SHAPE OPTIMIZATION PROBLEMS FOR EIGENVALUES OF ELLIPTIC OPERATORS

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ABSTRACT. We consider a general formulation for shape optimization problems involving the eigenvalues of the Laplace operator. Both the cases of Dirichlet and Neumann conditions on the free boundary are studied. We survey the most recent results concerning the existence of optimal domains, and list some conjectures and open problems. Some open problems are supported by efficient numerical computations.

Keywords: Shape optimization, Eigenvalues, Laplace operator, Finite elements method.

AMS Classification Numbers: 49J45, 49Q10, 49R50

1. INTRODUCTION

The study of shape optimization problems for the eigenvalues of an elliptic operator is a fascinating field that has strong relations with several applications as for instance the stability of vibrating bodies, the propagation of waves in composite media, the thermic insulation of conductors. In its mathematical formulation the problem consists in taking an elliptic operator (the Laplacian $-\Delta$ for instance) and considering its eigenvalues λ_k as functions of the domain Ω where the problem is solved; we have then the sequence

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \lambda_{k+1}(\Omega) \le \dots$$

where each $\lambda_k(\Omega)$ is counted with its multiplicity. Given a function $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ we consider the optimization problem

$$\min\left\{\Phi\big(\lambda(\Omega)\big)\ :\ \Omega \subset D,\ |\Omega| \le m\right\} \tag{1.1}$$

where $\lambda(\Omega)$ stands for the sequence $\big(\lambda_k(\Omega)\big)_{k\in\mathbb{N}}$, the design region D is fixed, and $m > 0$. The generic question we try to answer is: *does problem* (1.1) *have a solution*?

In Section 2 we survey the existence of optimal solutions for problem (1.1) when Dirichlet conditions are imposed on the free boundary $\partial\Omega$. We shall see that problem (1.1) is well posed in the following situations:

- when the competing domains Ω are subjected to some suitable geometrical restrictions;
- when the cost function Φ satisfies some suitable monotonicity assumptions;
- in some very special cases, like for instance the one where Φ depends only on the low part of the spectrum (e.g. on the first two eigenvalues $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$).

Most of the classical isoperimetric problems of the literature fit into these general frames.

In Section 3 we consider the same problem for the case of Neumann conditions on the free boundary $\partial\Omega$; we denote by $\mu_k(\Omega)$ the eigenvalues in this case. Due to the failure of the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ for non smooth sets and to the lack of monotonicity

$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \mu_k(\Omega_2) \le \mu_k(\Omega_1)$$

(which, on the contrary, holds in the Dirichlet case), the question of the existence of optimal solutions is much more delicate, and up to now only very few results are available. In fact, proving the existence of a solution for such shape optimization problems is related to the well understanding of the behavior of the Neumann spectrum on highly oscillating boundaries. This is a challenging question and it is largely open (see [15] for recent advances in this direction). We present some open problems and conjectures, partially supported by some numerical computations. In some particular cases of highly oscillating boundaries, these computations allow us to formally identify the problems with the limit spectra.

2. Dirichlet eigenvalues

In this section we deal with optimization problems for functionals involving the eigenvalues of elliptic operators when a Dirichlet condition is imposed on the free boundary.

2.1. **Eigenvalues in Hilbert spaces.** We start by recalling some basic facts about eigenvalues of operators in Hilbert spaces. If H is a Hilbert space and $R : H \to H$ is a bounded linear operator which is compact, self-adjoint and nonnegative, the spectrum of R consists only on eigenvalues, which can be ordered (counting their multiplicities):

$$0 \le \cdots \le \Lambda_{n+1}(R) \le \Lambda_n(R) \le \cdots \le \Lambda_1(R).$$

For every integer n, the eigenvalue $\Lambda_n(R)$ is given by the formula

$$\Lambda_n(R) = \min_{\phi_1,\dots,\phi_{n-1}\in H} \max_{\substack{|\phi| = 1 \\ (\phi,\phi_1) = \cdots = (\phi,\phi_{n-1}) = 0}} |R\phi|,$$

where $(\cdot,\cdot)$ is the scalar product of H and $|\cdot|$ is the corresponding norm. According to the Courant-Fischer theorem, for every integer n the following equality holds:

$$\Lambda_n(R) = \max_{E\in S_n} \min_{\phi\in E, |\phi|=1} |R\phi|,$$

where S_n denotes the family of all subspaces of H of dimension n. Moreover, the Rayleigh min-max formula provides the following equivalent formulation:

$$\Lambda_n(R) = \min_{\phi_1,\dots,\phi_{n-1}\in H} \max_{\substack{|\phi|=1 \\ (\phi,\phi_1)=\cdots=(\phi,\phi_{n-1})=0}} \frac{(R\phi,\phi)}{|\phi|^2}.$$

We recall the following general results from [19, Corollaries XI.9.3 and XI.9.4].

Theorem 2.1. *Let R_1, R_2 be compact, self-adjoint and non-negative operators on H. For every $m, n \geq 1$ we have*

(1) $\Lambda_{m+n-1}(R_1 + R_2) \leq \Lambda_m(R_1) + \Lambda_n(R_2)$,
(2) $\Lambda_{m+n-1}(R_1R_2) \leq \Lambda_m(R_1)\Lambda_n(R_2)$
(3) $|\Lambda_n(R_1) - \Lambda_n(R_2)| \leq |R_1 - R_2|_{\mathcal{L}(H)}$

where $|R|_{\mathcal{L}(H)}$ denotes the usual operator norm

$$|R|_{\mathcal{L}(H)} = \sup\{|R\phi| : |\phi| \leq 1\}.$$

2.2. **Eigenvalues of elliptic operators.** We are interested in the study of optimization problems for functions of eigenvalues of some elliptic operators. For simplicity we consider here the Laplace operator $-\Delta$, even if several conclusions can be extended to elliptic operators of a more general form. More precisely, we consider a bounded Lipschitz domain D of $\mathbb{R}^N$ and, for every quasi-open subset Ω of D, the elliptic operator $-\Delta$ defined on the Sobolev space $H^1_0(\Omega)$ of functions of $H^1_0(D)$ which vanish quasi-everywhere on $D \setminus \Omega$. The equation

$$-\Delta u = f \text{ in } \Omega \qquad u \in H^1_0(\Omega), \tag{2.1}$$

intended in its weak form

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega f\phi \, dx \qquad \forall \phi \in H^1_0(\Omega),$$

then provides the resolvent operator R_Ω which associates to every $f \in L^2(\Omega)$ the unique solution u. Since D is bounded, the Sobolev spaces $H^1_0(\Omega)$ are compactly embedded into $L^2(\Omega)$; consequently, R_Ω is well defined, compact, non-negative and self-adjoint. We define the eigenvalues $\lambda_k(\Omega)$ of the elliptic operator $-\Delta$ on $H^1_0(\Omega)$ by setting

$$\lambda_k(\Omega) = \frac{1}{\Lambda_k(R_\Omega)}.$$

In this way, for every $n \geq 1$ there exists $u \in H^1_0(\Omega) \setminus \{0\}$ such that

$$-\Delta u = \lambda_n(\Omega) u.$$

Moreover, the Rayleigh formula can be used, and we obtain

$$\begin{aligned}
\lambda_n(\Omega) &= \max_{\phi_1,\dots,\phi_{n-1}\in L^2(\Omega)} \min_{\substack{\phi \in H_0^1(\Omega)\setminus\{0\} \\ (\phi,\phi_1)=\dots=(\phi,\phi_{n-1})=0}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx}{\displaystyle\int_\Omega \phi^2\,dx} \\
&= \min_{E\in S_n} \max_{\phi\in E\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx}{\displaystyle\int_\Omega \phi^2\,dx}\,.
\end{aligned}$$

Here $(\cdot,\cdot)$ is the scalar product in $L^2(D)$ and S_n denotes the family of subspaces of $H_0^1(\Omega)$ of dimension n. As a direct consequence of the Rayleigh formula we obtain the following monotonicity result for the Dirichlet eigenvalues $\lambda_n(\Omega)$.

Proposition 2.2. *Let Ω_1, Ω_2 be two open (or quasi-open) sets of finite Lebesgue measure. If $\Omega_2 \subset \Omega_1$, then for every $n \geq 1$, $\lambda_n(\Omega_1) \leq \lambda_n(\Omega_2)$.*

At this stage we have obtained the eigenvalues

$$0 < \lambda_1(\Omega) \leq \dots \leq \lambda_n(\Omega) \leq \lambda_{n+1}(\Omega) \leq \dots \to +\infty,$$

and we denote by $\lambda(\Omega)$ the sequence $\big(\lambda_n(\Omega)\big)_{n\in\mathbb{N}}$. Since D is bounded, all eigenvalues $\lambda_k(\Omega)$ are bounded from below by $\lambda_1(D)$. Therefore, by Theorem 2.1, we have that $\lambda(\Omega_n) \to \lambda(\Omega)$ (i.e. $\lambda_k(\Omega_n) \to \lambda_k(\Omega)$ for every $k \in \mathbb{N}$) as soon as the resolvent operators $R_{\Omega_n} \to R_\Omega$ in the operator norm of $\mathcal{L}(L^2(D))$. In other words, we have that $\lambda(\Omega_n) \to \lambda(\Omega)$ whenever the domains Ω_n and Ω satisfy the condition:

$$f_n \to f \text{ weakly in } L^2(D) \quad \Rightarrow \quad u_{\Omega_n}(f_n) \to u_\Omega(f) \text{ in } L^2(D)$$

being $u_{\Omega_n}(f_n)$ and $u_\Omega(f)$ the solutions of (2.1) with Ω_n, f_n and Ω, f respectively.

Remark 2.3. It is possible to show (see for instance [7] and references therein) that, in order to obtain the convergence of the resolvent operators in the operator norm of $\mathcal{L}(L^2(D))$, it is enough to have the convergence of the solutions above only when $f_n = f = 1$.

Remark 2.4. It is important to stress that the convergence $\Omega_n \to \Omega$ defined above is not compact; in other words it is possible to construct sequences $(\Omega_n)_{n\in\mathbb{N}}$ of (smooth) subsets of D such that the corresponding solutions $u_{\Omega_n}(f)$ converge to some function $u(f)$ which is not of the form $u_\Omega(f)$ for any subset Ω of D. This is the essential reason why in general the shape optimization problems with Dirichlet conditions on the free boundary do not have a solution. A relaxation procedure is then needed to study the asymptotic behaviour of minimizing sequences, and the characterisation of the relaxed problems involves the study of equations like (2.1) with a lower order term which contains a measure of capacitary type. We do not want to detail here this delicate field, and we refer the interested reader to the book

[7] and to the several papers quoted therein. Just notice that the knowledge of the relaxed form is crucial for the most part of numerical methods of identifying optimal shapes.

2.3. **The optimization problem.** Given a function $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ we consider the optimization problem

$$\min \left\{ \Phi\big(\lambda(\Omega)\big) \ : \ \Omega \subset D, \ |\Omega| \leq m \right\} \tag{2.2}$$

where $\lambda(\Omega)$ is the spectrum defined above and $|\Omega|$ denotes the Lebesgue measure of Ω. In particular, we may consider optimization problems involving only the first k eigenvalues:

$$\min \left\{ \Phi\big(\lambda_1(\Omega), \dots, \lambda_k(\Omega)\big) \ : \ \Omega \subset D, \ |\Omega| \leq m \right\}$$

where $\Phi : \mathbb{R}^k \to [0, +\infty]$. Some special cases of the problem above are well known in the literature. For instance see [1, 2] for a detailed presentation of the results below:

- if $\Phi(\lambda) = \lambda_1$ and D is large enough (to contain a ball of volume m) then the optimal domain Ω for the problem (2.2) is any ball of volume m;
- if $\Phi(\lambda) = \lambda_2$ and D is large enough (to contain two disjoint balls of volume $m/2$ each) then the optimal domain Ω for the problem (2.2) is any array of two disjoint balls of volume $m/2$ each;
- if $\Phi(\lambda) = \lambda_1/\lambda_2$ and D is large enough (to contain a ball of volume m) then the optimal domain Ω for the problem (2.2) is any ball of volume m.

We stress that numerical computations (see for instance [22, 27]) give a proof to the fact that the optimal solution of problem (2.2) is not always an array of disjoint balls. This is the case for $\Phi(\lambda) = \lambda_5$ or $\Phi(\lambda) = \lambda_6$.

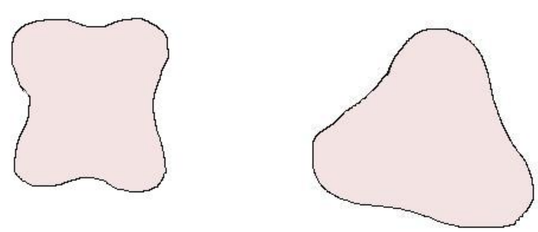

FIGURE 1. Numerical plot of the optimal sets for λ_5 and λ_6.

2.4. **Existence under geometrical constraints.** As stated in Remark 2.4, in general, without further restrictions on the class of admissible domains, or additional assumptions on the cost functionals, the optimization problem (2.2) does not admit a solution. In fact, a minimizing sequence for problem (2.2) always exists, and one may try to understand what is the "geometrical" behaviour of this sequence. The non existence issue is related to appearance of high oscillations and small holes during the minimization process. A particular attention has to be given in the case $D = \mathbb{R}^N$, where phenomena of concentration-compactness, dichotomy and vanishing type may appear.

In this subsection we list some classes of domains which, due to some geometrical constraints that are imposed, are compact for the convergence of resolvent operators. The compactness is obtained (see [7] for further details and references) by showing that, under suitable geometrical constraints, the convergence of resolvent operators is equivalent to the so called Hausdorff complementary convergence

$$\Omega_n \to \Omega \text{ in } H^c \quad \iff \quad \overline{D} \setminus \Omega_n \to \overline{D} \setminus \Omega \text{ in the Hausdorff sense.}$$

As it is well known, the H^c convergence is compact on the family of open subsets of D. The classes of domains in which the convergence of resolvent operators is equivalent to the H^c-convergence are the following (from the strongest constraints to the weakest ones).

- The class $\mathcal{A}_{convex}$ of all open convex subsets of D.
- The class $\mathcal{A}_{unif\ cone}$ of all open subsets of D satisfying a uniform exterior cone property (see Chenais [18]), i.e. such that for every point x_0 on the boundary of every $\Omega \in \mathcal{A}_{unif\ cone}$ there is a closed cone, with uniform height and opening, and with vertex in x_0, lying in the complement of Ω.
- The class $\mathcal{A}_{unif\ flat\ cone}$ of open subsets of D satisfying a uniform flat cone condition (see Bucur, Zolésio [12]), i.e. as above, but with the weaker requirement that the cone may be flat, that is of dimension $N-1$.
- The class $\mathcal{A}_{cap\ density}$ of all open subsets of D satisfying a uniform capacitary density condition (see [12]), i.e. such that there exist $c, r > 0$ such that for every $\Omega \in \mathcal{A}_{cap\ density}$, and for every $x \in \partial\Omega$, we have
$$\forall t \in (0, r) \quad \frac{\operatorname{cap}(B_{x,t} \setminus \Omega, B_{x,2t})}{\operatorname{cap}(B_{x,t}, B_{x,2t})} \geq c,$$
where $B_{x,s}$ denotes the ball of radius s centered at x.
- The class $\mathcal{A}_{unif\ Wiener}$ of all open subsets of D satisfying a uniform Wiener condition (see [13]), i.e. such that for every $\Omega \in \mathcal{A}_{unif\ Wiener}$ and for every point $x \in \partial\Omega$
$$\int_r^R \frac{\operatorname{cap}(B_{x,t} \setminus \Omega, B_{x,2t})}{\operatorname{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r, R, x) \quad \text{for every} \quad 0 < r < R < 1$$
where $g : (0,1) \times (0,1) \times D \to \mathbb{R}_+$ is fixed, such that for every $R \in (0,1)$ it is $\lim_{r \to 0} g(r, R, x) = +\infty$ locally uniformly on x.

Another interesting class, which is only of topological type and is not contained in any of the previous ones, was given by Šverák [25] and consists in the following.

- For $N = 2$, the class of all open subsets Ω of D for which the number of connected components of $\overline{D} \setminus \Omega$ is uniformly bounded.

As said above, the following inclusions can be established:

$$\mathcal{A}_{convex} \subset \mathcal{A}_{unif\ cone} \subset \mathcal{A}_{unif\ flat\ cone} \subset \mathcal{A}_{cap\ density} \subset \mathcal{A}_{unif\ Wiener}.$$

Putting together the previous results we obtain the following existence theorem.

Theorem 2.5. *Let $\mathcal{A}$ be one of the classes above and let $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ be a lower semicontinuous function, in the sense that*

$$\Phi(\lambda_\infty) \le \liminf_{n\to+\infty} \Phi(\lambda_n) \quad \text{whenever } \lambda_{n,k} \to \lambda_{\infty,k} \text{ for every } k \in \mathbb{N}. \tag{2.3}$$

Then the optimization problem

$$\min\left\{\Phi\big(\lambda(\Omega)\big) \ :\ \Omega \in \mathcal{A},\ |\Omega| \le m\right\} \tag{2.4}$$

admits a solution.

2.5. **Existence under monotonicity assumptions.** In this subsection we consider the optimization problem (2.2) without any further geometrical restrictions on the domains. This is the natural frame in which the classical isoperimetric problems are considered. In this case, in order to provide the existence of an optimal domain, some monotonicity conditions on the cost functional have to be assumed.

Definition 2.6. *We say that a function $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ is monotone nondecreasing if*

$$\lambda_k \le \nu_k \text{ for every } k \in \mathbb{N} \quad \Rightarrow \quad \Phi(\lambda) \le \Phi(\nu). \tag{2.5}$$

By using the monotonicity of Dirichlet eigenvalues seen in Proposition 2.2 it is possible to obtain the following existence result (first obtained by Buttazzo and Dal Maso in [16]).

Theorem 2.7. *Let $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ be a function which is lower semicontinuous in the sense of (2.3) and monotone in the sense of (2.5). Then the optimization problem (2.2) admits a solution.*

In particular, every function $\Phi : \mathbb{R}^k \to [0, +\infty]$, which is lower semicontinuous and monotone nondecreasing in each of its variables $(\lambda_1, \dots, \lambda_k)$, provides a minimization problem which admits an optimal solution. For instance, if $k \in \mathbb{N}$

$$\Phi(\lambda) = \lambda_k\ , \qquad \Phi(\lambda) = \sum_{i=1}^k \lambda_i\ , \qquad \Phi(\lambda) = \sum_{i=1}^k \lambda_i^2$$

give three examples of well posed optimization problems of the form (2.2), for every $m > 0$ and every design region D.

2.6. **The case of the first two eigenvalues.** In this subsection we consider the very special case of optimization problems involving only the first two eigenvalues of the Laplace operator. In other words, we deal with minimization problems of the form

$$\min\left\{\Phi\big((\lambda_1(\Omega),\lambda_2(\Omega)\big)\ :\ \Omega\subset D,\ |\Omega|\le m\right\} \tag{2.6}$$

It is convenient to introduce the *spot function* $s:\{\Omega\subset D\}\to\mathbb{R}^2$ defined by

$$s(\Omega)=\big(\lambda_1(\Omega),\lambda_2(\Omega)\big)$$

and its image

$$E=\left\{(x,y)\in\mathbb{R}^2\ :\ (x,y)=s(\Omega)\text{ for some }\Omega\subset D,\ |\Omega|\le m\right\}.$$

The optimization problem (2.6) can be then rewritten in the form

$$\min\left\{\Phi(x,y)\ :\ (x,y)\in E\right\}.$$

It becomes then crucial to know whether the set E is closed or not in $\mathbb{R}^2$. The following result has been proved in [8].

Theorem 2.8. *Let $m>0$ and assume that D is large enough to contain a ball of volume $2^{N-1}m$. Then the set E above is closed in $\mathbb{R}^2$. As a consequence, the optimization problem (2.6) admits a solution for every lower semicontinuous function $\Phi:\mathbb{R}^2\to[0,+\infty]$.*

We stress that in the theorem above, due to the very special form of the cost functional, no monotonicity assumptions are required. For instance, we may obtain the existence of an optimal domain which minimizes the quantity

$$\frac{\lambda_1^2(\Omega)+\lambda_2^2(\Omega)}{\lambda_1(\Omega)+\lambda_2(\Omega)}\,.$$

The numerical study of the set E, in the case $N=2$, has been performed by Wolf and Keller in [27] where the following picture for E is obtained (see Figure 2).

2.7. **Further remarks and open problems.** In the subsections above we have considered optimization problems for the Dirichlet eigenvalues of the Laplace operator, and we have seen that in several cases an optimal domain exists. However, some open questions remain, if some of the previous assumptions fail. Here we list some of them; a more complete list of open problems related to eigenvalues optimization can be found for instance in [1].

- A first important issue deals with the regularity of optimal shapes. It would be interesting to investigate about the regularity of the optimal domains of problems like (2.2). In fact, the general result of Theorem 2.7 only provides optimal domains which are quasi-open subsets of D. In the case when $\Phi(\lambda)=\lambda_1$, by using variational methods, it is known (see for instance [20]) that the minimizers are in fact open sets. Under the additional constraint of convexity for the

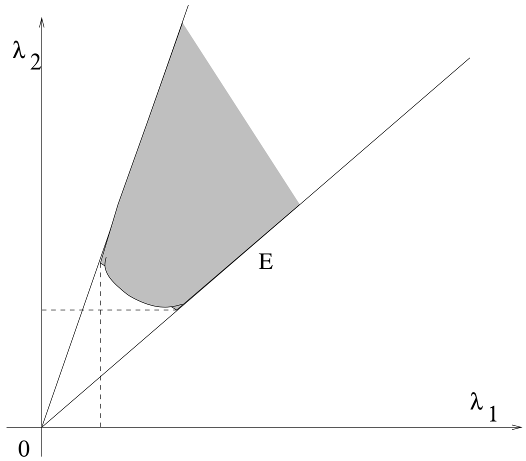

FIGURE 2. The set E.

competing domains, a C^1-regularity result for the optimal domains can be proved (see [6]).

- A second question is related to the fact that the design region D is assumed to be bounded. If we drop this condition, most of the existence results obtained above fail. However, in [10] it is proved that the problem

$$\min\left\{\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^N,\ |\Omega|=m\right\}$$

has a solution for $k=3$. In two dimensions, the disk is suspected to be the solution (see [1]); up to now, as far as we know, this is still a conjecture. The existence of an optimal domain for the problem above in the case $k\geq 4$ is not solved. Roughly speaking, if one proves the existence of bounded minimizers for the cases $k=3,\dots,M$, then it is possible to obtain the existence of a minimizer (bounded or unbounded) for the case $k=M+1$.

- Concerning Theorem 2.8, there are many other questions which can be raised. Is the set E convex? Is E still closed if the pair (λ_1,λ_2) is replaced by (λ_i,λ_j), or more generally if we consider the set

$$E_K=\Big\{(\lambda_i(\Omega))_{i\in K}\ :\ \Omega\subset D,\ |\Omega|\leq m\Big\}$$

where K is a given subset of positive integers? Are the sets Ω on the boundary of E smooth? When are they convex? If the design region is a general open set D, is the set E still closed? Or if the Laplace operator is replaced by another elliptic operator of the form

$$L=-\partial_i(a_{ij}\partial_j)+b_i\partial_i+c\ \ ?$$

- For every $k \geq 2$ prove (or disprove) that if Ω^* is a solution of
$$\min\left\{\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^N,\ |\Omega|\leq m\right\},$$
then $\lambda_k(\Omega^*)=\lambda_{k-1}(\Omega^*)$, i.e. on the optimal domain, the k-th eigenvalue is not simple, and equals the one of lower order. This happens for $k=2$ and in view on the conjectured optimum for λ_3, this should also hold when $k=3$. Moreover, the numerical computations of Oudet [22] for several values of k support the conjecture.

- **The clamped plate.** Rayleigh conjectured that the ball minimizes the first eigenvalue of the clamped plate $\Gamma_1(\Omega)$ among all domains of prescribed measure. For a bounded open set Ω, $\Gamma_1(\Omega)$ is defined by
$$\Gamma_1(\Omega)=\min_{u\in H^2_0(\Omega)u\neq 0}\frac{\int_\Omega|\Delta u|^2dx}{\int_\Omega|u|^2dx}.$$
The minimizer u of the Rayleigh quotient above satisfies the equation
$$\begin{cases}\Delta^2 u= & \Gamma_1(\Omega)\ u\ \text{ in }\ \Omega\\ u= & 0\ \ \partial\Omega\\ \frac{\partial u}{\partial n}= & 0\ \ \partial\Omega.\end{cases}$$
This problem was solved in dimension 2 by Nadirashvili [21] and in dimension 3 by Ashbaugh and Benguria [3], but is still open for $N\geq 4$. For a more detailed description of the problem we refer the reader to [1, 2].

- **The buckling load of a clamped plate.** Consider a uniform compressive force acting on the normal direction on every point of the boundary of a plate (in the two dimensional space). The critical buckling load is related to the following eigenvalue of the bi-Laplacian operator with Dirichlet boundary conditions:
$$\Lambda_1(\Omega)=\min_{u\in H^2_0(\Omega)u\neq 0}\frac{\int_\Omega|\Delta u|^2dx}{\int_\Omega|\nabla u|^2dx}.$$
This time, the minimizer u of the Rayleigh quotient above satisfies the equation
$$\begin{cases}\Delta^2 u= & -\Lambda_1(\Omega)\ \Delta u\ \text{ in }\ \Omega\\ u= & 0\ \ \partial\Omega\\ \frac{\partial u}{\partial n}= & 0\ \ \partial\Omega.\end{cases}$$
Pólya and Szegö conjectured in 1950 that the solution of the problem
$$\min_{|\Omega|=m}\Lambda_1(\Omega)$$
is the ball (see [1] for a detailed presentation of the problem). This problem is still open despite a series of partial results. We notice the idea of Willms and Weinberger [26] which, roughly speaking, gives a solution to the problem provided that one proves that a smooth

simply connected set solves the problem. The existence in the simply connected class was proved in [4], but the regularity of the optimum is still unknown.

3. Neumann eigenvalues

For every bounded open set Ω such that the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact the resolvent operator associated to the Neumann Laplacian fits into the general frame of Subsection 2.1. Then the spectrum of the Neumann Laplacian consists only on eigenvalues:

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \leq \mu_k(\Omega) \leq \ldots \to +\infty.$$

In this case, for every $k \in \mathbb{N}$, there exists $u_k \in H^1(\Omega) \setminus \{0\}$ such that, in the weak variational sense,

$$\begin{cases} - \triangle u_k = \mu_k(\Omega) u_k \text{ in } \Omega \\ \frac{\partial u_k}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

i.e. for every $\phi \in H^1(\Omega)$

$$\int_\Omega \nabla u_k \nabla \phi dx = \mu_k(\Omega) \int_\Omega u\phi dx.$$

3.1. **Existence results.**

Let $\Phi : \mathbb{R}^k_+ \mapsto \mathbb{R}$ and let $\mathcal{U}_{ad}$ be the family of smooth open subsets of a bounded design region D with prescribed measure. The smoothness is required for the compactness of the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$. The generic shape optimization problem we consider in this section is

$$\max_{\Omega \in \mathcal{U}_{ad}} \Phi(\mu_1(\Omega), \ldots, \mu_k(\Omega)). \tag{3.2}$$

Formally, we prefer in this section to write the generic shape optimization problem as a maximization problem, since the natural problem for the k-th eigenvalue μ_k is the maximization over domains of prescribed measure. On the contrary, the minimum of $\mu_k(\Omega)$ in $\mathcal{U}_{ad}$ is zero: in fact, to see this, is is enough to take an admissible set with k connected components. If we take the subclass of $\mathcal{U}_{ad}$ made by connected sets, the infimum is still zero, being every set approached by cracked membranes with a "dense" crack.

As examples of optimal shapes, we refer to Weinberger [24] and Szegö [23] for the following results:

(1) The ball is the unique solution of

$$\max\{\mu_2(\Omega) : \Omega \subseteq \mathbb{R}^N,\ \Omega \text{ smooth }, |\Omega| = m\}. \tag{3.3}$$

(2) The ball is the unique solution of

$$\min\{\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} : \Omega \subseteq \mathbb{R}^2,\ \Omega \, \text{simply connected and smooth }, |\Omega| = m\}. \tag{3.4}$$

The proof of these result relies on direct comparison between an arbitrary open set and the ball of the same measure. The reader is referred to [1] for a list of open problems with isoperimetric inequalities for eigenvalues of the Neumann Laplacian.

In order to prove the existence of a solution for the shape optimization problems involving the Neumann eigenvalues, we have in mind the direct methods of the calculus of variations, which rely on the continuity of the spectrum for particular geometric perturbations. Contrary to the case of Dirichlet boundary conditions, the behavior of the eigenvalues of the Neumann Laplacian for non-smooth variations of the boundary of the geometric domain is (almost) uncontrollable. Several facts can explain this phenomenon, the most important being the one that for a non-smooth domain Ω, the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ may not be compact. Consequently, on a perturbed domain the spectrum of the Neumann-Laplacian may not consist only on eigenvalues and a small geometric perturbation of a smooth boundary may produce a large essential spectrum.

For shape optimization problems it is convenient to introduce the following relaxed values which can be defined on arbitrary open sets. Let Ω be a bounded open set. We define

$$\mu_k(\Omega) = \inf_{E \in S_k(\Omega)} \sup_{\phi \in E \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla \phi|^2 dx}{\displaystyle\int_\Omega \phi^2 dx}, \tag{3.5}$$

where $S_k(\Omega)$ is the family of all linear spaces of $H^1(\Omega)$ of dimension k. Of course, if Ω is smooth enough, then $\mu_k(\Omega)$ coincides with the k^{th} eigenvalue of the Neumann Laplacian. If Ω is not smooth, e.g. with an infinite number of connected components, then $\mu_k(\Omega) = 0$. Nevertheless, for a shape optimization problem where the k-th eigenvalue has to be maximized, such domains are ruled out by maximizing sequences.

In order to solve the shape optimization problem (3.2) for eigenvalues the following steps may be considered:

(1) consider problem (3.2) on smooth domains;
(2) replace the eigenvalues by the *relaxed values* and relax the problem to non-smooth domains;
(3) prove the existence of the optimal shape for the relaxed shape optimization problem;
(4) prove the regularity of an optimal shape, and recover true eigenvalues for the optimal shape.

Continuity of the Neumann eigenvalues for geometric convergence of the domains is not well understood. We refer the reader to [15] for recent fine results into this direction. Roughly speaking, if no geometrical constraints are imposed on the competing domains, the continuity of eigenvalues with respect to the convergence of domains can not be expected. Nevertheless, we can give the following weaker result (whose proof can be found in [7, Theorem 7.4.7]).

Theorem 3.1. *Let D be a bounded design region in $\mathbb{R}^N$ and let $\Omega_n, \Omega \subseteq D$ such that*

$$\forall \varphi \in H^1(\Omega),\ \exists \varphi_n \in H^1(\Omega_n)$$

$$(1_{\Omega_n}\varphi_n, 1_{\Omega_n}\nabla\varphi_n) \longrightarrow (1_\Omega\varphi, 1_\Omega\nabla\varphi) \text{ strongly in } L^2(D) \times L^2(D, \mathbb{R}^N). \tag{3.6}$$

Then, for every $k \in \mathbb{N}^*$, we have

$$\mu_k(\Omega) \geq \limsup_{n\to\infty} \mu_k(\Omega_n). \tag{3.7}$$

This result allows us to prove the following. The family $\mathcal{A}_{convex}$ is defined in Section 2 and $\mathcal{A}_{\varepsilon\ cone}$ stands for the family of all open subsets of $\mathcal{D}$ satisfying an ε-cone condition (see [18]).

Theorem 3.2. *Let D be a bounded open set and let $\Phi : \mathbb{R}_+^k \mapsto \mathbb{R}$ be an upper semicontinuous function which is nondecreasing in each variable. Then, the problem*

$$\max_{\Omega\in\mathcal{U}_{ad}} \Phi(\mu_1(\Omega), \ldots, \mu_k(\Omega)) \tag{3.8}$$

has at least one solution for $\mathcal{U}_{ad}$ being either $\mathcal{A}_{convex}$ or $\mathcal{A}_{\varepsilon\ cone}$.

Proof. The existence of a solution is a direct consequence of Theorem 3.1 and of the result of Chenais [18] on the existence of uniformly bounded extension operators from $H^1(\Omega)$ to $H^1(D)$. Indeed, consider a maximizing sequence $(\Omega_n)_n$ either in the class of convex sets or in the class of sets satisfying a uniform cone condition. Up to extracting a subsequence, one can assume that Ω_n converges in the Hausdorff complementary topology to an open set Ω. In order to verify (3.6) for every $\varphi \in H^1(\Omega)$ one can define $\varphi_n = P_n\varphi|_{\Omega_n}$, where $P_n : H^1(\Omega_n) \to H^1(D)$ is the extension operator with norm bounded by a constant depending only on D and on the cone. Then property (3.6) follows, and Ω turns out to be a maximizer for problem (3.8). □

If in Theorem 3.2 one drops the geometric conditions, it is not clear that a maximizing sequence will satisfy (3.6). For the Neumann problem, we recall the following result [11], where $\mathcal{O}_l(D)$ denotes the class of all open subsets Ω of D such that Ω^c has at most l connected components.

Theorem 3.3. *Let $N = 2$ and $\{\Omega_n\}_{n\in\mathbb{N}} \in \mathcal{O}_l(D)$ be such that $\Omega_n \xrightarrow{H^c} \Omega$. Then condition (3.6) is satisfied if and only if $|\Omega_n| \to |\Omega|$.*

Theorem 3.3 cannot be applied directly for proving existence of solutions for problem (3.8) in $\mathcal{O}_l(D)$. In fact $\mathcal{O}_l(D)$ is compact for the H^c-topology, but a priori there is no argument ensuring that for a maximizing sequence one gets the condition $|\Omega_n| \to |\Omega|$.

Theorem 3.4. *Let $D \subseteq \mathbb{R}^2$ be a bounded design region, c, l, M positive constants, and let us denote*

$$\mathcal{U}_{ad} = \{\Omega : \Omega \in \mathcal{O}_l(D),\ |\Omega| = c,\ \mathcal{H}^1(\partial\Omega) \leq M\}.$$

Then problem (3.8) *has at least one solution.*

Proof. The proof relies on Theorem 3.3 under the observation that if $\Omega_n \in \mathcal{O}_l(D)$ is such that $\mathcal{H}^1(\partial\Omega_n) \leq M$ and the number of connected components of $\partial\Omega_n$ is uniformly bounded, then $\Omega_n \stackrel{H^c}{\to} \Omega$ implies that $|\Omega_n| \to |\Omega|$ (see for instance [17]).

Let us consider a maximizing sequence (Ω_n). A priori, if $\sharp\Omega^c \leq l$, it is not true that $\sharp\partial\Omega$ is finite. Nevertheless, if the number of the connected components of Ω is less than or equal to k then the number of the connected components of $\partial\Omega$ is less than or equal to $k + l - 1$. Note that, unless the functional F is trivial, it is enough to search the maximum only among domains Ω which have less than k connected components. Indeed, if $\sharp\Omega = k$ then $\mu_1(\Omega) = \cdots = \mu_k(\Omega) = 0$, hence F is maximal on such a set.

Up to a subsequence we can assume that $\Omega_n \xrightarrow{H^c} \Omega$. Since $\sharp(\partial\Omega_n) \leq k + l - 1$ we get that $\mathcal{H}^1(\partial\Omega) \leq M$. The properties of the H^c-convergence for sets with uniformly bounded perimeter give (see [14]) $\Omega \in \mathcal{U}_{ad}$. Theorems 3.1 and 3.3 give that $\mu_i(\Omega) \geq \limsup_{n\to\infty} \mu_i(\Omega_n)$. We conclude the proof by using the upper semicontinuity and the monotonicity of F. $\square$

Remark 3.5. For problems (3.3) and (3.4) existence of the solution holds into the class $\mathcal{O}_l(D)$ with $l = 1$ without imposing any additional constraint on the competing shapes. It is natural to conjecture the following:

Conjecture 3.6. *For $N = 2$, problem* (3.8) *has a solution in $\mathcal{O}_l(D)$, for every $l \in \mathbb{N}$.*

Note that the proof of Theorem 3.4 could be adapted to Conjecture 3.6 provided that the maximizing sequence does not lose measure, i.e. $|\Omega_n| \to |\Omega|$. Somehow, one should rule out the possibility of losing measure by using the fact that the sequence is maximizing for the shape functional F. Losing measure is related to highly oscillating boundaries. Roughly speaking, one may think that every loop of an oscillation introduces small eigenvalues and consequently rules out this set from the maximizing sequence.

In the next subsection, we exhibit some situations where this kind of phenomena is observed and precisely described by accurate numerical computations. Into a typical situation of "losing measure" sequence one intuitively observes an accumulation of the spectrum below a precise value depending on the shape. Although the behaviour of the spectrum is in general unknown, these computations give a precise hint for identification of the degenerate bilinear form which has among its critical values the limits of the eigenvalues on the competing shapes.

In the following subsection we study numerically the behavior of the Neumann spectrum on some particular situations of sequences of domains "losing measure".

3.2. **Asymptotic behavior in domains with multiple cracks.** We introduce a numerical method for the approximation of eigenpairs of the Neumann Laplace operator in domains with cracks which lose mass asymptotically (see [5] for details concerning this method). The method is based on a mixed variational formulation which allows us to set the problem in the entire domain (including the cracks) and to express the crack conditions as purely functional constraints which is easier to handle from the computational point of view. Moreover, this formulation permits to extend in an easy way the main approximation results of the spectral theory to the case of non Lipschitz domains.

3.2.1. *Geometric setting and variational formulation.* Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary Γ, and $(\gamma_i)_i$, $1 \leq i \leq I$ a given number of Lipschitz continuous curves in Ω without self-intersections, such that $\Omega_\gamma = \Omega \setminus \bigcup_{i=1}^I \gamma_i$ is connected. We assume that there exists a finite family Ω_j, $1 \leq j \leq J$ of pairwise disjoint Lipschitz open subsets of Ω, and a set of zero Lebesgue measure Σ not intersecting any of the Ω_j such that

$$\Omega = \bigcup_{j=1}^J \Omega_j \cup \Sigma, \quad \bigcup_{i=1}^I \gamma_i \subset \Sigma. \tag{3.9}$$

For every i, $1 \leq i \leq I$, γ_i is supposed to be a part of the boundary of two subdomains $\Omega_{j1(i)}$ and $\Omega_{j2(i)}$.

Due to the compactness of the injection $H^1(\Omega_\gamma) \subset L^2(\Omega_\gamma)$, the spectral theory of compact operators summarized in Section 2.1 can be applied. The spectrum consists only on eigenvalues which can be ordered into an increasing sequence (for the convenience we denote here $\mu_0 = 0$)

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \ldots,$$

each μ_i, being isolated and of finite multiplicity.

We will denote $L^2_0(\Omega_\gamma)$ the space of functions in $L^2(\Omega_\gamma)$ whose integral vanishes. We introduce the space

$$H(\mathrm{div}\ ,\Omega) = \left\{ \mathbf{q} \in L^2(\Omega)^2,\ \mathrm{div}\,\mathbf{q} \in L^2(\Omega) \right\},$$

and denote

$$\mathbf{X} = \{\mathbf{q} \in H(\mathrm{div}\ ,\Omega), \forall \varphi \in H^1(\Omega \setminus \gamma)\ \int_\Omega \mathbf{q} \nabla \varphi dx + \int_\Omega \mathrm{div}\ \mathbf{q} \varphi dx = 0\}.$$

The equality $\int_\Omega \mathbf{q}\nabla\varphi dx + \int_\Omega \mathrm{div}\ \mathbf{q}\varphi dx = 0$ for every $\varphi \in H^1(\Omega \setminus \gamma)$ represents an integral form of the Dirichlet conditions $\mathbf{q} \cdot n = 0$ on Γ and $\mathbf{q} \cdot n_i = 0$ on γ_i (see [5]).

Thus, we are led to a new view point of the mixed variational formulation of the eigenvalue problem which will allows us to work in the entire domain

$\Omega = \Omega_\gamma \cup \gamma$. The conditions $\mathbf{q} \cdot n_i^\pm = 0$ on γ_i are replaced by only one condition $\mathbf{q} \cdot n_i = 0$ (on one side of the crack).

In order to derive the mixed variational formulation of the problem, we set in $\mathcal{D}'(\Omega_\gamma)$

$$\mathbf{p} = \mathbf{grad}\ u, \quad \text{in } \Omega_\gamma. \tag{3.10}$$

Consequently the mixed variational formulation reads: find $\mu \in \mathbb{R}$, such that there exists a solution $(\mathbf{p}, u) \in \mathbf{X} \times L^2_0(\Omega)$ of

$$\begin{cases} \int_\Omega \mathbf{p}\,\mathbf{q}\,dx + \int_\Omega u \operatorname{div} \mathbf{q}\,dx = 0, \quad \forall \mathbf{q} \in \mathbf{X}, \\ \\ \int_\Omega \operatorname{div} \mathbf{p}\, v\,dx + \mu \int_\Omega u\,v\,dx = 0, \quad \forall v \in L^2_0(\Omega). \end{cases} \tag{3.11}$$

The main advantage of these formulation is that the new admissible solutions are defined in the entire domain $\Omega = \Omega_\gamma \cup \gamma$ and the restrictions imposed on the cracks are expressed as internal constraints prescribed on the given subset $\bigcup_i \gamma_i$ of Ω while the cracks γ_i, $1 \le i \le I$ are removed as "geometrical constraints" and transformed into purely functional one. This is suitable from both practical and approximation points of view.

It is proven in [5] that if μ, $(u, \mathbf{p})$ is an eigenpair for (3.11) then μ, and the restrictions of $(u, \mathbf{p})$ to the domain Ω_γ is an eigenpair of the initial problem and conversely. Moreover, replacing $\mu\, u$ by a source term f in the second line of (3.11), we obtain an associated source problem. Therefore, denoting by T, the operators $T : L^2_0(\Omega) \longrightarrow L^2_0(\Omega)$, defined by $Tf = u$ and $S : L^2_0(\Omega) \longrightarrow \mathbf{X}$ defined by $Sf = \mathbf{p}$ where $(\mathbf{p}, u)$ defined by the source problem are well defined. We get that T is self-adjoint and compact, and the following theorem holds.

Theorem 3.7. *The triplet $(\mathbf{p}, u, \mu)$ is an eigensolution of problem (3.11) if and only if $\mu T u = u$, $\mathbf{p} = \mathbf{grad}\ u$ (i.e. $\mathbf{p} = S(\mu u)$).*

Remark 3.8. In order to build an efficient discretization, we transform the problem (3.11) to have an unconstrained formulation. This is done by expressing the conditions on the cracks via Lagrange multipliers.

The discretization of the final variational problem is based on the finite elements method, the details and the complete analysis of the discretization is performed in [5]. In particular, choosing for the approximation spaces, the lower order Raviart-Thomas element for $\mathbf{p}$, the piecewise constants elements for u and piecewise constants on the cracks for the Lagrange multipliers, we obtain the main approximation results.

Theorem 3.9. *Assume that μ, $(u, \mathbf{p}, \Lambda)$ is an eigensolution of the continuous problem, with the algebraic multiplicity m for μ and assume that $u \in \prod_{i=1}^I H^1(\Omega_i)$, $\mathbf{p} \in (H^1(\Omega))^2$ and $\Lambda \in H^{\frac{1}{2}+s}(\gamma)$, $0 < s < \frac{1}{2}$. Let μ_{ih}, $i = 1, \dots, m$ be the eigenvalues associated to μ and obtained from the discrete problem. Then, the following error estimate holds*

$$|\mu - \mu_{ih}| \le Ch^{2s}, \quad i = 1, \dots, k, \tag{3.12}$$

where the constant C depends linearly on $(\|u\|_{H^1(\Omega_i)}, 1 \le i \le I, \|\mathbf{p}\|_{H^1(\Omega)^2})$.

3.2.2. *Numerical results.* We present some numerical results to show the asymptotic behaviour of the spectrum when the number of cracks increases (for a given geometry see Figure 3 for example). We also perform some numerical computations in the case of domains with fingers (see Figure 3) instead of cracks. For the implementation details we refer the reader to [5].

We also propose a formal limit eigenvalue problem for which, due to the rectangular geometry of the domains considered here, we are able to perform analytic computations throughout a standard separation of variables argument. Indeed, let $\Omega =]-1,1[^2 \setminus P$ denotes the computations domain, where P is either a union of finite number of vertical cracks (like γ in the previous section) or the complementary set of a finite union of vertical fingers (see Figure 3). We denote by ρ the constant density in $(-1,1)\times(0,1)$, so that $\rho = 1$ in the case of cracks and set $D = (-1,1)\times(-1,1)$ and $\omega = (-1,1)\times(-1,0)$, the part of Ω free from cracks or fingers. Starting from a formal asymptotic problem (as the number of cracks or fingers goes to $+\infty$) where eigenvalues are given by critical points of the following Rayleigh quotient

$$\frac{\int_\omega |\nabla u|^2 dV + \rho \int_{D\setminus\omega} |\partial_y u|^2 \, dV}{\int_\omega |u|^2 dV + \rho \int_{D\setminus\omega} |u|^2 \, dV}, \tag{3.13}$$

we obtain the asymptotic values μ_k as zeros of the equation

$$2\rho\sqrt{\mu}\tan(\sqrt{\mu})\Big(1+\exp\sqrt{k^2\pi^2-4\mu}\Big) = \sqrt{k^2\pi^2-4\mu}\Big(\exp\sqrt{k^2\pi^2-4\mu}-1\Big) \tag{3.14}$$

The results of these computations agree with the numerical computations as shown in Tables 1 and 2. In Table 1, we have plugged the eigenvalues up to 19 in the case of domain with fingers (Figure 3) with asymptotic density $\rho = 1/2$. The second line, respectively the third line, represent the values with 9 fingers, respectively 33 fingers. In the last line we have reported the values from the analytic computations using the asymptotic formula (3.14). Note that the analytic results are very close to those given by the numerical method but in the case with 9 fingers this property deteriorates rather fast (and becomes quickly divergent from $k = 11$). In the case with 33 fingers the two computations yield reasonably closer values. This is somehow expected as the exact computations correspond to a "limit problem". In the case of Table 2, we have reported the eigenvalues up to 19 for a domain with no cracks (second line), respectively 31 cracks (the third line) and 127 cracks (the fourth line), the last line still corresponds to the analytic values obtained from (3.14) with density $\rho = 1$. In this case the analytic values and the computed ones are closer even with few cracks.

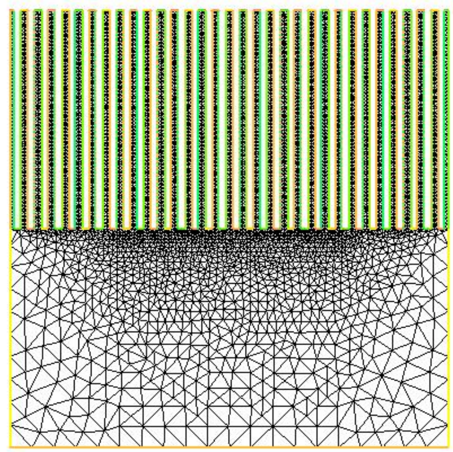

FIGURE 3. A domain with many fingers

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
9 fingers	1.1217	1.7555	1.9523	2.0504	2.1062	2.1387	2.1567	2.1619	2.4104
33 fingers	1.1231	1.7773	1.9846	2.0913	2.1572	2.2021	2.2346	2.2592	2.2785
∞	1.1265	1.7905	2.0013	2.1099	2.1770	2.2227	2.2559	2.2811	2.3009

μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}
3.7354	9.8693	9.9405	11.4940	15.6515	16.1131	17.9664	18.7367	19.1308	19.3333
2.2941	2.3067	2.3174	2.3262	2.3339	2.3403	2.3461	2.3509	2.3552	2.3588
2.3170	2.3301	2.3506	2.3587	2.3657	2.3719	2.3774	2.3823	2.3867	2.3906

TABLE 1

Acknowledgements. This work has been done during a visit of D. Bucur in Pisa and of G. Buttazzo in Metz. The authors wish to thank the Departments of Mathematics of the two Universities for the kind and warm hospitality.

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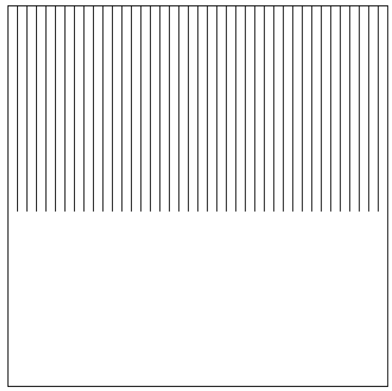

FIGURE 4. Example of multi-cracked domain

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0 crack	2.467	2.467	4.935	9.869	9.869	12.337	12.337	19.740	22.204
31 cracks	0.810	1.421	1.700	1.865	1.974	2.052	2.110	2.155	2.191
127 cracks	0.799	1.401	1.675	1.836	1.943	2.019	2.076	2.121	2.156
∞	0.7981	1.3989	1.6714	1.8324	1.9392	2.0153	2.0722	2.1164	2.1517

	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}
31	2.220	2.244	2.264	2.281	2.296	2.309	2.319	2.329	2.337	2.345
127	2.185	2.209	2.229	2.247	2.262	2.275	2.286	2.297	2.306	2.314
∞	2.1805	2.2045	2.2248	2.2422	2.2573	2.2705	2.2821	2.2924	2.3016	2.3099

TABLE 2

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