AN AREA FORMULA IN METRIC SPACES

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Abstract. We present an area formula for continuous mappings between metric spaces, under minimal regularity assumptions. In particular, we do not require any notion of differentiability. This is a consequence of a measure theoretic notion of Jacobian, defined as the density of a suitable "pull-back measure".

Let (X, d, μ) and (Y, ρ, ν) be two metric measure spaces, where μ is a Borel regular measure on X and ν is a Borel measure on Y. The terminology "measure" refers to a countably subadditive nonnegative set function, see 2.1.2 of [2]. We also assume that μ is finite on bounded sets and that there exists a μ *Vitali relation* V, 2.8.16 of [2].

The first point is the notion of "pull-back measure" with respect to a continuous mapping. To do this, we need the following important result, proved in 2.2.13 of [2]. *Let X be a complete and separable metric space and let $g : X \longrightarrow Y$ be continuous. Then for every Borel set $B \subset X$, we have that $g(B)$ is ν-measurable.*

Throughout, the above assuptions will constitute our underlying assumptions.

Definition 1 (Pull-back measure)**.** Let (X, d) be complete and separable, let $E \subset X$ be closed and let $f : E \longrightarrow Y$ be continuous. For each $S \subset E$, we set $\zeta(S) = \nu\big(f(S)\big)$. We denote by $f^*\nu$ the measure arising from the Caratheodory's construction applied with ζ defined on the family of Borel sets, according to 2.10.1 of [2]. We say that $f^*\nu$ is the *pull-back measure* of ν with respect to f. The measure $f^*\nu$ is automatically extended to the whole of X, by setting $f^*\nu(A) = f^*\nu(A \cap E)$ for any $A \subset X$.

In the sequel, E will stand for any closed subset of X. Notice that $f^*\nu$ is a Borel regular measure on E, as it follows by the Carathéodory construction.

Recall that the *multiplicity function* of $f : E \longrightarrow Y$ relative to A is defined as $N(f, A, y) = \#\big(A \cap f^{-1}(y)\big)$ for all $y \in Y$. For any Borel set $T \subset E$, Theorem 2.10.10 of [2] gives us the formula

$$f^*\nu(T) = \int_Y N(f, T, y)\, d\nu(y)\,. \tag{1}$$

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Remark 1. It is important to notice that when $f^*\nu$ is absolutely continuous with respect to μ and finite on bounded sets, standard arguments show that

$$f^*\nu(A) = \int_Y N(f, A, y)\, d\nu(y) \tag{2}$$

for any μ-measurable set $A \subset E$, extending (1) to μ-measurable sets.

We are now led to two notions of metric Jacobian.

Definition 2 (Metric Jacobian)**.** Let $f : E \longrightarrow Y$ be continuous and let $x \in E$. Then we introduce two *metric Jacobians* of f at x as follows

$$J_f(x) = (V) \limsup_{S \to x} \frac{\nu\big(f(S \cap E)\big)}{\mu(S)} \quad \text{and} \quad Jf(x) = (V) \limsup_{S \to x} \frac{f^*\nu(S)}{\mu(S)}\,. \tag{3}$$

From 2.8.16 of [2], for each $\overline{\mathbb{R}}$-valued function φ defined on a subset of V, we have

$$(V) \limsup_{S \to x} \varphi(S) = \lim_{\varepsilon \to 0^+} \sup \big\{ \varphi(S) : (x, S) \in V,\ S \in \mathrm{dmn}(\varphi),\ \mathrm{diam}(S) < \varepsilon \big\}\,,$$

where $\mathrm{dmn}(\varphi)$ denotes the domain of φ. It is understood that $(V)\lim$ and $(V)\liminf$ are introduced in analogous way.

In the sequel, we will present in two distinct theorems the metric area formula under slightly different assumptions, that depend on the notion of metric Jacobian we use. This essentially provides an *axiomatic approach* to the area formula in a metric setting, without appealing to any notion of differentiability.

Theorem 1 (Area formula I)**.** *Let $f : E \longrightarrow Y$ be continuous and assume that the pull-back $f^*\nu$ is finite on bounded sets and absolutely continuous with respect to μ. Then Jf is μ-a.e. finite and for all μ-measurable sets $A \subset E$, we have*

$$\int_A Jf(x)\, d\mu(x) = \int_Y N(f, A, y)\, d\nu(y)\,. \tag{4}$$

Proof. Under our assumptions, Theorem 2.9.7 of [2] shows that any μ-measurable set $A \subset X$ is also $f^*\nu$-measurable and the integral formula

$$f^*\nu(A) = \int_A \mathbf{D}(f^*\nu, \mu, V, x)\, d\mu(x)$$

holds, where $\mathbf{D}(f^*\nu, \mu, V, x)$ is the density of $f^*\nu$ with respect to μ and the Vitali relation V, see 2.9.1 of [2]. By definition of metric Jacobian, for any μ-measurable set $A \subset E$, we have $f^*\nu(A) = \int_A Jf(x)\, d\mu(x)$. Thus, formula (2) concludes the proof. $\square$

It should be apparent how in the previous theorem the regularity requirements on the mapping f are transfered to the pull-back measure $f^*\nu$. These conditions on $f^*\nu$ are satisfied in all known contexts concerning the area formula and represent the minimal regularity assumptions. For instance, they are clearly satisfied for mappings between stratified groups and then also between Euclidean spaces, [5].

Another known metric context is that of Lipschitz mappings from subsets of $\mathbb{R}^n$ to metric spaces, equipped with n-dimensional Hausdorff measures. Here an area formula for Lipschitz mappings from Euclidean spaces to metric spaces has been established with different notions of metric Jacobian, [3, 1]. In this framework an a.e. metric differentiability theorem is established for Lipschitz mappings and the metric Jacobians are clearly related to the so-called *metric differential*.

In the following example, we wish to present a special context where no reasonable a.e. metric differentiability theorem holds. Nevertheless, our metric area formula (4) holds, without referring to any differentiable structure.

Example 1. Let us consider the identity $I : (\mathbb{H}^1, d) \longrightarrow (\mathbb{H}^1, \rho)$ of the Heisenberg group, that has been constructed in [4]. Here d is a homogeneous distance of $\mathbb{H}^1$ and ρ is a left invariant distance of $\mathbb{H}^1$ that is not homogeneous. In the above mentioned work, it is proved that I is 1-Lipschitz and *nowhere metrically differentiable*, according to the notion of [3] extended to the group setting. We have the maximal oscillations

$$\limsup_{t\to0^+} \frac{\rho\big(I(x\delta_t z), I(x)\big)}{d(x\delta_t z, x)} = 1 \quad \text{and} \quad \liminf_{t\to0^+} \frac{\rho\big(I(x\delta_t z), I(x)\big)}{d(x\delta_t z, x)} = 0\,. \tag{5}$$

Let us equip $(\mathbb{H}^1, d)$ and $(\mathbb{H}^1, \rho)$ with the Hausdorff measure $\mathcal{H}^4_d$ and $\mathcal{H}^4_\rho$, respectively. Since $\mathcal{H}^4_d$ is doubling on $(\mathbb{H}^1, d)$, by Theorem 2.8.17 of [2], the *covering relation* of closed balls $\{(x, D_{x,r}) : x \in \mathbb{H}^1, r > 0\}$ form an $\mathcal{H}^4_d$ Vitali relation in $(\mathbb{H}^1, d)$. Furthermore, the injectivity of I gives $f^*\mathcal{H}^4_\rho(A) = \mathcal{H}^4_\rho(A) \le \mathcal{H}^4_d(A)$ for any $\mathcal{H}^4_d$-measurable set $A \subset \mathbb{H}^1$. Clearly $f^*\mathcal{H}^4_\rho$ satisfies the assumptions of Theorem 1, hence we have

$$\mathcal{H}^4_\rho(A) = \int_A JI(x)\,\mathcal{H}^4_d(x) \tag{6}$$

where for all $x \in \mathbb{H}^1$, we have

$$JI(x) = J_I(x) = \limsup_{r\to0^+} \frac{\mathcal{H}^4_\rho(D_{x,r})}{\mathcal{H}^4_d(D_{x,r})} = \limsup_{r\to0^+} \frac{\mathcal{H}^4_\rho(D_{0,r})}{\mathcal{H}^4_d(D_{0,r})} = c_0 < +\infty\,.$$

Then we have obtained $\mathcal{H}^4_\rho = c_0\,\mathcal{H}^4_d$ with $c_0 \ge 0$. If we knew that $\mathcal{H}^4_\rho$ is positive on open sets, then the previous equality would also follow by uniqueness of the Haar measure in a locally compact Lie group. This positivity of $\mathcal{H}^4_\rho$ does not seem a straightforward computation due to the strong oscillations of ρ with respect to d, according to (5). Notice that (6) does not refer to any notion of differentiability, although it turns out to be simple a change of variable formula formula for two different measures.

The next lemma is a simple variant of Lemma 2.9.3 in [2], where we replace the Borel regularity of the measure ζ with the absolute continuity with respect to μ.

Lemma 1. *Let ζ and μ be measures that are finite on bounded sets of X, where ζ is absolutely continuous with respect to μ. Then for any $\alpha > 0$ and any μ-measurable set $A \subset \Big\{x \in X \,\Big|\, (V) \liminf_{S\to x} \frac{\zeta(S)}{\mu(S)} < \alpha\Big\}$, we have $\zeta(A) \le \alpha\,\mu(A)$.*

The next version of the metric area formula uses the more manageable notion of metric Jacobian J_f, hence it requires some additional assumptions on f. Since often one can compare this metric notion of Jacobian with the one related to the differential, this theorem can be thought of as a unified approach to the area formula.

Theorem 2 (Area formula II). *Let $f : E \longrightarrow Y$ be continuous and assume that the pull-back $f^*\nu$ is finite on bounded sets and absolutely continuous with respect to μ. If $A \subset E$ is μ-measurable and there exist disjoint μ-measurable sets $\{E_i\}_{i\in\mathbb{N}}$ such that*

$$\mu\Big(E \setminus \bigcup_{i\in\mathbb{N}} E_i\Big) = 0,$$

$f_{|E_i}$ is injective for every $i \geq 1$ and $J_f(x) = 0$ for μ-a.e. $x \in E_0$, then we have

$$\int_A J_f(x)\, d\mu(x) = \int_Y N(f, A, y)\, d\nu(y)\,. \tag{7}$$

Proof. We can assume that any E_i is contained in E. Let us fix $\varepsilon > 0$ and consider a sequence of closed sets $C_i \subset E_i$ such that $\mu(E_i \setminus C_i) \leq \varepsilon 2^{-i}$ for any $i \in \mathbb{N}$. Let us set $f_i = f_{|C_i}$ and notice that for all $x \in C_i$ we have

$$J_{f_i}(x) = (V) \limsup_{S\to x} \frac{\nu\big(f(S \cap C_i)\big)}{\mu(S)} \leq (V) \limsup_{S\to x} \frac{\nu\big(f(S\cap E)\big)}{\mu(S)} = J_f(x).$$

By Corollary 2.9.9 of [2] applied to both $\mathbf{1}_{C_i}$ and $\mathbf{1}_{C_i}\, Jf$, it follows that for μ-a.e. $x \in C_i$, we have

$$(V) \lim_{S\to x} \frac{1}{\mu(S)} \int_S \mathbf{1}_{C_i}(z)\, \mathbf{D}(f^*\nu, \mu, V, z)\, d\mu(z) = Jf(x)\,, \tag{8}$$

$$(V) \lim_{S\to x} \frac{1}{\mu(S)} \int_S \mathbf{D}(f^*\nu, \mu, V, z)\, d\mu(z) = Jf(x)\,. \tag{9}$$

Now, for all $x \in C_i$ such that (8) and (9) hold, we have

$$\begin{aligned}
J_f(x) &= (V) \limsup_{S\to x} \frac{\nu\big(f(S\cap E)\big)}{\mu(S)} \\
&\leq (V) \limsup_{S\to x} \frac{f^*\nu(S\cap E)}{\mu(S)} = Jf(x) \\
&\leq (V) \limsup_{S\to x} \frac{\nu\big(f(S\cap E\cap C_i)\big)}{\mu(S)} + (V) \limsup_{S\to x} \frac{f^*\nu(S\cap E\setminus C_i)}{\mu(S)} \\
&\leq (V) \limsup_{S\to x} \frac{\nu\big(f_i(S\cap C_i)\big)}{\mu(S)} + (V) \limsup_{S\to x} \frac{f^*\nu(S\setminus C_i)}{\mu(S)} \\
&= (V) \limsup_{S\to x} \frac{\nu\big(f_i(S\cap C_i)\big)}{\mu(S)}\,.
\end{aligned}$$

The last equality follows by both (8) and (9), hence we get $J_f(x) = Jf(x) = J_{f_i}(x)$. These equalities hold a.e. in C_i for any $i \geq 1$. Let $B_1 = \cup_{i=1}^{\infty} C_i$ and let $A_1 = \cup_{i=1}^{\infty} E_i$.

Then we have $\mu(A_1 \setminus B_1) \le \varepsilon$, where we have shown that the previous equalities of metric Jacobians hold μ-a.e. in B_1. The arbitrary choice of ε allows for constructing an increasing sequence of Borel sets $B_i \subset A_1$ such that $\mu(A_1 \setminus B_n) \le \varepsilon/n$ for all $n \ge 1$. In particular, setting $B_\infty = \cup_{n=1}^\infty B_n$, we have that

$$\mu(A_1 \setminus B_n) \searrow \mu(A_1 \setminus B_\infty)$$

as $n \to \infty$ and this limit is zero. Thus, in view of formula (4), we get

$$f^*\nu(A \cap A_1) = \int_{A\cap A_1} Jf(x)\, d\mu(x) = \int_{A\cap A_1} J_f(x)\, d\mu(x)\,.$$

We have obtained the formula

$$f^*\nu(A) \quad = \quad \int_{A\cap A_1} J_f(x)\, d\mu(x) + f^*\nu(A\cap E_0)\,. \tag{10}$$

We have to show that $f^*\nu(A \cap E_0) = 0$. Let us consider for any $Z \subset X$ the "preimage measure" $f^\sharp\nu(Z) = \nu(f(Z))$ that is absolutely continuous with respect to μ. Since the set where $J_f > 0$ in E_0 is μ-negligible and $f^*\nu$ is absolutely continuous with respect to μ, it is not restrictive to assume that J_f everywhere vanishes on E_0. Now, for every $\epsilon > 0$ and every μ-measurable bounded set $F \subset E_0$, we get $f^\sharp\nu(F) \le \epsilon\mu(F)$, due to Lemma 1 applied with $\zeta = f^\sharp\nu$. This clearly implies $f^\sharp\nu(E_0) = \nu\big(f(E_0)\big) = 0$, hence (2) gives $f^*\nu(E_0) = 0$. Then (10) easily guides us to the conclusion. $\square$

Remark 2. The metric area formulae (4) and (7) can be extended to all nonnegative measurable mappings $u : A \longrightarrow [0, +\infty]$, obtaining

$$\int_A u(x)\, J_f(x)\; d\mu(x) = \int_Y \sum_{x\in f^{-1}(y)} u(x)\; d\nu(y)\,. \tag{11}$$

This follows by standard approximation arguments with measurable step functions.

Remark 3. Let (X, d) be a complete and separable metric space, let $\alpha > 0$, let (Y, ρ) be a metric space and consider the metric measure spaces $(X, d, \mathcal{H}^\alpha_d)$ and $(Y, \rho, \mathcal{H}^\alpha_\rho)$. Let $E \subset X$ be closed and let $f : E \longrightarrow Y$ be a Lipschitz mapping. We assume that

(1) $\mathcal{H}^\alpha_d$ is finite on bounded sets of X,

(2) for $\mathcal{H}^\alpha_d$-a.e. $x \in X$ the inequality $\liminf_{r\to 0^+} \dfrac{\mathcal{H}^\alpha(D_{x,r})}{r^\alpha} > 0$ holds.

These conditions easily imply that $C = \{(x, D_{x,r}) : x \in X,\, r > 0\}$ is $\mathcal{H}^\alpha_d$ Vitali relation, hence the areae formulae of Theorem 1 and Theorem 2 hold for f, where $\mu = \mathcal{H}^\alpha_d$ and $\nu = \mathcal{H}^\alpha_\rho$. It follows that the metric area formula of [5] follows as a special case of our Theorem 2.

Remark 4. Let E be a closed subset of $\mathbb{R}^n$, let (Y, ρ) be a metric space equipped with the measure $\mathcal{H}^n_\rho$ and let $f : E \longrightarrow Y$ be Lipschitz. The conditions of the previous remark are clearly satisfied with the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$. Then our area formulae hold, along with that of [1] and [3]. In particular, all the different notions of metric Jacobian that are involved in these formulae coincide $\mathcal{L}^n$-a.e.

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