j), 2\rho\big)$. Now, given i, j, let $\theta_{ij} \in \Theta$ be a path connecting x_i to x_j and having a cost minimal up to an error ρ, that is

$$\delta_\Sigma(\theta_{ij}) < d_\Sigma(x_i, x_j) + \rho \,.$$

We claim that the conclusion follows if for every $x \in \Omega$ there is a Borel map

$$\alpha_x : B(x, 2\rho) \to \Theta$$

such that $\alpha_x(y)$ is a path between x and y with length less than 4ρ and having $\mathscr{H}^1-$negligible intersection with Σ. Indeed, defining on each C_{ij} the function q_ε by the formula

$$q_\varepsilon(x, y) := \widehat{\alpha_{x_i}(x)} \cdot \theta_{ij} \cdot \alpha_{x_j}(y)$$

(where $\hat\theta(t) := \theta(1 - t)$), one has that q_ε is a Borel function; moreover, if ρ is sufficiently small, one gets (1.12) by the continuity of A. It suffices therefore to prove the existence of such an α_x (observe that Lemma 1.1 already provides a map satisfying all the required conditions except for the Borel property). For this purpose, we begin defining $\alpha_x(y)$ as the line segment between x and y. Since Σ is rectifiable, such a segment has $\mathscr{H}^1$-negligible intersection with Σ unless y is contained in one of countably many radii $\{R_k\}_{k\in\mathbb{N}}$ of the ball $B(x,2\rho)$. For each $k \in \mathbb{N}$, choose arbitrarily a two-dimensional halfplane Π_k containing R_k on its boundary; then, for $y \in R_k$, define $\alpha_x(y)$ as the half circle joining x to y and lying on Π_k. Arguing as before, it is clear that such a path has $\mathscr{H}^1$-negligible intersection with Σ except for countably many points $y \in R_k$. Finally, for each of these latter y, by Lemma 1.1 we may arbitrarily select a path $\alpha_x(y)$ connecting x to y which is shorter than 4ρ and has $\mathscr{H}^1-$negligible intersection with Σ. The resulting function α_x has the required properties and so the proof is completed. $\square$

Corollary 1.7. *For any $\varepsilon > 0$ there is a Borel function $q'_\varepsilon : \Omega \times \Omega \to \Theta$ such that $q'_\varepsilon(x,y)$ is a path connecting x with y and satisfying*

$$\mathscr{H}^1(q'_\varepsilon(x,y)) \le |y-x| + \varepsilon\,, \qquad \mathscr{H}^1(q'_\varepsilon(x,y)\cap\Sigma) = 0\,.$$

Proof. Consider the case when

$$A(s) = s\,, \qquad B(s) = \operatorname{diam}\Omega + 2\varepsilon$$

for every $s > 0$. By Lemma 1.1 it is clear that $d_\Sigma(x,y) = |y-x|$ and that $\delta_\Sigma(\theta) = \mathscr{H}^1(\theta)$ whenever $\mathscr{H}^1(\theta\cap\Sigma) = 0$. Apply now Proposition 1.6 to find a map q'_ε such that

$$\delta_\Sigma(q'_\varepsilon(x,y)) < d_\Sigma(x,y) + \varepsilon = |y-x| + \varepsilon\,.$$

If

$$\mathscr{H}^1(q'_\varepsilon(x,y)\cap\Sigma) > 0\,,$$

then

$$\delta_\Sigma(q'_\varepsilon(x,y)) \ge \operatorname{diam}\Omega + 2\varepsilon > |y-x| + \varepsilon\,,$$

and this gives a contradiction. Thus,

$$\mathscr{H}^1(q'_\varepsilon(x,y)\cap\Sigma) = 0$$

and, as a consequence,

$$\mathscr{H}^1(q'_\varepsilon(x,y)) = \delta_\Sigma(q'_\varepsilon(x,y)) < |y-x| + \varepsilon\,;$$

hence the thesis follows. $\square$

We see now an example, showing that the infimum in (1.5) may be not a minimum, and that δ_Σ may be not lower semicontinuous.

Example 1.8. Let Ω be the ball in $\mathbb{R}^2$ centered at the origin and with radius 2, let $\Sigma = [0,1]\times\{0\}$, $A(t) = t$ and $B(t) = 2t$; let moreover θ and θ_n be the paths connecting $(0,0)$ to $(1,0)$ given by

$$\theta(t) := (t,0)\,, \qquad \theta_n(t) := \left(t, \frac{1-|2t-1|}{n}\right)\,.$$

Then one has that θ_n converges to θ in (Θ, d_Θ), $\delta_\Sigma(\theta) = 2$, while $\delta_\Sigma(\theta_n) \to 1$: therefore, δ_Σ is not l.s.c. Moreover, it is clear that

$$d_\Sigma\big((0,0),(1,0)\big) = 1\,,$$

but $\delta_\Sigma(\sigma) > 1$ for each path $\sigma \in \Theta$ connecting $(0,0)$ and $(1,0)$. Hence, the infimum in (1.5) is not a minimum.

Since δ_Σ is not, in general, l.s.c., we compute now its relaxed envelope with fixed endpoints,

$$\bar{\delta}_\Sigma(\theta) := \inf\left\{\liminf_{n\to\infty} \delta_\Sigma(\theta_n) :\ \theta_n(0) = \theta(0),\, \theta_n(1) = \theta(1),\ \theta_n \xrightarrow{\Theta} \theta\right\}. \tag{1.13}$$

Notice that $\bar{\delta}_\Sigma \leq \delta_\Sigma$, and that the infimum in (1.13) is a minimum. Thanks to the standard properties of relaxed envelopes (see [**13**]), we are allowed to rewrite (1.5) obtaining

$$d_\Sigma(x,y) = \inf\left\{\bar{\delta}_\Sigma(\theta) :\ \theta \in \Theta,\, \theta(0) = x,\, \theta(1) = y\right\}. \tag{1.14}$$

Proposition 1.9. *The function $\bar{\delta}_\Sigma : \Theta \to \mathbb{R}^+$ is l.s.c.*

Proof. Let us take $\theta_n \to \theta$ in Θ: then, without loss of generality, we may assume

$$|\theta_n(0) - \theta(0)| \leq \frac{1}{n}\,, \qquad\qquad |\theta_n(1) - \theta(1)| \leq \frac{1}{n}\,.$$

Following (1.13), we choose $\hat{\theta}_n$ having the same endpoints as θ_n and such that

$$d_\Theta(\theta_n, \hat{\theta}_n) \leq \frac{1}{n}\,, \qquad\qquad \delta_\Sigma(\hat{\theta}_n) \leq \bar{\delta}_\Sigma(\theta_n) + \frac{1}{n}\,. \tag{1.15}$$

Take now, according to Lemma 1.1, two paths α_n and β_n connecting $\theta(0)$ with $\hat{\theta}_n(0)$ and $\hat{\theta}_n(1)$ with $\theta(1)$ respectively, with the properties

$$\begin{aligned} &\mathscr{H}^1(\alpha_n \setminus \Sigma) \leq \frac{2}{n}\,, &\qquad &\mathscr{H}^1(\beta_n \setminus \Sigma) \leq \frac{2}{n}\,, \\ &\mathscr{H}^1(\alpha_n \cap \Sigma) = 0\,, &\qquad &\mathscr{H}^1(\beta_n \cap \Sigma) = 0\,. \end{aligned} \tag{1.16}$$

Define then $\overline{\theta}_n := \alpha_n \cdot \hat{\theta}_n \cdot \beta_n$, so that $\{\overline{\theta}_n\}_{n\in\mathbb{N}}$ is a sequence of paths connecting $\theta(0)$ to $\theta(1)$ which still converges to θ. For any $n \in \mathbb{N}$, by (1.15) and (1.16) we have

$$\begin{aligned} \delta_\Sigma(\overline{\theta}_n) &= A\Big(\mathscr{H}^1\big(\overline{\theta}_n \setminus \Sigma\big)\Big) + B\Big(\mathscr{H}^1\big(\overline{\theta}_n \cap \Sigma\big)\Big) \\ &= A\Big(\mathscr{H}^1\big((\hat{\theta}_n \cup \alpha_n \cup \beta_n) \setminus \Sigma\big)\Big) + B\Big(\mathscr{H}^1\big(\hat{\theta}_n \cap \Sigma\big)\Big) \\ &\leq \delta_\Sigma(\hat{\theta}_n) + A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big) + 4/n\Big) - A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big)\Big) \\ &\leq \bar{\delta}_\Sigma(\theta_n) + 1/n + A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big) + 4/n\Big) - A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big)\Big). \end{aligned}$$

Since the paths $\{\theta_n\}$ have uniformly bounded lengths, by the uniform continuity of A in the bounded intervals and by (1.13) we infer

$$\bar{\delta}_\Sigma(\theta) \leq \liminf_{n\to\infty} \delta_\Sigma(\overline{\theta}_n) \leq \liminf_{n\to\infty} \bar{\delta}_\Sigma(\theta_n)\,,$$

so the proof is completed. □

COROLLARY 1.10. *The l.s.c. envelope of* δ_Σ *in* (Θ, d_Θ) *is* $\bar{\delta}_\Sigma$*.*

PROOF. The l.s.c. envelope of δ_Σ in (Θ, d_Θ) is lower than $\bar{\delta}_\Sigma$, as a direct consequence of the definition (1.13). On the other hand, it is the greatest l.s.c. function lower than δ_Σ, thus it is also greater than $\bar{\delta}_\Sigma$ by Proposition 1.9. □

COROLLARY 1.11. *The infimum in (1.14) is actually a minimum.*

PROOF. Let us choose x and y and take a minimizing sequence θ_n for (1.14): if the Euclidean lengths of θ_n (possibly up to a subsequence) are bounded, then the result immediately follows from the lower semicontinuity of $\bar{\delta}_\Sigma$ and by Ascoli–Arzelà Theorem. Otherwise, since Σ has finite length, it would follow that

$$\limsup \mathscr{H}^1(\theta_n \setminus \Sigma) = \infty\,;$$

in this case, take a path θ joining x to y with $\mathscr{H}^1$−negligible intersection with Σ and with finite length: since A is nondecreasing, this path provides the minimum in (1.14). □

More precisely, we see that one can somehow "pass to the limit" in Proposition 1.6. Throughout the monograph, we will call *geodesics* the paths θ such that

$$\bar{\delta}_\Sigma(\theta) = d_\Sigma\big(\theta(0), \theta(1)\big)\,.$$

COROLLARY 1.12. *There is a Borel function* $q : \Omega \times \Omega \to \Theta$ *such that* $q(x, y)$ *is a path connecting* x *to* y *with cost* $\bar{\delta}_\Sigma\big(q(x, y)\big) = d_\Sigma(x, y)$*.*

PROOF. Using the classical results in [**23**], it is sufficient to show that the subset G of Θ given by the geodesics is closed and there is at least one element of G connecting any couple of points in $\Omega \times \Omega$. The second fact follows from Corollary 1.11, while the closedness of G is a direct consequence of the lower semicontinuity of $\bar{\delta}_\Sigma$ and of the continuity of d_Σ. □

LEMMA 1.13. *For any* $\varepsilon > 0$*, there is a Borel function* $\alpha_\varepsilon : \Theta \to \Theta$ *such that for any* $\theta \in \Theta$ *one has*

$$\begin{aligned}
&\big(\alpha_\varepsilon(\theta)\big)(0) = \theta(0)\,, && \big(\alpha_\varepsilon(\theta)\big)(1) = \theta(1)\,, && d_\Theta(\alpha_\varepsilon(\theta), \theta) \leq \varepsilon\,,\\
&\mathscr{H}^1(\alpha_\varepsilon(\theta)) \leq \mathscr{H}^1(\theta) + \varepsilon\,, && \delta_\Sigma(\alpha_\varepsilon(\theta)) \leq \bar{\delta}_\Sigma(\theta) + \varepsilon\,.
\end{aligned}$$

PROOF. Our argument is quite similar to the one in Proposition 1.6: fixed $L > 0$ and fixed arbitrarily a path $\theta \in \Theta$ with $\mathscr{H}^1(\theta) \leq L$, we know by definition of $\bar\delta_\Sigma$ the existence of a path $\tilde\theta$ with

$$\begin{aligned} &\tilde\theta(0)=\theta(0)\,, && \tilde\theta(1)=\theta(1)\,, && d_\Theta(\tilde\theta,\theta)\leq\frac{\varepsilon}{4}\,,\\ &\mathscr{H}^1(\tilde\theta)\leq\mathscr{H}^1(\theta)+\frac{\varepsilon}{4}\,, && \delta_\Sigma(\tilde\theta)\leq\bar\delta_\Sigma(\theta)+\frac{\varepsilon}{4}\,. \end{aligned}\tag{1.17}$$

We take now a number $\delta\leq\varepsilon/8$ such that

$$A(s+4\delta)-A(s)\leq\frac{\varepsilon}{2}$$

for any $0\leq s\leq L$, which is possible by the continuity of A; moreover, since the Euclidean length and the map $\bar\delta_\Sigma$ are l.s.c., we can also assume that δ is so small that

$$\left\{\begin{aligned} &\mathscr{H}^1(\sigma)\geq\mathscr{H}^1(\theta)-\frac{\varepsilon}{4}\,,\\ &\bar\delta_\Sigma(\sigma)\geq\bar\delta_\Sigma(\theta)-\frac{\varepsilon}{4}\,, \end{aligned}\right. \qquad \text{whenever}\quad d_\Theta(\theta,\sigma)\leq\delta\,. \tag{1.18}$$

If we can define a Borel function $\alpha_\varepsilon: B_\Theta(\theta,\delta)\to\Theta$ as in the claim of this corollary, this will show the thesis: indeed, since the subset Θ_L of Θ made by the paths of Euclidean length bounded by L is compact, it can be covered by a finite number of balls $B_\Theta(\theta_i,\delta_i)$, so that we infer the existence of a Borel function $\alpha_\varepsilon:\Theta_L\to\Theta$ as in the claim; finally, it is immediate to conclude the thesis covering Θ with countably many sets Θ_{L_i} for a sequence $L_i\to\infty$. Summarizing, we can restrict our attention to a ball $B_\Theta(\theta,\delta)$.

Define now the Borel function $\beta_1: B_\Theta(\theta,\delta)\to\Theta$ as

$$\beta_1(\sigma):=q'_\delta\big(\sigma(0),\theta(0)\big)\,,$$

where q'_δ is as in Corollary 1.7: then $\beta_1(\sigma)$ is a path connecting $\sigma(0)$ with $\theta(0)$ such that

$$\mathscr{H}^1(\beta_1(\sigma)\cap\Sigma)=0\,,\qquad \mathscr{H}^1(\beta_1(\sigma))\leq|\sigma(0)-\theta(0)|+\delta\leq 2\delta\,. \tag{1.19}$$

Similarly, we let $\beta_2: B_\Theta(\theta,\delta)\to\Theta$ to be a Borel function such that $\beta_2(\sigma)$ is a path connecting $\theta(1)$ with $\sigma(1)$ satisfying

$$\mathscr{H}^1(\beta_2(\sigma)\cap\Sigma)=0\,,\qquad\qquad \mathscr{H}^1(\beta_2(\sigma))\leq 2\delta\,; \tag{1.20}$$

We finally define $\alpha_\varepsilon(\sigma):=\beta_1(\sigma)\cdot\tilde\theta\cdot\beta_2(\sigma)$: by construction, the map

$$B_\Theta(\theta,\delta)\ni\sigma\mapsto\alpha_\varepsilon(\sigma)\in\Theta$$

is Borel; moreover,

$$\alpha_\varepsilon(\sigma(0))=\sigma(0)\,,\qquad\qquad \alpha_\varepsilon(\sigma(1))=\sigma(1)\,.$$

In addition, minding (1.19), (1.20) and (1.17), we get

$$d_\Theta(\alpha_\varepsilon(\sigma),\sigma)\leq d_\Theta(\alpha_\varepsilon(\sigma),\tilde\theta)+d_\Theta(\tilde\theta,\theta)+d_\Theta(\theta,\sigma)\leq 2\delta+\frac{\varepsilon}{4}+\delta<\varepsilon\,.$$

Again by (1.19), (1.20), (1.17) and (1.18) one has

$$\mathscr{H}^1(\alpha_\varepsilon(\sigma))\leq 4\delta+\mathscr{H}^1(\tilde\theta)\leq 4\delta+\frac{\varepsilon}{4}+\mathscr{H}^1(\theta)\leq\mathscr{H}^1(\sigma)+\varepsilon\,.$$

Finally, by (1.19) and (1.20) we know that

$$\mathscr{H}^1(\alpha_\varepsilon(\sigma)\cap\Sigma)=\mathscr{H}^1(\tilde\theta\cap\Sigma)$$

so that again (1.19) and (1.20), together with (1.17) and (1.18), yield

$$\begin{aligned}\delta_\Sigma(\alpha_\varepsilon(\sigma)) &\le A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)+4\delta\big)+B\big(\mathscr{H}^1(\tilde\theta\cap\Sigma)\big)\\ &\le \delta_\Sigma(\tilde\theta)+A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)+4\delta\big)-A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)\big)\\ &\le \delta_\Sigma(\tilde\theta)+\frac{\varepsilon}{2}\le\bar\delta_\Sigma(\theta)+\frac{3}{4}\varepsilon\le\bar\delta_\Sigma(\sigma)+\varepsilon:\end{aligned}$$

hence, the proof is complete. □

Now, generalizing (1.8), set

$$\overline{C}_\Sigma(\eta):=\int_\Theta\bar\delta_\Sigma(\theta)\,d\eta(\theta)\,. \tag{1.21}$$

PROPOSITION 1.14. *The following equalities hold*

$$\begin{aligned}\inf\big\{C_\Sigma(\eta):\ \eta \textit{ is a t.p.m.}\big\}&=\min\{I_\Sigma(\gamma)\ :\ \gamma \textit{ is a transport plan}\}\\ &=\min\big\{\overline{C}_\Sigma(\eta):\ \eta \textit{ is a t.p.m.}\big\}\qquad\big(=MK(\Sigma)\big)\,.\end{aligned}\tag{1.22}$$

Before giving the proof, we point out the following remark.

REMARK 1.15. The equality (1.22) ensures the existence of at least one optimal transport plan $\gamma_{\rm opt}$ and one t.p.m. $\eta_{\rm opt}$ optimal with respect to $\overline{C}_\Sigma$, which satisfy the equality $I_\Sigma(\gamma_{\rm opt})=\overline{C}_\Sigma(\eta_{\rm opt})$. On the other hand, the infimum in (1.22) needs not to be achieved: for instance, just consider the situation of Example 1.8 with $f^+:=\delta_{(0,0)}$ and $f^-:=\delta_{(1,0)}$.
Concerning the equality between the two minima in (1.22), in particular, if $\gamma_{\rm opt}$ is an optimal transport plan then $q_\#\gamma_{\rm opt}$ is an optimal t.p.m., where q is defined in Corollary 1.12. Conversely, if $\eta_{\rm opt}$ is an optimal t.p.m. then $(p_0,p_1)_\#\eta_{\rm opt}$ is an optimal transport plan, where p_0 and p_1 are as in Definition 1.5.

OF PROPOSITION 1.14. First of all, note that the existence of some optimal transport plan is standard since the set of the transport plans is a bounded and weakly* closed –thus, weakly* compact– subset of $\mathcal{M}^+(\Omega\times\Omega)$ and I_Σ is l.s.c. on $\mathcal{M}(\Omega\times\Omega)$ with respect to the weak* topology –in fact, Proposition 1.3 shows that it is a continuous function.

Moreover, given a t.p.m. η, one can construct the associated transport plan $\gamma=(p_0,p_1)_\#\eta$, and from (1.5) we get $I_\Sigma(\gamma)\le C_\Sigma(\eta)$. On the other hand, given any transport plan γ and $\varepsilon>0$, we can define $\eta:=q_{\varepsilon\#}\gamma$ where q_ε is as in Proposition 1.6; we obtain $C_\Sigma(\eta)\le I_\Sigma(\gamma)+\varepsilon$, thus the first equality in (1.22) is established.

Concerning the second one, using (1.14) in place of (1.5) in the previous argument one gets

$$\min\{I_\Sigma(\gamma)\}\le\inf\{\overline{C}_\Sigma(\eta)\}\,.$$

But since $\overline{C}_\Sigma \le C_\Sigma$ (because $\bar\delta_\Sigma \le \delta_\Sigma$), it is also true that

$$\inf\{\overline{C}_\Sigma(\eta)\} \le \inf\{C_\Sigma(\eta)\}\,.$$

We derive $\min\{I_\Sigma(\gamma)\} = \inf\{\overline{C}_\Sigma(\eta)\}$, so to conclude we need only to prove that the last inf is a minimum. To this aim, it suffices to take an optimal transport plan $\gamma_{\rm opt}$ and to define $\eta := q_\#\gamma_{\rm opt}$ where q is as in Corollary 1.12: by definition of q, one has $\overline{C}_\Sigma(\eta) = I_\Sigma(\gamma_{\rm opt})$, so η minimizes $\overline{C}_\Sigma$ and the proof is achieved. □

A useful property, that we have already mentioned, is the fact that the *bounded subsets of* Θ (i.e., the subsets made by paths of uniformly bounded Euclidean length) are sequentially compact with respect to d_Θ. This becomes particularly helpful once we know that a t.p.m. is concentrated on a bounded subset of Θ, which is the argument of Corollary 1.17 below.

LEMMA 1.16. *If $A(s)$ is not constant for large s (for instance, if $A(s) \to \infty$ as $s \to \infty$), then there is a constant $L \in \mathbb{R}$ such that the Euclidean length $\mathscr{H}^1(\theta)$ of any geodesic θ is bounded by L. Otherwise, if $A(s)$ is constant for large s, it is still true that for any pair (x, y) of points in Ω there exists some geodesic of length bounded by L. In both cases, the constant L depends only on A, Ω and $\mathscr{H}^1(\Sigma)$ (but not on Σ).*

PROOF. Suppose first that $A(s)$ is not constant for large s, and let L be a sufficiently large number such that

$$A\big(L - \mathscr{H}^1(\Sigma)\big) > A\big(\operatorname{diam}\Omega + 1\big)\,.$$

Take now a path $\theta \in \Theta$ with $\mathscr{H}^1(\theta) > L$ and let $\hat\theta$, according to Lemma 1.1, be a path with length less than

$$|\theta(1) - \theta(0)| + 1 \le \operatorname{diam}\Omega + 1$$

connecting $\theta(0)$ to $\theta(1)$ and having $\mathscr{H}^1-$negligible intersection with Σ. Since

$$\mathscr{H}^1(\theta \setminus \Sigma) \ge \mathscr{H}^1(\theta) - \mathscr{H}^1(\Sigma) > L - \mathscr{H}^1(\Sigma)\,,$$

we immediately get $\bar\delta_\Sigma(\theta) > \bar\delta_\Sigma(\hat\theta)$, so that θ is not a geodesic and the first part of the proof is achieved.

Consider now the case when $A(s)$ is constant for large s, and let

$$L := \mathscr{H}^1(\Sigma) + \operatorname{diam}\Omega + 1\,.$$

Arguing exactly as in the first part of the proof, we see that for any path θ there is a path $\hat\theta$ with $\mathscr{H}^1(\hat\theta) \le \operatorname{diam}\Omega + 1$ and with $\bar\delta_\Sigma(\hat\theta) \le \bar\delta_\Sigma(\theta)$ (the only difference is that this time the strict inequality $\bar\delta_\Sigma(\hat\theta) < \bar\delta_\Sigma(\theta)$ in the case $\mathscr{H}^1(\theta) > L$ may be false). Hence, it is not true that *all* the geodesics have Euclidean length less than L, but that for any pair $(x, y) \in \Omega \times \Omega$ there is *at least one* geodesic between x and y of Euclidean length less than L. □

In what follows, we will say that a set $A \subseteq \Theta$ is *bounded in* Θ *by* L, where $L \in \mathbb{R}^+$, if $\mathscr{H}^1(\theta) \le L$ for any $\theta \in A$.

COROLLARY 1.17. *If $A(s)$ is not constant for large s then the support of any t.p.m. η which is optimal with respect to $\overline{C}_\Sigma$ is bounded in Θ by L, where L is as in the previous Lemma. Otherwise, if $A(s)$ is constant for large s, it is still true that there exists some optimal t.p.m. η the support of which is bounded in Θ by L.*

PROOF. Recall that, thanks to (1.22), any t.p.m. optimal with respect to $\overline{C}_\Sigma$ is concentrated in the set of all geodesics; this set is closed, as already noticed in Corollary 1.12, hence the whole support of any optimal t.p.m. is made by geodesics and the first part of the proof is trivial.

Concerning the second claim, we recall that Corollary 1.12 implies that the set G of all geodesics is a closed subset of Θ containing at least one path which connects any given pair of points in $\Omega \times \Omega$. The same property is true for the set

$$G_L := G \cap \left\{\theta \in \Theta :\, \mathscr{H}^1(\theta) \leq L\right\},$$

by the above lemma and since the Euclidean length is l.s.c. with respect to the distance in Θ. Therefore, arguing as in Corollary 1.12, we find a Borel map $\tilde{q} : \Omega \times \Omega \to \Theta$ such that $\tilde{q}(x, y)$ is a geodesic between x and y of Euclidean length less than L. This easily gives also the second part of the thesis: arguing as in Proposition 1.14, taken any optimal transport plan γ, one has that the t.p.m. $\tilde{q}_{\#}\gamma$ is as required. □

We present now a useful exact formula for $\bar{\delta}_\Sigma$.

PROPOSITION 1.18. *The following equality holds:*

$$\bar{\delta}_\Sigma(\theta) = J\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta \cap \Sigma)\big), \tag{1.23}$$

where the function $J\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is given by

$$J(a, b) := \inf\left\{A\big(a + l\big) + B\big(b - l\big) :\; 0 \leq l \leq b\right\}. \tag{1.24}$$

Before giving the proof, we shortly discuss the above formula.

REMARK 1.19. The meaning of (1.23), as one can understand comparing with (1.2), is that, roughly speaking, one can "walk on the railway": in other words, the cost $\bar{\delta}_\Sigma$ of some path θ is not necessarily given by the cost of moving by own means out of the network and by train along it, but moving by own means out of the network and possibly in some part of it, and by train along the remaining part. The basic idea of the proof is then easily imagined: instead of walking on the network, one can just walk very close to it, which is possible since the dimension N is larger than 1.

OF PROPOSITION 1.18. Set $a := \mathscr{H}^1(\theta \setminus \Sigma)$ and $b := \mathscr{H}^1(\theta \cap \Sigma)$, then take an arbitrary sequence θ_n of paths having the same endpoints as θ and converging to θ. It is known that

$$\mathscr{H}^1(\theta) \leq \liminf_{n\to\infty} \mathscr{H}^1(\theta_n), \tag{1.25}$$

$$\mathscr{H}^1(\theta \setminus \Sigma) \leq \liminf_{n\to\infty} \mathscr{H}^1(\theta_n \setminus \Sigma); \tag{1.26}$$

the first inequality is the classical lower semicontinuity of the length, the second is a recent generalization of the Gołąb theorem that we state in Theorem 2.6 (see also for instance [**11**] and [**24**]).

For a given $n \in \mathbb{N}$, assume that

$$\mathscr{H}^1(\theta_n \cap \Sigma) \geq \mathscr{H}^1(\theta \cap \Sigma) \ :$$

then, taking $l = 0$ in (1.24), we obtain

$$\begin{aligned} J(a,b) &\leq A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big) \\ &\leq A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\leq \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) - A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big) . \end{aligned} \tag{1.27}$$

On the other hand, if

$$\mathscr{H}^1(\theta_n \cap \Sigma) < \mathscr{H}^1(\theta \cap \Sigma)$$

then, taking

$$l := \mathscr{H}^1(\theta \cap \Sigma) - \mathscr{H}^1(\theta_n \cap \Sigma)$$

in (1.24), we obtain

$$\begin{aligned} J(a,b) &\leq A\big(\mathscr{H}^1(\theta) - \mathscr{H}^1(\theta_n \cap \Sigma)\big) + B\big(\mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\leq \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta) - \mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\qquad\qquad - A\big(\mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta_n \cap \Sigma)\big) . \end{aligned} \tag{1.28}$$

Recalling now that A is nondecreasing and continuous, combining (1.27) with (1.26) and (1.28) with (1.25) gives $J(a,b) \leq \liminf \delta_\Sigma(\theta_n)$, and therefore $J(a,b) \leq \bar{\delta}_\Sigma(\theta)$.

To prove the opposite inequality take $0 \leq l \leq \mathscr{H}^1(\theta \cap \Sigma)$ and let $\{\theta_n\}$ be, according to Corollary 1.2, a sequence of paths connecting $\theta(0)$ and $\theta(1)$ such that

$$\theta_n \to \theta \,, \quad \mathscr{H}^1(\theta_n) \to \mathscr{H}^1(\theta) \,, \quad \mathscr{H}^1(\theta_n \cap \Sigma) = \mathscr{H}^1(\theta \cap \Sigma) - l \ \forall n \in \mathbb{N} .$$

Hence, making use of the continuity of A, one gets

$$\delta_\Sigma(\theta_n) = A\big(\mathscr{H}^1(\theta_n) - (\mathscr{H}^1(\theta \cap \Sigma) - l)\big) + B\big(b - l\big) \xrightarrow[n \to \infty]{} A(a+l) + B(b-l) .$$

Thus for every $0 \leq l \leq b$ one has

$$\bar{\delta}_\Sigma(\theta) \leq A(a+l) + B(b-l) \,,$$

so the inequality $J(a,b) \geq \bar{\delta}_\Sigma(\theta)$ follows taking the infimum on l. $\square$

It is also convenient to introduce an auxiliary function, namely

$$D(a,b) := J(a, b-a) \,; \tag{1.29}$$

indeed, the above proposition tells us that

$$\bar{\delta}_\Sigma(\theta) = J\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta \cap \Sigma)\big) \,,$$

or equivalently that

$$\bar{\delta}_\Sigma(\theta) = D\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta)\big) . \tag{1.30}$$

In other words, we can express $\bar{\delta}_\Sigma(\theta)$ in terms of the length $\mathscr{H}^1(\theta\setminus\Sigma)$ outside of the network and of the length $\mathscr{H}^1(\theta\cap\Sigma)$ inside the network if we make use of J, or in terms of the length $\mathscr{H}^1(\theta\setminus\Sigma)$ out of the network and of the total length $\mathscr{H}^1(\theta)$ if we make use of D. The advantage of the second possibility, i.e. the advantage of (1.30) with respect to (1.23), is that the variables $\mathscr{H}^1(\theta\setminus\Sigma)$ and $\mathscr{H}^1(\theta)$ satisfy the useful $\liminf$ inequalities (1.25)–(1.26), while the same is not true for $\mathscr{H}^1(\theta\cap\Sigma)$; on the contrary, for $\mathscr{H}^1(\theta\cap\Sigma)$ the $\limsup$ inequality is true, as one can immediately deduce by Lemma 3.1. Another easy interesting property of both D and J is the following one.

PROPOSITION 1.20. *The functions J and D are nondecreasing in each of their variables.*

PROOF. Consider first J: take $b\geq 0$ and $a'\geq a\geq 0$; for $0\leq l\leq b$ one has $A(a+l)\leq A(a'+l)$, so by (1.24) one gets $J(a',b)\geq J(a,b)$ and thus J is nondecreasing in its first variable. Concerning the second one, take $a\geq 0$ and $b'\geq b\geq 0$: one has

$$A(a+l)+B(b'-l)\geq A(a+l)+B(b-l)\geq J(a,b)\qquad \forall\, 0\leq l\leq b\,;$$

on the other hand, one has

$$A(a+l)+B(b'-l)\geq A(a+b)+B(0)\geq J(a,b)\qquad \forall\, b\leq l\leq b'\,.$$

It follows that $J(a,b')\geq J(a,b)$, so J is nondecreasing also in its second variable.

Consider now D: first of all, we rewrite (1.29) in a more convenient way as

$$\begin{aligned} D(a,b) &= \inf\left\{A(a+l)+B(b-a-l):\ 0\leq l\leq b-a\right\} \\ &= \inf\left\{A(l)+B(b-l):\ a\leq l\leq b\right\}\,. \end{aligned} \tag{1.31}$$

Then, take $b\geq 0$ and $a'\geq a\geq 0$: if $a'\leq l\leq b$ then a fortiori $a\leq l\leq b$, hence one gets $D(a,b)\leq D(a',b)$ directly by (1.31), and consequently D is nondecreasing in its first variable. Finally, concerning the second one, take $a\geq 0$ and $b'\geq b\geq 0$: if $a\leq l\leq b$ then

$$A(l)+B(b'-l)\geq A(l)+B(b-l)\geq D(a,b)\,;$$

on the other hand, if $b\leq l\leq b'$ then

$$A(l)+B(b'-l)\geq A(b)+B(0)\geq D(a,b)\,.$$

It follows that $D(a,b')\geq D(a,b)$, so D is nondecreasing also in its second variable and the proof is completed. $\square$

CHAPTER 2

Optimal connected networks

In this chapter we consider the problem of finiding an optimal network under the additional constraint that the admissible networks are assumed connected. This extra connectedness assumption enables us to obtain the necessary compactness to ensure the existence of an optimal network. We recall that this case has been extensively studied in [**10, 11, 14, 18**], where several necessary conditions of optimality have been found; therefore we limit ourselves to briefly recall the main known results. In the last section of this chapter we show some numerical plots of the optimal networks in a slightly simpler situation, where instead of transporting f^+ onto f^-, the goal of the planner is to transport f^+ on the network in the most efficient way.

1. Optimization problem

The optimization problem we consider in this chapter is

$$\min\{\mathfrak{F}_L(\Sigma)\ :\ \Sigma \subseteq \Omega,\ \Sigma \text{ connected}\}\,, \tag{2.1}$$

where $\mathfrak{F}_L$ is the cost functional defined in (1.11) with $H(s) = 0$ for $s \leq L$, $H(s) = \infty$ otherwise. More generally, one may consider the optimization problem

$$\min\{MK(\Sigma)\ :\ \Sigma \subseteq \Omega,\ \Sigma \text{ connected}\} \tag{2.2}$$

where the function d_Σ appearing in (1.5) is now given by

$$\begin{aligned} d_\Sigma(x,y) := \inf \Big\{ &\Phi\big(\mathscr{H}^1(\theta\setminus\Sigma), \mathscr{H}^1(\theta\cap\Sigma), \mathscr{H}^1(\Sigma)\big)\ : \\ &\qquad\qquad \theta\in\Theta,\ \theta(0)=x,\ \theta(1)=y\Big\}\,, \end{aligned} \tag{2.3}$$

and $\Phi : (\mathbb{R}^+)^3 \to [0,+\infty]$ is a function satisfying the following conditions:

- Φ is lower semicontinuous;
- Φ is continuous in its first variable;
- Φ is nondecreasing in each of its variables, i.e.

$$a_1 \leq a_2,\ b_1 \leq b_2,\ c_1 \leq c_2 \quad \Longrightarrow \quad \Phi(a_1,b_1,c_1) \leq \Phi(a_2,b_2,c_2);$$

- $\Phi(a,b,c) \geq H(c)$ with $H(c) \to \infty$ as $c \to \infty$.

For instance, taking

$$\Phi(a,b,c) := A(a) + B(b) + \chi_L(c)\,,$$

where $\chi_L(c)$ is the function which takes the value zero if $c \leq L$ and ∞ otherwise, gives the problem (2.1).

Under the connectedness requirement, the existence of an optimal network always occurs, as the below theorem states.

THEOREM 2.1. *Under the assumptions above on the function Φ, the optimization problem (2.2) admits a solution Σ_{opt}.*

The proof of theorem above is given in [**11**]; let us sketch here its main steps. First of all we consider on the admissible class

$$\{\Sigma \subseteq \Omega,\ \Sigma \text{ connected}\}$$

the Hausdorff topology, generated by the distance

$$d_H(\Sigma_1, \Sigma_2) = \sup\{\text{dist}(x_1, \Sigma_2) + \text{dist}(x_2, \Sigma_1)\ :\ x_1 \in \Sigma_1,\ x_2 \in \Sigma_2\}$$

where by $\text{dist}(x, \Sigma)$ we denoted the minimal distance from the point x to the closed set Σ. It is well-known that the Hausdorff topology is compact and that the Hausdorff limit of a sequence of connected sets is still connected. Moreover, the following result, known as Gołąb theorem (see for instance [**3, 28**]) holds.

THEOREM 2.2 (Gołąb). *If Σ_n is a sequence of connected closed subsets of Ω which Hausdorff converges to a set Σ, then*

$$\mathscr{H}^1(\Sigma) \leq \liminf_{n\to\infty} \mathscr{H}^1(\Sigma_n)\,.$$

REMARK 2.3. More generally, the above result on lower semicontinuity of the one-dimensional Hausdorff measure $\mathscr{H}^1$ still holds if we assume that the number of connected components of Σ_n is a priori bounded by a fixed constant. On the other hand, it is very easy to find counterexamples which show that the Gołąb theorem fails if the connectedness assumption is completely removed.

By an approximation argument similar to the one we used in the proof of Proposition 1.18 we can compute the relaxed form of the distance d_Σ appearing in (2.3). In other words, if convenient, we may approximate a curve θ passing through the network Σ by a sequence of curves θ_n converging to θ uniformly and not passing through Σ. More precisely, the following result holds.

PROPOSITION 2.4. *For every closed connected subset Σ of Ω we have*

$$\begin{aligned} d_\Sigma(x,y) = \inf\Big\{ J\Big(\mathscr{H}^1(\theta\setminus\Sigma), \mathscr{H}^1(\theta\cap\Sigma), \mathscr{H}^1(\Sigma)\Big)\ :\ \\ \theta\in\Theta,\ \theta(0)=x,\ \theta(1)=y\Big\}, \end{aligned} \tag{2.4}$$

where the function J is given by

$$J(a,b,c) := \inf\{\Phi(a+t, b-t, c)\ :\ 0 \leq t \leq b\}\,.$$

It is now convenient to change variables and to replace the quantity

$$\mathscr{H}^1(\theta \cap \Sigma)$$

by

$$\mathscr{H}^1(\theta) - \mathscr{H}^1(\theta \setminus \Sigma)\,,$$

so that the function J is replaced by the new one

$$D(a,b,c) = J(a, b-a, c) \tag{2.5}$$

and formula (2.4) becomes

$$\begin{aligned} d_\Sigma(x,y) = \inf \Big\{ D\Big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta), \mathscr{H}^1(\Sigma)\Big) \ :& \\ \theta \in \Theta,\ \theta(0) = x,\ \theta(1) = y \Big\}&. \end{aligned} \tag{2.6}$$

The advantage of using the new function D in (2.5) consists in the following monotonicity and lower semicontinuity property (the proof of the first part is very similar to the one of Proposition 1.20).

PROPOSITION 2.5. *The function D defined in (2.5) is monotone nondecreasing and lower semicontinuous in each variable.*

The proof of the existence Theorem 2.1 uses also the following generalization of the Gołąb Theorem, the proof of which can be found in [**11**], also obtained by Dal Maso and Toader in [**24**] for different purposes, related to the study of models in fracture mechanics.

THEOREM 2.6. *Let X be a metric space, $\{\Gamma_n\}_{n\in\mathbb{N}}$ and $\{\Sigma_n\}_{n\in\mathbb{N}}$ be two sequences of compact subsets such that $\Gamma_n \to \Gamma$ and $\Sigma_n \to \Sigma$ in the Hausdorff sense, for some compact subsets Γ and Σ. Let us also suppose that Γ_n is connected for all $n \in \mathbb{N}$. Then*

$$\mathscr{H}^1(\Gamma \setminus \Sigma) \le \liminf_{n\to\infty} \mathscr{H}^1(\Gamma_n \setminus \Sigma_n)\,. \tag{2.7}$$

By using the above extension of the Gołąb theorem and the monotonicity result of Proposition 2.5 it is not difficult to obtain the following lower semicontinuity property for the function d_Σ.

PROPOSITION 2.7. *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in Ω such that $x_n \to x$ and $y_n \to y$. If $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a sequence of closed connected sets such that $\Sigma_n \to \Sigma$ in the Hausdorff sense, then*

$$d_\Sigma(x,y) \le \liminf_{n\to\infty} d_{\Sigma_n}(x_n, y_n)\,. \tag{2.8}$$

In particular, for every closed connected $\Sigma \subseteq \Omega$ the function d_Σ is lower semicontinuous on $\Omega \times \Omega$.

The proof of Theorem 2.1 can be now obtained by putting together the above results. In fact, if $\{\Sigma_n\}$ is a minimizing sequence of the optimization problem (2.2), by the compactness of the Hausdorff convergence we may assume, up to a subsequence, that $\Sigma_n \to \Sigma$ for a suitable closed connected

$\Sigma \subseteq \Omega$. The lower semicontinuity of the cost functional $MK(\Sigma)$ in (2.2) with respect to the Hausdorff convergence follows now by showing that the optimal transportation plans for the distances d_{Σ_n} weakly* converge in $\Omega\times\Omega$ to an optimal transportation plan for d_Σ. All the details of the argument above can be found in the paper [**11**], to which we refer the interested reader.

2. Properties of the optimal networks

In this section we collect the results of [**16, 17, 41**] regarding the case of optimization problem (2.1) when $A(t) = t$, $B(t) = 0$ and Σ is required to be connected.

First, we recall the following basic results regarding topological structure and regularity of optimal sets in this case (see [**16, 17, 41**]).

THEOREM 2.8. *Let in problem* (2.1) $A(t) = t$ *and* $B(t) = 0$, *and assume that* f^+, $f^- \ll \mathscr{L}^N$ *and* $f^+ \neq f^-$. *Then every closed connected* $\Sigma \subseteq \mathbb{R}^N$ *solving this problem has the following properties.*

- $\mathscr{H}^1(\Sigma) = L$.
- Σ *contains no closed curve (a homeomorphic image of* S^1*). In particular, if* $N = 2$, *then* $\mathbb{R}^2 \setminus \Sigma$ *is connected.*

If, moreover, f^+, $f^- \in L^p(\Omega)$ *where* $p = 4/3$ *when* $N = 2$ *and* $p = N/(N-1)$ *otherwise, then the following additional properties hold.*

- Σ *is Ahlfors regular in the sense that for every* $x \in \Sigma$ *and* $r < \operatorname{diam}\Sigma$ *one has*
$$cr \leq \mathscr{H}^1\big(\Sigma \cap B_r(x)\big) \leq Cr$$
for some constants $c > 0$ *and* $C > 0$ *independent both on* x *and on* $r > 0$.

To discuss deeper properties of optimal sets, we need to recall some topological notions from [**34**].

DEFINITION 2.9. *Let* Σ *be a topological space. We will say that* the order of the point $x \in \Sigma$ does not exceed $\mathfrak{n}$, *writing* $ord_x\Sigma \leq \mathfrak{n}$, *where* $\mathfrak{n}$ *is a cardinal, if for every* $\varepsilon > 0$ *there is an open subset* $U \subseteq \Sigma$ *such that* $x \in U$, $\operatorname{diam}(U) < \varepsilon$ *and* $\#\partial U \leq \mathfrak{n}$, $\#$ *standing for the cardinality of a set.*

The order of the point $x \in \Sigma$ *is said to be equal* $\mathfrak{n}$, *written* $ord_x\Sigma = \mathfrak{n}$, *if* $\mathfrak{n}$ *is the least cardinal for which* $ord_x\Sigma \leq \mathfrak{n}$.

If $ord_x\Sigma = \mathfrak{n}$, *with* $\mathfrak{n} \geq 3$, *then* x *will be called a* branching point *of* Σ, *while if* $ord_x\Sigma = 1$, *then* x *will be called an* endpoint *of* Σ.

The following results hold (see [**16, 49, 52**]).

THEOREM 2.10. *Let in problem* (2.1) $A(t) = t$ *and* $B(t) = 0$, *and assume that* $N = 2$, f^+, $f^- \in L^p(\Omega)$, $p = 4/3$ *and* $f^+ \neq f^-$. *Then every closed connected* $\Sigma \subseteq \mathbb{R}^N$ *solving this problem has the following properties.*

- $ord_x\Sigma \leq 3$ *for all* $x \in \Sigma$, *and, further, the number of endpoints and that of branching points (which hence are all triple points) is finite.*

- *the generalized mean curvature* $\mathcal{H}$ *of* Σ *defined to be the vector-valued distribution satisfying the relationship*
$$\langle \mathcal{H}, X\rangle := \int_\Sigma div^\Sigma X \, d\mathscr{H}^0 \qquad \textit{for all } X \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)\,,$$
is a Radon measure and $\mathcal{H}(\{x\}) = 0$ *whenever* $x \in \Sigma$ *is a branching point. This property may be interpreted as a "weak form" of the assertion that every branching point is a "regular tripod", i.e. a triple point where three smooth branches meet with angles of 120 degrees.*
- *for any point* $x \in \Sigma$ *there is a number* $r_0 > 0$ *such that for any arc* $\theta \subset \Sigma$ *starting at* x *we have*
$$\#\theta \cap \partial B_r(x) = 1$$
for any $r \in (0, r_0]$.
- *If* $f^\pm \in L^\infty(\Omega)$, *then for every branching point* $x \in \Sigma$ *there is* $\delta > 0$ *such that the set* $\Sigma \cap B_\delta(x)$ *consists of exactly three* $C^{1,1}$*-arcs starting at* x.
- *If* $f^\pm \in L^\infty(\mathbb{R}^2)$ *and* $x \in \Sigma$ *is not a branching point, then the Euclidean dimension of the set* $k^{-1}(x)$ *is at least* 1, *where* $k\colon \mathbb{R}^2 \to \Sigma$ *stands for the projection map on* Σ *defined a.e. on* $\mathbb{R}^2$. *In other words, in this case* $k^{-1}(x)$ *cannot reduce to a single point.*

3. Average distance problem

We consider in this section a slightly simpler situation which occurs when, instead of transporting f^+ onto f^-, the goal of the planner is to transport f^+ on Σ in the most efficient way. Such a problem, with $A(t) = t$ and $B(t) = 0$, is formulated as follows:
$$\min\Big\{ \int_\Omega \mathrm{dist}(x,\Sigma) f^+(x)\, dx \;:\; \Sigma \subset \Omega,\ \Sigma \text{ connected},\ \mathscr{H}^1(\Sigma) \le L) \Big\}$$
which corresponds to finding a network Σ_{opt} for which the *average distance* for a citizen to reach the closest point of Σ is minimal. Problems with the similar functional, with a fixed set Σ also intervene in the study of equilibrium configurations of growing sandpiles (see for instance [**21**]).

Notice that, denoting by $MK(\Sigma, f^+, f^-)$ the cost of transportation of f^+ onto f^- with network Σ, introduced in (1.9), when $A(t) = t$ and $B(t) = 0$, the average distance functional above is simply $MK(\Sigma, f^+, f^-)$ where f^- is any measure with support in Σ. Thus, the existence result and the necessary conditions of optimality seen in Chapter 2 still hold.

We provide below some numerical plots of the optimal network Σ_{opt} in the average distance case.

We end this section by pointing out the question of studying the regularity of optimal solutions Σ_{opt} under the connectedness assumption. Even if we assume that all the data are extremely smooth, this revealed to be a rather difficult issue, and very little is known, apart the Ahlfors regularity

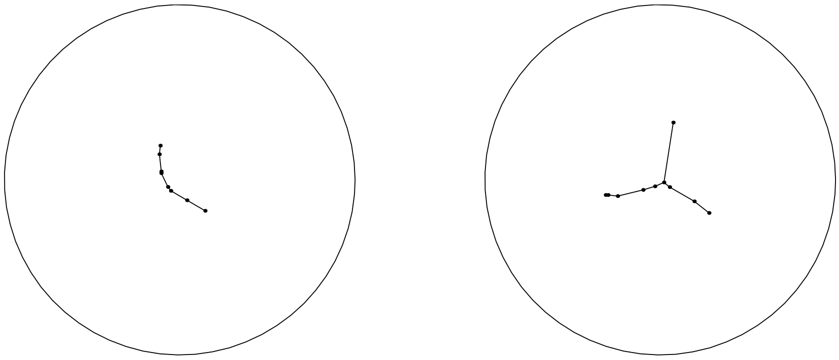

FIGURE 1. Plot of Σ_{opt} in a circle of radius 1 with $f^+ \equiv 1$ and $L = 0.5$ and $L = 1$.

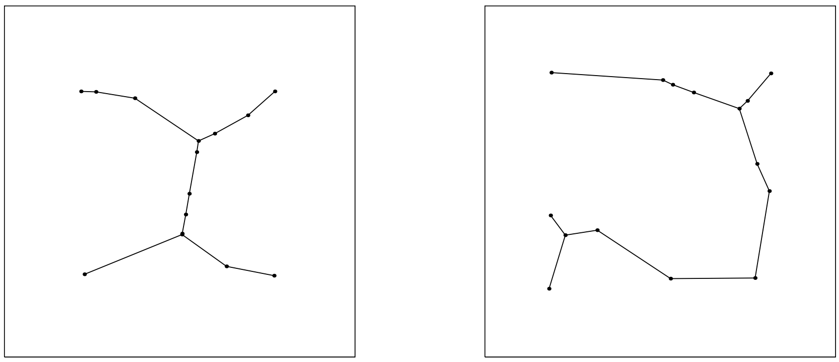

FIGURE 2. Plot of Σ_{opt} in a square of side 1 with $f^+ \equiv 1$ and $L = 1.5$ and $L = 2$.

seen in Theorem 2.8 above. Nevertheless, we expect that Σ_{opt} is the finite union of (at least) C^1 curves; in other words, we conjecture that Σ_{opt} is (at least) C^1 regular outside a finite number of branching points, any of which, in the case $N = 2$, is a 120 degrees regular tripod. For a more detailed discussion on the problem, as well as for a list of open problems and conjectures, we refer the interested reader to [**14, 16, 17**], where several others numerical computations providing the optimal sets Σ_{opt} are also made.

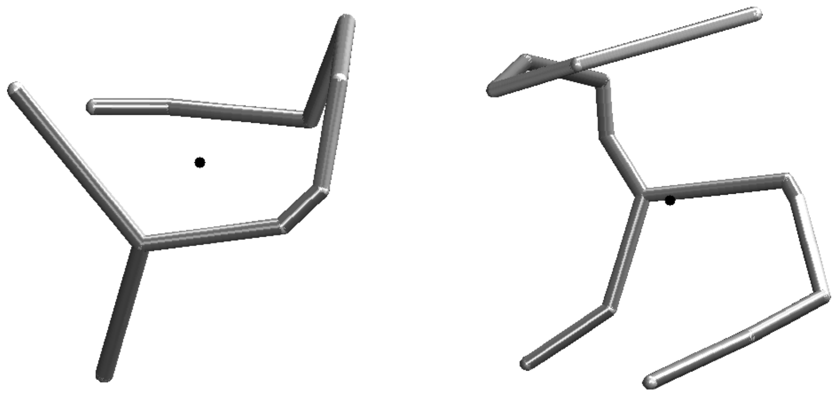

FIGURE 3. Plot of Σ_{opt} in a three dimensional ball of radius 1 with $f^+ \equiv 1$ and $L = 3$ and $L = 4$.

CHAPTER 3

Relaxed problem and existence of solutions

In the general setting that we are considering, the existence of solutions (i.e. Borel sets Σ minimizing the functional $\mathfrak{F}$ defined in (1.11)) may fail, as we will see in the example of Section 3; this is why we introduce a relaxed version of the problem. This relaxation is very convenient for a number of reasons: first of all it deals, instead of Borel sets, with Radon measures, which have a considerably better structure and nicer properties; moreover, it easily admits solutions and, as we will see, it is possible to study these solutions and to understand whether or not they are classical, i.e. of the form $\mathscr{H}^1 \llcorner \Sigma$ for a set Σ. Finally, we will give an interpretation of the meaning of the relaxed problem, and generalize all the definitions that we presented in Chapter 1 for the classical setting. In Section 1 we introduce the relaxed optimization problem and we show that it always admits a solution. In Section 2 we show some properties of the relaxed solutions, in particular that there is a solution $\mu \in \mathcal{M}^+(\Omega)$ such that for some rectifiable set Σ one has $\mu = a(x)\mathscr{H}^1 \llcorner \Sigma$, where $a : \Sigma \to [0,1]$ is some density; the relaxed solution μ is then classical if and only if $a(x) \equiv 1$ for $\mu-$a.e. $x \in \Omega$ and, in this case, μ corresponds to the rectifiable set $\{x : a(x) = 1\}$. In Section 3 we present a class of examples in which all the solutions are not classical. Finally, in Section 4 we show that there are classical solutions if the function D defined in (1.29) is concave in the first variable, and that all the solutions are classical if this concavity is strict.

1. Relaxed problem setting

First of all, we recall some abstract results.

LEMMA 3.1. *Let $\{C_n\}$ be a sequence of closed sets of some metric space X converging in the Hausdorff distance to C. Let moreover $\{\nu_n\} \subseteq \mathcal{M}^+(X)$ be a sequence of measures weakly* converging to $\nu \in \mathcal{M}^+(X)$. Then, one has*

$$\nu(C) \geq \limsup \nu_n(C_n)\,.$$

PROOF. Since $\{C_n\}$ converges in the Hausdorff distance to C, having fixed any $\varepsilon > 0$ one has $C_n \subseteq B_\varepsilon(C)$ for n large enough, where

$$B_\varepsilon(C) = \{x \in X : d_X(x, C) \leq \varepsilon\}\,;$$

therefore, since $B_\varepsilon(C)$ is a closed set, one has

$$\nu\big(B_\varepsilon(C)\big) \geq \limsup_{n\to\infty} \nu_n\big(B_\varepsilon(C)\big) \geq \limsup_{n\to\infty} \nu_n(C_n)\,.$$

The proof is completed noticing that $\nu\big(B_\varepsilon(C)\big) \searrow \nu(C)$ for $\varepsilon \searrow 0$. □

We remind that the $\Gamma - \liminf$ of a sequence $\{g_n\}$ of l.s.c. functions is defined as

$$\Gamma - \liminf_{n\to\infty} g_n(x) := \inf\big\{\liminf_{n\to\infty} g_n(x_n)\,:\, x_n \to x\big\}\,.$$

Moreover, we underline that an equivalent definition, very useful in the next proof, is the following one:

$$\Gamma - \liminf_{n\to\infty} g_n := \sup_{n\in\mathbb{N}} \Big(\mathrm{env}\big(\inf_{m\geq n} g_m\big)\Big)\,, \tag{3.1}$$

where $\mathrm{env}(\varphi)$ stands for the lower semicontinuous envelope of any function $\varphi : X \to \mathbb{R}$.

Lemma 3.2. *Let X be a topological space, $\{\nu_n\} \in \mathcal{M}^+(X)$ a sequence of measures weakly* converging to ν, and $\{g_n\} : X \to \mathbb{R}$ a sequence of l.s.c. functions. Then*

$$\int_X \Gamma - \liminf_{n\to\infty} g_n\, d\nu \leq \liminf_{n\to\infty} \int_X g_n\, d\nu_n\,.$$

Proof. Defining for simplicity

$$\tau_n := \mathrm{env}\big(\inf_{m\geq n} g_m\big)\,,$$

we fix $j \in \mathbb{N}$ and evaluate

$$\begin{aligned}\liminf_{n\to\infty} \int_X g_n\, d\nu_n &\geq \liminf_{n\to\infty} \int_X \big(\inf_{m\geq j} g_m\big)\, d\nu_n \\ &\geq \liminf_{n\to\infty} \int_X \mathrm{env}\big(\inf_{m\geq j} g_m\big)\, d\nu_n \\ &= \liminf_{n\to\infty} \int_X \tau_j\, d\nu_n \geq \int_X \tau_j\, d\nu.\end{aligned}$$

Since this is true for any $j \in \mathbb{N}$, by the Monotone Convergence Theorem and (3.1) the thesis follows. □

Now we notice a couple of properties of d_Σ and $\mathfrak{F}$: the first one is the lower semicontinuity of d_Σ with respect to the weak* convergence of the measures $\mathscr{H}^1 \llcorner \Sigma$; as a consequence, we can also show the lower semicontinuity of the functional $\mathfrak{F}$ with respect to the same convergence. Before showing these properties, we define formally the convergence we are going to use.

Definition 3.3. *Given a sequence $\{\Sigma_n\}$ of Borel sets of bounded length, we say that $\{\Sigma_n\}$* converges in the $\mathscr{H}^1$ sense *to Σ, shortly $\Sigma_n \xrightarrow{\mathscr{H}^1} \Sigma$, if the measures $\mathscr{H}^1 \llcorner \Sigma_n$ weakly* converge to the measure $\mathscr{H}^1 \llcorner \Sigma$ in $\mathcal{M}^+(\Omega)$.*

PROPOSITION 3.4. *Let* $\{\Sigma_n\} \xrightarrow{\mathscr{H}^1} \Sigma$ *and* $\theta_n \to \theta$ *uniformly: then*
$$\bar{\delta}_\Sigma(\theta) \leq \liminf_{n\to\infty} \bar{\delta}_{\Sigma_n}(\theta_n)\,.$$
Moreover, for each x, $y \in \Omega$
$$d_\Sigma(x,y) \leq \liminf_{n\to\infty} d_{\Sigma_n}(x,y)\,.$$

PROOF. Applying Lemma 3.1 with $C_n = \theta_n$, $C = \theta$, $\nu_n = \mathscr{H}^1 \llcorner \Sigma_n$ and $\nu = \mathscr{H}^1 \llcorner \Sigma$, one obtains
$$\mathscr{H}^1(\theta \cap \Sigma) \geq \limsup \mathscr{H}^1(\theta_n \cap \Sigma_n)\,.$$
Therefore, setting
$$a := \mathscr{H}^1(\theta \setminus \Sigma)\,, \quad b := \mathscr{H}^1(\theta \cap \Sigma)\,, \quad l_n := \mathscr{H}^1(\theta \cap \Sigma) - \mathscr{H}^1(\theta_n \cap \Sigma_n)\,,$$
one has that $l_n \leq b$ and $\liminf l_n \geq 0$. Recalling then formulas (1.23) and (1.24) for $\bar{\delta}_\Sigma$, and the facts that A and B are nondecreasing, we obtain
$$\begin{aligned}
\liminf_{n\to\infty} \bar{\delta}_{\Sigma_n}(\theta_n) &= \liminf_{n\to\infty} A\big(\mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta_n \cap \Sigma_n)\big) + B\big(\mathscr{H}^1(\theta_n \cap \Sigma_n)\big) \\
&= \liminf_{n\to\infty} A\Big(\mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta_n \cap \Sigma_n) + \mathscr{H}^1(\theta \cap \Sigma) + \mathscr{H}^1(\theta \setminus \Sigma) - \mathscr{H}^1(\theta)\Big) \\
&\qquad + B\Big(\mathscr{H}^1(\theta \cap \Sigma) - \big(\mathscr{H}^1(\theta \cap \Sigma) - \mathscr{H}^1(\theta_n \cap \Sigma_n)\big)\Big) \\
&= \liminf_{n\to\infty} A\big(a + l_n + \mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta)\big) + B\big(b - l_n\big) \\
&\geq \liminf_{n\to\infty} A\big(a + l_n\big) + B\big(b - l_n\big) \\
&\geq \inf\big\{A\big(a + l\big) + B\big(b - l\big) :\ 0 \leq l \leq b\big\} = J(a,b) = \bar{\delta}_\Sigma(\theta)\,,
\end{aligned}$$
where the first inequality is due to the fact that $a + l_n$ is bounded, and hence A is uniformly continuous, and to the inequality
$$\mathscr{H}^1(\theta) \leq \liminf \mathscr{H}^1(\theta_n)\,.$$
Thus the first part of the claim follows. Concerning the second part, we argue by contradiction: if it were not true that $d_\Sigma \leq \liminf d_{\Sigma_n}$, then there would exist a pair x, $y \in \Omega$ and $\varepsilon > 0$ such that
$$d_{\Sigma_n}(x,y) \leq d_\Sigma(x,y) - \varepsilon$$
for countably many $n \in \mathbb{N}$. According to Corollary 1.11 and Lemma 1.16, for any n we select a path θ_n with
$$\mathscr{H}^1(\theta_n) \leq L\,, \qquad\qquad \bar{\delta}_{\Sigma_n}(\theta_n) = d_{\Sigma_n}(x,y)\,.$$
Hence, up to a subsequence we can assume that $\theta_n \to \theta$ uniformly and that
$$\bar{\delta}_{\Sigma_n}(\theta_n) \leq d_\Sigma(x,y) - \varepsilon$$
for any n. The first part of the proof ensures that
$$\bar{\delta}_\Sigma(\theta) \leq d_\Sigma(x,y) - \varepsilon < d_\Sigma(x,y)\,,$$
which gives the desired contradiction and so also the second part of the claim follows. $\square$

The assertion below is a consequence of the previous proposition.

COROLLARY 3.5. *The functional $\mathfrak{F}$ is l.s.c. with respect the convergence in the $\mathscr{H}^1$ sense.*

PROOF. If $\{\Sigma_n\} \xrightarrow{\mathscr{H}^1} \Sigma$, then

$$\mathscr{H}^1(\Sigma) = \lim \mathscr{H}^1(\Sigma_n) < +\infty\,.$$

Further, by Proposition 3.4 one has

$$\bar{\delta}_\Sigma \leq \Gamma - \liminf \bar{\delta}_{\Sigma_n}\,,$$

and hence, by Lemma 3.2 one has

$$\overline{C}_\Sigma(\eta) \leq \liminf \overline{C}_{\Sigma_n}(\eta_n)$$

whenever $\eta_n \xrightarrow{*} \eta$ in the weak* sense of measures. According to Proposition 1.14, we may take η_n such that $MK(\Sigma_n) = \overline{C}_{\Sigma_n}(\eta_n)$; since by Corollary 1.17 one may assume that the supports of η_n are bounded in Θ by some constant L independent of n, minding that

$$\|\eta_n\|_{\mathcal{M}^+(\Theta)} = \|f^+\|_{\mathcal{M}^+(\Omega)} < +\infty\,,$$

we get $\eta_n \xrightarrow{*} \eta$ for some t.p.m. η up to a subsequence. Hence

$$MK(\Sigma) \leq \overline{C}_\Sigma(\eta) \leq \liminf_{n\to\infty} \overline{C}_{\Sigma_n}(\eta_n) = \liminf_{n\to\infty} MK(\Sigma_n)\,;$$

moreover, from (1.10) we deduce

$$H(\mathscr{H}^1(\Sigma)) \leq \liminf H\big(\mathscr{H}^1(\Sigma_n)\big)$$

and finally, by the definition (1.11) of the functional $\mathfrak{F}$, we deduce

$$\mathfrak{F}(\Sigma) \leq \liminf \mathfrak{F}(\Sigma_n)\,.$$

□

Let us finally introduce the relaxed version of the problem: we consider as relaxed admissible networks all the nonnegative measures $\mu \in \mathcal{M}^+(\Omega)$, identifying any classical network Σ with the measure $\mathscr{H}^1 \llcorner \Sigma$. In a standard way, we define then the relaxed functional (using the same symbol with a slight abuse of notation) as

$$\mathfrak{F}(\mu) := \inf\left\{\liminf_{n\to\infty} \mathfrak{F}(\Sigma_n) : \mathscr{H}^1 \llcorner \Sigma_n \xrightarrow{*} \mu\right\}\,, \tag{3.2}$$

so that the relaxed optimization problem now reads

$$\min\{\mathfrak{F}(\mu) : \mu \in \mathcal{M}^+(\Omega)\}\,. \tag{3.3}$$

It is worth remarking that, thanks to the lower semicontinuity proved in Corollary 3.5, one has $\mathfrak{F}(\Sigma) = \mathfrak{F}(\mathscr{H}^1 \llcorner \Sigma)$, which justifies the above abuse of notations.

We list now some properties of the relaxed functional $\mathfrak{F}$.

Proposition 3.6. *The following three properties hold:*

i) $\mathfrak{F}$ *is coercive, that is,* $\mathfrak{F}(\mu_n) \to \infty$ *if* $\|\mu_n\| \to \infty$;

ii) $\mathfrak{F}$ *is l.s.c. in* $\mathcal{M}^+(\Omega)$ *with respect to the weak* convergence;*

iii) $\min \{\mathfrak{F}(\mu) : \mu \in \mathcal{M}^+(\Omega)\} =$

$$\inf \{\mathfrak{F}(\Sigma) : \Sigma \text{ is a Borel set of finite length}\} .$$

Proof. This is a standard fact in relaxation theory, which follows for instance from Proposition 1.3.1 and 1.3.5 of [**13**]. □

2. Properties of relaxed minimizers

In this section we show some properties of an optimal measure μ, which exists thanks to Proposition 3.6; first of all, we establish the existence of an upper bound for the length of the paths contained in the support of optimal t.p.m.'s related to a minimizing sequence of sets Σ_n.

Lemma 3.7. *There is a constant* $L \geq 0$*, depending only on* A*,* Ω *and* H*, such that the following holds: given any minimizing sequence* $\{\Sigma_n\}$ *for* $\mathfrak{F}$*, for any* $n \in \mathbb{N}$ *sufficiently large there is a t.p.m.* η_n*, optimal for the functional* $\overline{C}_{\Sigma_n}$*, whose support is bounded in* Θ *by* L*. It is also possible to find a sequence of t.p.m.* $\{\eta_n\}$ *with supports bounded in* Θ *by* L*, such that*

$$C_{\Sigma_n}(\eta_n) \leq MK(\Sigma_n) + \varepsilon_n$$

with $\varepsilon_n \to 0$ *(in the following, such a sequence is referred to as "almost optimal with respect to* C_{Σ_n}*").*

Proof. Keeping in mind Corollary 1.17, to show the first assertion it is enough to check that, for each sequence $\{\Sigma_n\}$ minimizing $\mathfrak{F}$ and n large enough, the length $\mathscr{H}^1(\Sigma_n)$ is bounded by a constant depending only on A, Ω and H. To this aim, take $l \geq 0$ such that

$$H(l) - 1 \geq MK(\emptyset) = \mathfrak{F}(\emptyset) ,$$

and notice that l depends only on H, Ω and A but not on B. If $\mathscr{H}^1(\Sigma) \geq l$, one has

$$\mathfrak{F}(\Sigma) \geq H(\mathscr{H}^1(\Sigma)) \geq H(l) \geq \mathfrak{F}(\emptyset) + 1 \geq \min\{\mathfrak{F}(\mu) : \mu \in \mathcal{M}^+(\Omega)\} + 1 .$$

Therefore, if $\{\Sigma_n\}$ is a sequence minimizing $\mathfrak{F}$, then $\mathscr{H}^1(\Sigma_n) < l$ for n large enough.

To show the second assertion for a sequence of t.p.m.'s almost optimal with respect to C_{Σ_n}, we apply Lemma 1.13: taking, for each $n \in \mathbb{N}$, a t.p.m. η_n with support bounded in Θ by L and optimal with respect to $\overline{C}_{\Sigma_n}$, the t.p.m.'a $\alpha_{\varepsilon_n\#}\eta_n$ are almost optimal with respect to C_{Σ_n} and their supports are bounded in Θ by $L + \varepsilon_n$. □

Our next aim is to study the properties of the optimal measures μ: in particular, we want to determine whether or not a relaxed solution μ corresponds to a classical solution Σ.

DEFINITION 3.8. *A measure μ optimal for the relaxed functional $\mathfrak{F}$ is called* minimal *if any measure $\nu \leq \mu$, $\nu \neq \mu$ is not optimal for $\mathfrak{F}$.*

REMARK 3.9. Notice that, thanks to the lower semicontinuity and the coercivity of $\mathfrak{F}$, that we proved in Proposition 3.6, the existence of an optimal measure for $\mathfrak{F}$ is straightforward. In particular, the set of these optimal measures is a non-empty, bounded and weakly* closed subset of $\mathcal{M}^+(\Omega)$; therefore, since the map $\mu \mapsto \|\mu\|$ is weakly* l.s.c., among the optimal measures μ there are elements minimizing the norm $\|\mu\|$. Finally, an immediate application of Zorn's Lemma gives us the existence of minimal optimal measures; more precisely, for any optimal measure μ there exists a minimal optimal measure $\mu' \leq \mu$.

A first property that we are now able to show is that any minimal optimal measure is absolutely continuous with respect to $\mathscr{H}^1$ –in fact, in most of the cases *all* the optimal measures are minimal, see Proposition 3.21.

LEMMA 3.10. *Each minimal optimal measure is absolutely continuous with respect to $\mathscr{H}^1$.*

PROOF. We argue by contradiction assuming the existence of a set B such that $\mathscr{H}^1(B) = 0$ but $\mu(B) > 0$. Take a sequence $\{\Sigma_n\}$ of Borel sets such that

$$\mathscr{H}^1 \llcorner \Sigma_n =: \mu_n \stackrel{*}{\rightharpoonup} \mu$$

and $\mathfrak{F}(\Sigma_n) \to \mathfrak{F}(\mu)$, which is possible in view of the definition (3.2) of $\mathfrak{F}$. Fixed now $\varepsilon > 0$, by definition of the Hausdorff measure we can take countably many open balls B_i with radii r_i, such that $S_\varepsilon := \cup B_i$ contains B and $\sum r_i < \varepsilon$. Define then

$$\Sigma_{n,\varepsilon} := \Sigma_n \setminus S_\varepsilon\,, \qquad \mu_{n,\varepsilon} := \mathscr{H}^1 \llcorner \Sigma_{n,\varepsilon}\,.$$

Choose now an arbitrary path $\theta \in \Theta$: we claim the existence of a path $\alpha(\theta)$ having the same endpoints of θ and such that

$$\alpha(\theta) \setminus S_\varepsilon \subseteq \theta \setminus S_\varepsilon\,, \qquad \mathscr{H}^1(\alpha(\theta) \cap S_\varepsilon) \leq 2\varepsilon\,.$$

To show this fact, define $\theta_1 := \theta$ if $\theta \cap B_1 = \emptyset$; otherwise, fixed an arbitrary parametrization of θ, let t_1 and t_2 be the first and the last instant such that $\theta(t) \in \overline{B_1}$, and define θ_1 to be the path that equals θ in $[0, t_1] \cup [t_2, 1]$, and that is the line segment connecting $\theta(t_1)$ to $\theta(t_2)$ in $[t_1, t_2]$. In this way,

$$\theta_1 \setminus B_1 \subseteq \theta \setminus B_1\,, \qquad \mathscr{H}^1(\theta_1 \cap B_1) \leq 2r_1\,.$$

Note that the map $\theta \mapsto \theta_1$ is Borel since it is continuous on each of the sets

$$\{\theta \in \Theta\,:\, \theta \cap \overline{B_1} \neq \emptyset\}\,, \qquad \{\theta \in \Theta\,:\, \theta \cap \overline{B_1} = \emptyset\}\,.$$

In the same way, replacing θ by θ_1 and B_1 by B_2, we define θ_2 so that

$$\theta_2 \setminus B_2 \subseteq \theta_1 \setminus B_2\,, \qquad\qquad \mathscr{H}^1(\theta_2 \cap B_2) \le 2r_2\,.$$

Analogously, the map $\theta \mapsto \theta_2$ is continuous on each of the four sets

$$\{\theta \in \Theta\,:\, \theta \cap \overline{B_1} \neq \emptyset, \theta \cap \overline{B_2} \neq \emptyset\}\,, \quad \{\theta \in \Theta\,:\, \theta \cap \overline{B_1} = \emptyset, \theta \cap \overline{B_2} \neq \emptyset\}\,,$$
$$\{\theta \in \Theta\,:\, \theta \cap \overline{B_1} = \emptyset, \theta \cap \overline{B_2} \neq \emptyset\}\,, \quad \{\theta \in \Theta\,:\, \theta \cap \overline{B_1} = \emptyset, \theta \cap \overline{B_2} = \emptyset\}\,,$$

and hence is Borel. Iterating this procedure, we find a sequence $\{\theta_n\}$ of paths connecting $\theta(0)$ and $\theta(1)$ and, by construction and recalling that $\sum_{i\in\mathbb{N}} r_i \le \varepsilon$, we deduce that the paths θ_n uniformly converge to a path $\alpha(\theta)$ which equals θ outside of S_ε and with

$$\mathscr{H}^1(\alpha(\theta) \cap S_\varepsilon) \le \sum\nolimits_i 2r_i \le 2\varepsilon\,,$$

so that our claim is proved. Moreover, each map $\theta \mapsto \theta_n$ is Borel and hence so is their pointwise limit $\alpha : \Theta \to \Theta$ (in fact, Θ is the countable union of Borel sets on each of which the map α is continuous).

Applying now Lemma 3.7, we find a t.p.m. η_n whose support is bounded in Θ by L and which is optimal with respect to $\overline{C}_{\Sigma_n}$. Since A is uniformly continuous on $[0, L]$, we denote by ω its modulus of continuity. Then, for every $\theta \in \operatorname{spt}\eta_n$ one has by construction

$$\mathscr{H}^1(\alpha(\theta) \cap \Sigma_{n,\varepsilon}) \le \mathscr{H}^1(\theta \cap \Sigma_n)$$

and

$$\mathscr{H}^1(\alpha(\theta) \setminus \Sigma_{n,\varepsilon}) \le \mathscr{H}^1(\theta \setminus \Sigma_n) + 2\varepsilon\,,$$

so that

$$\bar{\delta}_{\Sigma_{n,\varepsilon}}(\alpha(\theta)) \le \bar{\delta}_{\Sigma_n}(\theta) + \omega(2\varepsilon)\,;$$

as a consequence,

$$\begin{aligned} \mathfrak{F}(\mu_{n,\varepsilon}) &\le \overline{C}_{\Sigma_{n,\varepsilon}}(\alpha_\#\eta_n) + H(\|\mu_{n,\varepsilon}\|) \le \overline{C}_{\Sigma_n}(\eta_n) + \omega(2\varepsilon) + H(\|\mu_n\|) \\ &= \mathfrak{F}(\mu_n) + \omega(2\varepsilon)\,, \end{aligned} \tag{3.4}$$

since $\mu_{n,\varepsilon} \le \mu_n$ by definition and H is nondecreasing. Since $\mu_n \stackrel{*}{\rightharpoonup} \mu$ and $\mu_{n,\varepsilon} \le \mu_n$, up to a subsequence we can assume that $\mu_{n,\varepsilon} \stackrel{*}{\rightharpoonup} \mu_\varepsilon$ with $\mu_\varepsilon \le \mu$. In particular,

$$\mu_\varepsilon(B) \le \mu_\varepsilon(S_\varepsilon) = 0\,,$$

since S_ε is open and $\mu_{n,\varepsilon}(S_\varepsilon) = 0$ for any $n \in \mathbb{N}$.

We select now a sequence $\varepsilon_j \searrow 0$, finding the measures μ_{ε_j} as shown above; we can choose the balls B_h related to each ε_j in such a way that $S_{\varepsilon_i} \subseteq S_{\varepsilon_j}$ whenever $i \ge j$: as a consequence, we have

$$\mu_{n,\varepsilon_i} \ge \mu_{n,\varepsilon_j} \qquad\qquad \text{whenever} \qquad\qquad i \ge j\,.$$

Hence, passing to a weak* limit as $n \to \infty$ (choosing a common subsequence of indices), we have $\mu_{\varepsilon_i} \ge \mu_{\varepsilon_j}$ for $i \ge j$. Therefore, $j \mapsto \mu_{\varepsilon_j}$ is an increasing sequence of measures bounded by μ and with the property that $\mu_{\varepsilon_j}(B) = 0$

for any j. We derive that μ_{ε_j} converge strongly as $j \to \infty$ to a measure $\bar{\mu}$ with

$$\bar{\mu} \leq \mu\,, \qquad \bar{\mu}(B) = 0\,, \qquad \mathfrak{F}(\bar{\mu}) \leq \mathfrak{F}(\mu)\,;$$

the latter property immediately follows by (3.4) recalling the lower semicontinuity of $\mathfrak{F}$, the fact that $\mathfrak{F}(\mu_n) \to \mathfrak{F}(\mu)$, and passing to the limit in (3.4) first as $n \to \infty$, and then as $\varepsilon \searrow 0$.

The measure $\bar{\mu}$ is then optimal because so is μ; but

$$\bar{\mu}(B) = 0 < \mu(B)\,,$$

hence $\bar{\mu} \neq \mu$; this contradicts the fact that μ is a minimal optimal measure. □

We are able now to prove a stronger result, namely, that the minimal optimal measures for $\mathfrak{F}$ are concentrated on one-dimensional sets.

Lemma 3.11. *Every minimal optimal measure μ for $\mathfrak{F}$ has the form*

$$\mu = \varphi \mathscr{H}^1 \llcorner \Sigma$$

for a suitable Borel set Σ and a Borel function $\varphi : \Sigma \to \mathbb{R}^+$.

Proof. Let μ be a minimal optimal measure, so that thanks to Lemma 3.10 we have $\mu \ll \mathscr{H}^1$. Thanks to Theorem 3.2 in [**50**], the thesis is achieved if

$$\theta_1^*(\mu, x) := \limsup_{\varepsilon \to 0} \frac{\mu\big(B(x,\varepsilon)\big)}{\varepsilon} > 0 \qquad \text{for } \mu\text{–a.e. } x\,. \tag{3.5}$$

We prove now (3.5) by contradiction; to this aim, take a sequence

$$\mu_n := \mathscr{H}^1 \llcorner \Sigma_n$$

such that

$$\mu_n \xrightarrow{*} \mu\,, \qquad \mathfrak{F}(\mu) = \lim \mathfrak{F}(\Sigma_n)\,.$$

If (3.5) does not hold, there is a constant C such that $\mu(X) > C > 0$, where

$$X := \{x : \theta_1^*(\mu, x) = 0\}\,.$$

Fixed now a small $\delta > 0$ (that we will eventually send to 0), we know that $\mu\big(X^\delta\big) > C$, defining

$$X^\delta := \{x : \theta_1^*(\mu, x) < \delta\}\,.$$

We deduce that $\mu\big(X^\delta_\varepsilon\big) > C$ for ε small enough (depending on δ), where

$$X^\delta_\varepsilon := \big\{x : \mu\big(\overline{B}(x,\varepsilon)\big) < \delta\varepsilon\big\}\,.$$

Note also that, since we used closed balls $\overline{B}(x,\varepsilon)$, the set X^δ_ε is open.

Keeping fixed δ and ε, for n large enough we define

$$\widetilde{\Sigma}_n := \Sigma_n \setminus X^\delta_\varepsilon$$

and $\tilde{\mu}_n := \mathscr{H}^1 \llcorner \widetilde{\Sigma}_n$. We aim to show the existence of some $K(\delta) \xrightarrow[\delta\to 0]{} 0$ such that

$$\mathfrak{F}(\tilde{\mu}_n) \leq \mathfrak{F}(\mu_n) + K(\delta)\,. \tag{3.6}$$

We claim that (3.6) implies the thesis. Indeed, let μ_δ be a weak* limit of a subsequence of $\{\tilde{\mu}_n\}$: since $\tilde{\mu}_n \leq \mu_n$ and $\mu_n \xrightarrow{*} \mu$, one has $\mu_\delta \leq \mu$; moreover, since $\tilde{\mu}_n(X_\varepsilon^\delta) = 0$ and X_ε^δ is open, we obtain $\mu_\delta(X_\varepsilon^\delta) = 0$ while $\mu(X_\varepsilon^\delta) > C$, so that

$$\|\mu_\delta\| \leq \|\mu\| - C\,.$$

As a consequence, a weak* limit $\bar{\mu}$ of a subsequence of $\{\mu_\delta\}$ as $\delta \searrow 0$ satisfies $\bar{\mu} \leq \mu$ and $\|\bar{\mu}\| \leq \|\mu\| - C$, hence $\bar{\mu} \neq \mu$. But if (3.6) is true, as in Lemma 3.10 $\mathfrak{F}(\bar{\mu}) \leq \mathfrak{F}(\mu)$ and this contradicts the fact that μ is a minimal optimal measure.

In order to get (3.6), since $\mathscr{H}^1\big(\widetilde{\Sigma}_n\big) \leq \mathscr{H}^1\big(\Sigma_n\big)$, recalling (1.11) and (1.22), it suffices to show that for n large enough (depending on δ) one has

$$d_{\widetilde{\Sigma}_n}(x,y) \leq d_{\Sigma_n}(x,y) + K(\delta) \qquad\qquad \forall\,(x,y) \in \Omega\times\Omega\,. \tag{3.7}$$

By Lemma 1.16, we get that the functions $d_{\widetilde{\Sigma}_n}$ and d_{Σ_n} are equi-uniformly continuous over $\Omega\times\Omega$, since their moduli of continuity can be estimated by means of the modulus of continuity of A in $[0,L]$. Therefore, by the boundedness of Ω, it suffices to verify the inequality (3.7) for a finite number of pairs (x_i,y_i), and hence for a given pair (x,y).

Let now $(x,y) \in \Omega\times\Omega$ be fixed and choose paths θ_n almost optimal between x and y with respect to Σ_n in the sense that

$$\delta_{\Sigma_n}(\theta_n) \leq d_{\Sigma_n}(x,y) + \frac{1}{n}\,.$$

It suffices to show that

$$\delta_{\widetilde{\Sigma}_n}(\theta_n) \leq \delta_{\Sigma_n}(\theta_n) + K(\delta) \tag{3.8}$$

for all n large enough. To show (3.8), we consider θ_n as parametrized by constant speed and we recall that by Lemma 1.16 one can assume $\mathscr{H}^1(\theta_n) \leq L$ for n large enough; hence, up to a subsequence, the paths θ_n uniformly converge to some path θ. We claim that there exists an $\bar{n} = \bar{n}(\delta,x,y)$ such that for every $t \in [0,1]$ and $n \geq \bar{n}$ one has

$$\mu_n\big(\overline{B}(\theta_n(t),\varepsilon/2)\big) \leq \mu\big(\overline{B}(\theta_n(t),\varepsilon)\big) + \delta\varepsilon\,. \tag{3.9}$$

In fact, for each fixed $\bar{t} \in [0,1]$ we have

$$\mu_n\Big(\overline{B}\Big(\theta_n(\bar{t}),\frac{4}{6}\,\varepsilon\Big)\Big) \leq \mu\Big(\overline{B}\Big(\theta_n(\bar{t}),\frac{5}{6}\,\varepsilon\Big)\Big) + \delta\varepsilon\,, \tag{3.10}$$

whenever $n \geq \bar{n}(\bar{t},\delta,x,y)$. Since the paths θ_n are parametrized by constant speed and have length bounded by L, then

$$\overline{B}\Big(\theta_n(t),\frac{3}{6}\,\varepsilon\Big) \subseteq \overline{B}\Big(\theta_n(\bar{t}),\frac{4}{6}\,\varepsilon\Big)\,,\quad \overline{B}\Big(\theta_n(\bar{t}),\frac{5}{6}\,\varepsilon\Big) \subseteq \overline{B}\Big(\theta_n(t),\varepsilon\Big)\,, \tag{3.11}$$

whenever

$$|t - \bar{t}| \le \frac{\varepsilon}{6L}\,.$$

From (3.10) and (3.11) we deduce the validity of (3.9) for all $n \ge \bar{n}(\bar{t}, \delta, x, y)$ and for all t satisfying

$$|t - \bar{t}| \le \frac{\varepsilon}{6L}\,.$$

Hence, by compactness of $[0,1]$, we get the validity of (3.9) for all $n \ge \bar{n}(\delta, x, y)$ (i.e. with $\bar{n}$ independent of t).

Take now $n \ge \bar{n}(\delta, x, y)$ and $t \in [0,1]$ such that $\theta_n(t) \in X_\varepsilon^\delta$: the definition of X_ε^δ and the above estimate ensure that

$$\mu_n\big(\overline{B}(\theta_n(t), \varepsilon/2)\big) \le 2\delta\varepsilon\,.$$

Setting then

$$D_t := [t - \varepsilon/(2v_n), t + \varepsilon/(2v_n)]\,,$$

where

$$v_n = \mathscr{H}^1(\theta_n) = |\theta_n'(t)|$$

for every $t \in [0,1]$, one has that $\mathscr{H}^1\big(\theta_n(D_t)\big) = \varepsilon$, while

$$\mathscr{H}^1 \llcorner \Sigma_n\big(\theta_n(D_t)\big) \le \mu_n\big(\overline{B}(\theta_n(t), \varepsilon/2)\big) \le 2\delta\varepsilon = 2\delta\mathscr{H}^1\big(\theta_n(D_t)\big)\,.$$

Now we can cover $\theta_n^{-1}\big(X_\varepsilon^\delta\big)$ by a finite number of intervals of the form of D_t in such a way that no instant $t \in [0,1]$ belongs to more than two such intervals. As a consequence, we immediately infer

$$\mathscr{H}^1(\theta_n) \ge \frac{\mathscr{H}^1 \llcorner \Sigma_n\big(\theta_n \cap X_\varepsilon^\delta\big)}{4\delta} \tag{3.12}$$

so, defining

$$a := \mathscr{H}^1(\theta_n \setminus \Sigma_n)\,, \qquad b := \mathscr{H}^1 \llcorner \Sigma_n\big(\theta_n \cap X_\varepsilon^\delta\big)\,, \qquad c := \mathscr{H}^1 \llcorner \widetilde{\Sigma}_n(\theta_n)\,,$$

we may summarize what obtained by the equalities

$$\delta_{\Sigma_n}(\theta_n) = A\big(a\big) + B\big(b + c\big)\,, \qquad \delta_{\widetilde{\Sigma}_n}(\theta_n) = A\big(a + b\big) + B\big(c\big)\,; \tag{3.13}$$

moreover,

$$a + b + c = v_n \le L\,.$$

Finally, (3.12) and the definition of a, b and c give

$$a + b + c \ge \frac{b}{4\delta}\,.$$

This implies that $b \le 4\delta\, L$ and hence, by (3.13) and by the uniform continuity of A on the compact set $[0, L] \subseteq \mathbb{R}$, we get

$$\begin{aligned} \delta_{\widetilde{\Sigma}_n}(\theta_n) - \delta_{\Sigma_n}(\theta_n) &= A\big(a + b\big) + B\big(c\big) - A\big(a\big) - B\big(b + c\big) \\ &\le A\big(a + b\big) - A\big(a\big) \le \omega_A(4\delta L)\,, \end{aligned}$$

where ω_A stands for the modulus of continuity of A over the interval $[0, L]$. Hence, (3.8) is proved with

$$K(\delta) := \omega_A(4\delta L)\,,$$

which therefore concludes the proof. $\square$

We can go now further, showing that the set Σ in last lemma can be chosen rectifiable.

LEMMA 3.12. *Any minimal optimal measure is concentrated on a rectifiable set Σ.*

PROOF. Let us take an arbitrary optimal measure β and a minimizing sequence $\beta_n = \mathscr{H}^1 \llcorner \Sigma_n$ for the functional $\mathfrak{F}$, i.e. $\beta_n \overset{*}{\rightharpoonup} \beta$ and $\mathfrak{F}(\beta_n) \to \mathfrak{F}(\beta)$. The strategy to achieve the conclusion will be to construct an optimal measure $\mu \leq \beta$ concentrated on a rectifiable set: by Definition 3.8 of minimality, this yields the thesis.

If $\mathfrak{F}(\beta) = \mathfrak{F}(\emptyset)$, there is nothing to prove. Otherwise,

$$K := \mathfrak{F}(\emptyset) - \mathfrak{F}(\beta) > 0\,,$$

so that for n sufficiently large one has

$$\mathfrak{F}(\emptyset) - \mathfrak{F}(\beta_n) > \frac{K}{2}\,.$$

According to Proposition 1.14, for n large enough we take a t.p.m. η_n such that

$$C_{\Sigma_n}(\eta_n) - MK(\Sigma_n) < \mathfrak{F}(\emptyset) - \mathfrak{F}(\beta_n) - K/2\,.$$

Therefore,

$$\begin{aligned}
&\int_\Theta A\big(\mathscr{H}^1(\theta)\big) - A\big(\mathscr{H}^1(\theta \setminus \Sigma_n)\big)\, d\eta_n \geq \\
&\qquad \geq \int_\Theta A\big(\mathscr{H}^1(\theta)\big) - \Big(A\big(\mathscr{H}^1(\theta \setminus \Sigma_n)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma_n)\big)\Big)\, d\eta_n \\
&\qquad = C_\emptyset(\eta_n) - C_{\Sigma_n}(\eta_n) \\
&\qquad \geq \mathfrak{F}(\emptyset) - C_{\Sigma_n}(\eta_n) > \mathfrak{F}(\beta_n) - MK(\Sigma_n) + K/2 \\
&\qquad = \mathfrak{F}(\Sigma_n) - MK(\Sigma_n) + K/2 \geq K/2\,.
\end{aligned} \tag{3.14}$$

Recalling that all the t.p.m.'s have unitary total mass, since

$$\|f^+\| = \|f^-\| = 1\,,$$

we deduce the existence of paths $\theta_n \in \operatorname{spt} \eta_n$ such that

$$A\big(\mathscr{H}^1(\theta_n)\big) - A\big(\mathscr{H}^1(\theta_n \setminus \Sigma_n)\big) \geq \frac{K}{2}\,. \tag{3.15}$$

By Lemma 3.7, since $\theta_n \in \operatorname{spt} \eta_n$ and the sequence $\{\eta_n\}$ is almost optimal with respect to C_{Σ_n}, we may assume that

$$\mathscr{H}^1(\theta_n) \leq L\,;$$

thus, thanks to the uniform continuity of A in $[0, L]$, from (3.15) we infer that

$$\begin{aligned}
\beta_n(\theta_n) = \mathscr{H}^1 \llcorner \Sigma_n(\theta_n) &= \mathscr{H}^1\big(\theta_n \cap \Sigma_n\big) = \mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta_n \setminus \Sigma_n) \\
&\geq \omega^{-1}(K/2)\,,
\end{aligned}$$

where ω stands for the modulus of continuity of A in $[0, L]$, and $\omega^{-1}(K/2)$ stands for the biggest number $x > 0$ such that $\omega(x) \leq K/2$ (this is a slight abuse of notations, since ω may be not *strictly* increasing). Since $\{\theta_n\}$ is a sequence of paths of Euclidean length bounded by L we may assume, up to a subsequence, that the paths θ_n uniformly converge to some path $\theta \in \Theta$, thus in particular the respective traces $\theta_n([0,1])$ converge in the Hausdorff distance to $C_1 = \theta([0,1])$. Again up to a subsequence, we assume that

$$\beta_n \llcorner \theta_n \xrightharpoonup{*} \mu_1$$

so that, since C_1 is closed, we deduce that μ_1 is concentrated in C_1: in fact, for every open set U such that $\operatorname{dist}(U, C_1) > 0$ one has

$$\mu_1(U) \leq \liminf_{n\to\infty} \beta_n \llcorner \theta_n(U) = 0 .$$

Thanks to Lemma 3.1, we have

$$\|\mu_1\| \geq \omega^{-1}(K/2) .$$

Noticing that $\mu_1 \leq \beta$, we obtain the estimates

$$\beta(C_1) \geq \mu_1(C_1) \geq \omega^{-1}(K/2) > 0 .$$

We want now to iterate the above argument: to this aim, we call

$$\widetilde{\Sigma}_n := \Sigma_n \setminus \theta_n$$

so that

$$\tilde{\beta}_n = \mathscr{H}^1 \llcorner \widetilde{\Sigma}_n \xrightharpoonup{*} \beta - \mu_1 .$$

We use now an argument similar to the previous one: if $\mathfrak{F}(\mu_1) \leq \mathfrak{F}(\beta)$, then μ_1 is an optimal measure and we conclude. Otherwise,

$$K' := \mathfrak{F}(\mu_1) - \mathfrak{F}(\beta) > 0 ,$$

so that for n sufficiently large

$$\rho_n := \mathfrak{F}(\beta_n \llcorner \theta_n) - \mathfrak{F}(\beta_n) - \frac{K'}{2} > 0 \ :$$

indeed, we know that $\mathfrak{F}(\beta_n) \to \mathfrak{F}(\beta)$ and by lower semicontinuity of $\mathfrak{F}$ one has

$$\liminf \mathfrak{F}(\beta_n \llcorner \theta_n) \geq \mathfrak{F}(\mu_1) .$$

Arguing as in (3.14), choosing the sequence $\{\eta_n\}$ almost optimal for C_{Σ_n} (η_n is possibly different from the ones of the first step), noticing that

$$\beta_n \llcorner \theta_n = \mathscr{H}^1 \llcorner (\Sigma_n \cap \theta_n)$$

and recalling that

$$\mathscr{H}^1(\Sigma_n) \geq \mathscr{H}^1(\Sigma_n \cap \theta_n)$$

for any n, one has the relationship

$$\begin{aligned}
C_{\Sigma_n \cap \theta_n}(\eta_n) - C_{\Sigma_n}(\eta_n) &\geq MK(\Sigma_n \cap \theta_n) - MK(\Sigma_n) - \rho_n \\
&= \mathfrak{F}(\Sigma_n \cap \theta_n) - H(\mathscr{H}^1(\Sigma_n \cap \theta_n)) - \mathfrak{F}(\Sigma_n) + H(\mathscr{H}^1(\Sigma_n)) - \rho_n \\
&\geq \mathfrak{F}(\Sigma_n \cap \theta_n) - \mathfrak{F}(\Sigma_n) - \rho_n = K'/2 .
\end{aligned}$$

Therefore, for n large enough, one has

$$\begin{aligned}
&\int_\Theta A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)+\mathscr{H}^1(\theta\cap\Sigma_n\setminus\theta_n)\big)-A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)\big)\,d\eta_n\\
&\quad\geq\int_\Theta\Big(A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)+\mathscr{H}^1(\theta\cap\Sigma_n\setminus\theta_n)\big)\\
&\qquad\qquad+B\big(\mathscr{H}^1(\theta\cap\Sigma_n)-\mathscr{H}^1(\theta\cap\Sigma_n\setminus\theta_n)\big)\Big)\\
&\qquad\qquad-\Big(A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)\big)+B\big(\mathscr{H}^1(\theta\cap\Sigma_n)\big)\Big)\,d\eta_n\\
&\quad=\int_\Theta\Big(A\big(\mathscr{H}^1(\theta\setminus(\Sigma_n\cap\theta_n))\big)+B\big(\mathscr{H}^1(\theta\cap(\Sigma_n\cap\theta_n))\big)\Big)-\delta_{\Sigma_n}(\theta)\,d\eta_n\\
&\quad=C_{\Sigma_n\cap\theta_n}(\eta_n)-C_{\Sigma_n}(\eta_n)\geq\frac{K'}{2}.
\end{aligned}$$

As before, we deduce the existence of a sequence of paths $\{\sigma_n\}\in\operatorname{spt}\eta_n$ of lengths bounded by L and with

$$A\big(\mathscr{H}^1(\sigma_n\setminus\Sigma_n)+\mathscr{H}^1(\sigma_n\cap\Sigma_n\setminus\theta_n)\big)-A\big(\mathscr{H}^1(\sigma_n\setminus\Sigma_n)\big)\geq\frac{K'}{2}\,,$$

so that

$$\tilde\beta_n(\sigma_n)=\mathscr{H}^1(\sigma_n\cap\Sigma_n\setminus\theta_n)\geq\omega^{-1}(K'/2)\,.$$

Again, by Lemma 3.1 we find a measure μ_2 concentrated in a curve $C_2\in\Theta$ of length bounded by L (C_2 is the Hausdorff limit of the traces $\sigma_n([0,1])$), such that

$$\tilde\beta_n\llcorner\sigma_n\xrightarrow{\;*\;}\mu_2\,,\qquad\text{and}\qquad\|\mu_2\|\geq\omega^{-1}(K'/2)\,.$$

Recalling that $\tilde\beta_n\xrightarrow{\;*\;}\beta-\mu_1$, we deduce $\mu_1+\mu_2\leq\beta$.

We iterate this argument, finding measures μ_i concentrated in curves C_i and such that

$$\hat\mu_n:=\mu_1+\mu_2+\cdots+\mu_n$$

is an increasing sequence of positive measures with $\hat\mu_n\leq\beta$; since $\|\beta\|<+\infty$, we deduce that $\hat\mu_n$ converges strongly to a measure μ_∞ with $\mu_\infty\leq\beta$. Now, by construction we have that

$$\big\|\hat\mu_{n+1}-\hat\mu_n\big\|=\|\mu_{n+1}\|\geq\omega^{-1}\Big(\big(\mathfrak{F}(\hat\mu_n)-\mathfrak{F}(\beta)\big)/2\Big)\,;$$

but since $\hat\mu_n\to\mu_\infty$ strongly, we infer that

$$\big\|\hat\mu_{n+1}-\hat\mu_n\big\|\longrightarrow0\,,$$

then

$$\omega^{-1}\Big(\big(\mathfrak{F}(\hat\mu_n)-\mathfrak{F}(\beta)\big)/2\Big)\longrightarrow0\,.$$

Keeping in mind that ω is the modulus of continuity of the function A in $[0,L]$, we infer that $\mathfrak{F}(\hat\mu_n)\to\mathfrak{F}(\beta)$ so that, by lower semicontinuity of $\mathfrak{F}$, the measure μ_∞ is an optimal measure less than β, as we were looking for. $\square$

Summarizing, we know that there exists an optimal measure μ concentrated in a $1-$rectifiable set Σ, and absolutely continuous with respect to $\mathscr{H}^1$. It follows that $\mu = \varphi\mathscr{H}^1 \llcorner \Sigma$, so we found a relaxed solution which is, in a certain sense, similar to the classical one: if $\varphi \equiv 1$, we have indeed a classical solution. We will prove in a moment that $\varphi \leq 1$: this is quite reasonable, since we can imagine that covering a path with $\varphi > 1$ has a cost strictly greater than covering the same path with $\varphi \equiv 1$: even though this argument is far from being formal, this is more or less the idea of the next proof. On the other hand, the case $\varphi < 1$ is not meaningless: in fact, following a path of length l on a network where $\varphi \equiv p \in [0,1]$ can be interpreted as making a length pl by train and the remaining $(1-p)l$ by own means, which may be, in some cases, better than if $\varphi \equiv 1$. More precisely, one can imagine that in some situation this is indeed the best case, which is exactly what we will prove in the example of Chapter 3. This intuitive discussion will become clearer in light of Proposition 3.16.

In the sequel, we will use for simplicity the following notation.

Definition 3.13. *We denote by $\mathcal{M}_1^+(\Omega)$ the set of those measures $\mu \in \mathcal{M}^+(\Omega)$ which can be represented as*

$$\mu = \varphi\mathscr{H}^1 \llcorner \Sigma \,,$$

where Σ is the union of countably many Lipschitz paths of uniformly bounded length, and

$$\varphi : \Sigma \to [0,1]$$

is a Borel function.

Definition 3.14. *Given a path $\theta \in \Theta$, and given two points of θ, namely $P = \theta(s)$ and $Q = \theta(t)$ with $t \geq s$, we will denote by $\widetilde{PQ}$ the path $\theta \llcorner [s,t] \in \Theta$; we will use this notation only when the particular path θ containing P and Q will be clear from the context.*

We show now, as anticipated, the existence of an optimal measure μ with $\varphi \leq 1$. In view of Definition 3.13, we can equivalently say that there exists an optimal measure $\mu \in \mathcal{M}_1^+(\Omega)$.

Theorem 3.15. *There is an optimal measure $\mu \in \mathcal{M}_1^+(\Omega)$; more precisely, each optimal measure μ which is minimal in the sense of Definition 3.8 belongs to $\mathcal{M}_1^+(\Omega)$.*

Proof. Let μ be a minimal optimal measure, which is concentrated in a rectifiable set

$$\Sigma = \cup_{i \in \mathbb{N}} \theta_i$$

where the paths $\theta_i \in \Theta$ have lengths bounded by a constant L by Lemmas 3.7 and 3.12. Let $\{\Sigma_n\}$ be a minimizing sequence in the definition of $\mathfrak{F}(\mu)$, that is,

$$\mu_n := \mathscr{H}^1 \llcorner \Sigma_n \overset{*}{\rightharpoonup} \mu$$

and $\mathfrak{F}(\mu_n) \to \mathfrak{F}(\mu)$. We claim that $\mu \in \mathcal{M}_1^+(\Omega)$: if it were false, there would be an index $i \in \mathbb{N}$ and a Lipschitz path

$$\Delta_0 = \widetilde{PQ} \subseteq \theta_i \subseteq \Sigma$$

such that $\mu(\Delta_0) > \mathscr{H}^1(\Delta_0)$.

In order to find a competitor to μ, that will lead to a contradiction, we fix an $\varepsilon > 0$ (that we will eventually lead to 0) and we let C_ε to stand for the open $\varepsilon-$tubular neighborhood of Δ_0, that is the set of those points of Ω the minimal distance of which from θ_i is strictly less than ε and is reached at some point of Δ_0. Notice that, up to an arbitrary small movement of P and Q, Δ_0 can be chosen in such a way that, for all $\varepsilon > 0$ except for countably many,

$$\mu\big(\partial C_\varepsilon\big) = 0\,, \qquad \text{and} \qquad \forall\, n \in \mathbb{N}\,,\ \mu_n\big(\partial C_\varepsilon\big) = 0\,. \tag{3.16}$$

In the rest of the proof, we deal only with numbers $\varepsilon > 0$ for which (3.16) holds.

We now set, for any n,

$$\Sigma_{n,\varepsilon} := \big(\Sigma_n \setminus C_\varepsilon\big) \cup \Delta_0, \qquad \mu_{n,\varepsilon} := \mathscr{H}^1 \llcorner \Sigma_{n,\varepsilon}\,. \tag{3.17}$$

In short, from every Σ_n we drop the whole tubular neighborhood C_ε, and then we add the curve Δ_0.

We want now to check that $\mathfrak{F}\big(\Sigma_{n,\varepsilon}\big)$ is not much larger than $\mathfrak{F}\big(\Sigma_n\big)$. To this aim, we fix a $\delta < \varepsilon^2$ and we define $D_\delta \subseteq C_\varepsilon$ the δ-tubular neighborhood of the part of Δ_0 having distance more than ε from both P and Q; formally,

$$D_\delta := \Big\{ x \in \Omega : \exists\, y \in \Delta_0,\ \mathscr{H}^1(\widetilde{Py}) > \varepsilon, \mathscr{H}^1(\widetilde{yQ}) > \varepsilon,$$
$$\operatorname{dist}(x, \Delta_0) = |y - x| < \delta \Big\}$$

(see Figure 1). Notice that, since μ is a finite measure, we have

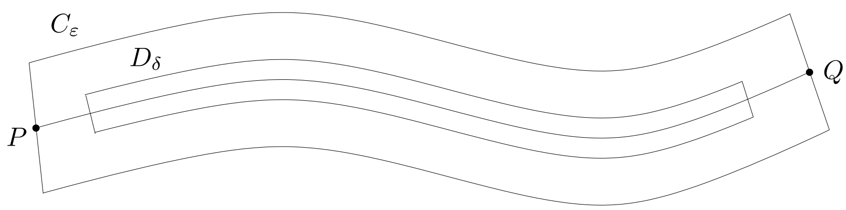

FIGURE 1. Construction of the proof of Theorem 3.15

$$\mu(C_\varepsilon) = \mu(\Delta_0) + K_1(\varepsilon)$$

with $K_1(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$ and, exactly as in (3.16), for all but countably many δ one has

$$\mu\big(\partial D_\varepsilon\big) = 0, \qquad \text{and} \qquad \forall\, n \in \mathbb{N},\ \mu_n\big(\partial D_\varepsilon\big) = 0\,. \tag{3.18}$$

Moreover,

$$\Sigma_\varepsilon := \cap_{\delta>0} D_\delta$$

is a Lipschitz path (since it is contained in Δ_0), namely the part of Δ_0 made by those points having distance more than ε from both P and Q: therefore, for any $\delta > 0$ one has $\mu(D_\delta) \geq \mu(\Sigma_\varepsilon)$. Since

$$\mu(\Sigma_\varepsilon) \nearrow \mu(\Delta_0) \qquad \text{as} \qquad \varepsilon \searrow 0\,,$$

it follows that

$$\mu\big(C_\varepsilon \setminus D_\delta\big) \leq \mu(C_\varepsilon) - \mu(\Sigma_\varepsilon) = K_1(\varepsilon) + \mu(\Delta_0) - \mu(\Sigma_\varepsilon) =: K_2(\varepsilon)\,,$$

where again $K_2(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. Recalling that $\mu_n \stackrel{*}{\rightharpoonup} \mu$, (3.16) and (3.18), we derive that

$$\mu_n\big(C_\varepsilon \setminus D_\delta\big) \to \mu\big(C_\varepsilon \setminus D_\delta\big)$$

as $n \to \infty$, and hence, for n large enough, one has

$$\mu_n\big(C_\varepsilon \setminus D_\delta\big) \leq 2K_2(\varepsilon)\,. \tag{3.19}$$

Using the projection map from the tubular neighborhood C_ε onto Δ_0, which is defined for all sufficiently small ε, we may define a Lipschitz map $\alpha_\delta : \Omega \to \Omega$ such that (recall that $\delta < \varepsilon^2$)

i) $\alpha_\delta \equiv \mathrm{Id} \quad \text{in } \Omega \setminus C_\varepsilon$;

ii) $\alpha_\delta(x) \in \Delta_0 \quad \forall x \in D_\delta$;

iii) for a.e. $x \in \Omega$, $|\nabla \alpha_\delta(x)| \leq \big(1 + 3\varepsilon\big)$;

iv) $|\alpha_\delta(x) - x| \leq \varepsilon \ \forall x \in \Omega$.

We want now to define a Borel map $r : \Theta \to \Theta$ such that $r(\theta)$ is a path with the same endpoints as θ and with the property that, if $\mathscr{H}^1(\theta) \leq L$ and n is large enough,

$$\bar\delta_{\Sigma_{n,\varepsilon}}\big(r(\theta)\big) \leq \delta_{\Sigma_n}(\theta) + K(\varepsilon) \tag{3.20}$$

for some $K(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. To do that, we set

$$r(\theta) := \theta_1 \cdot (\alpha_\delta \circ \theta) \cdot \theta_2\,,$$

where θ_1 and θ_2 are the segments connecting $\theta(0)$ with $\alpha_\delta(\theta(0))$ and $\alpha_\delta(\theta(1))$ with $\theta(1)$ respectively. Therefore, by construction and recalling the properties iii) and iv) above, one has

$$\mathscr{H}^1\big(r(\theta)\big) \leq 2\varepsilon + \mathscr{H}^1(\theta) + 3L\varepsilon = \mathscr{H}^1(\theta) + K_3(\varepsilon)\,, \tag{3.21}$$

where again $K_3(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. Write now $\theta = \theta_a \cup \theta_b \cup \theta_c$, where

$$\theta_a \subseteq \Omega \setminus C_\varepsilon\,, \qquad \theta_b \subseteq C_\varepsilon \setminus D_\delta\,, \qquad \theta_c \subseteq D_\delta\ :$$

we consider now separately θ_a, θ_b and θ_c. Concerning θ_a, by the property i) we have $\alpha_\delta(\theta_a)=\theta_a$, so that by (3.17)

$$\mathscr{H}^1\big(\alpha_\delta(\theta_a)\setminus\Sigma_{n,\varepsilon}\big)=\mathscr{H}^1\big(\theta_a\setminus\Sigma_n\big)\,. \tag{3.22}$$

Concerning θ_b, by (3.19) we know that

$$\mathscr{H}^1(\Sigma_n\cap\theta_b)=\mu_n(\theta_b)\leq 2K_2(\varepsilon)$$

then, also by property iii), one has

$$\begin{aligned}\mathscr{H}^1\big(\alpha_\delta(\theta_b)\setminus\Sigma_{n,\varepsilon}\big)&\leq(1+3\varepsilon)\mathscr{H}^1(\theta_b)\leq\mathscr{H}^1(\theta_b)+3L\varepsilon\\&\leq\mathscr{H}^1\big(\theta_b\setminus\Sigma_n\big)+2K_2(\varepsilon)+3L\varepsilon\,.\end{aligned} \tag{3.23}$$

Finally, concerning θ_c, we know by ii) that

$$\alpha_\delta(\theta_c)\subseteq\Delta_0\subseteq\Sigma_{n,\varepsilon}$$

so that

$$\mathscr{H}^1\big(\alpha_\delta(\theta_c)\setminus\Sigma_{n,\varepsilon}\big)=0\,. \tag{3.24}$$

Recalling now that

$$r(\theta)=\alpha_\delta(\theta_a)\cup\alpha_\delta(\theta_b)\cup\alpha_\delta(\theta_c)\cup\theta_1\cup\theta_2$$

and putting together (3.22), (3.23) and (3.24), we derive that

$$\mathscr{H}^1\big(r(\theta)\setminus\Sigma_{n,\varepsilon}\big)\leq\mathscr{H}^1(\theta\setminus\Sigma_n)+K_4(\varepsilon)\,,$$

where again $K_4\xrightarrow[\varepsilon\to 0]{}0$. As a consequence,

$$\begin{aligned}0\leq l:=&\Big(\mathscr{H}^1(\theta\setminus\Sigma_n)-\mathscr{H}^1\big(r(\theta)\setminus\Sigma_{n,\varepsilon}\big)+K_3(\varepsilon)+K_4(\varepsilon)\Big)\wedge\mathscr{H}^1(r(\theta)\cap\Sigma_{n,\varepsilon})\\&\leq\mathscr{H}^1(r(\theta)\cap\Sigma_{n,\varepsilon})\,.\end{aligned}$$

Making use of the formula for $\bar\delta_\Sigma$ given in Proposition 1.18 and also by (3.21), we deduce (recall that $\mathscr{H}^1(\theta)\leq L$ and denote as usual by ω the modulus of continuity of A in $[0,L]$)

$$\begin{aligned}\bar\delta_{\Sigma_{n,\varepsilon}}\big(r(\theta)\big)&\leq A\big(\mathscr{H}^1\big(r(\theta)\setminus\Sigma_{n,\varepsilon}\big)+l\big)+B\big(\mathscr{H}^1(r(\theta)\cap\Sigma_{n,\varepsilon})-l\big)\\&\leq A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)+K_3(\varepsilon)+K_4(\varepsilon)\big)\\&\qquad+B\Big(0\vee\Big(\mathscr{H}^1(r(\theta))-\mathscr{H}^1(\theta\setminus\Sigma_n)-K_3(\varepsilon)-K_4(\varepsilon)\Big)\Big)\\&\leq A\big(\mathscr{H}^1(\theta\setminus\Sigma_n)\big)+\omega\big(K_3(\varepsilon)+K_4(\varepsilon)\big)\\&\qquad+B\Big(0\vee\Big(\mathscr{H}^1(\theta\cap\Sigma_n)-K_4(\varepsilon)\Big)\Big)\\&\leq\delta_{\Sigma_n}(\theta)+\omega\big(K_3(\varepsilon)+K_4(\varepsilon)\big)\ :\end{aligned}$$

then we finally proved (3.20) with

$$K(\varepsilon):=\omega\big(K_3(\varepsilon)+K_4(\varepsilon)\big)\,.$$

Take now any t.p.m. η such that $\mathscr{H}^1(\theta) \leq L$ for all $\theta \in \operatorname{spt}\eta$. By construction $r_\# \eta$ is another t.p.m., so recalling Proposition 1.14 and thanks to (3.20) we derive

$$MK(\Sigma_{n,\varepsilon}) \leq \overline{C}_{\Sigma_{n,\varepsilon}}(r_\#\eta) \leq C_{\Sigma_n}(\eta) + K(\varepsilon)\,.$$

As we already noticed in Lemma 3.7, one can take such a t.p.m. η with $C_{\Sigma_n}(\eta)$ arbitrarily close to $MK(\Sigma_n)$, therefore we deduce

$$MK(\Sigma_{n,\varepsilon}) \leq MK(\Sigma_n) + K(\varepsilon)\,. \tag{3.25}$$

Due to (3.16),

$$\lim \mu_n(C_\varepsilon) = \mu(C_\varepsilon) \geq \mu(\Delta_0) > \mathscr{H}^1(\Delta_0)\,.$$

Hence, at least for n large enough, one has

$$\mathscr{H}^1(\Sigma_{n,\varepsilon}) < \mathscr{H}^1(\Sigma_n)$$

(recall the definition (3.17) of $\Sigma_{n,\varepsilon}$). Together with (3.25), this ensures that for n large enough one has

$$\mathfrak{F}(\Sigma_{n,\varepsilon}) \leq \mathfrak{F}(\Sigma_n) + K(\varepsilon)\,.$$

Up to a subsequence, we may assume $\mu_{n,\varepsilon} \xrightarrow[n\to\infty]{*} \mu_\varepsilon$. Thus, minding that

$$\mu_{n,\varepsilon} \leq \mu_n \vee (\mathscr{H}^1 \llcorner \Delta_0)\,,$$

we get

$$\mu_\varepsilon \leq \mu \vee (\mathscr{H}^1 \llcorner \Delta_0)\,.$$

In particular,

$$\mu_\varepsilon(\Omega \setminus C_\varepsilon) \leq \mu(\Omega \setminus C_\varepsilon)\,,$$

and also $\mu_\varepsilon(\partial C_\varepsilon) = 0$. On the other hand, the latter fact implies

$$\mu_\varepsilon(C_\varepsilon) = \lim_{n\to\infty} \mu_{n,\varepsilon}(C_\varepsilon) = \mathscr{H}^1(\Delta_0)\,,$$

while $\mu(C_\varepsilon) \geq \mu(\Delta_0)$. Therefore,

$$\begin{aligned} \|\mu_\varepsilon\| &\leq \mu_\varepsilon(\Omega \setminus C_\varepsilon) + \mathscr{H}^1(\Delta_0) \leq \|\mu\| - \mu(C_\varepsilon) + \mathscr{H}^1(\Delta_0) \\ &\leq \|\mu\| - \mu(\Delta_0) + \mathscr{H}^1(\Delta_0)\,. \end{aligned} \tag{3.26}$$

Finally, by the lower semicontinuity of $\mathfrak{F}$ and since $\mathfrak{F}(\mu_n) \to \mathfrak{F}(\mu)$, we know that

$$\mathfrak{F}(\mu_\varepsilon) \leq \liminf_{n\to\infty} \mathfrak{F}(\mu_{n,\varepsilon}) \leq \liminf_{n\to\infty} \mathfrak{F}(\mu_n) + K(\varepsilon) = \mathfrak{F}(\mu) + K(\varepsilon)\,.$$

Let us call now ν_0 a weak* limit of some sequence μ_{ε_j} with $\varepsilon_j \to 0$ for $j \to \infty$: since $K(\varepsilon_j) \to 0$ we have, again by the lower semicontinuity of $\mathfrak{F}$, that $\mathfrak{F}(\nu_0) \leq \mathfrak{F}(\mu)$, and hence ν_0 is optimal. On the other hand, since

$$\mu_\varepsilon \leq \mu \vee (\mathscr{H}^1 \llcorner \Delta_0)\,,$$

by (3.26) and the lower semicontinuity of the norm one has

$$\nu_0 \leq \mu \vee (\mathscr{H}^1 \llcorner \Delta_0)\,, \qquad \|\nu_0\| \leq \|\mu\| - \mu(\Delta_0) + \mathscr{H}^1(\Delta_0)\,. \tag{3.27}$$

Summarizing, starting from a Lipschitz path Δ_0 such that $\mu(\Delta_0) > \mathscr{H}^1(\Delta_0)$, we constructed a measure ν_0 which is optimal for $\mathfrak{F}$ and verifies (3.27). In the very same way, the construction above also works if Δ_0 is replaced by a finite union of disjoint Lipschitz paths. We are then going to define a sequence $\{\Delta_k\}_{k\in\mathbb{N}}$ of finite unions of disjoint Lipschitz paths contained in Δ_0: to this aim, for every integer k, we divide $\Delta_0 = \widetilde{PQ}$ in k subpaths Δ_i^k, $i = 1, \ldots, k$ of the same length $\mathscr{H}^1(\widetilde{PQ})/k$, and we define Δ_k as the union of all Δ_i^k for which

$$\mu(\Delta_i^k) > \mathscr{H}^1(\Delta_i^k)\,.$$

As already said, we can repeat the construction above with Δ_k in place of Δ_0, obtaining a measure ν_k which is optimal for $\mathfrak{F}$ and such that

$$\begin{aligned}&\nu_k \leq \mu \vee (\mathscr{H}^1 \llcorner \Delta_k)\,,\\ &\|\nu_k\| \leq \|\mu\| - \mu(\Delta_k) + \mathscr{H}^1(\Delta_k) \leq \|\mu\| - \mu(\Delta_0) + \mathscr{H}^1(\Delta_0)\,,\end{aligned} \tag{3.28}$$

where the last inequality follows from the fact that

$$\mu(\Delta_0 \setminus \Delta_k) \leq \mathscr{H}^1(\Delta_0 \setminus \Delta_k)\,.$$

Finally, passing to the limit as $k \to \infty$, there is a measure ν such that, up to a subsequence, $\nu_k \overset{*}{\rightharpoonup} \nu$. By lower semicontinuity the measure ν is optimal for $\mathfrak{F}$; moreover, writing $\mu = \varphi \mathscr{H}^1 \llcorner \Sigma$ according to Lemma 3.11, by construction one has

$$\mathscr{H}^1 \llcorner \Delta_k \overset{*}{\rightharpoonup} \mathscr{H}^1 \llcorner \{x \in \widetilde{PQ} : \ \varphi(x) > 1\}\,.$$

Hence, the limit in (3.28) gives

$$\nu \leq \mu \vee (\mathscr{H}^1 \llcorner \{\varphi(x) > 1\}) = \mu, \quad \|\nu\| \leq \|\mu\| - \mu(\Delta_0) + \mathscr{H}^1(\Delta_0) < \|\mu\|\,,$$

so we found the desired contradiction to the fact that μ is a minimal optimal measure. $\square$

We give now a useful representation of $\mathfrak{F}(\mu)$ for a measure $\mu \in \mathcal{M}_1^+(\Omega)$.

Proposition 3.16. *For any $\mu \in \mathcal{M}_1^+(\Omega)$ one has*

$$\mathfrak{F}(\mu) = MK(\mu) + H(\|\mu\|)\,, \tag{3.29}$$

where

$$MK(\mu) := \inf\big\{C_\mu(\eta) \ : \ \eta \textit{ is a t.p.m.}\big\}\,, \tag{3.30}$$

$$C_\mu(\eta) := \int_\Theta \delta_\mu(\theta)\,d\eta(\theta)\,, \tag{3.31}$$

$$\delta_\mu(\theta) := A\big(\mathscr{H}^1(\theta) - \mu(\theta)\big) + B\big(\mu(\theta)\big)\,. \tag{3.32}$$

Remark 3.17. We point out that equations (3.29), (3.30), (3.31) and (3.32) are the generalizations of (1.11), (1.9), (1.8) and (1.2) respectively. Note also that, in order for these equations to make sense, it is necessary that $\mu \in \mathcal{M}_1^+(\Omega)$, i.e. $\mu = \varphi\mathscr{H}^1 \llcorner \Sigma$ with $\varphi \leq 1$. Otherwise, if μ has parts of dimension lower than one, or with dimension one but density greater than one, then $\mathscr{H}^1(\theta) - \mu(\theta)$ could be negative, which is meaningless. On the other hand, if μ has parts of dimension greater than one then those parts have no effect in (3.32), and so we implicitly need to make use of Lemma 3.12. We also remark that the above result is extremely useful: in fact, it allows to evaluate the cost $\mathfrak{F}$ of any measure μ making use of the transport path measures, exactly as one does for the sets Σ, instead of working with optimal sequences of sets as we needed to do until now.

We may now prove Proposition 3.16.

OF PROPOSITION 3.16. Having taken (3.30), (3.31) and (3.32) as definitions, we need to establish (3.29).

First of all, it is useful to compute, exactly as in (1.13) and (1.21), the relaxed envelope $\bar\delta_\mu$ of δ_μ with fixed endpoints, and the corresponding generalized cost $\overline{C}_\mu$, as well as to generalize the definition (1.5) of the distance in Ω and of the cost I_Σ in the obvious way:

$$\begin{aligned}
\bar\delta_\mu(\theta) &:= \inf\left\{\liminf_{n\to\infty} \delta_\mu(\theta_n) :\ \theta_n(0)=\theta(0),\ \theta_n(1)=\theta(1),\ \theta_n \xrightarrow{\Theta} \theta\right\};\\
\overline{C}_\mu(\eta) &:= \int_\Theta \bar\delta_\mu(\theta)\, d\eta(\theta);\\
d_\mu(x,y) &:= \inf\left\{\delta_\mu(\theta) :\ \theta\in\Theta,\ \theta(0)=x,\ \theta(1)=y\right\};\\
I_\mu(\gamma) &:= \iint_{\Omega\times\Omega} d_\mu(x,y)\, d\gamma(x,y).
\end{aligned} \tag{3.33}$$

Then, as in Proposition 1.14, one shows that

$$\begin{aligned}
MK(\mu) &= \min\left\{I_\mu(\gamma) \ :\ \gamma \text{ is a transport plan}\right\}\\
&= \inf\left\{C_\mu(\eta) :\ \eta \text{ is a t.p.m.}\right\}\\
&= \min\left\{\overline{C}_\mu(\eta) :\ \eta \text{ is a t.p.m.}\right\};
\end{aligned} \tag{3.34}$$

moreover, as in Proposition 1.18 one has

$$\begin{aligned}
\bar\delta_\mu(\theta) &= \inf\left\{A\big(\mathscr{H}^1(\theta)-\mu(\theta)+l\big) + B\big(\mu(\theta)-l\big) :\ 0\le l\le \mu(\theta)\right\}\\
&= \inf\left\{A\big(\mathscr{H}^1(\theta)-l\big) + B\big(l\big) :\ 0\le l\le \mu(\theta)\right\}.
\end{aligned} \tag{3.35}$$

Consider now any sequence of closed sets $\{\Sigma_n\}$ such that $\mathscr{H}^1 \llcorner \Sigma_n \overset{*}{\rightharpoonup} \mu$: given any sequence of paths $\theta_n \to \theta$, we claim that

$$\bar\delta_\mu(\theta) \le \liminf_{n\to\infty} \bar\delta_{\Sigma_n}(\theta_n). \tag{3.36}$$

In fact, by Gołąb Theorem we know that

$$\mathscr{H}^1(\theta) \le \liminf \mathscr{H}^1(\theta_n),$$

and on the other hand by Lemma 3.1 one has

$$\mu(\theta) \ge \limsup \mathscr{H}^1 \llcorner \Sigma_n(\theta_n).$$

This gives (3.36), since recalling (3.35) we get

$$\bar\delta_\mu(\theta) = \inf\left\{A\big(\mathscr{H}^1(\theta)-l\big) + B\big(l\big) :\ 0\le l\le \mu(\theta)\right\}$$

$$
\begin{aligned}
&\le \liminf_{n\to\infty} \inf \big\{ A\big(\mathscr{H}^1(\theta) - l\big) + B\big(l\big) :\ 0 \le l \le \mu(\theta) \wedge \mathscr{H}^1(\theta_n)\big\} \\
&\le \liminf_{n\to\infty} \inf \big\{ A\big(\mathscr{H}^1(\theta_n) - l\big) + B\big(l\big) :\ 0 \le l \le \mu(\theta) \wedge \mathscr{H}^1(\theta_n)\big\} \\
&\le \liminf_{n\to\infty} \inf \big\{ A\big(\mathscr{H}^1(\theta_n) - l\big) + B\big(l\big) :\ 0 \le l \le \mu(\theta) \wedge \mathscr{H}^1 \llcorner \Sigma_n(\theta_n)\big\} \\
&= \liminf_{n\to\infty} \inf \big\{ A\big(\mathscr{H}^1(\theta_n) - l\big) + B\big(l\big) :\ 0 \le l \le \mathscr{H}^1 \llcorner \Sigma_n(\theta_n)\big\} \\
&= \liminf_{n\to\infty} \bar{\delta}_{\Sigma_n}(\theta).
\end{aligned}
$$

Hence, we proved the $\Gamma - \liminf$ inequality (3.36). Take now any sequence of t.p.m.'s η_n optimal for $\overline{C}_{\Sigma_n}$, and let η be a weak* limit (possibly, up to a subsequence) of $\{\eta_n\}$. By Lemma 3.2, we have

$$
\begin{aligned}
MK(\mu) \le \overline{C}_\mu(\eta) &= \int_\Theta \bar{\delta}_\mu(\theta)\, d\eta(\theta) \\
&\le \int_\Theta \liminf_{n\to\infty} \bar{\delta}_{\Sigma_n}(\theta)\, d\eta(\theta) \le \liminf_{n\to\infty} \int_\Theta \bar{\delta}_{\Sigma_n}(\theta)\, d\eta_n(\theta) \\
&= \liminf_{n\to\infty} \overline{C}_{\Sigma_n}(\eta_n) = \liminf_{n\to\infty} MK(\Sigma_n).
\end{aligned}
$$

Since it is also true that

$$
H(\|\mu\|) \le \liminf H\big(\mathscr{H}^1(\Sigma_n)\big)
$$

because H is l.s.c. and nondecreasing and since

$$
\|\mu\| \le \liminf \|\mathscr{H}^1 \llcorner \Sigma_n\| = \liminf \mathscr{H}^1(\Sigma_n)\,,
$$

it follows that

$$
MK(\mu) + H(\|\mu\|) \le \liminf_{n\to\infty} MK(\Sigma_n) + H(\|\Sigma_n\|) = \liminf_{n\to\infty} \mathfrak{F}(\Sigma_n)\,.
$$

Recalling the definition (3.2) of $\mathfrak{F}$ for general measures, we get

$$
\mathfrak{F}(\mu) \ge MK(\mu) + H(\|\mu\|)\,,
$$

so that the first inequality in (3.29) is shown.

We show now the opposite inequality: thanks to Lemma 3.19 below, there exists an optimal t.p.m. η for μ (that is, a t.p.m. η with $\overline{C}_\mu(\eta) = MK(\mu)$) with $\operatorname{spt}\eta$ contained in the subset of Θ made by all the paths of length bounded by L. Then, since $\mu \in \mathcal{M}_1^+(\Omega)$, we can write $\mu = \varphi \mathscr{H}^1 \llcorner \Sigma$, and we can consider Σ to be the countable union $\cup_i \theta_i$ where each θ_i is a Lipschitz path of length bounded by L. We define then Σ_n as follows: we take the paths θ_i for $i = 1, 2, \ldots, n$, we divide each of them in the union of n subpaths $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,n}$ of length $\mathscr{H}^1(\theta_i)/n$, and finally we let Σ_n be the union of n^2 closed connected subpaths, each of which is contained in $\theta_{i,j}$ for $(i,j) \in \{1, 2, \ldots, n\}^2$ and with length $\mu(\theta_{i,j})$. Note that this is possible thanks to the hypothesis $\varphi \le 1$, since consequently $\mathscr{H}^1(\theta_{i,j}) \ge \mu(\theta_{i,j})$.

One notices that the characteristic functions $\chi_{\Sigma_n} : \Sigma \to \mathbb{R}$ converge to φ weakly$*$ in $L^\infty(\Sigma)$, thus for any $\theta \in \Theta$ one has

$$\mathscr{H}^1 \llcorner \Sigma_n(\theta) = \int_\Sigma \chi_{\Sigma_n}(s)\chi_\theta(s)\, d\mathscr{H}^1(s) \to \int_\Sigma \varphi(s)\chi_\theta(s)\, d\mathscr{H}^1(s) = \mu(\theta)\,,$$

and hence

$$\limsup_{n\to\infty} \bar\delta_{\Sigma_n}(\theta) \le \bar\delta_\mu(\theta)\,.$$

We use now the Dominated Convergence Theorem, which is possible since, on the set of paths of length less than L, $\bar\delta_{\Sigma_n}$ is bounded by a constant that does not depend on n (indeed, $\bar\delta_{\Sigma_n}(\theta) \le A\big(\mathscr{H}^1(\theta)\big) \le A(L)$); we obtain that

$$\begin{aligned}\liminf_{n\to\infty} MK(\Sigma_n) &\le \liminf_{n\to\infty} \overline{C}_{\Sigma_n}(\eta)\\ &= \liminf_{n\to\infty} \int_\Theta \bar\delta_{\Sigma_n}(\theta)\, d\eta \le \int_\Theta \limsup_{n\to\infty} \bar\delta_{\Sigma_n}(\theta)\, d\eta \\ &\le \int_\Theta \bar\delta_\mu(\theta)\, d\eta = \overline{C}_\mu(\eta) = MK(\mu)\,.\end{aligned} \tag{3.37}$$

Since by construction

$$\mathscr{H}^1(\Sigma_n) = \|\mu\| \qquad\qquad \forall\, n \in \mathbb{N}$$

and $\mathscr{H}^1 \llcorner \Sigma_n \overset{*}{\rightharpoonup} \mu$, from (3.37) it follows

$$\begin{aligned}\mathfrak{F}(\mu) &\le \liminf_{n\to\infty} \mathfrak{F}(\Sigma_n) = \liminf_{n\to\infty} MK(\Sigma_n) + H\big(\mathscr{H}^1(\Sigma_n)\big)\\ &= \liminf_{n\to\infty} MK(\Sigma_n) + H(\|\mu\|)\\ &\le MK(\mu) + H(\|\mu\|)\,.\end{aligned}$$

The second inequality in (3.29) is then shown, so the proof is complete. $\square$

We prove now the generalizations to the relaxed functional $\mathfrak{F}$ of some properties that we already encountered for the original functional in Proposition 1.3, Lemma 1.16, Corollary 1.17 and Proposition 3.4. All these properties can be obtained in a very similar way as in the classical case. It is convenient to introduce for this purpose the new set of measures

$$\mathcal{M}^+_{\mu,0}(\Theta) := \operatorname{Arg\,min}\{\overline{C}_\mu(\eta) \,:\, \eta \text{ a t.p.m.}\}\,.$$

Lemma 3.18. *For any measure $\mu \in \mathcal{M}^+_1(\Omega)$, the following properties hold:*
i) the distance d_μ is continuous;
ii) there exists an optimal t.p.m.*, that is a t.p.m. $\eta_{\rm opt} \in \mathcal{M}^+_{\mu,0}(\Theta)$;*
iii) for any $\eta_{\rm opt} \in \mathcal{M}^+_{\mu,0}(\Theta)$, one has that $\operatorname{spt}\eta_{\rm opt}$ *is contained in the set of all geodesics with respect to d_μ.*

Proof. By construction, the function $\theta \mapsto \bar\delta_\mu(\theta)$ is lower semicontinuous: as a consequence, for any pair $(x, y) \in \Omega \times \Omega$ the distance $d_\mu(x, y)$, introduced in (3.33), can be equivalently defined as the minimum of $\bar\delta_\mu(\theta)$ among the paths $\theta \in \Theta$ connecting x and y; as in the classical case, we will call *geodesics* these minimizers. Recalling again the lower semicontinuity of $\bar{\delta}_\mu$, it follows also immediately that the set G of the geodesics is a closed subset of Θ, as well as the lower semicontinuity of d_μ. The proof of the upper semicontinuity of d_μ is identical to the classical case, making use of the continuity of A and of Lemma 1.1 (which clearly holds also when Σ is a countable union of paths of uniformly bounded length); hence, i) holds.

Concerning the existence of an optimal t.p.m. , this is also obtained exactly as in the classical case: the existence of an optimal transport plan $\gamma_{\rm opt}$, i.e. a measure γ minimizing $I_\mu(\gamma)$ as defined in (3.33), is again standard. Moreover, the closedness of G and the existence of geodesics for each pair in $\Omega \times \Omega$ ensures again the existence of a Borel map

$$q : \Omega \times \Omega \to \Theta$$

associating to any pair of points a geodesics connecting them; finally, the t.p.m. $q_\# \gamma_{\rm opt}$ is clearly an optimal t.p.m. thanks to (3.34); therefore, also ii) is shown.

Finally, the property iii) is proved again as in the classical case: thanks to (3.34), any optimal t.p.m. is concentrated in the set G of the geodesics; therefore, since this set is closed as already remarked, the whole support of any optimal t.p.m. is contained in G. $\square$

LEMMA 3.19. *Given any $\mu \in \mathcal{M}_1^+(\Omega)$ there exists a constant L depending only on A, Ω and $\|\mu\|$ such that for any pair $(x, y) \in \Omega \times \Omega$ the Euclidean length $\mathscr{H}^1(\theta)$ of some geodesic θ connecting x and y (in particular, of* each *geodesic if A is not constant for all values large enough) is bounded by L. In addition, there exists an optimal t.p.m. η for $\overline{C}_\mu$ (in particular,* each *optimal t.p.m. if A is not constant for all values large enough) such that* $\operatorname{spt} \eta$ *is contained in the set of paths θ with Euclidean length bounded by L.*

PROOF. For the first part, one can simply adapt the proof of Lemma 1.16 to this case; one only has to notice, as we already did, that Lemma 1.1 is true also in the case when Σ is the union of countably many paths of equibounded length.

The proof of the second part, also in light of Lemma 3.18-iii), can be made exactly as the proof of Corollary 1.17. $\square$

PROPOSITION 3.20. *Given $\mu \in \mathcal{M}_1^+(\Omega)$ and a sequence $\{\mu_n\} \in \mathcal{M}_1^+(\Omega)$ such that $\mu_n \stackrel{*}{\rightharpoonup} \mu$, one has that*

$$\bar{\delta}_\mu \leq \Gamma - \liminf \bar{\delta}_{\mu_n} ,$$

that is

$$\bar{\delta}_\mu(\theta) \leq \liminf \bar{\delta}_{\mu_n}(\theta_n)$$

whenever $\theta_n \to \theta$ uniformly. As a consequence, $d_\mu \leq \liminf d_{\mu_n}$.

PROOF. Take any sequence θ_n of paths uniformly converging to some path θ; therefore,

$$\mu(\theta) \geq \limsup \mu_n(\theta_n)$$

by Lemma 3.1, and on the other hand

$$\mathscr{H}^1(\theta) \leq \liminf \mathscr{H}^1(\theta_n)$$

by Gołąb Theorem. We can argue exactly as when we showed property (3.36) in the proof of Proposition 3.16 to obtain

$$\bar{\delta}_\mu(\theta) \leq \liminf \bar{\delta}_{\mu_n}(\theta_n)\,,$$

so the first part of the proof follows.

Concerning the second one, given any pair $(x, y) \in \Omega \times \Omega$, we can choose according to Lemma 3.19 paths θ_n having Euclidean length less than L and being geodesics with respect to d_{μ_n}, that is $\bar{\delta}_{\mu_n}(\theta_n) = d_{\mu_n}(x, y)$. As usual, we can assume up to a subsequence that $\theta_n \to \theta$ uniformly as elements of $C([0, 1], \Omega)$ and in the Hausdorff distance as closed subsets of Ω. Recalling the $\Gamma-$liminf inequality shown above we conclude, since

$$d_\mu(x, y) \leq \bar{\delta}_\mu(\theta) \leq \liminf_{n\to\infty} \bar{\delta}_{\mu_n}(\theta_n) = \liminf_{n\to\infty} d_{\mu_n}(x, y)\,.$$

□

We remark now what follows: in Lemmas 3.10 and 3.12 and Theorem 3.15, we showed the existence of some optimal measure respectively which is absolutely continuous with respect to $\mathscr{H}^1$, which is concentrated in some countable union of curves of uniformly bounded length, and which belongs to $\mathcal{M}_1^+(\Omega)$. In fact, we were able to prove that the above properties are satisfied by any minimal optimal measure. Roughly speaking, if μ is an optimal measure and $\mu' \leq \mu$ is minimal and optimal, we proved that the part $\mu - \mu'$ of the optimal measure is “useless”; formally, we can say that the part $\mu - \mu' \leq \mu$ is *useless* if $d_\mu \equiv d_{\mu'}$, and then $MK(\mu) = MK(\mu')$ and $\mathfrak{F}(\mu) \geq \mathfrak{F}(\mu')$. It is now quite easy to guess that, often, a stronger assertion than the one of the above lemmas is true, namely that *all* the optimal measures possess the desired properties: this holds whenever the presence of a useless part of an optimal measure is impossible, that is, when each optimal measure is minimal. This is certainly true, for instance, in the two situations considered in the next proposition.

Proposition 3.21. *All the optimal measures are minimal in each of the following cases:*

- *the function H is* strictly *increasing;* (3.38)
- *$B'(s) < A'(t)$ for every $s \in \mathbb{R}^+$, $0 \leq t \leq \operatorname{diam}(\Omega)$ and $f^+ - f^-$ is not concentrated on a set of finite $\mathscr{H}^1$-measure.* (3.39)

Proof. The first situation is clear: indeed, as noticed before, if there is a non-minimal optimal measure μ, and so a non-null useless part $\mu - \mu'$, then $MK(\mu) = MK(\mu')$; by the strict monotonicity of H, $H(\mu') < H(\mu)$, which leads to the contradiction $\mathfrak{F}(\mu') < \mathfrak{F}(\mu)$.

Let us then consider the case (3.39), and assume by contradiction the existence of a non-minimal optimal measure μ, and then the existence of an optimal measure μ' strictly less than μ: take then an optimal t.p.m. η and notice that the conclusion is proved if the set of the paths θ such that $\mu'(\theta) < \mathscr{H}^1(\theta)$ is not $\eta-$negligible. Indeed, in this case, we could replace $\mu - \mu'$ by the measure $\mathscr{H}^1 \llcorner \theta_0 - \mu \llcorner \theta_0$, for a suitable $\theta_0 \in \Theta$ satisfying

$$0 < \mathscr{H}^1(\theta_0) - \mu(\theta_0) \leq \|\mu - \mu'\| \,.$$

Hence, thanks to the assumption $B'(s) < A'(t)$, this would provide a strictly positive gain because an η-nonnegligible set of paths can be changed so as to strictly decrease their costs (this can be made rigorous arguing as in Lemma 5.9 below). We can then assume that for $\eta-$a.a. paths θ we have $\mu'(\theta) = \mathscr{H}^1(\theta)$ so that, denoting by Σ the set where μ' has $\mathscr{H}^1-$density equal to 1, one has $\mathscr{H}^1(\theta) = \mathscr{H}^1(\theta \cap \Sigma)$; the contradiction with (3.39) will now follow by proving that $f^+ - f^-$ is concentrated on a set of finite length. To this aim, we select paths $\theta_n \in \Theta$ almost maximizing

$$\mathscr{H}^1\Big(\theta \setminus \cup_{i=0}^{n-1} \theta_i\Big)$$

among all paths for which $\mathscr{H}^1(\theta \setminus \Sigma) = 0$, i.e.

$$\begin{aligned} &\mathscr{H}^1(\theta_n \setminus \Sigma) = 0 \,, \\ &\mathscr{H}^1\Big(\theta_n \setminus \cup_{i=0}^{n-1} \theta_i\Big) \geq \sup\Big\{\mathscr{H}^1\Big(\theta \setminus \cup_{i=0}^{n-1} \theta_i\Big),\ \mathscr{H}^1(\theta \setminus \Sigma) = 0\Big\} - \frac{1}{2^n} \,. \end{aligned}$$

Setting now $\Sigma' := \cup \theta_n$, up to adding to Σ the $\mathscr{H}^1-$negligible set $\cup_n \theta_n \setminus \Sigma$, we can assume that $\Sigma' \subseteq \Sigma$. Moreover, by construction of Σ', we have that

$$\begin{aligned} &\mathscr{H}^1(\Sigma') \leq \mathscr{H}^1(\Sigma) \leq \|\mu'\| < +\infty \,, \\ &\mathscr{H}^1(\theta \setminus \Sigma') = 0 \text{ for } \eta-\text{a.a. } \theta \in \Theta \,. \end{aligned} \tag{3.40}$$

Notice now that Σ' is a countable union of pairwise disjoint and pathwise connected components Σ_n (each Σ_n is the union of some of the paths θ_n). Now, we say that the pair (Σ_i, Σ_j) is *connectible* if there is a path $\theta_{ij} \in \Theta$ such that

$$\mathscr{H}^1(\theta_{ij} \setminus \Sigma') = 0 \,, \qquad \mathscr{H}^1(\theta_{ij} \cap \Sigma_i) > 0 \,, \qquad \mathscr{H}^1(\theta_{ij} \cap \Sigma_j) > 0 \,.$$

If (Σ_i, Σ_j) is connectible, adding to Σ' the $\mathscr{H}^1-$negligible set $\theta_{ij} \setminus \Sigma'$ we have that Σ_i and Σ_j belong now to the same connected component of Σ'. Iterating countably many times this procedure, we end up with a set Σ'' for which (3.40) still holds; but by construction Σ'' is made by countably many connected components (since so is Σ'), still denoted by Σ_i, and no pair (Σ_i, Σ_j) is connectible; again up to adding to Σ'' a $\mathscr{H}^1-$negligible set, we can also assume that all Σ_i are closed. As a consequence of (3.40) and of the fact that the different connected components of Σ'' are not connectible, $\eta-$a.e.

path θ is entirely contained in a single Σ_i; hence, recalling the definition of t.p.m., we deduce that $f^+ - f^-$ is concentrated on the set

$$\Big\{\theta(0) \cup \theta(1) : \ \exists i \in \mathbb{N}, \ \theta \subseteq \Sigma_i\Big\} \subseteq \Sigma'' ,$$

of finite length. The desired contradiction to (3.39) then is found, and the proof is completed. □

Summarizing, we know now the existence of optimal measures $\mu \in \mathcal{M}_1^+(\Omega)$: any such measure can be written as $\mu = \varphi \mathscr{H}^1 \llcorner \Sigma$ with Σ a rectifiable Borel set and $\varphi : \Sigma \to [0,1]$ a Borel function. In particular, the measure μ corresponds to a set if and only if

$$\varphi(x) = 1 \qquad\qquad \text{for } \mu\text{−a.e. } x \in \Omega.$$

Definition 3.22. *The set*

$$\mathcal{M}_2^+(\Omega) \subseteq \mathcal{M}_1^+(\Omega) \subseteq \mathcal{M}^+(\Omega)$$

is defined as the set of all measures $\mathscr{H}^1 \llcorner \Sigma$ for some set Σ contained in a countable union of Lipschitz paths of uniformly bounded length. Therefore, $\mu = \varphi \mathscr{H}^1 \llcorner \Sigma$ belongs to $\mathcal{M}_2^+(\Omega)$, if and only if $\varphi(x) = 1$ for μ−a.e. $x \in \Omega$.

As we discussed before Definition 3.13, one can expect the existence of optimal measures contained in $\mathcal{M}_1^+(\Omega) \setminus \mathcal{M}_2^+(\Omega)$. Indeed, as we anticipated, there are situations in which some optimal measure (or even all the optimal measures) are contained in $\mathcal{M}_1^+(\Omega) \setminus \mathcal{M}_2^+(\Omega)$: we give such an example in the next section. In that example, as in many others through the monograph, the situation is basically one-dimensional; this means that $f^\pm$, Σ and almost all the paths θ are contained in the same segment. However, recall that the ambient space is always at least two-dimensional, as we explained in the beginning of the monograph. We point the reader's attention to the importance of this fact: indeed, even in these "one-dimensional" situations, the fact that the ambient space has a higher dimension is crucial; otherwise, the setting of the problem should have been different in order to be meaningful. Roughly speaking, if the space is one-dimensional then one cannot walk "close to the railway" because there is no room to do that; thus, however expensive the train ticket is, the passenger has no choice and is forced to buy it.

3. Non-existence of classical solutions.

In this section we present a situation in which no optimal measure μ corresponds to a set, that is $\mu \notin \mathcal{M}_2^+(\Omega)$ for any optimal measure μ. More precisely, it will appear that whenever A and B are strictly convex, one should expect that no optimal measure corresponds to a set, so that the cases when there is an optimal measure $\mu \in \mathcal{M}_2^+(\Omega)$ will be exceptional.

To do that, we take

$$\Omega = [0,2]^2 \subseteq \mathbb{R}^2 ,$$

we assume that $A = B$ is a strictly convex and C^1 function, that $H(s) = 0$ for all $s \in [0,1]$ and $H(s) > 0$ for any $s > 1$, that

$$f^{\pm} \ll \mathscr{H}^1 \llcorner \big([0,2] \times \{0\}\big)\,,$$

and that

$$f^+\big([0,s] \times \{0\}\big) > f^-\big([0,s] \times \{0\}\big) \qquad \forall\, 0 < s < 2\,. \tag{3.41}$$

We will denote for brevity by f^+ and f^- also the densities of f^+ and f^- with respect to $\mathscr{H}^1$, and we will write for convenience $\bar z \equiv (z,0) \in \Omega$ for any real number $z \in [0,2]$; finally, we assume that f^+ and f^- are strictly positive inside $[0,2] \times \{0\}$. It is clear that all optimal measures μ are concentrated in $[0,2] \times \{0\}$: otherwise, projecting μ on $[0,2] \times \{0\}$ would provide a lower value in the cost $\mathfrak{F}$.

Take now an optimal measure

$$\mu = \varphi(x) \mathscr{H}^1 \llcorner [0,2] \in \mathcal{M}_1^+(\Omega)$$

and notice that, for any $x \le y$ in $[0,2]$ one has

$$\begin{aligned} d_\mu(\bar x, \bar y) &= \inf_{0 \le t \le \mu([\bar x, \bar y])} A\big(y - x - t\big) + B\big(t\big) \\ &= A\big(y - x - l\big) + A\big(l\big), \qquad \text{where} \qquad l = \frac{y-x}{2} \wedge \mu([\bar x, \bar y])\,. \end{aligned} \tag{3.42}$$

Since the map

$$t \mapsto A\big(y - x - t\big) + B\big(t\big)$$

has its absolute minimum at $t = (y-x)/2$, from (3.42) we get

$$d_\mu(\bar x, \bar y) \ge A\Big(\frac{y-x}{2}\Big) + B\Big(\frac{y-x}{2}\Big) = 2\,A\Big(\frac{y-x}{2}\Big)\,. \tag{3.43}$$

Note that in the above calculations we used the convexity of the function A; moreover, the inequality (3.43) holds strictly if and only if

$$\mu([\bar x, \bar y]) < \frac{y-x}{2}\,,$$

due to the *strict* convexity of A.

We present now a well-known property of the transport problems, namely the *cyclical monotonicity*, which is introduced and shown, for instance, in [**1, 27, 46**]: this property says that, whenever γ is an optimal transport plan,

$$d_\mu(p,q) + d_\mu(p',q') \le d_\mu(p,q') + d_\mu(p',q) \quad \forall (p,q),\, (p',q') \in \operatorname{spt}\gamma\,. \tag{3.44}$$

The following property is a consequence of the cyclical monotonicity.

Lemma 3.23. *In the hypotheses of this section, for any optimal transport plan γ one has*

$$x_1 < x_2 \Longrightarrow y_1 \le y_2 \qquad \text{whenever} \qquad (\bar x_1, \bar y_1),\, (\bar x_2, \bar y_2) \in \operatorname{spt}\gamma\,. \tag{3.45}$$

PROOF. First of all, we underline that since γ is an optimal transport plan then for any $(x,y) \in \operatorname{spt}\gamma$ one has $x \leq y$ thanks to (3.41): this is a standard and well-known fact, since otherwise one can easily find a contradiction with (3.45). Therefore, to show the claim of the lemma it is sufficient to take

$$0 \leq x_1 < x_2 < y \leq 2$$

and to show that

$$\frac{\partial d_\mu}{\partial y}(\overline{x_1}, \overline{y'}) > \frac{\partial d_\mu}{\partial y}(\overline{x_2}, \overline{y'})\,, \tag{3.46}$$

denoting by $\partial d_\mu/\partial y$ the partial derivative of $d_\mu(\bar{x},\bar{y})$ relative to the vector $(1,0)$ in the variable $\bar{y}$. To show (3.46), keeping in mind (3.42) we define

$$\begin{aligned} l_1 &:= \frac{y'-x_1}{2} \wedge \mu([\overline{x_1}, \overline{y'}])\,, & \tilde{l}_1 &:= y' - x_1 - l_1\,, \\ l_2 &:= \frac{y'-x_2}{2} \wedge \mu([\overline{x_2}, \overline{y'}])\,, & \tilde{l}_2 &:= y' - x_2 - l_2\,, \end{aligned}$$

so that

$$d_\mu(\overline{x_1}, \overline{y'}) = A\big(\tilde{l}_1\big) + B\big(l_1\big) \qquad \text{and} \qquad d_\mu(\overline{x_2}, \overline{y'}) = A\big(\tilde{l}_2\big) + B\big(l_2\big)\,. \tag{3.47}$$

Again using (3.42), recalling that $x_1 < x_2$ it is immediate to deduce that

$$\tilde{l}_1 \geq l_1 \qquad \text{and} \qquad \tilde{l}_2 \geq l_2\,; \tag{3.48}$$

$$l_1 \geq l_2\,, \qquad \tilde{l}_1 \geq \tilde{l}_2 \qquad \text{and at least one of the inequalities is strict:} \tag{3.49}$$

the last assertion is obvious since

$$l_1 + \tilde{l}_1 = y' - x_1 > y' - x_2 = l_2 + \tilde{l}_2\,.$$

Up to an error $o(\varepsilon)$ we can now write, using (3.42) and (3.49), the following estimates:

$$\begin{aligned} d_\mu(\overline{x_1}, \overline{y'+\varepsilon}) - d_\mu(\overline{x_1}, \overline{y'}) &= \varepsilon\big((1-c_1)\,A'(\tilde{l}_1) + c_1\,A'(l_1)\big) + o(\varepsilon)\,, \\ d_\mu(\overline{x_2}, \overline{y'+\varepsilon}) - d_\mu(\overline{x_2}, \overline{y'}) &= \varepsilon\big((1-c_2)\,A'(\tilde{l}_2) + c_2\,A'(l_2)\big) + o(\varepsilon)\,, \end{aligned} \tag{3.50}$$

where the constants c_1 and c_2 are defined by

$$c_1 := \left\{ \begin{array}{cl} \varphi(x) & \text{if } l_1 < \tilde{l}_1\,, \\ \varphi(x) \wedge 1/2 & \text{if } l_1 = \tilde{l}_1\,, \end{array} \right. \qquad c_2 := \left\{ \begin{array}{cl} \varphi(x) & \text{if } l_2 < \tilde{l}_2\,, \\ \varphi(x) \wedge 1/2 & \text{if } l_2 = \tilde{l}_2 \end{array} \right.$$

(notice that $0 \leq c_1,\, c_2 \leq 1$ since $\mu \in \mathcal{M}_1^+(\Omega)$). Finally, we conclude considering separately the three possibilities:

<u>*Case I.*</u> $c_1 = c_2\,.$

In this case, (3.46) follows directly from (3.50) recalling (3.47), (3.49) and the fact that A' is strictly increasing since A is strictly convex and C^1.

<u>*Case II.*</u> $c_1 < c_2\,.$

As in the first case we have

$$(1-c_2)\,A'(\tilde{l}_1) + c_2\,A'(l_1) > (1-c_2)\,A'(\tilde{l}_2) + c_2\,A'(l_2)\,;$$

with the hypothesis $c_1 < c_2$, by (3.48) we deduce

$$(1-c_1)\,A'(\tilde{l}_1)+c_1\,A'(l_1)\geq(1-c_2)\,A'(\tilde{l}_1)+c_2\,A'(l_1)\,,$$

and therefore (3.46) is proved also in this case.

Case III. $c_1 > c_2$.
Recalling the definition of c_1 and c_2, this case is possible only if $l_2 = \tilde{l}_2$; but then by (3.48) and (3.49) we infer that

$$\tilde{l}_1 \geq l_1 \geq \tilde{l}_2 = l_2\,,$$

so that (3.46) follows immediately by (3.50) and the fact that A' is strictly increasing. □

We point out now another well-known fact, i.e. that there is a unique optimal transport plan $\bar{\gamma}$ between f^+ and f^- fulfilling property (3.45), hence by Lemma 3.23 a unique optimal transport plan. Otherwise, there would be two optimal transport plans γ_1 and γ_2, a number $x \in [0,2]$ and two numbers $y_1 < y_2 \in [0,2]$ such that

$$(\bar{x},\bar{y}_1)\in\operatorname{spt}\gamma_1, \qquad\qquad (\bar{x},\bar{y}_2)\in\operatorname{spt}\gamma_2\setminus\operatorname{spt}\gamma_1\,.$$

By (3.45) and recalling that f^+ and f^- are assumed strictly positive in $[0,2]\times\{0\}$, it follows the existence of some $x' > x$ and $y_1 < y' < y_2$ such that $(\bar{x}',\bar{y}')\in\operatorname{spt}\gamma_1$. By the linearity of the cost (1.6) with respect to γ, it follows that also

$$\gamma=\frac{\gamma_1+\gamma_2}{2}$$

is an optimal transport plan; but $(\bar{x},\bar{y}_2)$ and $(\bar{x}',\bar{y}')$ are two pairs in $\operatorname{spt}\gamma$ against (3.45); therefore, the uniqueness of the optimal transport plan $\bar{\gamma}$ is established.

Set now $t(x) = y$ for $x \in [0,2]$, where $y \in [0,2]$ is the unique number such that

$$f^+([0,x]\times\{0\}) = f^-([0,y]\times\{0\})\,.$$

This is a well-defined continuous function thanks to the assumption that f^- is strictly positive, and moreover one has $t(0) = 0$, $t(2) = 2$ and $t(x) > x$ for $0 < x < 2$ by (3.41). The unique optimal transport plan $\bar{\gamma}$ is given by

$$\bar{\gamma} := (\mathrm{Id},\bar{t})_\# f^+\,,$$

where $\bar{t} : [0,2]\times\{0\}\to\Omega$ denotes the map $\bar{t}(\bar{x}) := \overline{t(x)}$. Moreover, from the equality

$$\int_0^x f^+(s)\,ds = \int_0^{t(x)} f^-(s)\,ds$$

and the strict positivity of f^-, one has that the derivative $t'(x)$ exists and is strictly positive for a.e. $x \in [0,2]$. In words, the transport plan moves the mass from each point

$$\bar{x}\in[0,2]\times\{0\}$$

to the corresponding point

$$\bar{t}(\bar{x}) = \overline{t(x)} \in [0,2] \times \{0\} \ :$$

the fact that, in this case, the unique optimal transport plan is $\bar{\gamma} = (\mathrm{Id}, \bar{t})_{\#} f^+$, is again well-known (see for instance [**1**, Theorem 3.1]). Note that we found the unique optimal transport plan without knowing anything about the measure μ: in other words, for any measure $\mu \in \mathcal{M}_1^+(\Omega)$, the same plan $\bar{\gamma}$ is the unique minimizer of

$$I_\mu(\gamma) = \iint_{\Omega\times\Omega} d_\mu(p,q)\, d\gamma(p,q)\,.$$

Thanks to (3.43) and the fact that $H(1) = 0$, the measure

$$\bar{\mu} := \frac{1}{2}\mathscr{H}^1 \llcorner \big([0,2] \times \{0\}\big)$$

is optimal, since

$$d_{\bar{\mu}}(\bar{x}, \bar{y}) = 2\, A\Big(\frac{y-x}{2}\Big)$$

for any $0 \le x \le y \le 2$. Then, we have found an optimal measure which is not classical, that is, an optimal measure

$$\bar{\mu} \in \mathcal{M}_1^+(\Omega) \setminus \mathcal{M}_2^+(\Omega)\,;$$

we want now to show more, that is, that there is no classical optimal measure $\mu \in \mathcal{M}_2^+(\Omega)$. First of all, we need to establish the next result.

Lemma 3.24. *In the assumptions of this chapter, if μ is an optimal measure then*

$$\mu\big([\bar{x}, \overline{t(x)}]\big) = \frac{t(x)-x}{2} \qquad\qquad \forall\, x \in [0,2]\,. \tag{3.51}$$

Proof. Recalling (3.43) and the fact that $\bar{\mu}$ is optimal, since μ is optimal we immediately know that the inequality

$$\mu\big([\bar{x}, \overline{t(x)}]\big) \ge \frac{t(x)-x}{2} \tag{3.52}$$

holds for any $0 \le x \le 2$; we suppose then the existence of a point $0 < x_0 < 2$ such that the strict inequality

$$\mu\big([\bar{x}_0, \overline{t(x_0)}]\big) - \frac{t(x_0)-x_0}{2} =: \rho > 0$$

holds, and we aim to find a contradiction. Let us define, starting from x_0, a two-sided sequence $\{x_z\}$ with $z \in \mathbb{Z}$ by induction setting $x_{h+1} = t(x_h)$ (this defines x_z for any $z \in \mathbb{Z}$ since t is continuous and strictly increasing thus invertible). Since t is strictly increasing in $(0,2)$, we have

$$\lim_{z\to\infty} x_z = 2, \qquad\qquad \lim_{z\to-\infty} x_z = 0\,.$$

By the fact that for any $z \in \mathbb{Z}$ the inequality (3.52) holds, for any $z < 0 < m$ we have

$$\mu\big([\bar{x}_z, \bar{x}_m]\big) \ge \frac{x_m - x_z}{2} + \rho\,.$$

Letting $m \to \infty$ and $z \to -\infty$, the preceding inequality ensures
$$\|\mu\| \geq 1 + \rho\,;$$
but then
$$H(\|\mu\|) > 0 = H(\|\bar{\mu}\|)\,,$$
hence $\mathfrak{F}(\bar{\mu}) < \mathfrak{F}(\mu)$ against the optimality of μ, and we found the desired contradiction. □

We can now prove, as a consequence of (3.51), the claim that we already stated before, that is, that no optimal measure is classical (i.e. belongs to $\mathcal{M}_2^+(\Omega)$).

Lemma 3.25. *In the assumptions of this chapter, there are no optimal measures $\mu \in \mathcal{M}_2^+(\Omega)$.*

Proof. Take any optimal measure μ, and let $x \in (0,2)$ be a point such that $\bar{x}$ is a point of density 1 of μ and the derivative $t'(x)$ exists. By (3.51), one has
$$\mu([\bar{x}, \overline{t(x)}]) = \frac{t(x) - x}{2}\,, \qquad \mu([\overline{x+\varepsilon}, \overline{t(x+\varepsilon)}]) = \frac{t(x+\varepsilon) - (x+\varepsilon)}{2}\,.$$
This implies
$$\mu\big([\overline{t(x)}, \overline{t(x+\varepsilon)}]\big) - \mu\big([\bar{x}, \overline{x+\varepsilon}]\big) = \frac{t(x+\varepsilon) - t(x) - \varepsilon}{2}\,.$$
But
$$\mu\big([\bar{x}, \overline{x+\varepsilon}\,]\big) = \varepsilon + o(\varepsilon)\,,$$
because $\bar{x}$ is a point of density 1; moreover
$$t(x+\varepsilon) = t(x) + \varepsilon t'(x) + o(\varepsilon)$$
since $t'(x)$ is defined. It follows
$$\mu\big([\overline{t(x)}, \overline{t(x) + \varepsilon t'(x)}]\big) = \frac{t'(x)+1}{2}\,\varepsilon + o(\varepsilon)\,,$$
and so the density of μ at $\overline{t(x)}$ is
$$\frac{t'(x)+1}{2\,t'(x)}\,.$$
Analogously, if the density of μ at $\bar{x}$ is 0, then the density at $\overline{t(x)}$ is given by
$$\frac{t'(x)-1}{2\,t'(x)}\,.$$
If the measure μ belongs to $\mathcal{M}_2^+(\Omega)$, then the density of μ at a.e. point $\bar{x} \in [0,2] \times \{0\}$ is either 0 or 1; since, as already pointed out, the derivative $t'(x)$ exists and is strictly positive at almost each $x \in [0,2]$, we deduce that it should be $t'(x) = 1$ for a.e. $x \in [0,2]$. But this would imply $t \equiv \mathrm{Id}$, hence $f^+ = f^-$, which gives a contradiction with (3.41), which concludes the proof. □

Summarizing, for any choice of f^+ and f^- as in the assumptions of this chapter all the optimal measures do not belong to $\mathcal{M}_2^+(\Omega)$, while for instance the measure

$$\bar{\mu} = \frac{\mathscr{H}^1 \llcorner \big([0,2]\times\{0\}\big)}{2} \in \mathcal{M}_1^+(\Omega) \setminus \mathcal{M}_2^+(\Omega)$$

is always optimal.

4. Existence of classical solutions

In the previous section, we noticed that the existence of classical solution (more precisely, of solutions $\mu \in \mathcal{M}_2^+(\Omega)$) may fail when A and B are convex; on the other hand, we show now that when the latter functions are concave, this existence is guaranteed. Indeed, we will prove Theorem 3.27, which provides the existence of optimal measures in $\mathcal{M}_2^+(\Omega)$ when the function D defined in (1.29) is concave in the first variable. In particular, in Lemma 3.26 below we show that the latter concavity condition holds, for instance, when both A and B are concave. In the following claim, we denote by $g'_\pm(s)$ the left and right derivative of any function $g:\mathbb{R}\to\mathbb{R}$ at s, which always exist when g is concave.

LEMMA 3.26. *If both A and B are concave, then the function D defined in (1.29) is concave; moreover, D is strictly concave if, in addition, either A or B is strictly concave and*

$$B'_+(0) < A'_-(\operatorname{diam}\Omega)\,.$$

PROOF. Assume that both A and B are concave. Then so is the function

$$l \mapsto A\left(\frac{a+\tilde{a}}{2}+l\right) + B\left(b-\frac{a+\tilde{a}}{2}-l\right). \tag{3.53}$$

As a consequence, the infimum of this function in the interval

$$0 \le l \le b - \frac{a+\tilde{a}}{2}$$

(which equals $D\big((a+\tilde{a})/2, b\big)$ by definition) is a minimum, and it is attained either at $l=0$ or at $l = b-(a+\tilde{a})/2$. In the first case, we have

$$\begin{aligned} D\left(\frac{a+\tilde{a}}{2}, b\right) &= A\left(\frac{a+\tilde{a}}{2}\right) + B\left(b - \frac{a+\tilde{a}}{2}\right) \\ &\ge \frac{A(a)+A(\tilde{a})+B(b-a)+B(b-\tilde{a})}{2} \\ &\ge \frac{D(a,b)+D(\tilde{a},b)}{2}\,. \end{aligned} \tag{3.54}$$

On the other hand, in the second case one has

$$D\left(\frac{a+\tilde{a}}{2}, b\right) = A(b) + B(0) \ge D(a,b)\vee D(\tilde{a},b) \ge \frac{D(a,b)+D(\tilde{a},b)}{2}\,. \tag{3.55}$$

Hence, we have proved the concavity of $D(\cdot, b)$, minding that it is a continuous function.

Concerning the strict concavity, notice that the first inequality in (3.54) is strict if either A or B is strictly concave; but the inequality (3.55) may fail to be strict even if both A and B are strictly concave. For instance, if $B'(s) > A'(l)$ for any s and l, then $D(\alpha, b) = A(b) + B(0)$ for any α, and then the concavity of $D(\cdot, b)$ is not strict. On the other hand, if $B'_+(0) < A'_-(\operatorname{diam}\Omega)$, then the infimum in (3.53) is always attained at $l = 0$: the meaning of this fact is that, if the ticket is sufficiently cheap, then for each path it is convenient to use as much as possible of the public transportation network, or in other words $\delta_\mu = \bar{\delta}_\mu$ for each measure μ. Since the inequality (3.54) is strict if either A or B is strictly concave (and both are concave) then $D(\cdot, b)$ is strictly concave and the proof is achieved. $\square$

THEOREM 3.27. *If the function $D(\cdot, b)$ is concave (resp. strictly concave) for any $b \in \mathbb{R}^+$, then there exists an optimal measure μ in $\mathcal{M}_2^+(\Omega)$ (resp. any optimal measure $\mu \in \mathcal{M}_1^+(\Omega)$ belongs to $\mathcal{M}_2^+(\Omega)$). In particular, by Lemma 3.26, there is an optimal measure $\mu \in \mathcal{M}_2^+(\Omega)$ if both A and B are concave; moreover, each optimal measure $\mu \in \mathcal{M}_1^+(\Omega)$ belongs to $\mathcal{M}_2^+(\Omega)$ if both A and B are concave, at least one of them is strictly concave and, in addition, $B'_+(0) < A'_-(\operatorname{diam}\Omega)$.*

PROOF. Assume the existence of an optimal measure

$$a\mathscr{H}^1 \llcorner \Sigma = \mu \in \mathcal{M}_1^+(\Omega) \setminus \mathcal{M}_2^+(\Omega)\,,$$

where, as usual, Σ is a rectifiable set and $a : \Sigma \to [0, 1]$ a Borel function; since $\mu \notin \mathcal{M}_2^+(\Omega)$ we have that $0 < a(x) < 1$ in a set of strictly positive length. As an immediate consequence, we can find $\varepsilon > 0$ and two sets Σ_1, Σ_2 such that

$$\begin{aligned} &\mu(\Sigma_1) = \mu(\Sigma_2) > 0\,, && \Sigma_1 \cap \Sigma_2 = \emptyset\,, \\ &\varepsilon < a(x) < 1 - \varepsilon \quad \text{for any } x \in \Sigma_1 \cup \Sigma_2\,. \end{aligned}$$

We define therefore

$$\mu_s := \mu + s\big(\mu \llcorner \Sigma_1 - \mu \llcorner \Sigma_2\big)\,;$$

by construction, for any $-\varepsilon < s < \varepsilon$ one has $\mu_s \in \mathcal{M}_1^+(\Omega)$ and $\|\mu_s\| = \|\mu\|$. Take now any $\theta \in \Theta$: since $s \mapsto \mu_s(\theta)$ is a linear map, then

$$s \mapsto D\big(\mathscr{H}^1(\theta) - \mu_s(\theta), \mathscr{H}^1(\theta)\big)$$

is concave or strictly concave if so is D in the first variable. By the definition (3.33) of $\bar{\delta}_\mu$, the equality

$$\bar{\delta}_\mu(\theta) = D\big(\mathscr{H}^1(\theta) - \mu(\theta), \mathscr{H}^1(\theta)\big)$$

holds; in particular, it can be obtained in the same way as formula (1.30). Therefore, for a generic path $\theta \in \Theta$ the map $s \mapsto \bar{\delta}_{\mu_s}(\theta)$ is concave or strictly concave. Take now an optimal t.p.m. η for the cost $\overline{C}_\mu$, and observe that

$$\text{the map } s \mapsto \overline{C}_{\mu_s}(\eta) \text{ is concave (or strictly concave) if so is } D(\cdot, b)\,. \tag{3.56}$$

We now consider the strictly concave case: by (3.56) and the fact that μ_s is defined for s in a neighborhood of zero we derive the existence of some $-\varepsilon < \bar{s} < \varepsilon$ such that $\overline{C}_{\mu_{\bar{s}}}(\eta) < \overline{C}_\mu(\eta)$. Since η is an optimal t.p.m. for $\overline{C}_\mu$, it follows

$$MK(\mu_{\bar{s}}) \le \overline{C}_{\mu_{\bar{s}}}(\eta) < \overline{C}_\mu(\eta) = MK(\mu)\,,$$

and since $\|\mu_{\bar{s}}\| = \|\mu\|$,

$$H(\|\mu_{\bar{s}}\| = H(\|\mu\|)$$

and hence $\mathfrak{F}(\mu_{\bar{s}}) < \mathfrak{F}(\mu)$ against the optimality of μ. This contradiction shows that *any optimal measure* $\mu \in \mathcal{M}_1^+(\Omega)$ belongs to $\mathcal{M}_2^+(\Omega)$ in the strictly concave case (in particular, this means that any optimal measure μ belongs to $\mathcal{M}_2^+(\Omega)$ under the assumptions of Proposition 3.21).

Let us consider now the concave case: it is still true by (3.56) that $s \mapsto \overline{C}_{\mu_s}(\eta)$ is concave for $-\varepsilon < s < \varepsilon$; hence, for any given optimal measure μ, either there exists some $\bar{s}$ such that $\overline{C}_{\mu_{\bar{s}}}(\eta) < \overline{C}_\mu(\eta)$, and then we conclude as in the previous case, or one has

$$\overline{C}_{\mu_s}(\eta) = \overline{C}_\mu(\eta) \qquad\qquad \text{for each } -\varepsilon < s < \varepsilon\,. \tag{3.57}$$

We denote by $\mathcal{N}$ the set of all the optimal measures belonging to $\mathcal{M}_1^+(\Omega)$, and consider the auxiliary problem of minimizing, over $\mathcal{N}$, the functional $\widetilde{\mathfrak{F}}$ corresponding to the choice

$$\widetilde{A}(t) := \sqrt{t}\,, \qquad\qquad \widetilde{B}(t) := 0\,.$$

Since $\mathfrak{F}$ is l.s.c. and coercive by construction, the set $\mathcal{N}$ is a bounded and weakly* closed subset of $\mathcal{M}^+(\Omega)$, hence it is also sequentially compact with respect to the weak* convergence. As a consequence, being $\widetilde{\mathfrak{F}}$ also l.s.c. and coercive, there is a measure μ minimizing $\widetilde{\mathfrak{F}}$ over $\mathcal{N}$.

As noticed before, since μ minimizes $\mathfrak{F}$ then (3.57) holds; thus all the measures μ_s belong to $\mathcal{N}$ for $-\varepsilon < s < \varepsilon$. By Lemma 3.26, $\widetilde{D}$ is strictly concave in its first variable; therefore, the above arguments give the analogue of (3.56), so that the map $s \mapsto \widetilde{\mathfrak{F}}(\mu_s)$ is strictly concave: hence, there is some $-\varepsilon < \bar{s} < \varepsilon$ such that $\widetilde{\mathfrak{F}}(\mu_{\bar{s}}) < \widetilde{\mathfrak{F}}(\mu)$. Since $\mu_{\bar{s}} \in \mathcal{N}$, this gives a contradiction with the fact that μ minimizes $\widetilde{\mathfrak{F}}$ over $\mathcal{N}$. So, we deduce the existence of *some optimal measure* μ belonging to $\mathcal{M}_2^+(\Omega)$. $\square$

CHAPTER 4

Topological properties of optimal sets

In Chapter 3 we proved the existence of relaxed solutions to our network optimization problem, and gave conditions on the existence of classical solutions. Under such conditions, it becomes important to study some qualitative properties of an optimal set Σ, and in particular its closedness and connectedness.

For this purpose we introduce the idea of "transiting mass", that is the total mass passing through a single point. Even though the case we will mainly concentrate our attention on is when $A(t) = t$ and $B(t) = 0$, for the sake of generality we first present this definition in a more general framework.

1. Transiting mass function

In order to define and study the properties of the "transiting mass", we need to introduce a suitable set Θ_μ of paths.

Definition 4.1. *For every measure $\mu \in \mathcal{M}_1^+(\Omega)$, we define Θ_μ as the subset of Θ made by all the paths $\theta \in \Theta$ satisfying*

$$J\big(\mathscr{H}^1(\theta) - \mu(\theta), \mu(\theta)\big) = A\big(\mathscr{H}^1(\theta) - \mu(\theta)\big) + B\big(\mu(\theta)\big)\,,$$

where J is given by (1.24).

In other words, Θ_μ is the set of those paths for which $\delta_\mu(\theta) = \bar{\delta}_\mu(\theta)$, i.e. the most convenient choice for a passenger traveling along θ is to use the network as much as possible. Note that $\Theta_\mu = \Theta$ if $A(t) = t$ and $B(t) = 0$, or more generally if $A \geq B$.

Definition 4.2. *Assume that A and B are C^1 functions and let η be a t.p.m., the support of which is made of paths of uniformly bounded lengths, and minimizing the cost $\overline{C}_\mu$. We define the* generalized transiting mass *as the function $\alpha_\eta : \Omega \to \mathbb{R}^+$ given by*

$$\alpha_\eta(x) := \int_{\Theta_\mu} \Big(A'\big(\mathscr{H}^1(\theta) - \mu(\theta)\big) - B'\big(\mu(\theta)\big)\Big)^+ \chi_\theta(x)\, d\eta(\theta)\,.$$

In particular, when $A(t) = t$ and $B(t) = 0$, we have

$$\alpha_\eta(x) = \eta\{\theta \in \Theta : \; x \in \theta\}\,, \tag{4.1}$$

which justifies the name of transiting mass function.

REMARK 4.3. Note that the function

$$v(\theta) := A'\big(\mathscr{H}^1(\theta) - \mu(\theta)\big) - B'\big(\mu(\theta)\big)$$

is non-negative for any $\theta \in \Theta_\mu$ such that $\mu(\theta) > 0$. Indeed, if $\mu(\theta) > 0$ and $v(\theta) < 0$ then the minimum in the definition (1.24) is not achieved at $l = \mu(\theta)$, contradicting the assumption $\theta \in \Theta_\mu$.

We now show the upper semicontinuity of α_η and the fact that every optimal Σ must be an upper level of α_η.

LEMMA 4.4. *The function α_η is u.s.c.*

PROOF. We write

$$\alpha_\eta(x) = \int_{\Theta_\mu} \chi_\theta(x) v^+(\theta)\, d\eta(\theta)\,,$$

and the thesis follows by Fatou's Lemma. Indeed,

$$\chi_\theta(x) \geq \limsup \chi_\theta(x_n)$$

whenever $x_n \to x$, since a path θ containing infinitely many x_n must also contain their limit x. Moreover, $\theta \mapsto v^+(\theta)$ is a positive Borel function uniformly bounded from above since A and B are of class C^1 and $\mathscr{H}^1(\theta)$ is uniformly bounded over $\operatorname{spt}\eta$ by assumption. $\square$

THEOREM 4.5. *Let $\mu \in \mathcal{M}_1^+(\Omega)$ be an optimal measure and let $\eta \in \mathcal{M}_{\mu,0}^+(\Theta)$ be an optimal t.p.m., the support of which is made of paths of uniformly bounded lengths. Then μ is an upper level of α_η, in the sense that*

$$\mathscr{H}^1 \llcorner \big\{\alpha_\eta(x) > r\big\} \leq \mu \leq \mathscr{H}^1 \llcorner \big\{\alpha_\eta(x) \geq r\big\}\,. \tag{4.2}$$

PROOF. We start defining

$$\mu_\varepsilon := \mu \vee (\mathscr{H}^1 \llcorner \Sigma_\varepsilon)$$

for a small number ε, where Σ_ε is a Borel set chosen in such a way that

$$\|\mu_\varepsilon - \mu\| = \varepsilon\,.$$

In order to evaluate $C_{\mu_\varepsilon}(\eta)$, we notice that for any $\theta \notin \Theta_\mu$, one has by definition $\bar\delta_{\mu_\varepsilon}(\theta) = \bar\delta_\mu(\theta)$ for $\varepsilon \leq \bar\varepsilon(\theta)$. Defining then

$$\Theta'_\varepsilon := \{\theta \in \Theta \setminus \Theta_\mu :\ \varepsilon \leq \bar\varepsilon(\theta)\}\,, \qquad \Theta''_\varepsilon := \big(\Theta \setminus \Theta_\mu\big) \setminus \Theta'_\varepsilon\,,$$

and recalling that $\mathscr{H}^1(\theta)$ is bounded for $\theta \in \operatorname{spt}\eta$, one finds

$$\begin{aligned} &\bar\delta_{\mu_\varepsilon}(\theta) = \bar\delta_\mu(\theta) \quad \text{on } \Theta'_\varepsilon\,, \qquad \bar\delta_{\mu_\varepsilon}(\theta) = \bar\delta_\mu(\theta) + O(\varepsilon) \quad \text{on } \Theta''_\varepsilon\,,\\ &\eta\big(\Theta''_\varepsilon\big) = o(1)\,, \end{aligned} \tag{4.3}$$

the second equality being true due to the fact that $A, B \in C^1$, and the third equality being true due to the fact that $\Theta''_\varepsilon \searrow \emptyset$ as $\varepsilon \to 0$ (all the infinitesimals here and below are to be intended for $\varepsilon \to 0$, and are always uniform with respect to θ). On the other hand, we claim that

$$\int_{\Theta_\mu} \big(\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta)\big)\, d\eta(\theta) = \int_{\Theta_\mu} -v^+(\theta)\big(\mu_\varepsilon(\theta) - \mu(\theta)\big)\, d\eta(\theta) + o(\varepsilon)\,. \quad (4.4)$$

Indeed, by definition of $\bar\delta_\mu$ we easily obtain that

$$\bar\delta_\mu(\theta) = A\big(\mathscr{H}^1(\theta) - \mu(\theta)\big) + B\big(\mu(\theta)\big)\,,$$

and that

$$\bar\delta_{\mu_\varepsilon}(\theta) = A\big(\mathscr{H}^1(\theta) - \mu(\theta) - \lambda(\theta)\big) + B\big(\mu(\theta) + \lambda(\theta)\big)$$

for some

$$0 \le \lambda(\theta) \le \mu_\varepsilon(\theta) - \mu(\theta) \le \varepsilon\,.$$

Thus, recalling that A and B are C^1 functions, we get

$$\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta) = -\lambda(\theta)\,\big(v(\theta) + o(1)\big) \quad (4.5)$$

as $\varepsilon \to 0^+$, the infinitesimal in the above formula being uniform with respect to θ because the function $\theta \mapsto \mathscr{H}^1(\theta)$ is uniformly bounded over $\operatorname{spt}\eta$ by Lemma 3.7 and $\mu(\theta) \le \|\mu\|$. Notice now that if $v(\theta) > 0$ then

$$\lambda(\theta) = \mu_\varepsilon(\theta) - \mu(\theta)$$

for ε sufficiently small depending on θ; analogously, if $v(\theta) < 0$, then $\lambda(\theta) = 0$ for ε small enough. Hence, the two integrands in formula (4.4) coincide at θ whenever ε is small enough (depending on θ), and are of order $O(\varepsilon)$ uniformly with respect to θ due to (4.5) and to the fact that $v(\theta)$ is bounded. Therefore, (4.4) follows.

Making use of (4.3) and (4.4), we can now evaluate

$$\begin{aligned}
\overline{C}_{\mu_\varepsilon}(\eta) - \overline{C}_\mu(\eta) &= \int_\Theta \big(\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta)\big)\, d\eta(\theta) \\
&= \int_{\Theta_\mu} \big(\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta)\big)\, d\eta(\theta) + \int_{\Theta\setminus\Theta_\mu} \big(\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta)\big)\, d\eta(\theta) \\
&= \int_{\Theta_\mu} -v^+(\theta)\big(\mu_\varepsilon(\theta) - \mu(\theta)\big)\, d\eta(\theta) + o(\varepsilon) + \int_{\Theta''_\varepsilon} \big(\bar\delta_{\mu_\varepsilon}(\theta) - \bar\delta_\mu(\theta)\big)\, d\eta(\theta) \\
&= \int_{\Theta_\mu} -v^+(\theta)\big(\mu_\varepsilon(\theta) - \mu(\theta)\big)\, d\eta(\theta) + o(\varepsilon) \\
&= \int_{\Theta_\mu} -v^+(\theta)\Big(\int_{\Sigma_\varepsilon} \chi_\theta(x)\, d(\mu_\varepsilon - \mu)(x)\Big)\, d\eta(\theta) + o(\varepsilon) \\
&= \int_{\Sigma_\varepsilon}\int_{\Theta_\mu} -v^+(\theta)\chi_\theta(x)\, d\eta(\theta)\, d(\mu_\varepsilon - \mu)(x) + o(\varepsilon) \\
&= -\int_{\Sigma_\varepsilon} \alpha_\eta(\theta)\, d(\mu_\varepsilon - \mu)(x) + o(\varepsilon)\,.
\end{aligned}$$

Analogously, we define

$$\mu'_\varepsilon := \mu - (\mu \llcorner \Sigma'_\varepsilon)$$

for a set Σ'_ε such that $\mu(\Sigma'_\varepsilon) = \varepsilon$. A similar computation as above allows to evaluate

$$\overline{C}_{\mu'_\varepsilon}(\eta) - \overline{C}_\mu(\eta) = \int_{\Sigma'_\varepsilon} \alpha_\eta(x)\, d(\mu - \mu'_\varepsilon)(x) + o(\varepsilon)\,. \tag{4.6}$$

Finally, we set

$$\mu''_\varepsilon = \mu \vee \mathscr{H}^1 \llcorner \Sigma_\varepsilon - \mu \llcorner \Sigma'_\varepsilon\,,$$

where Σ_ε and Σ'_ε are two sets as before and having empty intersection. We deduce

$$C_{\mu''_\varepsilon}(\eta) - C_\mu(\eta) = -\int_{\Sigma_\varepsilon} \alpha_\eta(x)\, d(\mu_\varepsilon - \mu)(x) + \int_{\Sigma'_\varepsilon} \alpha_\eta(x)\, d(\mu - \mu'_\varepsilon)(x) + o(\varepsilon)\,.$$

where

$$\mu_\varepsilon := \mu \vee \mathscr{H}^1 \llcorner \Sigma_\varepsilon\,, \qquad \mu'_\varepsilon := \mu - (\mu \llcorner \Sigma'_\varepsilon)\,.$$

We conclude by pointing out that, if Σ does not satisfy (4.2), then there exist two numbers $r > s \geq 0$, and two sets

$$\Sigma_\varepsilon \subseteq \{x \,:\, \alpha_\eta(x) \geq r\}\,, \qquad \Sigma'_\varepsilon \subseteq \{x \,:\, \alpha_\eta(x) \leq s\}$$

with $\mu(\Sigma'_\varepsilon) = \varepsilon$ and

$$\mu \llcorner \Sigma_\varepsilon \leq \mathscr{H}^1 \llcorner \Sigma_\varepsilon\,, \qquad \|\mathscr{H}^1 \llcorner \Sigma_\varepsilon - \mu \llcorner \Sigma_\varepsilon\| = \varepsilon$$

for $\varepsilon > 0$ small enough. Hence, with the above notation

$$-\int_{\Sigma_\varepsilon} \alpha_\eta(x)\, d(\mu_\varepsilon - \mu)(x) + \int_{\Sigma'_\varepsilon} \alpha_\eta(x)\, d(\mu - \mu'_\varepsilon)(x) \leq -\beta\varepsilon$$

for

$$\beta := \frac{r - s}{2} > 0$$

and for all $\varepsilon > 0$ small enough. Since $\|\mu\| = \|\mu''_\varepsilon\|$ and by the fact that η is an optimal t.p.m. for μ, this gives $\mathfrak{F}(\mu''_\varepsilon) < \mathfrak{F}(\mu)$ for ε small enough, against the optimality of μ. $\square$

REMARK 4.6. It is clear that among the numbers $r \geq 0$ satisfying (4.2) there is a maximum one.

LEMMA 4.7. *Given an optimal measure μ and an optimal t.p.m. η, for any $r \geq 0$ such that $\mu \geq \mathscr{H}^1 \llcorner S$, where*

$$S := \{\alpha_\eta(x) > r\}\,,$$

one has that S is contained in countably many geodesics $\theta \in \operatorname{spt}\eta$ (up to an $\mathscr{H}^1$−negligible set).

PROOF. Consider first the case $r > 0$. Take θ_1 which "almost maximizes"

$$\mathscr{H}^1(\theta \cap S)$$

among all paths $\theta \in \operatorname{spt}\eta$ in the sense that

$$\mathscr{H}^1(\theta \cap S) \leq 2\mathscr{H}^1(\theta_1 \cap S) \qquad \forall\, \theta \in \operatorname{spt}\eta\,.$$

Let us now define, inductively, $\theta_{h+1} \in \operatorname{spt}\eta$ as a path almost maximizing

$$\mathscr{H}^1\Big(\theta \cap \Big(S \setminus \bigcup_{j=1}^{h} \theta_j\Big)\Big)$$

in $\operatorname{spt}\eta$. We claim that S is $\mathscr{H}^1-$essentially contained in $\cup_{h\in\mathbb{N}}\theta_h$, that is,

$$\mathscr{H}^1(S \setminus S_0) = 0\,, \qquad \text{where} \qquad S_0 := S \cap \big(\cup_{h\in\mathbb{N}}\, \theta_h\big)\,.$$

Indeed, since $\alpha_\eta \geq r > 0$ inside S, assuming $\mathscr{H}^1(S\setminus S_0) > 0$ one would have

$$\begin{aligned} 0 < \int_{S\setminus S_0} \alpha_\eta(x)\, d\mathscr{H}^1(x) &= \int_{S\setminus S_0}\int_{\Theta_\mu} \chi_\theta(x) v^+(\theta)\, d\eta(\theta)\, d\mathscr{H}^1(x) \\ &= \int_{\Theta_\mu}\int_{S\setminus S_0} \chi_\theta(x) v^+(\theta)\, d\mathscr{H}^1(x)\, d\eta(\theta)\,. \end{aligned}$$

Therefore, there exists $\bar\theta \in \operatorname{spt}\eta$ such that

$$\int_{S\setminus S_0} \chi_\theta(x) v^+(\bar\theta)\, d\mathscr{H}^1(x) > 0\,,$$

and hence

$$\tau := \mathscr{H}^1(\theta \cap (S \setminus S_0)) > 0\,.$$

By construction, $\theta_h \neq \bar\theta$ for all $h \in \mathbb{N}$, and consequently for any $h \in \mathbb{N}$

$$\mathscr{H}^1\Big(\theta_{h+1} \cap \Big(S \setminus \bigcup_{j=1}^{h} \theta_j\Big)\Big) > \frac{\tau}{2}\,.$$

This would give $\mathscr{H}^1(S) = \infty$, which is a contradiction since by hypothesis $\mathscr{H}^1(S) \leq \|\mu\|$, thus the proof is achieved for the case $r > 0$.

If $r = 0$, note that $S := \cup_k S_k$, where

$$S_k := \Big\{\alpha_\eta(x) > \frac{1}{k}\Big\}$$

and $\mu \geq \mathscr{H}^1 \llcorner S$ implies $\mu \geq \mathscr{H}^1 \llcorner S_k$ for each $k \in \mathbb{N}$. Hence, as proved above, each S_k is contained in a countable number of geodesics of $\operatorname{spt}\eta$, and hence so is S. □

From now on, we will consider a particular situation, i.e.

$$A(t) = t\,, \qquad\qquad B(t) = 0\,. \tag{4.7}$$

This is the simplest case, and it has been already considered in many works with the further assumption that Σ is connected (among them [**16, 17**]). By the convexity of A and B and by Theorem 3.27, we already know that there exists an optimal measure $\mu = \mathscr{H}^1 \llcorner \Sigma \in \mathcal{M}_2^+(\Omega)$, even though there may be also optimal measures not contained in $\mathcal{M}_2^+(\Omega)$; indeed, one has $D(a, b) = a$, so that $D(\cdot, b)$ is concave but not strictly concave. Moreover, one has clearly $\bar\delta_\mu = \delta_\mu$ for any measure μ by definition of $\bar\delta_\mu$ given in (3.33). Hence by Lemma 3.18-ii) we infer the existence of optimal t.p.m.'s for C_μ, and not just for $\overline{C}_\mu$; more precisely, (4.7) implies that $C_\mu = \overline{C}_\mu$.

We underline that, since A and B are now fixed, the hypotheses of the next results will be only about H; until now, we have not yet given any hypothesis on H. This is due to the fact that, in all the proofs of the previous chapter, starting from an optimal measure μ we have always built a "competitor" μ' having the same total mass: therefore, since

$$H(\|\mu\|) = H(\|\mu'\|)\,,$$

the function H played no role. On the other hand, in the rest of the monograph we will often consider competitors having a different total mass, so that the hypotheses on H will become of primary importance.

Note that the hypothesis $B \equiv 0$ means that "moving by train is free of charge". Then, for people who need to move from/to places not so far from the network, the most convenient thing should be to reach the network, take a train until they arrive close to their destination, then reach the latter by own means.

We stress that, since we are assuming (4.7), one has $\Theta_\mu = \Theta$ for any μ, which makes the Definition 4.2 of the transiting mass more significant; in particular, as already remarked, α_η is given by

$$\alpha_\eta(x) = \eta\{\theta \in \Theta :\ x \in \theta\}\,.$$

Therefore, we can give the following useful calculation for $C_\mu(\eta) = \overline{C}_\mu(\eta)$, which shows that $\mu \mapsto C_\mu(\eta)$ is a constant plus the integral of a given function in $d\mu$.

$$\begin{aligned} C_\mu(\eta) &= \int_\Theta \big(\mathscr{H}^1(\theta) - \mu(\theta)\big)\, d\eta(\theta) = C_\emptyset(\eta) - \int_\Theta \mu(\theta)\, d\eta(\theta) \\ &= C_\emptyset(\eta) - \int_\Theta \Big(\int_\Omega \chi_\theta(x))\, d\mu(x)\Big)\, d\eta(\theta) \\ &= C_\emptyset(\eta) - \int_\Omega \Big(\int_\Theta \chi_\theta(x)\, d\eta(\theta)\Big)\, d\mu(x) \\ &= C_\emptyset(\eta) - \int_\Omega \alpha_\eta(x)\, d\mu(x)\,. \end{aligned} \tag{4.8}$$

We conclude this chapter with a couple of remarks: the first one is a consequence of Lemma 4.7 and of the above calculation of $C_\mu(\eta)$.

Remark 4.8. Let μ be an optimal measure and η be an optimal t.p.m. As an immediate consequence of Lemma 4.7, we obtain that the measure

$$\mu' = \mu \llcorner \big\{\alpha_\eta(x) > 0\big\}$$

is concentrated on countably many geodesics $\theta \in \operatorname{spt} \eta$. By (4.8), we obtain that

$$C_{\mu'}(\eta) = C_\mu(\eta) + \int_\Omega \alpha_\eta(x)\, d(\mu - \mu')(x) = C_\mu(\eta)\,.$$

Hence, minding that $\mu' \le \mu$, and thus $H(\|\mu'\|) \le H(\|\mu\|)$, one has that also μ' is an optimal measure and that η is an optimal t.p.m. also with respect to μ'. As discussed in Proposition 3.21, we obtain that $\mu' = \mu$ whenever either (3.38) or (3.39) occurs. Summarizing, in either of these cases we have obtained that *any* optimal measure μ is concentrated on countably many geodesics contained in the support of an optimal t.p.m.; otherwise, an optimal measure $\mu' < \mu$ is concentrated on countably many geodesics, and the remaining part $\mu - \mu'$ is completely useless.

REMARK 4.9. It is worth noticing that, if one is given a t.p.m. η which is known to be optimal with respect to some unknown optimal measure, there is an easy method to determine this measure, or more precisely to determine all the measures μ which are optimal and with respect to which η is an optimal t.p.m. Indeed, for any $l \geq 0$ define $\mathcal{M}_l$ to be the subset of $\mathcal{M}_1^+(\Omega)$ made by all those measures which are upper levels in the sense of (4.2) and with total mass equal to l. An immediate consequence of (4.8) and of the formula

$$\int_\Omega \alpha_\eta \, d\mu = \int_0^{+\infty} \mu\big(\{x : \alpha_\eta(x) \geq t\}\big) \, dt$$

is that for $\mu \in \mathcal{M}_l$ one has

$$C_\mu(\eta) = C_\emptyset(\eta) - \beta(l)$$

for a function $\beta(l)$ which depends on l but not on μ: more precisely, β is given by the following formula (to be intended in the weak sense, or in the classical sense up to countably many points)

$$\begin{cases} \beta(0) = 0\,, \\ \beta'(l) = \inf\big\{t : \mathscr{H}^1\big(\{x : \alpha_\eta(x) \geq t\}\big) \leq l\big\}\,. \end{cases} \tag{4.9}$$

Arguing as in Theorem 4.5, we know that for any measure $\mu \in \mathcal{M}_1^+(\Omega)$ with $\|\mu\| = l$ one has

$$C_\mu(\eta) \geq C_\emptyset(\eta) - \beta(l)\,,$$

and the inequality is strict if and only if $\mu \notin \mathcal{M}_l$. Notice also that the function $l \mapsto \beta(l)$ is concave thanks to (4.9); more precisely, the left and the right derivative of β at l coincide respectively with the maximum and minimum numbers $r^+(l)$ and $r^-(l)$ such that (4.2) holds for a measure $\mu \in \mathcal{M}_l$: hence, $r^\pm(l)$ depend on l but not on the choice of $\mu \in \mathcal{M}_l$, and $r^-(l_1) \geq r^+(l_2)$ whenever $l_1 < l_2$.

It is now clear that for any $\mu \in \mathcal{M}_l$ one has

$$\mathfrak{F}(\mu) \leq C_\emptyset(\eta) - \beta(l) + H(l)\,,$$

and the inequality is in fact an equality if and only if η is an optimal t.p.m. with respect to μ. Consider now the function $l \mapsto H(l) - \beta(l)$: since H and β are positive functions, $H(0) = \beta(0) = 0$, $H(l) \to \infty$ when $l \to \infty$ while $\beta(l)$ is bounded by $C_\emptyset(\eta)$ thanks to (4.8), we deduce that $\operatorname{Arg\,min}(H - \beta)$ is a non-empty compact subset L of $\mathbb{R}^+$. Since we know that η is an optimal t.p.m. for some optimal measure μ, we finally conclude that the optimal measures for which η is an optimal t.p.m. are precisely the measures belonging to some $\mathcal{M}_l$ with $l \in L$.

2. Ordered transport path measures

In this section, we introduce the notion of *ordered* transport path measures, which are t.p.m. with additional useful properties. First of all, we select an optimal measure $\mu \in \mathcal{M}_1^+(\Omega)$; then, we notice that by the lower semicontinuity of $\mathfrak{F}$ the set $\mathcal{M}^+_{\mu,0}(\Theta)$ of optimal t.p.m.'s is a non-empty, weakly* closed and convex subset of $\mathcal{M}^+(\Theta)$. We define then the following two sets.

Definition 4.10. *Let*

$$\mathcal{M}^+_{\mu,1}(\Theta) := \operatorname{Arg\,min}\Big\{ \int_\Theta \ell(\theta)\, d\eta(\theta) : \ \eta \in \mathcal{M}^+_{\mu,0}(\Theta)\Big\}, \tag{4.10}$$

$$\mathcal{M}^+_{\mu,2}(\Theta) := \operatorname{Arg\,max}\Big\{ \int_\Sigma \alpha_\eta(x)^2\, d\mathscr{H}^1(x) : \ \eta \in \mathcal{M}^+_{\mu,1}(\Theta)\Big\}, \tag{4.11}$$

where

$$\ell(\theta) := \int_0^1 |\theta'(t)|\, dt$$

is the parametric length *of θ. The elements of $\mathcal{M}^+_{\mu,2}(\Theta)$ will be called* ordered transport path measures.

Remark 4.11. Notice that, similarly to $\mathcal{M}^+_{\mu,0}(\Theta)$, also $\mathcal{M}^+_{\mu,1}(\Theta)$ and $\mathcal{M}^+_{\mu,2}(\Theta)$ are non-empty and weakly* closed subsets of $\mathcal{M}^+(\Theta)$. In fact, $\theta \mapsto \ell(\theta)$ is l.s.c., hence the map $\eta \mapsto \int_\Theta \ell(\theta)\, d\eta$ is l.s.c., which implies the claim concerning $\mathcal{M}^+_{\mu,1}(\Theta)$. Moreover, the map $\eta \to \alpha_\eta(x)$ is u.s.c. for any $x \in \Sigma$ by (4.1) since the set

$$\big\{\theta \in \Theta : \ x \in \theta\big\}$$

is closed, while

$$\alpha_\eta(x) \le \eta(\Theta) = \|f^+\|,$$

and hence the fact that $\mathcal{M}^+_{\mu,2}(\Theta)$ is weakly* closed and non-empty follows with the help of Fatou's lemma. Moreover, clearly both $\mathcal{M}^+_{\mu,0}(\Theta)$ and $\mathcal{M}^+_{\mu,1}(\Theta)$ (but not $\mathcal{M}^+_{\mu,2}(\Theta)$) are convex sets.

As we will see in the sequel, the elements of $\mathcal{M}^+_{\mu,2}(\Theta)$ have rather particular properties among the optimal t.p.m.'s. For this reason, in what follows we will restrict our attention to the ordered t.p.m.'s.

We notice first an easy consequence of Lemma 3.1.

Corollary 4.12. *For any t.p.m. $\eta \in \mathcal{M}^+_{\mu,0}(\Theta)$ there is a path σ maximizing*

$$\theta \mapsto \mu(\theta) = \mathscr{H}^1(\theta \cap \Sigma)$$

within $\operatorname{spt} \eta$.

Proof. Just take a maximizing sequence $\{\theta_n\}$, extract a subsequence converging to a path θ with respect to d_Θ (this is possible since all these paths have uniformly bounded Euclidean lengths), notice that $\theta \in \operatorname{spt}\eta$ because $\operatorname{spt}\eta$ is closed by definition, and finally apply Lemma 3.1 with $X := \Theta$, $C_n := \theta_n$ and $\nu := \mu$. □

We show now the following geometric property of the paths belonging to $\operatorname{spt}\eta$ with $\eta \in \mathcal{M}^+_{\mu,0}(\Theta)$.

Lemma 4.13. *Let $\eta \in \mathcal{M}^+_{\mu,0}(\Theta)$ and let $\theta \in \operatorname{spt}\eta$ be a path such that*

$$\mu\big(\theta([t_1,t_2])\big) = a > 0$$

for some $0 \leq t_1 < t_2 \leq 1$. Then for any $\varepsilon > 0$ there is $\rho > 0$ such that the following holds: given any $\tau \in \operatorname{spt}\eta \cap B_\Theta\big(\theta,\rho\big)$ and considering on τ a parametrization such that $|\tau(t_i) - \theta(t_i)| \leq \rho$ for $i = 1, 2$, one has

$$\mu\big(\tau([t_1,t_2])\big) > a - \varepsilon\,.$$

In particular, the above statement with $t_1 = 0$ and $t_2 = 1$ implies the lower semicontinuity of $\theta \mapsto \mu(\theta)$ in $\operatorname{spt}\eta$. Since the upper semicontinuity holds in the whole Θ by Lemma 3.1, we derive that the quantity of network taken by the paths, that is $\theta \mapsto \mu(\theta)$, is a continuous function in $\operatorname{spt}\eta$.

of Lemma 4.13. By the lower semicontinuity of the Euclidean length, we know that

$$\mathscr{H}^1(\theta_{|[t_1,t_2]}) \leq \mathscr{H}^1(\tau_{|[t_1,t_2]}) + \frac{\varepsilon}{2}\,, \tag{4.12}$$

provided that ρ is sufficiently small. We claim that if the assertion is not true, that is, if

$$\mu\big(\tau([t_1,t_2])\big) \leq a - \varepsilon\,, \tag{4.13}$$

then τ is not a geodesic between $\tau(0)$ and $\tau(1)$: since η is an optimal t.p.m., this will give a contradiction with Lemma 3.18-iii), hence concluding the proof.

To show the claim, define $\tilde\tau$ as the path that equals τ outside of $[t_1,t_2]$ and that is defined as follows between t_1 and t_2: a segment connecting $\tau(t_1)$ to $\theta(t_1)$, then the path $\theta_{|[t_1,t_2]}$ between $\theta(t_1)$ and $\theta(t_2)$, then the segment connecting $\theta(t_2)$ to $\tau(t_2)$. Figure 1 shows the situation, where τ is the upper path, θ the lower one, and $\tilde\tau$ the dark shaded one. Then, minding that by (4.13) one has

$$\delta_\mu(\tilde\tau_{|[t_1,t_2]}) \geq \mathscr{H}^1(\tau_{|[t_1,t_2]}) - a + \varepsilon\,, \tag{4.14}$$

we find

$$\begin{aligned}
\delta_\mu(\tilde\tau) &= \delta_\mu(\tilde\tau_{|[0,t_1]}) + \delta_\mu(\tilde\tau_{|[t_1,t_2]}) + \delta_\mu(\tilde\tau_{|[t_2,1]})\\
&\leq \delta_\mu(\tau_{|[0,t_1]}) + 2\rho + \delta_\mu(\theta_{|[t_1,t_2]}) + \delta_\mu(\tau_{|[t_2,1]})\\
&= \delta_\mu(\tau_{|[0,t_1]}) + 2\rho + \mathscr{H}^1(\theta_{|[t_1,t_2]}) - a + \delta_\mu(\tau_{|[t_2,1]})\\
&\leq \delta_\mu(\tau_{|[0,t_1]}) + 2\rho + \mathscr{H}^1(\tau_{|[t_1,t_2]}) + \varepsilon/2 - a + \delta_\mu(\tau_{|[t_2,1]}) \qquad \text{(by (4.12))}
\end{aligned}$$

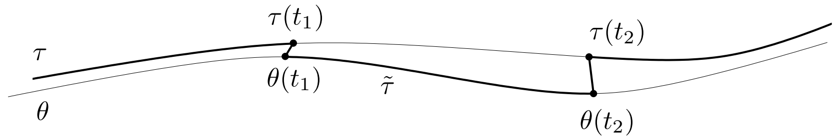

FIGURE 1. Situation in Lemma 4.13

$$\begin{aligned}&\le \delta_\mu(\tau_{|[0,t_1]}) + 2\rho + \delta_\mu(\tau_{|[t_1,t_2]}) + a - \varepsilon/2 - a + \delta_\mu(\tau_{|[t_2,1]}) \quad \text{(by (4.14))}\\ &= \delta_\mu(\tau) + 2\rho - \varepsilon/2 < \delta_\mu(\tau),\end{aligned}$$

provided $\rho < \varepsilon/4$, which, recalling that $\delta_\mu = \bar\delta_\mu$ thanks to (4.7), gives the desired contradiction. $\square$

COROLLARY 4.14. *In the hypotheses of Lemma 4.13, for any $\varepsilon > 0$ there is a $\rho > 0$ such that for any $\tau \in \operatorname{spt}\eta \cap B_\Theta(\theta, \rho)$ one has*

$$\mu\big(\tau([t_1,t_2]) \cap \theta([t_1,t_2])\big) > a - \varepsilon\,.$$

PROOF. Assume that the thesis is not true, i.e. there exists an $\varepsilon > 0$ such that for every $\rho > 0$ there is a $\tau_\rho \in \operatorname{spt}\eta \cap B_\Theta(\theta,\rho)$ with

$$\mu\big(\tau_\rho([t_1,t_2]) \cap \theta([t_1,t_2])\big) \le a - \varepsilon\,.$$

Then, by Lemma 4.13 then there exists a $\rho_1 > 0$ such that, taking

$$\tau_1 := \tau_{\rho_1} \in \operatorname{spt}\eta\,,$$

one has

$$\mu\big(\tau_1([t_1,t_2])\big) > a - \frac{\varepsilon}{2}\,,$$

so that

$$\mu\big(\tau_1([t_1,t_2]) \setminus \theta([t_1,t_2])\big) > \frac{\varepsilon}{2}\,.$$

Suppose now that for some $k \in \mathbb{N}$ and for each $n \le k$ we constructed a $\rho_n \le \rho_{n-1}$ and a path

$$\tau_n := \tau_{\rho_n} \in B_\Theta(\theta, \rho_{n-1}) \cap \operatorname{spt}\eta$$

such that

$$\mu\Big(\tau_n([t_1,t_2]) \setminus \big(\bigcup_{j=1}^{n-1} \tau_j([t_1,t_2]) \cup \theta([t_1,t_2])\big)\Big) > \frac{\varepsilon}{n+1}\,. \tag{4.15}$$

Mind that

$$\mu(\tau_\rho([t_1,t_2]) \setminus \theta([t_1,t_2])) \to 0 \qquad \text{as} \qquad \rho \to 0$$

since

$$\limsup_{\rho\to 0} \mu(\tau_\rho([t_1,t_2])) \le \mu(\theta([t_1,t_2]))$$

by Lemma 3.1. Therefore,

$$\mu\left((\tau_\rho([t_1,t_2])\setminus\theta([t_1,t_2]))\cap\left(\bigcup_{j=1}^k\tau_j([t_1,t_2])\right)\right)\to 0$$

as $\rho\to 0$. Then there exists a $\rho_{k+1}\le\rho_k$ such that, taking $\tau_{k+1}:=\tau_{\rho_{k+1}}$, one has

$$\mu\left((\tau_{k+1}([t_1,t_2])\setminus\theta([t_1,t_2]))\cap\left(\bigcup_{j=1}^k\tau_j([t_1,t_2])\right)\right)\le\frac{\varepsilon}{2}-\frac{\varepsilon}{k+2},$$

while

$$\mu\big(\tau_{k+1}([t_1,t_2])\big)>a-\frac{\varepsilon}{2}$$

by Lemma 4.13, so that

$$\begin{aligned}\mu\Big(\tau_{k+1}([t_1,t_2])\setminus\big(\bigcup_{j=1}^k\tau_j([t_1,t_2])\cup\theta([t_1,t_2])\big)\Big)&=\\ \mu(\tau_{k+1}([t_1,t_2]))-\mu(\tau_{k+1}([t_1,t_2])\cap\theta([t_1,t_2]))&\\ -\,\mu\Big((\tau_{k+1}([t_1,t_2])\setminus\theta([t_1,t_2]))\cap\big(\bigcup_{j=1}^k\tau_j([t_1,t_2])\big)\Big)&>\frac{\varepsilon}{k+2}.\end{aligned}$$

By induction we have found therefore two sequences $\rho_n\to 0$ and $\tau_n\to\theta$ such that (4.15) holds for all $n\in\mathbb{N}$. This leads to a contradiction since the harmonic series diverges, while $\|\mu\|<\infty$. □

We can now find a stronger consequence of the definition (4.10), namely that different paths cannot cross in different directions as the following lemma ensures.

LEMMA 4.15. *If $\eta\in\mathcal{M}^+_{\mu,1}(\Theta)$, then for no couple of paths θ, $\tau\in\operatorname{spt}\eta$ one has (with suitable parameterizations) $\theta(t_1)=\tau(t_2)$ and $\theta(t_2)=\tau(t_1)$ for two time instants $0\le t_1<t_2\le 1$.*

REMARK 4.16. Notice that this assertion says, in words, that if two different paths in $\operatorname{spt}\eta$ pass through the same two points, they must do it in the same order. In particular, taking $\theta=\tau$, one gets that each $\theta\in\operatorname{spt}\eta$ is an arc (an injective curve), which also implies that $\ell(\theta)=\mathscr{H}^1(\theta)$; in other words, there are no paths in $\operatorname{spt}\eta$ with loops.

Note that, as in Lemma 3.18-iii), Lemma 3.19, Lemma 4.13 and Corollary 4.14, the most important part of this assertion is that we can establish the announced property for *any* path $\theta\in\operatorname{spt}\eta$, and not only for η-almost any.

OF LEMMA 4.15. Assume by contradiction that there are θ and τ as in the assertion being proved. Hence, $\theta([t_1,t_2])$ and $\tau([t_1,t_2])$ are contained in Σ, since otherwise we would find a contradiction to the cyclical monotonicity property (3.44) with

$$x:=\theta(0),\qquad y:=\theta(1),\qquad x':=\tau(0),\qquad y':=\tau(1)$$

and the optimal transport plan $\gamma=(p_0,p_1)_\#\eta$ associated to η.

By Corollary 4.14 there is a $\rho > 0$ such that for any

$$\tilde\theta \in \operatorname{spt}\eta \cap B_\Theta\bigl(\theta,\rho\bigr)$$

and

$$\tilde\tau \in \operatorname{spt}\eta \cap B_\Theta\bigl(\tau,\rho\bigr)$$

one has, minding that $\mathscr{H}^1 \geq \mu$ and that $\theta([t_1,t_2])$ and $\tau([t_1,t_2])$ are contained in Σ, the relationships

$$\begin{aligned}
\mathscr{H}^1\bigl(\tilde\theta([t_1,t_2]) \cap \theta([t_1,t_2])\bigr) &\geq \mu\bigl(\tilde\theta([t_1,t_2]) \cap \theta([t_1,t_2])\bigr) > \frac{3}{4}\mu\bigl(\theta([t_1,t_2])\bigr)\\
&= \frac{3}{4}\mathscr{H}^1(\theta([t_1,t_2]))\,,\\
\mathscr{H}^1\bigl(\tilde\tau([t_1,t_2]) \cap \tau([t_1,t_2])\bigr) &\geq \mu\bigl(\tilde\tau([t_1,t_2]) \cap \tau([t_1,t_2])\bigr) > \frac{3}{4}\mu\bigl(\tau([t_1,t_2])\bigr)\\
&= \frac{3}{4}\mathscr{H}^1(\tau([t_1,t_2]))\,,
\end{aligned} \tag{4.16}$$

with suitable parameterizations of $\tilde\theta$ and $\tilde\tau$. Let $t_1^{\tilde\theta}$ and $t_2^{\tilde\theta}$ (resp. $t_1^{\tilde\tau}$ and $t_2^{\tilde\tau}$) be the first and the last instants in $[t_1,t_2]$ when θ (resp. τ) touches $\tilde\theta$ (resp. $\tilde\tau$). We denote then by $\hat\theta_1$ the shortest arc connecting $\theta(t_1)$ with

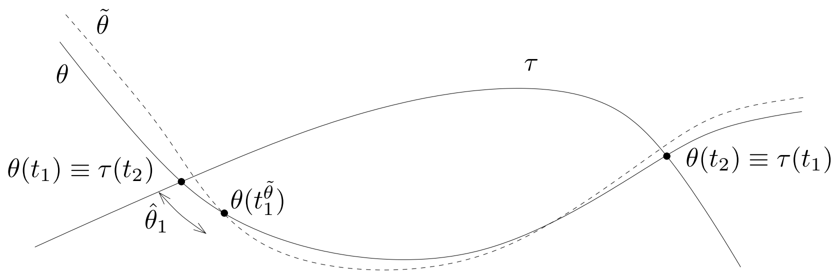

FIGURE 2. Situation in Lemma 4.15

$\theta(t_1^{\tilde\theta})$ inside $\theta([t_1,t_2])$, and by $\hat\theta_2$ the shortest arc connecting $\theta(t_2^{\tilde\theta})$ with $\theta(t_2)$ inside $\theta([t_1,t_2])$ (resp. by $\hat\tau_1$ the shortest arc connecting $\tau(t_1)$ with $\tau(t_1^{\tilde\tau})$ inside $\tau([t_1,t_2])$, and by $\hat\tau_2$ the shortest arc connecting $\tau(t_2^{\tilde\tau})$ with $\tau(t_2)$ inside $\tau([t_1,t_2])$): see Figure 2.

We now define in the obvious way $\alpha(\tilde\theta,\tilde\tau)$ connecting $\tilde\theta(0)$ with $\tilde\tau(1)$ and $\beta(\tilde\theta,\tilde\tau)$ connecting $\tilde\tau(0)$ with $\tilde\theta(1)$: formally,

$$\alpha(\tilde\theta,\tilde\tau) := \tilde\theta_{|[0,t_1]}\cdot\hat\theta_1\cdot\hat\tau_2\cdot\tilde\tau_{|[t_2,1]}\,,\qquad \beta(\tilde\theta,\tilde\tau) := \tilde\tau_{|[0,t_1]}\cdot\hat\tau_1\cdot\hat\theta_2\cdot\tilde\theta_{|[t_2,1]}\,.$$

In particular, the couple $(\alpha(\theta,\tau),\beta(\theta,\tau))$ is given by the formulae

$$\alpha = \theta_{|[0,t_1]}\cdot\tau_{|[t_2,1]}\,,\qquad\qquad \beta = \tau_{|[0,t_1]}\cdot\theta_{|[t_2,1]}\,.$$

The functions α,β are clearly Borel. Note also that

$$\delta_\mu(\tilde\theta) + \delta_\mu(\tilde\tau) = \delta_\mu(\alpha(\tilde\theta,\tilde\tau)) + \delta_\mu(\beta(\tilde\theta,\tilde\tau))\,, \tag{4.17}$$

since $\theta([t_1, t_2])$ and $\tau([t_1, t_2])$ are contained in Σ. On the other hand,

$$\ell(\alpha(\tilde{\theta}, \tilde{\tau})) + \ell(\beta(\tilde{\theta}, \tilde{\tau})) < \ell(\tilde{\theta}) + \ell(\tilde{\tau})\,. \tag{4.18}$$

in view of (4.16). We consider now two measures η_1 and η_2 concentrated on sufficiently small balls centered in θ and τ respectively, so that $\alpha(\tilde{\theta}, \tilde{\tau})$ and $\beta(\tilde{\theta}, \tilde{\tau})$ are defined for any $\tilde{\theta} \in \operatorname{spt} \eta_1$, $\tilde{\tau} \in \operatorname{spt} \eta_2$, and with the property that

$$\|\eta_1\| = \|\eta_2\| =: \varepsilon > 0\,, \qquad \eta_1 + \eta_2 \leq \eta\,.$$

Thanks to the classical results on the isomorphism of measure spaces (see Theorem 2 in [**40**], or also Theorem 9 [**47**, Chapter 15]) we can take two Borel maps $\varphi_i : [0, \varepsilon] \to \Theta$, $i = 1, 2$ so that $\varphi_{i\#}\mathscr{L}_\varepsilon = \eta_i$, where $\mathscr{L}_\varepsilon$ stands for the Lebesgue measure on $[0, \varepsilon]$. Let us then consider the maps $\hat{\alpha}$, $\hat{\beta} : [0, \varepsilon] \to \Theta$ given by

$$\hat{\alpha}(t) := \alpha(\varphi_1(t), \varphi_2(t))\,, \qquad \hat{\beta}(t) := \beta(\varphi_1(t), \varphi_2(t))\,,$$

and finally define

$$\tilde{\eta} := \eta - \eta_1 - \eta_2 + \hat{\alpha}_\#\mathscr{L}_\varepsilon + \hat{\beta}_\#\mathscr{L}_\varepsilon\,.$$

By construction $\tilde{\eta}$ is an admissible t.p.m. and is contained in $\mathcal{M}^+_{\mu,0}(\Theta)$ by (4.17); moreover

$$\int_\Theta \ell(\theta)\, d\tilde{\eta}(\theta) < \int_\Theta \ell(\theta)\, d\eta(\theta)$$

due to (4.18), which gives a contradiction with the definition (4.10). $\square$

Now, we see the importance of the maximization in (4.11), which provides a sort of "solidarity" between different paths in $\operatorname{spt} \eta$.

LEMMA 4.17. *If $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, given two points $x \neq y \in \Omega$ such that*

$$\eta\big(\{\sigma \in \Theta \,:\, x,\, y \in \sigma\}\big) > 0\,,$$

then $\eta-$a.e. path passing through both x and y contains the same subpath σ_{xy} between x and y.

PROOF. Denote

$$\Theta_{xy} := \big\{\sigma \in \Theta : \, x,\, y \in \sigma\big\} \subseteq \Theta\,,$$

and for each $\sigma \in \Theta_{xy} \cap \operatorname{spt} \eta$ let $\tilde{\sigma}$ be the subpath of σ connecting x and y. Due to the assumption (4.7), for any $\sigma \in \Theta_{xy} \cap \operatorname{spt} \eta$ the subpath $\tilde{\sigma}$ is a geodesic between x and y, so that $\delta_\mu(\tilde{\sigma})$ is a constant depending only on x and y (i.e. independent on σ). We obtain the thesis showing that, by the maximization in (4.11), one of these geodesics is chosen by η-a.e. path in Θ_{xy}. To this aim we first show that there is a constant c such that $\mathscr{H}^1(\tilde{\sigma}) = c$ for η-a.e. $\sigma \in \Theta_{xy}$. In fact, otherwise there would exist two constants $c_1 < c_2$ such that

$$\eta\big(\{\sigma \in \Theta_{xy} \,:\, \mathscr{H}^1(\tilde{\sigma}) = c_i\}\big) > 0\,, \qquad i = 1, 2\,.$$

Define the function $f\colon\, \Theta \to \Theta$ according to the relationship

$$f(\sigma) := \begin{cases} \sigma_1, & \sigma \in \Theta_{xy} \cap \operatorname{spt}\eta, \mathscr{H}^1(\tilde\sigma) = c_2\,, \\ \sigma, & \text{otherwise}\,, \end{cases} \tag{4.19}$$

where $\sigma_1 \in \Theta_{xy} \cap \operatorname{spt}\eta$ is some chosen path with $\mathscr{H}^1(\tilde\sigma) = c_1$. We have then that $\eta' := f_{\#}\eta$ is still an optimal t.p.m., but clearly

$$\int_\Theta \mathscr{H}^1(\theta)\, d\eta' < \int_\Theta \mathscr{H}^1(\theta)\, d\eta\,,$$

contradicting the assumption $\eta \in \mathcal{M}^+_{\mu,1}(\Theta)$ (recall that $\mathscr{H}^1(\theta) = \ell(\theta)$ for all $\theta \in \operatorname{spt}\eta$ according to Lemma 4.15).

Now, since both $\delta_\mu(\tilde\sigma)$ and $\mathscr{H}^1(\tilde\sigma)$ are η-a.e. constant for $\sigma \in \Theta_{xy}$, we get that so is

$$\mu(\tilde\sigma) = \mathscr{H}^1(\tilde\sigma) - \delta_\mu(\tilde\sigma)\,.$$

Choose now an arbitrary

$$\sigma' \in \Theta_{xy} \cap \operatorname{spt}\eta$$

and let $\hat\eta$ be the t.p.m. that leaves all the paths of $\operatorname{spt}\eta$ unchanged, except those connecting x and y, the latter being forced to follow $\tilde\sigma$ between x and y. Formally, we define the function $g\colon\, \Theta \to \Theta$ by setting $g(\sigma)$ for $\sigma \in \Theta_{xy} \cap \operatorname{spt}\eta$ to be the composition of the arc of σ between $\sigma(0)$ and x, $\tilde\sigma'$, and the arc of σ between y and $\sigma(0)$ (for all the other paths σ we set $g(\sigma) := \sigma$). A straightforward computation ensures that

$$\hat\eta := g_{\#}\eta \in \mathcal{M}^+_{\mu,1}(\Theta)\,,$$

while since $\alpha_{\hat\eta}$ is just a redistribution of α_η, one has using the strict convexity of $\alpha_\eta(x)^2$ that

$$\int_\Sigma \alpha_{\hat\eta}(z)^2\, d\mathscr{H}^1(z) < \int_\Sigma \alpha_\eta(z)^2\, d\mathscr{H}^1(z)\,,$$

unless η-a.e. path connecting x and y already follows a unique path, which concludes the proof. $\square$

Remark 4.18. Notice that the path σ_{xy} is well-defined only for such pairs (x, y) that the set Θ_{xy} in the above proof is not η-negligible. Moreover, if σ_{xy} is defined and z, $w \in \sigma_{xy}$, then σ_{zw} is defined as well and it is the subpath of σ_{xy} connecting z and w.

As the Example 4.19 below shows, the above result is not true for *all* the paths in $\operatorname{spt}\eta$ connecting x and y, but only for η-a.e.. This is in contrast with all the other properties enlisted before, which were true for all the paths in $\operatorname{spt}\eta$, and not just for η-almost all.

Example 4.19. Consider the following situation, shown in Figure 3: let

$$f^+ = \mathscr{H}^1 \llcorner A + \delta_x + M\delta_{x'} \qquad\qquad f^- = 2\delta_y + M\delta_{y'}\,;$$

here A is a segment of unit length, having x as a right endpoint. The

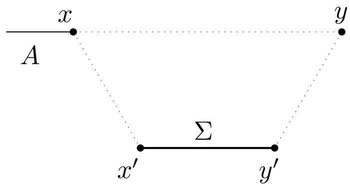

FIGURE 3. Situation in Example 4.19

geometry is chosen so that

$$|x-y| = |x-x'| + |y-y'|\,.$$

If M is large enough, it is easy to define H in such a way that the unique optimal measure is $\mu := \mathscr{H}^1 \llcorner \Sigma$, with $\Sigma := [x', y']$ being the line segment connecting x' to y', and that there is a unique optimal t.p.m. $\eta \in \mathcal{M}^+_{\mu,1}(\Theta)$ (hence $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$) which moves the mass as follows:

- $M\delta_{x'}$ goes on $M\delta_{y'}$ along Σ;
- δ_x goes on δ_y via the line segment $[x, y]$: using Σ would have the same cost but would contradict the fact that $\eta \in \mathcal{M}^+_{\mu,1}(\Theta)$, namely the minimization (4.10);
- $\mathscr{H}^1 \llcorner A$ goes on δ_y moving each point $z \in A$ along the line segment $[z, x']$, then following the network between x' and y', and finally along the line segment $[y', y]$.

By construction, the path θ connecting x to y and passing through Σ belongs to the support of η, since $\operatorname{spt}\eta$ is closed and $x \in A$. On the other hand σ_{xy} is the line segment $[x, y]$. This shows that the assertion of Lemma 4.17 cannot be strengthened replacing "η-a.e. path connecting x and y" with "each path in $\operatorname{spt}\eta$ connecting x and y". However, notice that $\sigma_{x'y'}$ coincides with the restriction of θ between x' and y', according to Lemma 4.20 below.

Nevertheless, we can show that a sort of *everywhere solidarity*, i.e. that *all* the paths in $\operatorname{spt}\eta$ (instead of η−almost all) follow the path σ_{xy} between x and y, is true where there is some railway. In the following, when a path $\theta \in \Theta$ and two points x, $y \in \theta$ are given, we will write $\widetilde{xy}$ to denote the subpath of θ connecting x to y.

LEMMA 4.20. *Let $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, let $\theta \in \operatorname{spt}\eta$ be a path such that $\mu(\theta) > 0$, and let x' and y' be such points that $\widetilde{x'y'}$ is the shortest subpath of θ satisfying $\mu\big(\widetilde{x'y'}\big) = \mu(\theta)$. Then, for any two points x, y belonging to the open subpath $\widetilde{x'y'}$, the path σ_{xy} introduced in Lemma 4.17 is well-defined and coincides with $\widetilde{xy}$.*

PROOF. Let us define

$$B := \left\{ z \in \widetilde{x'y'} :\ \liminf_{n\to\infty} \frac{\eta\big(\{\sigma \in B_\Theta\big(\theta, 1/n\big) : z \in \sigma\}\big)}{\eta\big(B_\Theta\big(\theta, 1/n\big)\big)} \le \frac{1}{2} \right\}$$

the set of "bad" points of $\widetilde{x'y'}$, i.e. those points in $\widetilde{x'y'}$ such that it happens that only few paths close to θ pass through them. Thanks to Corollary 4.14 we have $\mu(B) = 0$. Indeed, if $\mu(B) > 0$, then by definition there is a sequence $\rho_\nu \to 0$ and a constant $C > 0$ such that

$$\mu\left(\left\{z \in \widetilde{x'y'} : \frac{\eta\big(\{\sigma \in B_\Theta(\theta, \rho_\nu) : z \in \sigma\}\big)}{\eta\big(B_\Theta(\theta, \rho_\nu)\big)} > \frac{1}{2}\right\}\right) \leq \mu(\theta) - C$$

for all $\nu \in \mathbb{N}$, which contradicts Corollary 4.14. Therefore, for $\mathscr{H}^1$−a.e. point $z \in \widetilde{x'y'} \cap \Sigma$, more than 50% (with respect to η) of the paths which are close to θ pass through z. Take then two such points x and y: for any $\rho > 0$ the measure η of the paths in $B_\Theta\big(\theta, \rho\big)$ passing through both x and y is strictly positive. Then σ_{xy} is well-defined. We prove now $\sigma_{xy} = \widetilde{xy}$. Indeed, if it were not so, then one would have

$$d_\Theta\big(\widetilde{xy}, \sigma_{xy}\big) =: \rho > 0$$

which leads to a contradiction, since by Lemma 4.17, η-a.e. path passing through both x and y must follow σ_{xy}, which is impossible for the paths belonging to $B_\Theta\big(\theta, \rho/2\big)$, that are not η-negligible. Finally, since the points x and y can be chosen arbitrarily close to x' and y' by the definition of x' and y' and by the fact that $\mu(B) = 0$, the thesis follows by Remark 4.18. □

3. Closedness of optimal sets

In this section we derive the closedness of an optimal set Σ by the concavity of H. More precisely, we show that if H is concave (resp. strictly concave) then there exists a closed optimal set Σ (resp. *any* optimal measure corresponds to an essentially closed set). Moreover, we present an example showing that the concavity of H is crucial for such results. Recall that we are assuming that $A(t) = t$ and $B(t) = 0$ for all $t \in \mathbb{R}^+$.

Theorem 4.21. *If H is concave, then there exists a closed optimal set Σ. Moreover, if H is strictly concave then any optimal measure μ corresponds to an essentially closed optimal set (in particular, every optimal set is essentially closed).*

Proof. Take an optimal measure μ and a t.p.m. η minimizing C_μ, and define, in accordance with Theorem 4.5, $r > 0$ to be the maximal constant satisfying (4.2). By the maximality of r, for any $\varepsilon > 0$ the set

$$\Sigma_\varepsilon := \big\{\alpha_\eta(x) \geq r + \varepsilon\big\}$$

is such that $\mu - \mathscr{H}^1 \llcorner \Sigma_\varepsilon > 0$. Defining then $\mu_\varepsilon := \mathscr{H}^1 \llcorner \Sigma_\varepsilon$ and recalling (4.8), we immediately deduce

$$C_{\mu_\varepsilon}(\eta) \leq C_\mu(\eta) + \|\mu - \mu_\varepsilon\|\big(r + \varepsilon\big)\,. \tag{4.20}$$

On the other hand, since H is concave we can evaluate

$$H\big(\mathscr{H}^1(\Sigma_\varepsilon)\big) = H\big(\|\mu\| - (\|\mu - \mu_\varepsilon\|)\big) \leq H(\|\mu\|) - H'_-(\|\mu\|)\|\mu - \mu_\varepsilon\|\,, \quad (4.21)$$

where $H'_-(s)$ is the left derivative of H at $s \in \mathbb{R}^+$, which is well-defined since H is concave. Since $MK(\mu) = C_\mu(\eta)$ because η is an optimal t.p.m., $\mathfrak{F}(\mu) \leq \mathfrak{F}(\mu_\varepsilon)$ because μ is an optimal measure, and $MK(\mu_\varepsilon) \leq C_{\mu_\varepsilon}(\eta)$, we deduce by (4.20) and (4.21) that

$$r + \varepsilon \geq H'_-\big(\|\mu\|\big)\,.$$

Since ε is arbitrary, we obtain

$$H'_-\big(\|\mu\|\big) \leq r\,. \quad (4.22)$$

Let us define now

$$\overline{\Sigma} := \big\{\alpha_\eta(x) \geq r\big\}\,,$$

which is a closed set by the upper semicontinuity of α_η; notice that $\overline{\Sigma}$ contains the support of μ, but it may also be strictly bigger. Let also $\bar\mu := \mathscr{H}^1 \llcorner \overline{\Sigma}$; if $\|\bar\mu - \mu\| = 0$, then $\overline{\Sigma}$ is an optimal closed set and hence there is nothing to prove. Otherwise, again by (4.8) we know that

$$C_{\bar\mu}(\eta) = C_\mu(\eta) - \int_{\overline{\Sigma}} \alpha_\eta(x)\, d(\bar\mu - \mu)(x)\,.$$

By definition, $\alpha_\eta \geq r$ on $\operatorname{spt}(\bar\mu - \mu)$, so that the last estimate becomes

$$C_{\bar\mu}(\eta) \leq C_\mu(\eta) - r\,\|\bar\mu - \mu\|\,.$$

On the other hand, again by the concavity of H one has

$$H\big(\mathscr{H}^1(\overline{\Sigma})\big) = H\big(\|\bar\mu\|\big) \leq H(\|\mu\|) + H'_-(\|\mu\|)\,\|\bar\mu - \mu\|\,. \quad (4.23)$$

Summarizing, we obtain

$$\begin{aligned}\mathfrak{F}(\bar\mu) &\leq C_{\bar\mu}(\eta) + H\big(\mathscr{H}^1(\overline{\Sigma})\big) \leq C_\mu(\eta) + H(\|\mu\|) + (H'_-(\|\mu\|) - r)\,\|\bar\mu - \mu\| \\ &= \mathfrak{F}(\mu) + (H'_-(\|\mu\|) - r)\,\|\bar\mu - \mu\| \leq \mathfrak{F}(\mu)\,,\end{aligned} \quad (4.24)$$

the last inequality being valid in view of (4.22). Since μ is optimal, we have proved that also $\bar\mu = \mathscr{H}^1 \llcorner \overline{\Sigma}$ is an optimal measure and then that $\overline{\Sigma}$ is a closed optimal set: thus the existence of some closed optimal set is established.

Concerning the second part of the claim, if H is strictly concave and $\|\bar\mu - \mu\| > 0$ then the first inequality in (4.23) becomes strict; as a consequence, also the inequality (4.24) is strict and this gives a contradiction with the optimality of μ. Hence, we may conclude that $\mu = \bar\mu$ and therefore a generic optimal measure corresponds to a closed optimal set. $\square$

Recall that Theorem 3.27 already ensures that all the optimal measures are represented by sets if $D(\cdot, b)$ is strictly concave. The above Theorem 4.21 says that in the case $A(t) = t$, $B(t) = 0$, in which $D(a, b) = a$ is not strictly concave, it is still true that all the optimal measures are sets whenever H is strictly concave.

The following example shows that the hypothesis of concavity of H in Theorem 4.21 is essential.

EXAMPLE 4.22. Let

$$\Omega := [-2,2] \times [-1,1] \subseteq \mathbb{R}^2 ,$$

and let $a_n < b_n$ be such that all the open segments (a_n, b_n) are disjoint, and their union $D := \cup_j (a_j, b_j)$ contains the set of all the rational numbers in $[-1,1]$ and has length 1. Define now

$$f^+ := \sum_{n\in\mathbb{N}} 2^{-n} \delta_{(a_n,0)} , \qquad\qquad f^- := \sum_{n\in\mathbb{N}} 2^{-n} \delta_{(b_n,0)} ,$$

let θ_n be the line segment connecting $(a_n, 0)$ to $(b_n, 0)$, and let η be any t.p.m. related to f^+ and f^-. Since for any n one has

$$\begin{aligned} f^+\big([-2,a_n]\big) &= \sum_{\{j\,:\,a_j \le a_n\}} 2^{-j} = \sum_{\{j\,:\,a_j < b_n\}} 2^{-j} > \sum_{\{j\,:\,b_j < a_n\}} 2^{-j} \\ &= f^-\big([-2,a_n]\big) , \end{aligned}$$

there must be some $\theta \in \operatorname{spt} \eta$ connecting some $a_i \le a_n$ to some $b_j \ge b_n$. This implies that the projection on the first axis of the set

$$S_\eta = \bigcup_{\theta \in \operatorname{spt} \eta} \theta \subseteq \mathbb{R}^2$$

contains the whole segment $[a_n, b_n]$. Let now

$$\bar\eta := \sum_{n\in\mathbb{N}} 2^{-n}\, \delta_{\theta_n} ,$$

which is clearly a t.p.m. related to f^+ and f^-, let $\Sigma := \cup_{n\in\mathbb{N}}\theta_n$ and define the function H by the relationship

$$H(l) := \begin{cases} 0, & l \le 1 , \\ \infty, & l > 1 . \end{cases}$$

Obviously, H is not concave. Notice that $C_\Sigma(\bar\eta) = 0$ and that $H(\mathscr{H}^1(\Sigma)) = 0$, so that $\mathfrak{F}(\Sigma) = 0$. Consider now any $\mu \le \mathscr{H}^1$ such that $\mathfrak{F}(\mu) = 0$. One has then $\|\mu\| \le 1$ and $C_\mu(\eta) = 0$ for some t.p.m. η. This implies $\mu(\theta) \ge \mathscr{H}^1(\theta)$ for η-a.e. $\theta \in \Theta$, and hence, minding the semicontinuity of μ and $\mathscr{H}^1$ over Θ, actually for all $\theta \in \operatorname{spt}\eta$. Therefore, one has

$$\mu \llcorner \theta = \mathscr{H}^1 \llcorner \theta , \qquad\qquad \text{for all } \theta \in \operatorname{spt}\eta$$

(because $\mu \le \mathscr{H}^1$). Since for any n there is a path $\theta_n \in \operatorname{spt}\eta$ connecting some $a_i \le a_n$ with some $b_j \ge b_n$, then calling π the projection on the first axis one has

$$\begin{aligned} 1 &\ge \|\mu\| \ge \mu(\cup_n \theta_n) = \mathscr{H}^1(\cup_n\theta_n) \ge \mathscr{H}^1\big(\pi(\cup_n\theta_n)\big) \ge \mathscr{H}^1\big(\cup_n [a_n,b_n]\big) \\ &= \sum_n |b_n - a_n| = 1 , \end{aligned}$$

from which we derive that $\mu = \mathscr{H}^1 \llcorner \Sigma$. Summarizing, $\mathfrak{F}(\mu) > 0$ for any $\mu \neq \mathscr{H}^1 \llcorner \Sigma$. This shows that $\mathscr{H}^1 \llcorner \Sigma$ is the unique optimal measure, so that Σ is the unique (up to a an $\mathscr{H}^1$-negligible set) optimal set; however, this set is not essentially closed: indeed, it is an open set of length 1 dense in $[-1,1]$.

4. Number of connected components of optimal sets

As already pointed out, it would be interesting to understand if there is an optimal set Σ having finitely many, or at least countably many, connected components. We however can immediately notice that this is not always the case, according to the following counterexample similar to Example 4.22.

Example 4.23. Let $\Omega = [-2,2] \times [-1,1] \subseteq \mathbb{R}^2$. Consider now the following construction of a Cantor-type set in $[0,1]$ having strictly positive length. Given a sequence $\{p_n\}_{n\in\mathbb{N}}$ with $0 < p_n < 1/2$ for any n, we first divide $[0,1]$ in the three segments

$$[\alpha_{1,1}, \beta_{1,1}]\,, \qquad [\beta_{1,1}, \alpha_{1,2}]\,, \qquad [\alpha_{1,2}, \beta_{1,2}]$$

having length p_1, $1-2p_1$ and p_1 respectively, defining

$$\alpha_{1,1} := 0\,, \qquad \beta_{1,1} := p_1, \qquad \alpha_{1,2} := 1-p_1, \qquad \beta_{1,2} := 1\,.$$

We continue by induction, defining $\alpha_{n,m}$ and $\beta_{n,m}$ for any $n \in \mathbb{N}$ and any $1 \leq m \leq 2^n$ in such a way that any segment $[\alpha_{n,m}, \beta_{n,m}]$ is divided in the three segments

$$[\alpha_{n+1,2m-1}, \beta_{n+1,2m-1}]\,, \quad [\beta_{n+1,2m-1}, \alpha_{n+1,2m}]\,, \quad [\alpha_{n+1,2m}, \beta_{n+1,2m}]\,,$$

the lengths of which are in the ratio p_{n+1}, $1-2p_{n+1}$ and p_{n+1} respectively. We define then K_n as the closed set given by the union of the 2^n closed segments $[\alpha_{n,m}, \beta_{n,m}]$ and K as the intersection of the sets K_n. Notice that the classical Cantor set corresponds to the choice $p_n = 1/3$ for any n. Similarly to the Cantor set, K is a closed set containing more than countably many points and is totally disconnected, that is, all the connected components of K reduce to a single point. Moreover,

$$\mathscr{H}^1(K) = \lim_{n\to\infty} \mathscr{H}^1(K_n)$$

and

$$\mathscr{H}^1(K_{n+1}) = 2p_{n+1}\mathscr{H}^1(K_n)\,,$$

so that a suitable choice of the coefficients p_n allows $\mathscr{H}^1(K)$ to be any number in $[0,1)$. We consider then a sequence $\{p_n\}$ such that $\mathscr{H}^1(K) = 1/2$. Now, we define f^+ and f^- as follows:

$$f^+ := \sum_{n\in\mathbb{N}} \sum_{1\leq m\leq 2^n} 4^{-n}\delta_{(\alpha_{n,m},0)}\,, \qquad f^- := \sum_{n\in\mathbb{N}} \sum_{1\leq m\leq 2^n} 4^{-n}\delta_{(\beta_{n,m},0)}\,.$$

It is easily noticed that f^+ and f^- are two probability measures and that, setting $\theta_{n,m}$ to be the segment $[\alpha_{n,m}, \beta_{n,m}]$, the measure

$$\eta := \sum_{n\in\mathbb{N}} \sum_{1\leq m\leq 2^n} 4^{-n}\delta_{\theta_{n,m}}$$

is a t.p.m. related to f^+ and f^-. We set now

$$H(s) := \begin{cases} 0 & \text{for } 0 \leq s \leq 1/2\,, \\ \infty & \text{for } s > 1/2\,. \end{cases}$$

It is immediate to notice that any optimal measure μ must be supported in $[0,1]\times\{0\}$. Moreover, arguing as in Example 4.22, one has that given any measure μ supported in $[0,1]\times\{0\}$, all the paths $\theta_{n,m} \in \operatorname{spt}\eta'$ for every t.p.m. η', and hence, the t.p.m. η defined above is optimal for the cost C_μ. It is also clear that the function α_η achieves its maximum exactly in the set K. Hence, the measure $\mu := \mathscr{H}^1 \llcorner K$ is the unique optimal measure, by Theorem 4.5 and by the choice of H. Therefore, K is an optimal set which has more than countably many connected components, since it is totally disconnected and uncountable. The optimal sets are exactly those coinciding with K up to $\mathscr{H}^1-$negligible sets; it is easily noticed that any such set K' is still totally disconnected.

Remark 4.24. By a more careful construction, replacing Dirac masses in the above example by smooth functions with "small" supports, it is possible to obtain a counterexample as before with absolutely continuous measures $f^\pm$ with densities belonging even to C^∞.

We give now conditions for the existence of an optimal set having finite or countably many connected components: as in Theorem 4.21, a concavity hypothesis on H will be essential. First, we prove the following *a posteriori* result.

Lemma 4.25. *Let μ be an optimal measure, η an ordered optimal t.p.m. for μ, and assume that the constant r satisfying (4.2) is not unique. Then,*

$$\mu = \mathscr{H}^1 \llcorner \Sigma$$

for some closed set Σ such that $\Sigma\cap\theta$ has finitely many connected components for all $\theta \in \operatorname{spt}\eta$.

Proof. First of all we stress that, since the constant r in (4.2) is not unique, then μ corresponds to a set: indeed, (4.2) is satisfied by all $r_1 < r < r_2$, so that

$$\mathscr{H}^1\big(\{\alpha_\eta(x) = r\}\big) = 0$$

for any such r and thus $\mu = \mathscr{H}^1 \llcorner \Sigma$ with $\Sigma = \{\alpha_\eta(x) \geq r\}$. Notice that Σ is closed since, by Lemma 4.4, α_η is u.s.c..

Take now a single path $\theta \in \operatorname{spt}\eta$: we define σ as the shortest subpath of θ such that

$$\sigma\cap\Sigma = \theta\cap\Sigma\,.$$

We claim first that the function

$$\alpha_\eta \circ \sigma : t \in [0,1] \mapsto \alpha_\eta(\sigma(t))$$

has bounded variation. This amounts to proving that the weak derivative $(\alpha_\eta \circ \sigma)'$ is a finite measure, where the derivative $(\alpha_\eta \circ \sigma)'$ is intended as the weak limit of

$$\frac{\alpha_\eta(\sigma(s+\varepsilon)) - \alpha_\eta(\sigma(s))}{\varepsilon}$$

as $\varepsilon \to 0$. Define for this purpose Θ' to be the set of all $\alpha \in \operatorname{spt} \eta$ such that $\mu(\alpha \cap \sigma) > 0$. By Lemma 4.20 we deduce that, for η-a.e. $\alpha \in \Theta'$, defining $p(\alpha)$ and $q(\alpha)$ to be the first and the last point of α intersecting σ respectively, the subpath $\widetilde{pq}$ of α connecting $p(\alpha)$ and $q(\alpha)$ is entirely contained in σ. In other words, there are almost no paths entering in σ, then exiting, then entering again, according to what we called "solidarity" in Section 2. As an easy consequence, for any x and y in σ,

$$\alpha_\eta(y) - \alpha_\eta(x) = \eta\big(\{\alpha \in \Theta' :\ p(\alpha) \in \widetilde{xy}\}\big) - \eta\big(\{\alpha \in \Theta' :\ q(\alpha) \in \widetilde{xy}\}\big)\,,$$

that is,

$$\alpha_\eta' = p_\# \eta - q_\# \eta\,.$$

Hence, $(\alpha_\eta \circ \sigma)'$ is a finite (signed) measure and the claim is shown.

Now, take $r_1 > r_2$ such that (4.2) is satisfied both for r_1 and for r_2, and take a point $x \in \partial\big(\Sigma \cap \sigma\big)$ where ∂ stands for the boundary relative to σ. Since x belongs to the relative boundary of Σ in σ, then for any $\varepsilon > 0$ there is a point $z_\varepsilon \in \sigma \cap \Sigma$ and a point $w_\varepsilon \in \sigma \setminus \Sigma$ having distance less than ε from x. Using (4.2) with the points z_ε and w_ε, and the constants r_1 and r_2 respectively, we notice that $\alpha_\eta(z_\varepsilon) \geq r_1$ and that $\alpha_\eta(w_\varepsilon) \leq r_2$. Since ε is arbitrary, it follows that α_η has, at the point x, a jump of at least $r_1 - r_2$, i.e.,

$$\big|(\alpha_\eta \circ \sigma)'(\{x\})\big| \geq r_1 - r_2\,.$$

Since, as proved above, $\alpha_\eta \circ \sigma$ is a function of bounded variation, we have that $\partial\Sigma$ contains only finitely many points. Therefore, $\Sigma \cap \sigma$ has only a finite number of connected components and the proof is concluded. $\square$

COROLLARY 4.26. *If μ is an optimal measure, η is an ordered optimal t.p.m. for μ, and the constant r satisfying (4.2) is not unique, then $\mu = \mathscr{H}^1 \llcorner \Sigma$ for some closed set Σ having countably many connected components.*

PROOF. By the non-uniqueness assumption, (4.2) holds with some $r > 0$. Hence, by Lemma 4.7, μ is concentrated in the union of at most countably many geodesics, which are Lipschitz curves of uniformly bounded length. Therefore, Σ is contained in the (at most countable) union of θ_i, where each θ_i is contained in $\operatorname{spt} \eta$ and $\mathscr{H}^1(\Sigma \cap \theta_i) > 0$ for any i. The conclusion now follows directly from Lemma 4.25. $\square$

REMARK 4.27. Arguing as in Corollary 4.26, we can deduce that Σ has a finite number of connected components, instead of a countable number, whenever we know that Σ is contained in finitely many geodesics with respect to d_Σ. For instance, this is the case when the conditions of Theorem 5.1 hold: in this case Σ is contained in finitely many geodesics, and hence the non-uniqueness of r allows us to obtain that Σ has finitely many connected components.

We prove now a result ensuring the existence of at most countably many connected components of an optimal set Σ whenever H is concave.

THEOREM 4.28. *If H is concave, then there exists an optimal set Σ having at most countably many connected components. Moreover, if H is strictly concave, then any optimal measure μ corresponds to an optimal set having at most countably many connected components.*

PROOF. Consider an optimal measure μ, an ordered optimal t.p.m. η for C_μ and a constant r satisfying (4.2). We show the first part of the claim considering separately three possible cases.

<u>*Case I.*</u> $H'_+(\|\mu\|) < r$.

Let us write

$$H'_+(\|\mu\|) = r - \varepsilon$$

with some $\varepsilon > 0$: we claim that the measure

$$\tilde{\mu} := \mathscr{H}^1 \llcorner \{x \in \Omega : \, r - \varepsilon < \alpha_\eta(x)\}$$

coincides with μ. Indeed, clearly $\tilde{\mu} \geq \mu$. Moreover, if

$$\|\tilde{\mu} - \mu\| = \delta > 0 \,,$$

then by the concavity of H one has

$$H(\|\tilde{\mu}\|) - H(\|\mu\|) \leq (r - \varepsilon)\delta \,,$$

while on the other hand, by (4.8) we deduce

$$C_{\tilde{\mu}}(\eta) < C_\mu(\eta) - (r - \varepsilon)\delta \,.$$

Summarizing, we have $\mathfrak{F}(\tilde{\mu}) < \mathfrak{F}(\mu)$ which contradicts the optimality of μ. Therefore, we obtain that $\tilde{\mu} = \mu$, but then any constant between $r - \varepsilon$ and r suits for (4.2). Thus Corollary 4.26 ensures that μ corresponds to a set with countably many connected components.

<u>*Case II.*</u> $H'_-(\|\mu\|) > r$.

This case is analogous to the previous one: we write

$$H'_-(\|\mu\|) = r + \varepsilon$$

for some $\varepsilon > 0$ and set

$$\tilde{\mu} := \mathscr{H}^1 \llcorner \{x \in \Sigma : \, r + \varepsilon < \alpha_\eta(x)\} \,.$$

This time $\tilde{\mu} \leq \mu$, and exactly as before we find that $\tilde{\mu} = \mu$ since otherwise $\tilde{\mu}$ would be strictly better than the optimal measure μ. Again, the constant suitable for (4.2) is not unique and Corollary 4.26 allows to conclude.

<u>*Case III.*</u> $H'(\|\mu\|) = r$.

In this last case, we can assume that $\mu = \mathscr{H}^1 \llcorner \Sigma$ for some Borel set Σ

thanks to Theorem 3.27. Noticing that $r > 0$ since by hypothesis H is concave and unbounded, we may argue as in Corollary 4.26 and deduce that

$$\Sigma \subseteq \bigcup_i \theta_i$$

for countably many geodesics $\theta_i \in \operatorname{spt} \eta$. Denote by σ_i the smallest subpath of θ_i such that

$$\sigma_i \cap \Sigma = \theta_i \cap \Sigma\,,$$

and write Σ_i° to denote the interior part of $\Sigma \cap \sigma_i$ relative to σ_i. Define then the "boundary" $\partial\Sigma$ of Σ to be the union of the boundaries $\Sigma \cap \sigma_i$ relative to σ_i. Take then a point $x \in \partial\Sigma$ with $\alpha_\eta(x) \neq r$: arguing as at the end of the proof of Lemma 4.25, we deduce that it is a discontinuity point for α_η. But since α_η is a function of bounded variation on each σ_i, we know that it can have at most countably many discontinuity points, and hence we deduce that $\alpha_\eta(x) = r$ for $\mathscr{H}^1-$a.e. point of $\partial\Sigma$. Considering then $\Sigma^\circ := \Sigma \setminus \partial\Sigma$ and writing $\delta := \mathscr{H}^1(\Sigma \cap \partial\Sigma)$, one has

$$H(\mathscr{H}^1(\Sigma^\circ)) \leq H(\mathscr{H}^1(\Sigma)) - r\delta$$

again by the concavity of H. Moreover

$$C_{\Sigma^\circ}(\eta) = C_\Sigma(\eta) + r\delta\,,$$

and therefore we find $\mathfrak{F}(\Sigma^\circ) \leq \mathfrak{F}(\Sigma)$. Hence, Σ° is an optimal set. But for each i the set Σ_i° is open relative to θ_i and, since $\Sigma^\circ = \bigcup_i \Sigma_i^\circ$, then Σ° is a countable union of open subpaths of the curves θ_i. Therefore Σ° is an optimal set having at most countably many connected components.

We have then shown the first part of the Theorem. Consider now what happens if H is strictly concave: we need to prove that in this case the generic optimal measure μ corresponds to a set, and that this set has at most countably many connected components. If one of the two cases I or II above occurs, we have already shown that μ corresponds to a set with countably many connected components without any need of the strict concavity of H, hence there is nothing to prove. We need therefore to consider what happens in the case III. We affirm what follows:

If H is strictly concave and the case III above occurs, then

$$\mathscr{H}^1\big(\{\alpha_\eta(x) = r\}\big) = 0\,.$$

Notice that this claim will give the thesis: indeed, if

$$\mathscr{H}^1(\{\alpha_\eta(x) = r\}) = 0$$

then

$$\{\alpha_\eta(x) \geq r\} = \{\alpha_\eta(x) > r\}$$

in the $\mathscr{H}^1-$sense, so by (4.2) we deduce that $\mu = \mathscr{H}^1 \llcorner \Sigma$ with

$$\Sigma := \{\alpha_\eta(x) \geq r\}\,.$$

In particular, since we proved above that $\alpha_\eta(x) = r$ for all but countably many $x \in \partial\Sigma$, one has

$$\big\{\alpha_\eta(x) > r\big\} \subseteq \Sigma^\circ \subseteq \big\{\alpha_\eta(x) \geq r\big\},$$

where the first inclusion is true up to a countable (hence $\mathscr{H}^1$-negligible) set. Thus, one has

$$\Sigma = \big\{\alpha_\eta(x) \geq r\big\} = \big\{\alpha_\eta(x) > r\big\} = \Sigma^\circ$$

where the equalities are understood up to an $\mathscr{H}^1$-negligible set, and hence $\mu = \mathscr{H}^1 \llcorner \Sigma^\circ$ which gives the thesis.

We are then reduced to prove the above claim. For this purpose, we assume

$$\mathscr{H}^1(\big\{\alpha_\eta(x) = r\big\}) > 0$$

and we find a contradiction in the following two cases, at least one of which has to occur.

Case III.A. $\|\mathscr{H}^1 \llcorner \{\alpha_\eta(x) = r\} \wedge \mu\| = \delta > 0$.
In this case, define

$$\tilde{\mu} := \mu - \big(\mathscr{H}^1 \llcorner \{\alpha_\eta(x) = r\} \wedge \mu\big).$$

One has that

$$C_{\tilde{\mu}}(\eta) = C_\mu(\eta) + r\delta, \text{ and } H(\|\tilde{\mu}\|) < H(\|\mu\|) - r\delta,$$

because H is strictly concave and $\delta > 0$. This implies $\mathfrak{F}(\tilde{\mu}) < \mathfrak{F}(\mu)$ which is a contradiction.

Case III.B. $\|\mathscr{H}^1 \llcorner \{\alpha_\eta(x) = r\} - \mu \llcorner \{\alpha_\eta(x) = r\}\| = \delta > 0$.
In this case, define

$$\tilde{\mu} := \mathscr{H}^1 \llcorner \{\alpha_\eta(x) \geq r\}.$$

One has that

$$C_{\tilde{\mu}}(\eta) = C_\mu(\eta) - r\delta, \text{ and } H(\|\tilde{\mu}\|) < H(\|\mu\|) + r\delta,$$

because H is strictly concave and $\delta > 0$. This implies $\mathfrak{F}(\tilde{\mu}) < \mathfrak{F}(\mu)$ which is a contradiction. □

Remark 4.29. In the case when H is strictly concave, applying to an arbitrary optimal measure μ both Theorems 4.21 and 4.28 one finds that $\mu = \mathscr{H}^1 \llcorner \Sigma$ with Σ closed and $\mu = \mathscr{H}^1 \llcorner \Sigma'$ with Σ' having countably many connected components. Clearly, one has then

$$\mathscr{H}^1(\Sigma \Delta \Sigma') = 0,$$

but Σ and Σ' could be different. Indeed, consider the classical Cantor "middle third" set

$$C := [0,1] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i) \subseteq [0,1],$$

where (a_i, b_i) are the "middle third" intervals, and set

$$S := \bigcup_{i=1}^{\infty} (a_i, a_i + 8^{-i}) \cup (b_i - 8^{-i}, b_i)\,.$$

Then

$$\bar{S} = S \cup \bigcup_{i=1}^{\infty} \{a_i + 8^{-i}, b_i - 8^{-i}\} \cup C\,,$$

and hence, $\mathscr{H}^1(\bar{S}) = \mathscr{H}^1(S)$. Further, every singleton $\{x\}$ with

$$x \in C \setminus \bigcup\nolimits_i \{a_i, b_i\}$$

is a connected component of $\bar{S}$, and therefore, S has more than countably many connected components. Now, in a way similar to Example 4.30 below, one can find two measures f^+ concentrated on the set

$$\bigcup_{i=1}^{\infty} \{a_i\} \times \{0\}$$

and f^- concentrated on the set

$$\bigcup_{i=1}^{\infty} \{b_i\} \times \{0\}\,,$$

and a function H, such that the only optimal measure is given by

$$\mu = \mathscr{H}^1 \llcorner \Sigma = \mathscr{H}^1 \llcorner \bar{\Sigma}\,,$$

where $\Sigma := S \times \{0\}$. On the other hand, Σ has countably many connected components but is not closed, and its closure $\bar{\Sigma} = \bar{S} \times \{0\}$ has more than countably many connected components.

The following example similar to Examples 4.22 and 4.23 shows that the result of the above Theorem 4.28 is optimal in the sense that in some situations no optimal set has finitely many connected components, even if the function H is strictly concave.

Example 4.30. Let $\Omega \subseteq \mathbb{R}^2$ be a domain containing the segment $[0,1] \times \{0\}$, and define f^+ and f^- as follows:

$$f^+ := \frac{1}{2}\delta_{(0,0)} + \sum_{n \geq 1} \frac{1}{2^{n+1}} \delta_{(2/3^n, 0)}\,,$$

$$f^- := \frac{1}{2}\delta_{(1,0)} + \sum_{n \geq 1} \frac{1}{2^{n+2}} \big(\delta_{(1/3^n, 0)} + \delta_{(3/3^n, 0)}\big)\,.$$

We will choose later a strictly concave function H such that $H'(s) > 0$ for all $s > 0$. As in the previous examples it is easy to deduce that all the optimal measures must be supported in $[0,1] \times \{0\}$. Moreover, there is a t.p.m. which is optimal for all measures supported in $[0,1]\times\{0\}$, and hence for all optimal measures. Such a t.p.m. is given by the formula

$$\eta = \frac{1}{4}\,\delta_{[0,1]} + \sum_{n\geq 1} \frac{1}{2^{n+2}}\left(\delta_{[2\cdot 3^{-n},3\cdot 3^{-n}]} + \delta_{[2\cdot 3^{-n},1]} + \delta_{[0,3^{-n}]}\right),$$

where $\delta_{[p,q]}$ is the Dirac mass concentrated on the element of Θ given by the segment connecting $(p,0)$ to $(q,0)$ (note that for every measure μ supported in $[0,1]\times\{0\}$ there must be an optimal t.p.m. which moves f^+ along $[0,1]\times\{0\}$ from left to right, and that η is the unique t.p.m. with the latter property). One can then write down the function α_η, which is given by $\alpha_\eta(x,y)=0$ for any $y\neq 0$, and

$$\alpha_\eta(x,0) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2^{n+2}} & \text{for } 1/3^n \leq x \leq 2/3^n\,, \\ \dfrac{1}{2} + \dfrac{1}{2^{n+2}} & \text{for } 2/3^n \leq x \leq 3/3^n\,. \end{cases} \tag{4.25}$$

Keeping in mind Remark 4.9, and in particular the formula (4.9) for β, we deduce that

$$\beta'(s) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2^{n+2}} & \text{for } \frac{1-1/3^{n-1}}{2} \leq s \leq \frac{1-1/3^n}{2}\,, \\ \dfrac{1}{2} - \dfrac{1}{2^{n+2}} & \text{for } \frac{1+1/3^n}{2} \leq s \leq \frac{1+1/3^{n-1}}{2}\,. \end{cases}$$

Notice that β is concave according to Remark 4.9. We can now define H setting $H(0):=0$ and $H'(s)$ equal to

$$\begin{cases} \dfrac{1}{2} + \dfrac{1}{2^{n+2}} + \dfrac{3^n}{2^{n+2}}\Big(\dfrac{1-1/3^n}{2} - x\Big) & \text{for } \dfrac{1-1/3^{n-1}}{2} \leq s \leq \dfrac{1-1/3^n}{2}\,, \\ \dfrac{1}{2} - \dfrac{1}{2^{n+3}} - \dfrac{3^n}{2^{n+3}}\Big(x - \dfrac{1+1/3^n}{2}\Big) & \text{for } \dfrac{1+1/3^n}{2} \leq s \leq \dfrac{1+1/3^{n-1}}{2}\,. \end{cases}$$

According to the definition above, H is a strictly concave function. Moreover, one has $H'(s)<\beta'(s)$ for all $0\leq s<1/2$, while $H'(s)>\beta'(s)$ for all $s>1/2$. Therefore, the function $s\mapsto H(s)-\beta(s)$ has a unique minimum in $s=1/2$. Recalling Remark 4.9 and the fact that η is an optimal t.p.m. with respect to any optimal measure μ, we derive that the optimal measures are exactly the elements of $\mathcal{M}_{1/2}$. Finally, formula (4.25) for α_η ensures that the unique element of $\mathcal{M}_{1/2}$ is $\mu=\mathscr{H}^1\llcorner\Sigma$, where

$$\Sigma := \Big(\bigcup\nolimits_{n\in\mathbb{N}}\Big[\frac{2}{3^n},\frac{3}{3^n}\Big]\Big)\times\{0\}\,,$$

which clearly has countably many connected components. It is worth noting that $\overline{\Sigma}:=\Sigma\cup\{0\}$ is still an optimal set having countably many connected components and is also closed. However, there is no optimal set with only a finite number of connected components, since for every Borel set

$$\Sigma'\subseteq[0,1]\times\{0\}$$

with such property one has $\mathscr{H}^1(\Sigma\Delta\Sigma') > 0$.

CHAPTER 5

Optimal sets and geodesics in the two-dimensional case

This chapter is dedicated to the analysis of the two-dimensional case under the assumptions $A \equiv \mathrm{Id}$, $B \equiv 0$ and with no extra requirements on the function H. We will prove that in this case, if $f^+, f^- \in L^\infty(\Omega)$ (i.e. the measures f^+ and f^- are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ and their densities, that we still denote by f^+ and f^-, belong to $L^\infty(\Omega)$), then every optimal set Σ is contained in a finite number of geodesics with respect to d_Σ. Note that this result cannot be considered an improvement of Theorem 4.28, since we will not show Σ to have finitely many connected components, but only to be contained in a finite union of connected paths. In fact, in Example 4.23 we showed a situation in which the optimal set Σ is contained in the single geodesic $[0,1]\times\{0\}$, but is totally disconnected.

THEOREM 5.1. *If $A(t) = t$, $B(t) = 0$, $\Omega \subseteq \mathbb{R}^2$ and $f^\pm \in L^\infty(\Omega)$, then any optimal set Σ is contained in a finite number of Lipschitz paths of bounded lengths, which are geodesics with respect to Σ. More precisely, there exists an ordered optimal t.p.m. η such that Σ is contained in finitely many paths $\theta \in \operatorname{spt}\eta$.*

To prove this result, we first fix an optimal set Σ and denote $\mu := \mathscr{H}^1 \llcorner \Sigma$. Then, given any optimal ordered t.p.m. $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, we recall that by Remark 4.6 there is a maximal constant $r > 0$ satisfying (4.2), so that one has in particular

$$\left\{\alpha_\eta(x) \geq r\right\} \subseteq \Sigma \subseteq \left\{\alpha_\eta(x) > r\right\}.$$

Our first goal is to show the following result.

PROPOSITION 5.2. *With the above definitions one has $r > 0$.*

This is a crucial step to obtain Theorem 5.1. The proof of Proposition 5.2 is quite involved; we dedicate the whole next section to show that the assumption $r = 0$ leads to a contradiction.

1. Preliminary constructions

In this section we show that the hypothesis $r = 0$ leads to a contradiction. To do that, we perform a careful analysis of the consequences of this case.

We stress that all the results of this section strongly use the (contradictory) assumption $r = 0$, hence they may not hold true in general. Recall that throughout this section we assume fixed an optimal t.p.m. $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, so that $MK(\Sigma) = C_\Sigma(\eta)$.

LEMMA 5.3. *Under the (contradictory) assumption $r = 0$, given a path $\sigma \in \operatorname{spt}\eta$ such that $\mu(\sigma) > 0$, and defining x' and y' as in Lemma 4.20, the whole open arc $\widetilde{x'y'}$ is contained in Σ.*

PROOF. Recalling the proof of Lemma 4.20, we have that in particular

$$\alpha_\eta(z) > 0 \qquad\qquad \forall\, z \in \widetilde{x'y'}\ :$$

indeed, for any such z, an $\eta-$positive quantity of paths close to σ passes through z. By definition of r and since $r = 0$, it follows that

$$z \in \Sigma \qquad\qquad \forall\, z \in \widetilde{x'y'}\,.$$

So the claim follows. □

REMARK 5.4. We notice that Σ is contained in countably many paths belonging to $\operatorname{spt}\eta$: this follows by Remark 4.8 since in the hypotheses of Theorem 5.1 condition (3.39) holds. Therefore, by Lemma 5.3 we infer that Σ is made by countably many connected components $\{\Sigma_i\}_{i\in\mathbb{N}}$; in particular, any path in $\operatorname{spt}\eta$ may intersect at most one of the sets Σ_i.

It is well-known that $\mathscr{H}^1(\Sigma) = \mathscr{H}^1(\overline{\Sigma})$ whenever Σ is a connected set. Hence, denoting by Σ_i, $i = 1, 2, \dots$ the connected components of Σ, we may consider each Σ_i to be closed. Define now

$$\begin{aligned} &\Theta_i := \{\theta \in \operatorname{spt}\eta\ :\ \theta \cap \Sigma_i \neq \emptyset\}, && i = 1, 2, \dots\,, \\ &\Theta_0 := \operatorname{spt}\eta \setminus \cup_i \Theta_i \end{aligned}$$

(observe that all Θ_i, $i \neq 0$, are closed since so are assumed Σ_i). We obtain from Lemma 5.3 that all Θ_i are pairwise disjoint, so that one can write

$$\eta = \sum_{i=0}^{\infty} \eta_i, \ \text{ where } \eta_i := \eta \llcorner \Theta_i\,.$$

Note that each path $\theta \in \operatorname{spt}\eta_0$ is a line segment connecting $\theta(0)$ to $\theta(1)$ (since so is every $\theta \in \Theta_0$, because such a path by definition does not touch Σ and is a geodesic). On the other hand, every path $\theta \in \operatorname{spt}\eta_i$ is made by three parts: the first one is a line segment connecting $\theta(0)$ to the point $s(\theta) \in \Sigma_i$ where θ touches Σ for the first time, the second part is a path inside Σ_i, and the third is a line segment between a point of Σ and $\theta(1)$. Notice that the function $s : \operatorname{spt}\eta_i \to \Sigma_i$ is well-defined and measurable. It is also worth observing that $s(\theta)$ is one of the points of Σ_i of least distance from $\theta(0)$. We can then define a sort of "entry measure" by setting

$$\psi := s_\#\eta_i \in \mathcal{M}^+(\Sigma_i)\,.$$

Note that ψ measures the mass entering inside Σ_i, and the above definition makes sense only thanks to the (contradictory) assumption $r = 0$: in fact, in the above construction, we used in a crucial way Lemma 5.3 which implies that each path of $\operatorname{spt}\eta$ may intersect only one of the connected components Σ_i of Σ, while this property may be not true if $r > 0$.

LEMMA 5.5. *The measure ψ is non-atomic.*

PROOF. Suppose by contradiction that there exists a $z \in \Sigma_i$ such that

$$\psi(\{z\}) > 0 .$$

Since $r = 0$, we can take $\Sigma' \subseteq \Sigma$ such that $\mathscr{H}^1\big(\Sigma \setminus \Sigma'\big) = \varepsilon$ and

$$\alpha_\eta(x) < \frac{\psi(\{z\})}{2(2\pi + 1)} \qquad \text{for any } x \in \Sigma \setminus \Sigma' \tag{5.1}$$

(notice that we do not require $\Sigma \setminus \Sigma'$ to be contained in Σ_i). We define

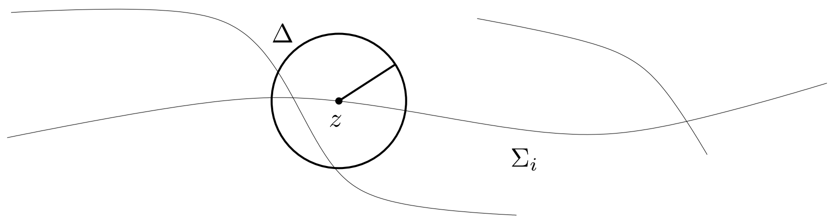

FIGURE 1. Definition of Δ in Lemma 5.5

now the set Δ, drawn in thick lines in Figure 1, as the union of the circle centered in z of radius $\varepsilon/(2\pi+1)$ and a radial segment connecting the circle to z. Then $\mathscr{H}^1(\Delta) = \varepsilon$, so that $\Sigma'' := \Sigma' \cup \Delta$ has the same length as Σ. We introduce now a new t.p.m. $\tilde\eta := \rho_\# \eta$ with the following Borel function $\rho : \Theta \to \Theta$:

- if $\theta \in \operatorname{spt}\eta_i$, $s(\theta) = z$ and
$$|\theta(0) - z| \geq \frac{\varepsilon}{2\pi + 1} ,$$
then $\rho(\theta)$ is the path following θ until it reaches the circle, then moving inside $\Delta \subseteq \Sigma''$ to reach the point z, and finally following again θ from z to $\theta(1)$;
- otherwise, $\rho(\theta) := \theta$.

Let us compute now $C_{\Sigma''}(\tilde\eta)$: for any path θ such that $\rho(\theta) = \theta$ one has

$$\delta_{\Sigma''}\big(\rho(\theta)\big) \leq \delta_{\Sigma'}(\theta) ,$$

while if θ belongs to the set

$$\Gamma := \{\theta :\, \rho(\theta) \neq \theta\} ,$$

then

$$\delta_{\Sigma''}\big(\rho(\theta)\big) = \delta_{\Sigma'}(\theta) - \frac{\varepsilon}{2\pi + 1} .$$

To verify the latter equality, just recall that each $\theta \in \Gamma$ between $\theta(0)$ and $s(\theta) = z$ is a line segment. We have

$$\begin{aligned} C_{\Sigma''}(\tilde\eta) &= \int_\Theta \delta_{\Sigma''}(\theta)\, d\rho_\# \eta(\theta) = \int_\Theta \delta_{\Sigma''}\big(\rho(\theta)\big)\, d\eta(\theta) \\ &\leq \int_\Theta \delta_{\Sigma'}(\theta)\, d\eta(\theta) - \frac{\varepsilon}{2\pi+1}\, \eta(\Gamma) \\ &= C_{\Sigma'}(\eta) - \frac{\varepsilon}{2\pi+1}\, \eta(\Gamma)\,. \end{aligned} \tag{5.2}$$

Now we compare $C_\Sigma(\eta)$ and $C_{\Sigma'}(\eta)$. By definition of Σ' and recalling (4.8) and (5.1), we obtain

$$\begin{aligned} C_{\Sigma'}(\eta) &= C_\emptyset(\eta) - \int_{\Sigma'} \alpha_\eta(x)\, d\mathscr{H}^1(x) \\ &= C_\emptyset(\eta) - \int_{\Sigma} \alpha_\eta(x)\, d\mathscr{H}^1(x) + \int_{\Sigma\setminus\Sigma'} \alpha_\eta(x)\, d\mathscr{H}^1(x) \\ &< C_\Sigma(\eta) + \frac{\psi(\{z\})}{2(2\pi+1)}\, \varepsilon\,. \end{aligned} \tag{5.3}$$

Finally, we need to estimate $\eta(\Gamma)$. By definition, we have

$$\eta(\Gamma) \geq \psi(\{z\}) - f^+\Big(B\big(z, \varepsilon/(2\pi+1)\big)\Big)\,,$$

and hence, since $f^+(\{z\}) = 0$, we can choose ε small enough in such a way that

$$\eta(\Gamma) > \frac{\psi(z)}{2}\,.$$

This, together with (5.2), (5.3) and the fact that $\mathscr{H}^1(\Sigma) = \mathscr{H}^1(\Sigma'')$, gives

$$\begin{aligned} \mathfrak{F}(\Sigma'') &\leq C_{\Sigma''}(\tilde\eta) + H\big(\mathscr{H}^1(\Sigma'')\big) \\ &\leq C_{\Sigma'}(\eta) - \frac{\varepsilon}{2\pi+1}\, \eta(\Gamma) + H\big(\mathscr{H}^1(\Sigma)\big) \\ &< C_\Sigma(\eta) + \frac{\psi(\{z\})}{2(2\pi+1)}\, \varepsilon - \frac{\varepsilon}{2\pi+1}\, \eta(\Gamma) + H\big(\mathscr{H}^1(\Sigma)\big) \\ &< C_\Sigma(\eta) + H\big(\mathscr{H}^1(\Sigma)\big) = \mathfrak{F}(\Sigma)\,, \end{aligned}$$

which leads to a contradiction since $\mu = \mathscr{H}^1 \llcorner \Sigma$ was supposed to be an optimal measure. $\square$

Remark 5.6. Notice that the proof of the above lemma did not use the hypothesis that the measures f^+, f^- belong to L^∞, but only that they be non-atomic.

We fix now arbitrarily two orthogonal axes $\hat e_1$ and $\hat e_2$ in $\mathbb{R}^2$. Let k: $\mathbb{R}^2 \to \Sigma_i$ stand for the Borel projection map onto Σ_i (the latter is well-known to be defined for $\mathscr{L}^2$-a.e., hence for $f^\pm$-a.e. $x \in \mathbb{R}^2$, namely, for all $x \in \mathbb{R}^2$, for which the distance function $x \mapsto \operatorname{dist}(x, \Sigma_i)$ is not differentiable).

Let $x \in \Sigma_i$ be such that the set $k^{-1}(x)$ belongs to some line l and is not reduced to a point. Let then

$$\omega(x) \in [-\pi/2, \pi/2)$$

stand for the angle between l and $\hat{e}_1$. We show now the following assertion.

LEMMA 5.7. *The entry angle* ω: $\Sigma_i \to [-\pi/2, \pi/2)$ *coincides* ψ*−a.e. with a Borel function.*

PROOF. Consider a point $y \in \Sigma_i$: the set $K(y)$ of all the points $x \in \Omega$ such that

$$\mathrm{dist}(x, \Sigma_i) = |x - y|$$

is convex, since it is the intersection of the half-planes

$$\big\{x \in \Omega : |x - y| \leq |x - z|\big\}$$

with $z \in \Sigma_i$. By construction, for any point x in the interior $U(y)$ of $K(y)$, the point y is the *unique* point of Σ_i where $\mathrm{dist}(x, \Sigma_i)$ is attained; therefore, all the sets $U(y)$ are disjoint. Finally, for any $y \in \Sigma_i$, we have:

- either $K(y)$ is a segment of nonzero length, and then $\omega(y)$ is well-defined;
- or $U(y)$ has a strictly positive Lebesgue measure, which may happen for at most countably many points y (and hence, on a ψ-negligible set by Lemma 5.5);
- or $K(y)$ reduces to a point, which happens only for such points $y \in \Sigma_i$, that if there is some $\theta \in \operatorname{spt}\eta$, $s(\theta) = y$, then $\theta(0) = y$ (recall that θ is a line segment between $\theta(0)$ and $s(\theta)$).

Therefore, the set $\Delta \subseteq \Sigma_i$ of the latter points has

$$\psi(\Delta) = f^+(\Delta) \leq f^+(\Sigma_i) = 0$$

since $f^+ \ll \mathscr{L}^2$. The Borel property for ω easily follows by taking a Borel selection $S : \Sigma_i \to \Theta_i$ such that

$$s(S(y)) = y \qquad\qquad \text{for } \psi - a.e.\ y \in \Sigma_i$$

and noticing that

$$\cos \omega(y) = \frac{\langle y - \big(S(y)\big)(0), \mathrm{e}_1\rangle}{\big|y - \big(S(y)\big)(0)\big|}$$

for ψ−a.e. y. □

In the sequel we will write $\omega(\theta)$ instead of $\omega(s(\theta))$. By the above lemma, in this way ω is defined for η_i-a.e. $\theta \in \Theta$. Summarizing, we know now that for ψ-a.e. point z of Σ_i there is a line passing through z and containing $\theta(0)$ for each path $\theta \in \operatorname{spt}\eta_i$ such that $s(\theta) = z$. The angle between the oriented segment $\overrightarrow{\theta(0)s(\theta)}$ and the direction $\hat{e}_1$, which we further will call "oriented entry angle", must be either equal to $\omega(\theta)$ or to $\omega(\theta) + \pi$. We will then say that a path $\theta \in \operatorname{spt}\eta_i$ "enters in Σ_i from the left" (resp. from the right), if the oriented entry angle associated to θ belongs to $[\pi/2, 3/2\pi)$ (resp. to

$[-\pi/2, \pi/2)$). We will then write $\eta_i = \eta_l + \eta_r$ where η–a.e. path in the support of η_l (resp. η_r) enters from the left (resp. from the right). As a consequence, we can write $\psi = \psi_l + \psi_r$, where

$$\psi_l := s_{\#}\eta_l\,, \qquad\qquad \psi_r := s_{\#}\eta_r\,.$$

We prove now the crucial property that $\psi_l = \psi_r$, that is for ψ–a.e. point of Σ_i the mass entering from the left and the mass entering from the right are equal.

LEMMA 5.8. *One has $\psi_l = \psi_r$.*

PROOF. Fix an arbitrary injective path $\tau \subseteq \Sigma_i$ and any point $x \in \tau$. Letting $\bar t$ be the instant such that $\tau(\bar t) = x$ and considering τ as parametrized at speed one (that is, $|\tau'| \equiv 1$, at least in a neighborhood of $\bar t$), we denote by x_σ the point $\tau(\bar t + \sigma)$, so that

$$\mathscr{H}^1\big(\widetilde{x x_\sigma}\big) = |\sigma|\,.$$

Choose now $\varepsilon > 0$, $N = N(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon, N) > 0$ and change Σ_i as follows. First of all, we cut the subpath $\widetilde{x_{-\varepsilon} x_\varepsilon}$ of τ joining $x_{-\varepsilon}$ and x_ε and shift it leftwards (that is, in the direction of $-\hat e_1$) by δ. Since in this way τ is divided in three parts, to keep the connectedness of τ we add two horizontal segments of length δ. Moreover, we notice that in this way Σ_i can become disconnected because the affluents entering in τ between $x_{-\varepsilon}$ and x_ε (*a priori*, there could be countably many of these affluents) may remain disconnected after this change. Hence, we choose $N-2$ of these affluents, namely those carrying more mass, and we connect them too with the new path τ, by means of $N-2$ horizontal segments of length δ. In this way, starting from Σ we have built a new set Σ' satisfying

$$\mathscr{H}^1\big(\Sigma'\big) \leq \mathscr{H}^1\big(\Sigma\big) + N\delta\,.$$

Figure 2 illustrates the above construction (the N segments added in Σ' are drawn darker than the rest of the figure). We want to compare now $MK\big(\Sigma'\big)$

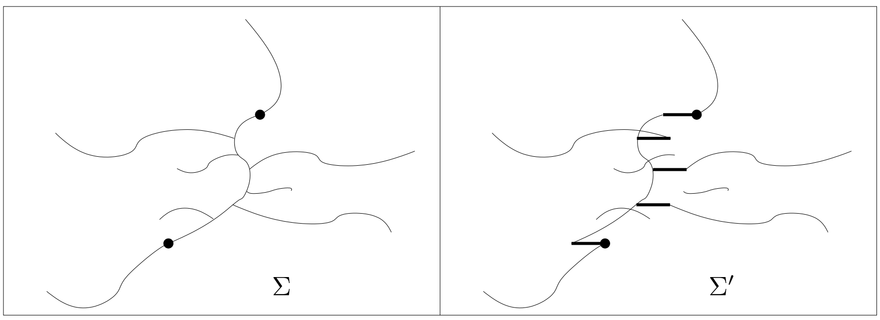

FIGURE 2. Construction of Σ' from Σ in Lemma 5.8

to $MK(\Sigma)$. To do that we use an argument similar to the one of Lemma 5.5, but a bit more involved: we choose an optimal t.p.m. $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, so that

$$MK(\Sigma) = C_\Sigma(\eta)$$

and we construct the new t.p.m. $\eta' = \rho_\# \eta$, where the map ρ: $\operatorname{spt}\eta \to \Theta$ is defined as follows. First of all, if θ does not intersect the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$, then simply $\rho(\theta) = \theta$, and hence

$$\delta_{\Sigma'}\big(\rho(\theta)\big) \leq \delta_\Sigma\big(\theta\big)\,. \tag{5.4}$$

Otherwise, we decompose $\theta = \theta_1 \cdot \theta_2 \cdot \theta_3$, where θ_1 is the part of θ between $\theta(0)$ and the first point of contact with the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$, θ_2 is the part of θ between its first and last point of contact with the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$, and, finally, θ_3 is the part of θ between its last point of contact with the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$ and $\theta(1)$. We write then $\rho(\theta) := \tilde\theta_1 \cdot \tilde\theta_2 \cdot \tilde\theta_3$, where the paths $\tilde\theta_i$ are defined as follows. The path $\tilde\theta_2$ is the leftwards shift by δ of the part of the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$ between $\theta_2(0)$ and $\theta_2(1)$, so that

$$\delta_{\Sigma'}\big(\tilde\theta_2\big) = \delta_\Sigma\big(\theta_2\big) = 0\,. \tag{5.5}$$

Concerning θ_1, there are four possibilities.

(i) If θ enters in $\widetilde{x_{-\varepsilon}x_\varepsilon}$ from $x_{-\varepsilon}$ or x_ε, then $\tilde\theta_1$ is defined as the composition of θ_1 with a horizontal segment of length δ connecting $\tilde\theta_1$ to $\tilde\theta_2$, and therefore

$$\delta_{\Sigma'}\big(\tilde\theta_1\big) = \delta_\Sigma\big(\theta_1\big)\,. \tag{5.6}$$

(ii) If θ enters in $\widetilde{x_{-\varepsilon}x_\varepsilon}$ from one of the biggest $N-2$ affluents, then we let $\tilde\theta_1$ to be again just the composition of θ_1 with a horizontal segment of length δ, and hence by construction of Σ' one still has

$$\delta_{\Sigma'}\big(\tilde\theta_1\big) = \delta_\Sigma\big(\theta_1\big)\,. \tag{5.7}$$

(iii) If θ enters in $\widetilde{x_{-\varepsilon}x_\varepsilon}$ from an affluent which is not among the biggest $N-2$, then we let $\tilde\theta_1$ to be still θ_1 composed with a horizontal segment of length δ connecting θ_1 to $\tilde\theta_2$, but this time the segment we added is not inside Σ', and therefore

$$\delta_{\Sigma'}\big(\tilde\theta_1\big) \leq \delta_\Sigma\big(\theta_1\big) + \delta\,. \tag{5.8}$$

(iv) If θ enters in Σ_i directly in the open arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$, then θ_1 is a line segment with endpoints $\theta(0)$ and $s(\theta) = \theta_2(0)$. In this case we simply let $\tilde\theta_1$ be the line segment between $\theta(0)$ and $\tilde\theta_2(0)$. By a straightforward computation we can compare the lengths of θ_1 and $\tilde\theta_1$ by means of the entry angle ω (recall that the latter is defined in $s(\theta)$ for η-a.e. θ): in fact,

$$\begin{aligned}\delta_{\Sigma'}\big(\tilde\theta_1\big) &\leq \overline{\theta(0)\tilde\theta_2(0)} = \overline{\theta(0)\theta_2(0)} \pm \delta\cos\omega\big(s(\theta)\big) + \delta o_\theta(1)\\ &= \delta_\Sigma\big(\theta_1\big) \pm \delta\cos\omega\big(s(\theta)\big) + \delta o_\theta(1)\,,\end{aligned} \tag{5.9}$$

where the sign is "−" if θ enters from the left and "+" if θ enters from the right, and where for any θ the quantity $o_\theta(1)$ vanishes as $\delta \to 0$ (not necessarily uniformly in θ) and $|o_\theta(1)| \leq 1$.

The way to define $\tilde{\theta}_3$ is completely symmetric.

Recalling the definition of $\rho(\theta)$ and the estimates (5.4)–(5.9), one can immediately notice that

$$\begin{aligned} MK(\Sigma') &\leq C_{\Sigma'}(\eta') \\ &\leq C_\Sigma(\eta) + \delta\Big(M_N - \int_{\widetilde{x_{-\varepsilon}x_\varepsilon}} \cos\omega(x)d(\psi_l - \psi_r)(x) + \int_{\Theta_{\mathrm{iv}}} o_\delta(1)\,d\eta\Big) \\ &= MK(\Sigma) + \delta\Big(M_N - \int_{\widetilde{x_{-\varepsilon}x_\varepsilon}} \cos\omega(x)d(\psi_l - \psi_r)(x) + \int_{\Theta_{\mathrm{iv}}} o_\delta(1)\,d\eta\Big)\,. \end{aligned}$$

Here M_N stands for the mass of the paths entering in $\widetilde{x_{-\varepsilon}x_\varepsilon}$ from the "small" affluents, plus the mass of the paths exiting from them, Θ_{iv} is the set of paths as in **(iv)**, and the last equality is due to the fact that $MK(\Sigma) = C_\Sigma(\eta)$ in view of the optimality of η. We claim that

$$\lambda := \int_{\widetilde{x_{-\varepsilon}x_\varepsilon}} \cos\omega(x)d(\psi_l - \psi_r)(x) \leq 0\,. \tag{5.10}$$

Indeed, assume by contradiction that $\lambda > 0$; by construction, for any fixed ε, one has $M_N \to 0$ as $N \to \infty$, so we may fix an integer N large enough such that $M_N < \lambda/3$. Moreover, since by the Dominated Convergence Theorem

$$\int_{\Theta_{\mathrm{iv}}} o_\theta(1)\,d\eta$$

tends to zero as $\delta \to 0$, we may take δ small enough so that

$$\int_{\Theta_{\mathrm{iv}}} o_\theta(1)\,d\eta < \lambda/3\,.$$

Therefore

$$MK(\Sigma') \leq MK(\Sigma) - C\delta$$

for some constant $C > 0$ which depends only on ε. Summarizing, if (5.10) is not true then starting from the set Σ we would be able to find, for δ arbitrarily small a set Σ' satisfying

$$\mathscr{H}^1(\Sigma') \leq \mathscr{H}^1(\Sigma) + N\delta\,, \qquad MK(\Sigma') \leq MK(\Sigma) - C\delta$$

with suitable constants N, $C > 0$. But this leads to conclude that $r \geq C/N$, which is impossible since we are assuming $r = 0$. In fact, if $r < C/N$, then one can find, for δ sufficiently small, a subset

$$\Delta \subseteq \Sigma' \cap \{\alpha_\eta < C/N\}$$

of length $N\delta$, so that by (4.8) for the set $\Sigma'' := \Sigma' \setminus \Delta$ one would have $MK(\Sigma'') < MK(\Sigma)$ while $\mathscr{H}^1(\Sigma'') \leq \mathscr{H}^1(\Sigma)$, against the optimality of Σ. Therefore, (5.10) holds.

In the very similar way, moving the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$ rightwards rather than leftwards we conclude the opposite inequality, and therefore

$$\int_{\widetilde{x_{-\varepsilon}x_\varepsilon}} \cos\omega(x)d(\psi_l - \psi_r)(x) = 0\,.$$

Analogously, moving the arc upwards and downwards, we obtain

$$\int_{\widetilde{x_{-\varepsilon}x_\varepsilon}} \sin\omega(x)d(\psi_l - \psi_r)(x) = 0\,.$$

This concludes the proof since the arc $\widetilde{x_{-\varepsilon}x_\varepsilon}$ was arbitrarily chosen in Σ. $\square$

LEMMA 5.9. *There exists a constant C such that for all sufficiently small ε there exists a set Δ satisfying*

$$\mathscr{H}^1(\Delta) = \varepsilon\,, \qquad\qquad MK(\Sigma\cup\Delta) \le MK(\Sigma) - C\varepsilon^2\,.$$

PROOF. Since $f^+(\Sigma) = 0$, there exists a connected component Σ_i of the optimal set Σ and a point $x \in \operatorname{spt}(\pi_{0\#}\eta)\setminus\Sigma_i$, where η is an optimal t.p.m. associated to Σ. Take a path $\theta\in\operatorname{spt}\eta_i$ with $\theta(0) = x$, and denote

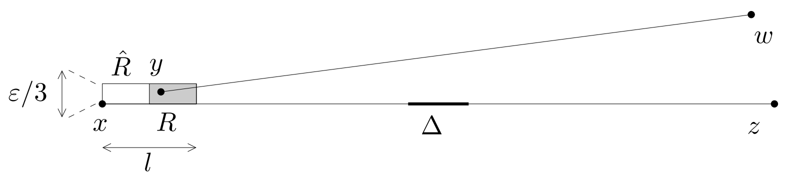

FIGURE 3. Construction in Lemma 5.9

$$z := s(\theta) \in \Sigma_i\,.$$

Consider now a thin rectangle $\widehat{R}$ with sides $l \ll \overline{xz}$ and $\varepsilon/3 \ll l$ having a corner in x and a long side contained in the segment xz. Divide then $\widehat{R}$ in two rectangles of sides $l/2$ and $\varepsilon/3$ and let R be the right one: in Figure 3, the rectangle $\widehat{R}$ is the wide one, while R is its shaded right half. Let now

$$\Sigma' := \Sigma\cup\Delta\,,$$

where $\Delta = zw$ is a segment of length ε contained in the middle of the segment xz.

Take now

$$y \in R\cap\operatorname{spt}(\pi_{0\#}\eta_i)\,,$$

a path $\tau\in\operatorname{spt}\eta_i$ with $\tau(0) = y$, and let $w := s(\tau)$. By the optimality of η and since Σ_i is connected, we already know that $\overline{xz}\le\overline{xw}$ as well as $\overline{yw}\le\overline{yz}$. Denote by y' the projection of y onto xz. Then, whenever l is small enough with respect to $\overline{xz}$, and ε is sufficiently small so that y' belongs to the segment xz and does not belong to Δ, one has the series of estimates

$$d_{\Sigma'}(y,z) \le \frac{\varepsilon}{3} + |y'-z| - \varepsilon$$

$$
\begin{aligned}
&= \frac{\varepsilon}{3} + |x-z| - |x-y'| - \varepsilon \\
&= |x-z| - |x-y'| - \frac{2\varepsilon}{3} \\
&\leq |x-w| - |x-y'| - \frac{2\varepsilon}{3} \\
&\leq |x-y| + |y-w| - |x-y'| - \frac{2\varepsilon}{3} \\
&\leq |x-y'| + \frac{\varepsilon}{3} + |y-w| - |x-y'| - \frac{2\varepsilon}{3} \\
&= |y-w| - \frac{\varepsilon}{3} = d_\Sigma(y,w) - \frac{\varepsilon}{3}\,.
\end{aligned}
$$

To conclude, it suffices now to notice that the area of R is proportional to ε (since x, θ and $l \ll \overline{xz}$ have been fixed), and, moreover, up to a suitable choice of x, θ and l, we can assume that $\pi_{0\#}\eta_i(R) \geq K\mathscr{L}^2(R)$ with a small constant K not depending on ε (here we use the assumption $f \in L^\infty$). We define $\eta' := \rho_\#\eta$, where $\rho(\theta) := \theta$ unless $\theta \in \operatorname{spt}\eta_i$ and $\theta(0) \in R$, in which case $\rho(\theta)$ replaces the line segment $\theta(0)s(\theta)$ by the composition of three segments $\theta(0)z$, zw and $ws(\theta)$. The conclusion now follows because

$$
\begin{aligned}
MK(\Sigma \cup \Delta) \leq C_{\Sigma\cup\Delta}(\eta') &\leq C_\Sigma(\eta) - K\mathscr{L}^2(R)\,\frac{\varepsilon}{3} \\
&= MK(\Sigma) - \frac{Kl}{18}\,\varepsilon^2\,.
\end{aligned}
\tag{5.11}
$$

□

REMARK 5.10. An inspection of the proof of Lemma 5.9 shows that the constant C can be estimated from below as

$$
C \geq \frac{f^+(x)\mathrm{dist}(x,\Sigma)}{C_0}
$$

where C_0 is a geometric constant depending only on $\operatorname{diam}\Omega$ and x is any Lebesgue point of f^+.

Finally, we can find a contradiction using Proposition 5.11 below, which will show that the assumption $r = 0$ cannot hold true and hence conclude the proof of Proposition 5.2.

PROPOSITION 5.11. *There exists a set $\widetilde{\Sigma}$ such that*

$$
\mathscr{H}^1\big(\widetilde{\Sigma}\big) = \mathscr{H}^1\big(\Sigma\big)\,, \qquad\qquad MK\big(\widetilde{\Sigma}\big) < MK\big(\Sigma\big)
$$

(so that $\mathfrak{F}\big(\widetilde{\Sigma}\big) < \mathfrak{F}\big(\Sigma\big)$ against the optimality of Σ).

PROOF. Thanks to Lemma 4.15, Σ_i can be endowed with the partial order generated by the relation given by $x \leq y \in \Sigma_i$ whenever there is a path $\theta \in \operatorname{spt}\eta_i$ passing through x and, afterwards, through y. Therefore, for almost all ε small enough (almost all is intended with respect to the Lebesgue measure on $\mathbb{R}$), it is possible to find a point x_ε such that

$$
\mathscr{H}^1(X_\varepsilon) = \varepsilon \qquad \text{where} \quad X_\varepsilon := \big\{x \in \Sigma_i :\ x \leq x_\varepsilon\big\}\,,
\tag{5.12}
$$

so that

$$\big\{x \in \Sigma_i :\ x \leq x_\varepsilon\big\}$$

is contained in the ball of center x_ε and radius ε. By Lemma 5.7, again for almost all ε we know that the entry angle $\omega(x_\varepsilon)$ is well-defined. Moreover, as a consequence of Lemma 5.9, we may prove that at least a mass $C\varepsilon$ passes through X_ε: in other words,

$$\psi(X_\varepsilon) = s_\# \eta_i(X_\varepsilon) \geq C\varepsilon\,, \tag{5.13}$$

the constant C being that of Lemma 5.9, thus independent of ε. Indeed, applying Lemma 5.9 find the set Δ, and denote by η' the t.p.m. built in the proof of the lemma: by (5.11) we know that

$$C_{\Sigma\cup\Delta}(\eta') \leq MK(\Sigma) - C\varepsilon^2\,.$$

If we now set

$$\Sigma' := \big(\Sigma \cup \Delta\big) \setminus X_\varepsilon\,,$$

which satisfies $\mathscr{H}^1(\Sigma') \leq \mathscr{H}^1(\Sigma)$, then by construction of η' one clearly has

$$C_{\Sigma'}(\eta') \leq C_{\Sigma\cup\Delta}(\eta') + \varepsilon\psi(X_\varepsilon)\,.$$

Hence, by the optimality of Σ we conclude

$$\begin{aligned} MK(\Sigma) \leq MK(\Sigma') \leq C_{\Sigma'}(\eta') &\leq C_{\Sigma\cup\Delta}(\eta') + \varepsilon\psi(X_\varepsilon) \\ &\leq MK(\Sigma) - C\varepsilon^2 + \varepsilon\psi(X_\varepsilon)\,, \end{aligned}$$

thus (5.13) is established. Let us now set

$$K := \sqrt{\frac{C}{8\pi\|f^+\|_{L^\infty}}}\,.$$

Since $f^+\big(B(x_\varepsilon, K\sqrt{\varepsilon})\big) \leq C\varepsilon/8$, while by (5.13)

$$\begin{aligned} f^+\Big(\big\{x \in \Omega : \exists\,\theta \in \operatorname{spt}\eta_i, \theta(0) = x,\ s(\theta) \in X_\varepsilon\big\}\Big) \\ \geq \eta\Big(\big\{\theta \in \operatorname{spt}\eta_i : s(\theta) \in X_\varepsilon\big\}\Big) = \psi(X_\varepsilon) \geq C\varepsilon\,, \end{aligned}$$

recalling also Lemma 5.8 we deduce that ε can be chosen with the additional property that there are two paths θ_P and θ_Q in $\operatorname{spt}\eta_i$ entering in Σ_i respectively from the left and from the right in such a way that

$$s(\theta_P) = s(\theta_Q) = x_\varepsilon$$

and both the points $P = \theta_P(0)$ and $Q = \theta_Q(0)$ are outside the ball $B\big(x_\varepsilon, K\sqrt{\varepsilon}\big)$. For simplicity, we assume that $x_\varepsilon \equiv (0,0)$ and that $\omega(x_\varepsilon) = 0$.

Since each path enters in Σ_i at a point of least distance from its first extreme, Σ_i intersects neither the open ball $B(P, \overline{Px_\varepsilon})$ nor the open ball $B(Q, \overline{Qx_\varepsilon})$. Thus, X_ε is constrained in the shaded region of Figure 4. We claim now that the entry angle $\omega(x)$ is "small" for most of the points $x \in X_\varepsilon$; more precisely one has

$$\psi\bigg(\bigg\{x \in X_\varepsilon :\ |\omega(x)| > 2\,\frac{\sqrt{\varepsilon}}{K}\bigg\}\bigg) \leq \frac{C}{8}\,\varepsilon\,. \tag{5.14}$$

In fact, take a point $x \in X_\varepsilon$ such that

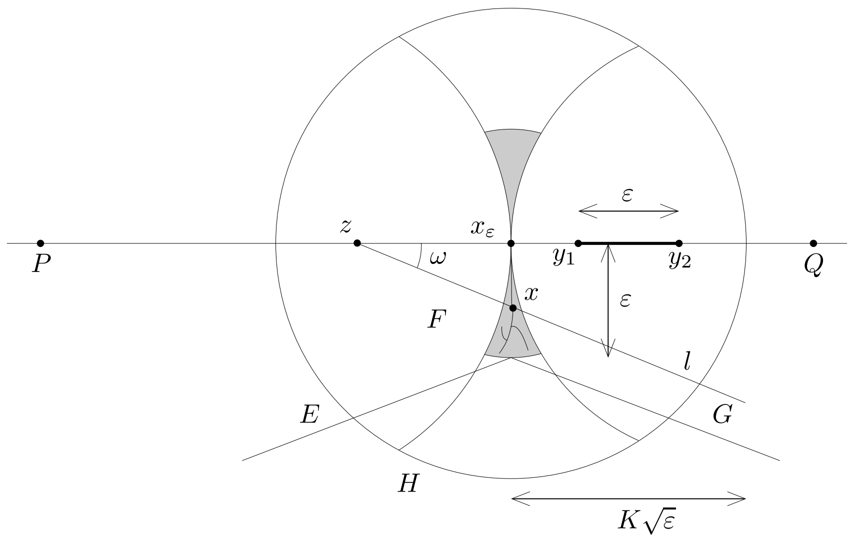

FIGURE 4. Situation in Proposition 5.11

$$|\omega(x)| > 2\frac{\sqrt{\varepsilon}}{K} :$$

we will say that $x \in X_L$ if x is above x_ε and $\omega(x) > 0$ or if x is below x_ε and $\omega(x) < 0$ (the figure shows this second possibility), while we will say that $x \in X_R$ otherwise. By construction, if $x \in X_L$ (resp. $x \in X_R$) then the line l passing through x with slope $\omega(x)$ intersects the horizontal axis at a point z in the interior of the ball

$$B\left(x_\varepsilon, K\sqrt{\frac{\varepsilon}{2}}\right),$$

and z is in the left (resp. right) of x_ε. Since for any $\theta \in \operatorname{spt} \eta_i$, $s(\theta)$ is the point of Σ_i closest to $\theta(0)$, the segment Px_ε cannot intersect the segment $\theta(0)x$ whenever $\theta \in \operatorname{spt} \eta_i$ and $\theta(0) = x$. For any such θ, the point $\theta(0)$ belongs to the line l by definition of the entry angle $\omega(x)$, therefore if $x \in X_L$ (resp. $x \in X_R$) and θ enters at x from the left (resp. from the right), then $\theta(0)$ belongs to the segment zx, thus to the ball $B(x_\varepsilon, K\sqrt{\varepsilon})$. We have then

$$\begin{aligned}\psi_l(X_L) + \psi_r(X_R) &= \eta_i\Big(\big\{\theta \in \operatorname{spt} \eta_l : s(\theta) \in X_L\big\} \cup \big\{\theta \in \operatorname{spt} \eta_r : s(\theta) \in X_R\big\}\Big)\\ &\leq f^+\Big(B\Big(x_\varepsilon, K\sqrt{\varepsilon/2}\Big)\Big) \leq \frac{C}{16}\,\varepsilon\,.\end{aligned}$$

Recalling that $\psi_l = \psi_r$ by Lemma 5.8, we deduce

$$\begin{aligned}\psi(X_\varepsilon) &= \psi_l(X_L) + \psi_r(X_L) + \psi_r(X_R) + \psi_l(X_R)\\ &\leq 2\big(\psi_l(X_L) + \psi_r(X_R)\big) \leq \frac{C}{8}\,\varepsilon\end{aligned}$$

and thus (5.14) is proved.

In order to define the competitor $\widetilde{\Sigma}$ to Σ, we divide $\mathbb{R}^2$ in four regions; since these regions are symmetric with respect to the horizontal axis, we draw only the bottom half in Figure 4 to avoid confusion. The region F is the ball $B(x_\varepsilon, K\sqrt{\varepsilon})$; the region H is made by such points $\xi \notin F$ below (resp. above) the horizontal axis, such that the angle between the segment connecting ξ to the point $(0,-\varepsilon)$ (resp. $(0,\varepsilon)$) and the horizontal axis has the absolute value greater than $\sqrt{\varepsilon}/2K$. The regions E and G are the left and the right side of the remaining part of $\mathbb{R}^2$. For simplicity of notation we denote by $m(E)$ the value

$$\eta\big(\big\{\theta \in \eta_i :\ \theta(0) \in E\big\}\big)\,,$$

and we adopt the analogous notation for the regions F, G and H. Further, without loss of generality, we assume $m(G) \geq m(E)$.

We now define the competitor $\widetilde{\Sigma}$: consider the segment y_1y_2 (drawn in bold in the figure), where

$$y_1 \equiv \Big(\frac{K}{2}\sqrt{\varepsilon} - \frac{\varepsilon}{2}, 0\Big)\,, \qquad\qquad y_2 \equiv \Big(\frac{K}{2}\sqrt{\varepsilon} + \frac{\varepsilon}{2}, 0\Big)\,;$$

we let then

$$\widetilde{\Sigma} := \big(\Sigma \setminus X_\varepsilon\big) \cup y_1y_2\,.$$

By (5.12), we have $\mathscr{H}^1\big(\widetilde{\Sigma}\big) \leq \mathscr{H}^1\big(\Sigma\big)$, so we only need to check that $MK\big(\widetilde{\Sigma}\big) < MK\big(\Sigma\big)$ to find the desired contradiction.

As in the last lemmas, to show that $MK\big(\widetilde{\Sigma}\big) < MK\big(\Sigma\big)$ we will provide a suitable t.p.m. $\eta' := \rho_\#\eta$ with $C_{\widetilde{\Sigma}}(\eta') < C_\Sigma(\eta)$. In order to construct the map $\rho : \Theta \to \Theta$, we first set $\rho(\theta) := \theta$ for every

$$\theta \notin \Theta_\varepsilon := \{\theta \in \operatorname{spt}\eta_i,\, s(\theta) \in X_\varepsilon\}\,.$$

For $\theta \notin \Theta_\varepsilon$, of course we have $\delta_{\widetilde{\Sigma}}\big(\rho(\theta)\big) \leq \delta_\Sigma(\theta)$. Concerning now the case $\theta \in \Theta_\varepsilon$, if $\theta(0) \in F$ we again let $\rho(\theta) := \theta$, so that

$$\theta(0) \in F \Longrightarrow \delta_{\widetilde{\Sigma}}\big(\rho(\theta)\big) \leq \delta_\Sigma(\theta) + \varepsilon\,. \tag{5.15}$$

If $\theta(0) \in H$, we still let $\rho(\theta) := \theta$ and by construction, we have

$$\theta(0) \in H \Longrightarrow \delta_{\widetilde{\Sigma}}\big(\rho(\theta)\big) \leq \delta_\Sigma(\theta) + \varepsilon\,. \tag{5.16}$$

We call now

$$\hat{t} = \hat{t}(\theta) := \max\{t :\ \theta(t) \in X_\varepsilon\}\,.$$

If $\theta(0) \in E$, we set $\rho(\theta)$ to be the path obtained joining the line segment $\theta(0)\theta(\hat{t})$ to the curve $\theta \llcorner [\hat{t}, 1]$. By definition of the set E, a simple geometric estimates ensures

$$\begin{aligned}\theta(0) \in E \Longrightarrow \delta_{\widetilde{\Sigma}}\big(\rho(\theta)\big) &\leq \delta_\Sigma(\theta) + \overline{\theta(0)\theta(\hat{t})} - \overline{\theta(0)s(\theta)} \\ &\leq \delta_\Sigma(\theta) + \varepsilon \sin\big(\sqrt{\varepsilon}/2K\big) + O(\varepsilon^{3/2}) \\ &= \delta_\Sigma(\theta) + o(\varepsilon)\,,\end{aligned} \tag{5.17}$$

where $o(\varepsilon)$ depends only on ε and not on θ. Finally, if $\theta(0) \in G$, we define $\rho(\theta)$ as the path joining the line segments $\theta(0)y_2$, y_2y_1, and $y_1\theta(\hat{t})$ to the curve $\theta \llcorner [\hat{t}, 1]$. Another easy geometric argument gives

$$\theta(0) \in G \Longrightarrow \delta_{\widetilde{\Sigma}}\big(\rho(\theta)\big) \leq \delta_\Sigma(\theta) - \varepsilon + O(\varepsilon^{3/2}) \leq \delta_\Sigma(\theta) - \varepsilon + o(\varepsilon)\,. \tag{5.18}$$

Now, (5.15)–(5.18) together yield

$$\begin{aligned}MK\big(\widetilde{\Sigma}\big) \leq MK\big(\Sigma\big) &+ m(F)\,\varepsilon + m(H)\,\varepsilon \\ &+ m(E)\,o(\varepsilon) - m(G)\,\varepsilon + m(G)\,o(\varepsilon)\,.\end{aligned} \tag{5.19}$$

By (5.13), we know that

$$m(E) + m(F) + m(G) + m(H) \geq C\varepsilon\,.$$

By (5.14), we know that

$$m(H) \leq \frac{1}{8}\,C\varepsilon\,;$$

by definition, we know that

$$m(F) \leq \frac{1}{8}\,C\varepsilon\,.$$

Hence, since we are assuming $m(G) \geq m(E)$, we find

$$m(G) \geq \frac{3}{8}\,C\varepsilon\,.$$

Therefore, (5.19) gives us that

$$MK\big(\widetilde{\Sigma}\big) \leq MK\big(\Sigma\big) + \varepsilon\left(\frac{1}{4}\,C - \frac{3}{8}\,C + Co(1)\right),$$

and then, provided ε was chosen sufficiently small, we get $MK\big(\widetilde{\Sigma}\big) < MK\big(\Sigma\big)$, hence the required contradiction that completes the proof. $\square$

Now, it is immediate to conclude the validity of Proposition 5.2

OF PROPOSITION 5.2. If $r = 0$, then the claims of Lemmas 5.3, 5.5, 5.7, 5.8, 5.9 and of Proposition 5.11 hold true; hence, there is a set $\widetilde{\Sigma}$ with $\mathfrak{F}(\widetilde{\Sigma}) < \mathfrak{F}(\Sigma)$, against the optimality of Σ. $\square$

2. Proof of the main result

We can finally give the proof of Theorem 5.1, which makes use of the technical results of Section 2 and of the fact that $r > 0$ for any $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$ which we have shown in the above section.

To prove Theorem 5.1 we need to select a special t.p.m. $\eta \in \mathcal{M}^+_2(\Omega)$. To this aim, let us first arbitrarily fix a function $\alpha : \Sigma \to \mathbb{R}^+$ such that there exists some $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$ for which $\alpha_\eta = \alpha$. We solve then another auxiliary maximization problem.

Lemma 5.12. *Defining* $F_1(\eta) := \max\{\mu(\theta), \theta \in \operatorname{spt}\eta\}$, *we have that the set*

$$\mathcal{M}^+_{\mu,3,1}(\Theta) := \operatorname{Arg\,max}\Big\{F_1(\eta) :\ \eta \in \mathcal{M}^+_{\mu,2}(\Theta),\ \alpha_\eta = \alpha\Big\} \tag{5.20}$$

is non-empty and convex.

We recall that the functional $F_1(\eta)$ is well-defined for all $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$ by Corollary 4.12.

of Lemma 5.12. Consider a maximizing sequence $\{\eta_n\} \subseteq \mathcal{M}^+_{\mu,2}(\Theta)$ for the functional F_1 under the constraint $\alpha_{\eta_n} = \alpha$, and define

$$\eta := \sum_{n\in\mathbb{N}} 2^{-n}\, \eta_n\,.$$

By Corollary 4.12, there is a path θ_n maximizing the length

$$\theta \mapsto \mu(\theta) = \mathscr{H}^1(\theta \cap \Sigma)$$

within $\operatorname{spt}\eta_n$. Since the Euclidean length of the paths θ_n is bounded thanks to Lemma 3.19 and since Ω is bounded, up to a subsequence we can assume that $\theta_n \to \theta$ for some path $\theta \in \Theta$. Notice that, since $\operatorname{spt}\eta$ is closed and contains θ_n by construction, we obtain $\theta \in \operatorname{spt}\eta$. Since we already know that $\theta \mapsto \mu(\theta)$ is an u.s.c. function, we obtain

$$\limsup_{n\to\infty} F_1(\eta_n) = \limsup_{n\to\infty} \mu(\theta_n) \le \mu(\theta) \le F_1(\eta)\,.$$

Therefore, $\eta \in \mathcal{M}^+_{\mu,3,1}(\Theta)$ and we have proved that the set $\mathcal{M}^+_{\mu,3,1}(\Theta)$ is non-empty. The convexity of this set follows immediately by noticing that a convex combination of elements of $\mathcal{M}^+_{\mu,3,1}(\Theta)$ is a measure, the support of which contains all the supports of the elements. □

Notice that the existence of solutions for the maximization problem (5.20) cannot be shown in the same way as done for $\mathcal{M}^+_{\mu,1}(\Theta)$ and $\mathcal{M}^+_{\mu,2}(\Theta)$ in Remark 4.11, where one had to deal with integral functionals and therefore it was sufficient to check the lower (or upper) semicontinuity of the integrands. Recall that in those cases, moreover, the standard direct methods of the calculus of variations also gave the weak* closedness of $\mathcal{M}^+_{\mu,1}(\Theta)$ and $\mathcal{M}^+_{\mu,2}(\Theta)$. On the contrary, it is fundamental to notice that (in general) $\mathcal{M}^+_{\mu,3,1}(\Theta)$ *is not weakly* closed*; indeed, if $\eta_n \overset{*}{\rightharpoonup} \eta$, $\theta_n \in \operatorname{spt}\eta_n$

and $\theta_n \to \theta$, then in general it is not true that $\theta \in \operatorname{spt}\eta$. For this reason, to extract a particular t.p.m. in $\mathcal{M}^+_{\mu,3,1}(\Theta)$ in the sequel we will not take a weak* limit of some optimizing sequence, since it may happen that this limit does not belong to $\mathcal{M}^+_{\mu,3,1}(\Theta)$; instead, we will repeat the method already used in the proof of Lemma 5.12, which only needs the convexity of $\mathcal{M}^+_{\mu,3,1}(\Theta)$ in place of the weak* closedness.

In the sequel, we will often use a t.p.m. $\eta \in \mathcal{M}^+_{\mu,3,1}(\Theta)$ together with a path θ_1 providing the maximum in F_1; therefore, it is useful to set

$$\Pi_1 := \Big\{(\eta,\theta_1):\ \eta \in \mathcal{M}^+_{\mu,3,1}(\Theta),\ \mu(\theta_1) = F_1(\eta)\}\Big\}.$$

Let us now prove a "second-order" version of Corollary 4.12.

Lemma 5.13. *For any $(\eta,\theta_1) \in \Pi_1$ there exists a $\tau \in \Theta$ maximizing*

$$\tau \mapsto \mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)(\tau)$$

within $\operatorname{spt}\eta$.

Proof. Fix $(\eta,\theta_1) \in \Pi_1$, take a sequence $\{\tau_n\}$ maximizing

$$\tau \mapsto \mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)(\tau)$$

within $\operatorname{spt}\eta$, and assume without loss of generality (thanks to Lemma 3.19) that $\tau_n \to \tau$ in Θ. As in Corollary 4.12, we conclude applying Lemma 3.1 with $X := \Theta$, $C_n := \theta_n$ and $\nu = \tilde\mu := \mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)$; indeed, the lemma gives

$$\tilde\mu(\tau) \geq \limsup_{n\to\infty} \tilde\mu(\tau_n),$$

and this clearly concludes the proof. □

We define

$$\begin{aligned}
F_2(\eta,\theta_1) &:= \max\{\mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)(\tau),\ \tau \in \operatorname{spt}\eta\};\\
\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta) &:= \operatorname{Arg\,max}\Big\{F_2(\eta,\theta_1),\ (\eta,\theta_1) \in \Pi_1\Big\};\\
\widetilde\Pi_2 &:= \Big\{(\eta,\theta_1,\tilde\theta_2):\ (\eta,\theta_1) \in \widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta),\\
&\qquad \tilde\theta_2 \in \operatorname{Arg\,max}\{\mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)(\tau),\ \tau \in \operatorname{spt}\eta\}\Big\};\\
\Pi_2 &:= \operatorname{Arg\,max}\Big\{\mu(\theta_2),\ (\eta,\theta_1,\theta_2) \in \widetilde\Pi_2\Big\};\\
\mathcal{M}^+_{\mu,3,2}(\Theta) &:= \Big\{\eta \in \mathcal{M}^+_{\mu,3,1}(\Theta):\ \exists\,\theta_1,\ \theta_2 \in \Theta, (\eta,\theta_1,\theta_2) \in \Pi_2\Big\}.
\end{aligned}$$

We underline the meaning of the sets we defined. Recall that our aim is to cover Σ with paths, possibly in an efficient way; therefore, we defined $\mathcal{M}^+_{\mu,3,1}(\Theta)$ to be the set of the optimal measures $\eta \in \mathcal{M}^+_{\mu,2}(\Theta)$, having α_η fixed, the longest path of which inside Σ has the maximal possible length. The elements of Π_1 are the pairs of such particular optimal measures and their paths using Σ as much as possible. To proceed covering Σ, we find now the "second longest path", namely the path in $\operatorname{spt}\eta$ having the maximal length inside $\Sigma\setminus\theta_1$. Hence, $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$ are the pairs $(\eta,\theta_1)\in\Pi_1$ maximizing this new length that is possible to gain by this second longest path, and $\widetilde{\Pi}_2$ are the corresponding triples given by the pairs

$$(\eta,\theta_1)\in\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$$

together with a second path $\tilde{\theta}_2$ maximizing the new length. For reasons that will become clear during the proof of Theorem 5.1, among those triples we have the convenience to make a further choice: for all the triples

$$(\eta,\theta_1,\theta_2)\in\widetilde{\Pi}_2$$

the length

$$\mathscr{H}^1 \llcorner \big(\Sigma\setminus\theta_1\big)(\theta_2)$$

is the same, and it is the maximal possible; we then select those triples for which

$$\mu(\theta_2)=\mathscr{H}^1 \llcorner \Sigma(\theta_2)$$

is maximal. In words, among the different paths maximizing the new length that one can gain by θ_2, we choose those maximizing also the total length $\mu(\theta_2)$. Those particular triples are collected in Π_2, and finally $\mathcal{M}^+_{\mu,3,2}(\Theta)$ consists of the optimal measures η which belong to Π_2 together with some pair of paths $(\theta_1,\,\theta_2)\in\operatorname{spt}\eta$.

We now extend Lemma 5.12 to $\mathcal{M}^+_{\mu,3,2}(\Theta)$.

Lemma 5.14. *The set $\mathcal{M}^+_{\mu,3,2}(\Theta)$ is non-empty and convex.*

Proof. We first show that $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$ is non-empty: let $(\eta_n,\theta_{1,n})$ be a maximizing sequence for the functional F_2 in Π_1. In particular, according to Lemma 5.13, one has

$$F_2(\eta_n,\theta_{1,n})=\mathscr{H}^1 \llcorner(\Sigma\setminus\theta_{1,n})(\tilde{\theta}_{2,n})$$

for some $\tilde{\theta}_{2,n}\in\operatorname{spt}\eta_n$. Then, as in Lemma 5.12, we define

$$\eta:=\sum_{n\in\mathbb{N}}2^{-n}\eta_n\,,$$

and we let θ_1 and $\tilde{\theta}_2$ be the limits in Θ of $\theta_{1,n}$ and $\tilde{\theta}_{2,n}$ respectively (up to a subsequence). We claim that

$$(\eta,\theta_1,\tilde{\theta}_2)\in\widetilde{\Pi}_2\,, \tag{5.21}$$

hence in particular $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$ contains (η,θ_1), so it is not empty.

To show (5.21), we notice that $\eta\in\mathcal{M}^+_{\mu,2}(\Theta)$ and $\alpha_\eta=\alpha$ as in Lemma 5.12; moreover,

$$\mu(\theta_1)\geq\limsup_{n\to\infty}\mu(\theta_{1,n})$$

by upper semicontinuity (recall Lemma 3.1). Then, since in this case all the paths $\theta_{1,n}$ have the same, maximal, length in Σ, we deduce that the above inequality is actually an equality, so that $(\eta, \theta_1) \in \Pi_1$. To show (5.21), we only need to prove that

$$\limsup_{n\to\infty} \mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_{1,n}\big)(\tilde\theta_{2,n}) \leq \mathscr{H}^1 \llcorner \big(\Sigma \setminus \theta_1\big)(\tilde\theta_2)\,,$$

which is equivalent to

$$\limsup_{n\to\infty} \mu\big(\tilde\theta_{2,n} \setminus \theta_{1,n}\big) - \mu\big(\tilde\theta_2 \setminus \theta_1\big) \leq 0\,.$$

In view of the inclusion

$$\big(A \setminus B\big) \setminus \big(C \setminus D\big) \subseteq \big(A \setminus C\big) \cup \big(D \setminus B\big)\,,$$

one has

$$\begin{aligned} \mu\big(\tilde\theta_{2,n} \setminus \theta_{1,n}\big) - \mu\big(\tilde\theta_2 \setminus \theta_1\big) &\leq \mu\Big(\big(\tilde\theta_{2,n} \setminus \theta_{1,n}\big) \setminus \big(\tilde\theta_2 \setminus \theta_1\big)\Big) \\ &\leq \mu\big(\tilde\theta_{n,2} \setminus \tilde\theta_2\big) + \mu\big(\theta_1 \setminus \theta_{n,1}\big)\,, \end{aligned} \tag{5.22}$$

so that it suffices to show that the lim sup of the right hand-side is zero. Concerning $\mu\big(\tilde\theta_{n,2} \setminus \tilde\theta_2\big)$, it is immediately seen that its lim sup is zero thanks to Lemma 3.1 with $\nu = \mathscr{H}^1 \llcorner \big(\Sigma \setminus \tilde\theta_2\big)$. In fact,

$$\mu\big(\tilde\theta_{n,2} \setminus \tilde\theta_2\big) = \nu(\tilde\theta_{n,2})$$

and of course $\nu(\tilde\theta_2) = 0$. Concerning $\mu\big(\theta_1 \setminus \theta_{n,1}\big)$, by the equality

$$\big|\mu(A) - \mu(B)\big| = \big|\mu(A \setminus B) - \mu(B \setminus A)\big|\,,$$

valid for any measure μ and sets A and B, one has

$$\big|\mu(\theta_1) - \mu(\theta_{n,1})\big| = \big|\mu(\theta_1 \setminus \theta_{n,1}) - \mu(\theta_{n,1} \setminus \theta_1)\big|\,.$$

As we already noticed,

$$\mu\big(\theta_{n,1}\big) = \mu\big(\theta_1\big) \qquad\qquad \forall\, n \in \mathbb{N}\,;$$

moreover,

$$\mu\big(\theta_{n,1} \setminus \theta_1\big) \to 0$$

exactly as previously done for $\tilde\theta_2$ and $\tilde\theta_{n,2}$. Then, it follows that also

$$\mu\big(\theta_1 \setminus \theta_{n,1}\big) \to 0$$

so that the lim sup of the right hand-side in (5.22) is zero, thus (5.21) follows: as noticed before, this gives that $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$ is non-empty.

Moreover, $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$ is also convex, as one can immediately deduce exactly as done for $\mathcal{M}^+_{\mu,3,1}(\Theta)$ at the end of Lemma 5.12.

Concerning $\mathcal{M}^+_{\mu,3,2}(\Theta)$, the convexity can be deduced again in the very same way as for $\mathcal{M}^+_{\mu,3,1}(\Theta)$ and $\widetilde{\mathcal{M}}^+_{\mu,3,2}(\Theta)$. To show that $\mathcal{M}^+_{\mu,3,2}(\Theta)$ is non-empty, we will check that so is Π_2. In fact, it suffices again to take a sequence

$$\big\{(\eta_n, \theta_{1,n}, \theta_{2,n})\big\} \subseteq \widetilde\Pi_2$$

such that $\theta_{2,n}$ is a maximizing sequence for μ. Then define, possibly up to a subsequence,

$$\eta := \sum_{n\in\mathbb{N}} 2^{-n}\eta_n$$

and θ_1 and θ_2 to be the limits in Θ of $\theta_{1,n}$ and $\theta_{2,n}$ respectively. By the same arguments as before we have

$$(\eta, \theta_1, \theta_2) \in \widetilde{\Pi}_2\,,$$

and moreover

$$\mu(\theta_2) \geq \limsup \mu(\theta_{2,n})$$

by upper semicontinuity again. Then $\Pi_2 \neq \emptyset$ as $(\eta, \theta_1, \theta_2) \in \Pi_2$ and the proof is concluded. $\square$

We define now by induction, for any $n \in \mathbb{N}$,

$$\begin{aligned}
F_n(\eta,\theta_1,\dots,\theta_{n-1}) &:= \max\{\mathscr{H}^1 \llcorner \big(\Sigma \setminus (\theta_1\cup\dots\cup\theta_{n-1})\big)(\tau),\ \tau\in\operatorname{spt}\eta\}\\
\widetilde{\mathcal{M}}^+_{\mu,3,n}(\Theta) &:= \operatorname{Arg\,max}\Big\{F_n(\eta,\theta_1,\dots,\theta_{n-1}) :\ (\eta,\theta_1,\dots,\theta_{n-1})\in\Pi_{n-1}\Big\};\\
\widetilde{\Pi}_n &:= \Big\{(\eta,\theta_1,\dots\theta_{n-1},\tilde{\theta}_n) :\ (\eta,\theta_1,\dots,\theta_{n-1})\in\widetilde{\mathcal{M}}^+_{\mu,3,n}(\Theta),\\
&\qquad\qquad \mathscr{H}^1 \llcorner \big(\Sigma \setminus (\theta_1\cup\dots\cup\theta_{n-1})\big)(\tilde{\theta}_n) = F_n(\eta,\theta_1,\dots,\theta_{n-1})\Big\};\\
\Pi_n &:= \operatorname{Arg\,max}\Big\{\mu(\theta_n) :\ (\eta,\theta_1,\dots,\theta_n)\in\widetilde{\Pi}_n\Big\};\\
\mathcal{M}^+_{\mu,3,n}(\Theta) &:= \Big\{\eta\in\mathcal{M}^+_{\mu,3,n-1} :\ \exists\theta_1,\,\dots,\,\theta_n\in\Theta :\ (\eta,\theta_1,\,\dots,\,\theta_n)\in\Pi_n\Big\}\,.
\end{aligned}$$

Further, set

$$\begin{aligned}
\Pi_\infty &:= \Big\{(\eta,\theta_1,\dots,\theta_n,\dots) :\ \forall n\in\mathbb{N},\ (\eta,\theta_1,\dots,\theta_n)\in\mathcal{M}^+_{\mu,3,n}\Big\};\\
\mathcal{M}^+_{\mu,3,\infty}(\Theta) &:= \Big\{\eta\in\mathcal{M}^+_{\mu,2}(\Theta) :\ \exists\theta_1,\,\dots,\,\theta_n,\dots\in\Theta :\\
&\qquad\qquad (\eta,\theta_1,\,\dots,\,\theta_n,\dots)\in\Pi_\infty\Big\}\,.
\end{aligned}$$

We have the following result.

LEMMA 5.15. *All the sets $\mathcal{M}^+_{\mu,3,n}(\Theta)$ are non-empty and convex, as well as $\mathcal{M}^+_{\mu,3,\infty}(\Theta)$.*

PROOF. The first part of the statement can be obtained exactly as in Lemma 5.14 via an inductive procedure. Concerning the second part, it can also be obtained in a similar way: take a sequence $\{\eta_n\} \in \mathcal{M}^+_{\mu,2}(\Theta)$ with $\alpha_{\eta_n} = \alpha$ and paths $\theta_{m,n} \in \Theta$ with $m \leq n$ such that for any $n \in \mathbb{N}$ one has

$$(\eta_n, \theta_{1,n}, \dots, \theta_{n,n}) \in \Pi_n\,.$$

By a standard diagonal argument, we can assume that for any $m \in \mathbb{N}$ one has $\theta_{m,n} \to \theta_m$ as $n \to \infty$, and we define as usual

$$\eta := \sum_{n\in\mathbb{N}} 2^{-n}\eta_n\,;$$

then, arguing as in the above lemmas we derive that

$$(\eta, \theta_1, \dots, \theta_m) \in \mathcal{M}^+_{\mu,3,m} \qquad\qquad \forall\, m \in \mathbb{N}\,,$$

so that

$$(\eta, \theta_1, \dots, \theta_n, \dots) \in \Pi_\infty\,.$$

Hence, the fact that $\mathcal{M}^+_{\mu,3,\infty}(\Theta)$ is non-empty is established. The convexity of $\mathcal{M}^+_{\mu,3,\infty}(\Theta)$ is straightforward. Indeed, for any

$$\eta_1,\, \eta_2 \in \mathcal{M}^+_{\mu,3,\infty}(\Theta)$$

and any $0 < t < 1$ one has

$$\operatorname{spt}\big(t\eta_1 + (1-t)\eta_2\big) \supseteq \operatorname{spt}\eta_1\,.$$

Thus, if $(\eta, \theta_1, \dots, \theta_n, \dots) \in \Pi_\infty$, one has also

$$\big(t\eta_1 + (1-t)\eta_2, \theta_1, \dots, \theta_n, \dots\big) \in \Pi_\infty$$

by definition of Π_∞. $\square$

of Theorem 5.1. Take an optimal set Σ and a t.p.m. $\eta \in \mathcal{M}^+_{\mu,3,\infty}(\Theta)$ with

$$(\eta, \theta_1, \dots, \theta_n, \dots) \in \Pi_\infty$$

for a suitable sequence $\{\theta_n\} \subseteq \operatorname{spt}\eta$. The thesis will be achieved once shown that

$$\Sigma \subseteq \theta_1 \cup \theta_2 \cup \dots \cup \theta_n$$

for some $n \in \mathbb{N}$. If this is not true, then one has $\mu(\theta_i) > 0$ for all $i \in \mathbb{N}$. We define now x_i and y_i those points of θ_i such that $\widetilde{x_i y_i}$ is the shortest subpath of θ_i with $\mu(\widetilde{x_i y_i}) = \mu(\theta_i)$. Now, the proof follows in five steps.

Step I. *For any $i \in \mathbb{N}$, there is at least a mass $r > 0$ passing through x_i (resp. y_i) and following the path θ_i for a while after x_i (resp. before y_i).* Fix $i \in \mathbb{N}$, and define

$$\begin{aligned}\Theta_i := \Big\{\theta \in \operatorname{spt}\eta :\ &\exists\, t_1 < t_2,\, t'_1 < t'_2,\, \theta(t_1) = x_i,\\ &\theta(t_2) \neq x_i,\, \theta\llcorner[t_1, t_2] = \theta_i\llcorner[t'_1, t'_2]\Big\},\\ \widetilde{\Theta}_i := \Big\{\theta \in \operatorname{spt}\eta :\ &\exists\, t_1 < t_2,\, t'_1 < t'_2,\, \theta(t_2) = y_i,\\ &\theta(t_1) \neq y_i,\, \theta\llcorner[t_1, t_2] = \theta_i\llcorner[t'_1, t'_2]\Big\}\,,\end{aligned}$$

so that Θ_i are precisely those paths passing through x_i and then following θ_i for at least a while, and $\widetilde{\Theta}_i$ are the paths passing through y_i having followed θ_i for a while. The claim of this step is then $\eta(\Theta_i) \geq r$ and $\eta(\widetilde{\Theta}_i) \geq r$.

By definition of x_i and y_i, we have a sequence $\{z_n\}$ of points contained in $\widetilde{x_i y_i}$, converging to x_i and belonging to Σ. By (4.2), one has $\alpha_\eta(z_n) \geq r$, which can be rewritten as

$$\eta\bigl(\Theta_{z_n}\bigr) \geq r\,, \tag{5.23}$$

where

$$\Theta_z := \{\theta \in \Theta :\ z \in \theta\}$$

for any $z \in \Omega$. Suppose now by contradiction that

$$\eta\bigl(\Theta_i\bigr) = r - \delta$$

for some $\delta > 0$; then, by (5.23) one has

$$\eta\bigl(\Theta_{z_n} \setminus \Theta_i\bigr) > \delta \qquad\qquad \forall n \in \mathbb{N}\,. \tag{5.24}$$

We remark now that, by construction and thanks to Lemma 4.20,

$$\Theta_i = \bigcup\nolimits_{j\in\mathbb{N}} \bigcap\nolimits_{n\geq j} \Theta_{z_n} \tag{5.25}$$

up to negligible sets. Moreover, again by Lemma 4.20,

$$\bigl\{\Theta_{z_1} \cap \Theta_{z_n}\bigr\}$$

is an $\eta-$essentially decreasing sequence (more precisely, the sequence of sets $\Theta_{z_1} \cap \Theta_{z_n} \cap \operatorname{spt}\eta$ is decreasing). Therefore, also the sequence

$$\bigl\{(\Theta_{z_1} \cap \Theta_{z_n}) \setminus \Theta_i\bigr\}$$

is $\eta-$essentially decreasing and its intersection must be $\eta-$negligible by (5.25). This implies the existence of an integer $n > 1$ such that

$$\eta\bigl((\Theta_{z_1} \cap \Theta_{z_n}) \setminus \Theta_i\bigr) < \eta\bigl(\Theta_{z_1} \setminus \Theta_i\bigr) - \delta\,.$$

So we assume without loss of generality (up to redefining the sequence $\{z_n\}$) that

$$\eta\bigl((\Theta_{z_1} \cap \Theta_{z_2}) \setminus \Theta_i\bigr) < \eta\bigl(\Theta_{z_1} \setminus \Theta_i\bigr) - \delta\,.$$

(the right hand-side of the above inequality is a positive number by (5.24). Applying (5.24) to $n = 2$ gives

$$\eta\bigl((\Theta_{z_1} \cup \Theta_{z_2}) \setminus \Theta_i\bigr) = \eta\bigl(\Theta_{z_2} \setminus \Theta_i\bigr) + \eta\bigl(\Theta_{z_1} \setminus \Theta_i\bigr) - \eta\bigl((\Theta_{z_1} \cap \Theta_{z_2}) \setminus \Theta_i\bigr) > 2\delta\,.$$

Arguing in the same way, for any $n \in \mathbb{N}$ we can obtain

$$\eta\Bigl((\Theta_{z_1} \cup \Theta_{z_2} \cup \cdots \cup \Theta_{z_n}) \setminus \Theta_i\Bigr) > n\delta\,.$$

Since of course this is not possible for $n > \|\eta\|/\delta$, the contradiction shows that $\eta(\Theta_i) \geq r$; in the very same way, $\eta(\widetilde{\Theta}_i) \geq r$).

Step II. *If $x_i \neq x_j$ (resp. $y_i \neq y_j$), one has $\Theta_i \cap \Theta_j = \emptyset$ (resp. $\widetilde{\Theta}_i \cap \widetilde{\Theta}_j = \emptyset$).* Assume that this is not true, so take $\theta \in \Theta_i \cap \Theta_j$, with $j > i$ and with $x_i \neq x_j$. Suppose that θ passes first through x_j, than through x_i. Then, since $\theta \in \Theta_i$ there is an interval $[t_1, t_2]$ such that

$$\theta(t_1) = \theta_i(t_1) = x_i$$

and

$$\theta \llcorner [t_1, t_2] \equiv \theta_i \llcorner [t_1, t_2]\,.$$

Moreover, since θ has passed through x_j before passing through x_i (and recalling that $\theta \in \Theta_j$) we have

$$\mu\big(\theta([0,t_1])\big) > 0 = \mu\big(\theta_i([0,t_1])\big)$$

by construction. We aim now to find a competitor to η proving that $\eta \notin \Pi_i$, finding then a contradiction. The idea is to modify θ and θ_i: more precisely, we define

$$\tilde\theta := \theta \llcorner [0,t_1] \cdot \theta_i \llcorner [t_1,1]$$

as the path which follows θ from $\theta(0)$ to x_i, and then follows θ_i from x_i to $\theta_i(1)$, and analogously we define

$$\tilde\theta_i := \theta_i \llcorner [0,t_1] \cdot \theta \llcorner [t_1,1]$$

as the path following θ_i from $\theta_i(0)$ to x_i and θ from x_i to $\theta(1)$. Then, we may define a modification $\tilde\eta$ of η as follows: we take an $\eta-$positive quantity of mass contained in a neighborhood of θ_i in Θ and passing through x_i (this is possible thanks to Step I), and an equal mass contained in a neighborhood of θ in Θ and passing through x_i (this is possible by Corollary 4.14 and Lemma 4.17). As in the proof of Lemma 4.15 we define $\tilde\eta$ swapping the ways of the paths when they meet at x_i, and leaving all the other paths unchanged. Therefore, $\tilde\theta$ and $\tilde\theta_i$ belong to $\operatorname{spt}\tilde\eta$. It is immediately seen that

$$C_\Sigma(\tilde\eta) = C_\Sigma(\eta)\,,$$

so that $\tilde\eta$ is still an optimal t.p.m. Moreover, $\theta_1, \theta_2, \dots, \theta_{i-1}$ belong to $\operatorname{spt}\tilde\eta$, provided we have chosen the neighborhoods of θ and θ_i in the construction above small enough. Recall also that

$$\Sigma \cap \theta_i \subsetneq \Sigma \cap \tilde\theta\ :$$

therefore, we obtain that

$$(\tilde\eta, \theta_i, \theta_2, \dots, \theta_{i-1}, \tilde\theta) \in \widetilde\Pi_i\ :$$

moreover, since $\mu(\tilde\theta) > \mu(\theta_i)$, we have a contradiction with the assumption

$$(\eta, \theta_i, \theta_2, \dots, \theta_{i-1}, \theta_i) \in \Pi_i\,.$$

Summarizing, we finally found a contradiction assuming that θ passes through x_j before than through x_i.

If θ passes through x_i before passing through x_j, a completely similar argument gives again a contradiction at the $j-$th step: in fact, an analogous change of the t.p.m. allows to assume that $\operatorname{spt}\eta$ contains the path

$$\tilde\theta := \theta \llcorner [0,t_1] \cdot \theta_j \llcorner [t_1,1]$$

which follows θ between $\theta(0)$ and x_j, and θ_j between x_j and $\theta_j(1)$. Therefore, at the $j-$th stage, it would be more convenient to select $\tilde\theta$ instead of θ_j, since

$$\Sigma \cap \tilde\theta \supsetneq \Sigma \cap \theta_j\ :$$

we again found a contradiction, so that the property

$$\Theta_i \cap \Theta_j = \emptyset$$

is proved. Concerning

$$\widetilde{\Theta}_i \cap \widetilde{\Theta}_j = \emptyset\,,$$

the argument is of course exactly the same. Then also this step is concluded.

Step III. If $x_i = x_j$ (resp. $y_i = y_j$), then either $\Theta_i \cap \Theta_j = \emptyset$ or $\Theta_i = \Theta_j$ (resp. either $\widetilde{\Theta}_i \cap \widetilde{\Theta}_j = \emptyset$ or $\widetilde{\Theta}_i = \widetilde{\Theta}_j$).

Also for this step, we will only proof the first case, being the second completely similar. To achieve the step, we will show that if $\Theta_i \cap \Theta_j \neq \emptyset$, then θ_i and θ_j coincide for a while after x_i; formally, this means that if $x_i = x_j$ and $\Theta_i \cap \Theta_j \neq \emptyset$ then one has

$$\theta_i([s,t]) = \theta_j([s',t']) \supsetneq \{x_i\} \tag{5.26}$$

with

$$\theta_i(s) = \theta_j(s') = x_i = x_j\,.$$

This will immediately conclude the step by definition of Θ_i and Θ_j, so it is enough to establish (5.26). To this aim, suppose that it is false: then there must exist a path $\bar\theta \in \Theta_i \cap \Theta_j$, and $\bar\theta$ should pass twice from x_i: once in order to follow θ_i for a while, and one to follow θ_j for another while. But this would mean that the path $\bar\theta$ contains a loop, which is impossible thanks to Remark 4.16. Notice that here it is crucial that the claims of Lemma 4.15 and of Remark 4.16 have been established for all the paths $\theta \in \operatorname{spt}\eta$, and not just for $\eta-$almost all.

Step IV. If $\Theta_i = \Theta_j$ and $\widetilde{\Theta}_i = \widetilde{\Theta}_j$ (and so $x_i = x_j$, $y_i = y_j$ by Step II), then θ_i and θ_j coincide between x_i and y_i.

By Step III we already know that, for suitable t_h, t'_h, $1 \leq h \leq 4$, one has

$$\begin{aligned} &\theta_i \llcorner [t_1,t_2] = \theta_j \llcorner [t'_1,t'_2]\,, && \theta_i \llcorner [t_3,t_4] = \theta_j \llcorner [t'_3,t'_4]\,,\\ &\theta_i(t_1) = x_i, && \theta_i(t_4) = y_i\,. \end{aligned}$$

On the other hand, applying twice Lemma 4.20 we know that

$$\theta_i \llcorner [t_2,t_3] = \sigma_{\theta_i(t_2)\theta_i(t_3)} = \sigma_{\theta_j(t'_2)\theta_j(t'_3)} = \theta_j \llcorner [t'_2,t'_3]\,,$$

and so θ_i and θ_j coincide in the whole part between x_i and y_i.

Step V. Conclusion.

Since $\|\eta\| = 1$, Steps I, II and III ensure that there can be at most $1/r$ different sets Θ_i and at most $1/r$ different sets $\widetilde{\Theta}_i$, hence at most $1/r^2$ different pairs $(\Theta_i, \widetilde{\Theta}_i)$. Since Σ is trivially contained in the countable union of the arcs $\widetilde{x_i y_i}$, and since by Step IV these arcs are in fact a finite number, we have concluded the proof. $\square$

Acknowledgments

This work was conceived during the meeting *Giornate di Lavoro in "Calculus of Variations and Geometric Measure Theory"* held in Levico Terme (Italy), and was carried on thanks to the project "Calcolo delle Variazioni" (PRIN 2004) of the Italian Ministry of Education. The work of the third author was partially supported by the italian GNAMPA–INDAM.

Bibliography

[1] L. Ambrosio, Lecture Notes on Optimal Transport Problems, in "Mathematical Aspects of Evolving Interfaces", Lecture Notes in Mathematics, LNM **1812**, Springer (2003), 1–52.

[2] L.Ambrosio & A.Pratelli, Existence and stability results in the L^1 theory of optimal transportation, in "Optimal Transportation and Applications", Lecture Notes in Mathematics, LNM **1813**, Springer (2003), 123–160.

[3] L. Ambrosio & P. Tilli, Topics on analysis in metric spaces, Oxford Lecture Series in Mathematics and its Applications, vol. **25**, Oxford University Press, Oxford (2004).

[4] V. Bangert, Minimal measures and minimizing closed normal one-currents, Geom. Funct. Anal., **9** (1999), 413–427.

[5] M. Beckmann, T. Puu, Spatial economics: density, potential and flow, Elsevier Science Ltd., 1985.

[6] M. Bernot, V. Caselles, & J.-M. Morel, Traffic plans, Publ. Mat., **49** (2) (2005), 417–451.

[7] M. Bernot, V. Caselles & J.-M. Morel, Are there infinite irrigation trees?, J. Mathematical Fluid Mechanics, 8 (3) (2006), 311–332.

[8] M. Bernot, V. Caselles & J.-M. Morel., The structure of branched transportation networks, preprint (2006).

[9] S. Bhaskaran & F.J.M. Salzborn, Optimal design of gas pipeline networks, J. Oper. Res. Society **30** (1979), 1047–1060.

[10] A. Brancolini, Problemi di Ottimizzazione in Teoria del Trasporto e Applicazioni, degree thesis, Università di Pisa, Pisa (2002). Available at the web page http://cvgmt.sns.it .

[11] A. Brancolini & G. Buttazzo, Optimal networks for mass transportation problems, ESAIM Control Optim. Calc. Var., **11** (2005), 88–101.

[12] A. Brancolini, G. Buttazzo & F. Santambrogio, Path functionals over Wasserstein spaces, J. Eur. Math. Soc. **8** (2006), 415–434.

[13] G. Buttazzo, Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations, Pitman Res. Notes Math. Ser. **207**, Longman, Harlow (1989).

[14] G. Buttazzo, E. Oudet & E. Stepanov, Optimal transportation problems with free Dirichlet regions, in "Variational Methods for Discontinuous Structures", Cernobbio 2001, Progress in Nonlinear Differential Equations **51**, Birkhäuser Verlag, Basel (2002), 41–65.

[15] G. Buttazzo, A. Pratelli & E. Stepanov, Optimal pricing policies for public transportation networks, SIAM J. Optimization, **16** (3) (2006), 826–853.

[16] G. Buttazzo & E. Stepanov, On regularity of transport density in the Monge-Kantorovich problem, SIAM J. Control Optim., **42** (3) (2003), 1044–1055.

[17] G. Buttazzo & E. Stepanov, Minimization problems for average distance functionals, in "Calculus of Variations: topics from the mathematical heritage of E. De Giorgi", Quaderni di Matematica II Università di Napoli, vol. **14**, Aracne Editrice, Roma (2004), 47–84.

[18] G. Buttazzo & E. Stepanov, Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (5) **2** (2003), 631–678.

[19] Y. Brenier, The dual least action problem for an ideal, incompressible fluid, Arch. Rational Mech. Anal., **122** (4) (1993), 323–351.

[20] L. Caffarelli, M. Feldman & R.J. McCann, Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs, J. Amer. Math. Soc., **15** (2002), 1–26.

[21] P. Cannarsa & P. Cardaliaguet, Representation of equilibrium solutions to the table problem for growing sandpiles, J. Eur. Math. Soc. (JEMS), **6**(4) (2004), 435–464.

[22] V. Caselles & J.-M. Morel, Irrigation, in Variational methods for discontinuous structures, Vol. **51** of *Progr. Nonlinear Differential Equations Appl.*, pages 81–90, Birkhäuser, Basel, 2002.

[23] C. Castaing & M. Valadier, Convex analysis and measurable multifunctions, Lecture Notes in Mathematics, **580**, Springer (1977).

[24] G. Dal Maso & R. Toader, A Model for the quasi-static growth of brittle fractures: existence and approximation results, Arch. Rational Mech. Anal., **162** (2002), 101–135.

[25] G. Devillanova & S. Solimini, On the dimension of an irrigable measure, preprint (2006).

[26] L.C. Evans, Partial Differential Equations and Monge–Kantorovich Mass Transfer, Current Developments in Mathematics (1997), 65–126.

[27] L.C. Evans & W. Gangbo, Differential Equations Methods for the Monge–Kantorovich Mass Transfer Problem, Memoirs of the A.M.S., Vol. **137**, Number 653, (1999).

[28] K.J. Falconer, The geometry of fractal sets, Cambridge University Press (1985).

[29] M. Feldman & R.J. McCann, Uniqueness and transport density in Monge's mass transportation problem, Calc. Var. P.D.E. **15** (2002), 81–113.

[30] E.N. Gilbert, Minimum cost communication networks, Bell System Tech. J. **46** (1967), 2209–2227.

[31] L.V. Kantorovich, On mass transportation, Doklady Acad. Sci. USSR. **37** (7–8) (1942), 227–229 (in Russian).

[32] L.V. Kantorovich, On a problem of Monge, Uspekhi Mat. Nauk. **3** (1948), 225–226 (in Russian).

[33] L.V. Kantorovich, Economic models of best use of resources, Acad. Sci USSR, 1960 (in Russian).

[34] C. Kuratowski, Topologie, vol. **1**, Państwowe Wydawnictwo Naukowe, Warszawa (1958).

[35] D.H. Lee, Low cost drainage networks, Networks **6** (1976), 351–371.

[36] F. Maddalena, J.M. Morel & S. Solimini, A variational model of irrigation patterns, Interfaces Free Bound **5** (4) (2003), 391–415.

[37] J.-M. Morel & F. Santambrogio, Comparison of distances between measures, Appl. Math. Lett., to appear.

[38] F. Morgan & R. Bolton, Hexagonal economic regions solve the location problem, Amer. Math. Monthly **109** (2) (2002), 165–172.

[39] S.J.N. Mosconi & P. Tilli, Γ-convergence for the irrigation problem, J. Convex Anal. **12** (1) (2005), 145–158.

[40] J.C. Oxtoby, Homeomorphic measures in metric spaces, Proc. Amer. Math. Soc., **24** (1970), 419–423.

[41] E. Paolini & E. Stepanov, Qualitative properties of maximum distance minimizers and average distance minimizers in $\mathbf{R}^n$, J. Math. Sciences (N.Y.), **122** (3) (2004), 105–122.

[42] E. Paolini & E. Stepanov, Optimal transportation networks as flat chains, Interfaces Free Bound **8** (2006) 393–436.

[43] E. Paolini & E. Stepanov, Connecting measures by means of branched transportation networks at finite cost, preprint (2006).

[44] A. Pratelli, Existence of optimal transport maps and regularity of the transport density in mass transportation problems, Ph.D. Thesis, Scuola Normale Superiore, Pisa, Italy (2003). Avalaible on http://cvgmt.sns.it/ .

[45] A. Pratelli, Equivalence between some definitions for the optimal mass transport problem and for the transport density on manifolds, Ann. Mat. Pura Appl., **184** (2005), no. 2, 215–238.

[46] S.T. Rachev & L. Rüschendorf, Mass Transportation Problems, Springer–Verlag (1998).

[47] H.L. Royden, Real Analysis (2nd edition), Macmillan, (1968).

[48] F. Santambrogio, Optimal channel networks, landscape function and branched transport, preprint (2006).

[49] F. Santambrogio & P. Tilli, Blow-up of optimal sets in the irrigation problem, Journal of Geometric Analysis, **15** (3) (2005), 343–362.

[50] L. Simon, Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, Volume **3** (1983).

[51] S.K. Smirnov, Decomposition of solenoidal vector charges into elementary solenoids and the structure of normal one-dimensional currents, St. Petersburg Math. J., **5** (4) (1994), 841–867.

[52] E. Stepanov, Partial geometric regularity of some optimal connected transportation networks, J. Math. Sciences (N.Y.), **132** (4) (2006), 522–552.

[53] V.N. Sudakov, Geometric Problems in the Theory of Infinite–Dimensional Probability Distributions, Proc. of the Steklov Institute of Mathematics, **141** (1979).

[54] A. Suzuki & Z. Drezner, The p-center location, Location science **4** (1-2) (1996), 69–82.

[55] A. Suzuki & A. Okabe, Using Voronoi diagrams, in *Facility location: a survey of applications and methods*, Z. Drezner editor, Springer series in operations research, pages 103–118. Springer Verlag, 1995.

[56] Q. Xia, Optimal paths related to transport problems, Communications in Contemporary Math., **5** (2) (2003), 251–279.

[57] Q. Xia, Interior regularity of optimal transport paths, Calc. Var. Partial Diff. Equations, **20** (3) (2004), 283–299.