BOLZA PROBLEMS WITH DISCONTINUOUS LAGRANGIANS AND LIPSCHITZ–CONTINUITY OF THE VALUE FUNCTION

ANDREA DAVINI

Abstract. We study the local Lipschitz–continuity of the value function v associated with a Bolza Problem in presence of a Lagrangian $L(x, q)$, convex and uniformly superlinear in q, but only Borel–measurable in x. Under these assumptions, the associated integral functional is not lower semicontinuous with respect to the suitable topology which assures the existence of minimizers, so all results known in literature fail to apply. Yet, the Lipschitz regularity of v does not depend on the existence of minimizers. In fact, it is enough to control the derivatives of quasi–minimal curves, but the problem is non–trivial due to the general growth conditions assumed here on $L(x, \cdot)$. We propose a new approach, based on suitable reparameterization arguments, to obtain suitable *a priori* estimates on the Lipschitz constants of quasi–minimizers. As a consequence of our analysis, we derive the Lipschitz–continuity of v and a compactness result for value functions associated with sequences of locally equi–bounded discontinuous Lagrangians.

1. Introduction

1.1. **Description of the problem and main results.** A typical issue in Partial Differential Equations is that of proving the local Lipschitz–continuity in $(0, +\infty) \times \mathbb{R}^N$ of the *value function*

$$v(t, x) := \inf \left\{ u(\gamma(0)) + \int_0^t L(s, \gamma(s), \dot\gamma(s))\,\mathrm{d}s \;:\; \gamma \in W^{1,1}\big([0, t], \mathbb{R}^N\big),\ \gamma(t) = x \right\}$$

associated with a Lagrangian $L : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to (-\infty, +\infty]$ and to a possibly discontinuous *initial cost* $u : \mathbb{R}^N \to (-\infty, +\infty]$. This is an important step when one is interested to show that v is a solution, in a suitable generalized sense, of the equation

$$\partial_t u + H(x, Du) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^N, \tag{1}$$

where H is the Hamiltonian associated with L through the Fenchel transform. When L is continuous, it is well known that v is a solution of (1) in the *viscosity sense* (see e.g. [3, 4, 26]). For discontinuous (and autonomous) Lagrangians, a PDE interpretation of the value function has been provided by Dal Maso and Frankowska in [18, 19]. By making use of the so called contingent derivatives, the authors prove that v satisfies the Hamilton–Jacobi equation in a suitable generalized sense, and characterize it as the unique solution of the associated Cauchy problem with initial datum u when the latter is lower semicontinuous.

The study of discontinuous Hamilton–Jacobi equations is a field of growing attention both from a theoretical viewpoint as well as for the applications, see [6, 8, 10, 11, 12, 28, 30, 33]. It is related to the study of geodesic distances, of some discontinuous control problems, of combustion phenomena in nonhomogeneous media and of geometric optic propagation in presence of layers, see [5, 23, 27]. The analysis we will develop here supplies the tools to prove representation formulas for generalized solutions of time–dependent measurable Hamilton–Jacobi equations in the spirit of [12]. This issue will be discussed in the forthcoming paper [9].

The Lipschitz–continuity of v is strictly related to the regularity of solutions to the *Bolza Problem*

$$\min\left\{u(\gamma(0))+\int_0^t L(s,\gamma(s),\dot\gamma(s))\,\mathrm{d}s\ :\ \gamma\in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(t)=x\right\}, \tag{2}$$

and to the possibility of finding some *a priori* estimates on the Lipschitz constants of minimizers. Clearly, any solution of (2) is also a Lagrangian minimizer with respect to its boundary conditions.

The study of regularity properties of Lagrangian minimizers is a classical topic in the Calculus of Variations. The first results were obtained by Tonelli in the late 20's [35, 36] for real–valued smooth Lagrangians $L(s,x,q)$, coercive and strictly convex in q. More recently, Tonelli's results have been generalized by Clarke and Vinter [17] to the case of measurable, locally bounded integrands $L(s,x,q)$, locally Lipschitz in x, convex and uniformly superlinear in q. By using the tools of non–smooth analysis, the classical Euler–Lagrange necessary condition becomes a differential inclusion.

The autonomous case has been widely studied. The results of [17] have been extended by Ambrosio, Ascenzi and Buttazzo in [2] to the case of a locally bounded Lagrangian $L(x,q)$, lower semicontinuous in x, convex and uniformly superlinear in q. In [18], Dal Maso and Frankowska succeeded to prove the same results without assuming neither semicontinuity in x nor convexity in q. They also obtain some uniform estimates on the Lipschitz constant of the minimizers, which are used to prove that the associated value function v is locally Lipschitz in $(0,+\infty)\times\mathbb{R}^N$, provided problem (2) admits solutions for every $(t,x)\in(0,+\infty)\times\mathbb{R}^N$.

Here we will be concerned with the case of a Borel–measurable Lagrangian $L:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, locally bounded with respect to (x,q), convex and uniformly superlinear in q. The growth conditions assumed on L can be restated in the following equivalent form:

$$\alpha(|q|)\le L(x,q)\le\beta(x,|q|)\qquad\text{for every } (x,q)\in\mathbb{R}^N\times\mathbb{R}^N,$$

where $\alpha(\cdot)$ and $\beta(x,\cdot)$ are superlinear functions from $[0,+\infty)$ to $\mathbb{R}$, with β locally bounded on $\mathbb{R}^N\times[0,+\infty)$ (cf. Lemma 2.3). The model example of Lagrangians included in this class are the ones of the form

$$L(x,q)=F(q)+n(x)$$

with $F(\cdot)$ convex and superlinear, and $n(\cdot)$ Borel–measurable and bounded.

The main result we prove is the local Lipschitz–continuity of the value function v in $(0,+\infty)\times\mathbb{R}^N$. Several Lipschitz–regularity results for the value function associated with a discontinuous Lagrangian, depending on the continuity properties enjoyed by the initial cost, are given in Section 4. Moreover, a compactness result holding for sequences of value functions is derived in Section 4.2 as a consequence of what proved in [20] (cf. Theorem 3.19). This kind of result essentially relies on the fact that all the Lipschitz estimates we provide do not depend explicitly on the Lagrangian, but only on the way $L(x,q)$ grows when $|q|\to+\infty$, i.e. on the functions α,β.

We remark that all results of the paper hold, with the obvious changes of notation, if $\mathbb{R}^N$ is replaced by a connected smooth Riemannian manifold $\mathcal{M}$ without boundary. Proofs can be rephrased by using local coordinates. When $\mathcal{M}$ is compact, some additional information on the Lipschitz continuity of the value function is deduced.

With respect to the literature quoted above, the key new point consists in dealing with a case when the minimizers of the Bolza problem do not exist in general. This gives rise to serious technical difficulties, since all arguments known in literature exploit the existence of minimizers to derive an information on their Lipschitz constants, and this in turn gives the desired regularity of v via a rather standard argument. The same reasoning however works as soon as we provide suitable integral estimates on the derivatives of quasi–optimal curves for $v(t,x)$,([1]) depending with some uniformity with respect to $(t,x) \in (0,+\infty)\times\mathbb{R}^N$. The problem is however non–trivial due to the general growth conditions assumed here on $L(x,\cdot)$, in particular to the fact that functions $\alpha(\cdot), \beta(x,\cdot)$ may have different growths for $|q| \to +\infty$.

The novelty of our approach relies on an unusual way of employing the DuBois–Raymond condition, which motivates the introduction of a distinguished family of Lipschitz curves parameterized in a special way (cf. Definition 3.12). The core of our arguments consists in proving that, in the formula defining $v(t,x)$, it is not restrictive to consider only curves belonging to this family (see Section 3). Once this is established, it is rather easy to obtain the *a priori* estimates on the Lipschitz constants of quasi–optimal curves for $v(t,x)$ that are needed to derive the desired regularity of the value function.

The analysis outlined above is carried out through suitable reparameterization techniques which use in an essential way the fact that L is autonomous and convex in q. The argument on which they are based was originally introduced in [22] and subsequently developed in [21] in the case of a continuous Lagrangian, but its use for the kind of problems studied herein seems new. A substantial effort is furthermore made to extend the techniques to the measurable setting and to gather the necessary information needed in the case at issue.

We end this discussion by mentioning that a possible alternative way to attack the problem would be to find a relaxed formulation of (2) in order to apply the results of [18]. The difficulty here is proving that the relaxation of the functional $\gamma \mapsto \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s$ admits an integral representation on $W^{1,1}\big([0,t],\mathbb{R}^N\big)$. The results proved in [1] assure that this approach actually works if the Lagrangian enjoys the following growth conditions in q:

$$|q|^p \le L(x,q) \le \Lambda\,(1+|q|^p) \qquad \text{for every } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N,$$

for some $p > 1$ and $\Lambda > 0$. To extend such results to the more general cases here considered, we would encounter difficulties similar to the ones previously described. As a matter of fact, a technical adaptation of the arguments here employed allows us to generalize the results of [1] to a wider class of abstract and integral functionals of autonomous type, that includes in particular the ones considered in this paper. This issue is specifically studied in [20].

1.2. **Strategy of the proof.**

To study the problem, we find convenient to introduce the function

$$S(y,x,t) := \inf\left\{ \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \,:\, \gamma \in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(0)=y,\ \gamma(t)=x \right\}. \quad (3)$$

[1] Namely, curves that realize the value $v(t,x)$ in (2), up to an addition of a suitably small positive constant.

defined for every $(y, x, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty)$, and to express v in the following equivalent form:

$$v(t, x) = \inf_{y \in \mathbb{R}^N} \Big(u(y) + S(y, x, t) \Big) \qquad \text{for every } (t, x) \in (0, +\infty) \times \mathbb{R}^N.$$

To conclude, it is enough to prove that S is locally Lipschitz in $\mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty)$. This would immediately follow from [18] if we were able to prove that minimizing curves for $S(y, x, t)$ exist for every $(y, x, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty)$. Unfortunately, this need not be true in our case. In fact, the superlinearity of $L(x, \cdot)$ and the Dunford–Pettis Theorem (see [7, Chapter 2]) actually imply that every minimizing sequence for $S(y, x, t)$ admits a subsequence uniformly converging to some limit curve, but the lack of continuity of L does not guarantee that the associated integral functional is lower semicontinuous for the convergence at issue (classical results by Olech [29] and Ioffe [25] assure that this is true if the Lagrangian is lower semicontinuous in x and convex in q), so the standard *direct method of the Calculus of Variations* fails to apply (see [7]).

Yet, existence of minimizers is not necessary to derive the desired regularity for S. Indeed, a fairly standard argument (see, for instance, [18, Proof of Theorem 4.4]) shows that S is locally Lipschitz as soon as we provide some *a priori* estimates on the Lipschitz constants of quasi–minimizers for $S(y, x, t)$, with some uniformity with respect to $(y, x, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty)$.(2) When the existence of a minimizer γ is postulated, as in [18], or assured by the assumptions made on L, as in [2], this can be deduced from the fact that γ satisfies the DuBois–Raymond necessary condition, namely there exists a constant $a \in \mathbb{R}$ such that

$$L(\gamma(s), \dot\gamma(s)) = \langle \dot\gamma(s), p \rangle - a \qquad \text{for every } p \in \partial_q L(\gamma(s), \dot\gamma(s)) \tag{4}$$

for almost every $s \in [0, t]$. Using the superlinearity of $L(x, \cdot)$, it is then easy to show that a is locally bounded with respect to (y, x, t) and this provides the desired control on the Lipschitz constant of γ.

Even if this reasoning cannot be applied in our case, we however notice that condition (4), which is crucial to get the desired estimates, provides just an information on the parameterization of the curve: when γ is action–minimizing, its parameterization must obey to an optimality condition.

The idea we develop here is to separate the issue of parameterization from that of minimizing the action. This is achieved by first considering a minimization problem with fixed support: we fix a Lipschitz curve $\gamma : [0, \ell] \to \mathbb{R}^N$ parameterized by arc–length, *the support*, and we try to solve the following problem

$$\min \left\{ \int_0^t L(\xi, \dot\xi)\, \mathrm{d}s \,:\, \xi \in [\gamma]_t \right\} \tag{5}$$

for every $t > 0$, where $[\gamma]_t$ denotes the family of absolutely continuous curves $\xi : [0, t] \to \mathbb{R}^N$ obtained through a reparameterization of γ.(3) Here, the crucial remark

2An *ε–minimizer* for $S(y, x, t)$ is a curve $\gamma \in W^{1,1}\big([0, t], \mathbb{R}^N\big)$ with $\gamma(0) = y$, $\gamma(t) = x$ such that

$$\int_0^t L(\gamma(s), \dot\gamma(s))\, \mathrm{d}s < S(y, x, t) + \varepsilon.$$

We say that γ is a *quasi–minimizer* or it is *quasi–optimal* for $S(y, x, t)$ if it is an ε–minimizer with $\varepsilon > 0$ suitably small.

3That is, $\xi = \gamma \circ \varphi$ on $[0, t]$ for some absolutely continuous map $\varphi : [0, t] \to [0, \ell]$ surjective and non–decreasing (cf. Definition 3.9).

is the following: any solution of (5) satisfies (4) for some $a \in \mathbb{R}$; conversely, any $\xi \in [\gamma]_t$ satisfying (4) for some $a \in \mathbb{R}$ is a solution to (5) (cf. proof of Theorem 3.15).

However, existence of minimizers of (5) is not clear. A possible way to tackle the problem might be to apply the standard direct methods to

$$\min\left\{\mathcal{G}(\psi)\,:\,\psi \in W^{1,\infty}([0,\ell],\mathbb{R}),\ \psi(0)=0,\ \psi(\ell)=t\right\},$$

where $\mathcal{G}$ is defined as

$$\mathcal{G}(\psi) := \int_0^\ell g(s,\psi'(s))\,\mathrm{d}s$$

and

$$g(s,u) := \begin{cases} L\left(\gamma(s),\dfrac{\dot\gamma(s)}{u}\right)u & \text{if } u>0 \\[2ex] \liminf\limits_{v\to 0^+} L\left(\gamma(s),\dfrac{\dot\gamma(s)}{v}\right)v & \text{if } u\le 0. \end{cases}$$

The functional $\mathcal{G}$ enjoys some nice properties, such as convexity and sequential lower semicontinuity with respect to the uniform convergence of equi–Lipschitz functions, but the lack of coercivity makes this approach non–trivial.

The idea exploited here is different. First, we introduce the notion of *a–Lagrangian parameterization* (cf. Definition 3.12), which amounts to requiring that the curve satisfies (4). Then we consider the multifunction $T_\gamma(\cdot)$ defined on $\mathbb{R}$ by

$$T_\gamma(a) := \{t>0\,:\,[\gamma]^\flat(a,t) \text{ is non empty}\}\qquad \text{for every } a\in\mathbb{R},$$

where $[\gamma]^\flat(a,t)$ denotes the subset of $[\gamma]_t$ consisting of a–Lagrangian bi–Lipschitz reparameterizations of γ, and we remark that, by what previously observed, the relation $t \in T_\gamma(a)$ implies that problem (5) admits a solution in $[\gamma]^\flat(a,t)$. Our attention is then addressed to establish the relevant properties of the multifunction $T_\gamma(\cdot)$, with particular interest to its range $\bigcup_{a\in\mathbb{R}} T_\gamma(a)$ (see Proposition 3.13). When this coincides with $(0,+\infty)$, we conclude that problem (5) is solvable for every $t>0$. In particular, (5) has a minimizer belonging to $[\gamma]^\flat(a,t)$ for some $a\in\mathbb{R}$, and its Lipschitz constant can be estimated by some $\kappa_a \in \mathbb{R}$ depending on a and on the kind of growth conditions assumed on L only. However, our analysis reveals that the range of $T_\gamma(\cdot)$ may actually be a bounded interval of the form $(0,T)$. In this instance, a solution to (5) exists if $t\le T$. For $t>T$, the minimum in (5) is only an infimum, in general; nevertheless, we are able to prove that this value can be obtained by minimizing the action over the family of κ_{c_γ}–Lipschitzian reparameterizations of γ, where κ_{c_γ} is a positive constant that can be estimated in terms of the growth conditions assumed on L (see Theorem 3.15).

This information is used to get the sought *a priori* estimates on the Lipschitz constants of quasi–minimizers (see Lemma 3.2): since any absolutely continuous curve from $[0,t]$ to $\mathbb{R}^N$ belongs to $[\gamma]_t$ for a suitable choice of the Lipschitz curve $\gamma:[0,\ell]\to\mathbb{R}^N$ (cf. Lemma 3.11), a quasi–minimizer for $S(y,x,t)$ can be always assumed to be κ_a–Lipschitz continuous, for some $a\in\mathbb{R}$. By using the superlinearity of $L(x,\cdot)$, the constant a is last estimated with some uniformity with respect to (y,x,t).

1.3. **Plan of the article.** Section 2 contains the main notation and assumptions, together with some well known propositions that will be needed in the rest of the paper.

The properties of the function S are studied in Section 3. In Section 3.1 some preliminary results are collected. The definition of a–Lagrangian reparameterization and the reparameterization arguments are presented in Section 3.2. Here, the main properties of the multifunction $T_\gamma(\cdot)$ are established and used to study an action–minimization problem with fixed support. The information gathered are then exploited to derive the sought *a priori* estimates on the Lipschitz constants of quasi–minimizers (cf. Lemma 3.2), which is all we need to prove Theorem 3.1. To simplify the exposition, the Lagrangian L is initially assumed locally bounded in q, uniformly with respect to x. The consequent extension to the case of Lagrangians locally bounded in (x, q) is easily derived in Section 3.3 via a localization argument (see Theorem 3.17). Here a sequential compactness result for locally equi–bounded discontinuous Lagrangians, established in [20], is recalled for later use.

The main results of the paper are derived in Section 4 as a simple application of the preceding analysis. Section 4.1 contains several Lipschitz–regularity results for the value function associated with a discontinuous Lagrangian, depending on the continuity properties assumed on the initial cost. An extension to the case when $\mathbb{R}^N$ is replaced by a compact and connected smooth Riemannian manifold $\mathcal{M}$ without boundary is also provided. Last, Section 4.2 contains a compactness result for the value functions associated with sequences of locally equi–bounded discontinuous Lagrangians.

2. Notation and standing assumptions

We write below a list of symbols used throughout this paper.

N	an integer number
$B_r(x)$	the open ball in $\mathbb{R}^N$ of radius r centered at x
B_r	the open ball in $\mathbb{R}^N$ of radius r centered at 0
$\mathbb{S}^{N-1}$	the $(N-1)$–dimensional unitary sphere of $\mathbb{R}^N$
$\mathcal{H}^k$	k–dimensional Hausdorff measure
$\langle \cdot, \cdot \rangle$	the scalar product in $\mathbb{R}^N$
$[u]$	the integer part of $u \in \mathbb{R}$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathcal{P}(\mathbb{R}_+)$	the family of subsets of $\mathbb{R}_+$
$\mathrm{UC}(\mathbb{R}^N)$	the space of uniformly continuous real functions on $\mathbb{R}^N$
$\mathrm{Lip}(\mathbb{R}^N)$	the space of Lipschitz–continuous real functions on $\mathbb{R}^N$

Given a subset U of $\mathbb{R}^k$, we denote by $\overline{U}$ its closure. We furthermore say that U is *compactly contained* in a subset V of $\mathbb{R}^k$ if $\overline{U}$ is compact and contained in V. If E is a Lebesgue measurable subset of $\mathbb{R}^k$, we denote by $|E|$ its k–dimensional Lebesgue measure, and we say that E is *negligible* whenever $|E| = 0$. The characteristic function of E is denoted by χ_E. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^k$ if it holds up to a negligible subset of $\mathbb{R}^k$. The Euclidean norm of $u \in \mathbb{R}^k$ is denoted by $|u|$.

Given a measurable vector–valued function $f : E \to \mathbb{R}^m$, we write $\|f\|_\infty$ to mean

$\left(\sum_{i=1}^k \|f_i\|_{L^\infty(E)}\right)^{1/2}$, where f_i and $\|f_i\|_{L^\infty(E)}$ denote the i-th component of f and the L^∞-norm of f_i, respectively.

Let $X \subseteq \mathbb{R}^k$ and $\mathcal{B}(X)$ the family of all Borel subsets of X. A multifunction Γ from X to compact subsets of $\mathbb{R}$ is said to be *Borel–measurable* (cf. [14]) if

$$\{x \in X \,:\, \Gamma(x) \cap U \neq \emptyset\} \in \mathcal{B} \qquad \text{for every open set } U \subseteq \mathbb{R}.$$

We say that Γ is *upper semicontinuous* at x if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Gamma(z) \subseteq \Gamma(x) + (-\varepsilon, \varepsilon) \qquad \text{for all } z \in B_\delta(x) \cap X.$$

When $k = 1$, we say that Γ is *non–decreasing on* X if

$$\sup \Gamma(x) \leq \inf \Gamma(y) \qquad \text{for every } x, y \in X \text{ with } x < y.$$

We say that Γ is *non–increasing on* X if the multifunction $-\Gamma(\cdot)$ is non–decreasing on X.

For a function $g : \mathbb{R}^k \to (-\infty, +\infty]$, we denote by $\mathrm{dom}(g)$ its effective domain; i.e., the subset of $\mathbb{R}^k$ where g is finite valued. We will say that g is superlinear if

$$\lim_{|x| \to +\infty} \frac{g(x)}{|x|} = +\infty.$$

For a convex function f from $\mathbb{R}^k$ to $\mathbb{R}$, we will denote by $\partial f(x)$ the *subdifferential* of f at x, defined as

$$\partial f(x) := \{p \in \mathbb{R}^k \,:\, f(y) \geq f(x) + \langle p, y - x\rangle \quad \text{for every } y \in \mathbb{R}^k\}.$$

The set $\partial f(x)$ is closed and convex, and the multifunction $x \mapsto \partial f(x)$ is upper semicontinuous on $\mathbb{R}^k$. We furthermore have (see [31]):

Proposition 2.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be convex. Then f is locally Lipschitz in $\mathbb{R}^k$. More precisely, for every $x_0 \in \mathbb{R}^k$ and $r, \delta > 0$, we have*

$$|f(x) - f(y)| \leq |x - y| \, \frac{2}{\delta} \sup_{B_{r+\delta}(x_0)} f \qquad \text{for every } x, y \in B_r(x_0).$$

In particular, $\partial f(x) \subset (2 \sup_{B_{r+1}} f)\, \overline{B}_1$ for every $x \in B_r$.

Given a function $f : \mathbb{R}^k \to \mathbb{R}$, we define its conjugate $f^* : \mathbb{R}^k \to (-\infty, +\infty]$ as follows:

$$f^*(x) := \sup_{y \in \mathbb{R}^k} \{\langle x, y\rangle - f(y)\} \qquad \text{for every } x \in \mathbb{R}^k.$$

We record for later use the following well known facts (cf. [31, Theorem 23.5])

Proposition 2.2. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be superlinear and convex. Then f^* is locally bounded and convex on $\mathbb{R}^k$. Moreover,*

$$f(x) = f^{**}(x) := \sup_{y \in \mathbb{R}^k} \{\langle x, y\rangle - f^*(y)\} \qquad \text{for every } x \in \mathbb{R}^k.$$

The following conditions on $x, x^* \in \mathbb{R}^k$ are equivalent to each other:

- (i) $f(x) + f^*(x^*) \leq \langle x, x^*\rangle$;
- (ii) $f(x) + f^*(x^*) = \langle x, x^*\rangle$;
- (iii) $x^* \in \partial f(x)$;
- (iv) $x \in \partial f^*(x^*)$.

By a modulus we mean a nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, vanishing and continuous at 0.

We denote by $W^{1,1}\big([0,t],\mathbb{R}^N\big)$ the space of absolutely continuous curves from the interval $[0,t]$ to $\mathbb{R}^N$. We recall that a curve $\gamma:[a,b]\to\mathbb{R}^N$ is said to be *parameterized by arc–length* if $|\dot\gamma(s)|=1$ for almost every $s\in(a,b)$.

Throughout the paper, α,β will always denote two functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ that are convex, non–decreasing and superlinear, namely

$$\lim_{h\to+\infty}\frac{\alpha(h)}{h}=\lim_{h\to+\infty}\frac{\beta(h)}{h}=+\infty.$$

We will denote by L a function from $\mathbb{R}^N\times\mathbb{R}^N$ to $\mathbb{R}$ such to satisfy the following assumptions:

(L1) $\quad L$ is Borel–measurable on $\mathbb{R}^N\times\mathbb{R}^N$,

(L2) $\quad \alpha(|q|)\le L(x,q)\le\beta(|q|)\qquad$ for all $(x,q)\in\mathbb{R}^N\times\mathbb{R}^N$,

(L3) $\quad L(x,\cdot)\quad$ is convex for every $x\in\mathbb{R}^N$.

Up to adding a constant to it, it is not restrictive, by (L2), to assume that L is positive. This will be systematically done in the sequel. We also point out that the second inequality in (L2) is equivalent to requiring that

$$\sup\{L(x,q)\,:\,(x,q)\in\mathbb{R}^N\times B_R\}<+\infty\qquad\text{for any } R>0.$$

In fact, the following holds.

Lemma 2.3. *Let U be an open subset of $\mathbb{R}^k$ and $L:U\times\mathbb{R}^k\to\mathbb{R}_+$ such that*

$$\sup\{L(x,q)\,:\,x\in U,\ |q|\le n\}<+\infty\qquad\textit{for every } n\in\mathbb{N}.$$

Then there exists a function $\beta:\mathbb{R}_+\to\mathbb{R}_+$, convex and non–decreasing, such that

$$L(x,q)\le\beta(|q|)\qquad\textit{for every } (x,q)\in U\times\mathbb{R}^k.$$

Proof. Set

$$a_n:=\sup\{L(x,q)\,:\,x\in U,\ |q|\le n\}\qquad\text{for each } n\in\mathbb{N},$$

and

$$f(h):=\sum_{n=1}^{\infty}a_n\,\chi_{[n-1,n)}(h)\qquad\text{for every } h\ge0.$$

As $L(x,q)\le f(|q|)$ for every $(x,q)\in U\times\mathbb{R}^k$, it will be enough to prove the statement for f. For each $n\in\mathbb{N}$, choose $m_n:=\max\{2a_n/(n-1),\,a_n-a_{n-1}\}$ and set

$$\beta(h):=\sup_{n\in\mathbb{N}}\{\,a_{n+1}+m_{n+1}(h-n)\,\}\qquad\text{for every } h\ge0.$$

By definition, each map $h\mapsto a_{n+1}+m_{n+1}(h-n)$ is greater or equal than f on $[n-1,n)$ and less than 0 on $[0,n/2)$, hence

$$f(h)\le\beta(h)<+\infty\qquad\text{for every } h\ge0.$$

The remainder of the assertion follows as β is the supremum of a family of convex and increasing functions. $\square$

To any L satisfying assumptions (L1)–(L3), we associate the function $S(y,x,t)$ defined through (3), for every $(y,x,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$. It is easy to check that the function S enjoys the following inequalities

$$t\,\alpha\left(\frac{|y-x|}{t}\right) \leq S(y,x,t) \leq t\,\beta\left(\frac{|y-x|}{t}\right) \qquad \text{on } \mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty). \tag{6}$$

Later on in the paper, condition (L2) will be relaxed to cover the case of a Lagrangian L locally bounded with respect to (x,q), i.e. such that

$$\sup\{L(x,q)\,:\,(x,q) \in B_R \times B_R\} < +\infty \qquad \text{for any } R > 0.$$

By Lemma 2.3, this amounts to replacing condition (L2) with the following one:

(L2′) $\quad \alpha(|q|) \leq L(x,q) \leq \beta_n(|q|) \qquad$ for all $(x,q) \in B_n \times \mathbb{R}^N$ and $n \in \mathbb{N}$,

where $(\beta_n)_{n\in\mathbb{N}}$ is a family of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

3. The Key Results

The goal of the analysis we are about to present is that of proving the local Lipschitz continuity of the function S associated via (3) with a Lagrangian satisfying assumptions (L1), (L2), (L3). As previously noticed, condition (L2) amounts to requiring that the function $L(x,\cdot)$ is superlinear and locally bounded on $\mathbb{R}^N$, uniformly with respect to x. This growth condition will be relaxed at the end of this section in order to consider the case of Lagrangians locally bounded with respect to (x,q). The precise statement of the result that we will establish is the following.

Theorem 3.1. *Let $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ be an autonomous Lagrangian satisfying conditions (L1)–(L3). Then the associated function S defined through* (3) *is locally Lipschitz in $\mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$. More precisely, for every $M > 0$ there exists $K = K(M,\alpha,\beta)$ such that*

$$S \quad \text{is } K\text{–Lipschitz continuous in } \overline{C_M},$$

where $C_M := \{(y,x,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)\,:\,|x-y| < M\,t\}$.

Theorem 3.1 can be proved via a rather standard argument as soon as we derive some a *a priori* estimates on the Lipschitz constant of quasi–optimal curves parameterized in $[0,t]$ and connecting y to x, for every $(y,t,x) \in C_M$. This information can be derived from the following Lemma:

Lemma 3.2. *Let $x, y \in \mathbb{R}^N$ and $t > 0$ such that $S(y,x,t) < M\,t$. Then there exists a constant $\kappa = \kappa(M,\alpha,\beta)$ such that*

$$S(y,x,t) = \inf\left\{\int_0^t L(\xi,\dot\xi)\,\mathrm{d}s\,:\,\xi(0)=y,\ \xi(t)=x,\ \|\dot\xi\|_\infty \leq \kappa\right\}.$$

The proof of Lemma 3.2 is quite delicate and relies on a careful analysis on the role played by reparameterizations. It will be carried out in the next two subsections. Before that, let us show how Lemma 3.2 can be used to prove Theorem 3.1.

Proof of Theorem 3.1. For a fixed $M > 0$, choose (y_1, t_1, x_1) and (y_2, t_2, x_2) in C_M, and set

$$h := |t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|, \qquad s_0 := \frac{t_1 - t_2}{2} + h.$$

Since C_M is convex, it suffices to prove the statement locally, namely for small values of h. Choose $h < t_2/2$ so that $s_0 < t_1/2$. Fix $\varepsilon > 0$ and let $\gamma_1 \in W^{1,1}\big([0, t_1], \mathbb{R}^N\big)$ be an ε–minimizer connecting y_1 to x_1. As $S(y_1, x_1, t_1) < t_1\, \beta(M)$, by Lemma 3.2 we can assume $\|\dot\gamma\|_\infty \le \kappa$ for some constant $\kappa = \kappa(M, \alpha, \beta)$. Choose $u_1, v_1 \in \mathbb{R}^N$ so that

$$\gamma_1(s_0) = y_2 + h u_1, \qquad \gamma_1(t_1 - s_0) = x_2 + h v_1,$$

and note that $|u_1|, |v_1| < 1 + 2\kappa$. Define a curve $\gamma_2 : [0, t_2] \to \mathbb{R}^N$ connecting y_2 to x_2 by setting:

$$\gamma_2(s) := \begin{cases} y_2 + s u_1 & \text{if } s \in [0, h], \\ \gamma_1(s_0 + s - h) & \text{if } s \in [h, t_2 - h] \\ x_2 + (t_2 - s) v_1 & \text{if } s \in [t_2 - h, t_2] \end{cases}$$

Recalling that L is positive, we get

$$\begin{aligned} S(y_2, x_2, t_2) - S(y_1, x_1, t_1) &\le \int_0^{t_2} L(\gamma_2, \dot\gamma_2)\, \mathrm{d}s - \int_0^{t_1} L(\gamma_1, \dot\gamma_1)\, \mathrm{d}s \ + \varepsilon \\ &\le \int_0^{h} L(\gamma_2, u_1)\, \mathrm{d}s + \int_{t_2 - h}^{t_2} L(\gamma_2, u_2)\, \mathrm{d}s \ + \varepsilon \le 2\,\beta(1 + 2\kappa)\, h \ + \varepsilon, \end{aligned}$$

so, setting $\widetilde K := 2\,\beta(1 + 2\kappa)$, we obtain

$$S(y_2, x_2, t_2) - S(y_1, x_1, t_1) \le \widetilde K \ (|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|) + \varepsilon.$$

As ε is arbitrary, the conclusion follows at once by interchanging the roles of (y_1, t_1, x_1) and (y_2, t_2, x_2) and by setting $K := \sqrt{2N + 1}\, \widetilde K$. □

3.1. **Preliminary tools.** Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the Hamiltonian associated with L through the Fenchel transform, namely

$$H(x, p) := \max_{q \in \mathbb{R}^N} \{\langle p, q\rangle - L(x, q)\}.$$

The function H is Borel–measurable, and $H(x, \cdot)$ is convex and superlinear for every $x \in \mathbb{R}^N$. For every $a \in \mathbb{R}$, set

$$\sigma_a(x, q) := \max\{\langle q, p\rangle \ : \ H(x, p) \le a\} \qquad \text{for every } q \in \mathbb{R}^N,\ a \in \mathbb{R},$$

where we agree that $\sigma_a(x, q) = -\infty$ whenever $a < -L(x, 0) = \min_{\mathbb{R}^N} H(x, \cdot)$.

Proposition 3.3. *For any $a \in \mathbb{R}$, the following properties hold:*

(i) $\sigma_a(x, \lambda\, q) = \lambda\, \sigma_a(x, q)$ *for every $(x, q) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\lambda > 0$;*

(ii) $L(x, q) \ge \sigma_a(x, q) - a$ *for every $(x, q) \in \mathbb{R}^N \times \mathbb{R}^N$.*

Proof. Assertion (i) is clear by definition. To prove (ii), we recall that L is the Fenchel transform of H (cf. Proposition 2.2), hence

$$L(x,q) = \max_{p \in \mathbb{R}^N} \{ \langle p, q \rangle - H(x,p) \} \geq \max_{H(x,p) \leq a} \{ \langle p, q \rangle - H(x,p) \} = \sigma_a(x,q) - a, \quad (7)$$

as claimed. $\square$

For any $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$ and $a \in \mathbb{R}$, we set

$$\Lambda_a(x,q) := \{ \lambda \in [0,+\infty) \, : \, L(x, \lambda\, q) = \sigma_a(x, \lambda\, q) - a \}, \quad (8)$$

and

$$\underline{\lambda}_a(x,q) := \inf \Lambda_a(x,q), \qquad \overline{\lambda}_a(x,q) := \sup \Lambda_a(x,q).$$

We agree that $\underline{\lambda}_a(x,q) = \overline{\lambda}_a(x,q) = 0$ whenever $\Lambda_a(x,q) = \emptyset$, that is, when $a < -L(x,0)$.

We define the following functions:

$$\alpha_*(u) := \max_{\lambda \in \mathbb{R}} \{ u\lambda - \alpha(|\lambda|) \}, \quad \beta_*(u) := \max_{\lambda \in \mathbb{R}} \{ u\lambda - \beta(|\lambda|) \} \qquad \text{for every } u \in \mathbb{R},$$

and we remark that they are convex and superlinear as $\alpha(|\cdot|)$ and $\beta(|\cdot|)$ are so. For every $a \in \mathbb{R}$, set

$$R_a := \max \{ |u| \, : \, \beta_*(u) \leq a \} \quad (9)$$

and

$$\kappa_a := 2 \max \{ \alpha_*(u) \, : \, |u| \leq R_a + 1 \}. \quad (10)$$

The following compactness result holds.

Lemma 3.4. *Let $a \in \mathbb{R}$. Then*

$$\Lambda_a(x,q) \subseteq [0, \kappa_a] \qquad \textit{for every } (x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}.$$

Proof. From the definition of α_* and β_* we obtain

$$\beta_*(|p|) \leq H(x,p) \leq \alpha_*(|p|) \qquad \text{for all } (x,p) \in \mathbb{R}^N \times \mathbb{R}^N, \quad (11)$$

in particular

$$\{ p \in \mathbb{R}^N \, : \, H(x,p) \leq a \} \subseteq \overline{B}_{R_a} \qquad \text{for every } x \in \mathbb{R}^N. \quad (12)$$

Pick up $(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. From (7) and Proposition 2.2 we derive that $\lambda \in \Lambda_a(x,q)$ if and only if $\lambda\, q \in \partial_p H(x,p)$ for some $H(x,p) \leq a$. In particular

$$\Lambda_a(x,q) \subseteq \left\{ |v| \, : \, v \in \partial_p H(x,p) \quad \text{for some } p \in \overline{B}_{R_a} \right\},$$

and the conclusion follows at once in view of (11) and of Proposition 2.1. $\square$

We now fix $(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$ and we examine the properties of the multifunction $a \mapsto \Lambda_a(x,q)$. Proposition 2.2 yields that

$$L(x, \lambda\, q) = \langle p, \lambda\, q \rangle - H(x,p) \qquad \text{for any } p \in \partial_q L(x, \lambda\, q), \quad (13)$$

for any given $\lambda \in \mathbb{R}$. In view of Proposition 3.3–(ii), we infer that $\lambda \in \Lambda_a(x,q)$ if and only if $a \in H(x, \partial_q L(x, \lambda\, q))$.

We start by considering the set–valued map $A(\lambda) := H(x, \partial_q L(x, \lambda\, q))$ on $[0,+\infty)$, which is the inverse of $a \mapsto \Lambda_a(x,q)$, in the sense of set–valued analysis (see [32, Chapter 5]). Indeed, note that

$$\Lambda_a(x,q) = \{ \lambda \in [0,+\infty) \, : \, a \in H(x, \partial_q L(x, \lambda\, q)) \}. \quad (14)$$

Proposition 3.5. *Let $A(\cdot)$ as above. The following facts hold.*

(i) For any $\lambda \in \mathbb{R}$

$$A(\lambda) = [\underline{a}(\lambda), \overline{a}(\lambda)] \quad \textit{for some } -L(x,0) \le \underline{a}(\lambda) \le \overline{a}(\lambda) < +\infty.$$

Moreover

$$A(0) = \{ -L(x,0) \}, \qquad \lim_{\lambda \to +\infty} \underline{a}(\lambda) = +\infty.$$

(ii) The set–valued map $A(\cdot)$ is upper semicontinuous on $[0, +\infty)$. In particular, $\underline{a}(\cdot)$ is lower semicontinuous and $\overline{a}(\cdot)$ is upper semicontinuous on $[0, +\infty)$.

(iii) The set–valued map $\lambda \mapsto A(\lambda)$ is non–decreasing on $[0, +\infty)$.

(iv) $\displaystyle\bigcup_{\lambda \ge 0} A(\lambda) = [-L(x,0), +\infty)$.

Proof. The function $f(\lambda) := L(x, \lambda\, q)$ is convex and superlinear, hence its conjugate f^* is so. We claim that $A(\lambda) = f^*(\partial f(\lambda))$ for every $\lambda \ge 0$. Indeed, by Proposition 2.2 we know that

$$f^*(\partial f(\lambda)) = \lambda\, \partial f(\lambda) - f(\lambda) \qquad \text{for any } \lambda \ge 0.$$

By classical result of non–smooth analysis (cf. [16, Theorem 2.3.10]), we also know that $\partial f(\lambda) = \langle \partial_q L(x, \lambda\, q), q \rangle$, hence the above equality becomes

$$f^*(\partial f(\lambda)) = \langle \partial_q L(x, \lambda\, q), \lambda\, q \rangle - L(x, \lambda\, q) \qquad \text{for any } \lambda \ge 0,$$

and the right–hand side term coincides with $A(\lambda)$ by (13), as claimed.

Let us now prove the above stated properties of $A(\cdot)$. As f is convex, its subdifferential $\partial f(\lambda)$ is a compact interval of $\mathbb{R}$, so the same is true for $A(\lambda)$. The equality $A(0) = \{-L(x,0)\}$ is an immediate consequence of (13), while the other assertion follows by superlinearity of f^* and f. That proves (i). The upper semicontinuity of $A(\cdot)$ comes from the fact that the multifunction $\lambda \mapsto \partial f(\lambda)$ is upper semicontinuous and f^* is continuous. The remainder of (ii) follows from that by definition of $\underline{a}(\cdot)$, $\overline{a}(\cdot)$.

Let us prove (iii). Since f^* and f are convex, the multimappings $u \mapsto \partial f^*(u)$ and $\lambda \mapsto \partial f(\lambda)$ are non–decreasing on $\mathbb{R}$. By superlinearity, we get in particular

$$\bigcup_{\lambda \ge 0} \partial f(\lambda) = [\underline{u}(0), +\infty) \qquad \text{with } \underline{u}(0) \in \partial f(0).$$

By duality (cf. Proposition 2.2), $0 \in \partial f^*(\underline{u}(0))$, so the monotonicity of $\partial f^*(\cdot)$ yields that f^* is non–decreasing on $[\underline{u}(0), +\infty)$.

Item (iv) comes from (ii) and (iii). □

Example 3.6. Take a Lagrangian of the form $L(x,q) = F(q) + n(x)$ for every $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$, with $F(\cdot)$ convex and superlinear, and $n(\cdot)$ Borel–measurable and bounded. We have

$$A(\lambda) = F^*\left(\partial F(\lambda\, q)\right) - n(x) \qquad \text{for every } \lambda \ge 0,$$

for any fixed $(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. When for instance $F(q) = |q|^2/2$, it reduces to

$$A(\lambda) = |\lambda\, q|^2/2 - n(x).$$

We use this information to prove a result that will be crucial for our future analysis.

Proposition 3.7. *Let $(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. The following facts hold.*

(i) For any $a \geq -L(x,0)$, we have

$$\Lambda_a(x,q) = [\underline{\lambda}_a(x,q), \overline{\lambda}_a(x,q)] \qquad \textit{for some} \quad 0 \leq \underline{\lambda}_a(x,q) \leq \overline{\lambda}_a(x,q) < +\infty.$$

Moreover,

$$\underline{\lambda}_{-L(x,0)}(x,q) = 0, \qquad \lim_{a \to +\infty} \underline{\lambda}_a(x,q) = +\infty.$$

(ii) The set–valued map $a \mapsto \Lambda_a(x,q)$ is upper semicontinuous and non–decreasing on $[-L(x,0), +\infty)$.

(iii) $$\underline{\lambda}_a(x,q) = \sup_{b<a} \overline{\lambda}_b(x,q) \qquad \textit{for any } a > -L(x,0)$$

and

$$\overline{\lambda}_a(x,q) = \inf_{b>a} \underline{\lambda}_b(x,q) \qquad \textit{for any } a \geq -L(x,0).$$

(iv) $$\underline{\lambda}_a(x,q) \geq \frac{a + L(x,0)}{2R_a} \quad \textit{for any } a > -L(x,0), \textit{ with } R_a \textit{ defined by (9).}$$

Proof. We recall that $\Lambda_a(x,q) = \{\lambda \geq 0 \,:\, a \in A(\lambda)\}$. The monotonicity and coercivity property of the set–valued map $a \mapsto \Lambda_a(x,q)$ is a consequence of Proposition 3.5, while the equality $\underline{\lambda}_{-L(x,0)}(x,q) = 0$ is apparent by definition (8). In particular, $\Lambda_a(x,q)$ is a bounded interval for any $a \geq -L(x,0)$.

To prove the upper semicontinuity of $a \mapsto \Lambda_a(x,q)$, we need to show that, for each pair of sequences $(a_n)_n$ and $(\lambda_n)_n$ such that $a_n \to a \in \mathbb{R}$, $\lambda_n \to \lambda \in \mathbb{R}$ and $\lambda_n \in \Lambda_{a_n}(x,q)$ for every $n \in \mathbb{N}$, we have $\lambda \in \Lambda_a(x,q)$. That easily follows by the upper semicontinuity of $A(\cdot)$ (in fact, it is equivalent, cf. [32, Theorem 5.7]). In particular, this implies that $\Lambda_a(x,q)$ is closed for any $a \geq -L(x,0)$.

Assertion (iii) immediately follows from the monotone and semicontinuous character of the map $a \mapsto \Lambda_a(x,q)$.

Let us prove (iv). Choose $a > -L(x,0)$ and set $\lambda := \underline{\lambda}_a(x,q)$. By Proposition 3.3–(ii) we get

$$\sigma_a(x,\lambda q) = L(x,\lambda q) + a \geq \sigma_{-L(x,0)}(x,\lambda q) + a + L(x,0),$$

hence, by (12),

$$a + L(x,0) \leq \lambda \left(\sigma_a(x,q) - \sigma_{-L(x,0)}(x,q)\right) \leq \lambda \left(R_a + R_{-L(x,0)}\right) |q|,$$

and the statement follows as $R_{-L(x,0)} < R_a$ by definition. □

Example 3.8. Let $L(x,q) := |q|^2/2 + n(x)$ for every $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$, with $n(\cdot)$ Borel–measurable and bounded. For any fixed $(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$, we have

$$\Lambda_a(x,q) = \left\{ \frac{1}{|q|} \sqrt{2\,(a + n(x))} \right\} \qquad \text{for every } a \geq -n(x).$$

3.2. **Optimal reparameterizations.** Let us now consider a Lipschitz curve γ defined on a bounded interval $J := [0, \ell]$.

Definition 3.9. A curve ξ defined on a bounded interval $[0,t]$ is said to be a *reparameterization* of γ if there exists an absolutely continuous map $\varphi : [0,t] \to [0,\ell]$, surjective and non–decreasing, such that

$$\xi = \gamma \circ \varphi \qquad \text{on } [0,t].$$

We furthermore say that ξ is a *(bi–)Lipschitz reparameterization* of γ if φ is a (bi–) Lipschitz homeomorphism.

Remark 3.10. For reasons that will be clear soon, we want to allow a reparameterization to stop at a point for some time. This accounts for the choice of the unusual definition given above.

We introduce the following notation:

$$\begin{array}{rcl} [\gamma]_t & := & \{\xi \in W^{1,1}\big([0,t], \mathbb{R}^N\big) \,:\, \xi \text{ is a reparameterization of } \gamma\,\} \\ [\gamma]^\flat_t & := & \{\xi \in W^{1,1}\big([0,t], \mathbb{R}^N\big) \,:\, \xi \text{ is a bi–Lipschitz reparameterization of } \gamma\,\}. \end{array}$$

The following lemma comes from classical results of analysis in metric spaces (see e.g. [24, Section VII.2]). We give a proof for the reader's convenience.

Lemma 3.11. *Let $\xi \in W^{1,1}\big([0,t], \mathbb{R}^N\big)$. Then there exists a Lipschitz curve γ, defined on a bounded interval $[0,\ell]$, such that $\xi \in [\gamma]_t$. We can furthermore assume that γ is parameterized by arc–length.*

Proof. Let $\varphi(s) := \int_0^s |\dot\xi(\varsigma)|\,\mathrm{d}\varsigma$ for every $s \in [0,t]$, and set $\ell := \varphi(t)$. Clearly, the map $\varphi : [0,t] \to [0,\ell]$ is absolutely continuous, surjective and non–decreasing. We claim that the statement holds true with $\gamma(s) := \xi(\varphi^{-1}(s))$ for every $s \in [0,\ell]$.
Indeed, it is easy to see that γ is well defined. Pick now a pair of points a, b in $[0,\ell]$ with $a < b$. By the monotone character of φ we have $\varphi^{-1}(a) = [A_-, A_+]$, $\varphi^{-1}(b) = [B_-, B_+]$ for some $A_- \le A_+ < B_- \le B_+$. Moreover

$$|\gamma(b) - \gamma(a)| \le \mathcal{H}^1\big(\gamma([a,b])\big) = \mathcal{H}^1\big(\xi([A_-, B_+])\big) = \int_{A_-}^{B^+} |\dot\xi(\varsigma)|\,\mathrm{d}\varsigma = b - a,$$

which yields that γ is 1–Lipschitz continuous. From the fact that $\int_0^\ell |\dot\gamma(\varsigma)|\,\mathrm{d}\varsigma = \mathcal{H}^1\big(\gamma([0,\ell])\big) = \ell$, we finally get that γ is parameterized by arc–length. $\square$

A further step in the analysis is carried out by picking up some special reparameterizations of the curve γ.

Definition 3.12. A curve ξ defined on a bounded interval $[0,t]$ is said to have an *a–Lagrangian parameterization* if

$$L(\xi(s), \dot\xi(s)) = \sigma_a(\xi(s), \dot\xi(s)) - a \qquad \text{for a.e. } s \in [0,t],\ a \in \mathbb{R}.$$

For any $a \in \mathbb{R}$ and $t > 0$, we define

$$[\gamma](a,t) := \{\xi \in [\gamma]_t \,:\, \xi \text{ has an } a\text{–Lagrangian parameterization}\,\},$$
$$[\gamma]^\flat(a,t) := \{\xi \in [\gamma]^\flat_t \,:\, \xi \text{ has an } a\text{–Lagrangian parameterization}\,\}.$$

Now assume γ is parameterized by arc–length, and let

$$c_\gamma := \operatorname{ess\,sup}_{s\in J} \; -L\left(\gamma(s),0\right)$$

We define a multifunction $T_\gamma : (c_\gamma,+\infty) \to \mathcal{P}(\mathbb{R}_+)$ by setting

$$T_\gamma(a) := \{t>0 \,:\, [\gamma]^\flat(a,t) \text{ is non–empty}\}.$$

The properties of the multifunction $T_\gamma(\cdot)$ are stated below.

Proposition 3.13. *Let γ and $T(\cdot) := T_\gamma(\cdot)$ as above. The following facts hold.*

(i) For any $a > c_\gamma$, $T(a)$ is a compact interval in $(0,+\infty)$, namely

$$T(a) := [\underline{T}\,(a), \overline{T}\,(a)] \qquad \text{for some} \quad \overline{T}\,(a) \geq \underline{T}\,(a) > 0.$$

(ii) The multifunction $T(\cdot)$ is non–decreasing and upper semicontinuous on $(c_\gamma,+\infty)$. Moreover $\quad \inf_{a>c_\gamma} \overline{T}\,(a) = 0$.

(iii) Let $\underline{T}(c_\gamma) := \sup_{a>c_\gamma} \overline{T}(a)$. If $\underline{T}(c_\gamma)$ is finite, then $[\gamma](c_\gamma, \underline{T}(c_\gamma)) \neq \emptyset$.

In particular, for any $0 < t \leq \underline{T}(c_\gamma)$ with $t < +\infty$, there exists $a \geq c_\gamma$ such that γ admits an a–Lagrangian Lipschitz reparameterization on $[0,t]$.

We first prove an auxiliary lemma.

Lemma 3.14. *Let $\gamma : [0,\ell] \to \mathbb{R}^N$ be a Lipschitz curve parameterized by arc–length and $a \in \mathbb{R}$. The following facts hold true.*

(i) For every $t>0$ and $\xi \in [\gamma]_t$, the map $\sigma_a(\xi(\cdot),\dot\xi(\cdot))$ is Lebesgue–measurable on $[0,t]$, and

$$\int_0^t \sigma_a(\xi(s),\dot\xi(s))\,\mathrm{d}s = \int_0^\ell \sigma_a(\gamma(s),\dot\gamma(s))\,\mathrm{d}s. \tag{15}$$

(ii) The maps $\underline{\lambda}_a(\gamma(\cdot),\dot\gamma(\cdot))$, $\overline{\lambda}_a(\gamma(\cdot),\dot\gamma(\cdot))$ are Lebesgue–measurable on $[0,\ell]$.

Proof. Take $t>0$ and $\xi \in [\gamma]_t$. Since the map $s \mapsto (\xi(s),\dot\xi(s))$ is Lebesgue measurable, in order to prove *(i)* it is enough to show that the function σ_a is Borel–measurable on $\mathbb{R}^N \times \mathbb{R}^N$. To this aim, let $(p_n)_n$ and $(\lambda_n)_n$ be two dense sequences in $\mathbb{R}^N$ and $(0,+\infty)$, respectively. The Borel–measurable character of σ_a follows at once by noticing that

$$\sigma_a(x,q) = \inf_k \left(\sup_n \left\{ \langle p_n, q\rangle\, \vartheta_{E_n^k}(x) \right\} \right) \qquad \text{for every } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $E_n^k := \{x \in \mathbb{R}^N \,:\, H(x,p_n) \leq a + 1/k\}$ and $\vartheta_{E_n^k}(\cdot)$ denotes the function identically 1 on E_n^k and $-\infty$ elsewhere. Equality (15) is a consequence of the fact that $\sigma_a(x,\cdot)$ is positively 1–homogeneous.

Let us prove *(ii)*. Since the map $s \mapsto (\gamma(s),\dot\gamma(s))$ takes values in $\mathbb{R}^N \times \mathbb{S}^{N-1}$ for a.e. $s \in [0,\ell]$, it suffices to show that the functions $\underline{\lambda}_a$, $\overline{\lambda}_a$ are Borel–measurable on $\mathbb{R}^N \times \mathbb{S}^{N-1}$. Let us show the statement for $\underline{\lambda}_a$. For each $n \in \mathbb{N}$, set

$$F_n := \{(x,q) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \,:\, H(x, \partial_q L(x,\lambda_n q)) \cap (-\infty,a) \neq \emptyset\},$$

which is Borel measurable for the multifunction $(x,q) \mapsto H(x,\partial_q L(x,\lambda_n q))$ is so. The assertion follows by noticing that $\underline{\lambda}_a(x,q) = \sup_n \lambda_n\, \chi_{F_n}(x,q)$ on $\mathbb{R}^N \times \mathbb{S}^{N-1}$, in view of (14) and of Proposition 3.7. The analogous statement for $\overline{\lambda}_a$ can be proved in a similar way. $\square$

Proof of Proposition 3.13. (i) Fix $a > c_\gamma$, and set

$$\underline{\lambda}_a(\varsigma) := \underline{\lambda}_a(\gamma(\varsigma), \dot\gamma(\varsigma)), \quad \overline{\lambda}_a(\varsigma) := \overline{\lambda}_a(\gamma(\varsigma), \dot\gamma(\varsigma)) \qquad \text{for a.e. } \varsigma \in [0, \ell].$$

Let

$$\underline{T}(a) := \int_0^\ell \frac{1}{\overline{\lambda}_a(\varsigma)}\, d\varsigma, \quad \overline{T}(a) := \int_0^\ell \frac{1}{\underline{\lambda}_a(\varsigma)}\, d\varsigma.$$

Such quantities are well defined, positive real values, thanks to Proposition 3.7-(iv) and to the measurable character of $\underline{\lambda}_a(\cdot)$, $\overline{\lambda}_a(\cdot)$. To show that they belong to $T(a)$, we will prove the existence of two curves $\underline{\gamma}_a$, $\overline{\gamma}_a$, defined on $\big(0, \underline{T}(a)\big)$ and $\big(0, \overline{T}(a)\big)$, respectively, which are a–Lagrangian bi–Lipschitz reparameterizations of γ. To this aim, let us define

$$\underline{f}_a(s) := \int_0^s \frac{1}{\overline{\lambda}_a(\varsigma)}\, d\varsigma, \quad \overline{f}_a(s) := \int_0^s \frac{1}{\underline{\lambda}_a(\varsigma)}\, d\varsigma \qquad \text{for any } s \in [0, \ell],$$

and set

$$\underline{\varphi}_a := (\underline{f}_a)^{-1}, \qquad \overline{\varphi}_a := \left(\overline{f}_a\right)^{-1},$$

defined on $\big(0, \underline{T}(a)\big)$ and $\big(0, \overline{T}(a)\big)$, respectively. As

$$\dot{\underline{\varphi}}_a(\tau) = \overline{\lambda}_a(\underline{\varphi}_a(\tau)), \qquad \dot{\overline{\varphi}}_a(\tau) = \underline{\lambda}_a(\overline{\varphi}_a(\tau)) \qquad \text{for a.e. } \tau,$$

we immediately derive that $\underline{\varphi}_a$ and $\overline{\varphi}_a$ are order–preserving bi–Lipschitz diffeomorphisms. Let us set

$$\underline{\gamma}_a := \gamma \circ \underline{\varphi}_a \quad \text{on } \big(0, \underline{T}(a)\big), \qquad \overline{\gamma}_a := \gamma \circ \overline{\varphi}_a \quad \text{on } \big(0, \overline{T}(a)\big).$$

Since

$$\dot{\underline{\gamma}}_a(\cdot) := \overline{\lambda}_a(\underline{\varphi}_a(\cdot))\ \dot\gamma(\underline{\varphi}_a(\cdot)) \qquad \text{a.e. on } \big(0, \underline{T}(a)\big)$$

and

$$\dot{\overline{\gamma}}_a(\cdot) := \underline{\lambda}_a(\overline{\varphi}_a(\cdot))\ \dot\gamma(\overline{\varphi}_a(\cdot)) \qquad \text{a.e. on } \big(0, \overline{T}(a)\big),$$

we conclude that the curves $\underline{\gamma}_a$ and $\overline{\gamma}_a$ has an a–Lagrangian parameterization by the very definition of $\overline{\lambda}_a$ and $\underline{\lambda}_a$.

In order to prove that $[\underline{T}(a), \overline{T}(a)] \subseteq T(a)$, we will show that

$$\delta\, \underline{T}(a) + (1-\delta)\, \overline{T}(a) \in T(a) \qquad \text{for any } \delta \in (0,1). \tag{16}$$

Fix $\delta \in (0,1)$, and set

$$\delta(\varsigma) := \frac{\delta\, \overline{\lambda}_a(\varsigma)}{\delta\, \overline{\lambda}_a(\varsigma) + (1-\delta)\, \underline{\lambda}_a(\varsigma)}, \qquad \lambda(\varsigma) := \delta(\varsigma)\, \underline{\lambda}_a(\varsigma) + (1-\delta(\varsigma))\, \overline{\lambda}_a(\varsigma)$$

for almost every $\varsigma \in [0, \ell]$, and

$$f(s) := \int_0^s \frac{1}{\lambda(\varsigma)}\, d\varsigma \quad \text{for } s \in [0, \ell], \qquad \varphi := f^{-1} \quad \text{on } [0, f(\ell)].$$

Since $\delta(\varsigma) \in [0,1]$ for almost every $\varsigma \in [0, \ell]$, we get that $\lambda_a(\varsigma) \in \Lambda_a(\gamma(\varsigma), \dot\gamma(\varsigma))$ for almost every $\varsigma \in [0, \ell]$, in particular φ is an order–preserving bi–Lipschitz diffeomorphism. Arguing as above, we see that the curve $\gamma_a := \gamma \circ \varphi$ is an a–Lagrangian bi–Lipschitz reparameterization of γ on $\big(0, f(\ell)\big)$, so $f(\ell) \in T(a)$. Now it is easy to check, by definition of $\delta(\cdot)$, that $f(\ell) = \delta\,\overline{T}\,(a) + (1-\delta)\,\underline{T}\,(a)$. That proves (16) as δ was arbitrarily chosen in $(0,1)$.

Let us now prove that $T(a) \subseteq [\underline{T}\,(a), \overline{T}\,(a)]$. Let $T \in T(a)$ and $\widetilde{\gamma} := \gamma \circ \varphi$ be an a–Lagrangian reparameterization of γ for some order–preserving bi–Lipschitz diffeomorphism $\varphi : [0,T] \to [0,\ell]$. Then

$$\dot{\varphi}(\tau) \in \Lambda_a\left(\gamma(\varphi(\tau)), \dot{\gamma}(\varphi(\tau))\right) \qquad \text{for a.e. } \tau \in (0,T).$$

Let $f := \varphi^{-1}$. We have

$$T = f(\ell) = \int_0^\ell \dot{f}(\varsigma)\,\mathrm{d}\varsigma = \int_0^\ell \frac{1}{\dot{\varphi}(f(\varsigma))}\,\mathrm{d}\varsigma,$$

and since $\dot{\varphi}(f(\varsigma)) \in \Lambda_a(\gamma(\varsigma), \dot{\gamma}(\varsigma)) = [\underline{\lambda}_a(\varsigma), \overline{\lambda}_a(\varsigma)]$ for a.e. $\varsigma \in [0,\ell]$, we clearly get $T \in [\underline{T}\,(a), \overline{T}\,(a)]$.

(ii) Let $b > a > c_\gamma$. Then $\underline{\lambda}_b(\varsigma) \geq \overline{\lambda}_a(\varsigma)$ for almost every $\varsigma \in [0,\ell]$, hence $\overline{T}(b) \leq \underline{T}(a)$. That proves that $T(\cdot)$ is a non–increasing multifunction. To prove that $T(\cdot)$ is u.s.c. on $(c_\gamma, +\infty)$, it will be enough to show that

$$\underline{T}(a) = \sup_{b>a} \overline{T}(b), \qquad \overline{T}(a) = \inf_{b<a} \underline{T}(b) \qquad \text{for any } a > c_\gamma.$$

This actually follows as a simple application of the Monotone Convergence Theorem and by the monotonicity poperties of $\underline{\lambda}_a$, $\overline{\lambda}_a$ (cf. Proposition 3.7-(iii)). The last assertion holds by definition of $\underline{T}(a)$ since $\sup_{a>c_\gamma} \overline{\lambda}_a(\varsigma) = +\infty$ for almost every $\varsigma \in [0,\ell]$.

(iii) Let $\underline{T}(c_\gamma)$ be finite. Arguing as in (i), we may find a non–increasing sequence of Borel–measurable maps $\lambda_n : [0,\ell] \to [0,+\infty)$ such that, for each $n \in \mathbb{N}$,

$$T_n = \int_0^\ell \frac{1}{\lambda_n(\varsigma)}\,\mathrm{d}\varsigma \qquad \text{and} \qquad \lambda_n(\varsigma) \in \Lambda_{c_\gamma + 1/n}(\gamma(\varsigma), \dot{\gamma}(\varsigma)) \qquad \text{for a.e. } \varsigma \in [0,\ell],$$

with $\sup_n T_n = \underline{T}(c_\gamma)$. Set

$$\lambda(\varsigma) = \inf_n \lambda_n(\varsigma) \qquad \text{for every } \varsigma \in [0,\ell].$$

Then $\lambda(\cdot)$ is measurable and $\lambda(\varsigma) \in \Lambda_{c_\gamma}(\gamma(\varsigma), \dot{\gamma}(\varsigma))$ for almost every $\varsigma \in [0,\ell]$. Moreover the Monotone Convergence Theorem yields

$$\underline{T}(c_\gamma) = \sup_{n\in\mathbb{N}}\; T_n = \sup_{n\in\mathbb{N}} \int_0^\ell \frac{1}{\lambda_n(\varsigma)}\,\mathrm{d}\varsigma = \int_0^\ell \frac{1}{\lambda(\varsigma)}\,\mathrm{d}\varsigma,$$

in particular the map

$$f(s) := \int_0^s \frac{1}{\lambda(\varsigma)}\,\mathrm{d}\varsigma$$

is increasing and absolutely continuous on $[0,\ell]$. A c_γ–Lagrangian Lipschitz reparametrization of γ defined on $[0, \underline{T}(c_\gamma)]$ can be now obtained by setting $\widetilde{\gamma} := \gamma \circ \varphi$ with $\varphi := (f)^{-1}$ on $[0, \underline{T}(c_\gamma)]$.

Last, the fact that the multifunction is upper semicontinuous, monotone and convex–set–valued implies that

$$\bigcup_{a>c_\gamma} T(a) = \big(0, \underline{T}(c_\gamma)\big)$$

and this is enough to obtain the remainder of the statement. □

We now seek for an optimal reparameterization of γ on the interval $[0, t]$, for any given $t \in (0, +\infty)$. Such a reparameterization does not exist in general, as we will see. In any case, however, we are able to derive an estimate on the Lipschitz constants of quasi–optimal reparameterizations. This is a crucial step for our study.

Theorem 3.15. *Let $\gamma : [0, \ell] \to \mathbb{R}^N$ be a Lipschitz curve parameterized by arc–length. Then, for every $t \in (0, +\infty)$, there exists $a \geq c_\gamma$ such that*

$$\inf_{\xi \in [\gamma]_t} \int_0^t L(\xi, \dot{\xi}) \, \mathrm{d}s = \inf_{\xi \in [\gamma]_t} \left\{ \int_0^t L(\xi, \dot{\xi}) \, \mathrm{d}s \; : \; \|\dot{\xi}\|_\infty \leq \kappa_a \right\} = \int_0^\ell \sigma_a(\gamma, \dot{\gamma}) \, \mathrm{d}s - a\, t,$$

with κ_a given by (10). *The above infimum is a minimum whenever $t \leq \underline{T}_\gamma(c_\gamma)$, and is in particular attained by some curve belonging to $[\gamma]^\flat(a, t)$ with $a > c_\gamma$ when $t < \underline{T}_\gamma(c_\gamma)$.*

Proof. By Proposition 3.3 and Lemma 3.14, we get

$$\int_0^t L(\xi, \dot{\xi}) \, \mathrm{d}s \geq \int_0^t \Big(\sigma_a(\xi, \dot{\xi}) - a \Big) \, \mathrm{d}s = \int_0^\ell \sigma_a(\gamma, \dot{\gamma}) \, \mathrm{d}s - a\, t \tag{17}$$

for any $a \geq c_\gamma$ and $\xi \in [\gamma]_t$, and (17) is an equality whenever $\xi \in [\gamma](a, t)$. The assertion for $t \leq \underline{T}_\gamma(c_\gamma)$ hence follows in force of Proposition 3.13 and Lemma 3.4.

Let us now assume $t > \underline{T}_\gamma(c_\gamma)$ and set $h := t - \underline{T}_\gamma(c_\gamma)$. Let $\xi \in [\gamma](c_\gamma, \underline{T}_\gamma(c_\gamma))$. By definition of c_γ, there exists, for each $n \in \mathbb{N}$, $s_n \in \big(0, \underline{T}_\gamma(c_\gamma)\big)$ such that

$$c_\gamma + L(\xi(s_n), 0) < \frac{1}{n}.$$

To ease notation, we will write c_n in place of $-L(\xi(s_n), 0)$. We define

$$\xi_n(s) := \begin{cases} \xi(s) & \text{if } s \in (0, s_n] \\ \xi(s_n) & \text{if } s \in [s_n, s_n + h] \\ \xi(s - h) & \text{if } s \in [s_n + h, t). \end{cases}$$

We have

$$\int_0^t L(\xi_n, \dot{\xi}_n) \, \mathrm{d}s = \int_0^{\underline{T}_\gamma(c_\gamma)} L(\xi, \dot{\xi}) \, \mathrm{d}s - h\, c_n = \int_0^{\underline{T}_\gamma(c_\gamma)} \sigma_{c_\gamma}(\xi, \dot{\xi}) \, \mathrm{d}s - \underline{T}_\gamma(c_\gamma)\, c_\gamma$$
$$-h\, c_n = \int_0^\ell \sigma_{c_\gamma}(\gamma, \dot{\gamma}) \, \mathrm{d}s - c_\gamma\, t + h\, (c_\gamma - c_n) < \int_0^\ell \sigma_{c_\gamma}(\gamma, \dot{\gamma}) \, \mathrm{d}s - c_\gamma\, t + \frac{h}{n}.$$

Taking (17) into account, we derive

$$\int_0^\ell \sigma_{c_\gamma}(\gamma, \dot{\gamma}) \, \mathrm{d}s - c_\gamma\, t \leq \int_0^t L(\xi_n, \dot{\xi}_n) \, \mathrm{d}s < \int_0^\ell \sigma_{c_\gamma}(\gamma, \dot{\gamma}) \, \mathrm{d}s - c_\gamma\, t + \frac{h}{n},$$

and we conclude letting $n \to +\infty$. □

Remark 3.16. If in Theorem 3.15 the Lagrangian is assumed lower semicontinuous in x, we can furthermore say that, for every $t > 0$,

$$\inf_{\xi \in [\gamma]_t} \int_0^t L(\xi, \dot{\xi}) \, \mathrm{d}s = \min \left\{ \int_0^t L(\xi, \dot{\xi}) \, \mathrm{d}s \; : \; \xi \in [\gamma](a, t) \right\}$$

for some constant $a \geq c_\gamma$. This can be proved by considering, in place of $T_\gamma(\cdot)$, the set–valued map defined as

$$T^*_\gamma(a) := \{t > 0 \,:\, [\gamma](a,t) \text{ is non–empty}\}$$

for every $a \geq c^*_\gamma := \sup_{s\in J} -L(\gamma(s),0)$. The multifunction $T^*_\gamma(\cdot)$ agrees with $T_\gamma(\cdot)$ on $(c^*_\gamma, +\infty)$. Indeed, the inequality

$$\underline{\lambda}_a(\gamma(s),\dot\gamma(s)) \geq \frac{a - c^*_\gamma}{2R_a} \qquad \text{for a.e. } s \in [0,\ell],$$

which holds true by Proposition 3.7, implies that $[\gamma](a,t) = [\gamma]^\flat(a,t)$ for every $a > c^*_\gamma$ and $t > 0$ (cf. the argument showing that $T(a) \subseteq [\underline{T}(a), \overline{T}(a)]$ in the proof of Proposition 3.13). On the other hand, we always have

$$T^*_\gamma(c^*_\gamma) = [\underline{T}_\gamma(c^*_\gamma), +\infty) \tag{18}$$

when $T(c^*_\gamma)$ is finite, and that is enough to get the statement in view of (17).

To prove (18), let ξ be a curve belonging to $[\gamma](c^*_\gamma, \underline{T}(c^*_\gamma))$ (which does exist in force of Proposition 3.13) and take $s_0 \in [0, \underline{T}_\gamma(c^*_\gamma)]$ such that $L(\xi(s_0),0) = -c^*_\gamma$. Such an s_0 always exists by the upper semicontinuity of $-L(\gamma(\cdot),0)$ on $[0,\ell]$. For every $h > 0$, define $\xi_h : [0, \underline{T}(c^*_\gamma) + h] \to \mathbb{R}^N$ as

$$\xi_h(s) := \begin{cases} \xi(s) & \text{if } s \in [0, s_0] \\ \xi(s_0) & \text{if } s \in [s_0, s_0 + h] \\ \xi(s-h) & \text{if } s \in [s_0 + h, \underline{T}_\gamma(c^*_\gamma) + h]. \end{cases}$$

It is easily seen that ξ_h is a c^*_γ–Lagrangian reparameterization of γ. This shows that

$$\underline{T}_\gamma(c^*_\gamma) + h \in T^*_\gamma(c^*_\gamma) \qquad \text{for every } h > 0,$$

as claimed.

With the aid of the results obtained so far, we can now prove Lemma 3.2.

Proof of Lemma 3.2 Choose $\overline{n} \in \mathbb{N}$ such that $M/\alpha(\overline{n}) < 1/2$ and set

$$A = A(\overline{n}) := \max\{\alpha_*(u) \,:\, |u| \leq 2\,\beta(\overline{n}+1)\},$$

where $\alpha_*(u) := \max_{\lambda\in\mathbb{R}}\{\lambda u - \alpha(|\lambda|)\}$. We claim that the statement holds true with $\kappa := \kappa_A$ defined according to (10).

Indeed, pick a curve $\xi \in W^{1,1}\big([0,t], \mathbb{R}^N\big)$ such that

$$\int_0^t L(\xi,\dot\xi)\,\mathrm{d}s < M\,t,$$

and let $\gamma : [0,\ell] \to \mathbb{R}^N$ be a Lipschitz curve, parameterized by arc–length, such that $\xi \in [\gamma]_t$, according to Lemma 3.11. In view of Theorem 3.15, up to choosing a different ξ in $[\gamma]_t$ without increasing the action, we can always assume that either $\|\dot\xi\|_\infty \leq \kappa_{c_\gamma}$, or $\xi \in [\gamma]^\flat(a,t)$ for some $a > c_\gamma$. In the first case, we note that

$$c_\gamma \leq \alpha_*(0),$$

for $-L(x,0) \leq -\alpha(0) \leq \alpha_*(0)$ for every $x \in \mathbb{R}^N$. The claim follows by definition of κ_A as $\alpha_*(0) \leq A$.

Let us instead assume that ξ belongs to $[\gamma]^\flat(a,t)$ for some $a > c_\gamma$. In particular, $|\dot\xi(s)| \neq 0$ almost everywhere on $[0,t]$. Set $J := \{s \in [0,t] \, : \, 0 < |\dot\xi(s)| < \overline{n}\}$. We have

$$M\,t > \int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \geq \int_0^t \alpha(|\dot\xi|)\,\mathrm{d}s \geq \alpha(\overline{n})\,\big|[0,t]\setminus J\,\big|,$$

hence $|J| > t/2$. Pick up a differentiability point $\overline{s} \in J$ for ξ. By the fact that ξ has an a–Lagrangian parameterization we derive that

$$a \in H\Big(\xi(\overline{s}), \partial_q L(\xi(\overline{s}), \dot\xi(\overline{s})\,)\Big)\,,$$

in particular $a \leq A$ by Proposition 2.1. As $|\dot\xi(s)| \in \Lambda_a(\xi(s), \dot\xi(s)/|\dot\xi(s)|)$ for a.e. $s \in [0,t]$, the claim follows in force of Lemma 3.4 since $\kappa_a \leq \kappa_A$ by definition (10). □

3.3. **Further extensions.** Let us now consider a Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ which satisfies, in place of (L2), condition (L2$'$) for some family $(\beta_n)_{n\in\mathbb{N}}$ of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$; i.e., which is uniformly superlinear in q, and locally bounded in $\mathbb{R}^N \times \mathbb{R}^N$. It is easy to generalize Theorem 3.1 as follows.

Theorem 3.17. *Let $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ be an autonomous Lagrangian satisfying conditions (L1), (L2$'$),(L3). Then the associated function S defined through* (3) *is locally Lipschitz in $\mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$. More precisely, for every $M, r > 0$ there exist a constant $K = K\big(M, r, \alpha, (\beta_n)_n\big)$ such that*

$$S \quad \text{is } K\text{–Lipschitz continuous in } \overline{C_M(r)},$$

where $C_M(r) := \{(y,t,x) \in B_r \times B_r \times (0,r) \, : \, |x-y| < M\,t\,\}$.

Proof. For every $n \in \mathbb{N}$, let us denote by S_n the function associated with the Lagrangian $L_n(x,q) := L(x,q)\,\chi_{B_n}(x) + \beta_n(|q|)\,\chi_{\mathbb{R}^N\setminus B_n}(x)$ through (3). We claim that, for every $M, r > 0$, there exists an index $k = k\big(M, r, \alpha, (\beta_n)_n\big)$ such that

$$S = S_k \quad \text{on } C_M(r). \tag{19}$$

Clearly, that is enough to conclude in force of Theorem 3.1.

Let us fix $M, r > 0$. We first notice that

$$S(y,x,t) < r\,\beta_m\,(M) \qquad \text{for any } (y,t,x) \in C_M(r), \tag{20}$$

where $m := [r] + 1$. Let γ be a curve in $W^{1,1}\big([0,t], \mathbb{R}^N\big)$ connecting y to x such to be quasi–optimal for $S(y,x,t)$. By (20), it is not restrictive to assume that

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s < r\,\beta_m\,(M)\,,$$

in particular

$$\int_0^t |\dot\gamma|\,\mathrm{d}s < r\Big(\alpha_1 + \beta_m(M)\Big),$$

with $\alpha_1 > 0$ such that $\alpha(|q|) \geq |q| - \alpha_1$ for any $q \in \mathbb{R}^N$. As γ has end–points lying in B_r, we deduce that γ is entirely contained in the open ball B_k with

$$k := \big[r\big(1 + \alpha_1 + \beta_m(M)\big)\big] + 1.$$

Thus

$$S(y,x,t) = \inf \left\{ \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \;:\; \gamma(0)=y,\ \gamma(t)=x,\ \gamma([0,t]) \subset B_k \right\}$$

for every $(y,t,x) \in C_M(r)$, and claim (19) follows at once as L coincide with L_k on $B_k \times \mathbb{R}^N$. $\square$

Remark 3.18. Theorem 3.17 still holds if, in place of condition (L3), L satisfies the following weaker convexity assumption:

(L3)′ for every for every $t_1 < t_2$ in $\mathbb{R}$ and $\gamma \in W^{1,\infty}\big((t_1,t_2),\mathbb{R}^N\big)$

$$\lambda \mapsto L(\gamma(s), \lambda\,\dot\gamma(s)) \quad \text{is convex on } \mathbb{R} \qquad \text{for a.e. } s \in (t_1,t_2).$$

The reparameterization techniques and the approach described above can be in fact adapted to this setting. The lack of convexity of L in q give raise to some technical difficulties. For instance, it is no longer true that L is the Fenchel transform of H. These obstructions can be overcome by computing the Fenchel transform of L along straight lines of any fixed direction, and by accordingly modifying the definition of σ_a. For the details, see [20].

We conclude this section by recording a result proved in [20] that we will need later. Let us denote by $\mathcal{L} = \mathcal{L}\big(\alpha, (\beta_n)_n\big)$ the family of Lagrangians satisfying assumptions (L1), (L2)′, (L3)′, where α and β_n, $n \in \mathbb{N}$, are fixed convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let $\Sigma = \Sigma\big(\alpha, (\beta_n)_n\big)$ be the space of functions S on $\mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$ associated through (3) with Lagrangians belonging to $\mathcal{L}$. We endow Σ with the metric induced by the uniform convergence on compact subset of $\mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$. The following result holds (see [20]):

Theorem 3.19. *The space of functions Σ is compact; i.e., every sequence $(S_k)_k$ in Σ admits as subsequence which converges to some element S of Σ, locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N \times (0,+\infty)$.*

Remark 3.20. The Lagrangian L associated with the limit function S trough (3) can be obtained by "differentiation" as follows:

$$L(x,q) = \lim_{h\to 0^+} \frac{S(x, x+hq, h)}{h} \qquad \text{for every } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{21}$$

As proved in [20], L is continuous in q for every x, and convex for almost every x. However, the convexity of $L(x,\cdot)$ for every $x \in \mathbb{R}^N$ is not assured, even if the approximating functions S_k are associated with Lagrangians convex in q.

4. Main Theorems

4.1. **Lipschitz regularity of the value function.** We now use the information gathered so far to prove some regularity properties of the value function $v : (0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined as

$$v(t,x) := \inf \left\{ u(\gamma(0)) + \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \;:\; \gamma \in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(t)=x \right\}, \tag{22}$$

where $u : \mathbb{R}^N \to (-\infty, +\infty]$, $u \not\equiv +\infty$, and L is a Lagrangian satisfying assumptions (L1)–(L3). The above formula can be equivalently restated as

$$v(t,x) = \inf_{y \in \mathbb{R}^N} \Big(u(y) + S_y(t,x) \Big), \tag{23}$$

where $S_y(t,x)$ stands for the function $S(y,x,t)$ associated with L through (3). In what follows, the notation g^+ will denote the positive part of a function $g : \mathbb{R}^k \to [-\infty, +\infty]$, namely $g^+(x) := \max\{g(x), 0\}$ for every $x \in \mathbb{R}^N$.

Theorem 4.1. *Let v be defined by (22) for some $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ satisfying conditions (L1)–(L3). If $u \not\equiv +\infty$ and u is bounded from below, then $v(t,x)$ is locally Lipschitz in $(0,+\infty) \times \mathbb{R}^N$, and*

$$\lim_{t \to 0^+} \| \, (v(t,\cdot) - u)^+ \, \|_{L^\infty(\mathrm{dom}\, u)} = 0. \tag{24}$$

In particular, $\lim\limits_{t \to 0^+} v(t,x) \le u(x)$ for every $x \in \mathbb{R}^N$. Moreover

- *(i) if u is either bounded or in $UC(\mathbb{R}^N)$, then, for any $t_0 > 0$, there exists a constant $K_{t_0} = K(t_0, u, \alpha, \beta)$ such that*

$$v(t,x) \qquad \text{is } K_{t_0}\text{–Lipschitz in} \quad [t_0, +\infty) \times \mathbb{R}^N;$$

- *(ii) if $u \in \mathrm{Lip}(\mathbb{R}^N)$, then there exists a constant $K = K(u, \alpha, \beta)$ such that*

$$v(t,x) \qquad \text{is } K\text{–Lipschitz in} \quad [0, +\infty) \times \mathbb{R}^N.$$

Proof. Pick a point $x_0 \in \mathrm{dom}(u)$ and plug $y = x_0$ in the expression at the right–hand side of (23). We get

$$v(t,x) \le u(x_0) + t\, \beta \left(\frac{|x - x_0|}{t} \right). \tag{25}$$

Any $y \in \mathbb{R}^N$ which is t–optimal for $v(t,x)$ satisfies

$$u(y) + S_y(t,x) \le u(x_0) + t \left(\beta \left(\frac{|x - x_0|}{t} \right) + 1 \right),$$

and that yields, for $y \ne x$,

$$\frac{\alpha\left(\frac{|x-y|}{t}\right)}{\frac{|x-y|}{t}} \le \frac{u(x_0) - u(y)}{|x - y|} + \left(\beta \left(\frac{|x - x_0|}{t} \right) + 1 \right) \frac{t}{|x - y|}. \tag{26}$$

When (t,x) varies in an open set U compactly contained in $(0,+\infty) \times \mathbb{R}^N$, inequality (26) is certainly false if $(y,t,x) \notin \overline{C_M}$ for a suitably large $M = M(U, \|(-u)^+\|_\infty, \alpha, \beta)$. Hence

$$v(t,x) = \inf_y \left\{ u(y) + S_y(t,x) \; : \; (y,t,x) \in \overline{C_M} \, \right\} \qquad \text{for every } (t,x) \in U,$$

and the assertion follows as v is the infimum of a family of equi–Lipschitz functions, in force of Theorem 3.1. Inequality (25) with $x_0 = x$ immediately gives (24) whenever $x \in \mathrm{dom}(u)$.

Items (i) and (ii) can be proved analogously by putting x in place of x_0 in (26) and by choosing as U the sets $(t_0, +\infty) \times \mathbb{R}^N$ and $(0, +\infty) \times \mathbb{R}^N$, respectively. For the case $u \in \mathrm{UC}(\mathbb{R}^N)$, we also make use of the fact that, for any such u, there exists $\varepsilon > 0$ such that

$$|u(x) - u(y)| < \frac{|x-y|}{\varepsilon} \qquad \text{for every } x, y \in \mathbb{R}^N \text{ with } |x-y| > \varepsilon.$$

□

Let us now assume that the Lagrangian L satisfies, in place of (L2), condition (L2′) for some family $(\beta_n)_{n\in\mathbb{N}}$ of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. We provide the following generalization of Theorem 4.1.

Theorem 4.2. *Let v be defined by (22) for some $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ satisfying conditions (L1), (L2′),(L3). If $u \not\equiv +\infty$ and u is bounded from below, then*

$$\lim_{t\to 0^+} \|\, (v(t,\cdot) - u)^+ \,\|_{L^\infty(B_r \cap \operatorname{dom} u)} = 0 \qquad \textit{for every } r > 0.$$

In particular, $\lim\limits_{t\to 0^+} v(t,x) \leq u(x)$ for every $x \in \mathbb{R}^N$. Moreover, $v(t,x)$ is locally Lipschitz in $(0,+\infty)\times\mathbb{R}^N$. More precisely, for every open set U compactly contained in $(0,+\infty)\times\mathbb{R}^N$, there exists a constant $K = K\big(U, u, \alpha, (\beta_n)_n\big)$ such that

$$v(t,x) \qquad \textit{is } K\text{–}\textit{Lipschitz in } U.$$

The proof is omitted, for it can be easily recovered by arguing as above and by using Theorem 3.17 in place of Theorem 3.1.

Last, we want to point out that all results of this paper can be easily extended to the case when $\mathbb{R}^N$ is replaced by a connected smooth Riemannian manifold $\mathcal{M}$ without boundary. In this case, the Lagrangian L is defined on the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and satisfies assumptions (L1), (L2) or (L2′), (L3), with $T\mathcal{M}$, $\|\cdot\|_x$ and $\mathcal{M}$ in place of $\mathbb{R}^N\times\mathbb{R}^N$, $|\cdot|$ and $\mathbb{R}^N$, respectively. ([4]) When $\mathcal{M}$ is compact, Theorem 4.1 can be partially improved as follows.

Proposition 4.3. *Let $\mathcal{M}$ be a compact and connected smooth Riemannian manifold without boundary, and $L : T\mathcal{M} \to \mathbb{R}_+$ an autonomous Lagrangian satisfying conditions (L1)–(L3), with $T\mathcal{M}$, $\|\cdot\|_x$ and $\mathcal{M}$ in place of $\mathbb{R}^N\times\mathbb{R}^N$, $|\cdot|$ and $\mathbb{R}^N$, respectively. Let v be defined by (22) with $u \not\equiv +\infty$ and bounded from below. Then, for any $t_0 > 0$, there exists a constant $K_{t_0} = K(t_0, \alpha, \beta)$ such that*

$$v(t,x) \qquad \textit{is } K_{t_0}\text{–}\textit{Lipschitz in} \quad [t_0,+\infty)\times\mathcal{M}.$$

Remark 4.4. The point is that the constant K_{t_0} appearing above is independent of the initial cost u, provided v is a well defined real function on $(0,+\infty)\times\mathcal{M}$. This is actually guaranteed by the conditions $u \not\equiv +\infty$ and $\inf\limits_{\mathcal{M}} u > -\infty$.

Proof. Let us denote by $\delta_{\mathcal{M}}$ the distance on $\mathcal{M}$ induced by its Riemannian metric. For any $M > 0$, set

$$C_M := \{(y,x,t) \in \mathcal{M}\times\mathcal{M}\times(0,+\infty) \,:\, \delta_{\mathcal{M}}(x,y) < M\,t\}.$$

Since $\mathcal{M}$ is compact, for any $t_0 > 0$ there exists M_{t_0}, depending on t_0 and on the diameter of $\mathcal{M}$ only, such that $\mathcal{M}\times\mathcal{M}\times[t_0,+\infty) \subset C_{M_{t_0}}$. Hence

$$v(t,x) = \inf_y \big\{u(y) + S_y(t,x) \,:\, (y,x,t) \in C_{M_{t_0}}\big\} \quad \text{for any } (t,x) \in [t_0,+\infty)\times\mathcal{M},$$

[4] $\|\cdot\|_x$ denotes the Riemannian norm on $T_x\mathcal{M}$, for every $x \in \mathcal{M}$.

and the assertion follows as v is the infimum of a family of equi–Lipschitz functions, in force of Theorem 3.1. $\square$

4.2. **A compactness result for value functions.** Let $\mathcal{L} = \mathcal{L}\big(\alpha, (\beta_n)_n\big)$ be defined as in Section 3.3, and let $(L_k)_k$ be a sequence of Lagrangians belonging to $\mathcal{L}$ and convex in q. For each $k \in \mathbb{N}$, let

$$v_k(t,x) := \inf\left\{u_k(\gamma(0)) + \int_0^t L_k(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ :\ \gamma \in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(t) = x\right\}$$

for every $(t,x) \in (0,+\infty)\times\mathbb{R}^N$, where u_k is a function from $\mathbb{R}^N$ to $(-\infty,+\infty]$ with $u_k \not\equiv +\infty$. With the aid of Theorem 3.19, we can prove the following result.

Theorem 4.5. *Let $(v_k)_k$ defined as above, and suppose one of the following conditions holds:*

(a) the functions u_k are equi–bounded from below on $\mathbb{R}^N$;
(b) the functions u_k are equi–uniformly continuous on $\mathbb{R}^N$.

Then, up to subsequences, $(v_k)_k$ locally uniformly converges on $(0,+\infty)\times\mathbb{R}^N$ to the function v defined as

$$v(t,x) := \inf\left\{u_*(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ :\ \gamma \in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(t) = x\right\},$$

where L is a Lagrangian belonging to $\mathcal{L}$ and u_* is the function defined as

$$u_*(x) := \inf\left\{\liminf_k u_k(x_k) \ :\ x_k \to x\right\} \qquad \textit{for every } x \in \mathbb{R}^N.$$

Proof. Let us denote by S_k the function associated with L_k through (3). By Theorem 3.19 we know that, up to subsequences, S_k converge to S, locally uniformly on $\mathbb{R}^N\times\mathbb{R}^N\times(0,+\infty)$. Let L be the element of $\mathcal{L}$ derived from S through (21). For every $M, r > 0$, the functions S_k are equi–Lipschitz continuous. Moreover, for every open set U compactly contained in $(0,+\infty)\times\mathbb{R}^N$ there exists a constant M independent of k such that

$$v_k(t,x) = \inf_y\left\{u_k(y) + S_k(y,x,t) \ :\ (y,t,x) \in \overline{C_M(r)}\right\} \quad \text{for every } (t,x) \in U,$$

where r is sufficiently large positive number such that $U \subset (0,r)\times B_r$. To see this, argue as in the proof of Theorem 4.1 and note that M can be estimated in terms of $\sup_k \|(-u_k)^+\|_\infty$ or of the continuity modulus shared by the functions $(u_k)_k$. In particular, the functions v_k are equi–Lipschitz continuous on U. By Ascoli–Arzelà Theorem, the proof then reduces to show that

$$\lim_k v_k(t,x) = v(t,x) \qquad \text{for every } (t,x) \in U.$$

Let us first prove that

$$v(t,x) \geq \limsup_k v_k(t,x) \qquad \text{for every } (t,x) \in U. \tag{27}$$

Chose $y \in \mathbb{R}^N$ and let $y_k \to y$ such that $u_k(y_k)$ converge to $u_*(y)$. We have

$$u_*(y) + S(y,x,t) = \lim_k u_k(y_k) + S_k(y_k,x,t) \geq \limsup_k v_k(t,x),$$

and (27) follows by taking the infimum of the above inequality for all $y \in \mathbb{R}^N$.
Next, let us prove that

$$\liminf_k v_k(t,x) \geq v(t,x) \qquad \text{for every } (t,x) \in U. \tag{28}$$

For each $k \in \mathbb{N}$, take y_k such that $(y_k, t, x) \in \overline{C_M(r)}$ and

$$v_k(t,x) + \frac{1}{k} \geq u_k(y_k) + S_k(y_k, x, t).$$

By possibly considering a subsequence, we can assume that $(y_k)_k$ converges to some point $y \in \mathbb{R}^N$. We infer that

$$\liminf_k v_k(t,x) \geq \liminf_k u_k(y_k) + S(y,x,t) \geq u_*(y) + S(y,x,t)$$

and (28) follows. □

References

[1] M. Amar, G. Bellettini, S. Venturini, Integral representation of functionals defined on curves of $W^{1,p}$. *Proc. Roy. Soc. Edinburgh,* **128A** (1998), 193–217.

[2] L. Ambrosio, O. Ascenzi, G. Buttazzo, Lipschitz Regularity for Minimizers of Integral Functionals with Higly discontinuous Integrands. *J. Math. Anal. Appl.* **142**, no. 2 (1989), 301–316.

[3] M. Bardi, I. Capuzzo Dolcetta, Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] G. Barles, Solutions de viscositè des équations de Hamilton–Jacobi. Mathmatiques & Applications, 17. Springer–Verlag, Paris, 1994.

[5] G. Barles, J.M.Roquejoffre, Large time behaviour of fronts governed by eikonal equations. *Interfaces Free Bound.* **5** (2003), 83–102.

[6] A. Briani, A. Davini, Monge solutions for discontinuous Hamiltonians. *ESAIM Control Optim. Calc. Var.,* **11** (2005), no. 2, 229–251.

[7] G. Buttazzo, M. Giaquinta, S. Hildebrandt, One–dimensional variational problems. An introduction. Oxford Lecture Series in Mathematics and its Applications, **15**. The Clarendon Press, Oxford University Press, New York, 1998.

[8] L. Caffarelli, M. G. Crandall, M. Kocan, A. Swiech, On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.* **49**, no. 4, (1996), 365-397.

[9] F. Camilli, A. Davini, A. Siconolfi, *in preparation.*

[10] F. Camilli, An Hopf-Lax formula for a class of measurable Hamilton-Jacobi equations. *Nonlinear Anal.* **57** (2004), no. 2, 265–286.

[11] F. Camilli, A. Siconolfi, Hamilton–Jacobi equations with measurable dependence on the state variable. *Adv. Differential Equations* **8** (2003), 733-768.

[12] F. Camilli, A. Siconolfi, Time–dependent measurable Hamilton–Jacobi equations. *Comm. Partial Differential Equations*, **30** (2005), no. 4-6, 813–847.

[13] F. Camilli, A. Siconolfi, Effective Hamiltonian and homogenization of measurable Eikonal equations. *Arch. Ration. Mech. Anal.*, to appear.

[14] G. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions. Springer–Verlag, Berlin, 1977.

[15] L. Cesari, Optimization Theory and Applications. Springer–Verlag, New York, 1983.

[16] F.H. Clarke, Optimization and nonsmooth analysis. Wiley, New York, 1983.

[17] F.H. Clarke, R.B. Vinter, Regularity properties of solutions to the basic problem in the Calculus of Variations. *Trans. Amer. Math. Soc.* **289** (1985), 73–98.
[18] G. Dal Maso, H. Frankowska, Autonomous integral functionals with discontinuous nonconvex integrands: Lipschitz regularity of minimizers, DuBois-Reymond necessary conditions, and Hamilton-Jacobi equations. *Appl. Math. Optim.* **48** (2003), no. 1, 39–66.
[19] G. Dal Maso, H. Frankowska, Value functions for Bolza problems with discontinuous Lagrangians and Hamilton-Jacobi inequalities. *ESAIM Control Optim. Calc. Var.* **5** (2000), 369–393.
[20] A. Davini, Integral representation of abstract functionals of autonomous type, *Preprint* (2006), (available at `http://cvgmt.sns.it/cgi/get.cgi/papers/dav06/`).
[21] A. Davini, A. Siconolfi, A generalized dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations. *SIAM J. Math. Anal.*, Vol. 38, No. 2, pp. 478–502.
[22] A. Fathi, A. Siconolfi, PDE aspects of Aubry–Mather theory for continuous convex Hamiltonians. *Calc. Var. Partial Differential Equations* **22** (2005), no. 2, 185–228.
[23] G. Galbraith, Extended Hamilton-Jacobi characterization of value functions in optimal control. *SIAM J. Control Optim.* **39** (2000), 281–305.
[24] M. Giaquinta, G. Modica, Analisi Matematica, Vol. 3. Pitagora Editrice, Bologna, 2000.
[25] A.D. Ioffe, On the lower semicontinuity of integral functionals I. *SIAM J. Control Optim.* **15** (1977), 521–538.
[26] P. L. Lions, Generalized solutions of Hamilton Jacobi equations. Research Notes in Mathematics, 69. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
[27] G. Namah, J.M.Roquejoffre, The "hump" effect in solid propellant combustion. *Interfaces Free Bound.* **2** (2000), 449–467.
[28] R.T. Newcomb, J. Su, Eikonal equations with discontinuities. *Differential Integral Equations* **8** (1995), no. 8, 1947–1960
[29] C. Olech, Weak lower semicontinuity of integral functionals. Existence theorem issue. *J. Optim. Theory Appl.* **19** (1976), 3–16.
[30] D. Ostrov, Solutions of Hamilton-Jacobi equations and scalar conservation laws with discontinuous space-time dependance. *J. Diff. Eq.* **182** (2002), 51–77.
[31] R.T. Rockafellar, Convex Analysis. Princeton Mathematical Series, No. 28. Princeton University Press, 1970.
[32] R.T. Rockafellar, R. J.-B. Wets, Variational Analysis. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 317. Springer-Verlag, Berlin, 1998.
[33] P. Soravia, Boundary value problems for Hamilton-Jacobi equations with discontinuous Lagrangian. *Indiana Univ. Math. J.*, **51** no. 2, (2002), 451–477.
[34] T. Strömberg, On viscosity solutions of irregular Hamilton-Jacobi equations. *Arch. Math.* (Basel) **81** (2003), no. 6, 678–688.
[35] L. Tonelli, Sur une méthòde directe du calcul de variations. *Rend. Circ. Mat. Palermo* **39** (1915), 223–264.
[36] L. Tonelli, Fondamenti di Calcolo delle Variazioni. Vol. 1 (1921), Vol. 2 (1923), Zanichelli, Bologna.

Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Belzoni, 7, 35100 Padova, Italy ([5])

E-mail address: `andrea.davini@ens-lyon.fr`

[5]Current address: Ecole Normale Supérieure, Départment de Mathématiques, 46, allée d'Italie, 69364 Lyon Cedex 7, France