On sequences of maps with finite energies in trace spaces between manifolds

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Abstract. *We deal with mappings defined between Riemannian manifolds that belong to trace spaces of Sobolev functions. Such mappings are equipped with a natural energy, equivalent to the fractional norm. We study the class of Cartesian currents that arise as weak limits of sequences of mappings with equibounded energies. Under suitable topological assumptions on the domain and target manifolds, we prove a density property of graphs of smooth maps. As a consequence, we discuss the corresponding relaxed energy. For mappings with values into the sphere, an explicit formula for the relaxed energy is obtained.*

Contents

1 Introduction

In the last decades there has been a growing interest in variational problems for vector valued mappings with geometric constrains, as e.g. for mappings defined between smooth manifolds isometrically embedded in Euclidean spaces. A central role is played by the analysis of the Dirichlet integral

$$\mathbf{D}(u) := \frac{1}{2}\int_{B^n} |Du(x)|^2\, dx$$

of maps $u : B^n \to \mathbb{R}^3$, where B^n is the unit ball in $\mathbb{R}^n$, that are constrained to take values into the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$. The problem is naturally set in the Sobolev class $W^{1,2}(B^n, \mathbb{S}^2)$, where

$$W^{1,p}(B^n, \mathbb{S}^2) := \{u \in W^{1,p}(B^n, \mathbb{R}^3) \,:\, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\}\,, \quad p > 1\,.$$

It is well known that during the process of relaxation in $W^{1,2}(B^n, \mathbb{S}^2)$, the functional $\mathbf{D}(u)$ may produce energy concentration. In the physical model $n = 3$, the weak $W^{1,2}$-limits of sequences of smooth maps u_k

from B^3 into $\mathbb{S}^2$ with equibounded Dirichlet energies, $\sup_k \mathbf{D}(u_k) < \infty$, may in general be Sobolev maps $u \in W^{1,2}(B^3, \mathbb{S}^2)$ with point singularities that are positioned in correspondence with "holes" in the graph of u, as e.g. for the map $u(x) := x/|x|$.

In the same spirit as Lebesgue's relaxed area functional, the *relaxed Dirichlet energy* of maps $u \in W^{1,2}(B^n, \mathbb{S}^2)$ is defined by

$$\widetilde{\mathbf{D}}(u) := \inf\{\liminf_{k\to\infty} \mathbf{D}(u_k) \mid \{u_k\} \subset W^{1,2}(B^n, \mathbb{S}^2) \cap C^1\,,\ u_k \to u \text{ strongly in } L^2(B^n, \mathbb{R}^3)\}\,.$$

By Schoen-Uhlenbeck density theorem [29], in low dimension $n = 2$ we always have $\mathbf{D}(u) = \widetilde{\mathbf{D}}(u)$. Moreover, as shown by Bethuel-Brezis-Coron [3] in dimension $n = 3$, see also Giaquinta-Modica-Souček [9], and by Tarp [30] in higher dimension $n \geq 4$, for every $u \in W^{1,2}(B^n, \mathbb{S}^2)$, where $n \geq 3$, we have

$$\widetilde{\mathbf{D}}(u) = \mathbf{D}(u) + 4\pi\, L(u)\,, \qquad L(u) \geq 0\,, \tag{1.1}$$

the gap $L(u)$ being the measure of the *minimal integral connection* of the singularities of u.

More precisely, the singularities of a map $u \in W^{1,2}(B^n, \mathbb{S}^2)$ are represented by an $(n-3)$-dimensional current $\mathbb{P}(u) \in \mathcal{D}_{n-3}(B^n)$, and correspond to "holes" in the graph of u. For example, if $n = 3$ and $u(x) = x/|x|$, we have $\mathbb{P}(u) = -\delta_0$, where δ_0 is the unit Dirac mass at the origin. Then $L(u)$ may be seen as the mass of the mass-minimizing integer multiplicity (say i.m.) rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = \mathbb{P}(u)$.

The theory of *Cartesian currents*, introduced by Giaquinta-Modica-Souček in 1989, and extensively studied in the monograph [10], has revealed to be a satisfactory approach to deal with such kind of geometric problems, especially in higher dimension n.

The naïve idea is to look at a map u in $W^{1,2}(B^n, \mathbb{S}^2)$ as a current G_u carried by its graph. The *Dirichlet energy* is extended to the class of i.m. rectifiable currents in $\mathcal{R}_n(B^n \times \mathbb{S}^2)$ that naturally arise as weak limits of sequences of graphs G_{u_k} of smooth maps $u_k : B^n \to \mathbb{S}^2$ with equibounded Dirichlet energies. The weak limit currents T describe the energy concentration phenomenon, and preserve geometric invariants as the degree. Any element T in the corresponding class of Cartesian currents is given by the sum of a graph G_{u_T}, for some $u_T \in W^{1,2}(B^n, \mathbb{S}^2)$, and a vertical current of the type $L_T \times [\![\, \mathbb{S}^2 \,]\!]$, where L_T is an i.m. rectifiable current of codimension 2 in B^n (that is, a sum of signed unit Dirac masses in the 2-dimensional case, and of oriented lines in the physically relevant 3-dimensional case). This L_T closes the holes created by the singularities, i.e., $(\partial L_T) \llcorner B^n = -\,\mathbb{P}(u)$. If e.g. $n = 3$ and $u_T(x) = x/|x|$, then L_T may be given by the current integration of 1-forms in B^3 over an oriented rectifiable arc with initial point at the boundary ∂B^3 and final point at the origin of B^3. As to the Dirichlet energy, we have

$$\mathbf{D}(T) = \mathbf{D}(u_T) + 4\pi \cdot \mathbf{M}(L_T) \quad \text{if} \quad T = G_{u_T} + L_T \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Therefore, the relaxed energy (1.1) may be seen as the *minimum* of the Dirichlet energy $\mathbf{D}(T)$ computed among all the Cartesian currents as above and such that the corresponding function u_T agrees with u.

Since $T \mapsto \mathbf{D}(T)$ is lower semicontinuous with respect to the weak convergence as currents, using a density property of smooth graphs, the representation formula (1.1) for the relaxed energy has been extended to the case of $W^{1,2}$-mappings from B^n into $\mathbb{R}^N$ that take values into more general submanifolds $\mathcal{Y}$ of $\mathbb{R}^N$, see [14] and [15].

For $p > 2$ non integer, $p \notin \mathbb{Z}$, in [21] it is shown that a map in $W^{1,p}(B^n, \mathbb{S}^2)$ belongs to the sequential weak $W^{1,p}$-closure of smooth maps in $W^{1,p}(B^n, \mathbb{S}^2) \cap C^1$ if and only if it belongs to the strong $W^{1,p}$-closure of smooth maps from B^n into $\mathbb{S}^2$. Moreover, as to the relaxed $W^{1,p}$-energy, defined by

$$\widetilde{\mathbf{D}}_p(u) = \inf\left\{\liminf_{k\to\infty} \frac{1}{p^{p/2}} \int_{B^n} |Du_k|^p\, dx \,:\, \{u_k\} \subset C^1(B^n, \mathbb{S}^2)\,,\ u_k \to u \quad \text{strongly in } L^p\right\},$$

for $2 < p < 3$ and for u in $W^{1,p}(B^n, \mathbb{S}^2)$ we have, see [17, Sec. 5.6],

$$\widetilde{\mathbf{D}}_p(u) = \begin{cases} \dfrac{1}{p^{p/2}} \displaystyle\int_{B^n} |Du|^p\, dx & \text{if } \mathbb{P}(u) = 0 \\ +\infty & \text{if } \mathbb{P}(u) \neq 0 \end{cases} \tag{1.2}$$

where $\mathbb{P}(u) \in \mathcal{D}_{n-3}(B^n)$ is the current of the singularities of u.

We finally remark that in the case of integer exponents $p \geq 3$, the situation is much more complicated due to the fact that the higher order homotopy groups of the 2-sphere are not all trivial, e.g., $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. If $h : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map, i.e., the map that generates the third homotopy group of $\mathbb{S}^2$, and $u : B^4 \to \mathbb{S}^2$ is given by $u(x) := h(x/|x|)$, then u belongs to $W^{1,3}(B^4, \mathbb{S}^2)$ and $\mathbb{P}(u) = 0$. However, the *topological singularity* at the origin is relevant, even if it cannot be treated by means of a homological theory as above, compare [23].

Much less is known if one consider variational problems concerning *fractional Sobolev classes* of maps defined between manifolds. We address to the bibliography in [27] for a related list of recent results. One of the aims of this paper is to discuss the relaxed energy of mappings defined between manifolds that are *traces* of Sobolev mappings. In order to discuss the content of the paper, we first fix some notation.

TRACE SPACES. We let $\mathcal{X}$ and $\mathcal{Y}$ be two smooth, connected, compact, oriented Riemannian manifolds that are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^N$, respectively. We shall equip $\mathcal{X}$ and $\mathcal{Y}$ with the metric induced by the Euclidean norms on the ambient spaces, and we let $n := \dim \mathcal{X}$. We shall also assume that the target manifold $\mathcal{Y}$ is without boundary. The domain manifold $\mathcal{X}$ may have a (possibly empty) smooth boundary $\partial\mathcal{X}$, a manifold of dimension $n-1$, the standard cases being $\mathcal{X} = B^n$, the unit n-ball, or $\mathcal{X} = \mathbb{S}^n$, the unit n-sphere.

For the sake of simplicity, in the sequel we shall always denote

$$W^{1/p} := W^{1-1/p,p}, \qquad p > 1\,,$$

and we recall, see e.g. [1], that for any real exponent $p > 1$ the fractional Sobolev space $W^{1/p}(\mathcal{X})$ is the Banach space of real valued functions u in $L^p(\mathcal{X})$ which have finite $W^{1/p}$-seminorm

$$|u|^p_{1/p,\mathcal{X}} := \int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p-1}}\, d\mathcal{H}^n(x)\, d\mathcal{H}^n(y) < \infty\,,$$

where $\mathcal{H}^k$ is the k-dimensional *Hausdorff measure*, endowed with the norm

$$\|u\|^p_{1/p,\mathcal{X}} := \|u\|^p_{L^p(\mathcal{X})} + |u|^p_{1/p,\mathcal{X}}\,. \tag{1.3}$$

$W^{1/p}(\mathcal{X}, \mathbb{R}^N)$ is the space of vector valued maps $u = (u^1, \ldots, u^N)$ such that $u^j \in W^{1/p}(\mathcal{X})$ for every $j = 1, \ldots, N$. If $\mathcal{X} = \partial\mathcal{M}$ for some smooth manifold $\mathcal{M}$, e.g., $\mathcal{X} = \mathbb{S}^n$, then $W^{1/p}(\partial\mathcal{M}, \mathbb{R}^N)$ can be characterized as the space of functions u that are *traces* on $\partial\mathcal{M}$ of functions U in the Sobolev space $W^{1,p}(\mathcal{M}, \mathbb{R}^N)$. More generally, since $\mathcal{X} \subset \mathbb{R}^l$, denoting by $\mathcal{C}^{n+1}$ the $(n+1)$-dimensional "cylinder"

$$\mathcal{C}^{n+1} := \mathcal{X} \times I \subset \mathbb{R}^l \times \mathbb{R}\,, \qquad I := [0,1]\,,$$

$W^{1/p}(\mathcal{X}, \mathbb{R}^N)$ can be seen as the space of functions u that are traces on $\mathcal{X} \times \{0\}$ of functions U in the Sobolev space $W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$. Since $\mathcal{Y} \subset \mathbb{R}^N$, we also let

$$W^{1/p}(\mathcal{X}, \mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X}, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{X}\}\,.$$

Our model case will be $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, the unit $(\mathfrak{p}-1)$-sphere in $\mathbb{R}^{\mathfrak{p}}$, with $\mathfrak{p} \geq 2$, where the positive integer $\mathfrak{p}$ in this paper will always denote

$$\mathfrak{p} := [p] \qquad \text{the integer part of } p\,.$$

We thus have

$$W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1}) := \{u \in W^{1/p}(\mathcal{X}, \mathbb{R}^{\mathfrak{p}}) \,:\, |u(x)| = 1 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{X}\}\,.$$

THE $\mathcal{E}_{1/p}$-ENERGY. In the sequel, instead of working with the $W^{1/p}$-norm (1.3), we shall work with the equivalent energy $\mathcal{E}_{1/p}(u)$ defined as follows. For $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ we define the *extension* of u

$$\operatorname{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$$

as the Hölder continuous function which minimizes the *p-energy integral*

$$\mathbf{D}_p(U) := \frac{1}{p^{p/2}} \int_{\mathcal{C}^{n+1}} |DU(x,t)|^p \, d\mathcal{H}^{n+1}(x,t) \tag{1.4}$$

among all functions $U \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ that agree with u on $\mathcal{X} \times \{0\}$. We also set

$$\mathcal{E}_{1/p}(u) := \mathbf{D}_p(\mathrm{Ext}(u)) \, ,$$

so that clearly $\mathcal{E}_{1/p}(u) \simeq ||u||_{1/p,\mathcal{X}}$. More precisely, by uniform convexity, and since $\mathcal{Y}$ is compact, it is readily checked that for maps in $W^{1/p}(\mathcal{X}, \mathcal{Y})$ the strong convergence $u_k \to u$ in $W^{1/p}$ is equivalent to the convergence in energy, i.e., $u_k \to u$ in L^p with $\mathcal{E}_{1/p}(u_k) \to \mathcal{E}_{1/p}(u)$. We shall also denote by Ω a generic open set $\Omega \subset \mathcal{X}$, and

$$\mathcal{E}_{1/p}(u, \Omega) := \mathbf{D}_p(\mathrm{Ext}(u), \Omega \times I) = \frac{1}{p^{p/2}} \int_{\Omega \times I} |D \,\mathrm{Ext}(u)|^p \, d\mathcal{H}^{n+1} \, . \tag{1.5}$$

THE RELAXED ENERGY. If $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ for some $p > 1$, for any open set $\Omega \subset \mathcal{X}$ we define

$$\widetilde{\mathcal{E}}_{1/p}(u, \Omega) := \inf\{\liminf_{k\to\infty} \mathcal{E}_{1/p}(u_k, \Omega) \mid \{u_k\} \subset W^{1/p}(\Omega, \mathcal{Y}) \cap C^1 \, , \; u_k \to u \text{ strongly in } L^p(\Omega, \mathbb{R}^N)\} \, , \tag{1.6}$$

where $\mathcal{E}_{1/p}(u, \Omega)$ is given by (1.5), and we let

$$\widetilde{\mathcal{E}}_{1/p}(u) := \widetilde{\mathcal{E}}_{1/p}(u, \mathcal{X}) \, .$$

In order to discuss the relaxed energy, we first introduce the class

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X}, \mathcal{Y}) \quad | \quad \text{there exists } \{u_k\} \subset C^\infty(\mathcal{X}, \mathcal{Y}) \text{ such that } u_k \to u \text{ strongly in } W^{1/p}\} \, . \tag{1.7}$$

It is well-known, see [2, 4], that

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) = W^{1/p}(\mathcal{X}, \mathcal{Y}) \qquad \text{if} \quad p \geq n+1 \, . \tag{1.8}$$

Moreover, for low exponents $1 < p < 2$, the equality (1.8) holds true in any dimension n, see [27, Cor. 4]. Therefore, in the case of low dimension $n \leq p - 1$, i.e., $n < \mathfrak{p} := [p]$, or low exponent $1 < p < 2$, we have

$$\widetilde{\mathcal{E}}_{1/p}(u) = \mathcal{E}_{1/p}(u) \qquad \forall \, u \in W^{1/p}(\mathcal{X}, \mathcal{Y}) \, . \tag{1.9}$$

On the other hand, in case of higher dimension $n > p - 1 \geq 1$, in general the strict inclusion

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) \subsetneq W^{1/p}(\mathcal{X}, \mathcal{Y}) \tag{1.10}$$

holds. More precisely, Bethuel [2] noticed that if $\pi_{\mathfrak{p}-1}(\mathcal{Y}) \neq 0$, and $n > p - 1 \geq 1$, i.e., $n \geq \mathfrak{p} \geq 2$, even in the case $\mathcal{X} = B^n$ or $\mathcal{X} = \mathbb{S}^n$, there exist functions $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ which cannot be approximated in $W^{1/p}$ by sequences of smooth maps in $W^{1/p}(\mathcal{X}, \mathcal{Y})$. For this reason, in the sequel we shall assume

$$n := \dim(\mathcal{X}) \geq \mathfrak{p} := [p] \geq 2 \, .$$

MAPS INTO THE SPHERE. In the model case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, and for open sets $\Omega \subset \mathbb{R}^n$, Hang-Lin [19] defined the *singularity* of a map u in $W^{1/\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}-1})$ as the $(n-\mathfrak{p})$-dimensional current in Ω, say $J_u \in \mathcal{D}_{n-\mathfrak{p}}(\Omega)$, acting on compactly supported smooth $(n-\mathfrak{p})$-forms $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\Omega)$ as

$$J_u(\phi) := \frac{1}{|B^{\mathfrak{p}}|} \int_{\Omega \times [0,1]} d\widetilde{\phi} \wedge U^{\#}(dy^1 \wedge \cdots \wedge dy^{\mathfrak{p}}) \, , \tag{1.11}$$

where $|B^{\mathfrak{p}}|$ is the measure of the unit $\mathfrak{p}$-ball, $U := \mathrm{Ext}(u) \in W^{1,\mathfrak{p}}(\Omega \times [0,1], \mathbb{R}^{\mathfrak{p}})$, and $d\widetilde{\phi}$ is the differential of the $(n-\mathfrak{p})$-form in $\Omega \times [0,1]$ given by $\widetilde{\phi} := \phi \wedge \eta$, for some smooth decreasing cut-off function $\eta : [0,1] \to [0,1]$ such that $\eta(t) = 1$ for $t \in [0, 1/4]$ and $\eta(t) = 0$ for $t \in [3/4, 1]$.

They showed that the minimal integral connection of the singularities is bounded in terms of the $W^{1/\mathfrak{p}}$-seminorm of u. More precisely, for every map u in $W^{1/\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$ there exists an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}+1}(\Omega\times[0,1])$ such that

$$(\partial L)\llcorner(\Omega\times[0,1[)=J_u \quad\text{and}\quad \mathbf{M}(L)\le c\,|u|_{1/\mathfrak{p},\Omega}\,,$$

where $c=c(n,\mathfrak{p})>0$ is an absolute constant.

This result was extended in [27] to the class $W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$, for general domain manifolds $\mathcal{X}$ as above. We also notice that the integral connection L can be chosen with support in $\mathcal{X}$, provided that its mass is bounded by an extra term depending on the L^p-norm of the extension $\mathrm{Ext}(u)$. More precisely, setting for $\Gamma\in\mathcal{D}_{n-\mathfrak{p}}(\Omega)$, where $\Omega\subset\mathbb{R}^n$,

$$m_{i,\Omega}(\Gamma):=\inf\{\mathbf{M}(L)\mid L\in\mathcal{R}_{n-\mathfrak{p}+1}(\Omega)\,,\quad (\partial L)\llcorner\Omega=\Gamma\}\,,$$

for every $u\in W^{1/p}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$ we have, see [27],

$$m_{i,\Omega}(J_u)\le c\left(|u|_{1/p}+\|\,\mathrm{Ext}(u)\|^p_{L^p}\right)<\infty\,. \tag{1.12}$$

In this paper, in analogy with the representation formula (1.1) about the relaxed Dirichlet energy of mappings into the 2-sphere, we shall characterize the relaxed energy of maps $u\in W^{1/\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$, where $\Omega\subset\mathbb{R}^n$ is a bounded open set, in terms of the minimal connection of their singular set, also showing a different feature in the case $p>2$ non-integer.

To this purpose, we shall introduce the class $\mathrm{cart}^{1/\mathfrak{p}}(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$ of currents in $\mathcal{D}_n(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$ that naturally arise as weak limits of sequences of graphs of smooth maps $u_k:\Omega\to\mathbb{S}^{\mathfrak{p}-1}$ with equibounded $\mathcal{E}_{1/\mathfrak{p}}$-energies. Similarly to the case $\mathfrak{p}=2$ from [11], every *Cartesian current* $T\in\mathrm{cart}^{1/\mathfrak{p}}(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$ can be written as

$$T=G_{u_T}+L_T\times[\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!]\,, \tag{1.13}$$

where G_{u_T} is the current carried by the graph of a map $u_T\in W^{1/\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$ and L_T is an i.m. rectifiable current in $\mathcal{R}_{n-\mathfrak{p}+1}(\Omega)$. Now, if G_u is the current integration on the graph $\mathcal{G}_u$ of a smooth $W^{1/\mathfrak{p}}$-map $u:\Omega\to\mathbb{S}^{\mathfrak{p}-1}$, since the boundary $\partial\mathcal{G}_u$ is supported in $\partial\Omega\times\mathbb{S}^{\mathfrak{p}-1}$, by Stoke's theorem we have

$$\partial G_u(\omega):=G_u(d\omega)=\int_{\mathcal{G}_u}d\omega=\int_{\partial\mathcal{G}_u}\omega=0$$

for every compactly supported smooth form $\omega\in\mathcal{D}^{n-1}(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$, whereas condition $\partial G_{u_k}(\omega)=0$ is clearly preserved by the weak convergence as currents. As a consequence, T also satisfies the *null-boundary condition*

$$(\partial T)\llcorner\Omega\times\mathbb{S}^{\mathfrak{p}-1}=0 \tag{1.14}$$

that, according to (1.11), reads as

$$(\partial L_T)\llcorner\Omega=(-1)^{\mathfrak{p}-1}J_{u_T}\,.$$

The $\mathcal{E}_{1/\mathfrak{p}}$-energy is naturally extended to currents T in $\mathrm{cart}^{1/\mathfrak{p}}(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$, see (1.13), by setting

$$\mathcal{E}_{1/\mathfrak{p}}(T)=\mathcal{E}_{1/\mathfrak{p}}(u_T,\Omega)+|B^{\mathfrak{p}}|\cdot\mathbf{M}(L_T)\,,$$

so that the functional $T\mapsto\mathcal{E}_{1/\mathfrak{p}}(T)$ turns out to be lower semicontinuous with respect to the weak convergence as currents. Moreover, we shall prove the following strong density property:

Theorem 1.1 *For every $T\in\mathrm{cart}^{1/\mathfrak{p}}(\Omega\times\mathbb{S}^{\mathfrak{p}-1})$ there exists a sequence of smooth $W^{1/\mathfrak{p}}$-maps $u_k:\Omega\to\mathbb{S}^{\mathfrak{p}-1}$ such that G_{u_k} weakly converges to T and $\mathcal{E}_{1/\mathfrak{p}}(u_k,\Omega)\to\mathcal{E}_{1/\mathfrak{p}}(T)$ as $k\to\infty$.*

As a consequence of (1.12) and of Theorem 1.1, we shall then prove:

Theorem 1.2 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. For every integers $n\ge\mathfrak{p}\ge2$ and for every $u\in W^{1/\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$ the relaxed energy satisfies*

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u,\Omega)=\mathcal{E}_{1/\mathfrak{p}}(u,\Omega)+|B^{\mathfrak{p}}|\cdot m_{i,\Omega}(J_u)<\infty\,.$$

In terms of currents, this means that the relaxed energy is equal to the minimum of the $\mathcal{E}_{1/\mathfrak{p}}$-energy of the Cartesian currents with corresponding function u_T equal to u, i.e.,

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u,\Omega) = \inf\{\mathcal{E}_{1/\mathfrak{p}}(T) \mid T \in \mathrm{cart}^{1/\mathfrak{p}}(\Omega \times \mathbb{S}^{\mathfrak{p}-1}) \,,\ u_T = u \ \text{ in } (1.13)\} \,.$$

Moreover, for non-integer exponents, similarly to (1.2) we shall prove

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, where $n \geq \mathfrak{p} := [p]$, and $p > 2$ non-integer. Then for every $u \in W^{1/p}(\Omega, \mathbb{S}^{\mathfrak{p}-1})$ we have*

$$\widetilde{\mathcal{E}}_{1/p}(u,\Omega) = \begin{cases} \mathcal{E}_{1/p}(u,\Omega) & \text{if } J_u = 0 \\ +\infty & \text{if } J_u \neq 0 \,. \end{cases}$$

Density properties. In order to extend Theorems 1.1, 1.2, and 1.3 to more general domain and target manifolds, we first recall some density results for $W^{1/p}$-mappings between manifolds.

Due to (1.10), in the case $n \geq \mathfrak{p}$ Bethuel introduced in [2] the classes $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ and $\mathcal{R}^0_{1/p}(\mathcal{X},\mathcal{Y})$, given by all the maps $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^r \Sigma_i \,, \qquad r \in \mathbb{N} \,, \tag{1.15}$$

where Σ_i is a smooth $(n-\mathfrak{p})$-dimensional subset of $\mathcal{X}$ with smooth boundary, if $n \geq \mathfrak{p}+1$, and Σ_i is a point if $n = \mathfrak{p}$. The following density property holds true:

Theorem 1.4 *For every $1 < p < n+1$, where $n = \dim(\mathcal{X})$, the class $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ is sequentially dense in $W^{1/p}(\mathcal{X},\mathcal{Y})$.*

In the case $n = p = 2$, Theorem 1.4 was proved in [28], compare also [4], for $\mathcal{X} = \mathbb{S}^2$ and with $\mathcal{Y} = \mathbb{S}^1$, the standard unit circle. For $p = 2$, it was extended in [12] to the case $\mathcal{X} = B^n$ or $\mathbb{S}^n$, in higher dimension $n \geq 2$ and for general target manifolds $\mathcal{Y}$, see also [17]. A complete proof in the general case is given in [26]. Moreover, denoting by $\pi_k(\mathcal{Y})$ the k^{th} free homotopy group of $\mathcal{Y}$, in [26] we also proved:

Proposition 1.5 *If $n \leq p < n+1$, and $p > 1$, then $H^{1/p}_S(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y})$ if and only if $\pi_{n-1}(\mathcal{Y}) = 0$.*

In case of higher dimension $n > p$, i.e., $n \geq \mathfrak{p}+1$, following observations by Hang-Lin [20], we showed in [26] that the possibly non-trivial topology of the domain manifold $\mathcal{X}$ plays a role. To this purpose, we recall that $\mathcal{X}$ is said to satisfy the *k-extension property with respect to $\mathcal{Y}$,* where $k \in \mathbb{N}$, if for any given CW-complex X on $\mathcal{X}$, denoting by X^k its k-dimensional skeleton, any continuous map $f : X^{k+1} \to \mathcal{Y}$ is such that its restriction to X^k can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. In [26] we obtained the following characterization, that we state here for the case $\partial\mathcal{X} = \emptyset$.

Theorem 1.6 *If $n > p > 1$, smooth maps in $C^\infty(\mathcal{X},\mathcal{Y})$ are sequentially dense in $W^{1/p}(\mathcal{X},\mathcal{Y})$, i.e., $H^{1/p}_S(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y})$, if and only if $\pi_{\mathfrak{p}-1}(\mathcal{Y}) = 0$ and $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathcal{Y}$.*

In particular, in the standard case $\mathcal{X} = \mathbb{S}^n$ we obtain that for any $n+1 > p > 1$, *smooth maps in $C^\infty(\mathbb{S}^n,\mathcal{Y})$ are sequentially dense in $W^{1/p}$, i.e., $H^{1/p}_S(\mathbb{S}^n,\mathcal{Y}) = W^{1/p}(\mathbb{S}^n,\mathcal{Y})$, if and only if $\pi_{\mathfrak{p}-1}(\mathcal{Y}) = 0$.* Moreover, for low exponents $1 < p < 2$ we recover the equality (1.8) in any dimension n. Finally, if the manifold $\mathcal{X}$ has a non-zero boundary, we recall that analogous density results hold for maps in $W^{1/p}(\mathcal{X},\mathcal{Y})$ with prescribed boundary data.

Other new results. In this paper, Theorems 1.1, 1.2, and 1.3 are then extended to open subsets Ω of manifolds $\mathcal{X}$ as above, provided that $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathbb{S}^{\mathfrak{p}-1}$, if $n := \dim(\mathcal{X}) \geq \mathfrak{p}+1$. In particular, in dimension $n \geq \mathfrak{p}+1$ one may assume that $\mathcal{X}$ is $\mathfrak{p}$-*connected*, i.e., that the k-dimensional free homotopy groups $\pi_k(\mathcal{X})$ are trivial, for $k = 0, 1, \ldots, \mathfrak{p}$.

In order to deal with more general target manifolds $\mathcal{Y}$, we shall denote by $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ the *spherical subgroup* of the *singular homology group* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$, see [10, Vol. II, Sec. 5.4.2] and Definition 4.3 below.

We shall discuss the relaxed $W^{1/p}$-energy of maps defined between smooth, connected, compact, oriented Riemannian manifolds $\mathcal{X}$ and $\mathcal{Y}$, isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^N$, respectively, that satisfy the following hypotheses:

(H_1) both $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$ and the quotient space $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})/\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ are torsion-free;

(H_2) in the case $\mathfrak{p}=2$, the first homotopy group $\pi_1(\mathcal{Y})$ is commutative; in the case $\mathfrak{p}\geq 3$, alternatively, $\pi_1(\mathcal{Y})=0$ and the *Hurewicz homomorphism* from the $(\mathfrak{p}-1)^{th}$ free homotopy group $\pi_{\mathfrak{p}-1}(\mathcal{Y})$ onto the $(\mathfrak{p}-1)^{th}$ real homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})$ is injective;

(H_3) if $n:=\dim(\mathcal{X})\geq \mathfrak{p}+1$, then $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathcal{Y}$.

As in the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, we shall first introduce the class $\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, see [13] for the case $\mathfrak{p}=2$ and $\mathcal{X}=B^n$. Roughly speaking, a Cartesian current $T\in\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ is a semi-current in $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$, i.e., its action is well-defined on compactly supported smooth n-forms in $\mathcal{X}\times\mathcal{Y}$ with at most $(\mathfrak{p}-1)$ differentials in the vertical $\mathcal{Y}$-variables. Moreover, T satisfies a *null-boundary condition* that is analogous to (1.14), and can be decomposed as $T=G_{u_T}+S_T$, where G_{u_T} is the current carried by the graph of some map $u_T\in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$. Also, the second term S_T is a "completely vertical" semi-current, i.e., its action is non-zero only on n-forms in $\mathcal{X}\times\mathcal{Y}$ with exactly $(\mathfrak{p}-1)$ differentials in the $\mathcal{Y}$-variables. By (H_1), and according to (1.13), it turns out that S_T is a finite sum of Cartesian products of currents of the type $L_s\times\gamma_s$, for some i.m. rectifiable currents $L_s\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$, where the equivalence classes $[\gamma_s]$ generate the spherical homology group $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$.

The $\mathcal{E}_{1/\mathfrak{p}}$-energy is naturally extended to $\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ in such a way that $\mathcal{E}_{1/\mathfrak{p}}(T)=\mathcal{E}_{1/\mathfrak{p}}(u_T)$ if $T=G_{u_T}$, and the functional $T\mapsto\mathcal{E}_{1/\mathfrak{p}}(T)$ is lower semicontinuous with respect to the weak convergence as currents. Moreover, if (H_2) and (H_3) are satisfied, we shall prove the following density property:

Theorem 1.7 *For every $T\in\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ there exists a sequence of smooth $W^{1/\mathfrak{p}}$-maps $u_k:\mathcal{X}\to\mathcal{Y}$ such that G_{u_k} weakly converges to T as $k\to\infty$ and*

$$\limsup_{k\to\infty}\mathcal{E}_{1/\mathfrak{p}}(u_k)\leq\big(2c(\mathfrak{p},\mathcal{Y})-1\big)\cdot\mathcal{E}_{1/\mathfrak{p}}(T)<\infty\,,$$

where $c(\mathfrak{p},\mathcal{Y})\geq 1$ is an absolute constant, only depending on $\mathfrak{p}$ and $\mathcal{Y}$.

As a consequence of Theorem 1.7 and of a result from [27] concerning the minimal connection of the singularities of maps in $W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$, see Theorem 4.9 below, we shall then prove that *for any integers $n\geq\mathfrak{p}\geq 2$ and for every $u\in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ the relaxed energy $\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u)$ is always finite.* Roughly speaking, the gap between the relaxed energy and the energy $\mathcal{E}_{1/\mathfrak{p}}(u)$ is bounded from above by an absolute constant times the minimal connection of the singularities of u, Proposition 6.4.

If $\mathfrak{p}=2$, the optimal constant $c(2,\mathcal{Y})$ in Theorem 1.7 is always equal to one. By lower semicontinuity of the $\mathcal{E}_{1/p}$-energy, this yields that in the case $\mathfrak{p}=2$ we find a smooth sequence $u_k:\mathcal{X}\to\mathcal{Y}$ such that G_{u_k} weakly converges to T and $\mathcal{E}_{1/2}(u_k)\to\mathcal{E}_{1/2}(T)$ as $k\to\infty$, compare [13] for the case $\mathcal{X}=B^n$. In particular, for $\mathfrak{p}=2$ we obtain an *explicit formula* for the relaxed energy $\widetilde{\mathcal{E}}_{1/2}(u)$, Corollary 6.5.

Finally, for non integer exponents $p>2$, and $n\geq\mathfrak{p}:=[p]$, we shall extend Theorem 1.3, showing that for every $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ we have

$$\widetilde{\mathcal{E}}_{1/p}(u)=\begin{cases}\mathcal{E}_{1/p}(u) & \text{if } \mathbb{P}(u)=0\\ +\infty & \text{if } \mathbb{P}(u)\neq 0\,,\end{cases}$$

where $\mathbb{P}(u)$ is the $(n-\mathfrak{p})$-current that describes the *homological singularities* of u, compare Sec. 4 below.

Plan of the paper. Sections 2 and 3 are dedicated to the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, see Theorems 1.1, 1.2, and 1.3. In Sec. 4, we recall from [27] the definition of current G_u carried by the graph of a function $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$, and discuss the *homological singularities* of u. In Sec. 5, we introduce the class of Cartesian currents in $\mathcal{X}\times\mathcal{Y}$ with finite $W^{1/\mathfrak{p}}$-energy, collecting their main properties. In Sec. 6, using the density property of graphs of smooth maps in $\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, see Theorem 1.7, we discuss the relaxed $\mathcal{E}_{1/p}$-energy of mappings in $W^{1/p}(\mathcal{X},\mathcal{Y})$ for general target manifolds. In order to prove Theorem 1.7, in Sec. 7 we show

how to approximate a *spherical* $(\mathfrak{p}-1)$-*cycle of* $\mathcal{Y}$. We shall first focus on the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, for which we shall give an optimal energy estimate, yielding to Theorem 1.1. Sec. 8 is then dedicated to give a sketch of the proof of the density theorems 1.1 and 1.7 in the easier cases of low dimension $n:=\dim(\mathcal{X})=\mathfrak{p}-1$ and $n=\mathfrak{p}$. Moreover, in the higher dimension $n\geq\mathfrak{p}+1$, using an idea from [16], we shall reduce the proof of the density theorem to Theorem 8.4 below. Since the proof of Theorem 8.4 is more technical, but it cannot be easily deduced from previously published results, it has been postponed to the Appendix.

2 Mappings into the sphere and singularities

In this section we collect some properties of mappings in $W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$, for which we refer to [27]. Here $\mathbb{S}^{\mathfrak{p}-1}$ is the unit $(\mathfrak{p}-1)$-sphere of $\mathbb{R}^{\mathfrak{p}}$ and $\mathfrak{p}:=[p]$ is the integer part of p, where $p\geq 2$. Due to (1.8), we shall then assume $n:=\dim(\mathcal{X})\geq\mathfrak{p}$. We first recall the notion of current carried by the graph of a $W^{1/p}$-map u and of singularities of u. Both of them are defined by means of the extension map $\mathrm{Ext}(u)\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$.

Currents carried by graphs. If U is a Sobolev map in $W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^{\mathfrak{p}})$, the $(n+1)$-current G_U in $\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}}$ carried by its graph is defined (in an approximate sense) by $G_U:=(Id\bowtie U)_{\#}[\![\mathcal{C}^{n+1}]\!]$, where $(Id\bowtie U)(z):=(z,U(z))$, compare [10]. For example, if $\omega=\gamma\wedge\eta\in\mathcal{D}^{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})$, where $\gamma\in\mathcal{D}^{n+1-h}(\mathcal{C}^{n+1})$, $\eta\in\mathcal{D}^h(\mathbb{R}^N)$, and $0\leq h\leq\min\{n+1,\mathfrak{p}\}$, we have

$$G_U(\gamma\wedge\eta)=[\![\mathcal{C}^{n+1}]\!]((Id\bowtie U)^{\#}(\gamma\wedge\eta))=[\![\mathcal{C}^{n+1}]\!](\gamma\wedge U^{\#}\eta)=\int_{\mathcal{C}^{n+1}}\gamma\wedge U^{\#}\eta\,. \tag{2.1}$$

Actually, G_U is an *integer multiplicity* (say i.m.) *rectifiable* current in $\mathcal{R}_{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})$, with finite *mass*,

$$\mathbf{M}(G_U):=\sup\{G_U(\omega)\mid\omega\in\mathcal{D}^{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}}),\ \|\omega\|\leq 1\}<\infty\,,$$

where $\|\omega\|$ is the *comass* norm of ω. In fact, by using the parallelogram inequality we infer that

$$\mathbf{M}(G_U)\leq C\,(\mathcal{H}^n(\mathcal{X})+\mathbf{D}_p(U))$$

for some absolute constant $C=C(n,p,\mathcal{X})>0$, not depending on U.

Definition 2.1 *To any map* $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ *we associate an* n*-current* G_u *in* $\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$ *by setting*

$$G_u:=(-1)^{n-1}(\partial G_U)\llcorner((\mathcal{X}\times\{0\})\times\mathbb{R}^{\mathfrak{p}})\qquad\text{on}\quad\mathcal{D}^n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})\,, \tag{2.2}$$

where $U:=\mathrm{Ext}(u)\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^{\mathfrak{p}})$.

Remark 2.2 The above definition holds true for every $u\in W^{1/p}(\mathcal{X},\mathbb{R}^{\mathfrak{p}})\cap L^\infty$, yielding to a semi-current G_u in $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^{\mathfrak{p}})$, see Sec. 4 below. However, if $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$, by Federer's support theorem [5], we infer that the current G_u actually belongs to $\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$. Notice that in general G_u is not i.m. rectifiable, even for $\mathcal{X}=B^n$ and in low dimension $n=\mathfrak{p}$, and for $p=2$.

We recall that the *boundary* of a current $\widetilde{T}$ in $\mathcal{D}_{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})$ is defined, by duality, for every compactly supported smooth n-form ω in $\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}}$ by

$$\partial\widetilde{T}(\omega):=\widetilde{T}(d\omega)\,,\qquad\omega\in\mathcal{D}^n(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})\,,$$

yielding to an n-current $\partial\widetilde{T}$ in $\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}}$. If $\widetilde{T}=G_U$ for some Sobolev map $U\in W^{1,p}(\mathcal{X},\mathbb{R}^{\mathfrak{p}})$, we have

$$\partial G_U(\omega)=0\qquad\forall\,\omega\in\mathcal{D}^n(\mathrm{int}(\mathcal{C}^{n+1})\times\mathbb{R}^{\mathfrak{p}})\,. \tag{2.3}$$

In fact, this condition is readily checked if U is smooth, by Stoke's theorem, and holds true in general by a standard density argument, as $\mathfrak{p}=[p]$.

In a similar way, the boundary of a current $T\in\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$ is well defined by

$$\partial T(\omega):=T(d\omega)\,,\qquad\omega\in\mathcal{D}^{n-1}(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})\,.$$

Notice that for maps $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ in general the boundary ∂G_u is non zero.

Finally, we recall that the weak convergence on currents is defined in a dual sense. Namely, a sequence $\{T_k\}\subset\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$ weakly converges to $T\in\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$, say $T_k\rightharpoonup T$, if $T_k(\omega)\to T(\omega)$ for every test form $\omega\in\mathcal{D}^n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$.

SINGULARITIES. Let $\omega_{\mathbb{S}^{\mathfrak{p}-1}}$ denote the *normalized volume* $(\mathfrak{p}-1)$*-form* in $\mathbb{S}^{\mathfrak{p}-1}$

$$\omega_{\mathbb{S}^{\mathfrak{p}-1}}:=\frac{1}{\alpha_{\mathfrak{p}}}\sum_{j=1}^{\mathfrak{p}}(-1)^{j-1}y^j\,dy^1\wedge\cdots\wedge dy^{j-1}\wedge dy^{j+1}\wedge\cdots\wedge dy^{\mathfrak{p}}\,, \tag{2.4}$$

where $\alpha_{\mathfrak{p}}:=\mathcal{H}^{\mathfrak{p}-1}(\mathbb{S}^{\mathfrak{p}-1})$, so that $[\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!](\omega_{\mathbb{S}^{\mathfrak{p}-1}})=\int_{\mathbb{S}^{\mathfrak{p}-1}}\omega_{\mathbb{S}^{\mathfrak{p}-1}}=1$. Moreover, let $\pi:\mathcal{X}\times\mathbb{R}^{\mathfrak{p}}\to\mathcal{X}$ and $\widehat{\pi}:\mathcal{X}\times\mathbb{R}^{\mathfrak{p}}\to\mathbb{R}^{\mathfrak{p}}$ denote the orthogonal projections onto the two factors.

Definition 2.3 *The* singularities *of a map u in $W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ are represented by the $(n-\mathfrak{p})$-dimensional current $\mathbf{P}(u)$ in $\mathcal{D}_{n-\mathfrak{p}}(\mathcal{X})$ defined by*

$$\mathbf{P}(u)(\phi):=(-1)^{\mathfrak{p}}\,\partial G_u(\pi^{\#}\phi\wedge\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}})\,,\qquad \phi\in\mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,.$$

We write explicitly the action of $\mathbf{P}(u)$, recovering the definition (1.11) of singularities introduced by Hang-Lin [19], for $\mathcal{X}$ an open set of $\mathbb{R}^n$. We recall that $\eta:[0,1]\to[0,1]$ is a smooth decreasing cut-off function such that $\eta(t)=1$ for $t\in[0,1/4]$ and $\eta(t)=0$ for $t\in[3/4,1]$.

Proposition 2.4 *For every $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ and every $\phi\in\mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$ we have*

$$\mathbf{P}(u)(\phi)=\frac{1}{|B^{\mathfrak{p}}|}\int_{\mathcal{C}^{n+1}}d\widetilde{\phi}\wedge U^{\#}(dy^1\wedge\cdots\wedge dy^{\mathfrak{p}})\qquad\forall\,\phi\in\mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,,$$

where $U:=\mathrm{Ext}(u)$ and $\widetilde{\phi}\in\mathcal{D}^{n-\mathfrak{p}}(\mathcal{C}^{n+1})$ is given by $\widetilde{\phi}:=\phi\wedge\eta$.

PROOF: We can assume that $U:=\mathrm{Ext}(u)$ takes values into the closure $\overline{B}^{\mathfrak{p}}$ of the unit $\mathfrak{p}$-ball. We then denote by $\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}}$ a $(\mathfrak{p}-1)$-form in $\mathcal{D}^{\mathfrak{p}-1}(\mathbb{R}^{\mathfrak{p}})$ that agrees with the right-hand side of (2.4) on $\overline{B}^{\mathfrak{p}}$. By the definition (2.2) and by (2.3) we have

$$\begin{aligned}\partial G_U(\pi^{\#}d\widetilde{\phi}\wedge\widehat{\pi}^{\#}\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}}) &= (-1)^{n-1}G_u(\pi^{\#}(d_x\widetilde{\phi}+d_t\widetilde{\phi})_{|t=0}\wedge\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}})\\ &= (-1)^{n-1}G_u(\pi^{\#}d\phi\wedge\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}})\,.\end{aligned}$$

Since $d\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}}=\widehat{\pi}^{\#}d\omega_{\mathbb{S}^{\mathfrak{p}-1}}=0$, we compute

$$\begin{aligned}\mathbf{P}(u)(\phi) &= (-1)^{\mathfrak{p}}G_u(d\pi^{\#}\phi\wedge\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}})=(-1)^{\mathfrak{p}}G_u(\pi^{\#}d\phi\wedge\widehat{\pi}^{\#}\omega_{\mathbb{S}^{\mathfrak{p}-1}})\\ &= (-1)^{n-\mathfrak{p}+1}\partial G_U(\pi^{\#}d\widetilde{\phi}\wedge\widehat{\pi}^{\#}\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}})=G_U(\pi^{\#}d\widetilde{\phi}\wedge d\widehat{\pi}^{\#}\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}})\\ &= G_U(\pi^{\#}d\widetilde{\phi}\wedge\widehat{\pi}^{\#}d\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}})\,.\end{aligned} \tag{2.5}$$

Therefore, it suffices to observe that since $\alpha_{\mathfrak{p}}=\mathfrak{p}\,|B^{\mathfrak{p}}|$ and $U(\mathcal{C}^{n+1})\subset\overline{B}^{\mathfrak{p}}$, then

$$U^{\#}\,d\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}}=U^{\#}\omega_{B^{\mathfrak{p}}}\,,\qquad \omega_{B^{\mathfrak{p}}}:=\frac{1}{|B^{\mathfrak{p}}|}\,dy^1\wedge\cdots\wedge dy^{\mathfrak{p}}\,,$$

and recall the action (2.1) of the current G_U. □

CARTESIAN MAPS. Of course, a map $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ has zero homological singularities, i.e., satisfies $\mathbf{P}(u)=0$, if and only if the current G_u associated to its graph has no inner boundary, i.e.,

$$\partial G_u=0\quad\text{on}\quad\mathcal{D}^{n-1}(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})\,. \tag{2.6}$$

For this reason, we give the following

Definition 2.5 *Let $\mathfrak{p}:=[p]\geq 2$. A map $u:\mathcal{X}\to\mathbb{S}^{\mathfrak{p}-1}$ is said to be a* Cartesian map *in* $\mathrm{cart}^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ *if u belongs to $W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ and satisfies the null-boundary condition* (2.6).

Trivially, condition $\mathbf{P}(u)=0$ holds true if u is smooth, say Lipschitz. Moreover, since the null-boundary condition (2.6) is preserved by the weak convergence in $\mathcal{D}_n(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$, and the strong convergence $u_k\to u$ in $W^{1/p}(\mathcal{X},\mathbb{R}^{\mathfrak{p}})$ yields the weak convergence $G_{u_k}\rightharpoonup G_u$, according to (1.7) we immediately obtain that

$$H_S^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\subset \mathrm{cart}^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\,.$$

Moreover, in [27] we showed that under suitable hypotheses on $\mathcal{X}$, *every map in* $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ *can be approximated by sequences of smooth maps in* $W^{1/p}$.

Theorem 2.6 *Let $p\geq 2$ and $n\geq\mathfrak{p}:=[p]$. If $n\geq\mathfrak{p}+1$, assume that $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathbb{S}^{\mathfrak{p}-1}$. Then*

$$H_S^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})= \mathrm{cart}^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\,. \tag{2.7}$$

We recall that $\mathcal{X}$ is said to be $\mathfrak{p}$-*connected* if $\pi_k(\mathcal{X})=0$ for $k=0,\dots,\mathfrak{p}$. In particular, we have:

Proposition 2.7 *If $n:=\dim(\mathcal{X})\geq\mathfrak{p}+1$, assume that $\mathcal{X}$ is $\mathfrak{p}$-connected. Then* (2.7) *holds for every $n\geq\mathfrak{p}$.*

Since both B^n and $\mathbb{S}^n$ are $\mathfrak{p}$-connected for $n\geq\mathfrak{p}+1$, we then obtain:

Corollary 2.8 *If $\mathcal{X}=B^n$ or $\mathbb{S}^n$, then* (2.7) *holds for every $n\geq\mathfrak{p}$.*

Proof of Proposition 2.7: In [27, Thm. 3.2] we showed that every map $u\in\mathrm{cart}^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ is the strong $W^{1/p}$-limit of a sequence of maps in $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\cap\mathrm{cart}^{1/p}$. Therefore, by a diagonal argument it suffices to show that $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\cap\mathrm{cart}^{1/p}$ is contained in $H_S^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$.

Let now $u\in\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})\cap\mathrm{cart}^{1/p}$ and let X be a cubeulation of $\mathcal{X}$ in *dual position* with respect to u, compare [27], so that the $(\mathfrak{p}-1)$-skeleton $X^{\mathfrak{p}-1}$ is disjoint from the $(n-\mathfrak{p})$-dimensional singular set $\Sigma(u)$ of u, see (1.15). By [26, Thm. 3], it suffices to show that the restriction $u_{|X^{\mathfrak{p}-1}}$ has a continuous extension $g:\mathcal{X}\to\mathbb{S}^{\mathfrak{p}-1}$. To this purpose, since the restriction of u to each $(\mathfrak{p}-1)$-simplex of $X^{\mathfrak{p}-1}$ has zero degree, we infer that $u_{|X^{\mathfrak{p}-1}}$ has a continuous extension $f:X^{\mathfrak{p}}\to\mathbb{S}^{\mathfrak{p}-1}$. This gives the claim, for $n=\mathfrak{p}$. If $n\geq\mathfrak{p}+1$, since $\mathcal{X}$ is $\mathfrak{p}$-connected, arguing as in [32, Sec. 6], we find a continuous map $\phi:\mathcal{X}\to\mathcal{X}$ homotopic to the identity map and such that the restriction $\phi_{|X^{\mathfrak{p}}}$ is constant. Then $f\circ\phi$ is homotopic to f and $f\circ\phi_{|X^{\mathfrak{p}}}$ is constant. Whence, $f_{|X^{\mathfrak{p}}}$ can be extended to a continuous map, as required. □

Integral connections. The current $\mathbf{P}(u)$ carrying the singularities of any map $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ is an *integral flat chain*.

Proposition 2.9 *Let $n\geq\mathfrak{p}\geq 2$. For every $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ there exists an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that*

$$(\partial L)\llcorner\mathrm{int}(\mathcal{X})=\mathbf{P}(u)\quad\textit{and}\quad \mathbf{M}(L)\leq C\left(\mathcal{E}_{1/p}(u)+\|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right),$$

where $C>0$ is an absolute constant, not depending on u, and $\mathrm{Ext}(u)\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ is the extension of u.

This property (and its local version) was first proved by Hang-Lin [19] for $p\geq 2$ integer and for $\mathcal{X}=\mathbb{R}^n$. In [27] we gave a proof based on the previous definitions. Notice that the extra term $\|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}$ in the above formula can be removed if we require that the integral connection L belongs to $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$, as in [19]. Finally, a result similar to Proposition 2.9 holds true for mappings with prescribed boundary data, compare [27].

3 The relaxed energy of mappings into the sphere

In this section we discuss the relaxed $\mathcal{E}_{1/p}$-energy for mappings into the $(\mathfrak{p}-1)$-sphere. We shall always assume $n:=\dim(\mathcal{X})\geq\mathfrak{p}:=[p]\geq 2$. Moreover, on account of Theorem 2.6, Proposition 2.7, and Corollary 2.8, if

$n \geq \mathfrak{p}+1$, we shall *assume that* $\mathcal{X}$ *is* $\mathfrak{p}$*-connected or*, more generally, that $\mathcal{X}$ *satisfies the* $(\mathfrak{p}-1)$*-extension property with respect to* $\mathbb{S}^{\mathfrak{p}-1}$.

We recall that by (1.6), for every open set $\Omega \subset \mathcal{X}$ and $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ we have

$$\widetilde{\mathcal{E}}_{1/p}(u,\Omega) := \inf\{\liminf_{k\to\infty} \mathcal{E}_{1/p}(u_k,\Omega) \mid \{u_k\} \subset W^{1/p}(\Omega, \mathbb{S}^{\mathfrak{p}-1}) \cap C^1\,,\ u_k \to u \text{ strongly in } L^p(\Omega, \mathbb{R}^{\mathfrak{p}})\}\,,$$

where $\mathcal{E}_{1/p}(u,\Omega)$ is given by (1.5), and we let $\widetilde{\mathcal{E}}_{1/p}(u) := \widetilde{\mathcal{E}}_{1/p}(u,\mathcal{X})$. We first consider the case of integer exponents $\mathfrak{p} \geq 2$. We shall then consider the case $p > 2$ non integer.

Real and integral mass. Recall:

Definition 3.1 *For every* $\Gamma \in \mathcal{D}_{n-\mathfrak{p}}(\Omega)$ *we denote by*

$$\begin{aligned} m_{r,\Omega}(\Gamma) &:= \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-\mathfrak{p}+1}(\Omega)\,, \quad (\partial D) \llcorner \Omega = \Gamma\} \\ m_{i,\Omega}(\Gamma) &:= \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-\mathfrak{p}+1}(\Omega)\,, \quad (\partial L) \llcorner \Omega = \Gamma\} \end{aligned}$$

the real mass *and* integral mass *of* Γ *relative to* Ω*, respectively.*

In case $m_{i,\Omega}(\Gamma) < \infty$, the infimum is always attained, i.e., there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-\mathfrak{p}+1}(\Omega)$ such that $(\partial L)\llcorner\Omega = \Gamma$ and $\mathbf{M}(L) = m_{i,\Omega}(\Gamma)$. Any such current is called an *integral minimal connection for the mass* of Γ *allowing connections to the boundary* of Ω.

Cartesian currents. Proposition 2.9 yields that for every $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ we have

$$m_{i,\mathrm{int}(\mathcal{X})}(\mathbf{P}(u)) \leq C\,\|\operatorname{Ext}(u)\|^{\mathfrak{p}}_{W^{1,\mathfrak{p}}(\mathcal{C}^{n+1})} < \infty\,. \tag{3.1}$$

This motivates us to introduce the following

Definition 3.2 *A current* $T \in \mathcal{D}_n(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *is said to be a* Cartesian current *in* $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *if*

$$\partial T = 0 \quad \text{on forms in} \quad \mathcal{D}^{n-1}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1}) \tag{3.2}$$

and T *can be decomposed as*

$$T = G_{u_T} + L_T \times [\![\mathbb{S}^{\mathfrak{p}-1}]\!] \tag{3.3}$$

for some map $u_T \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *and some i.m. rectifiable current* L_T *in* $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$.

In fact, from Definition 2.3 and the null boundary condition (3.2), it is readily checked that for every $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ we have

$$(\partial L_T) \llcorner \mathrm{int}(\mathcal{X}) = (-1)^{\mathfrak{p}-1}\mathbf{P}(u_T)\,. \tag{3.4}$$

As a consequence, denoting for every $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ by

$$\mathcal{T}_u^{1/\mathfrak{p}} := \{T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^2) \mid u_T = u \text{ in (3.3)}\}$$

the subclass of Cartesian currents T with corresponding function u_T equal to u, it turns out that

$$\mathcal{T}_u^{1/\mathfrak{p}} = \{G_u + L \times [\![\mathbb{S}^{\mathfrak{p}-1}]\!] \,:\, L \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})\,, \quad (\partial L) \llcorner \mathrm{int}(\mathcal{X}) = (-1)^{\mathfrak{p}-1}\mathbf{P}(u)\}\,. \tag{3.5}$$

Remark 3.3 By (3.1) we thus obtain that *for every* $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *the class* $\mathcal{T}_u^{1/\mathfrak{p}}$ *is non-empty.*

We finally notice that according to Definition 2.5, for every $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ we have

$$G_u \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1}) \iff u \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})\,.$$

The $\mathcal{E}_{1/\mathfrak{p}}$-energy on currents. We equip the class $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ with the $\mathcal{E}_{1/\mathfrak{p}}$*-energy* defined for every T as in (3.3) and every open set $\Omega \subset \mathcal{X}$ by

$$\mathcal{E}_{1/\mathfrak{p}}(T, \Omega \times \mathbb{S}^{\mathfrak{p}-1}) := \mathcal{E}_{1/\mathfrak{p}}(u_T, \Omega) + |B^{\mathfrak{p}}| \cdot \mathbf{M}(L_T \llcorner \Omega)\,,$$

see (1.5), where $|B^{\mathfrak{p}}|$ is the measure of the $\mathfrak{p}$-dimensional unit ball. We also let

$$\mathcal{E}_{1/\mathfrak{p}}(T) := \mathcal{E}_{1/\mathfrak{p}}(T, \mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1}) = \mathcal{E}_{1/\mathfrak{p}}(u_T) + |B^{\mathfrak{p}}| \cdot \mathbf{M}(L_T)\,. \tag{3.6}$$

In the following sections we shall deduce the following semicontinuity and closure-compactness properties:

Proposition 3.4 *We have, see Sec. 5 below:*

i) *the energy functional* $T \mapsto \mathcal{E}_{1/\mathfrak{p}}(T)$ *is sequentially lower semicontinuous in* $\text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *with respect to the weak convergence in* $\mathcal{D}_n(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$*;*

ii) *the class* $\text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *contains the* $\mathcal{D}_n$*-weak limit points of sequences* $\{G_{u_k}\}$ *of currents carried by graphs of smooth maps in* $W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *with equibounded energies,* $\sup_k \mathcal{E}_{1/\mathfrak{p}}(u_k) < \infty$*;*

iii) *the class* $\text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *is closed with respect to the weak* $\mathcal{D}_n$*-convergence along sequences* $\{T_k\}$ *with equibounded energies,* $\sup_k \mathcal{E}_{1/\mathfrak{p}}(T_k) < \infty$*;*

iv) *if* $\{T_k\} \subset \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *satisfies* $\sup_k \mathcal{E}_{1/\mathfrak{p}}(T_k) < \infty$*, possibly passing to a subsequence* T_k *weakly converges in* $\mathcal{D}_n(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *to some current* T *in* $\text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$.

A DENSITY RESULT. Finally, we shall prove the following strong density property:

Theorem 3.5 *For every* $T \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *there exists a sequence of smooth* $W^{1/\mathfrak{p}}$*-maps* $u_k : \mathcal{X} \to \mathbb{S}^{\mathfrak{p}-1}$ *such that* $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{D}_n(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ *as* $k \to \infty$ *and*

$$\lim_{k\to\infty} \mathcal{E}_{1/\mathfrak{p}}(u_k) = \mathcal{E}_{1/\mathfrak{p}}(T)\,.$$

Remark 3.6 Definition 3.2 holds true even in dimension $n = \mathfrak{p} - 1$. In this case, L_T is a finite sum of signed unit Dirac masses, and actually G_u belongs to $\text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ for every $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$. Moreover, Proposition 3.4 and Theorem 3.5 continue to hold.

A REPRESENTATION FORMULA. As a consequence of Proposition 3.4 and Theorem 3.5, we now obtain the following representation formula for the relaxed energy, in the case $\mathfrak{p} \geq 2$ integer.

Proposition 3.7 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *the relaxed energy* $\widetilde{\mathcal{E}}_{1/p}(u)$ *is finite. Moreover, we have*

$$\begin{aligned} \widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) &= \inf\{\mathcal{E}_{1/\mathfrak{p}}(T) \mid T \in \mathcal{T}_u^{1/\mathfrak{p}}\} \\ &= \mathcal{E}_{1/p}(u) + |B^{\mathfrak{p}}| \cdot m_{i,\text{int}(\mathcal{X})}(\mathbf{P}(u))\,. \end{aligned} \tag{3.7}$$

PROOF: Let $T \in \mathcal{T}_u^{1/\mathfrak{p}}$, see Remark 3.3, and let $\{u_k\}$ be the approximating sequence given by Theorem 3.5. Since the weak convergence as currents yields the $L^{\mathfrak{p}}$-convergence of u_k to the $W^{1/\mathfrak{p}}$-function u_T corresponding to T, see (3.3), and $u_T = u$, we have that $u_k \to u$ in $L^{\mathfrak{p}}(\mathcal{X}, \mathbb{R}^{\mathfrak{p}})$, hence $\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) \leq \mathcal{E}_{1/\mathfrak{p}}(T) < \infty$. The inequality "$\leq$" in the first line of (3.7) follows by taking the infimum on $T \in \mathcal{T}_u^{1/\mathfrak{p}}$. Conversely, for every sequence $\{u_k\} \subset W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1}) \cap C^\infty$ such that $u_k \to u$ strongly in $L^{\mathfrak{p}}(\mathcal{X}, \mathbb{R}^{\mathfrak{p}})$, with $\sup_k \mathcal{E}_{1/\mathfrak{p}}(u_k) < \infty$, by closure-compactness, and possibly passing to a subsequence, the currents $G_{u_k} \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ weakly converge to some $T \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$. We again have $u_T = u$, whence $T \in \mathcal{T}_u^{1/\mathfrak{p}}$. Moreover, by lower semicontinuity, we have

$$\mathcal{E}_{1/\mathfrak{p}}(T) \leq \liminf_{k\to\infty} \mathcal{E}_{1/\mathfrak{p}}(G_{u_k}) = \liminf_{k\to\infty} \mathcal{E}_{1/\mathfrak{p}}(u_k)\,,$$

which yields the inequality "$\geq$" in the first line of (3.7), by the arbitrariness of $\{u_k\}$. The second equality follows from the definition of $\mathcal{E}_{1/p}$-energy and from (3.5) and Definition 3.1. □

A LOCAL FORMULA. A local version of Proposition 2.9 holds true, concerning the integral mass of the singularity $\mathbf{P}(u)$ relative to any open set $\Omega \subset \mathcal{X}$, see Definition 3.1. We have, see [27]:

Corollary 3.8 *For every* $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *and every open set* $\Omega \subset \mathcal{X}$ *we have that*

$$m_{i,\Omega}(\mathbf{P}(u) \llcorner \Omega) \leq C\, \|\,\text{Ext}(u)\|^p_{W^{1,p}(\Omega\times[0,1])} < \infty\,.$$

Moreover, Proposition 3.4 and Theorem 3.5 hold true for Cartesian currents in $\Omega \times \mathbb{S}^{\mathfrak{p}-1}$. As a consequence, similarly to Proposition 3.7 we readily obtain:

Proposition 3.9 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *and every open set* $\Omega \subset \mathcal{X}$ *we have*

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u, \Omega) = \mathcal{E}_{1/p}(u, \Omega) + |B^{\mathfrak{p}}| \cdot m_{i,\Omega}(\mathbf{P}(u) \llcorner \Omega) < \infty\,.$$

THE CASE p NON-INTEGER. A different feature occurs if the exponent p is non-integer. We first recall that if $1 < p < 2$, by (1.8) we have (1.9) in any dimension n. Arguing as in [17, Sec. 5.6] we prove:

Proposition 3.10 *Let $n \geq \mathfrak{p}$ and $p > 2$ non-integer. Then for every $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ we have*

$$\widetilde{\mathcal{E}}_{1/p}(u) = \begin{cases} \mathcal{E}_{1/p}(u) & \text{if } \mathbf{P}(u) = 0 \\ +\infty & \text{if } \mathbf{P}(u) \neq 0\,, \end{cases}$$

where $\mathbf{P}(u) \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X})$ is the current of the singularities of u, see Definition 2.3.

PROOF: If $\mathbf{P}(u) = 0$, then $u \in \text{cart}^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$, and the assertion follows from Theorem 2.6. Conversely, we now show that

$$\widetilde{\mathcal{E}}_{1/p}(u) < \infty \Longrightarrow \mathbf{P}(u) = 0\,.$$

Let $\{u_k\} \subset C^\infty(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ satisfy $\sup_k \mathcal{E}_{1/p}(u_k) < \infty$ and $u_k \to u$ strongly in $L^p(\mathcal{X}, \mathbb{R}^{\mathfrak{p}})$. Since $\sup_k \mathcal{E}_{1/\mathfrak{p}}(G_{u_k}) < \infty$, possibly passing to a subsequence, by Proposition 3.4 the currents $G_{u_k} \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$ weakly converge to some current $T \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathbb{S}^{\mathfrak{p}-1})$. Write T as in (3.3), where $L_T \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ satisfies (3.4) and, of course, $u_T = u$, hence $T \in \mathcal{T}_u^{1/\mathfrak{p}}$. In order to prove that $\mathbf{P}(u) = 0$, it then suffices to show that $L_T = 0$.

Since we use a local argument, without loss of generality we may and do assume $\mathcal{X} = B^n$. Suppose by contradiction that $\mathbf{M}(L_T) > 0$, and let $\mathcal{L}$ be a compact subset of the set of points of L_T which have non-zero density, $\mathcal{L}$ with positive $\mathcal{H}^{n-\mathfrak{p}+1}$-measure. For $x \in \mathcal{L}$, we denote by Π_x the intersection with B^n of the $(\mathfrak{p}-1)$-dimensional plane containing x and orthogonal to the approximate tangent $(n-\mathfrak{p}+1)$-space to $\mathcal{L}$ at x. Since $\mathcal{L}$ is compact, for $\mathcal{H}^{n-\mathfrak{p}+1}$-a.e. $x \in \mathcal{L}$, the $(\mathfrak{p}-1)$-dimensional restrictions of G_{u_k} to $\Pi_x \times \mathbb{S}^{\mathfrak{p}-1}$ yield (possibly passing to a subsequence) a sequence of graphs of smooth functions $u_{k|\Pi_x} : \Pi_x \to \mathbb{S}^{\mathfrak{p}-1}$ with equibounded $\mathcal{E}_{1/p}$-energies. Moreover, let $U_k := \text{Ext}(u_{k|\Pi_x}) \in W^{1/p}(\Pi_x \times I, \mathbb{R}^{\mathfrak{p}})$ be the $W^{1,p}$-extension of $u_{k|\Pi_x}$, so that

$$\sup_k \int_{\Pi_x \times I} |DU_k|^p \, d\mathcal{H}^{\mathfrak{p}} \leq \widetilde{C} < \infty\,.$$

For $r > 0$ small, let $D_r(x) := B_r^n(x) \cap \Pi_x$ denote the $(\mathfrak{p}-1)$-dimensional disk of radius r, centered at x and contained in Π_x. Since $p > \mathfrak{p}$, the Hölder inequality yields

$$\begin{aligned} \int_{D_r(x)\times I} |DU_k|^{\mathfrak{p}} \, d\mathcal{H}^{\mathfrak{p}} &\leq \Big(\int_{D_r(x)\times I} |DU_k|^p \, d\mathcal{H}^{\mathfrak{p}} \Big)^{\mathfrak{p}/p} \cdot \mathcal{H}^{\mathfrak{p}}(D_r(x) \times I)^{1-\mathfrak{p}/p} \\ &\leq \widetilde{C}\, r^{(\mathfrak{p}-1)(p-\mathfrak{p})/p}\,, \end{aligned} \tag{3.8}$$

where the exponent $(\mathfrak{p}-1)(p-\mathfrak{p})/p > 0$. On the other hand, by a slicing argument, and by Remark 3.6, the $(\mathfrak{p}-1)$-dimensional currents $G_{u_{k|\Pi_x}}$ have to converge "near" the point x to the graph $G_{u_{|\Pi_x}}$ of the restriction $u_{|\Pi_x}$ plus a vertical part of the type $\sigma\delta_x \times [\![\mathbb{S}^{\mathfrak{p}-1}]\!]$, where σ is some non-zero integer. By lower semicontinuity, this is in contradiction with (3.8), for $r > 0$ small enough, hence $L_T = 0$, as required. □

4 Graphs of $W^{1/p}$-maps and singularities

In order to extend the results of the previous section to more general target manifolds $\mathcal{Y}$, in this section we recall from [27] the definition of current G_u carried by the graph of a function $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$. We shall then discuss the *homological singularities* of u. As before, we assume $n := \dim(\mathcal{X}) \geq \mathfrak{p} := [p] \geq 2$. We also set $M := \dim(\mathcal{Y})$, and recall that $\mathcal{Y} \subset \mathbb{R}^N$.

$\mathcal{D}_{k,r}$-CURRENTS. Every compactly supported smooth differential k-form $\omega \in \mathcal{D}^k(\mathcal{X} \times \mathcal{Y})$, where $k \leq n$, splits as a sum $\omega = \sum_{j=0}^{\underline{k}} \omega^{(j)}$, $\underline{k} := \min(k, M)$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. For fixed $r = 1, \ldots, \underline{k}$ we denote by $\mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$ the subspace of $\mathcal{D}^k(\mathcal{X} \times \mathcal{Y})$ of k-forms of the type $\omega = \sum_{j=0}^{r} \omega^{(j)}$, and by $\mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ the dual space of $\mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$. Of course we have $\mathcal{D}_{k,k} = \mathcal{D}_k$, the space of all k-currents. Moreover, a sequence $\{T_k\} \subset \mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ is said to converges *weakly in* $\mathcal{D}_{k,r}$, say $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{k,r}(\mathcal{X}\times\mathcal{Y})$. The class $\mathcal{D}_{k,r}(\mathcal{X}\times\mathcal{Y})$ is closed under the weak convergence in $\mathcal{D}_{k,r}$. A similar notation holds by replacing $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{C}^{n+1}$ and $\mathbb{R}^N$, respectively.

Example 4.1 If $U \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$, then G_U is a well defined $(n+1,\mathfrak{p})$-current in $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1}\times\mathbb{R}^N)$ and, in an approximate sense, $G_U := (Id \bowtie U)_\#[\![\mathcal{C}^{n+1}]\!]$. Formula (2.1) continues to hold, where this time $0 \le h \le \min\{n+1, M, \mathfrak{p}\}$. Setting moreover

$$\|G_U\| := \sup\{G_U(\omega) \mid \omega \in \mathcal{D}^{n+1,\mathfrak{p}}(\mathcal{C}^{n+1}\times\mathbb{R}^N),\ \|\omega\| \le 1\}\,,$$

where $\|\omega\|$ is the *comass* norm of ω, by using the parallelogram inequality we infer that

$$\|G_U\| \le C\,(\mathcal{H}^n(\mathcal{X}) + \mathbf{D}_p(U)) < \infty$$

for some absolute constant $C = C(n,p,\mathcal{X}) > 0$, not depending on U. Similarly to Definition 2.1, we set:

Definition 4.2 *To any map* $u \in W^{1/p}(\mathcal{X},\mathbb{R}^N)\cap L^\infty$ *we associate an* $(n,\mathfrak{p}-1)$*-current* G_u *in* $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^N)$ *by setting*

$$G_u := (-1)^{n-1}(\partial G_U) \llcorner ((\mathcal{X}\times\{0\})\times\mathbb{R}^N) \qquad on \quad \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^N)\,, \tag{4.1}$$

where $U := \mathrm{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$*, see Example* 4.1.

In particular, if $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$, by Federer's support theorem [5] we infer that *the current* G_u *in Definition* 4.2 *belongs to* $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$.

Boundaries. The exterior differential of forms in $\mathcal{X}\times\mathcal{Y}$ splits into a horizontal and a vertical differential, $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x\omega)$ defines a horizontal boundary operator $\partial_x : \mathcal{D}_{k,r}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{D}_{k-1,r}(\mathcal{X}\times\mathcal{Y})$. However, the vertical differential $d_y\omega$ of any form $\omega \in \mathcal{D}^{k-1,r}(\mathcal{X}\times\mathcal{Y})$ belongs to $\mathcal{D}^{k,r}(\mathcal{X}\times\mathcal{Y})$ if and only if $d_y\omega^{(r)} = 0$. Therefore, for every $T \in \mathcal{D}_{k,r}(\mathcal{X}\times\mathcal{Y})$ the vertical boundary operator $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{k-1,r}(\mathcal{X}\times\mathcal{Y})$, where

$$\mathcal{Z}^{k,r}(\mathcal{X}\times\mathcal{Y}) := \{\omega \in \mathcal{D}^{k,r}(\mathcal{X}\times\mathcal{Y}) \mid d_y\omega^{(r)} = 0\}\,.$$

For future use, we also set

$$\mathcal{B}^{k,r}(\mathcal{X}\times\mathcal{Y}) := \{\omega \in \mathcal{D}^{k,r}(\mathcal{X}\times\mathcal{Y}) \mid \exists\,\eta \in \mathcal{D}^{k-1,r-1}(\mathcal{X}\times\mathcal{Y})\ :\ \omega^{(r)} = d_y\eta\}$$

and

$$\mathcal{H}^{k,r}(\mathcal{X}\times\mathcal{Y}) := \frac{\mathcal{Z}^{k,r}(\mathcal{X}\times\mathcal{Y})}{\mathcal{B}^{k,r}(\mathcal{X}\times\mathcal{Y})}\,,$$

and recall, see e.g. [17, Prop. 4.23], that

$$\mathcal{H}^{k,r}(\mathcal{X}\times\mathcal{Y}) \simeq \mathcal{D}^{k-r}(\mathcal{X}) \otimes \mathcal{H}^r_{dR}(\mathcal{Y})\,, \tag{4.2}$$

where $\mathcal{H}^r_{dR}(\mathcal{Y})$ is the r-th *de Rham cohomology group*.

Spherical cycles. The current $\mathbb{P}(u)$ carrying the singularities of maps $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ only depends on the *spherical subgroup* $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ of $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$.

Definition 4.3 *We say that an integral* $(\mathfrak{p}-1)$*-cycle* $C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ *is* of spherical type *if its homology class contains a Lipschitz image of the* $(\mathfrak{p}-1)$*-sphere* $\mathbb{S}^{\mathfrak{p}-1}$*, i.e., if there exist a* $(\mathfrak{p}-1)$*-cycle* $Z \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$*, an i.m. rectifiable* $\mathfrak{p}$*-current* $R \in \mathcal{R}_{\mathfrak{p}}(\mathcal{Y})$*, and a Lipschitz function* $\phi : \mathbb{S}^{\mathfrak{p}-1} \to \mathcal{Y}$*, such that*

$$C - Z = \partial R \qquad and \qquad Z = \phi_\#[\![\mathbb{S}^{\mathfrak{p}-1}]\!]\,.$$

Moreover, denote

$$\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}) := \{[\gamma] \in \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}) \mid \exists\ \phi \in \mathrm{Lip}(\mathbb{S}^{\mathfrak{p}-1},\mathcal{Y})\ \mid\ \phi_\#[\![\mathbb{S}^{\mathfrak{p}-1}]\!] \in [\gamma]\}\,.$$

By the assumption (H_1), we infer that we may find a family $\gamma_1,\dots,\gamma_{\widetilde{s}}$ of integral cycles (with finite mass) in $\mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ such that the corresponding equivalence classes generate the spherical subgroup $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$, i.e.,

$$\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}) = \left\{ \sum_{s=1}^{\widetilde{s}} n_s\,[\gamma_s] \mid n_s \in \mathbb{Z} \right\}.$$

We shall denote by $[\sigma^1],\dots,[\sigma^{\widetilde{s}}]$ a dual basis of *spherical* $(\mathfrak{p}-1)$*-forms* in $\mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y})$, so that $\gamma_s(\sigma^r) = \delta_{sr}$.

Remark 4.4 In the case $\mathfrak{p}=2$, every integral 1-cycle in $\mathcal{Z}_1(\mathcal{Y})$ is of $\mathbb{S}^1$-type.

THE HOMOLOGICAL SINGULARITIES OF $W^{1/p}$-MAPS. Let $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$. For each $s=1,\dots,\widetilde{s}$, we define the current $\mathbb{P}_s(u) \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X})$ by setting for any $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$

$$\mathbb{P}_s(u)(\phi) := (-1)^{\mathfrak{p}}\partial G_u(\phi \wedge \sigma^s)\,, \qquad \phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,. \tag{4.3}$$

Formally, we have

$$\mathbb{P}_s(u) := (-1)^{\mathfrak{p}}\,(-1)^{(n-\mathfrak{p})(\mathfrak{p}-1)}\pi_{\#}((\partial G_u) \llcorner \widehat{\pi}^{\#}\sigma^s)\,,$$

where $\pi : \mathcal{X}\times\mathbb{R}^N \to \mathcal{X}$ and $\widehat{\pi} : \mathcal{X}\times\mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projections onto the first and second factor, respectively. Moreover, it turns out that $\mathbb{P}_s(u)$ does not depend on the representative in the cohomology class $[\sigma^s]$.

Also, for every s we define the current $\mathbb{D}_s(u) \in \mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$ by

$$\mathbb{D}_s(u)(\gamma) := G_U(\widetilde{\gamma} \wedge d\widehat{\sigma}^s)\,, \qquad \gamma \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})\,. \tag{4.4}$$

Here $U := \mathrm{Ext}(u)$, $\widetilde{\gamma} := \gamma\wedge\eta \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$, and $\widehat{\sigma}^s \in \mathcal{D}^{\mathfrak{p}-1}(\mathbb{R}^N)$ satisfies $i^{\#}\widehat{\sigma}^s = \sigma^s$, where $i : \mathcal{Y} \hookrightarrow \mathbb{R}^N$ is the injection map. This time, we formally have

$$\mathbb{D}_s(u) := (-1)^{\mathfrak{p}(n-\mathfrak{p}+1)}\,\pi_{\#}(G_U \llcorner \widehat{\pi}^{\#} d\widehat{\sigma}^s)\,.$$

Moreover, $\mathbb{D}_s(u)$ is a current of *finite mass* in $\mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$, as $U = \mathrm{Ext}(u)$ is a $W^{1,p}$-function and $d\widehat{\sigma}^s \in \mathcal{D}^{\mathfrak{p}}(\mathbb{R}^N)$, with $\mathfrak{p} = [p]$, see Example 4.1.

It turns out that for every $\omega \in \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$ the action $\partial G_u(\omega)$ only depends on the cohomology class of ω. Therefore, the boundary current ∂G_u induces a functional $(\partial G_u)_*$ on $\mathcal{H}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$,

$$(\partial G_u)_*(\omega + \mathcal{B}^{n-1,\mathfrak{p}-1}) := \partial G_u(\omega + \mathcal{B}^{n-1,\mathfrak{p}-1}) = \partial G_u(\omega)\,, \quad \omega \in \mathcal{Z}^{n-1,\mathfrak{p}-1}\,.$$

By (4.2) the *homology map* $(\partial G_u)_*$ is uniquely represented as an element of $\mathcal{D}_{n-\mathfrak{p}}(\mathcal{X};\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R}))$. We thus set

$$\mathbb{P}(u) := (\partial G_u)_* \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X};\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R}))\,.$$

More explicitly, we find that

$$\mathbb{P}(u) = (-1)^{\mathfrak{p}} \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u) \otimes [\gamma_s]\,. \tag{4.5}$$

The above notation reduces to Definition 2.3 in the model case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$. Moreover, for general target manifolds $\mathcal{Y}$, similarly to Proposition 2.4 we obtain that for every $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$

$$\mathbb{P}_s(u) = \partial\,\mathbb{D}_s(u) \quad \text{on} \quad \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X}) \qquad \forall\, s\,. \tag{4.6}$$

Therefore, we have

$$\mathbb{P}_s(u)(\phi) = \mathbb{D}_s(u)(d\phi) = G_U(d\widetilde{\phi}\wedge d\widehat{\sigma}^s) = \int_{\mathcal{C}^{n+1}} d\widetilde{\phi}\wedge U^{\#}(d\widehat{\sigma}^s)$$

for every $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$. Finally, if $u \in R^{\infty}_{1/p}(\mathcal{X},\mathcal{Y})$ we obtain that $\mathbb{P}(u)$ is an *integral flat chain.*

Proposition 4.5 *If* $u \in \mathcal{R}^{\infty}_{1/p}(\mathcal{X},\mathcal{Y})$*, then* $\mathbb{P}_s(u)$ *is i.m. rectifiable in* $\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X})$ *for every* $s=1,\dots,\widetilde{s}$.

Cartesian maps. A map $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ has zero homological singularities, i.e., it satisfies $\mathbb{P}(u)=0$, if and only if the current G_u associated to its graph has no inner boundary:

$$\partial G_u = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}), \qquad \mathfrak{p} := [p] \geq 2 . \tag{4.7}$$

For this reason, we extend Definition 2.5 as follows:

Definition 4.6 *A map $u : \mathcal{X} \to \mathcal{Y}$ is said to be a* Cartesian map *in $\text{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ if u belongs to $W^{1/p}(\mathcal{X},\mathcal{Y})$ and satisfies the null-boundary condition* (4.7).

In the sequel we shall tacitly assume that $\dim(\mathcal{Y}) \geq \mathfrak{p}-1$ and that the homology group $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ is non-trivial, so that in general $\mathbb{P}(u)\neq 0$, i.e., the strict inclusion $\text{cart}^{1/p}(\mathcal{X},\mathcal{Y}) \subsetneq W^{1/p}(\mathcal{X},\mathcal{Y})$ holds. As in the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, since the null-boundary condition (4.7) is preserved by the weak convergence in $\mathcal{D}_{n,\mathfrak{p}-1}$, and the strong convergence $u_k \to u$ in $W^{1/p}(\mathcal{X},\mathcal{Y})$ yields the weak convergence $G_{u_k} \rightharpoonup G_u$ in $\mathcal{D}_{n,\mathfrak{p}-1}$, according to (1.7) we obtain again that

$$H^{1/p}_S(\mathcal{X},\mathcal{Y}) \subset \text{cart}^{1/p}(\mathcal{X},\mathcal{Y}) .$$

Under suitable hypotheses on $\mathcal{X}$ and $\mathcal{Y}$, in [27] we showed that *every map in $\text{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ can be approximated by sequences of smooth maps in $W^{1/p}$.*

Theorem 4.7 *Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy the hypotheses (H_1), (H_2) and (H_3). For every $n \geq \mathfrak{p} \geq 2$ we have*

$$\text{cart}^{1/p}(\mathcal{X},\mathcal{Y}) = H^{1/p}_S(\mathcal{X},\mathcal{Y}) .$$

Remark 4.8 Even in the case $\mathcal{X}=B^n$, and $n=\mathfrak{p}$, the hypothesis (H_2) on the commutativity of the first homotopy group, in the case $\mathfrak{p}=2$, or on the injectivity of the Hurewicz maps, in the case $\mathfrak{p}\geq 3$, cannot be dropped from the statement of Theorem 4.7, compare [27].

Integral connections. Assume now that $\mathcal{X}$ and $\mathcal{Y}$ satisfy the hypotheses (H_1) and (H_2). Similarly to Proposition 2.9, in [27] we proved the following upper bound for the integral connection of the singularities.

Theorem 4.9 *Let $n \geq \mathfrak{p} := [p] \geq 2$ and $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$. Then for every $s=1,\dots,\widetilde{s}$ there exists an i.m. rectifiable current $L_s \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that $(\partial L_s) \llcorner \text{int}(\mathcal{X}) = \mathbb{P}_s(u)$ and the mass*

$$\mathbf{M}(L_s) \leq C\left(\mathcal{E}_{1/p}(u) + \|\,\text{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right),$$

where $C>0$ is an absolute constant, not depending on u.

A similar result is satisfied in the case with prescribed boundary data. Moreover, according to Definition 3.1, the following local version holds true:

Corollary 4.10 *For every $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ and every open set $\Omega \subset \mathcal{X}$ we have*

$$m_{i,\Omega}(\mathbb{P}_s(u)\llcorner\Omega) \leq C\,\|\,\text{Ext}(u)\|^p_{W^{1,p}(\Omega\times[0,1])} \qquad \forall\, s=1,\dots\widetilde{s} .$$

In the case $n=\mathfrak{p}$, or $n\geq\mathfrak{p}=2$, we finally have:

Proposition 4.11 *If $n=\mathfrak{p}$ or $\mathfrak{p}=2$,* Theorem 4.9 *and* Corollary 4.10 *hold true for the whole classes of maps in $W^{1/p}(\mathcal{X},\mathcal{Y})$.*

5 Cartesian currents with finite $W^{1/\mathfrak{p}}$-energy

In this section we introduce the class of Cartesian currents with finite $W^{1/\mathfrak{p}}$-energy, see Definitions 5.2 and 5.10, collecting their main properties. We assume $n := \dim(\mathcal{X}) \geq \mathfrak{p}-1 \geq 1$, with $\mathfrak{p}$ integer, and that $\mathcal{X}$ and $\mathcal{Y}$ satisfy the assumptions (H_1), (H_2), and (H_3). We shall extend analogous results proved in [13] in the case $\mathfrak{p}=2$ and $\mathcal{X}=B^n$. Therefore, we shall omit almost all the proofs, as they can be readily deduced by using arguments very similar to the corresponding ones given in [13] for $\mathfrak{p}=2$, see also [17, Ch. 6].

Integral flat cycles. We shall make use of the following

Definition 5.1 *We say that an i.m.* $(\mathfrak{p}-1)$*-cycle* $C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ *is an* integral flat cycle *if there exists an i.m. rectifiable current* $R \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ *such that* $\partial R = C$*.*

This definition has to do with the *relative integral homology* classes $\mathcal{H}_{\mathfrak{p}-1}(A, B; \mathbb{Z})$ in the case $A = \mathcal{Y}$ and $B = \partial\mathcal{Y} = \emptyset$. In fact, compare [10, Vol. I, Sec. 5.4.1], one has

$$\mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z}) = \{C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y}) \mid C \text{ is an integral flat } (\mathfrak{p}-1)\text{-cycle}\}$$

whereas, on account of Federer's flatness theorem,

$$\mathcal{B}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z}) = \{\partial W \mid W \in \mathcal{R}_{\mathfrak{p}}(\mathcal{Y})\}\,.$$

Therefore, an element q in $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$ is an equivalence class of integral flat $(\mathfrak{p}-1)$-cycles of $\mathcal{Y}$, where

$$C \sim Z \quad \Longleftrightarrow \quad \exists\, W \in \mathcal{R}_{\mathfrak{p}}(\mathcal{Y})\,:\, C - Z = \partial W\,.$$

It turns out that $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$ is isomorphic to the integral homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$. We shall then denote by $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$ the subgroup of $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$ that is isomorphic to $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$. Correspondingly, in the sequel an integral flat $(\mathfrak{p}-1)$-cycle is said to be of spherical type if its homology class belongs to $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$. Moreover, we denote by $[\widetilde{\gamma}_1], \ldots, [\widetilde{\gamma}_{\widetilde{s}}]$ a family of generators of $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$, in such a way that $\widetilde{\gamma}_s = \gamma_s + \partial R_s$ for some $R_s \in \mathcal{R}_{\mathfrak{p}}(\mathcal{Y})$, and hence $\widetilde{\gamma}_s(\sigma^r) = \gamma_s(\sigma^r) = \delta_{sr}$, as $\sigma^r \in \mathcal{Z}^{\mathfrak{p}-1}(\mathcal{Y})$ is closed.

Finally, we notice that for every s there exists a *mass minimizing* i.m. rectifiable current $R \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ such that ∂R belongs to the homology class $[\widetilde{\gamma}_s]$. Therefore, we shall denote

$$\widetilde{M}_s := \min\{\mathbf{M}(R) \mid R \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)\,,\ \partial R \in [\widetilde{\gamma}_s]\}\,, \qquad s = 1, \ldots, \widetilde{s}\,. \tag{5.1}$$

$\mathcal{E}_{1/\mathfrak{p}}$-GRAPHS AND EXTENSIONS. We now give the following

Definition 5.2 *Let* $T \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$*. We say that* T *is in* $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X} \times \mathcal{Y})$ *if*

$$\partial T = 0 \quad on \quad \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y}) \tag{5.2}$$

and T *can be decomposed as*

$$T = G_{u_T} + S_T\,, \quad S_T := \sum_{q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})} \mathbb{L}_q \times C_q\,, \quad on\ \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})\,. \tag{5.3}$$

Here $u_T \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$ *and* $\mathbb{L}_q$ *is an i.m. rectifiable current in* $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$*,* $\mathbb{L}_q := \tau(\mathcal{L}_q, 1, \overrightarrow{\mathcal{L}}_q)$*, where the* $(n-\mathfrak{p}+1)$*-rectifiable sets* $\mathcal{L}_q$ *are pairwise disjoint. Finally,* C_q *is an integral flat* $(\mathfrak{p}-1)$*-cycle of spherical type in the homology class* q*, according to Definition* 5.1*.*

Remark 5.3 Notice that S_T is *completely vertical*, i.e., $S_T(\omega) = 0$ if $\omega^{(\mathfrak{p}-1)} = 0$. Moreover, currents in $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X} \times \mathcal{Y})$ are defined in a homological sense. More precisely, the decomposition (5.3) does not depend on the choice of the representative C_q in the homology class q. In fact, if $\widetilde{C}_q - C_q = \partial W$ for some $W \in \mathcal{R}_{\mathfrak{p}}(\mathcal{Y})$, then for every $\phi \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})$ and $\sigma \in \mathcal{D}^{\mathfrak{p}-1}(\mathcal{Y})$

$$(\mathbb{L}_q \times \widetilde{C}_q)(\phi \wedge \sigma) - (\mathbb{L}_q \times C_q)(\phi \wedge \sigma) = \mathbb{L}_q(\phi) \cdot W(d_y\sigma)\,,$$

hence it is non-zero if and only if $d_y\sigma \neq 0$, i.e., if and only if $\phi \wedge \sigma \notin \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$.

Similarly to [11], we extend currents in $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X} \times \mathcal{Y})$ to suitable currents in $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$.

Definition 5.4 *Let* $T \in \mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X} \times \mathcal{Y})$ *be such that* (5.3) *holds. We denote* extension of T*, say* $\widetilde{T} = \mathrm{Ext}(T)$*, the current* $\widetilde{T}$ *in* $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ *such that*

$$\widetilde{T} = (-1)^{n-1}\Big(G_{U_T} + (-1)^{\mathfrak{p}} \sum_{q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})} \mathbb{L}_q \times R_q\Big)\,, \tag{5.4}$$

where $U_T = \mathrm{Ext}(u_T) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ *and* R_q *is the i.m. rectifiable current of least mass among all currents in* $\mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ *such that* $Z_q = \partial R_q$ *is in the homology class* q*, see Remark* 5.3*.*

Remark 5.5 Even if R_q minimizes the mass among all currents in $\mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ such that ∂R_q is in the homology class q, a priori the mass of ∂R_q is not finite. Moreover, by an isoperimetric theorem, see e.g. [17, Thm. 1.101], the infimum of the mass of the R_q's is bounded from below by a positive constant $C_{\mathcal{Y}} > 0$, only depending on $\mathcal{Y}$, i.e.,

$$\inf\{\mathbf{M}(R_q) \mid R_q \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)\,,\ \partial R_q \in q\,,\ q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})\,,\ q \text{ non trivial}\} \geq C_{\mathcal{Y}} > 0\,. \tag{5.5}$$

In particular, according to (5.1) we have $\widetilde{M}_s \geq C_{\mathcal{Y}}$ for every s.

Remark 5.6 From Definition 5.4 and (4.1) we infer that the restriction to $(\mathcal{X}\times\{0\})\times\mathcal{Y}$ of the *boundary* of the current $\widetilde{T}$ in (5.4) is equal to T on forms in $\mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$. In fact, for every $q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})$

$$\partial(\mathbb{L}_q \times R_q) = \partial\,\mathbb{L}_q \times R_q + (-1)^{n-\mathfrak{p}+1}\,\mathbb{L}_q \times \partial R_q$$

and hence, since $\partial R_q = C_q \in q$, whereas $\partial\,\mathbb{L}_q \times R_q = 0$ on $\mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^N)$, we have

$$(-1)^{n-1+\mathfrak{p}}\partial(\mathbb{L}_q \times R_q) = \mathbb{L}_q \times C_q \qquad \text{on} \qquad \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})\,.$$

Remark 5.7 In the model case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, the class $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$ agrees with $\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathbb{S}^{\mathfrak{p}-1})$, and the extension of $T = G_{u_T} + L_T \times [\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!]$ is the current $\widetilde{T}$ in $\mathcal{D}_{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})$ given by

$$\widetilde{T} = (-1)^{n-1}\Big(G_{U_T} + (-1)^{\mathfrak{p}} L_T \times [\![\, B^{\mathfrak{p}}\,]\!]\Big)\,, \qquad U_T = \text{Ext}(u_T) \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^{\mathfrak{p}})\,. \tag{5.6}$$

THE $\mathcal{E}_{1/\mathfrak{p}}$-ENERGY. If $\widetilde{T} \in \mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1}\times\mathbb{R}^N)$, we denote by $\mathbf{D}_{\mathfrak{p}}(\widetilde{T})$ the *parametric polyconvex extension* of the $\mathfrak{p}$-energy (1.4) on $\widetilde{T}$, compare [10, Vol. II, Sec. 1.2.4]. It turns out that $\widetilde{T} \mapsto \mathbf{D}_{\mathfrak{p}}(\widetilde{T})$ *is sequentially weak-$\mathcal{D}_{n+1,\mathfrak{p}}$ lower semicontinuous on the class* $\{\widetilde{T} = \text{Ext}(T) \mid T \in \mathcal{E}_{1/\mathfrak{p}}\text{-graph}(\mathcal{X}\times\mathcal{Y})\}$. Moreover, if $\widetilde{T} \in \mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1}\times\mathbb{R}^N)$ satisfies (5.4), we have

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}) = \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{\mathcal{C}^{n+1}} |DU_T|^{\mathfrak{p}}\,d\mathcal{H}^{n+1} + \sum_{q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})} \mathbf{M}(\mathbb{L}_q)\cdot\mathbf{M}(R_q)\,, \tag{5.7}$$

with $\mathbf{D}_{\mathfrak{p}}(G_U) = \mathbf{D}_{\mathfrak{p}}(U)$ if $\widetilde{T} = G_U$ for some $U \in W^{1,\mathfrak{p}}(\mathcal{C}^{n+1},\mathbb{R}^N)$. We shall refer to (5.7) as the $\mathfrak{p}$-*energy* of $\widetilde{T}$.

Definition 5.8 *Let T be in $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X}\times\mathcal{Y})$, so that* (5.3) *holds. The $\mathcal{E}_{1/\mathfrak{p}}$-energy $\mathcal{E}_{1/\mathfrak{p}}(T)$ of T is defined as the $\mathfrak{p}$-energy $\mathbf{D}_{\mathfrak{p}}(\widetilde{T})$ of the extension $\widetilde{T} := \text{Ext}(T)$, see* (5.4) *and* (5.7).

Moreover, for any open set $\Omega \subset \mathcal{X}$, any $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$, and $T \in \mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X}\times\mathcal{Y})$, we shall denote

$$\begin{aligned}
\mathcal{E}_{1/p}(u,\Omega) \;&:=\; \mathbf{D}_p(\text{Ext}(u),\Omega\times I)\,,\\
\mathcal{E}_{1/\mathfrak{p}}(T,\Omega\times\mathcal{Y}) \;&:=\; \mathbf{D}_{\mathfrak{p}}(\text{Ext}(T),\Omega\times I\times\mathbb{R}^N)\\
&=\; \mathbf{D}_{\mathfrak{p}}(U_T,\Omega\times I) + \sum_{q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})} \mathbf{M}(\mathbb{L}_q \llcorner \Omega)\cdot\mathbf{M}(R_q)\,.
\end{aligned} \tag{5.8}$$

Finally, if $T = G_u$ for some $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ and $U = \text{Ext}(u)$, we let

$$\text{Ext}(G_u) := (-1)^{n-1}G_U\,, \quad \mathcal{E}_{1/\mathfrak{p}}(G_u) := \mathbf{D}_{\mathfrak{p}}(G_U) = \mathbf{D}_{\mathfrak{p}}(U)\,.$$

Remark 5.9 Of course, if $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$ and $\widetilde{T} \in \mathcal{D}_{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^{\mathfrak{p}})$ satisfies (5.6), we have

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}) = \mathbf{D}_{\mathfrak{p}}(U_T) + |B^{\mathfrak{p}}|\cdot\mathbf{M}(L_T)\,,$$

hence Definition 5.8 reduces to (3.6).

CARTESIAN CURRENTS. We now give the following

Definition 5.10 *Let $T \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$. We say that T is a* Cartesian current *in* $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ *if T belongs to* $\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X}\times\mathcal{Y})$ *and the $\mathcal{E}_{1/\mathfrak{p}}$-energy $\mathcal{E}_{1/\mathfrak{p}}(T)$ of T is finite, see Definitions* 5.2 *and* 5.8.

If $T = G_u$ for some $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$, we infer that $G_u \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ if and only if u is a Cartesian map in $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$, see Definition 4.6. Moreover, for $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$ we obtain the class of Cartesian currents in Definition 3.2.

Now, if $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, since $\widetilde{\gamma}_s(\sigma^r) = \delta_{sr}$, we may decompose

$$T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T) \times \widetilde{\gamma}_s \qquad \text{on} \quad \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})\,, \tag{5.9}$$

where $\mathbb{L}_s(T) \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ only depends on the cohomology class of σ^s and is defined by

$$\mathbb{L}_s(T)(\phi) := S_T(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\sigma^s)\,, \qquad \phi \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})\,,$$

S_T being the second component of T in (5.3). As a consequence, on account of (4.3) and (5.2), similarly to (3.4) we infer that

$$\partial\, \mathbb{L}_s(T) \llcorner \mathrm{int}(\mathcal{X}) = (-1)^{\mathfrak{p}-1}\, \mathbb{P}_s(u_T) \qquad \forall\, s = 1,\ldots,\widetilde{s}\,. \tag{5.10}$$

According to (5.1), we thus infer that for every open set $\Omega \subset \mathcal{X}$

$$\mathcal{E}_{1/\mathfrak{p}}(T, \Omega\times\mathcal{Y}) = \mathcal{E}_{1/\mathfrak{p}}(u,\Omega) + \sum_{s=1}^{\widetilde{s}} \widetilde{M}_s \cdot \mathbf{M}(\mathbb{L}_s(T) \llcorner \Omega)\,. \tag{5.11}$$

Remark 5.11 Conversely, if $T \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$ satisfies (5.2) and (5.9), with $u_T \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ and $\mathbb{L}_s(T) \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$, arguing as in [7], see also [17, Prop. 4.6.6], we may decompose T as in (5.3), whereas $\mathcal{E}_{1/\mathfrak{p}}(T) < \infty$, whence T belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$.

The weak convergence. We say that $\{T_k\} \subset \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$ converges to $T \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$ weakly in $\mathcal{Z}_{n,\mathfrak{p}-1}$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$. Moreover, if $\{T_k\} \subset \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ we say that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ if $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,\mathfrak{p}-1}$ and in addition $\sup_k \mathcal{E}_{1/\mathfrak{p}}(T_k) < \infty$.

Remark 5.12 For future use, we observe that if $u \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$, by Theorem 4.7 we find a smooth sequence $\{u_k\} \subset W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y}) \cap C^1$ that strongly converges to u in the $W^{1/\mathfrak{p}}$-sense as $k \to \infty$. The dominated convergence theorem yields that $G_{u_k} \rightharpoonup G_u$ weakly in $\mathcal{Z}_{n,\mathfrak{p}-1}$, with $\mathcal{E}_{1/\mathfrak{p}}(u_k) \to \mathcal{E}_{1/\mathfrak{p}}(u)$.

The $(\mathfrak{p}-1)$-dimensional case. Definition 5.10 is motivated by the following facts that hold true in low dimension $n := \dim(\mathcal{X}) = \mathfrak{p}-1$.

Theorem 5.13 *Let $n = \mathfrak{p}-1$ and $\{u_k\} \subset C^1(\mathcal{X},\mathcal{Y})$ be a sequence of smooth maps with $\sup_k |u_k|_{1/\mathfrak{p}} < \infty$. Possibly passing to a subsequence, $G_{u_k} \rightharpoonup T$ weakly in* $\mathrm{cart}^{1/\mathfrak{p}}$ *to some current $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$.*

Therefore the class $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ contains the weak limits in $\mathrm{cart}^{1/\mathfrak{p}}$ of sequences of graphs of smooth maps with equibounded $\mathcal{E}_{1/\mathfrak{p}}$-energies. Moreover, the following lower semicontinuity property holds.

Proposition 5.14 *Let $n = \mathfrak{p}-1$ and $\{u_k\} \subset C^1(\mathcal{X},\mathcal{Y})$ be a smooth sequence with $\sup_k |u_k|_{1/\mathfrak{p}} < \infty$ and such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{Z}_{\mathfrak{p}-1,\mathfrak{p}-1}$ to some current $T \in \mathcal{E}_{1/\mathfrak{p}}$-*graph*$(\mathcal{X}\times\mathcal{Y})$. Then T belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ and*

$$\mathcal{E}_{1/\mathfrak{p}}(T) \le \liminf_{k\to\infty} \mathcal{E}_{1/\mathfrak{p}}(G_{u_k})\,.$$

Finally, the following closure theorem holds.

Theorem 5.15 *Let $n = \mathfrak{p}-1$. The class $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ is closed under the sequential weak convergence in $\mathrm{cart}^{1/\mathfrak{p}}$.*

Density results. In the sequel of this paper we shall prove the following weak density result.

Theorem 5.16 *Assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy (H_1), (H_2), and (H_3). Let $n\geq\mathfrak{p}\geq 2$ integers. Then for every $T\in\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\}\subset C^\infty(\mathcal{X},\mathcal{Y})$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ as $k\to\infty$ and*

$$\limsup_{k\to\infty}\mathcal{E}_{1/\mathfrak{p}}(u_k)\leq\,\big(2c(\mathfrak{p},\mathcal{Y})-1\big)\cdot\mathcal{E}_{1/\mathfrak{p}}(T)\,,$$

where $c(\mathfrak{p},\mathcal{Y})\geq 1$ is an absolute constant, only depending on $\mathfrak{p}$ and $\mathcal{Y}$.

In the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, the optimal constant $c(\mathfrak{p},\mathbb{S}^{\mathfrak{p}-1})$ is equal to one for every $\mathfrak{p}\geq 2$, by Proposition 7.1 below. Moreover, if $\mathfrak{p}=2$, the optimal constant $c(2,\mathcal{Y})$ is always equal to one, see Remark 7.4 below. As a consequence, by the lower semicontinuity of the $\mathcal{E}_{1/p}$-energy, we immediately obtain the following strong density result, that reduces to Theorem 3.5, if $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, and to the density result from [13], if $\mathfrak{p}=2$ and $\mathcal{X}=B^n$.

Theorem 5.17 *Under the hypotheses of Theorem 5.16, if $\mathfrak{p}=2$ or if $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$, then for every $T\in\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\}\subset C^\infty(\mathcal{X},\mathcal{Y})$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ as $k\to\infty$ and*

$$\lim_{k\to\infty}\mathcal{E}_{1/\mathfrak{p}}(u_k)=\mathcal{E}_{1/\mathfrak{p}}(T)\,.$$

Closure-compactness. Taking into account the closure theorem 5.15 and the approximation theorem 5.16, we obtain in any dimension $n\geq\mathfrak{p}$ the following closure property.

Theorem 5.18 *In any dimension $n\geq\mathfrak{p}$, the class $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ is closed under the sequential weak convergence in $\mathrm{cart}^{1/\mathfrak{p}}$.*

Proof: Let T be the weak $\mathcal{Z}_{n,\mathfrak{p}-1}$-limit of a sequence $\{T_k\}\subset\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ with $\sup_k\mathcal{E}_{1/\mathfrak{p}}(T_k)<\infty$. By Theorem 5.16 and a diagonal argument, we may and do assume that each T_k is equal to G_{u_k} for some smooth map $u_k\in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$. As in the proof of [13, Thm. 2.12], writing T as in (5.9), and on account of Remark 5.11, we reduce to show that the $\mathbb{L}_s(T)$'s are flat chains. This property is obtained arguing in a way similar to the proof of [13, Thm. 2.15]. We omit any further detail. □

As a consequence of the closure property, we finally obtain the following lower semicontinuity and compactness properties in any dimension $n\geq\mathfrak{p}-1$.

Proposition 5.19 *Let $\{T_k\}\subset\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ be such that $T_k\rightharpoonup T$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ to some current $T\in\mathcal{E}_{1/\mathfrak{p}}$-graph$(\mathcal{X}\times\mathcal{Y})$. Then $T\in\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ and*

$$\mathcal{E}_{1/\mathfrak{p}}(T)\leq\liminf_{k\to\infty}\mathcal{E}_{1/\mathfrak{p}}(T_k)\,. \tag{5.12}$$

Proposition 5.20 *Let $\{T_k\}\subset\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ be such that $\sup_k\mathcal{E}_{1/\mathfrak{p}}(T_k)<\infty$. Possibly passing to a subsequence, we have that $T_k\rightharpoonup T$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ to some current $T\in\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$.*

The previous properties clearly reduce to Proposition 3.4 in the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$.

6 The relaxed $W^{1/p}$-energy

In this section we discuss the relaxed $\mathcal{E}_{1/p}$-energy (1.6) of mappings in $W^{1/p}(\mathcal{X},\mathcal{Y})$ for general target manifolds. We shall assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy the hypotheses (H_1), (H_2), and (H_3). We first consider the case $p=\mathfrak{p}\geq 2$ integer.

The case p integer. Following the notation from Sec. 5, for any $u\in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ we let

$$\mathcal{T}_u^{1/\mathfrak{p}}:=\{T\in\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})\mid u_T=u\} \tag{6.1}$$

denote the subclass of Cartesian currents T for which the corresponding function $u_T\in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ in (5.3) agrees with u. On account of (5.10) and Remark 5.11, we actually have

$$\mathcal{T}_u^{1/\mathfrak{p}}=\Big\{G_u+\sum_{s=1}^{\widetilde{s}}L_s\times\widetilde{\gamma}_s\,:\,L_s\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})\,,\quad(\partial L_s)\llcorner\mathrm{int}(\mathcal{X})=(-1)^{\mathfrak{p}-1}\,\mathbb{P}_s(u)\quad\forall\,s\Big\}\,. \tag{6.2}$$

Moreover, we observe:

Proposition 6.1 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ *we have*

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) < \infty \iff \mathcal{T}_u^{1/\mathfrak{p}} \neq \emptyset\,.$$

PROOF: Assume that $\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) < \infty$, and let $\{u_k\} \subset W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y}) \cap C^1$ be such that $u_k \to u$ strongly in $L^{\mathfrak{p}}(\mathcal{X},\mathbb{R}^N)$, with $\sup_k \mathcal{E}_{1/\mathfrak{p}}(u_k) < \infty$. By Proposition 5.20, possibly passing to a subsequence, the currents $G_{u_k} \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ weakly converge in $\mathrm{cart}^{1/\mathfrak{p}}$ to some $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$. Since the weak convergence as currents yields the $L^{\mathfrak{p}}$-convergence of u_k to the $W^{1/\mathfrak{p}}$-function u_T corresponding to T, see (5.3), we have $u_T = u$, whence $T \in \mathcal{T}_u^{1/\mathfrak{p}}$. Conversely, let $T \in \mathcal{T}_u^{1/\mathfrak{p}}$, and let $\{u_k\}$ be the approximating sequence given by Theorem 5.16. Since $u_k \to u_T = u$ in $L^{\mathfrak{p}}(\mathcal{X},\mathbb{R}^N)$, we obtain that $\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) < \infty$. □

We now show that the relaxed energy is always finite.

Theorem 6.2 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ *the relaxed energy* $\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) < \infty$.

PROOF: Let $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ and let $\{u_k\} \subset \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ be strongly converging to u in $W^{1/\mathfrak{p}}$, see Theorem 1.4. Possibly passing to a subsequence, we can assume that

$$\mathcal{E}_{1/\mathfrak{p}}(u_k) \le 2\,\mathcal{E}_{1/\mathfrak{p}}(u) \quad \text{and} \quad \|\,\mathrm{Ext}(u_k)\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(\mathcal{C}^{n+1})} \le 2\,\|\,\mathrm{Ext}(u)\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(\mathcal{C}^{n+1})} \qquad \forall\, k\,.$$

Therefore, by Theorem 4.9 for every k and every $s = 1,\dots,\widetilde{s}$ we can find an i.m. rectifiable current $L^k_s \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that $(\partial L^k_s) \llcorner \mathrm{int}(\mathcal{X}) = \mathbb{P}_s(u_k)$ and the mass

$$\mathbf{M}(L^k_s) \le 2\,C\left(\mathcal{E}_{1/p}(u) + \|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right).$$

According to (6.2), the current

$$T_k := G_{u_k} + (-1)^{\mathfrak{p}-1}\sum_{s=1}^{\widetilde{s}} L^k_s \times \widetilde{\gamma}_s$$

belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, and actually $T_k \in \mathcal{T}^{1/\mathfrak{p}}_{u_k}$, with energy

$$\mathcal{E}_{1/\mathfrak{p}}(T_k) = \mathcal{E}_{1/\mathfrak{p}}(u_k) + \sum_{s=1}^{\widetilde{s}} \widetilde{M}_s \cdot \mathbf{M}(L^k_s)\,,$$

where the numbers $\widetilde{M}_s$ are given by (5.1), so that by the previous estimates

$$\mathcal{E}_{1/\mathfrak{p}}(T_k) \le 2\left(1 + C\sum_{s=1}^{\widetilde{s}} \widetilde{M}_s\right)\left(\mathcal{E}_{1/\mathfrak{p}}(u) + \|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right) < \infty\,.$$

By applying Theorem 5.16 to T_k, for each k we find a sequence $\{u^k_h\}_h \subset C^\infty(\mathcal{X},\mathcal{Y})$ such that $G_{u^k_h} \rightharpoonup T_k$ as $h \to \infty$ weakly in $\mathrm{cart}^{1/\mathfrak{p}}$ and

$$\limsup_{h\to\infty} \mathcal{E}_{1/\mathfrak{p}}(u^k_h) \le \left(2c(\mathfrak{p},\mathcal{Y}) - 1\right)\cdot \mathcal{E}_{1/\mathfrak{p}}(T_k)\,.$$

Since $u^k_h \to u_k$ as $h \to \infty$ strongly in $L^{\mathfrak{p}}(\mathcal{X},\mathbb{R}^N)$, by a diagonal argument we find a sequence $\{v_k\} \subset C^\infty(\mathcal{X},\mathcal{Y})$ that converges to u in $L^{\mathfrak{p}}$ and satisfies

$$\sup_k \mathcal{E}_{1/\mathfrak{p}}(v_k) \le 2\left(2c(\mathfrak{p},\mathcal{Y}) - 1\right)\left(1 + C\sum_{s=1}^{\widetilde{s}} \widetilde{M}_s\right)\left(\mathcal{E}_{1/\mathfrak{p}}(u) + \|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right) < \infty\,.$$

This yields the assertion. □

Remark 6.3 As a consequence of Proposition 6.1 and Theorem 6.2, for every $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ and every $s = 1,\dots,\widetilde{s}$ we infer that $\mathbb{P}_s(u)$ is an *integral flat chain*, i.e., $m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u)) < \infty$, compare Definition 3.1.

We finally obtain the following upper bound for the relaxed energy:

Proposition 6.4 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ *we have*

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) \le \big(2c(\mathfrak{p},\mathcal{Y})-1\big)\cdot\Big(\mathcal{E}_{1/\mathfrak{p}}(u)+\sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u))\Big)\,,$$

where the numbers $\widetilde{M}_s$ *are given by* (5.1)*, and* $c(\mathfrak{p},\mathcal{Y})$ *by Theorem* 5.16*.*

Proof: Let $L_s \in \mathcal{R}_{n-p+1}(\mathcal{X})$ be such that $(\partial L_s)\llcorner\,\mathrm{int}(\mathcal{X}) = \mathbb{P}_s(u)$ and $\mathbf{M}(L_s) = m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u)) < \infty$ for every $s = 1,\dots,\widetilde{s}$, see Remark 6.3. We apply Theorem 5.16 to the Cartesian current $T \in \mathcal{T}_u^{1/\mathfrak{p}}$ given by

$$T = G_u + (-1)^{\mathfrak{p}-1}\sum_{s=1}^{\widetilde{s}} L_s \times \widetilde{\gamma}_s\,,$$

and find $\{u_k\}\subset C^\infty(\mathcal{X},\mathcal{Y})$ such that $u_k \to u$ strongly in $L^{\mathfrak{p}}(\mathcal{X},\mathbb{R}^N)$, with

$$\limsup_{k\to\infty}\mathcal{E}_{1/\mathfrak{p}}(u_k) \le \big(2c(\mathfrak{p},\mathcal{Y})-1\big)\cdot\mathcal{E}_{1/\mathfrak{p}}(T)\,,$$

whereas by (5.11)

$$\mathcal{E}_{1/\mathfrak{p}}(T) = \mathcal{E}_{1/\mathfrak{p}}(u) + \sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u))\,.$$

This proves the assertion. □

By lower semicontinuity, as in the proof of Proposition 3.7 we infer that for every $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u) \ge \mathcal{E}_{1/\mathfrak{p}}(u) + \sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u))\,. \tag{6.3}$$

Therefore, in the case $\mathfrak{p}=2$, since $c(2,\mathcal{Y})=1$, we obtain the explicit formula:

Corollary 6.5 *For every* $u \in W^{1/2}(\mathcal{X},\mathcal{Y})$ *we have*

$$\widetilde{\mathcal{E}}_{1/2}(u) = \mathcal{E}_{1/2}(u) + \sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u))\,.$$

Remark 6.6 We conjecture that the equality sign holds true in (6.3) for every integer $\mathfrak{p}\ge 2$. Of course, this would be obtained if we could show that the constant $c(\mathfrak{p},\mathcal{Y})\ge 1$ in Theorem 5.16 is equal to one, see Remark 7.4 below.

A local formula. We may also obtain a local version of the density theorem 5.16, i.e., with $\mathcal{X}$ replaced by any open set $\Omega\subset\mathcal{X}$. As a consequence, we readily infer the following local estimates, compare Proposition 3.9:

Proposition 6.7 *For every* $u \in W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$ *and every open set* $\Omega\subset\mathcal{X}$ *we have*

$$\widetilde{\mathcal{E}}_{1/\mathfrak{p}}(u,\Omega) \le \big(2c(\mathfrak{p},\mathcal{Y})-1\big)\cdot\Big(\mathcal{E}_{1/\mathfrak{p}}(u,\Omega)+\sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\Omega}(\mathbb{P}_s(u)\llcorner\Omega)\Big) < \infty\,.$$

In particular, for $\mathfrak{p}=2$ *we have*

$$\widetilde{\mathcal{E}}_{1/2}(u,\Omega) = \mathcal{E}_{1/2}(u,\Omega) + \sum_{s=1}^{\widetilde{s}}\widetilde{M}_s\cdot m_{i,\Omega}(\mathbb{P}_s(u)\llcorner\Omega)\,.$$

THE CASE p NON-INTEGER. If $1 < p < 2$, by (1.8) we have (1.9) in any dimension n. For $p > 2$ non-integer, as in Proposition 3.10 we obtain:

Proposition 6.8 *Let $n \geq \mathfrak{p} = [p]$, with $p > 2$ non-integer. Then for every $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ we have*

$$\widetilde{\mathcal{E}}_{1/p}(u) = \begin{cases} \mathcal{E}_{1/p}(u) & \text{if } \mathbb{P}(u) = 0 \\ +\infty & \text{if } \mathbb{P}(u) \neq 0\,, \end{cases}$$

where $\mathbb{P}(u) := (\partial G_u)_* \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X}; \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R}))$, see (4.5).

PROOF: We follow the proof of Proposition 3.10, to which we refer for further details. If $\mathbb{P}(u) = 0$, then $u \in \text{cart}^{1/p}(\mathcal{X},\mathcal{Y})$, and the assertion follows from Theorem 4.7. Conversely, if $\widetilde{\mathcal{E}}_{1/p}(u) < \infty$, and $\{u_k\} \subset C^1(\mathcal{X},\mathcal{Y})$ satisfies $\sup_k \mathcal{E}_{1/p}(u_k) < \infty$ and $u_k \to u$ strongly in $L^p(\mathcal{X},\mathbb{R}^N)$, by Proposition 5.20, possibly passing to a subsequence the currents $G_{u_k} \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ weakly converge in $\text{cart}^{1/\mathfrak{p}}$ to some current $T \in \text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$. Write T as in (5.9), where $\mathbb{L}_s(T) \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ satisfies (5.10) and, of course, $u_T = u$, hence $T \in \mathcal{T}_u^{1/\mathfrak{p}}$. By (4.5), condition $\mathbb{P}(u) = 0$ holds true if we show by contradiction that $\mathbb{L}_s(T) = 0$ for every s. To this purpose, assume $\mathcal{X} = B^n$, and let $\mathcal{L}$ be a compact subset of the union of the set of points of $\mathbb{L}_s(T)$ which have non-zero density, $\mathcal{L}$ with positive $\mathcal{H}^{n-\mathfrak{p}+1}$-measure. For $x \in \mathcal{L}$, we use the notation from Proposition 3.10, and obtain again (3.8), whereas the $(\mathfrak{p}-1)$-dimensional currents $G_{u_{k|\Pi_x}}$ have to converge "near" the point x to the graph $G_{u_{|\Pi_x}}$ plus a vertical part of the type $\delta_x \times C$, for some non-trivial integral flat $(\mathfrak{p}-1)$-cycle $C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$. On account of the isoperimetric property (5.5), and by lower semicontinuity, Proposition 5.19, this is in contradiction with (3.8), for $r > 0$ sufficiently small, hence $\mathbb{L}_s(T) = 0$ for every s, as required. □

7 Weak approximation of spherical cycles

In this section we show how to approximate spherical $(\mathfrak{p}-1)$-cycles of $\mathcal{Y}$. We shall first focus on the model case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, and fix some notation. To this purpose, we choose $n = \mathfrak{p}-1$ and denote

$$\begin{aligned} &B^{\mathfrak{p}} := \{(x,t) \in \mathbb{R}^{\mathfrak{p}-1}\times\mathbb{R} \mid |x|^2 + t^2 < 1\}\,, \quad && B^+ := \{(x,t) \in \overline{B}^{\mathfrak{p}} \mid t \geq 0\}\,, \\ &\partial^+ B := \{(x,t) \in \partial B^{\mathfrak{p}} \mid t > 0\}\,, && J := \partial B^+ \setminus \partial^+ B = \overline{B}^{\mathfrak{p}-1} \times \{0\}\,. \end{aligned}$$

Proposition 7.1 *Let $P \in \mathbb{S}^{\mathfrak{p}-1}$ and $q \in \mathbb{Z}$. For every $\varepsilon > 0$ there exists a smooth $W^{1,\mathfrak{p}}$-map $U_\varepsilon : B^+ \to \overline{B}^{\mathfrak{p}}$ satisfying the following properties:*

i) *the restriction $f_\varepsilon := U_{\varepsilon|J}$ is a smooth map $f_\varepsilon : J \to \mathbb{S}^{\mathfrak{p}-1}$;*

ii) *the restriction $U_{\varepsilon|\partial^+ B}$ is constantly equal to P;*

iii) *$U_{\varepsilon\#}[\![B^+]\!] = q\,[\![B^{\mathfrak{p}}]\!]$ and $f_{\varepsilon\#}[\![J]\!] = q\,[\![\mathbb{S}^{\mathfrak{p}-1}]\!]$;*

iv) *$\mathbf{D}_{\mathfrak{p}}(U_\varepsilon, B^+) \leq |q|\cdot|B^{\mathfrak{p}}| + \varepsilon$.*

PROOF: We divide the proof in three steps.

STEP 1: A CONFORMAL MAP INTO THE DISC. Consider the complex map

$$h(z) := \frac{1 - i\bar{z}}{\bar{z} - i}\,, \quad z \in \mathbb{C}\setminus\{-i\}\,.$$

It is readily checked that h is a biholomorphic map between the half-space $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ and the unit disc $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$, and that $h(z) \to -i$ as $|z| \to +\infty$. Setting $z = \rho + it$ for $\rho \in \mathbb{R}$ and $t > 0$, we have $h(z) = f(\rho,t) + i\,g(\rho,t)$ where

$$f(\rho,t) := \frac{2\rho}{\Delta(\rho,t)}\,, \quad g(\rho,t) := \frac{1-(\rho^2+t^2)}{\Delta(\rho,t)}\,, \quad \Delta(\rho,t) := \rho^2 + (t+1)^2\,.$$

Therefore, the following Cauchy-Riemann equations are satisfied:

$$f_{,\rho} = -g_{,t} = \frac{2}{\Delta(\rho,t)^2}\left((t+1)^2 - \rho^2\right), \quad f_{,t} = g_{,\rho} = \frac{4\rho(t+1)}{\Delta(\rho,t)^2}\,.$$

In particular, the mapping $\Phi(\rho,t) := (f,g)(\rho,t)$ is conformal, i.e.,

$$\langle D_i\Phi(\rho,t), D_j\Phi(\rho,t)\rangle_{\mathbb{R}^2} = \delta_{ij}\,\frac{4}{\Delta(\rho,t)^2}\,, \quad 1 \le i \le j \le 2\,.$$

We also observe that the trace on $t=0$ of Φ satisfies

$$\Phi(\rho,0) = \Big(\frac{2\rho}{1+\rho^2}, \frac{1-\rho^2}{1+\rho^2}\Big), \quad |\Phi(\rho,0)| = 1 \quad \forall\, \rho \in \mathbb{R}\,.$$

In the half-space

$$\mathbb{R}^{\mathfrak{p}}_+ := \{(x,t) \in \mathbb{R}^{\mathfrak{p}-1}\times\mathbb{R} \mid t>0\}$$

we now define the map $U : \mathbb{R}^{\mathfrak{p}}_+ \to B^{\mathfrak{p}}$ by

$$U(x,t) := \Big(\frac{2}{|x|^2+(t+1)^2}\,x, \frac{1-(|x|^2+t^2)}{|x|^2+(t+1)^2}\Big)$$

or, equivalently,

$$U(x,t) := \Big(f(|x|,t)\,\frac{x}{|x|}, g(|x|,t)\Big)\,.$$

Using that

$$(f_{,\rho}(\rho,t))^2 + (f_{,t}(\rho,t))^2 = \Big(\frac{f(\rho,t)}{\rho}\Big)^2 = \frac{4}{\Delta(\rho,t)^2}\,,$$

it turns out that U is a conformal map, as

$$\langle D_iU(x,t), D_jU(x,t)\rangle_{\mathbb{R}^{\mathfrak{p}}} = \delta_{ij}\,\frac{4}{\Delta(|x|,t)^2}\,, \quad 1 \le i \le j \le \mathfrak{p}\,.$$

This yields that the $\mathfrak{p}$-dimensional Jacobian $J_{\mathfrak{p}}(DU)$ is equal to $\mathfrak{p}^{-\mathfrak{p}/2}\,|DU|^{\mathfrak{p}}$. As a consequence, since U is a.e. one to one and onto, by the area formula we infer that

$$\mathbf{D}_{\mathfrak{p}}(U,\mathbb{R}^{\mathfrak{p}}_+) := \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{\mathbb{R}^{\mathfrak{p}}_+} |DU(x,t)|^{\mathfrak{p}}\,dx\,dt = \int_{\mathbb{R}^{\mathfrak{p}}_+} J_{\mathfrak{p}}(DU)\,dx\,dt = |B^{\mathfrak{p}}|\,.$$

Moreover, $U(x,t)$ converges to the south pole $P_S := (0_{\mathbb{R}^{\mathfrak{p}-1}}, -1)$ as $|(x,t)| \to +\infty$.

We finally observe that the trace on $t=0$ of U satisfies

$$U(x,0) = \Big(\frac{2}{1+|x|^2}\,x, \frac{1-|x|^2}{1+|x|^2}\Big), \quad |U(x,0)| = 1 \quad \forall\, x \in \mathbb{R}^{\mathfrak{p}-1}\,,$$

hence it agrees with the inverse $\sigma^{-1} : \mathbb{R}^{\mathfrak{p}-1} \to \mathbb{S}^{\mathfrak{p}-1}$ of the stereographic projection map σ of the unit sphere $\mathbb{S}^{\mathfrak{p}-1}$ onto $\mathbb{R}^{p-1}$ from the south pole P_S. We recall, see [10, Vol. I, Sec. 4.1.1], that the map $(-1)^{\mathfrak{p}-1}\,\sigma^{-1}$ is an orientation preserving conformal diffeomorphism, provided that $\mathbb{S}^{\mathfrak{p}-1}$ is equipped with the natural orientation induced from the outward unit normal; in particular, $\sigma^{-1}_{\#}[\![\,\mathbb{R}^{\mathfrak{p}-1}\,]\!] = (-1)^{\mathfrak{p}-1}[\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!]$.

Step 2: a modification of the conformal map into the disc. Following [10, Vol. II, Sec. 4.1.1], write

$$\sigma^{-1}(x) = \left(\frac{x}{|x|}\,\sin\theta(|x|), -\cos\theta(|x|)\right), \qquad x \in \mathbb{R}^{\mathfrak{p}-1}\,,$$

where $\theta(\rho)$, for $\rho > 0$, is the *angular distance*, i.e. the geodesic distance of $\sigma^{-1}(\partial B^{\mathfrak{p}-1}_\rho)$ from the south pole P_S. For $\varepsilon > 0$ we set

$$\theta_\varepsilon(\rho) := \begin{cases} \theta(\rho) & \text{if} \quad \rho < R_\varepsilon \\ \varepsilon\,(2R_\varepsilon - \rho)/R_\varepsilon & \text{if} \quad R_\varepsilon \le \rho \le 2R_\varepsilon \\ 0 & \text{if} \quad \rho > 2R_\varepsilon\,, \end{cases}$$

where $R_\varepsilon := \theta^{-1}(\varepsilon)$, and we define $\widetilde{\varphi}_\varepsilon : \mathbb{R}^{\mathfrak{p}-1} \to \mathbb{S}^{\mathfrak{p}-1}$ by

$$\widetilde{\varphi}_\varepsilon(x) := \left(\frac{x}{|x|} \sin\theta_\varepsilon(|x|), -\cos\theta_\varepsilon(|x|) \right), \qquad x \in \mathbb{R}^{\mathfrak{p}-1} .$$

The function $\widetilde{\varphi}_\varepsilon$ is Lipschitz-continuous, with $\widetilde{\varphi}_\varepsilon(x) = \sigma^{-1}(x)$ for $|x| < R_\varepsilon$ and $\widetilde{\varphi}_\varepsilon(x) \equiv P_S$ for $|x| > 2R_\varepsilon$, and the image current satisfies

$$\widetilde{\varphi}_{\varepsilon\#}[\![\mathbb{R}^{\mathfrak{p}-1}]\!] = \widetilde{\varphi}_{\varepsilon\#}[\![B^{\mathfrak{p}-1}_{2R_\varepsilon}]\!] = (-1)^{\mathfrak{p}-1}[\![\mathbb{S}^{\mathfrak{p}-1}]\!] .$$

Moreover, for our purposes we recall that if $R_\varepsilon < |x| < 2R_\varepsilon$ we have $|D\widetilde{\varphi}_\varepsilon(x)| \le c(\mathfrak{p})\, \varepsilon / R_\varepsilon$.

We notice that

$$|U(x,t) - P_S| = \frac{2}{\sqrt{\Delta(|x|,t)}} ,$$

i.e., U maps the set of points of $\mathbb{R}^{\mathfrak{p}}_+$ that are distant d to the south pole P_S into the set of points of $B^{\mathfrak{p}}$ that are distant $2/d$ to P_S, for every $d > 1$. We then denote

$$\Omega_\rho := \{ (x,t) \in \mathbb{R}^{\mathfrak{p}}_+ \mid \Delta(|x|,t) < 1 + \rho^2 \} , \quad \rho > 0 ,$$

and define $V_\varepsilon(x,t) := U(x,t)$ on Ω_{R_ε} whereas $V_\varepsilon(x,t) \equiv P_S$ on $\mathbb{R}^{\mathfrak{p}}_+ \setminus \Omega_{2R_\varepsilon}$. In order to extend in a smooth way V_ε to the whole of $\mathbb{R}^{\mathfrak{p}}_+$, we observe that for $R_\varepsilon < |x| < 2R_\varepsilon$ we have

$$|\widetilde{\varphi}_\varepsilon(x) - P_S| = \delta_\varepsilon(|x|) := \sqrt{2(1 - \cos\theta_\varepsilon(|x|))} .$$

We thus define V_ε on $\Omega_{2R_\varepsilon} \setminus \Omega_{R_\varepsilon}$ by

$$V_\varepsilon(x,t) := U\Big(P_S + \frac{2}{\delta_\varepsilon(|x|)} \, \frac{(x, t+1)}{\sqrt{\Delta(|x|,t)}} \Big) .$$

We now check that for $(x,t) \in \Omega_{2R_\varepsilon} \setminus \Omega_{R_\varepsilon}$

$$|DV_\varepsilon(x,t)| \le c(\mathfrak{p}) \, \frac{\varepsilon^{1/2}}{R_\varepsilon} , \tag{7.1}$$

where $c(\mathfrak{p}) > 0$ is an absolute constant, possibly depending on $\mathfrak{p}$. In fact, changing coordinates by translation of the origin to the south pole P_S, we write

$$U(x,t) = 2 \, \frac{(x,T)}{|(x,T)|^2} + P_S , \quad T = t + 1 ,$$

and for $1 + R_\varepsilon^2 < |(x,T)|^2 < 1 + 4R_\varepsilon^2$ we have

$$V_\varepsilon(x,t) = \delta_\varepsilon(|x|) \, \frac{(x,T)}{|(x,T)|} + P_S .$$

Since $|\delta'_\varepsilon(\rho)| \le c\, |\theta'_\varepsilon(\rho)| = c\, \varepsilon / R_\varepsilon$, whereas $|\delta_\varepsilon(\rho)| \le c \sqrt{\theta_\varepsilon(\rho)} \le c\, \varepsilon^{1/2}$ and $R_\varepsilon > 1$, this yields (7.1).

By conformality, and since $U : B^+ \to B^{\mathfrak{p}}$ is a.e. one to one, we have

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\Omega_{R_\varepsilon}} |DV_\varepsilon|^{\mathfrak{p}} \, dx\, dt = \int_{\Omega_{R_\varepsilon}} |J_{\mathfrak{p}} U| \, dx\, dt = \mathcal{H}^{\mathfrak{p}}(U(\Omega_{R_\varepsilon})) \le |B^{\mathfrak{p}}| ,$$

whereas by using (7.1)

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\Omega_{2R_\varepsilon} \setminus \Omega_{R_\varepsilon}} |DV_\varepsilon|^{\mathfrak{p}} \, dx\, dt \le c(\mathfrak{p}) \, |\Omega_{2R_\varepsilon}| \, \frac{\varepsilon^{\mathfrak{p}/2}}{R_\varepsilon^{\mathfrak{p}}} \le c(\mathfrak{p}) \, \varepsilon^{\mathfrak{p}/2} .$$

This yields that $V_\varepsilon(x,t) \equiv P_S$ on $\partial\Omega_{2R_\varepsilon}$ and

$$\mathbf{D}_{\mathfrak{p}}(V_\varepsilon, \Omega_{2R_\varepsilon}) \le |B^{\mathfrak{p}}| + c(\mathfrak{p}) \, \varepsilon^{\mathfrak{p}/2} .$$

STEP 3: THE APPROXIMATING MAP. We now observe that we can suitably re-parameterize the function $(-1)^{\mathfrak{p}-1} V_\varepsilon$ on the half disc $B^{\mathfrak{p}}_{2R_\varepsilon} \cap \{(x,t) \mid t > 0\}$ by means of a bilipschitz map Ψ_{R_ε} with Lipschitz constants $\mathrm{Lip}(\Psi_{R_\varepsilon}), \mathrm{Lip}(\Psi^{-1}_{R_\varepsilon}) \leq 1 + o(\varepsilon)$. Using that in dimension $\mathfrak{p}$ the $\mathfrak{p}$-energy is invariant under homothetic dilatations $(\widetilde{x}, \widetilde{t}) = \lambda\,(x,t)$, $\lambda > 0$, we readily obtain the map $U_\varepsilon : B^+ \to \overline{B}^{\mathfrak{p}}$ satisfying the required properties i)–iv), in the case $q = 1$ and $P = (-1)^{\mathfrak{p}-1} P_S$. For a general $P \in \mathbb{S}^{\mathfrak{p}-1}$, it suffices to compose U_ε on the right with a rotation of $\mathbb{S}^{\mathfrak{p}-1}$ that maps the pole $(-1)^{\mathfrak{p}-1} P_S$ into P. Finally, for a general $q \in \mathbb{Z}$, we consider $|q|$ copies of the function $\mathrm{sgn}(q)\, \widetilde{U}_\varepsilon$, where $\widetilde{U}_\varepsilon$ is the function obtained in the case $q = 1$, each copy "re-parameterized" in a suitable subdomain D_i of B^+, obtained by means of a dilatation $\lambda\,(x,t)$, with $0 < \lambda \ll 1$, plus a translation in the x-variables, so that the D_i's are pairwise disjoint and all contained in B^+, and set $U_\varepsilon \equiv P$ elsewhere on B^+. □

APPROXIMATING SPHERICAL $(\mathfrak{p}-1)$-CYCLES. We would like to have a similar result to Proposition 7.1 for integral flat spherical $(\mathfrak{p}-1)$-cycles in $\mathcal{Y}$. To this purpose, we first give the following

Remark 7.2 We set

$$\mathcal{Y}_\varepsilon := \overline{U_\varepsilon(\mathcal{Y})}\,,$$

where $U_\varepsilon(A) := \{y \in \mathbb{R}^N \mid \mathrm{dist}(y, A) < \varepsilon\}$ is the ε-neighborhood of $A \subset \mathbb{R}^N$. Since $\mathcal{Y}$ is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $\mathrm{Lip}(\Pi_\varepsilon) \leq (1 + c\,\varepsilon) \to 1^+$ as $\varepsilon \to 0^+$.

By adapting the proof of [13, Prop. 4.6], we readily obtain:

Proposition 7.3 *Let $C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ be an integral flat $(\mathfrak{p}-1)$-cycle of spherical type and let $R \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ be the i.m. rectifiable current of least mass in $\mathbb{R}^N$ such that $\partial R = C$. Let $P \in \mathcal{Y}$ be a given point. Then there exists a sequence of Lipschitz functions $f_\varepsilon : B^+ \to \mathbb{R}^N$ such that $f_{\varepsilon|\partial^+ B} \equiv P$, $f_\varepsilon(J) \subset \mathcal{Y}_{2\varepsilon}$, $f_{\varepsilon\#}[\![B^+]\!] \rightharpoonup R$ and $f_{\varepsilon\#}[\![J]\!] \rightharpoonup C$ weakly in $\mathcal{D}_{\mathfrak{p}}(\mathbb{R}^N)$ and $\mathcal{D}_{\mathfrak{p}-1}(\mathcal{Y}_{\varepsilon_0})$, respectively, and*

$$\lim_{\varepsilon \to 0} A(f_\varepsilon, B^+) = \mathbf{M}(R)\,,$$

where $A(f_\varepsilon, B^+)$ is the $\mathfrak{p}$-dimensional mapping area *of f_ε, i.e., $A(f_\varepsilon, B^+) := \int_{B^+} |J_{\mathfrak{p}} f_\varepsilon|$.*

Remark 7.4 Now, for $\mathfrak{p} = 2$, by Morrey's ε-conformality theorem [25, Thm. 2.1], in [13] we defined a suitable orientation preserving diffeomorphism $\Psi_\varepsilon : B^+ \to B^+$ such that, if $\widetilde{f}_\varepsilon := f_\varepsilon \circ \Psi_\varepsilon$, then

$$\mathbf{D}_2(\widetilde{f}_\varepsilon, B^+) \leq (1+\varepsilon)\, A(\widetilde{f}_\varepsilon, B^+) = (1+\varepsilon)\, A(f_\varepsilon, B^+)\,.$$

This yields that *for $\mathfrak{p} = 2$ we find f_ε as in Proposition* 7.3 *such that*

$$\lim_{\varepsilon \to 0} \mathbf{D}_2(f_\varepsilon, B^+) = \mathbf{M}(R)\,.$$

However, we do not have an analogous of Morrey's ε-conformality theorem for $\mathfrak{p} \geq 3$. This yields that we are not able to find a strong approximation theorem for integral flat $(\mathfrak{p}-1)$-cycles of spherical type in the case $\mathfrak{p} \geq 3$. For this reason, we now obtain a weak estimate, arguing in a different way.

Proposition 7.5 *Let $\mathfrak{p} \geq 3$ integer. Let $q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$ and let $R_q \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ be the i.m. rectifiable current of least mass in $\mathbb{R}^N$ such that $\partial R_q \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ belongs to q. Also, let $P \in \mathcal{Y}$ be a given point. Then there exists a Lipschitz function $f_q : B^+ \to \mathbb{R}^N$ such that $f_{q|\partial^+ B} \equiv P$, $f_q(J) \subset \mathcal{Y}$, $f_{q\#}[\![J]\!] \in q$, and*

$$\mathbf{D}_{\mathfrak{p}}(f_q, B^+) \leq c(\mathfrak{p}, \mathcal{Y})\, \mathbf{M}(R_q)\,,$$

where $c(\mathfrak{p}, \mathcal{Y}) \geq 1$ is an absolute real constant, only depending on $\mathfrak{p}$ and $\mathcal{Y}$.

PROOF: We divide the proof into three steps.

STEP 1: For $(x,t) \in \mathbb{R}^{\mathfrak{p}-1} \times \mathbb{R}$, denote

$$\Omega := \{(x,t) \in B^{\mathfrak{p}} \mid 1/2 < |(x,t)| < 1\,, t > 0\}\,,$$

so that $\Omega \subset B^+$, and let

$$\partial^+\Omega := \{(x,t) \in \partial\Omega \mid t > 0\}\,, \qquad \partial^0\Omega := \partial\Omega \setminus \partial^+\Omega\,.$$

We now define a suitable Lipschitz map $F : \overline{\Omega} \to \overline{B}^{\mathfrak{p}}$ such that

$$F(\partial^0\Omega) \subset \mathbb{S}^{\mathfrak{p}-1}\,, \quad F_{|\partial^0\Omega\#}[\![\,\partial^0\Omega\,]\!] = [\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!]\,, \quad F_{|\Omega\#}[\![\,\Omega\,]\!] = [\![\,B^{\mathfrak{p}}\,]\!]\,.$$

Such a function is generated by the mapping $(y_1, y_2) \mapsto (z_1, z_2)$ given by

$$(z_1, z_2) = \frac{2}{\pi}\,\phi \cdot \big(\sin(2\pi(1-\rho)), \cos(2\pi(1-\rho))\big)\,,$$

where in polar coordinates we have set

$$(y_1, y_2) = \rho\,(\sin\phi, \cos\phi)\,, \quad 0 < \phi < \pi/2\,, \quad 1/2 < \rho < 1\,.$$

More precisely, we define $F(x,t)$ for $(x,t) \in \Omega$ by

$$F(x,t) := \frac{2}{\pi}\,\phi(x,t)\,\Big(\sin(2\pi(1-\rho(x,t)))\,\frac{x}{|x|}, \cos(2\pi(1-\rho(x,t)))\Big)\,,$$

where

$$\phi(x,t) := \arctan\Big(\frac{|x|}{t}\Big)\,, \quad \rho(x,t) := |(x,t)|\,.$$

Step 2: For $s = 1, \ldots, \widetilde{s}$, since $\widetilde{\gamma}_s \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$, there exists a Lipschitz map $f_s \in \mathrm{Lip}(\mathbb{S}^{\mathfrak{p}-1}, \mathcal{Y})$ such that $f_{s\#}[\![\,\mathbb{S}^{\mathfrak{p}-1}\,]\!] \in [\widetilde{\gamma}_s]$. Let $\Phi_s \in \mathrm{Lip}(\overline{B}^{\mathfrak{p}}, \mathbb{R}^N)$ be such that $\Phi_{s|\partial B^{\mathfrak{p}}} = f_s$ and

$$\mathbf{D}_{\mathfrak{p}}(\Phi_s) \le \inf\{\mathbf{D}_{\mathfrak{p}}(\Phi) \mid \Phi \in \mathrm{Lip}(\overline{B}^{\mathfrak{p}}, \mathbb{R}^N)\,,\ \Phi_{|\partial B^{\mathfrak{p}}} = f_s\} + 1\,.$$

The Lipschitz function $\Psi_s : \overline{\Omega} \to \mathbb{R}^N$ given by $\Psi_s(x,t) := (\Phi_s \circ F)(x,t)$ satisfies:

$$\Psi_s(\partial^0\Omega) \subset \mathcal{Y}\,, \quad \Psi_{s|\partial^0\Omega\#}[\![\,\partial^0\Omega\,]\!] \in [\widetilde{\gamma}_s]\,, \quad \Psi_{s|\Omega\#}[\![\,\Omega\,]\!] = \Phi_{s\#}[\![\,B^{\mathfrak{p}}\,]\!] \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)\,.$$

For $0 < \lambda \le 1$ and for any $X \subset \mathbb{R}^{\mathfrak{p}}$ we set

$$\lambda\,X := \{\lambda\,(x,t) \mid (x,t) \in X\}$$

and define $\Psi^{\pm}_{\lambda,s} : \lambda\,\overline{\Omega} \to \mathbb{R}^N$ by

$$\Psi^{\pm}_{\lambda,s}(x,t) := \Psi_s(\lambda^{-1}(\pm x_1, x_2, \ldots, x_{\mathfrak{p}-1}, t))\,,$$

so that we have

$$\Psi^{\pm}_{\lambda,s}(\partial^0\Omega) \subset \mathcal{Y}\,, \quad \Psi^{\pm}_{\lambda,s|\partial^0\Omega\#}[\![\,\partial^0\Omega\,]\!] \in \pm[\widetilde{\gamma}_s]\,, \quad \Psi^{\pm}_{\lambda,s|\Omega\#}[\![\,\Omega\,]\!] = \pm\Phi_{s\#}[\![\,B^{\mathfrak{p}}\,]\!] \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)\,.$$

Moreover, it is readily checked that for every $0 < \lambda \le 1$

$$\mathbf{D}_{\mathfrak{p}}(\Psi^{\pm}_{\lambda,s}, \lambda\,\Omega) = \mathbf{D}_{\mathfrak{p}}(\Psi_s, \Omega) < \infty\,.$$

Finally, denoting by

$$\partial^e\Omega := \{(x,t) \in \partial^+\Omega \,:\, |x| = 1\}\,, \quad \partial^i\Omega := \{(x,t) \in \partial^+\Omega \,:\, |x| = 1/2\}$$

the "exterior" and "interior" part of the boundary $\partial^+\Omega$, respectively, and accordingly

$$\partial^e\lambda\,\Omega := \lambda\,\partial^e\Omega\,, \qquad \partial^i\lambda\,\Omega := \lambda\,\partial^i\Omega\,,$$

for every $0 < \lambda \le 1$ we have

$$\Psi^{\pm}_{\lambda,s}(\partial^e\lambda\Omega) = P^e_s := f_s(0_{\mathbb{R}^{\mathfrak{p}-1}}, +1)\,, \quad \Psi^{\pm}_{\lambda,s}(\partial^i\lambda\Omega) = P^i_s := f_s(0_{\mathbb{R}^{\mathfrak{p}-1}}, -1)\,.$$

The target manifold $\mathcal{Y}$ being connected, we also let $\Gamma_{P,Q}$ denote a geodesic arc between any two points P and Q in $\mathcal{Y}$, and let $\mathcal{L}(\Gamma_{P,Q})$ be the length of $\Gamma_{P,Q}$. If $\Gamma_{P,Q}:[0,\mathcal{L}(\gamma_{P,Q})]\to\mathcal{Y}$ also denotes the arc-length parameterization of $\Gamma_{P,Q}$, with initial point P and final point Q, setting $\widetilde{\Psi}_{P,Q}:\overline{\Omega}\to\mathcal{Y}$ by

$$\widetilde{\Psi}_{P,Q}(x,t):=\Gamma_{P,Q}\big(2\,\mathcal{L}(\Gamma_{P,Q})\cdot(1-|(x,t)|)\big)\,,$$

and defining $\widetilde{\Psi}_{\lambda,P,Q}:\lambda\,\Omega\to\mathcal{Y}$ by

$$\widetilde{\Psi}_{\lambda,P,Q}(x,t):=\widetilde{\Psi}_{P,Q}(\lambda^{-1}(x,t))\,,$$

for every $0<\lambda\leq 1$ we have $\widetilde{\Psi}_{\lambda,P,Q|\partial^e\lambda\,\Omega}\equiv P$, $\widetilde{\Psi}_{\lambda,P,Q|\partial^i\lambda\,\Omega}\equiv Q$, and

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{\Psi}_{\lambda,P,Q},\lambda\,\Omega)=\mathbf{D}_{\mathfrak{p}}(\widetilde{\Psi}_{P,Q},\Omega)\leq C(\mathfrak{p})\,\mathcal{L}(\Gamma_{P,Q})^{\mathfrak{p}}\,,$$

where $C(\mathfrak{p})>0$ is an absolute constant, whereas

$$\widetilde{\Psi}_{\lambda,P,Q\#}[\![\,\lambda\,\Omega\,]\!]=0\in\mathcal{D}_{\mathfrak{p}}(\mathbb{R}^N)\,,\quad \widetilde{\Psi}_{\lambda,P,Q|\partial^0\lambda\,\Omega\#}[\![\,\partial^0\lambda\,\Omega\,]\!]=0\in\mathcal{D}_{\mathfrak{p}-1}(\mathcal{Y})\,.$$

Finally, we let $\Omega_k:=2^{-k}\,\Omega$, for $k\in\mathbb{N}$, and observe that the family $\{\Omega_k\}_k$ is pairwise disjoint and satisfies $\cup_k\overline{\Omega}_k=\overline{B}^+$.

Step 3: Now, for any $q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})$, we have

$$q=\sum_{s=1}^{\widetilde{s}}n_s^q\,[\widetilde{\gamma}_s]$$

for some integers $n_s^q\in\mathbb{Z}$. Without loss of generality, we may and do assume that $n_s^q\neq 0$ if and only if $s=1,\dots,s_q$, for some $s_q\in\{1,\dots,\widetilde{s}\}$.

We first define $f^q=\widetilde{\Psi}_{P,P_1^e}$ on $\overline{\Omega}_0=\overline{\Omega}$. For $k=2j-1$, where $j=1,\dots,|n_1^q|$, we then set $f^q=\Psi^{\pm}_{2^{-k},1}$ on $\overline{\Omega}_k$, where the orientation $\pm$ is chosen according to the sign of n_1^q. If $|n_1^q|\geq 2$, we also let $f^q=\widetilde{\Psi}_{2^{-k},P_1^i,P_1^e}$ on $\overline{\Omega}_k$, for $k=2h$ and $h=1,\dots,|n_1^q|-1$.

Setting $A_1:=\bigcup_{k=1}^{2|n_1^q|}\overline{\Omega}_{k-1}$ and $\partial^0A_1:=\bigcup_{k=1}^{2|n_1^q|}\partial^0\Omega_{k-1}$, this way we have defined f^q on A_1 so that

$$f^q_{\#}[\![\,\partial^0A_1\,]\!]\in n_1^q\,[\widetilde{\gamma}_1]\,,\quad \mathbf{D}_{\mathfrak{p}}(f^q,A_1)\leq|n_1^q|\,\mathbf{D}_{\mathfrak{p}}(\Psi_1,\Omega)+c(\mathfrak{p})\,[\mathcal{L}(\Gamma_{P,P_1^e})^{\mathfrak{p}}+(|n_1^q|-1)\,\mathcal{L}(\Gamma_{P_1^i,P_1^e})^{\mathfrak{p}}]\,,$$

where $c(\mathfrak{p})>1$ is an absolute constant.

We then proceed as above, by iteration on $s=2,\dots,s_q$. For $k=K_s:=2\sum_{l=1}^{s-1}|n_l^q|$, we define $f^q=\widetilde{\Psi}_{2^{-k},P^i_{s-1},P^e_s}$ on $\overline{\Omega}_k$. Then, for $k=K_s+2j-1$, where $j=1,\dots,|n_s^q|$, we set $f^q=\Psi^{\pm}_{2^{-k},s}$ on $\overline{\Omega}_k$, where the orientation $\pm$ is chosen according to the sign of n_s^q. If $|n_s^q|\geq 2$, we also let $f^q=\Psi_{2^{-k},P_s^i,P_s^e}$ on $\overline{\Omega}_k$, for $k=K_s+2h$ and $h=1,\dots,|n_s^q|-1$.

Setting $A_s:=\bigcup_{k=1}^{2|n_s^q|}\overline{\Omega}_{k+K_s-1}$ and $\partial^0A_s:=\bigcup_{k=1}^{2|n_s^q|}\partial^0\Omega_{k+K_s-1}$, we have defined f^q on A_s so that

$$f^q_{\#}[\![\,\partial^0A_s\,]\!]\in n_s^q\,[\widetilde{\gamma}_s]\,,\quad \mathbf{D}_{\mathfrak{p}}(f^q,A_s)\leq|n_s^q|\,\mathbf{D}_{\mathfrak{p}}(\Psi_s,\Omega)+c(\mathfrak{p})\,[\mathcal{L}(\Gamma_{P^i_{s-1},P^e_s})^{\mathfrak{p}}+(|n_s^q|-1)\,\mathcal{L}(\Gamma_{P_s^i,P_s^e})^{\mathfrak{p}}]\,.$$

Setting $f^q\equiv P^i_{s_q}$ on $B^+\setminus\bigcup_{s=1}^{s_q}A_s$, we infer that $f^q:B^+\to\mathbb{R}^N$ is a Lipschitz function satisfying $f_{q|\partial^+B}\equiv P$, $f_q(J)\subset\mathcal{Y}$, $f_{q\#}[\![\,J\,]\!]\in q$, and

$$\mathbf{D}_{\mathfrak{p}}(f_q,B^+)\leq\sum_{s=1}^{\widetilde{s}}|n_s^q|\,\mathbf{D}_{\mathfrak{p}}(\Psi_s,\Omega)+c(\mathfrak{p})\sum_{s=1}^{s_q}[\mathcal{L}(\Gamma_{P^i_{s-1},P^e_s})^{\mathfrak{p}}+(|n_s^q|-1)\,\mathcal{L}(\Gamma_{P_s^i,P_s^e})^{\mathfrak{p}}]\,,$$

where we have set $P_0^i:=P$. Recalling (5.1), we now set

$$c_1(\mathfrak{p},\mathcal{Y}):=\sup_{1\leq s\leq\widetilde{s}}\frac{\mathbf{D}_{\mathfrak{p}}(\Psi_s,\Omega)}{\widetilde{M}_s}\,,\quad c_2(\mathfrak{p},\mathcal{Y}):=\sup_{1\leq s\leq\widetilde{s}}\frac{\mathcal{L}(\Gamma_{P_s^i,P_s^e})^{\mathfrak{p}}}{\widetilde{M}_s}\,,$$

$$c_3(\mathfrak{p},\mathcal{Y}) := \sup_{1\le s\le \widetilde{s}} \sup_{l\neq s} \frac{\mathcal{L}(\Gamma_{P^i_l,P^e_s})^{\mathfrak{p}}}{\widetilde{M}_s}\,, \quad c_4(\mathfrak{p},\mathcal{Y}) := \sup_{1\le s\le \widetilde{s}} \sup_{P\in\mathcal{Y}} \frac{\mathcal{L}(\Gamma_{P,P^e_s})^{\mathfrak{p}}}{\widetilde{M}_s}\,.$$

All the constant $c_i(\mathfrak{p},\mathcal{Y})$ are finite and only depend on $\mathfrak{p}$ and $\mathcal{Y}$, by the smoothness and compactness of $\mathcal{Y}$. Since $\mathbf{M}(R_q) = \sum_{s=1}^{\widetilde{s}} |n^q_s|\,\widetilde{M}_s$, we obtain the assertion by setting $c(\mathfrak{p},\mathcal{Y}) := c(\mathfrak{p})\cdot\sum_{i=1}^4 c_i(\mathfrak{p},\mathcal{Y})$. □

8 The density theorem

The rest of the paper is dedicated to give the proof of the density theorem 5.16. We first give a proof in the case of low dimension $n := \dim(\mathcal{X}) = \mathfrak{p}-1$. We then consider the easier case of dimension $n = \mathfrak{p}$. Finally, in higher dimension $n \ge \mathfrak{p}+1$, we show how the proof is reduced to the validity of Theorem 8.4 below, that will be proved in the Appendix.

DENSITY IN DIMENSION $n = \mathfrak{p}-1$. In this case, it suffices that $\mathcal{Y}$ satisfies (H_1). In particular, no topological conditions on $\mathcal{Y}$ as in (H_2) is required.

SKETCH OF THE PROOF: Recall from Sec. 5 that if $n = \mathfrak{p}-1$ any current $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$ has the form

$$T = G_{u_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times C_i \qquad \text{on} \qquad \mathcal{Z}^{\mathfrak{p}-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})\,,$$

where δ_x is the Dirac mass in x, $x_i \in \mathcal{X}$, and the $C_i \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ are integral flat $(\mathfrak{p}-1)$-cycles of spherical type and with non trivial homology. Moreover, we choose the representative C_i in such a way that if $R_i \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ minimizes the mass among all the i.m. rectifiable currents $R \in \mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ such that $Z = \partial R$ is in the homology class of C_i, then $C_i = \partial R_i$. The $\mathcal{E}_{1/\mathfrak{p}}$-energy of T is then defined as the $\mathfrak{p}$-energy of its extension $\widetilde{T}$, compare Definition 5.4, i.e.,

$$\mathcal{E}_{1/\mathfrak{p}}(T) := \mathbf{D}_{\mathfrak{p}}(\widetilde{T}) = \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathcal{C}^{\mathfrak{p}}} |Du_T|^{\mathfrak{p}}\, d\mathcal{H}^{\mathfrak{p}} + \sum_{i=1}^{i_0} \mathbf{M}(R_i)\,,$$

where $\widetilde{T} = \mathrm{Ext}(T) := (-1)^{\mathfrak{p}} G_{U_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times R_i$, with $U_T := \mathrm{Ext}(u_T) \in W^{1,\mathfrak{p}}(\mathcal{C}^{\mathfrak{p}},\mathbb{R}^N)$.

Adapting an argument by Schoen-Uhlenbeck [29], as in [2, Sec. 2.1] we can find a sequence of smooth maps $U_k : \mathcal{C}^{\mathfrak{p}} \to \mathbb{R}^N$ such that $U_k \to \mathrm{Ext}(u)$ strongly in $W^{1,\mathfrak{p}}(\mathcal{C}^{\mathfrak{p}},\mathbb{R}^N)$ and for which there exists a positive number $t_0 > 0$ such that $U_k(\mathcal{X}\times[0,t_0]) \subset \mathcal{Y}_{\varepsilon_0}$ for every k. In particular the traces $u_k := \mathbf{T}(U_k)$ belong to $W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y}_{\varepsilon_0})$ and $u_k \to u$ in $W^{1/\mathfrak{p}}(\mathcal{X},\mathbb{R}^N)$.

Since we use a local argument, without loss of generality we assume that $\mathcal{X} = B^{\mathfrak{p}-1}$ and $\mathcal{C}^{\mathfrak{p}} := B^{\mathfrak{p}-1} \times [0,1]$. Arguing exactly as in [13, Thm. 4.1] for the case $\mathfrak{p} = 2$, it suffices to apply Proposition 7.5 in order to approximate each vertical part $\delta_{x_i} \times C_i$ by paying an amount of energy estimated by $c(\mathfrak{p},\mathcal{Y})\,\mathbf{M}(R_i)$, where $c(\mathfrak{p},\mathcal{Y}) > 0$ is the absolute constant in Proposition 7.5. □

Remark 8.1 In the case $\mathfrak{p} = 2$, by Proposition 7.3 and Remark 7.4 we have $c(2,\mathcal{Y}) = 1$, whence we obtain the strong approximation with energy, Theorem 5.17. Finally, in the model case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, we have

$$T = G_{u_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times [\![\, \mathbb{S}^{\mathfrak{p}-1} \,]\!]\,.$$

Using Proposition 7.1, we obtain the strong density of smooth maps as in Theorem 3.5 (for $n = \mathfrak{p}-1$).

DENSITY IN DIMENSION $n = \mathfrak{p}$. In this case, for every $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ we first obtain that $\mathbb{P}(u)$ is an *integral flat chain.*

Proposition 8.2 *Let* $n = \mathfrak{p}$. *Let* $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ *and* $\{u_k\}$ *be a sequence in* $\mathcal{R}^{\infty}_{1/p}(\mathcal{X},\mathcal{Y})$ *that strongly converges in* $W^{1/p}$ *to* u*, see Theorem* 1.4*. Then we have:*

(i) $\mathbf{M}(\mathbb{D}_s(u_k) - \mathbb{D}_s(u)) \to 0$ *as* $k \to \infty$ *for each* $s = 1, \dots, \widetilde{s}$, *see* (4.4);

(ii) *there exists* $L \in \mathcal{R}_1(\mathcal{X}; \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$, *with* $\mathbf{M}(L) < \infty$, *such that* $\mathbb{P}(u) = (\partial L) \llcorner \mathrm{int}(\mathcal{X})$; *in particular* $\mathbb{P}(u)$ *is an integral flat chain;*

(iii) *if* $L^s_{u_k,u}$ *denotes an i.m. rectifiable current in* $\mathcal{R}_1(\mathcal{X})$ *of least mass such that*

$$(\partial L^s_{u_k,u}) \llcorner \mathrm{int}(\mathcal{X}) = \mathbb{P}_s(u) - \mathbb{P}_s(u_k)\,, \qquad s = 1, \dots, \widetilde{s}\,, \tag{8.1}$$

then $\mathbf{M}(L^s_{u_k,u}) \to 0$ *as* $k \to \infty$;

(iv) *if* $\partial\mathcal{X} = \emptyset$, *or* $u = \varphi$ *on* $\partial\mathcal{X}$ *for some smooth* $W^{1/p}$*-map* $\varphi : \overline{\mathcal{X}} \to \mathcal{Y}$, *then for each* $s = 1, \dots, \widetilde{s}$ *there exist points* $a_i, b_i \in \mathcal{X}$ *such that*

$$\mathbb{P}_s(u) = \sum_{i=1}^{\infty} (\delta_{a_i} - \delta_{b_i})\,, \qquad \sum_{i=1}^{\infty} \mathrm{dist}_{\mathcal{X}}(a_i, b_i) < \infty\,,$$

where $\mathrm{dist}_{\mathcal{X}}$ *is the geodesic distance in* $\mathcal{X}$.

PROOF: The proof of property (i) is similar to the one in [13, Prop. 1.4], and holds true even in higher dimension $n \geq \mathfrak{p}+1$. As to the rest of the proof, similarly to [10, Vol. II, Sec. 4.2.5], we observe that $\mathbb{P}_s(u_k)$ is an i.m. rectifiable current in $\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X})$. Theorem 4.9 yields $m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) < \infty$. By Federer's theorem [6], for $n = \mathfrak{p}$ we then infer that

$$m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) = m_{r,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) \qquad \forall\, s = 1, \dots, \widetilde{s}\,, \tag{8.2}$$

see Definition 3.1. Therefore, (i) and (4.6) give $m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k) - \mathbb{P}_s(u)) \to 0$, and the claims follow. □

In the case $n = \mathfrak{p}$, we thus have a simple proof of Theorems 5.16, 5.17, and 3.5.

SKETCH OF THE PROOF: Let $\{u_k\}$ in $\mathcal{R}^\infty_{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$ be strongly converging to u_T in $W^{1/\mathfrak{p}}$, see Theorem 1.4. On account of Proposition 8.2, let $L^s_{u_k,u_T}$ be given by (8.1), and set

$$T_k := G_{u_k} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}^s_k \times \widetilde{\gamma}_s\,, \qquad \mathbb{L}^s_k := (-1)^{\mathfrak{p}} L^s_{u_k,u_T} + \mathbb{L}_s(T)\,.$$

By (5.10) we have

$$\partial\, \mathbb{L}^s_k = (-1)^{\mathfrak{p}-1}\, \mathbb{P}_s(u_k)\,,$$

which is a finite sum of Dirac masses. Moreover, see (4.3), we have

$$\partial G_{u_k} = (-1)^{\mathfrak{p}} \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u_k) \times \widetilde{\gamma}_s \qquad \text{on} \quad \mathcal{Z}^{\mathfrak{p}-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})\,,$$

whence T_k belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$, see Remark 5.11. Since $\mathbf{M}(\mathbb{L}^s_k - \mathbb{L}_s(T)) = \mathbf{M}(L^s_{u_k,u_T}) \to 0$, we infer that $T_k \rightharpoonup T$ and $\mathcal{E}_{1/\mathfrak{p}}(T_k) \to \mathcal{E}_{1/\mathfrak{p}}(T)$. Since the mass of the boundary of $\mathbb{L}^s_k$ is finite for every s and k, we then may proceed in four steps along the lines of the proof of [13, Thm. 6.1]. Roughly speaking, in Step 1 we reduce to the case of a current T such that u_T has a finite number of point singularities. In Step 2, we reduce to the case of T such that the $\mathbb{L}_s(T)$'s are (curvilinear) polyhedral 1-chains. In Step 3, we reduce to the case of T as in (5.3) such that each $\mathbb{L}_q$ is the union of a finite number of oriented curvilinear segments S which only intersect at boundary points, the supports of the $\mathbb{L}_q$'s are pairwise disjoint, and $u_T \in \mathcal{R}^0_{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$ is locally Lipschitz on $\mathcal{X} \setminus \bigcup_q \mathrm{spt}\, \partial\, \mathbb{L}_q$. In Step 4 we use a dipole-type construction based on Proposition 7.5 to approximate the vertical parts $\mathbb{L}_q \times C_q$, and reduce to a current $T = G_u$ for some Cartesian map $u \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$. By using Theorem 4.7 and Remark 5.12, and by a diagonal argument, we prove Theorem 5.16. Finally, in the case $\mathfrak{p} = 2$, or $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, the dipole-type construction is based on the argument of Proposition 7.3 and Remark 7.4, or Proposition 7.1, respectively, yielding to the strong density of graphs of smooth maps, Theorems 5.17 and 3.5. □

Remark 8.3 It is in the last step, when applying Theorem 4.7 to remove the homologically trivial point singularities, that we make use of the additional hypothesis (H_2). We recall that such hypothesis cannot be removed, even in the case $\mathcal{X} = B^n$, compare Remark 4.8.

We also observe that in the case $\mathfrak{p} = 2$, since Theorem 4.9 yields $m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) < \infty$, Hardt-Pitts' theorem [22] gives (8.2) and hence the statements (ii) and (iii) in Proposition 8.2, for any $n \geq 2$. Therefore, a similar proof holds in any dimension $n \geq 2$, compare [13, Sec. 7]. However, even in the simple case $\mathcal{X} = B^n$, if $\mathfrak{p} \geq 3$, and $n \geq \mathfrak{p} + 1$, the proof of Theorem 5.16 cannot be obtained by a simple adaptation of the above argument. In this case, in fact, we do not know whether (8.2) holds true, compare [24, 31], or even if

$$m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) \leq c \cdot m_{r,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k))$$

for some absolute constant $c > 0$, not depending on u_k, a weaker condition that would give the assertion of Proposition 8.2, and hence a simple proof of the density theorem.

Density in dimension $n \geq \mathfrak{p} + 1$. In order to give a general proof of Theorem 5.16, we first recall that any $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$ satisfies (5.3), where $u = u_T \in W^{1/2}(\mathcal{X}, \mathcal{Y})$ and the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ with multiplicity one such that, writing $\mathbb{L}_q = \tau(\mathcal{L}_q, 1, \tau_q)$, the $(n - \mathfrak{p} + 1)$-rectifiable sets $\mathcal{L}_q := \mathrm{set}(\mathbb{L}_q)$ are pairwise disjoint. Moreover, the $\mathcal{E}_{1/\mathfrak{p}}$-energy of T is given by (5.7), where R_q minimizes the mass among all currents in $\mathcal{R}_\mathfrak{p}(\mathbb{R}^N)$ such that ∂R_q is in the homology class q. We denote by μ_T the finite Radon measure on $\mathcal{X}$ given for every Borel set $B \subset \mathcal{X}$ by

$$\mu_T(B) := \sum_{q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})} \mathbf{M}(R_q) \cdot \mathbf{M}(\mathbb{L}_q \llcorner B)\,, \tag{8.3}$$

and we set

$$\mathcal{E}_{1/\mathfrak{p}}(T, B) := \mathcal{E}_{1/\mathfrak{p}}(u_T, B) + \mu_T(B)\,, \tag{8.4}$$

where $\mathcal{E}_{1/\mathfrak{p}}(u_T, B)$ is defined as in (1.5), so that $\mathcal{E}_{1/\mathfrak{p}}(T, \mathcal{X}) = \mathcal{E}_{1/\mathfrak{p}}(T, \mathcal{X} \times \mathcal{Y})$.

We finally denote by $\mathbf{F}(T)$ the *flat norm*

$$\mathbf{F}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})\,,\ \mathbf{F}(\omega) \leq 1\}\,,$$

where for $\omega \in \mathcal{Z}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$

$$\mathbf{F}(\omega) := \max\left\{ \sup_{z \in \mathcal{X} \times \mathcal{Y}} \|\omega(z)\|\,,\ \sup_{z \in \mathcal{X} \times \mathcal{Y}} \|d\omega(z)\| \right\}.$$

As $|T(\omega)| \leq \mathbf{F}(T)\,\mathbf{F}(\omega)$, we infer that $T_k \rightharpoonup T$ weakly in $\mathcal{Z}_{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$ provided that $\mathbf{F}(T_k - T) \to 0$.

Similarly to [16], the proof of Theorem 5.16 is based on the following density theorem, the proof of which is postponed to the Appendix.

Theorem 8.4 *Let $\varepsilon \in (0, 1/2)$ and $k \in \mathbb{N}^+$. For every $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$ we can find a current $\widehat{T} \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$ such that*

$$\mathcal{E}_{1/\mathfrak{p}}(\widehat{T}) \leq \mathcal{E}_{1/\mathfrak{p}}(T) + \varepsilon^k + \big(c(\mathfrak{p}, \mathcal{Y}) - 1\big)\,\mu_T(\mathcal{X})\,,$$
$$\mathbf{F}(\widehat{T} - T) \leq \varepsilon^k \qquad \text{and} \qquad \mu_{\widehat{T}}(\mathcal{X}) \leq \frac{1}{2} \cdot \mu_T(\mathcal{X})\,,$$

where $c(\mathfrak{p}, \mathcal{Y}) \geq 1$ is the absolute constant given by Proposition 7.5. Moreover, $c(2, \mathcal{Y}) = 1$ for every $\mathcal{Y}$ and $c(\mathfrak{p}, \mathbb{S}^{\mathfrak{p}-1}) = 1$ for every $\mathfrak{p} \geq 2$.

In fact, by Theorem 8.4, for every $\varepsilon > 0$ we find a current $T_\varepsilon \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$ such that $\mathcal{E}_{1/\mathfrak{p}}(T_\varepsilon) \leq \mathcal{E}_{1/\mathfrak{p}}(T) + 2\varepsilon + 2\big(c(\mathfrak{p}, \mathcal{Y}) - 1\big)\,\mu_T(\mathcal{X})$, whereas $\mu_{T_\varepsilon}(\mathcal{X}) = 0$ and $\mathbf{F}(T_\varepsilon - T) \leq 2\varepsilon$. Using a diagonal argument on ε, we then find a sequence $\{T_k\} \subset \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$ that weakly converges to T in $\mathrm{cart}^{1/\mathfrak{p}}$ with

$$\limsup_{k \to \infty} \mathcal{E}_{1/\mathfrak{p}}(T_k) \leq \mathcal{E}_{1/\mathfrak{p}}(T) + 2\,\big(c(\mathfrak{p}, \mathcal{Y}) - 1\big)\,\mu_T(\mathcal{X}) \leq \big(2c(\mathfrak{p}, \mathcal{Y}) - 1\big)\,\mathcal{E}_{1/\mathfrak{p}}(T)$$

and such that $\mu_{T_k}(\mathcal{X}) = 0$. Therefore, T_k agrees with the current G_{u_k} for some $u_k \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$, see (4.1), where u_k belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$. By Theorem 4.7, and by means of a diagonal argument, we obtain Theorem 5.16. Finally, since $c(2, \mathcal{Y}) = 1$ for every $\mathcal{Y}$ and $c(\mathfrak{p}, \mathbb{S}^{\mathfrak{p}-1}) = 1$ for every $\mathfrak{p}$, by lower semicontinuity we obtain Theorems 5.17 and 3.5. $\square$

A Appendix

The appendix is dedicated to the proof of Theorem 8.4.

A.1 The density theorem, Part I

First, by the definition (8.3) we have $\mu_T = \theta_T\,\mathcal{H}^{n-\mathfrak{p}+1} \llcorner \mathcal{L}_T$, where $\mathcal{L}_T$ is the $(n-\mathfrak{p}+1)$-rectifiable set

$$\mathcal{L}_T := \bigcup_{q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})} \mathrm{set}(\mathbb{L}_q)\,, \qquad \mathcal{H}^{n-\mathfrak{p}+1}(\mathcal{L}_T) < \infty\,. \tag{A.1}$$

Moreover, the density θ_T is the positive $\mathcal{H}^{n-\mathfrak{p}+1} \llcorner \mathcal{L}_T$-summable function $\theta_T : \mathcal{L}_T \to [C_{\mathcal{Y}}, +\infty)$, where $C_{\mathcal{Y}} > 0$ is given by (5.5), defined by

$$\theta_T(x) := \mathbf{M}(R_q) \qquad \text{if} \quad x \in \mathrm{set}(\mathbb{L}_q)\,. \tag{A.2}$$

On account of Definition 5.4 and Remark 5.6, for every open set $\Omega \subset \mathcal{X}$ we shall denote by $\mathcal{F}(T,\Omega)$ the class of currents $\overline{T} \in \mathcal{D}_{n+1,\mathfrak{p}}((\Omega\times I)\times\mathbb{R}^N)$ such that $(-1)^{n-1}\partial\overline{T} = T$ on $\mathcal{Z}^{n,\mathfrak{p}-1}((\Omega\times\{0\})\times\mathcal{Y})$ and $\overline{T} = G_U$ on $\mathcal{D}^{n+1,\mathfrak{p}}((\Omega\times]0,1])\times\mathbb{R}^N)$ for some "smooth" map $U \in W^{1,\mathfrak{p}}(\Omega\times]0,1],\mathbb{R}^N)$. In this case, we shall write

$$\overline{T} = \overline{T}_U \in \mathcal{F}(T,\Omega)\,, \qquad \mathbf{D}_{\mathfrak{p}}(\overline{T},\Omega) := \mathbf{D}_{\mathfrak{p}}(\overline{T},(\Omega\times I)\times\mathbb{R}^N)\,. \tag{A.3}$$

Slicing properties. Let $T \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, and write $T = G_{u_T} + S_T$, see (5.3). Similarly to the case of normal currents, for every point $x_0 \in \mathcal{X}$ and for a.e. radius $r \in (0, r_0)$, where $r_0 = r_0(x) > 0$ is sufficiently small, in dependence of x, the *sliced current*

$$\langle T, d_{x_0}, r\rangle = \langle G_{u_T}, d_{x_0}, r\rangle + \langle S_T, d_{x_0}, r\rangle\,,$$

where $d_{x_0}(x,y) = \delta_{x_0}(x) := \mathrm{dist}_{\mathcal{X}}(x_0,x)$, is a well-defined Cartesian current in $\mathrm{cart}^{1/\mathfrak{p}}(\partial B_r(x_0)\times\mathcal{Y})$, where $B_r(x_0)$ denotes the *geodesic ball* of radius r centered at x_0, and $\partial B_r(x_0)$ its boundary. More precisely, we have

$$\langle G_{u_T}, d_{x_0}, r\rangle(\omega) = (-1)^{n-1}\langle \partial G_{U_T}, \widetilde{d}_{x_0}, r\rangle(\omega)\,, \qquad \omega \in \mathcal{D}^{n-1,\mathfrak{p}-1}(\partial B_r(x_0)\times\mathcal{Y})\,,$$

where $U_T = \mathrm{Ext}(u_T)$ and $\widetilde{d}_{x_0}(x,t,y) := \delta_{x_0}(x)$, so that the restriction $U_{T|\partial B_r(x_0)\times I}$ of U_T to $\partial B_r(x_0)\times I$ is a Sobolev function in $W^{1,\mathfrak{p}}(\partial B_r(x_0)\times I,\mathbb{R}^N)$. Also, by (5.3),

$$\langle S_T, d_{x_0}, r\rangle = \sum_{q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})} \langle \mathbb{L}_q, \delta_{x_0}, r\rangle \times C_q \qquad \text{on} \quad \mathcal{Z}^{n-1,\mathfrak{p}-1}(\partial B_r(x_0)\times\mathcal{Y})\,.$$

As a consequence, we infer that for every Borel set $B \subset \mathcal{X}$ the $\mathcal{E}_{1/\mathfrak{p}}$-energy of $\langle T, d_{x_0}, r\rangle$ on $B\times\mathcal{Y}$ is bounded by

$$\mathcal{E}_{1/\mathfrak{p}}(\langle T, d_{x_0}, r\rangle, B\times\mathcal{Y}) \le \mathbf{D}_{\mathfrak{p}}(U_{T|\partial B_r(x_0)\times I}, B) + \sum_{q\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y},\emptyset;\mathbb{Z})} \mathbf{M}(R_q)\cdot\mathbf{M}(\langle \mathbb{L}_q, \delta_{x_0}, r\rangle \llcorner B)\,, \tag{A.4}$$

where

$$\mathbf{D}_{\mathfrak{p}}(U_{T|\partial B_r(x_0)\times I}, B) := \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{(\partial B_r(x_0)\cap B)\times I} |D_\tau U_{T|\partial B_r(x_0)\times I}|^{\mathfrak{p}}\, d\mathcal{H}^n\,,$$

the derivative D_τ being computed w.r.t. an orthonormal frame τ tangential to $\partial B_r(x_0)\times I$. We also let

$$\mathcal{E}_{1/\mathfrak{p}}(\langle T, d_{x_0}, r\rangle) := \mathcal{E}_{1/\mathfrak{p}}(\langle T, d_{x_0}, r\rangle, \partial B_r(x_0)\times\mathcal{Y})\,.$$

Remark A.1 On account of Remark 7.2, for $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$ we denote by

$$B_{\mathcal{Y}}(y,\varepsilon) := \overline{B}^N(y,\varepsilon) \cap \mathcal{Y}$$

the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at y, so that we have $\Pi_\varepsilon(\overline{B}^N(y,\varepsilon)) = B_{\mathcal{Y}}(y,\varepsilon)$. Moreover, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \to B_{\mathcal{Y}}(y,\varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \overline{B}^N(y,\varepsilon) \\ \varepsilon\, \dfrac{z-y}{|z-y|} & \text{if} \quad z \in \mathbb{R}^N \setminus \overline{B}^N(y,\varepsilon) \end{cases} \tag{A.5}$$

so that $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\operatorname{Lip}\Psi_{(y,\varepsilon)} = \operatorname{Lip}\Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

As in [8, Sec. 4], Theorem 8.4 is based on the following local arguments, the proof of which is postponed to the next section. First, Proposition A.2, we show how to "deform" a current T, satisfying suitable energy estimates on the boundary of a ball, into a current satisfying a bound on the oscillation, see (A.8). Secondly, Propositions A.3 and A.4, we use a local approximation argument. In the sequel we shall denote by $c > 0$ an absolute constant, possibly varying from line to line.

Projecting the image of a current. For $n \geq \mathfrak{p}+1$, and $\mathfrak{p} \geq 2$, we set

$$B^n_\rho := B^n(0,\rho)\,, \qquad x = (\widetilde{x},\widehat{x}) \in \mathbb{R}^{n-\mathfrak{p}+1} \times \mathbb{R}^{\mathfrak{p}-1}\,, \qquad D_\rho := B^{n-\mathfrak{p}+1}(0,\rho)\,. \tag{A.6}$$

Proposition A.2 *Let $0 < R < \delta < 1$ and $T \in \operatorname{cart}^{1/\mathfrak{p}}(B^n_\delta \times \mathcal{Y})$ be such that $0 < \theta_T(0_{\mathbb{R}^n}) < \infty$, see (A.2), and let $U_T := \operatorname{Ext}(u_T)$. There exists a positive constant $\sigma_0 \in (0,1)$, only depending on n, $\mathfrak{p}$, on the isoperimetric constant $C_{\mathcal{Y}}$ in (5.5), and on $\theta_T(0_{\mathbb{R}^n})$, for which the following holds true. Assume that*

$$\begin{aligned} &\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_0,R\rangle, (\partial B^n_R \setminus (\overline{D}_R \times \{0_{\mathbb{R}^{\mathfrak{p}-1}}\})) \times \mathcal{Y}) \leq c\,\sigma\,\theta_T(0_{\mathbb{R}^n}) R^{n-\mathfrak{p}}\,, \\ &\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_0,R\rangle) \leq c\,\theta_T(0_{\mathbb{R}^n})\, R^{n-\mathfrak{p}}\,, \\ &\int_{\partial B^n_R \times [0,R]} |U_T(x)-y_0|^{\mathfrak{p}}\, d\mathcal{H}^n \leq c\,\sigma\, R^n \end{aligned} \tag{A.7}$$

for some $y_0 \in \mathcal{Y}$ and for some $\sigma > 0$ smaller than σ_0. Let $\varepsilon_\sigma := c_1\,\sigma^{1/(6\mathfrak{p})}$, where $c_1 = c_1(n,\mathfrak{p}) > 0$ is an absolute constant, and let $m \in \mathbb{N}^+$ be the integer part of $\sigma^{-2\alpha(n,\mathfrak{p})}$, where $\alpha(n,\mathfrak{p}) := 1/(4\mathfrak{p}(n-1)) > 0$. Then we can find a Cartesian current $\widehat{T} \in \operatorname{cart}^{1/\mathfrak{p}}((B^n_R \setminus \overline{B}^n_r) \times \mathcal{Y})$, where $r := R(1-1/m)$, and a current $\overline{T} = \overline{T}_U \in \mathcal{F}(\widehat{T}, B^n_R \setminus \overline{B}^n_r)$, see (A.3), such that the following facts hold:

- *(a) $\langle \widehat{T},d_0,R\rangle = \langle T,d_0,R\rangle$ and $\langle \widehat{T},d_0,r\rangle = (\psi_{R,r} \bowtie \Psi_{(y_0,\varepsilon_\sigma)})_\#\langle T,d_0,R\rangle$, where $\psi_{R,r}(x) := rx/R$ and $\Psi_{(y_0,\varepsilon_\sigma)}(z) := \Pi_{\varepsilon_\sigma} \circ \xi_{(y_0,\varepsilon_\sigma)}$, see (A.5), so that $\operatorname{spt}\langle \widehat{T},d_0,r\rangle \subset \partial B^n_r \times B_{\mathcal{Y}}(y_0,\varepsilon_\sigma)$;*
- *(b) the smooth function $U \in W^{1,\mathfrak{p}}((B^n_R \setminus \overline{B}^n_r) \times]0,1], \mathbb{R}^N)$ corresponding to $\overline{T}_U$ satisfies*

$$U((B^n_R \setminus \overline{B}^n_r) \times [0,R/m]) \subset \overline{B}^N(y_0,\varepsilon_\sigma)\,; \tag{A.8}$$

- *(c) $\widehat{T}$ has small energy on $B^n_R \setminus B^n_r$, i.e.,*

$$\mathcal{E}_{1/\mathfrak{p}}(\widehat{T}, B^n_R \setminus B^n_r) \leq \mathbf{D}_{\mathfrak{p}}(\overline{T}, (B^n_R \setminus B^n_r) \times I) \leq c\,\sigma^{\alpha(n,\mathfrak{p})} R^{n-\mathfrak{p}+1}\,; \tag{A.9}$$

- *(d) finally, the flat norm*

$$\mathbf{F}((\widehat{T} - G_{y_0}) \llcorner (B^n_R \setminus \overline{B}^n_r) \times \mathcal{Y}) \leq c\,\sigma^{\alpha(n,\mathfrak{p})} R^{n-\mathfrak{p}+1}\,; \tag{A.10}$$

  where G_{y_0} is the graph current of the constant map y_0.

Approximation on a ball. Using the notation from (A.6), let

$$B^+_r := \{(\widehat{x},t) \in B^{\mathfrak{p}}_r \mid \widehat{x} \in \mathbb{R}^{\mathfrak{p}-1},\ t > 0\}\,, \qquad B^+ := B^+_1\,.$$

Moreover, let $y(\widetilde{x}) := (r - |\widetilde{x}|_{\mathbb{R}^{n-\mathfrak{p}+1}})$ denote the (signed) distance of $\widetilde{x} \in \mathbb{R}^{n-\mathfrak{p}+1}$ from the boundary of the $(n-\mathfrak{p}+1)$-disk D_r, and for $\delta > 0$ small let

$$\phi_\delta(x,t) := (\widetilde{x}, \varphi_\delta(y(\widetilde{x}))\,\widehat{x}, \varphi_\delta(y(\widetilde{x}))\,t)\,, \quad \widetilde{x} \in D_r\,, \quad (\widehat{x},t) \in B^+\,, \quad \varphi_\delta(y) := \min\{y,\delta\}\,, \tag{A.11}$$

so that $\Omega_\delta := \phi_\delta(D_r \times B^+)$ is a small neighborhood of the interior of the disk $D_r \times \{0_{\mathbb{R}^\mathfrak{p}}\}$ in $B^n_R \times I$. Finally, let

$$\widetilde{\Omega}_\delta := \phi_\delta(D_r \times \overline{B}^+_{1/2}) = \{(\widetilde{x}, \widehat{x}, t) \mid \widetilde{x} \in D_r\,,\ (\widehat{x}, t) \in \overline{B}^+_{1/2}\,,\ \rho \le \varphi_\delta(y(\widetilde{x}))/2\}\,, \tag{A.12}$$

where in the sequel $\rho := |(\widehat{x}, t)|_{\mathbb{R}^\mathfrak{p}}$, and

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0_{\mathbb{R}^\mathfrak{p}}\})\,, \qquad \mathbb{R}^n_x := \mathbb{R}^n \times \{0\}\,.$$

Proposition A.3 *Let $T \in \mathrm{cart}^{1/\mathfrak{p}}(B^n_r \times \mathcal{Y})$, so that (5.3) holds, with $\mathcal{X} = B^n_r$. Let $\overline{T} = \overline{T}_U \in \mathcal{F}(T, B^n_r)$, see (A.3), where the smooth map $U \in W^{1,\mathfrak{p}}(B^n_r \times]0,1], \mathbb{R}^N)$ satisfies*

$$U(B^n_r \times [0, \delta_0]) \subset \overline{B}^N(y_0, \varepsilon_\sigma)$$

for some fixed $\delta_0 > 0$ and $y_0 \in \mathcal{Y}$, where $\varepsilon_\sigma = c_1\,\sigma^{1/(6\mathfrak{p})}$, for $\sigma > 0$ small. Moreover, assume that $D_r \times \{0_{\mathbb{R}^{\mathfrak{p}-1}}\} \subset \mathrm{set}(\mathbb{L}_q)$ for some $q \in \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}, \emptyset; \mathbb{Z})$, and let $R_q \in \mathcal{R}_\mathfrak{p}(\mathbb{R}^N)$ be such that $\partial R_q \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ belongs to q. Then, for $\delta > 0$ small enough, we can find a current $\widehat{T} \in \mathrm{cart}^{1/\mathfrak{p}}((B^n_r \setminus (\widetilde{\Omega}_\delta \cap \mathbb{R}^n_x)) \times \mathcal{Y})$ and a current $\widetilde{T} = \widetilde{T}_V \in \mathcal{F}(\widehat{T}, B^n_r \setminus (\widetilde{\Omega}_\delta \cap \mathbb{R}^n_x))$, see (A.3), satisfying the following properties:

i) $\partial \widetilde{T} = \partial(\overline{T} \llcorner (B^n_r \times I) \times \mathbb{R}^N) - [\![\partial \widetilde{\Omega}_\delta]\!] \times \delta_{y_0} - [\![\partial D_r \times \{0_{\mathbb{R}^\mathfrak{p}}\}]\!] \times R_q$;

ii) $\mathbf{D}_\mathfrak{p}(\widetilde{T}, ((B^n_r \times I) \setminus \widetilde{\Omega}_\delta) \times \mathbb{R}^N) \le \mathbf{D}_\mathfrak{p}(T, ((B^n_r \times I) \setminus \Omega_\delta) \times \mathbb{R}^N) + c\,\sigma\, r^{n-\mathfrak{p}+1} + c\,\mu_T(\Omega_{(r,\delta)} \cap \mathbb{R}^n_x)$;

iii) $\mathbf{F}((\widehat{T} - T) \llcorner (B^n_r \setminus (\widetilde{\Omega}_\delta \cap \mathbb{R}^n_x)) \times \mathcal{Y}) \le c\,\sigma\, r^{n-\mathfrak{p}+1}$.

THE DIPOLE CONSTRUCTION. We shall also make use of the following dipole argument, the proof of which is omitted, being very similar to the one given e.g. in [13], on account of Proposition 7.5.

Proposition A.4 *Let q, R_q, and $c(\mathfrak{p}, \mathcal{Y})$ be as in Proposition 7.5, and let $y_0 \in \mathcal{Y}$ be a given point. For every $\sigma > 0$, there exists a function $V_\sigma \in W^{1,\mathfrak{p}}(\widetilde{\Omega}_\delta, \mathbb{R}^N)$, with $\delta > 0$ sufficiently small, smooth on $\widetilde{\Omega}_\delta \setminus \mathbb{R}^n_x$ and with trace $v_\sigma := \mathbf{T}(V_\sigma) \in W^{1/\mathfrak{p}}((\widetilde{\Omega}_\delta \cap \mathbb{R}^n_x), \mathcal{Y})$, such that*

$$\mathbf{D}_\mathfrak{p}(V_\sigma, \widetilde{\Omega}_\delta) \le c\,\sigma\, r^{n-\mathfrak{p}+1} + c(\mathfrak{p}, \mathcal{Y}) \cdot \mathcal{H}^{n-\mathfrak{p}+1}(D_r) \cdot \mathbf{M}(R_q) \tag{A.13}$$

and

$$\partial G_{V_\sigma} = [\![\partial \widetilde{\Omega}_\delta]\!] \times \delta_{y_0} + [\![\partial D_r \times \{0_{\mathbb{R}^\mathfrak{p}}\}]\!] \times R_q\,. \tag{A.14}$$

Of course, by Proposition 7.3 and Remark 7.4 we have $c(2, \mathcal{Y}) = 2$. Moreover, in the case $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, using this time Proposition 7.1, we obtain the following

Proposition A.5 *Let $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$ and $y_0 \in \mathbb{S}^{\mathfrak{p}-1}$ be a given point. For every integer $q \in \mathbb{Z}$ and every $\sigma > 0$, there exists a function $V_\sigma \in W^{1,\mathfrak{p}}(\widetilde{\Omega}_\delta, \mathbb{R}^\mathfrak{p})$, with $\delta > 0$ sufficiently small, smooth on $\widetilde{\Omega}_\delta \setminus \mathbb{R}^n_x$ and with trace $v_\sigma := \mathbf{T}(V_\sigma) \in W^{1/\mathfrak{p}}((\widetilde{\Omega}_\delta \cap \mathbb{R}^n_x), \mathbb{S}^{\mathfrak{p}-1})$, such that*

$$\mathbf{D}_\mathfrak{p}(V_\sigma, \widetilde{\Omega}_\delta) \le c\,\sigma\, r^{n-\mathfrak{p}+1} + \mathcal{H}^{n-\mathfrak{p}+1}(D_r) \cdot |q| \cdot |B^\mathfrak{p}|$$

and

$$\partial G_{V_\sigma} = [\![\partial \widetilde{\Omega}_\delta]\!] \times \delta_{y_0} + q\, [\![\partial D_r \times \{0_{\mathbb{R}^\mathfrak{p}}\}]\!] \times [\![B^\mathfrak{p}]\!]\,.$$

We shall finally make use of the following

Lemma A.6 *For every $U \in W^{1,\mathfrak{p}}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ and for $\mathcal{H}^n$-a.e. point $x \in \mathcal{X}$ we have*

$$\lim_{\rho \to 0^+} \fint_{B^n(x,\rho) \times I} |DU(z)|^\mathfrak{p}\, d\mathcal{H}^{n+1} = \int_I |DU(x,t)|^\mathfrak{p}\, dt < \infty\,.$$

We are now ready to give the

Proof of Theorem 8.4: We divide the proof in four steps.

Step 1: Applying arguments as for instance in the proof of Federer [5, 4.2.19], by [5, 3.2.29] there exists a countable family $\mathcal{G}$ of $(n-\mathfrak{p}+1)$-dimensional C^1-submanifolds $\mathcal{M}_j$ of $\mathcal{X}$ such that μ_T-almost all of $\mathcal{X}$ is covered by $\mathcal{G}$. Let $\sigma\in(0,1)$ to be fixed. By the Vitali-Besicovitch theorem, and by the properties of the class $\text{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$, we can find a countable disjoint family of closed geodesic balls $B_j:=\overline{B}(p_j,r_j)$, contained in $\mathcal{X}$ and centered at points p_j in $\mathcal{L}_T$, and numbers $\sigma_j\in(0,\sigma)$, satisfying the properties listed below. In the sequel we shall denote by $c>0$ an absolute constant, possibly varying from line to line, which is independent of σ, of the σ_j's, and of the radii r_j of the balls B_j.

i) It holds $\mu_T(\mathcal{X}\setminus\bigcup_j B_j)=0$.

ii) For every j there is a manifold $\mathcal{M}_j$ of $\mathcal{G}$ such that the center p_j of B_j belongs to $\mathcal{M}_j$.

iii) Since $\mathcal{H}^{n-\mathfrak{p}+1}(\mathcal{L}_T)<\infty$ and $\theta_T:\mathcal{L}_T\to[C_{\mathcal{Y}},+\infty)$ is $\mathcal{H}^{n-\mathfrak{p}+1}\llcorner\mathcal{L}_T$-summable, see (5.5), then

$$\sum_{j=1}^\infty {r_j}^{n-\mathfrak{p}+1}\le c\,\mathcal{H}^{n-\mathfrak{p}+1}(\mathcal{L}_T)<\infty\,,\qquad \sum_{j=1}^\infty \theta_T(p_j)\,{r_j}^{n-\mathfrak{p}+1}\le c\,\mu_T(\mathcal{X})<\infty\,. \tag{A.15}$$

iv) Setting $\sigma_j:=\min\{\sigma,\sigma(p_j)\}$, where $\sigma(p_j)$ is the positive constant given by Proposition A.2 in correspondence of $\theta_T(p_j)$, we can find a number $t=t_j\in(1/2,1)$ such that

$$\mu_T(B(p_j,r_j)\setminus(B(p_j,t_jr_j)\cap\mathcal{M}_j))\le\sigma_j\,\mu_T(B(p_j,r_j))\qquad\forall\,j\,.$$

v) According to (A.1), we have $\mathcal{M}_j\subset\text{set}(\mathbb{L}_{q_j})$ for some $q_j\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$.

vi) All the p_j's are Lebesgue points of $U_T:=\text{Ext}(u_T)$ with Lebesgue values $U_T(p_j)=u_T(p_j)=z_j\in\mathcal{Y}$, and by a slicing argument

$$\int_{\partial B(p_j,t_jr_j)\times[0,t_jr_j]}|U_T(x)-z_j|^{\mathfrak{p}}\,d\mathcal{H}^n\le c\,\sigma_j\,{r_j}^n\,. \tag{A.16}$$

vii) Using a blow-up argument at p_j in the x-variables, we may and do assume that the current $S_j:=[\![\,B_j\,]\!]\times\delta_{z_j}+[\![\,\mathcal{M}_j\,]\!]\times q_j$ has small flat distance from T on $B_j\times\mathcal{Y}$, i.e.,

$$\mathbf{F}((S_j-T)\llcorner B_j\times\mathcal{Y})\le c\,\sigma_j\,{r_j}^{n-\mathfrak{p}+1}\,. \tag{A.17}$$

viii) By a slicing argument, we may and will assume that for some $R\in(tr_j,2tr_j)$ the current $\langle T,d_{p_j},tr_j\rangle$ belongs to $\text{cart}^{1/\mathfrak{p}}$ and satisfies

$$\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_{p_j},tr_j\rangle,\partial B(p_j,tr_j)\setminus\mathcal{M}_j)\le\frac{c}{r_j}\,\mathcal{E}_{1/\mathfrak{p}}(T,B(p_j,R)\setminus\mathcal{M}_j)\,.$$

Also, by the construction, and by using iv) and Lemma A.6, we may assume that both

$$\mu_T(B(p_j,\rho)\setminus(B(p_j,t_j\rho)\cap\mathcal{M}_j))\le\sigma_j\cdot\mu_T(B(p_j,\rho)) \tag{A.18}$$

and

$$\mu_T(B(p_j,\rho))\le c\,\theta_T(p_j)\,\rho^{n-\mathfrak{p}+1}\,,\qquad \mathbf{D}_{\mathfrak{p}}(U_T,B(p_j,\rho)\times I)\le c\,\rho^n\int_I|DU_T(p_j,t)|^{\mathfrak{p}}\,dt<\infty \tag{A.19}$$

hold true for any $0<\rho<2r_j$. Therefore, taking r_j small so that $r_j^{\mathfrak{p}-1}\int_I|DU_T(p_j,t)|^{\mathfrak{p}}\,dt\le\sigma_j\,\theta_T(p_j)$, we readily obtain that

$$\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_{p_j},t_jr_j\rangle,\partial B(p_j,t_jr_j)\setminus\mathcal{M}_j)\le c\,\sigma_j\,\theta_T(p_j)\,{r_j}^{n-\mathfrak{p}}\,. \tag{A.20}$$

ix) Using a similar slicing argument and (A.19), we may and do assume that

$$\mathcal{E}_{1/\mathfrak{p}}(\langle T, d_{p_j}, tr_j\rangle) \le c\,\theta_T(p_j)\, {r_j}^{n-\mathfrak{p}} . \tag{A.21}$$

x) Since $\theta_T(p_j)$ is the $(n-\mathfrak{p}+1)$-dimensional density of μ_T at p_j, and $p_j \in \operatorname{set}(\mathbb{L}_{q_j})$, we also may and do assume that

$$|\mu_T(B_j) - \mathbf{M}(R_{q_j}) \cdot \omega_{n-\mathfrak{p}+1}\, {r_j}^{n-\mathfrak{p}+1}| \le \sigma \cdot \omega_{n-\mathfrak{p}+1}\, {r_j}^{n-\mathfrak{p}+1} , \tag{A.22}$$

where $\omega_{n-\mathfrak{p}+1} := |B^{n-\mathfrak{p}+1}|$.

xi) There exists a suitable bilipschitz homeomorphism ψ_σ from $\mathcal{X}$ onto itself, with $\operatorname{Lip}\psi_\sigma \le 2$ and $\operatorname{Lip}\psi_\sigma^{-1} \le 2$, such that ψ_σ maps bijectively B_j onto B_j, with $\psi_{\sigma|\partial B_j} = Id_{|\partial B_j}$, for all j, and ψ_σ is equal to the identity outside the union of the balls B_j.

xii) We have $\psi_\sigma(B(p_j, t_j r_j) \cap \mathcal{M}_j) = B(p_j, \rho_j) \cap (p_j + \operatorname{Tan}(\mathcal{M}_j, p_j))$ and $\psi_\sigma(\partial B(p_j, t_j r_j)) = \partial B(p_j, \rho_j)$ for every j, where $\rho_j \in (r_j/2, r_j)$ and $\operatorname{Tan}(\mathcal{M}_j, p_j)$ is the tangent $(n-\mathfrak{p}+1)$-space to $\mathcal{M}_j$ at p_j.

Step 2: If $\widetilde{T} := \operatorname{Ext}(T) \in \mathcal{D}^{n+1,\mathfrak{p}}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$, setting for any j

$$\begin{array}{rcl} \widetilde{T}_j^\sigma &:=& (\psi_\sigma \bowtie Id_{I\times\mathbb{R}^N})_\# \widetilde{T} \llcorner (\operatorname{int}(B_j) \times I \times \mathbb{R}^N) , \\ T_j^\sigma &:=& (-1)^{n-1} \widetilde{T}_j^\sigma \llcorner (\operatorname{int}(B_j) \times \mathcal{Y}) \qquad \text{on} \quad \mathcal{Z}^{n,\mathfrak{p}-1}(\operatorname{int}(B_j) \times \mathcal{Y}) , \end{array}$$

T_j^σ belongs to $\operatorname{cart}^{1/\mathfrak{p}}(\operatorname{int}(B_j)\times\mathcal{Y})$, and $\widetilde{T}_j^\sigma = G_{U_j^\sigma}$ on $\operatorname{int}(B_j)\times]0,1]\times\mathbb{R}^N$, where $U_j^\sigma := (U_T \circ \psi_\sigma^{-1})_{|\operatorname{int}(B_j)\times I}$ belongs to $W^{1,\mathfrak{p}}(\operatorname{int}(B_j) \times I, \mathbb{R}^N)$, whereas $\mu_{T_j^\sigma} = \psi_{\sigma\#}(\mu_T \llcorner \operatorname{int}(B_j))$. Moreover, by (A.20), (A.21), and (A.16) we infer that T_j^σ and U_j^σ satisfy the hypotheses (A.7) of Proposition A.2, where $y_0 = z_j \in \mathcal{Y}$ is the Lebesgue value of U_T at p_j, with $p_j = 0$, $\delta = r_j$ and $R = \rho_j$, i.e.,

$$\begin{gathered} B_j = \overline{B}_\delta^n , \quad B(p_j, \rho_j) = B_R^n , \\ B(p_j, \rho_j) \cap (p_j + \operatorname{Tan}(\mathcal{M}_j, p_j)) = \overline{D}_R \times \{0_{\mathbb{R}^{\mathfrak{p}-1}}\} . \end{gathered}$$

Denote $A(p_j) := B(p_j, \rho_j) \setminus \overline{B}(p_j, \delta_j)$, where $\delta_j := \rho_j(1-1/m_j)$, and m_j is the integer part of $\sigma_j^{-2\alpha(n,\mathfrak{p})}$. Proposition A.2 yields a Cartesian current $T_j \in \operatorname{cart}^{1/\mathfrak{p}}(A(p_j) \times \mathcal{Y})$, and $\widetilde{T}_j \in \mathcal{F}(T_j, A(p_j))$, see (A.3), such that by (A.9) and (A.10), and since $\sigma_j \le \sigma$ for every j, we have

$$\mathbf{D}_\mathfrak{p}(\widetilde{T}_j, A(p_j) \times I) \le c\,\sigma^{\alpha(n,\mathfrak{p})} r_j^{n-\mathfrak{p}+1} , \tag{A.23}$$

$$\mathbf{F}((T_j - G_{z_j}) \llcorner (A(p_j) \times \mathcal{Y})) \le c\,\sigma^{\alpha(n,\mathfrak{p})}\, {r_j}^{n-\mathfrak{p}+1} , \tag{A.24}$$

whereas the support $\operatorname{spt}\langle T_j, d_{p_j}, \delta_j\rangle \subset \partial B(p_j, \delta_j) \times B_\mathcal{Y}(z_j, \varepsilon_\sigma)$. Moreover,

$$T_j = (-1)^{n-1} \partial \widetilde{T}_j \quad \text{on} \quad \mathcal{Z}^{n,\mathfrak{p}-1}((A(p_j) \times \{0\}) \times \mathcal{Y}) ,$$

where $\widetilde{T}_j = G_{U_j}$ on $\mathcal{D}^{n+1,\mathfrak{p}}((A(p_j)\times]0,1]) \times \mathbb{R}^N)$ for some "smooth" map $U_j \in W^{1,\mathfrak{p}}(A(p_j)\times]0,1], \mathbb{R}^N)$ with image

$$U_j(A(p_j) \times [0, h_j]) \subset \overline{B}^N(z_j, \varepsilon_\sigma) , \tag{A.25}$$

compare (A.8), for a positive $h_j \in]0,1[$.

Using Remark A.1, we now define $\psi_j(x) := p_j + \dfrac{\delta_j}{\rho_j}(x - p_j)$ and

$$\breve{T}_j := (\psi_j \bowtie \Psi_{(z_j,\varepsilon_\sigma)})_\# (T_j^\sigma \llcorner B(p_j, \rho_j) \times \mathcal{Y}) ,$$

so that we have $\breve{T}_j \in \operatorname{cart}^{1/\mathfrak{p}}(B(p_j, \delta_j), \mathcal{Y})$ with $\operatorname{spt}\breve{T}_j \subset \overline{B}(p_j, \delta_j) \times B_\mathcal{Y}(z_j, \varepsilon_\sigma)$, and by Proposition A.2

$$\partial(T_j - \breve{T}_j) \llcorner (\partial B(p_j, \delta_j) \times \mathcal{Y}) = 0 . \tag{A.26}$$

Write $\breve{T}_j = G_{v_j} + S_{\breve{T}_j}$, so that $v_j \in W^{1/\mathfrak{p}}(B(p_j, \delta_j), B_\mathcal{Y}(z_j, \varepsilon_\sigma))$, and let $V_j \in W^{1,\mathfrak{p}}((B(p_j, \delta_j) \times I), \mathbb{R}^N)$ be the function that minimizes the $\mathfrak{p}$-energy among all the smooth maps satisfying the following properties:

(a) $(-1)^{n-1}\partial G_{V_j} = \breve{T}_j$ on $\mathcal{Z}^{n,\mathfrak{p}-1}((B(p_j,\delta_j)\times\{0\})\times\mathcal{Y})$;

(b) $\partial(G_{V_j}-G_{U_j})\llcorner((\partial B(p_j,\delta_j)\times]0,1])\times\mathbb{R}^N)=0$;

(c) $V_j(B(p_j,\delta_j)\times[0,h_j])\subset\overline{B}^N(z_j,\varepsilon_\sigma)$.

These condition are not in contradiction, on account of (A.25). Moreover, comparing the $\mathfrak{p}$-energy of V_j with the one of a suitable reparametrization of U_j, see (A.23), and observing that $T_{V_j}\in\mathcal{F}(\breve{T}_j,B(p_j,\delta_j))$, according to the notation (A.3), it is readily checked that

$$\mathbf{D}_\mathfrak{p}(T_{V_j},B(p_j,\delta_j)\times I)\leq c\,c(\sigma)\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1}+c(\sigma)\,\mathcal{E}_{1/\mathfrak{p}}(T_j,B(p_j,\rho_j)\times\mathcal{Y})\,, \tag{A.27}$$

where $c(\sigma)\to 1^+$ as $\sigma\to 0^+$, whereas by (A.17), the definition of ψ_σ, ψ_j, and the triangular inequality

$$\mathbf{F}((\breve{T}_j-T)\llcorner B(p_j,\delta_j)\times\mathcal{Y})\leq c\,\sigma\,r_j^{n-\mathfrak{p}+1}\,. \tag{A.28}$$

Step 3: By (A.25) we infer that $\breve{T}_j$ and V_j satisfy the hypotheses of Proposition A.3, with $B(p_j,\delta_j)$ instead of B^n_r, $y_0=z_j$, and $q=q_j$, that yields a current $\widehat{T}_j^\sigma\in\mathcal{D}^{n+1,\mathfrak{p}}(((B(p_j,\delta_j)\times I)\setminus\widetilde{\Omega}^j_\delta)\times\mathbb{R}^N)$, where $\widetilde{\Omega}^j_\delta$ is defined similarly to (A.12), but in correspondence of $B(p_j,\delta_j)$.

Moreover, by applying Proposition A.4, with $B(p_j,\delta_j)$ instead of B^n_r and $q_0=q_j$, we find a suitable function $V_j^\sigma\in W^{1,\mathfrak{p}}(\widetilde{\Omega}^j_\delta,\mathbb{R}^N)$. Setting then

$$\overline{S}_j^\sigma:=\widehat{T}_j^\sigma+G_{V_j^\sigma}\,,\qquad \breve{S}_j^\sigma:=(-1)^{n-1}\partial\overline{S}_j^\sigma\llcorner(B(p_j,\delta_j)\times\mathcal{Y})\,,$$

(A.14) and i) in Proposition A.3 yield that $\breve{S}_j^\sigma\in\mathrm{cart}^{1/\mathfrak{p}}(B(p_j,\delta_j)\times\mathcal{Y})$ and also, using (A.26),

$$\partial(\breve{S}_j^\sigma-T_j)\llcorner(\partial B(p_j,\delta_j)\times\mathcal{Y})=0\,, \tag{A.29}$$

whereas $\overline{S}_j^\sigma\in\mathcal{F}(\breve{S}_j^\sigma,B(p_j,\delta_j))$, see (A.3). Moreover, by (A.13) we have

$$\mathbf{D}_\mathfrak{p}(V_j^\sigma,\widetilde{\Omega}^j_\delta)\leq c\,\sigma\,{\delta_j}^{n-\mathfrak{p}+1}+c(\mathfrak{p},\mathcal{Y})\cdot\mathcal{H}^{n-\mathfrak{p}+1}(D_{r_j})\cdot\mathbf{M}(R_{q_j})\,.$$

Therefore, since $\delta_j\in(r_j/2,r_j)$, by (A.22) we get

$$\mathbf{D}_\mathfrak{p}(V_j^\sigma,\widetilde{\Omega}^j_\delta)\leq c\,\sigma\,{r_j}^{n-\mathfrak{p}+1}+c(\mathfrak{p},\mathcal{Y})\cdot\mu_T(B_j)\,. \tag{A.30}$$

Finally, using (A.18) to estimate the last term in the right-hand side of ii) in Proposition A.3, by (A.27) and (A.30) we obtain

$$\begin{aligned}\mathcal{E}_{1/\mathfrak{p}}(\breve{S}_j^\sigma,B(p_j,\delta_j)\times\mathcal{Y}) &\leq c(\sigma)\,\mathcal{E}_{1/\mathfrak{p}}(T_j^\sigma,B(p_j,\delta_j)\times\mathcal{Y})\\ &+ c\,c(\sigma)\,\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1}+\big(c(\mathfrak{p},\mathcal{Y})-1+c\,\sigma\big)\,\mu_T(\overline{B}(p_j,r_j))\,,\end{aligned} \tag{A.31}$$

whereas by (A.17), (A.28), and the cited propositions we have

$$\mathbf{F}((\breve{S}_j^\sigma-T)\llcorner B(p_j,\delta_j)\times\mathcal{Y})\leq c\,\sigma\,r_j^{n-\mathfrak{p}+1}\,. \tag{A.32}$$

Step 4: We now define

$$\widetilde{S}_j^\sigma:=\overline{S}_j^\sigma+\widetilde{T}_j+\widetilde{T}_j^\sigma\llcorner(B(p_j,r_j)\setminus B(p_j,\rho_j))\times\mathbb{R}^N$$

and $S_j^\sigma:=(-1)^{n-1}\partial\widetilde{S}_j^\sigma\llcorner((B(p_j,r_j)\times\{0\})\times\mathcal{Y})$, so that by (A.29), and by the construction, S_j^σ belongs to $\mathrm{cart}^{1/\mathfrak{p}}(\mathrm{int}(B_j)\times\mathcal{Y})$. By (A.23), (A.27), and (A.31) we obtain that

$$\begin{aligned}\mathcal{E}_{1/\mathfrak{p}}(S_j^\sigma,B_j\times\mathcal{Y}) &\leq c(\sigma)\,\mathcal{E}_{1/\mathfrak{p}}(T_j^\sigma,B(p_j,\delta_j)\times\mathcal{Y})\\ &+ c\,c(\sigma)\,\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1}+\big(c(\mathfrak{p},\mathcal{Y})-1+c\,\sigma\big)\,\mu_T(B_j)\,.\end{aligned} \tag{A.33}$$

Moreover, by (A.10), (A.24), and (A.32), we deduce that for ε, δ small enough

$$\mathbf{F}((S_j^\sigma-T_j^\sigma)\llcorner B_j\times\mathcal{Y})\leq c\,\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1}\,. \tag{A.34}$$

Setting now

$$\widetilde{T}_j^{(\sigma)} := (\psi_\sigma^{-1} \bowtie \mathrm{Id}_{I\times\mathbb{R}^N})_\# \widetilde{S}_j^\sigma \llcorner \mathrm{int}(B_j) \times I \times \mathbb{R}^N , \qquad T_j^{(\sigma)} := (\psi_\sigma^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\# S_j^\sigma \llcorner \mathrm{int}(B_j) \times \mathcal{Y} ,$$

by (A.33) we infer that for every j

$$\begin{array}{rcl} \mathcal{E}_{1/\mathfrak{p}}(T_j^{(\sigma)}, \mathrm{int}(B_j)\times\mathcal{Y}) & \leq & c(\sigma)\,\mathcal{E}_{1/\mathfrak{p}}(T, B(p_j,\delta_j)\times\mathcal{Y}) \\ & + & \big(c(\mathfrak{p},\mathcal{Y}) - 1 + c\,\sigma\big)\,\mu_T(B_j) + c\,c(\sigma)\,\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1} \end{array} \tag{A.35}$$

whereas by (A.34)

$$\mathbf{F}((T_j^{(\sigma)} - T)\llcorner B_j \times \mathcal{Y}) \leq c\,\sigma^{\alpha(n,\mathfrak{p})}\,{r_j}^{n-\mathfrak{p}+1} . \tag{A.36}$$

We finally define $T^\sigma \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$ by

$$T^\sigma := \sum_{j=1}^\infty T_j^{(\sigma)} + T \llcorner \big(\mathcal{X} \setminus \bigcup_{j=1}^\infty \mathrm{int}(B_j)\big) \times \mathcal{Y} .$$

We have $T^\sigma \in \mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}\times\mathcal{Y})$; by (A.15) and (A.35), for $\sigma > 0$ small, so that $c(\sigma)$ is near to 1, we obtain

$$\mathcal{E}_{1/\mathfrak{p}}(T^\sigma) \leq \mathcal{E}_{1/\mathfrak{p}}(T) + \big(c(\mathfrak{p},\mathcal{Y}) - 1 + c\,\sigma\big)\,\mu_T(\mathcal{X}) + c\,\sigma^{\alpha(n,\mathfrak{p})}\,\mathcal{H}^{n-\mathfrak{p}+1}(\mathcal{L}_T) ,$$

so that if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu_T) > 0$ is small, we have

$$\mathcal{E}_{1/\mathfrak{p}}(T^\sigma) \leq \mathcal{E}_{1/\mathfrak{p}}(T) + \varepsilon^k + \big(c(\mathfrak{p},\mathcal{Y}) - 1\big)\,\mu_T(\mathcal{X}) .$$

Moreover, by (A.18), taking σ small, the above construction yields that

$$\begin{array}{rcl} \mu_{T^\sigma}(\mathcal{X}) & \leq & c \displaystyle\sum_{j=1}^\infty \mu_T(B(p_j,r_j) \setminus (B(p_j, t r_j) \cap \mathcal{M}_j)) + \mu_T(\mathcal{X}\setminus\mathcal{L}_T) \\ & \leq & c\,\sigma\,\mu_T(\mathcal{X}) < \dfrac{1}{2}\cdot\mu_T(\mathcal{X}) . \end{array}$$

Finally, by (A.36) and (A.15) we have

$$\begin{array}{rcl} \mathbf{F}(T^\sigma - T) & \leq & \displaystyle\sum_{j=1}^\infty \mathbf{F}((T_j^{(\sigma)} - T)\llcorner B_j \times \mathcal{Y}) \\ & \leq & c\,\sigma^{\alpha(n,\mathfrak{p})} \displaystyle\sum_{j=1}^\infty {r_j}^{n-\mathfrak{p}+1} < \varepsilon^k , \end{array}$$

if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu_T, n, \mathfrak{p}) > 0$ is small. Theorem 8.4 follows by taking $\widetilde{T} = T^\sigma$ for $\sigma > 0$ small. □

A.2 The density theorem, Part II

In this section we conclude the proof of Theorem 8.4, by proving the local arguments stated in the previous section.

Proof of Proposition A.2: We use an argument reminiscent of the one used in [18, Step 3] and [8, Prop. 4.6]. For a given cubeulation of the boundary ∂B_R^n, we shall denote by Σ_R^k the family of k-faces of the skeleton, and by $\widetilde{\Sigma}_R^k$ the subset of k-faces that do not intersect $\overline{D}_R \times \{0\}$. Moreover, we let $\widetilde{T} := \mathrm{Ext}(T)$, so that the slices satisfy

$$(-1)^{n-1}\langle \widetilde{T}, \widetilde{d}_0, R\rangle = \langle T, d_0, R\rangle \qquad \text{on} \quad \mathcal{Z}^{n-1,\mathfrak{p}-1}(\partial B_R^n \times \mathcal{Y}) ,$$

and denote the restriction of the slices to the k-skeleton of the grid by

$$T_R^k := \langle T, d_0, R\rangle \llcorner \Sigma_R^k \times \mathcal{Y} , \qquad \widetilde{T}_R^k := \langle \widetilde{T}, \widetilde{d}_0, R\rangle \llcorner (\Sigma_R^k \times I) \times \mathbb{R}^N .$$

Finally, for simplicity we will denote

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,\Sigma_R^k) := \mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,(\Sigma_R^k\times I)\times\mathbb{R}^N)$$
$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,\widetilde{\Sigma}_R^k) := \mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,(\widetilde{\Sigma}_R^k\times I)\times\mathbb{R}^N)\,.$$

To our purposes, we want that $T_R^{\mathfrak{p}-2}$ agrees with the graph of a continuous $W^{1/\mathfrak{p}}$-map with image contained in the geodesic ball $B_{\mathcal{Y}}(y,\varepsilon_\sigma)$, and the image of $\Sigma_R^{\mathfrak{p}-2}\times[0,R/q]$ by the extension map U_T of u_T is contained in the ball $\overline{B}^N(y,\varepsilon_\sigma)$.

We thus choose a suitable cubeulation of ∂B_R^n made of small $(n-1)$-dimensional "cubes" of side R/m, i.e., each one bilipschitz homeomorphic to the $(n-1)$-cube $[-R/m,R/m]^{n-1}$ by linear maps f_i such that $\|Df_i\|_\infty\le K$, $\|Df_i^{-1}\|_\infty\le K$, where $K>0$ is an absolute constant. Since $\overline{D}_R\times\{0\}$ has dimension $n-\mathfrak{p}+1$, we may and do assume that $\Sigma_R^{\mathfrak{p}-2}\cap(\overline{D}_R\times\{0\})=\emptyset$, i.e., $\Sigma_R^{\mathfrak{p}-2}=\widetilde{\Sigma}_R^{\mathfrak{p}-2}$, and that the following estimates hold for every $k=\mathfrak{p}-2,\dots,n-1$:

$$\int_{\Sigma_R^k\times[0,R]}|U_T-y_0|^{\mathfrak{p}}\,d\mathcal{H}^{k+1}\le c\left(\frac{m}{R}\right)^{n-1-k}\int_{\partial B_R^n\times[0,R]}|U_T-y_0|^{\mathfrak{p}}\,d\mathcal{H}^n\,; \tag{A.37}$$

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,\Sigma_R^k)\le c\left(\frac{m}{R}\right)^{n-1-k}\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_0,R\rangle)\,; \tag{A.38}$$

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,\widetilde{\Sigma}_R^k)\le c\left(\frac{m}{R}\right)^{n-1-k}\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_0,R\rangle,\partial B_R^n\setminus(\overline{D}_R\times\{0_{\mathbb{R}^{\mathfrak{p}-1}}\}))\,. \tag{A.39}$$

By the first inequality in (A.7) we infer that (A.39) yields

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^k,\widetilde{\Sigma}_R^k)\le c\,\sigma\,\theta_T(0_{\mathbb{R}^n})\,R^{k-\mathfrak{p}+1}\,m^{n-1-k}\,. \tag{A.40}$$

In particular, for $k=\mathfrak{p}-1$, if

$$\sigma^{1/3}\,m^{n-\mathfrak{p}}<1/\theta_T(0_{\mathbb{R}^n})\,, \tag{A.41}$$

condition that satisfied for $\sigma>0$ small by the choice of m, we infer that

$$\mathbf{D}_{\mathfrak{p}}(\widetilde{T}_R^{\mathfrak{p}-1},\widetilde{\Sigma}_R^{\mathfrak{p}-1})\le c\,\sigma^{2/3}\,.$$

Due to (5.5), taking $\sigma>0$ small, it turns out that $T_R^{\mathfrak{p}-1}$ has no vertical part in $\widetilde{\Sigma}_R^{\mathfrak{p}-1}\times\mathcal{Y}$, hence agrees with the graph of a $W^{1/\mathfrak{p}}$-map from $\widetilde{\Sigma}_R^{\mathfrak{p}-1}$ with values into $\mathcal{Y}$. We now prove the following

Claim: *The image of $\Sigma_R^{\mathfrak{p}-2}\times[0,R/m]$ by $U_T:=\mathrm{Ext}(u_T)$ is contained in the ball $\overline{B}^N(y_0,\varepsilon_\sigma)$.*

Proof of the claim: Since $\Sigma_R^{\mathfrak{p}-2}=\widetilde{\Sigma}_R^{\mathfrak{p}-2}$, the inequalities (A.40) and (A.41) yield that

$$\int_{\Sigma_R^{\mathfrak{p}-2}\times I}|DU_{T|\Sigma_R^{\mathfrak{p}-2}\times I}|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\le c\,\sigma^{2/3}\,\frac{m}{R}\,. \tag{A.42}$$

Let Q be any $(\mathfrak{p}-2)$-cube of side R/m in $\Sigma_R^{\mathfrak{p}-2}$. By applying the Sobolev embedding theorem to the $W^{1,\mathfrak{p}}$-function $U_{T|\Sigma_R^{\mathfrak{p}-2}\times I}$ on the $(\mathfrak{p}-1)$-dimensional "cube" $Q\times[0,R/m]$, and by a scaling argument, we obtain that

$$\sup_{z_1,z_2\in Q\times[0,R/m]}\frac{|U_T(z_1)-U_T(z_2)|^{\mathfrak{p}}}{|z_1-z_2|}\le c\int_{Q\times[0,R/m]}|DU_{T|\Sigma_R^{\mathfrak{p}-2}\times I}|^{\mathfrak{p}}d\mathcal{H}^{\mathfrak{p}-1}\,.$$

By (A.42), the diameter of $Q\times[0,R/m]$ being smaller than $c\,(R/m)$, this yields that the *oscillation* of U_T on each $(\mathfrak{p}-1)$-cube $Q\times[0,R/m]$ is bounded by $c\,\sigma^{2/3\mathfrak{p}}$, and hence by $\varepsilon_\sigma/2$, for $\sigma>0$ small.

Moreover, by (A.37) and the third inequality in (A.7), we have

$$\int_{\Sigma_R^{\mathfrak{p}-2}\times[0,R/m]}|U_T-y_0|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\le c\,\sigma\,R^n\left(\frac{m}{R}\right)^{n-\mathfrak{p}+1}=c\,\sigma\,R^{\mathfrak{p}-1}\,m^{n-\mathfrak{p}+1}\,.$$

For every $(\mathfrak{p}-2)$-cube Q in $\Sigma_R^{\mathfrak{p}-2}$ as above, since $\mathcal{H}^{\mathfrak{p}-1}(Q\times[0,R/m])\geq c\,(R/m)^{\mathfrak{p}-1}$, we obtain

$$\fint_{Q\times[0,R/m]}|U_T-y_0|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\leq c\,\sigma\,m^n\,.$$

Since $n-1\geq\mathfrak{p}\geq 2$, we infer that $(c\,\sigma\,m^n)^{1/\mathfrak{p}}<\varepsilon_\sigma/2$ for $\sigma>0$ small, thus we find a point $z\in Q\times[0,R/m]$ such that $|U_T(z)-y_0|\leq\varepsilon_\sigma/2$. The claim follows. □

Using arguments reminiscent of [18], we now construct the current $\widetilde{T}$ satisfying the above properties. More precisely, we first define $\Phi_m:\partial B_R^n\to\partial B_r^n$ by $\Phi_m(x):=(1-m^{-1})\,x$ and $\pi_{(R,r)}:B_R^n\setminus B_r^n\to\partial B_R^n$ by $\pi_{(R,r)}(x):=R\,x/|x|$. Setting for $k=\mathfrak{p}-1,\dots,n-1$

$$\mathcal{M}^k_{(R,r)}:=\pi^{-1}_{(R,r)}(\Sigma_R^{k-1})\,,\qquad\mathcal{N}^k_{(R,r)}:=\mathcal{M}^k_{(R,r)}\cup\Sigma^k_R\cup\Phi_m(\Sigma^k_R)\,,$$

it turns out that $\mathcal{N}^k_{(R,r)}$ is the k-skeleton of a family of n-dimensional "cubes" $\{C_i\}_{i=1}^{i_0}$, where $i_0=c\,m^{n-1}$, each one bilipschitz homeomorphic to the n-cube $[-R/m,R/m]^n$ by linear maps $\widetilde{f}_i$ such that $\|D\widetilde{f}_i\|_\infty\leq K$, $\|D\widetilde{f}_i^{-1}\|_\infty\leq K$, where $K>0$ is an absolute constant. We also let $\mathcal{N}^n_{(R,r)}:=\bigcup_{i=1}^{i_0}C_i=\overline{B}^n_R\setminus B^n_r$.

We first define a $\mathfrak{p}$-current $S^{\mathfrak{p}}$ on $(\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}\times I)\times\mathbb{R}^N$ by setting

$$S^{\mathfrak{p}}:=\begin{cases}\widetilde{T}^{\mathfrak{p}-1}_R & \text{on}\quad(\Sigma_R^{\mathfrak{p}-1}\times I)\times\mathbb{R}^N\\ H_\#([\![0,1]\!]\times\widetilde{T}^{\mathfrak{p}-2}_R) & \text{on}\quad(\mathcal{M}^{\mathfrak{p}-1}_{(R,r)}\times I)\times\mathbb{R}^N\,,\end{cases}$$

where $H:[0,1]\times\partial B_R^n\times I\times\mathbb{R}^N\to(B^n_R\setminus B^n_r)\times\mathbb{R}^N$ is the affine homotopy map

$$H(\lambda,x,t,y):=\Big(\lambda r\frac{x}{|x|}+(1-\lambda)x,t,y\Big)\,.$$

Moreover, if F is any $(\mathfrak{p}-1)$-face of $\Phi_m(\Sigma_R^{\mathfrak{p}-1})$, we define $S^{\mathfrak{p}}$ on $(F\times I)\times\mathbb{R}^N$ as the graph of the $W^{1,\mathfrak{p}}$-map $v_F:F\times I\to\mathbb{R}^N$

$$v_F(x,t):=\begin{cases}\dfrac{m}{R}\,t\,U_T\big(\Phi_m^{-1}(x),t\big)+\Big(1-\dfrac{m}{R}\,t\Big)\,\xi_{(y_0,\varepsilon_\sigma)}\circ U_T\big(\Phi_m^{-1}(x),t\big) & \text{if}\quad 0\leq t\leq\dfrac{R}{m}\\ U_T\big(\Phi_m^{-1}(x),t\big) & \text{if}\quad\dfrac{R}{m}\leq t\leq 1\,.\end{cases}$$

Recall that $\widetilde{T}^{\mathfrak{p}-1}_R$ agrees with the $\mathfrak{p}$-current carried by the graph of the $W^{1,\mathfrak{p}}$-map $U_{T|\Sigma_R^{\mathfrak{p}-1}\times I}$, and the restriction $U_{T|\Sigma_R^{\mathfrak{p}-2}\times[0,R/m]}$ is a continuous function with image contained in $\overline{B}^N(y_0,\varepsilon_\sigma)$, by the claim. Therefore, for every $x\in\partial F$ we have

$$v_F(x,t)=U_T\big(\Phi_m^{-1}(x),t\big)\qquad\forall\,t\in[0,1]\,,$$

whereas for $t=0$ we have

$$v_F(x,0)=\Psi_{(y_0,\varepsilon_\sigma)}\circ u_T\big(\Phi_m^{-1}(x)\big)\in B_{\mathcal{Y}}(y_0,\varepsilon_\sigma)\quad\forall\,x\in F\,,\quad\forall\,F\in\Phi_m(\Sigma_R^{\mathfrak{p}-1})\,.$$

As a consequence, for every $(\mathfrak{p}-1)$-face F of $\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}$, the restriction $S^{\mathfrak{p}}_F$ of $S^{\mathfrak{p}}$ to $(F\times I)\times\mathbb{R}^N$ agrees with the graph of a $W^{1,\mathfrak{p}}$-map, with no interior boundary, so that if F_1 and F_2 intersect in a common $(\mathfrak{p}-2)$-face L, we have

$$S^{\mathfrak{p}}_{F_1}\llcorner(L\times I)\times\mathbb{R}^N=-S^{\mathfrak{p}}_{F_2}\llcorner(L\times I)\times\mathbb{R}^N\,.$$

Most importantly, we have

$$\operatorname{spt}(S^{\mathfrak{p}}\llcorner(\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}\times[0,R/m])\times\mathbb{R}^N)\subset(\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}\times[0,R/m])\times\overline{B}^N(y_0,\varepsilon_\sigma)\,.\tag{A.43}$$

Moreover, the energy estimate

$$\mathbf{D}_{\mathfrak{p}}(S^{\mathfrak{p}},(\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}\times I)\times\mathbb{R}^N)\le c\left(\mathbf{D}_{\mathfrak{p}}(\widetilde{T}^{\mathfrak{p}-1}_R,\Sigma^{\mathfrak{p}-1}_R)+\frac{R}{m}\,\mathbf{D}_{\mathfrak{p}}(\widetilde{T}^{\mathfrak{p}-2}_R,\Sigma^{\mathfrak{p}-2}_R)\right) \tag{A.44}$$

holds. In fact, $\operatorname{Lip}\xi_{(y_0,\varepsilon_\sigma)}\le 1$, see (A.5), so that by the Poincaré inequality

$$\begin{aligned}
&\left(\frac{m}{R}\right)^{\mathfrak{p}}\int_{F\times[0,R/m]}|U_T\big(\Phi_m^{-1}(x),t\big)-\xi_{(y_0,\varepsilon_\sigma)}\circ U_T\big(\Phi_m^{-1}(x),t\big)|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\\
&\quad\le c\left(\frac{m}{R}\right)^{\mathfrak{p}}\int_{\Phi_m^{-1}(F)\times[0,R/m]}|U_T-\xi_{(y_0,\varepsilon_\sigma)}\circ U_T|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\\
&\quad\le c\int_{\Phi_m^{-1}(F)\times[0,R/m]}|DU_T-D\big(\xi_{(y_0,\varepsilon_\sigma)}\circ U_T\big)|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\\
&\quad\le c\int_{\Phi_m^{-1}(F)\times I}|DU_T|^{\mathfrak{p}}\,d\mathcal{H}^{\mathfrak{p}-1}\,,
\end{aligned}$$

that clearly yields (A.44). Finally, since $\Phi_m(\Sigma^{\mathfrak{p}-1}_R)$ contains $c\,m^{n-1}$ such $(\mathfrak{p}-1)$-faces F, for $\sigma>0$ small we have the $L^{\mathfrak{p}}$-estimate

$$\sum_{F\in\Phi_m(\Sigma^{\mathfrak{p}-1}_R)}\|v_F-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(F\times[0,R/m])}\le c\,\varepsilon_\sigma{}^{\mathfrak{p}}\,m^{n-1}\left(\frac{R}{m}\right)^{\mathfrak{p}}. \tag{A.45}$$

Arguing by iteration on the dimension $k=\mathfrak{p},\dots,n$, we now define a $(k+1)$-current S^{k+1} on $(\mathcal{N}^k_{(R,r)}\times I)\times\mathbb{R}^N$. Using (A.43) for the case $k=\mathfrak{p}$, at the previous step we have defined S^k on $(\mathcal{N}^{k-1}_{(R,r)}\times I)\times\mathbb{R}^N$ in such a way that

$$\operatorname{spt}(S^k\llcorner((\mathcal{N}^{k-1}_{(R,r)}\times[0,R/m])\times\mathbb{R}^N))\subset(\mathcal{N}^{k-1}_{(R,r)}\times[0,R/m])\times\overline{B}^N(y_0,\varepsilon_\sigma)\,. \tag{A.46}$$

If Q is any k-face of $\mathcal{N}^k_{(R,r)}$, we distinguish three cases.

Case 1: If Q is contained in Σ^k_R we set

$$S^{k+1}\llcorner((Q\times I)\times\mathbb{R}^N):=\widetilde{T}^k_R\llcorner((Q\times I)\times\mathbb{R}^N)\,.$$

Case 2: If Q is contained in $\Phi_m(\Sigma^k_R)$ we first define the k-current T^k on $Q\times\mathcal{Y}$ by

$$T^k\llcorner(Q\times\mathcal{Y}):=(\psi_{R,r}\bowtie\Psi_{(y_0,\varepsilon_\sigma)})_\#\big(T^k_R\llcorner(Q\times\mathcal{Y})\big)\,.$$

Case 3: If Q is contained in $\mathcal{M}^k_{(R,r)}$, we first define the k-current T^k on $Q\times\mathcal{Y}$ by means of a "radial" extension of the boundary datum. More precisely, we set

$$T^k\llcorner(Q\times\mathcal{Y}):=\widehat{H}_\#(\llbracket 0,1\rrbracket\times(S^k\llcorner((\partial Q\times\{0\})\times\mathcal{Y})))\,, \tag{A.47}$$

where $\widehat{H}(\lambda,x,y):=(\lambda x_Q+(1-\lambda)x,y)$ and x_Q is the barycenter of Q.

Remark A.7 The current T^k given by the radial extension (A.47) may have a non-zero boundary of the type $\delta_{x_Q}\times S_Q$, as we test with $(k-1)$-forms in $\mathcal{Z}^{k-1,\mathfrak{p}-1}$. In the case $k=\mathfrak{p}$, S_Q is the integral flat $(\mathfrak{p}-1)$-cycle in $\mathcal{Y}$ given by $u_{T\#}\llbracket F\rrbracket-\Psi_{(y_0,\varepsilon_\sigma)}\circ u_{T\#}\llbracket F\rrbracket$, where F is the $(\mathfrak{p}-1)$-face of Q that intersects the boundary ∂B^n_R. However, by the construction we infer that S_Q is the boundary of a current R_Q in $\mathcal{R}_{\mathfrak{p}}(\mathbb{R}^N)$ with mass lower than an absolute constant times the $\mathfrak{p}$-energy of the extension $U_{T|F\times I}$, which is small with σ. Therefore, by (5.5) we infer that for $\sigma>0$ small the $(\mathfrak{p}-1)$-cycle S_Q is homologically trivial in $\mathcal{Y}$, whence

$$\partial T^{\mathfrak{p}}\llcorner(\operatorname{int}(Q)\times\mathcal{Y})=0\qquad\text{on}\quad\mathcal{Z}^{\mathfrak{p}-1,\mathfrak{p}-1}(\operatorname{int}(Q)\times\mathcal{Y})\,.$$

On the other hand, in the case $k\ge\mathfrak{p}+1$ no extra-boundary is produced in the radial extension (A.47), as currents of the type $\delta_x\times S$, where $S\in\mathcal{Z}_{k-1}(\mathcal{Y})$, are always zero when tested on forms in $\mathcal{Z}^{k-1,\mathfrak{p}-1}$.

In the Cases 2 and 3, we then define S^{k+1} on $(Q\times I)\times\mathbb{R}^N$ as the current G_{v_Q} carried by the graph of the $W^{1,\mathfrak{p}}$-map $v_Q : Q\times I\to\mathbb{R}^N$ that minimizes the $\mathfrak{p}$-energy among all maps v in $W^{1,\mathfrak{p}}(Q\times I,\mathbb{R}^N)$ satisfying the following conditions as to the boundary and the image:

i) $\partial G_v \llcorner ((\partial Q\times I)\times\mathbb{R}^N) = S^k \llcorner ((\partial Q\times I)\times\mathbb{R}^N)$;

ii) $\partial G_v \llcorner (Q\times\{0\})\times\mathbb{R}^N = T^k$;

iii) $v(Q\times[0,R/m])\subset \overline{B}^N(y_0,\varepsilon_\sigma)$.

Notice that conditions i) and iii) are not in contradiction, due to (A.46). Moreover, in the Case 3 the following energy estimate is readily checked:

$$\mathbf{D}_{\mathfrak{p}}(S^{k+1},(Q\times I)\times\mathbb{R}^N)\le C\,\frac{R}{m}\,\mathbf{D}_{\mathfrak{p}}(S^k,(\partial Q\times I)\times\mathbb{R}^N)\,, \tag{A.48}$$

whereas in the Cases 1 and 2 we have

$$\mathbf{D}_{\mathfrak{p}}(S^{k+1},(Q\times I)\times\mathbb{R}^N)\le C\,\mathbf{D}_{\mathfrak{p}}(\widetilde{T}^k_R\llcorner((Q\times I)\times\mathbb{R}^N))\,. \tag{A.49}$$

Once we have iterated the previous construction for $k=\mathfrak{p},\dots,n$, we set for every n-cube $C_i\in\mathcal{N}^n_{(R,r)}$

$$\overline{T}\llcorner((C_i\times I)\times\mathbb{R}^N)=S^{n+1}\llcorner((C_i\times I)\times\mathbb{R}^N)\,.$$

By (A.48) and (A.49) we obtain a bound of the $\mathfrak{p}$-energy of $\overline{T}$ on $((B^n_R\setminus B^n_r)\times I)\times\mathbb{R}^N$ in terms of

$$c\sum_{k=\mathfrak{p}-1}^{n-1}\left(\frac{R}{m}\right)^{n-k}\mathbf{D}_{\mathfrak{p}}(\widetilde{T}^k_R,\Sigma^k_R)+c\left(\frac{R}{m}\right)^{n-\mathfrak{p}+1}\mathbf{D}_{\mathfrak{p}}(S^{\mathfrak{p}},(\mathcal{N}^{\mathfrak{p}-1}_{(R,r)}\times I)\times\mathbb{R}^N)\,,$$

whence, on account of (A.44), in terms of

$$c\sum_{k=\mathfrak{p}-2}^{n-1}\left(\frac{R}{m}\right)^{n-k}\mathbf{D}_{\mathfrak{p}}(\widetilde{T}^k_R,\Sigma^k_R)$$

and hence, using (A.38), by $c\,(R/m)\,\mathcal{E}_{1/\mathfrak{p}}(\langle T,d_0,R\rangle)$ and definitively, by the second inequality in (A.7), and taking σ small so that $\sigma^{\alpha(n,\mathfrak{p})}<1/\theta_T(0_{\mathbb{R}^n})$, in terms of the right-hand side of (A.9).

As to the flat distance, we observe that

$$\mathbf{F}((\widehat{T}-G_{y_0})\llcorner(B^n_R\setminus\overline{B}^n_r)\times\mathcal{Y})\le c\left(\mathbf{D}_{\mathfrak{p}}(\overline{T},(B^n_R\setminus B^n_r)\times I)\times\mathbb{R}^N)+\|U_{\overline{T}}-y_0\|_{L^1((B^n_R\setminus B^n_r)\times[0,R/m])}\right),$$

where $U_{\overline{T}}$ is the $W^{1,\mathfrak{p}}$-function corresponding to $\overline{T}$, i.e., such that $\overline{T}=G_{U_{\overline{T}}}$ in $((B^n_R\setminus B^n_r)\times]0,1])\times\mathbb{R}^N$. Since $R/m<1$, we have

$$\|U_{\overline{T}}-y_0\|^{\mathfrak{p}}_{L^1((B^n_R\setminus B^n_r)\times[0,R/m])}\le\|U_{\overline{T}}-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}((B^n_R\setminus B^n_r)\times[0,R/m])}\,.$$

Moreover, by the construction, arguing as above it is readily checked that

$$\begin{aligned}\|U_{\overline{T}}-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}((B^n_R\setminus B^n_r)\times[0,R/m])}\ &\le\ c\sum_{k=\mathfrak{p}-2}^{n-1}\left(\frac{R}{m}\right)^{n-k}\|U_T-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(\Sigma^k_R\times[0,R/m])}\\ &+\ \left(\frac{R}{m}\right)^{n-\mathfrak{p}+1}\sum_{F\in\Phi_m(\Sigma^{\mathfrak{p}-1}_R)}\|v_F-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(F\times[0,R/m])}\,.\end{aligned}$$

Therefore, by (A.37) and (A.45)

$$\|U_{\overline{T}}-y_0\|^{\mathfrak{p}}_{L^{\mathfrak{p}}((B^n_R\setminus B^n_r)\times[0,R/m])}\le c\int_{\partial B^n_R\times[0,R]}|U_T(x)-y_0|^{\mathfrak{p}}\,d\mathcal{H}^n+c\,{\varepsilon_\sigma}^{\mathfrak{p}}\,R^{n+1}\,m^{-2}\,.$$

Finally, since $c\,\sigma R^n + c\,\varepsilon_\sigma{}^{\mathfrak{p}}\, R^{n+1}\, m^{-2} \le c\,\varepsilon_\sigma\, R^{n-\mathfrak{p}+1}$, by (A.9) and the third inequality in (A.7) we obtain (A.10). □

PROOF OF PROPOSITION A.3: For $0<\delta\ll\delta_0$, let $\psi_\delta : \Omega_\delta \setminus \widetilde{\Omega}_\delta \to \Omega_{(r,\delta)}$ be the bijective map

$$\psi_\delta(\widetilde{x},\widehat{x},t) := \left(\widetilde{x}, \left(2 - \frac{\varphi_\delta(y(\widetilde{x}))}{\rho}\right)(\widehat{x},t)\right).$$

Similarly to [17, Sec. 5.5], letting

$$\widetilde{T}_1 := ((\psi_\delta)^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^N})_\#(\overline{T}_U \llcorner (\mathrm{int}(\Omega_{(r,\delta)}) \times \mathbb{R}^N))$$

we see that $\widetilde{T}_1$ agrees on $\mathrm{int}(\Omega_\delta\setminus\widetilde{\Omega}_\delta)\times\mathbb{R}^N$ with the graph of the $W^{1,\mathfrak{p}}$-map $W := U\circ\psi_\delta : (\Omega_\delta\setminus\widetilde{\Omega}_\delta)\to\mathbb{R}^N$. Moreover, setting $T_1 := (-1)^{n-1}\partial\widetilde{T}_1$ on $\mathcal{Z}^{n,\mathfrak{p}-1}(((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x)\times\mathcal{Y})$, we have $T_1\in\mathrm{cart}^{1/\mathfrak{p}}(((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x)\times\mathcal{Y})$ and

$$\mu_{T_1}((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x) \le \mu_T(\mathrm{int}(\Omega_{(r,\delta)})\cap\mathbb{R}^n_x).$$

We now define $w : (\Omega_\delta\setminus\widetilde{\Omega}_\delta)\to\mathbb{R}^N$ by

$$w(x,t) := \left(\frac{2\rho}{\varphi_\delta(y(\widetilde{x}))} - 1\right)\cdot W(x,t) + \left(2 - \frac{2\rho}{\varphi_\delta(y(\widetilde{x}))}\right)\cdot y_0 .$$

Since the image of $\Omega_\delta\setminus\widetilde{\Omega}_\delta$ by W is contained in $\overline{B}^N(y_0,\varepsilon_\sigma)$, and ε_σ is small with σ, we infer that the energy $\mathbf{D}_\mathfrak{p}(w,\Omega_\delta\setminus\widetilde{\Omega}_\delta)$ is small if δ and σ are small. Moreover, the restriction of w to $(\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x$ belongs to $W^{1/\mathfrak{p}}((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x,\mathcal{Y})$.

We then may and do define a current $\widetilde{T}_2\in\mathcal{D}^{n+1,\mathfrak{p}}(\mathrm{int}(\Omega_\delta\setminus\widetilde{\Omega}_\delta)\times\mathbb{R}^N)$ that agrees on $(\mathrm{int}(\Omega_\delta\setminus\widetilde{\Omega}_\delta)\setminus\mathbb{R}^n_x)\times\mathbb{R}^N$ with the graph of the $W^{1,\mathfrak{p}}$-map w and satisfies the energy estimate

$$\mathbf{D}_\mathfrak{p}(\widetilde{T}_2,(\mathrm{int}(\Omega_\delta\setminus\widetilde{\Omega}_\delta)\setminus\mathbb{R}^n_x)\times\mathbb{R}^N) \le c\,\sigma\, r^{n-\mathfrak{p}+1} + c\,\mu_T(\Omega_{(r,\delta)}\cap\mathbb{R}^n_x),$$

taking δ small, and the boundary condition

$$\partial\widetilde{T}_2 = \partial(\overline{T}\llcorner\Omega_\delta\times\mathbb{R}^N) - [\![\,\partial\widetilde{\Omega}_\delta\,]\!]\times\delta_{y_0} - [\![\,\partial D_r\times\{0_{\mathbb{R}^\mathfrak{p}}\}\,]\!]\times R_q .$$

Notice that, setting $T_2 := (-1)^{n-1}\partial\widetilde{T}_2$ on $\mathcal{Z}^{n,\mathfrak{p}-1}(((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x)\times\mathcal{Y})$, the current T_2 belongs to $\mathrm{cart}^{1/\mathfrak{p}}(((\Omega_\delta\setminus\widetilde{\Omega}_\delta)\cap\mathbb{R}^n_x)\times\mathcal{Y})$. We finally set

$$\widetilde{T} := \overline{T}_U\llcorner((B^n_r\times I)\setminus\mathrm{int}(\Omega_\delta))\times\mathbb{R}^N + \widetilde{T}_2\llcorner(\mathrm{int}(\Omega_\delta)\setminus\widetilde{\Omega}_\delta)\times\mathbb{R}^N .$$

Properties i)–iii) in Proposition A.3 readily follow, for $\delta>0$ small. □

PROOF OF LEMMA A.6: Similarly to e.g. [10, Vol. I, Sec. 3.1.1], but this time making use of the "maximal" function

$$\widetilde{M}(u)(x) := \sup_{r>0} ⨍_{B^n(x,\rho)\times I} |u|\,d\mathcal{H}^{n+1},$$

we obtain that for every $u\in L^\mathfrak{p}(\mathbb{R}^n\times I)$

$$\mathcal{H}^n(\{x\in\mathbb{R}^n \mid \widetilde{M}(u)(x)>t\}) \le \frac{6^n}{t}\int_{\Omega_t}\int_I |u(x,y)|\,dy\,dx,$$

where $\Omega_t := \{x\in\mathbb{R}^n \mid \int_I |u(x,y)|\,dy > t/2\}$. This yields that $\widetilde{M}(u)(x)$ is finite $\mathcal{H}^n$-a.e.. Moreover, for $\mathcal{H}^n$-a.e. x we have

$$\left|\int_I u(x,y)\,dy\right| \le \widetilde{M}(u)(x) \qquad\text{and}\qquad \lim_{\rho\to 0}\int_I ⨍_{B^n(x,\rho)} |u(x,t)-u(z,t)|\,dz\,dy = 0 .$$

This clearly yields the assertion, by choosing $u := D_iU^j$. □

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