Relaxation of variational functionals with piecewise constant growth conditions

Domenico Mucci

We study the lower semicontinuous envelope of variational functionals given by $\int f(x,Du)\,dx$ *for smooth functions* u, *and equal to* $+\infty$ *elsewhere, under nonstandard growth conditions of* (p,q)*-type: namely, we assume that*

$$|z|^{p(x)} \le f(x,z) \le L(1+|z|^{p(x)})\,.$$

If the growth exponent is piecewise constant, i.e., $p(x)\equiv p_i$ *on each set of a smooth partition of the domain, we prove measure and representation property of the relaxed functional. We then extend the previous results by considering* $p(x)$ *uniformly continuous on each set of the partition. We finally give an example of energy concentration in the process of relaxation.*

The aim of this paper is the study of measure property and integral representation of the $L^1(\Omega;\mathbf{R}^N)$-lower semicontinuous envelope of variational functionals of the type

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(x,Du(x))\,dx & \text{if } u\in C^1(\Omega;\mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbf{R}^N) \end{cases} \tag{0.1}$$

where Ω is an open subset of $\mathbf{R}^n$ and f is a non-negative Borel function defined on $\Omega\times\mathbf{R}^{nN}$ and satisfying a nonstandard growth condition.

Under the assumption of p-growth

$$|z|^p \le f(x,z) \le L(1+|z|^p) \tag{0.2}$$

existence and integral representation of the lower semicontinuous envelope was proved in [9].

In case of nonstandard (p,q)-growth

$$|z|^p \le f(x,z) \le L(1+|z|^q)\,, \qquad q>p>1\,, \tag{0.3}$$

introduced by Marcellini [23] in the context of regularity theory for minimizers, dealing with the passage to the limit for variational problems, Zhikov [26] provided several results, in the context of Γ-convergence, when $N=1$ and $f(x,\cdot)$ is convex.

Actually, the difference between the space of coercivity, $W^{1,p}(\Omega;\mathbf{R}^N)$, and the smaller space of boundedness, $W^{1,q}(\Omega;\mathbf{R}^N)$, is responsible for the presence of the so called Lavrentiev effect, due to the lack of $W^{1,q}$-density of smooth functions in $W^{1,p}(\Omega;\mathbf{R}^N)$, see [27]. For example, let $\Omega=B_1$, the unit ball in $\mathbf{R}^2$, $N=1$ and

$$f(x,Du) := \begin{cases} |Du|^p & \text{if} \quad x_1x_2>0 \\ |Du|^q & \text{if} \quad x_1x_2<0 \end{cases} \qquad x=(x_1,x_2)\in B_1 \tag{0.4}$$

where $1<p<2<q$. Zhikov showed that if $u: B_1\to\mathbf{R}$ is given by

$$u(x) := \begin{cases} x_1/\|x\| & \text{if} \quad x_1>0 \text{ and } x_2>0 \\ 0 & \text{if} \quad x_1<0 \text{ and } x_2>0 \\ -x_2/\|x\| & \text{if} \quad x_1<0 \text{ and } x_2<0 \\ 1 & \text{if} \quad x_1>0 \text{ and } x_2<0 \end{cases} \qquad \|x\| := \sqrt{x_1{}^2+x_2{}^2}\,, \tag{0.5}$$

then u has finite (p,q)-energy, $\int_{B_1} f(x,Du)\,dx<+\infty$, but it cannot exist a smooth sequence $\{u_j\}\subset C^1(B_1)$, or in $W^{1,q}(B_1)$, which converges to u in (p,q)-energy, i.e., such that $\int_{B_1} f(x,Du_j)\,dx \to \int_{B_1} f(x,Du)\,dx$.

In the context of cavitation and related theories, described by functionals of the type (0.1), if the integrand f is satisfying a q-growth condition from above, the measure representation of the relaxed functional with respect to the weak $W^{1,p}$ convergence is obtained in [18], [6] and [1], assuming $z \mapsto f(x,z)$ convex and $p > q - q/n$. As to regularity of minimizers of relaxed functionals with (p,q)-growth see [16].

A borderline case lying between (0.2) and (0.3) is the one of $p(x)$-growth

$$|z|^{p(x)} \leq f(x,z) \leq L(1 + |z|^{p(x)}) , \qquad p(x) > 1 . \tag{0.6}$$

This kind of growth was first considered by Zhikov in the context of homogeneization, see [29], and in recent years the subject gained importance by providing variational models for many problems from Mathematical Physics: for instance, dealing with electrostatic fields in which conductivity depends on the intensity of the field, or thermal equilibrium in composite nonlinearly conductive materials, Zhikov's thermistor problem [28], or, more recently, the mathematical theory of electrorheological fluids developed by Rajagopal and Růžička, see [25].

In this paper, we will suppose the growth exponent $p(x)$ to be piecewise constant in a suitable way, the simplest non trivial example being the one described in (0.4). This corresponds to the physical model of a conductor made by different homogeneous materials, compare [2] for a regularity result in this context.

More precisely, we will suppose the open set Ω to be partitioned by an at most countable family of open sets $\{\Omega_i\}$ with smooth boundary, so that the transition set

$$\Sigma := \Omega \setminus \bigcup_{i=1}^{+\infty} \Omega_i \tag{0.7}$$

is negligible, i.e., $|\Sigma| = 0$. We will suppose $p(x)$ to be constant on each Ω_i

$$p(x) \equiv p_i > 1 \qquad \text{if } x \in \Omega_i \, , \quad \forall \, i \, , \tag{0.8}$$

and that it is locally finite the number of different phases p_i of $p(x)$.

Since we will make use of the localization method, we define

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x, Du(x)) \, dx & \text{if } u \in C^1 \left(A; \mathbf{R}^N\right) \\ +\infty & \text{elsewhere on } L^1 \left(\Omega; \mathbf{R}^N\right) \end{cases} \tag{0.9}$$

for every open subset A of Ω, and denote by $\overline{F}(\cdot, A)$ the lower semicontinuous envelope of $F(\cdot, A)$ with respect to the $L^1\left(\Omega; \mathbf{R}^N\right)$ topology, given for all $u \in L^1\left(\Omega; \mathbf{R}^N\right)$ by

$$\overline{F}(u,A) := \sup \left\{ G(u) \mid G \text{ is } L^1\left(\Omega; \mathbf{R}^N\right)\text{-l.s.c. and } G(u) \leq F(u,A) \right\} . \tag{0.10}$$

We will show measure and representation property of the relaxed functional (0.10), under the hypotheses (0.8), (0.9) and (0.6).

We will then weaken condition (0.8), considering the more general case of growth exponents which are uniformly continuous on each set Ω_i, provided the following estimate about the modulus of continuity $\omega(R)$ of $p(x)$ holds locally on each set Ω_i:

$$\limsup_{R \to 0^+} \omega(R) \, \log(1/R) < +\infty \, . \tag{0.11}$$

This condition was introduced by Zhikov to prove higher integrability of the gradient of minimizers of functionals with $p(x)$-growth, see [27]; note that (0.11) is sharp since, in general, dropping it causes the loss of any type of regularity, see [28]. Regularity results in this and in related contexts are obtained in [3], [4], [2] and [16].

Moreover, condition (0.11) seems to play a central role in the theory of functionals with $p(x)$-growth since Zhikov proved in [28] that such functionals exhibit the Lavrentiev phenomenon if (0.11) is violated, see the cited example in (0.4) and (0.5).

On the other side, in [1] it is proved that the singular part of the measure representation of relaxed functionals with growth (0.6) disappears if (0.11) holds true.

Finally, in [11] it is shown Γ-compactness and integral representation of the Γ-limit of integral functionals with $p(x)$-growth, provided a local the estimate of the type (0.11) holds for the modulus of continuity of $p(x)$. A crucial role here is played by the density result in the class $W^{1,p(x)}(\Omega;\mathbf{R}^N)$, see Proposition 4.2.

This paper is organized as follows. After giving notation and statements in Sec. 1, and preliminary results in Sec. 2, in Sec. 3 we will prove (Theorem 1.8) that, for every function $u \in L^1(\Omega;\mathbf{R}^N)$, the relaxed functional $\overline{F}(u,\cdot)$ satisfies the so called measure property. Thanks to the standard p_i-growth condition on each Ω_i, see (0.8), we will then represent (Theorem 1.9) the measure $\overline{F}(u,\cdot)$, writing its absolute continuous part as the integral of a quasi-convex function, satisfying the same $p(x)$-growth condition (0.6), plus a singular measure with support concentrated in the transition set Σ, see (0.7).

In Sec. 4 we will extend the previous results to growth exponents which are uniformly continuous on each set Ω_i, provided the estimate (0.11) about the modulus of continuity of $p(x)$ holds locally on each set Ω_i, see Theorems 1.13 and 1.14. The proof is modeled on the case of $p(x)$ piecewise constant, taking account of a Rellich's type result for $W^{1,p(x)}$ functions (Lemma 4.1), of the density of smooth maps in the class $W^{1,p(x)}$ (Proposition 4.2) and of an integral representation result for local functionals with $p(x)$-growth (Theorem 4.3), for the proof of which we refer to [11].

Finally, in Sec. 5, starting from Zhikov's example (0.4), we will prove existence of energy concentration in the process of relaxation (Example 1.15), showing that, if $1 < p < 2 < q < 2p$ (and $n = 2$) the singular part of the measure $\overline{F}(u,\cdot)$ is an atomic mass of infinite energy supported in the origin $0_{\mathbf{R}^2}$, provided u behaves like the function in (0.5).

We will show, in conclusion, that the measure property of the relaxed functional may fail if the localization method is not performed in the suitable way, see Remark 1.16.

Acknowledgements: The author thanks N. Fusco for suggesting this investigation and also E. Acerbi and G. Mingione for reading the paper.

1 Notation and statements

In the sequel Ω is a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary and $\mathcal{A}$ is the family of its open subsets; similarly, if $B \in \mathcal{A}$, we denote $\mathcal{A}(B)$ the family of open subsets of B; by $A \subset\subset B$ we mean that the closure $\overline{A}$ of A is a compact set contained in B, and by $\mathcal{A}_0$ we denote the class of all $A \in \mathcal{A}$ such that $A \subset\subset \Omega$; a similar notation is given for $\mathcal{A}_0(B)$. If $A', A \in \mathcal{A}$ with $A' \subset\subset A$, a cut-off function between A' and A is a smooth function $\varphi \in C_0^\infty(\Omega)$ with $0 \le \varphi \le 1$, such that $\operatorname{spt}\varphi \subset A$ and $\varphi \equiv 1$ on A'. Also, $B_\delta(x)$ denotes the ball of radius $\delta > 0$ centred at $x \in \mathbf{R}^n$ and $B_\delta := B_\delta(0_{\mathbf{R}^n})$. As usual, $\mathbf{R}^{nN}$ is identified with the set of real valued $(N \times n)$-matrices. Finally, for every $0 \le k \le n$ we denote by $\mathcal{H}^k$ is the k-dimensional Hausdorff measure on $\mathbf{R}^n$, see [17], so that $\mathcal{H}^n(E) = \operatorname{meas}(E) = |E|$ is the Lebesgue measure of E.

We will consider non-negative variational functionals $F : L^1(\Omega;\mathbf{R}^N) \to [0,+\infty]$ of the type

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, Du(x))\,dx & \text{if } u \in C^1(\Omega;\mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbf{R}^N) \end{cases}$$

where $f : \Omega \times \mathbf{R}^{nN} \to [0,+\infty)$ is a Borel measurable function satisfying a nonstandard growth condition of the form

$$\alpha|z|^{p(x)} \le f(x,z) \le a(x) + \beta|z|^{p(x)} \tag{1.1}$$

for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$, where $0 < \alpha \le \beta < +\infty$ and $a(x) \in L^1(\Omega)$, with $a(x) \ge 0$. We are interested in the study of the relaxed functional of F with respect to the strong $L^1(\Omega;\mathbf{R}^N)$ convergence, i.e., the lower semicontinuous envelope of F with respect to the $L^1(\Omega;\mathbf{R}^N)$ topology, given for all $u \in L^1(\Omega;\mathbf{R}^N)$ by

$$\overline{F}(u) := \inf\{\liminf_{k\to+\infty} F(u_k) \mid \{u_k\} \subset L^1(\Omega;\mathbf{R}^N), \quad u_k \to u \text{ in } L^1(\Omega;\mathbf{R}^N)\}.$$

The growth exponent $p(x) > 1$ is supposed to be piecewise constant, locally bounded and discontinuous on a transition set Σ of null measure, with Σ sufficiently smooth. More precisely, we introduce the following assumptions.

Definition 1.1 A family $\{\Omega_i\}$ is a locally finite regular partition of an open set Ω if each Ω_i is an open set with Lipschitz boundary, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$,

$$\Omega = \Sigma \cup \bigcup_{i=1}^{+\infty} \Omega_i \,, \tag{1.2}$$

where $|\Sigma| = 0$, and if $A \in \mathcal{A}_0$ yields $A \cap \Omega_i = \emptyset$ except for a finite number of indices.

Definition 1.2 A function $p : \Omega \to (1, +\infty)$ is a regular piecewise constant exponent if there exist a locally finite regular partition $\{\Omega_i\}$ of Ω and, for every i, a constant $p_i > 1$ such that

$$p(x) \equiv p_i > 1 \qquad \forall\, x \in \Omega_i \,, \quad \forall\, i \,. \tag{1.3}$$

Remark 1.3 If $p(x)$ is a regular piecewise constant exponent, by (1.3) we infer that $p(x)$ is locally finite and distant from 1, i.e., taking essential infimum and supremum,

$$1 < \inf_{x \in A} p(x) \leq \sup_{x \in A} p(x) < +\infty \qquad \forall A \in \mathcal{A}_0 \,. \tag{1.4}$$

Actually, growth condition (1.1) leads us to consider the class of Sobolev functions with summability of the $p(x)$ power of the gradient, which coincides with the standard one if $p(x) \equiv p \geq 1$ on Ω. Due to (1.3), for every open set $A \in \mathcal{A}$, we first introduce on the Lebesgue space $L^1\,(A; \mathbf{R}^N)$ the $L^{p(x)}$ norm

$$\|u\|_{L^{p(x)}(A;\mathbf{R}^N)} := \sum_{i=1}^{+\infty} \|u_{|A_i}\|_{L^{p_i}(A_i;\mathbf{R}^N)}$$

where A_i, here and in the sequel, denotes the intersection of $A \in \mathcal{A}$ with Ω_i, i.e., $A_i := A \cap \Omega_i$ for every i. We then introduce the class

$$L^{p(x)}\,(A; \mathbf{R}^N) := \{u \in L^1(A; \mathbf{R}^N) \mid \|u\|_{L^{p(x)}(A;\mathbf{R}^N)} < +\infty\}$$

of measurable functions with $p(x)$ summability. Similarly, we introduce on the Sobolev space $W^{1,1}\,(A; \mathbf{R}^N)$ the $W^{1,p(x)}$ norm

$$\|u\|_{W^{1,p(x)}(A;\mathbf{R}^N)} := \|u\|_{L^{p(x)}(A;\mathbf{R}^N)} + \|Du\|_{L^{p(x)}(A;\mathbf{R}^{nN})}$$

and the corresponding class

$$W^{1,p(x)}\,(A; \mathbf{R}^N) := \{u \in W^{1,1}(A; \mathbf{R}^N) \mid \|u\|_{W^{1,p(x)}(A;\mathbf{R}^N)} < +\infty\}$$

for every $A \in \mathcal{A}$. Therefore, if $u \in W^{1,p(x)}\,(A; \mathbf{R}^N)$, by (1.3) the restriction $u_{|A_i}$ of u to A_i is in the Sobolev space $W^{1,p_i}(A_i; \mathbf{R}^N)$ whereas, if p is the smallest of the exponents p_i for which $A_i \neq \emptyset$ (and $|A| < +\infty$), then $W^{1,p(x)}\,(A; \mathbf{R}^N) \subset W^{1,p}\,(A; \mathbf{R}^N)$. Moreover, we set

$$W^{1,p(x)}_{\mathrm{loc}}\,(A; \mathbf{R}^N) := \{u : A \to \mathbf{R}^N \mid u_{|B} \in W^{1,p(x)}(B; \mathbf{R}^N) \quad \forall B \in \mathcal{A},\ B \subset\subset A\}$$

and we say that $u_k \to u$ in $W^{1,p(x)}_{\mathrm{loc}}\,(A; \mathbf{R}^N)$ if $\|u_k - u\|_{W^{1,p(x)}(B;\mathbf{R}^N)} \to 0$ for every $B \in \mathcal{A}$, $B \subset\subset A$. Finally, we will denote

$$C^{1,p(x)}\,(A; \mathbf{R}^N) := \{u \in C^1\,(A; \mathbf{R}^N) \mid \|u\|_{W^{1,p(x)}(A;\mathbf{R}^N)} < +\infty\}$$

the class of smooth functions whose $W^{1,p(x)}$ norm is bounded. In the sequel, the target space $\mathbf{R}^N$ will be omitted when it is clear from the context, for example inside proofs. Finally, if $X = L^{p(x)}$, $W^{1,p(x)}$ or $C^{1,p(x)}$, then it is easy to show that $X\,(A; \mathbf{R}^N)$ is always a convex set. Due to (1.4), $X\,(A; \mathbf{R}^N)$ becomes a vector space if $A \in \mathcal{A}_0$.

To show measure property and integral representation of the relaxed functional we make use of the localization method, which consists in considering at the same time the dependence on the function and on the open set. To this aim, in the sequel we will always consider non negative variational functionals $F : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ of the form

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x,Du(x))\,dx & \text{if } u\in C^{1,p(x)}(A;\mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbf{R}^N) \end{cases} \tag{1.5}$$

for any open set $A\in\mathcal{A}$, where $f:\Omega\times\mathbf{R}^{nN}\to[0,+\infty)$ is a Borel function satisfying growth condition (1.1). Also, for every $A\in\mathcal{A}$, we denote by $\overline{F}(\cdot,A)$ the relaxed functional of $F(\cdot,A)$ with respect to the strong $L^1(\Omega;\mathbf{R}^N)$ convergence, see (0.10), given for all $u\in L^1(\Omega;\mathbf{R}^N)$ by

$$\overline{F}(u,A) := \inf\{\liminf_{k\to+\infty} F(u_k,A) \mid \{u_k\}\subset L^1(\Omega;\mathbf{R}^N)\,,\quad u_k\to u \text{ in } L^1(\Omega;\mathbf{R}^N)\}\,. \tag{1.6}$$

Remark 1.4 Since each sequence $\{u_k\}\subseteq L^1(A;\mathbf{R}^N)$ converging to u in $L^1(A;\mathbf{R}^N)$ can be extended to a sequence $L^1(\Omega;\mathbf{R}^N)$-converging to u, if $\overline{F}(u,A)<+\infty$, by (1.5) we have

$$\overline{F}(u,A) = \inf\{\liminf_{k\to+\infty}\int_A f(x,Du_k(x))\,dx \mid \{u_k\}\subset C^{1,p(x)}(A;\mathbf{R}^N)\,,\quad u_k\to u \text{ in } L^1(A;\mathbf{R}^N)\}\,.$$

We now recall some well known facts about set functions.

Definition 1.5 A function $\alpha:\mathcal{A}\to[0,+\infty]$ is called an increasing set function if $\alpha(\emptyset)=0$ and $\alpha(A)\le\alpha(B)$ if $A\subseteq B$. An increasing set function α is said to be subadditive if

$$\alpha(A\cup B)\le\alpha(A)+\alpha(B)$$

for all $A,B\in\mathcal{A}$, and it is said to be superadditive if

$$\alpha(A\cup B)\ge\alpha(A)+\alpha(B)$$

for all $A,B\in\mathcal{A}$ with $A\cap B=\emptyset$; finally α is said to be inner regular if for all $A\in\mathcal{A}$

$$\alpha(A)=\sup\{\alpha(B)\mid B\in\mathcal{A}\,,\quad B\subset\subset A\}\,.$$

Remark 1.6 Since $f\ge0$, then $\overline{F}(u,\cdot)$ is an increasing set function for every $u\in L^1(\Omega;\mathbf{R}^N)$. Moreover, by definition of relaxation one directly obtains that $\overline{F}(u,\cdot)$ is superadditive. Finally, we will denote by $\overline{F}_-(u,\cdot)$ the inner regular envelope of $\overline{F}(u,\cdot)$, given by

$$\overline{F}_-(u,C) := \sup\{\overline{F}(u,B)\mid B\in\mathcal{A}\,,\ B\subset\subset C\} \tag{1.7}$$

for every $C\in\mathcal{A}$, so that $\overline{F}(u,\cdot)$ is inner regular if $\overline{F}(u,\cdot)\equiv\overline{F}_-(u,\cdot)$ on $\mathcal{A}$.

We will apply the following criterion due to De Giorgi-Letta [14], compare also [7, 10.2].

Proposition 1.7 (Measure property criterion) *Let $\alpha:\mathcal{A}\to[0,+\infty]$ be an increasing set function. Then the following statements are equivalent:*

i) *α is the trace on $\mathcal{A}$ of a Borel measure on Ω;*

ii) *α is subadditive, superadditive and inner regular;*

iii) *the set function $\widetilde{\alpha}(E):=\inf\{\alpha(A)\mid A\in\mathcal{A},\ E\subset A\}$ defines a Borel measure on Ω.*

The first result of this paper is the following

Theorem 1.8 *Let $p:\Omega\to(1,+\infty)$ be a regular piecewise constant exponent satisfying* (1.4)*, according to Definition* 1.2*. Let $F:L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ be a variational functional of the type* (1.5)*, where $f:\Omega\times\mathbf{R}^{nN}\to[0,+\infty]$ is a Borel function satisfying growth condition* (1.1) *for a.e. $x\in\Omega$ and all $z\in\mathbf{R}^{nN}$, for some $0<\alpha\le\beta<+\infty$ and $a(x)\in L^1(\Omega)$, with $a(x)\ge0$. Then, for every function $u\in L^1(\Omega;\mathbf{R}^N)$, the $L^1(\Omega;\mathbf{R}^N)$-relaxed functional $\overline{F}(u,\cdot)$, see* (1.6)*, is the trace on $\mathcal{A}$ of a Borel measure on Ω.*

Since on each Ω_i the integrand f satisfies a standard p_i growth condition, see (1.3), as a consequence we are able to prove a representation result for the relaxed functional. To this aim, we first recall that a continuous function $g : \mathbf{R}^{nN} \to \mathbf{R}$ is called quasi-convex in the sense of Morrey [24] if for every $z \in \mathbf{R}^{nN}$, every bounded open set A of $\mathbf{R}^n$ and every function $\phi \in C_0^1(A; \mathbf{R}^N)$ we have

$$|A|\, g(z) \leq \int_A g(z + D\phi(x))\, dx\,.$$

A Carathéodory function $f : \Omega \times \mathbf{R}^{nN} \to [0, +\infty]$ is called quasi-convex if $z \to f(x,z)$ is quasi-convex for a.e. $x \in \Omega$. Moreover, the quasi-convex envelope $Qf(x,z)$ (with respect to z) of a Borel function $f(x,z)$ is the greatest Carathéodory function which is quasi-convex in z and less than or equal to $f(x,z)$ for a.e. $x \in \Omega$, see [12]. For quasi-convex functionals with $p(x)$-growth we refer to [4].

Theorem 1.9 *Under the hypotheses of Theorem* 1.8*, for each open set $A \in \mathcal{A}$ we have*

$$\overline{F}(u, A) = \begin{cases} \displaystyle\int_A \varphi(x, Du(x))\, dx + \mu(u, A) & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N) \end{cases} \tag{1.8}$$

where $\varphi : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$ is a quasi-convex function satisfying growth condition (1.1) *for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$, and $\mu(u, \cdot)$, for any $u \in L^1(\Omega; \mathbf{R}^N)$, is a non negative Borel measure on Ω concentrated in the transition set Σ of $p(x)$, see* (1.2)*. Finally, if $u \in L^1(\Omega; \mathbf{R}^N)$ is such that $u_{|A} \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ for some $A \in \mathcal{A}$, then $\mu(u, A) = 0$ in* (1.8) *if there exists a sequence of smooth functions $\{u_k\} \subset C^1(A; \mathbf{R}^N)$ such that $u_k \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$.*

Arguing as in [8, 4.4.5], we finally obtain the following

Corollary 1.10 *Under the hypotheses of Theorem* 1.9*, if f is a Carathéodory function, or $f(x, \cdot)$ is upper semicontinuous in $\mathbf{R}^{nN}$ for a.e. $x \in \Omega$, then we have $\varphi(x,z) = Qf(x,z)$, the quasi-convex envelope of f.*

In Sec. 4 we will then extend the previous results to a more general class of growth exponents. We first introduce the following assumptions.

Definition 1.11 A function $p : \Omega \to (1, +\infty)$ is a regular piecewise continuous exponent if there exist a locally finite regular partition $\{\Omega_i\}$ of Ω, see Definition 1.1, and, for every i, a uniformly continuous function $p_i : \Omega_i \to [1, +\infty)$ such that $p(x) = p_i(x)$ for every $x \in \Omega_i$.

Remark 1.12 If $p(x)$ is a regular piecewise continuous exponent satisfying

$$\inf_{x \in \Omega_i} p_i(x) > 1 \qquad \forall\, i \tag{1.9}$$

then (1.4) holds again.

In the following results, we will then consider variational functionals of the type (1.5) defined on the class of functions given by

$$\begin{aligned} L^{p(x)}(A; \mathbf{R}^N) &:= \{u : A \to \mathbf{R}^N \mid \int_A |u|^{p(x)}\, dx < +\infty\}\,, \\ W^{1,p(x)}(A; \mathbf{R}^N) &:= \{u \in L^{p(x)}(A; \mathbf{R}^N) \mid Du \in L^{p(x)}(A; \mathbf{R}^{nN})\}\,, \\ W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) &:= \{u : A \to \mathbf{R}^N \mid u_{|B} \in W^{1,p(x)}(B; \mathbf{R}^N) \quad \forall\, B \in \mathcal{A}\,,\ B \subset\subset A\}\,, \\ C^{1,p(x)}(A; \mathbf{R}^N) &:= C^1(A; \mathbf{R}^N) \cap W^{1,p(x)}(A; \mathbf{R}^N) \end{aligned}$$

for every $A \in \mathcal{A}$, where $p(x)$ is a regular piecewise continuous exponent. We also say that a sequence $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ converges to $u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ strongly in $W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ if

$$\lim_{j \to +\infty} \int_B (|u_j - u|^{p(x)} + |Du_j - Du|^{p(x)})\, dx = 0$$

for every $B \in \mathcal{A}_0$, $B \subset\subset A$. For the general properties of the function spaces $L^{p(x)}$ and $W^{1,p(x)}$ we refer to [21] and [15]. We first extend Theorem 1.8 by the following

Theorem 1.13 *Let* $p:\Omega\to(1,+\infty)$ *be a regular piecewise continuous exponent satisfying* (1.9)*, according to Definition* 1.11*. Let* $F: L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ *be a variational functional of the type* (1.5)*, where* $f:\Omega\times\mathbf{R}^{nN}\to[0,+\infty]$ *is a Borel function satisfying growth condition* (1.1) *for a.e.* $x\in\Omega$ *and all* $z\in\mathbf{R}^{nN}$*, for some* $0<\alpha\le\beta<+\infty$ *and* $a(x)\in L^1(\Omega)$*, with* $a(x)\ge 0$*. Then the conclusions of Theorem* 1.8 *hold again.*

Moreover, if the growth exponent satisfies a local estimate (0.11) about the modulus of continuity on each Ω_i, we are able to extend Theorem 1.9 as follows.

Theorem 1.14 *Under the hypotheses of Theorem* 1.13*, suppose that for every* i *the function* $p_i:\Omega\to(1,+\infty)$ *satisfy the following local estimate about the modulus of continuity:*

$$\forall\, A\in\mathcal{A}_0\,,\ A\subset\subset\Omega_i\,,\quad \exists\,\gamma_A>0:\quad |p_i(x)-p_i(y)|\le\frac{\gamma_A}{|\log|x-y||}\quad\forall\, x,y\in A\,,\quad 0<|x-y|<\frac{1}{2}\,. \tag{1.10}$$

Then all the conclusions of Theorem 1.9 *hold again.*

Notwithstanding, existence of energy concentration in general holds. More precisely, there are functions $u\in W^{1,p(x)}(\Omega;\mathbf{R}^N)$ such that $\mu(u,A)\ge C>0$ in (1.8) for every open set $A\in\mathcal{A}$ which contains a given point x_0 of the transition set Σ. The simplest examples are obtained by taking $p(x)$ so that it takes at least four different values p_i, which are not in increasing order, around the saddle point x_0, compare (1.11) where $x_0=0_{\mathbf{R}^2}$. In fact, Zhikov in [27] showed that a Lavrentiev phenomenon occurs for the minimum of energy of the functional $\int|Du|^{p(x)}\,dx$ in the unit ball B_1 in $\mathbf{R}^2$, where

$$p(x):=\begin{cases} p & \text{if}\quad x_1x_2>0\\ q & \text{if}\quad x_1x_2<0\end{cases}\qquad x=(x_1,x_2)\in B_1 \tag{1.11}$$

and $1<p<2<q$. More precisely, he considered the function $u_0:B_1\to\mathbf{R}$ given by

$$u_0(x):=\begin{cases} x_1/\|x\| & \text{if}\quad x_1>0\text{ and }x_2>0\\ 0 & \text{if}\quad x_1<0\text{ and }x_2>0\\ -x_2/\|x\| & \text{if}\quad x_1<0\text{ and }x_2<0\\ 1 & \text{if}\quad x_1>0\text{ and }x_2<0\end{cases}\qquad \|x\|:=\sqrt{{x_1}^2+{x_2}^2}\,, \tag{1.12}$$

which clearly belongs to $W^{1,p(x)}(B_1)$ if $p(x)$ is given by (1.11) and $p<2$. He essentially showed that if $q>2$ it cannot be find a sequence of smooth functions $W^{1,p(x)}$-converging to the function u_0 given by (1.12), compare the last assertion in Theorem 1.9. This depends on the fact that since $p(x)=q>n=2$ on the subset $\widetilde{B}:=\{x\in B_1\mid x_1x_2<0\}$, if u_0 were $W^{1,p(x)}$-approximable by smooth functions, then the restriction of u_0 to $\widetilde{B}$ ought to have a continuous extension to the closure of $\widetilde{B}$, which is impossible, see Remark 5.3. Following this argument, in the sequel we show existence of energy concentration in the process of relaxation.

Example 1.15 Let $\Omega=B_1$, the unit ball of $\mathbf{R}^2$, $n=2$, $N=1$; let $p(x)$ be given by (1.11) with $1<p<2<q<2p$ and let $f(x,z):=|z|^{p(x)}$ for a.e. $x\in B_1$ and all $z\in\mathbf{R}^{nN}$. Since $f(x,\cdot)=|\cdot|^{p(x)}$ is convex, then $Qf(x,z)=f(x,z)$ and hence Theorem 1.9 and Corollary 1.10 yield (1.8) with $\varphi(x,z)=|z|^{p(x)}$. Moreover, the singular measure $\mu(u,A)$ is obtained as follows. If $u\in L^1(\Omega)$ is such that $u_{|A}\in W^{1,p(x)}_{\rm loc}(A)$ for some open set $A\in\mathcal{A}(B_1)$ with $0_{\mathbf{R}^2}\in A$, we set

$$\lambda_1:=\lim_{\substack{\rho\to 0^+\\ \theta\in(\pi/2,\pi)}}\widetilde{u}(\rho,\theta)\qquad\text{and}\qquad \lambda_2:=\lim_{\substack{\rho\to 0^+\\ \theta\in(3\pi/2,2\pi)}}\widetilde{u}(\rho,\theta) \tag{1.13}$$

where $\widetilde{u}(\rho,\theta):=u(\rho\cos\theta,\rho\sin\theta)$ in polar coordinates. Note that the limits in (1.13) exist uniformly in θ since if $q>2$ and B is any bounded open subset of $\mathbf{R}^2$ with Lipschitz boundary, then $W^{1,q}(B)\subset C^{0,\alpha}(\overline{B})$ with $\alpha=1-2/q$. Then, for all $A\in\mathcal{A}(B_1)$ and $u\in W^{1,p(x)}_{\rm loc}(A)$ we will compute in Sec. 5

$$\mu(u,A)=\begin{cases} 0 & \text{if}\quad 0_{\mathbf{R}^2}\notin A\\ \chi^{\lambda_1}_{\lambda_2} & \text{if}\quad 0_{\mathbf{R}^2}\in A\end{cases} \tag{1.14}$$

where λ_1 and λ_2 are defined by (1.13) and

$$\chi_{\lambda_2}^{\lambda_1} := \begin{cases} 0 & \text{if} \quad \lambda_1 = \lambda_2 \\ +\infty & \text{if} \quad \lambda_1 \neq \lambda_2 . \end{cases}$$

In particular, if u_0 is given by (1.12), since $\lambda_1 = 0$ and $\lambda_2 = 1$ there is energy concentration in the origin, i.e.,

$$\overline{F}(u_0, A) = +\infty \qquad \forall\, A \in \mathcal{A} \quad \text{such that} \quad 0_{\mathbf{R}^2} \in A . \tag{1.15}$$

Remark 1.16 We finally make some comments about the definition of the localized functional $F(u, A)$ in (1.5), showing that a wrong localization procedure may cause lack of measure property of the relaxed functional.

For any Borel function $f : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$ as in Theorem 1.8, denote by $G : L^1(\Omega; \mathbf{R}^N) \times \mathcal{A} \to [0, +\infty]$ the variational functional

$$G(u, A) := \begin{cases} \displaystyle\int_A f(x, Du(x))\, dx & \text{if } u \in C^{1,p(x)}(\Omega; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N) \end{cases} \tag{1.16}$$

for every $A \in \mathcal{A}$, and by $\overline{G}(\cdot, A)$ the $L^1(\Omega; \mathbf{R}^N)$-relaxed functional of $G(\cdot, A)$ given by

$$\overline{G}(u, A) := \inf\{\liminf_{k \to +\infty} G(u_k, A) \mid \{u_k\} \subset L^1(\Omega; \mathbf{R}^N), \quad u_k \to u \text{ in } L^1(\Omega; \mathbf{R}^N)\} . \tag{1.17}$$

If $F(u, A)$ is given by (1.5), we obviuosly have $F(u, A) \leq G(u, A)$ and hence

$$\overline{F}(u, A) \leq \overline{G}(u, A) \qquad \forall\, u \in L^1(\Omega; \mathbf{R}^N),\ A \in \mathcal{A} . \tag{1.18}$$

Moreover, equality in (1.18) may fail if A does not have smooth boundary.

It can be shown that by using (1.16) in the localization method, the relaxed functional (1.17) does not satisfy the measure property. Taking in fact $\Omega = B_1$, $n = 2$, $N = 1$, $p(x)$ given by (1.11) with $1 < p < 2 < q < 2p$ and $f(x, z) := |z|^{p(x)}$, see Example 1.15, we will prove in Sec. 3 that if $u_0 \in W^{1,p(x)}(B_1)$ is given by (1.12), then (1.15) holds. On the other side, if $\Sigma := \{x \in B_1 \mid x_1 x_2 = 0\}$ is the transition set of $p(x)$, see (1.2), since $|\Sigma| = 0$, by definition (1.16) we clearly have

$$\overline{G}(u, A) = \overline{G}(u, A \setminus \Sigma) \qquad \forall\, u \in L^1(B_1),\ A \in \mathcal{A} . \tag{1.19}$$

Moreover, it can be shown (see Lemma 2.6) that if $B \in \mathcal{A}_0$ and $B \subset\subset B_1 \setminus \Sigma$, then $\overline{G}(u, B) = \int_B |Du|^{p(x)}\, dx$ for every function $u \in W^{1,p(x)}(B)$.

Suppose now by contradiction that $\overline{G}(u, \cdot)$ is a measure on $\mathcal{A}$. Then, by (1.19) and inner regularity, see Proposition 1.7, we would have

$$\begin{aligned} \overline{G}(u, A) &= \sup\{\overline{G}(u, B) \mid B \in \mathcal{A}_0 ,\ B \subset\subset A \setminus \Sigma\} \\ &= \sup\{\int_B |Du|^{p(x)}\, dx \mid B \in \mathcal{A}_0 ,\ B \subset\subset A \setminus \Sigma\} = \int_A |Du|^{p(x)}\, dx < +\infty \end{aligned}$$

for every $A \in \mathcal{A}$ and $u \in W^{1,p(x)}(A)$. This cannot hold for $u = u_0$ given by (1.12), for example, since by (1.18) and (1.15) we have $\overline{G}(u_0, A) = +\infty$ if $0_{\mathbf{R}^2} \in A$, whereas $u_0 \in W^{1,p(x)}(B_1)$. On the other side, it is easy to infer that the proof of Proposition 3.1 (which yields inner regularity) does not hold in case the localized functional is defined by (1.16), see Remark 3.2.

2 Preliminary results

In this section we provide some preliminary results. We will always suppose that $p(x)$ and $\{\Omega_i\}$ satisfy the hypotheses of Theorem 1.8. Since we need a compactness result for $W^{1,p(x)}$ functions (Lemma 2.2), we first make the following

Remark 2.1 Let Q be a closed n-rectangle of $\mathbf{R}^n$ with sides parallel to the coordinate directions. If $Q \subset \Omega$, setting for $\delta > 0$

$$Q_\delta := x_0 + \prod_{j=1}^{n} \delta\,[-\lambda_j, \lambda_j]\,, \qquad x_0 \in \Omega\,,\ \lambda_j > 0\ \forall\, j = 1, \ldots, n\,,$$

we choose x_0 and λ_j so that $Q_1 = Q$ and $Q_\delta \subset \Omega$ for every $\delta < \overline{\delta}$, for some $\overline{\delta} > 1$. Since $\{\Omega_i\}$ is a locally finite regular partition, see Definition 1.1, for any such δ we have that $Q_\delta \cap \Omega_i$ is nonempty for a finite number of indices i and hence, for a.e. $0 < \delta < \overline{\delta}$, the set $Q_\delta \cap \Omega_i$ has Lipschitz boundary for every i. Let us now fix $A', A \in \mathcal{A}$ with $A' \subset\subset A$. Taking the interior of an n-polyrectangle P, with sides parallel to the coordinates directions, such that $A' \subset\subset P \subset A$, and possibly moving the faces of P, by the previous argument it is possible to find an open set $\widetilde{A} \in \mathcal{A}_0$ with $A' \subset\subset \widetilde{A} \subset\subset A$ such that if $\widetilde{A}_i := \widetilde{A} \cap \Omega_i$, which is nonempty for finitely many i, then all the $\widetilde{A}_i$ have Lipschitz boundary. In a similar way, if $B \in \mathcal{A}$ is such that $B \cap \Omega_i$ has Lipschitz boundary for every i, we can also find $\widehat{A} \in \mathcal{A}_0$ with $A' \subset\subset \widehat{A} \subset\subset A$ such that $(\widehat{A} \cap B) \cap \Omega_i$ has Lipschitz boundary for every i.

This fact will allow us to use in a successfull way the following easy consequence of Rellich's theorem.

Lemma 2.2 *Let $p : \Omega \to (1, +\infty)$ be a regular piecewise constant growth exponent. Let $A \in \mathcal{A}_0$ be such that $A_i := A \cap \Omega_i$ has Lipschitz boundary for every i. Moreover, let $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ and $u \in L^1(A; \mathbf{R}^N)$ be such that $u_j \to u$ in $L^1(A; \mathbf{R}^N)$ and*

$$\sup_{j \in \mathbf{N}} \int_A |Du_j(x)|^{p(x)}\, dx < +\infty\,.$$

Then $\|u_j - u\|_{L^{p(x)}(A; \mathbf{R}^N)} \to 0$ as $j \to +\infty$.

Proof: since $\sup_j \|Du_j\|_{L^{p_i}(A_i)} < +\infty$ for every i, where $p_i > 1$ is given by (1.3), by Rellich's theorem we have that $\|u_j - u\|_{L^{p_i}(A_i)} \to 0$ and hence the assertion, since A_i is nonempty for finitely many i. $\square$

Remark 2.3 With a similar argument it can be shown that for every $A \in \mathcal{A}$

$$W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) = \{u : A \to \mathbf{R}^N \mid \int_B |Du|^{p(x)}\, dx < +\infty \quad \forall B \in \mathcal{A},\ B \subset\subset A\}\,. \tag{2.1}$$

Next we recall a semicontinuity result by Ioffe [20].

Theorem 2.4 *Let A be a bounded open set of $\mathbf{R}^n$ and let $g : A \times \mathbf{R}^N \times \mathbf{R}^{nN} \to [0, +\infty)$ be a Carathéodory function such that $g(x, u, \cdot)$ is convex for every $u \in \mathbf{R}^N$ and for a.e. $x \in A$. Then the functional*

$$G(u) := \int_A g(x, u(x), Du(x))\, dx$$

is lower semicontinuous on $W^{1,1}(A; \mathbf{R}^N)$ with respect to the weak convergence in $W^{1,1}(A; \mathbf{R}^N)$.

Taking account of (1.4), if F and $\overline{F}$ are given by (1.5) and (1.6), we then infer the following

Lemma 2.5 *Suppose* (1.4) *holds and let $A \in \mathcal{A}_0$ and $u \in L^1(\Omega; \mathbf{R}^N)$ be such that $\overline{F}(u, A) < +\infty$. Then $u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N)$ and*

$$\int_A |Du|^{p(x)}\, dx \leq \frac{1}{\alpha}\, \overline{F}(u, A) < +\infty\,.$$

Proof: let $p := \inf_A p(x) > 1$ be given by (1.4) and $\{u_k\} \in L^1(\Omega)$ be such that $\sup_k F(u_k, A) < +\infty$, $u_k \to u$ in $L^1(\Omega)$ and $F(u_k, A) \to \overline{F}(u, A) < +\infty$ as $k \to +\infty$. By (1.5) one has $\{u_{k|A}\} \subset C^{1,p(x)}(A)$ and hence by (1.1)

$$\sup_k \int_A |Du_k|^p\, dx \leq \sup_k \int_A (1 + |Du_k|^{p(x)})\, dx \leq |A| + \frac{1}{\alpha} \sup_k F(u_k, A) < +\infty\,.$$

Then, possibly passing to a subsequence we have that $Du_k \rightharpoonup Du$ weakly in $L^p(A)$ and hence $u_k \rightharpoonup u$ weakly in $W^{1,1}(A)$. Since $g(x,z) := |z|^{p(x)}$ is convex in z for a.e. $x \in \Omega$, by Theorem 2.4 and (1.1)

$$\alpha \int_A |Du|^{p(x)}\, dx \le \liminf_{k\to+\infty} \alpha \int_A |Du_k|^{p(x)}\, dx \le \lim_{k\to+\infty} \int_A f(x, Du_k)\, dx = \overline{F}(u,A) < +\infty$$

and hence we obtain the assertion by (2.1). $\square$

We now prove a representation result, stated in Remark 1.16, about the l.s.c. envelope $\overline{G}$ of the localized functional G, defined by (1.17) and (1.16), respectively, under the hypotheses of Example 1.15.

Lemma 2.6 *Let $\Omega = B_1$, $n = 2$, $N = 1$, $p(x)$ be given by (1.11), with $1 < p < 2 < q < 2p$, $f(x,z) := |z|^{p(x)}$ and $\Sigma := \{x \in B_1 \mid x_1 x_2 = 0\}$. If $A \in \mathcal{A}_0$, with $A \subset\subset B_1 \setminus \Sigma$, then for every function $u \in L^1(B_1)$ with $u_{|A} \in W^{1,p(x)}(A)$ we have*

$$\overline{G}(u,A) = \int_A |Du(x)|^{p(x)}\, dx < +\infty\,.$$

PROOF: for $i = 1, \ldots, 4$ let $A_i := A \cap \Omega_i$, where

$$\Omega_i := \{x = (\rho\cos\theta, \rho\sin\theta) \in B_1 \mid 0 \le \rho < 1 \text{ and } (i-1)\pi/2 < \theta < i\pi/2\}\,.$$

Since $p(x)$ is constant on each Ω_i, we can find smooth sequences $\{u_k^i\} \subset C^{1,p(x)}(A_i)$ converging to u in $L^1(A_i)$ and such that

$$\lim_{k\to+\infty} \int_{A_i} |Du_k^i|^{p(x)}\, dx = \int_{A_i} |Du|^{p(x)}\, dx \qquad \forall\, i = 1, \ldots, 4\,. \tag{2.2}$$

Moreover, since $A_i \subset\subset \Omega_i$, we can easily construct a sequence $\{u_k\} \subset C^{1,p(x)}(B_1)$ converging to u in $L^1(B_1)$ and such that $u_k \equiv u_k^i$ on A_i for every $i = 1, \ldots, 4$. Then by (2.2)

$$\overline{G}(u,A) \le \liminf_{k\to+\infty} \int_A |Du_k|^{p(x)}\, dx = \sum_{i=1}^{4} \lim_{k\to+\infty} \int_{A_i} |Du_k^i|^{p(x)}\, dx = \int_A |Du|^{p(x)}\, dx < +\infty\,.$$

On the other side, since $\overline{G}(u,A) < +\infty$, for every sequence $\{v_k\} \subset C^{1,p(x)}(B_1)$ converging to u in $L^1(B_1)$ and such that $\sup_k \int_A |Dv_k|^{p(x)}\, dx < +\infty$, by an argument similar to Lemma 2.5 we infer

$$\int_A |Du|^{p(x)}\, dx \le \liminf_{k\to+\infty} \int_A |Dv_k|^{p(x)}\, dx$$

and hence, taking the infimum on $\{v_k\}$ in the right-hand side, by (1.17) we conclude with

$$\int_A |Du|^{p(x)}\, dx \le \overline{G}(u,A)$$

and finally with the assertion. $\square$

To apply the localization method in Sec. 3, we need the following results.

Lemma 2.7 *Under the hypotheses of Theorem 1.8, let $A, A', B \in \mathcal{A}$, $A' \subset\subset A$, and $\varphi \in C_0^\infty(\Omega)$ be a cut-off function between A' and A. Then for every $u, v \in L^1(\Omega; \mathbf{R}^N)$, with $u_{|A} \in C^{1,p(x)}(A; \mathbf{R}^N)$ and $v_{|B} \in C^{1,p(x)}(B; \mathbf{R}^N)$, we have that $\varphi u + (1-\varphi)v$ belongs to $L^1(\Omega; \mathbf{R}^N)$ and to $C^{1,p(x)}(A' \cup B; \mathbf{R}^N)$.*

PROOF: clearly $\varphi u + (1-\varphi)v \in L^1(\Omega) \cap C^1(A' \cup B)$, whereas the convexity of $|\cdot|^{p(x)}$ yields that $\|\varphi u + (1-\varphi)v\|_{L^{p(x)}(A'\cup B)} < +\infty$. Moreover, by setting $K := \operatorname{spt}\varphi$ and $q := \sup_K p(x) < +\infty$, which is finite by (1.4), we have

$$\begin{aligned}\int_{A'\cup B} |D(\varphi u + (1-\varphi)v)|^{p(x)}\, dx &= \int_{A'\cup B} |\varphi Du + (1-\varphi)Dv + D\varphi \otimes (u-v)|^{p(x)}\, dx \\ &\le \int_A |Du|^{p(x)}\, dx + \int_B |Dv|^{p(x)}\, dx + \int_{(B\cap K)\setminus A'} |\cdot|^{p(x)}\, dx =: I + II + III\,.\end{aligned}$$

Then the terms I and II are finite, whereas

$$III \leq 4^{q-1} \int_{(B\cap K)\setminus A'} [(|Du|^{p(x)} + |Dv|^{p(x)}) + (1+\|D\varphi\|_\infty^q)(|u|^{p(x)} + |v|^{p(x)})]\,dx$$

which is finite, since $(B\cap K)\setminus A' \subset A\cap B$, and the proof is complete. □

Due to growth condition (1.1), we then infer the following fundamental $L^{p(x)}$ estimate.

Lemma 2.8 (Fundamental estimate) *Under the hypotheses of Theorem* 1.8*, for all open sets $A, A', B \in \mathcal{A}$, with $A' \subset\subset A$, and for every $\sigma > 0$, there exists a constant $M_\sigma > 0$, depending on α, β, $a(x)$ and $p(x)$, such that for every $u, v \in L^1(\Omega; \mathbf{R}^N)$ there exists a cut-off function φ between A' and A such that*

$$F(\varphi u + (1-\varphi)v, A'\cup B) \leq (1+\sigma)(F(u,A) + F(v,B)) + M_\sigma \int_{A\cap B} |u-v|^{p(x)}\,dx + \sigma\,. \tag{2.3}$$

Proof: we can suppose the right-hand side of (2.3) to be finite and hence, by (1.5), that $u \in C^{1,p(x)}(A)$ and $v \in C^{1,p(x)}(B)$. Then Lemma 2.7 gives $\varphi u + (1-\varphi)v \in L^1(\Omega)\cap C^{1,p(x)}(A'\cup B)$ for every cut-off φ between A' and A. The proof is then a readaptation of [7, 12.2], to which we refer for the following notations. Taking in fact $p(x)$ instead of p, $0<\nu<\delta$, $2\delta := \mathrm{dist}(A', \partial A)$, $0 < r < \delta - \nu$, $B_r^\nu := \{x \in B \mid r < \mathrm{dist}(x, A') < r+\nu\}$ and finally $q := \sup_{x\in B_0^\delta} p(x)$, which is finite by (1.4) since $B_0^\delta \subset\subset A$, we let φ be a cut-off between $\{x \in A \mid \mathrm{dist}(x,A') < r\}$ and $\{x\in A \mid \mathrm{dist}(x,A') < r+\nu\}$, with $\|D\varphi\|_\infty \leq 2/\nu$. Since by (1.5) all functionals involved in the following are integrals, by (1.1) one easily obtains (for $\nu < 2$)

$$F(\varphi u + (1-\varphi)v, A'\cup B) \leq F(u,A) + F(v,B) + \mu(B_r^\nu) + \beta\,\frac{2^{2q-1}}{\nu^q} \int_{A\cap B} |u-v|^{p(x)}\,dx$$

where

$$\mu(E) := (1+\beta\,2^{q-1}) \int_E (a(x) + |Du|^{p(x)} + |Dv|^{p(x)})\,dx\,.$$

Now, for every $m \in \mathbf{N}$ there exists $k = k_m \in \{1,\ldots,m\}$ such that, by (1.1), since $B_0^\delta \subset A\cap B$,

$$\mu(B_{\delta(k-1)/m}^{\delta/m}) \leq \frac{1}{m}\,\mu(B_0^\delta) \leq \frac{(1+\beta\,2^{q-1})}{m} \left(\int_{A\cap B} a(x)\,dx + \frac{1}{\alpha}\,(F(u,A) + F(v,B))\right)\,.$$

It then suffices to take

$$m \geq \frac{(1+\beta\,2^{q-1})}{\sigma}\cdot \max\left\{\int_{A\cap B} a(x)\,dx\,,\ \frac{1}{\alpha}\right\}\,, \qquad \nu = \frac{\delta}{m}\,, \qquad r = \delta\,\frac{k_m - 1}{m}$$

and hence obtain (2.3) with $M_\sigma = \beta 2^{2q-1} m^q \delta^{-q}$. □

Thanks to Lemma 2.2, we finally make use of the fundamental $L^{p(x)}$ estimate to prove a weak subadditivity property for the set function $\overline{F}(w,\cdot)$.

Lemma 2.9 (Weak subadditivity) *Under the hypotheses of Theorem* 1.8*, for every $w \in L^1(\Omega; \mathbf{R}^N)$ we have*

$$\overline{F}(w, A'\cup B) \leq \overline{F}(w,A) + \overline{F}(w,B) \tag{2.4}$$

for every $A', A \in \mathcal{A}$, with $A' \subset\subset A$, and every $B \in \mathcal{A}$ such that $B\cap\Omega_i$ has Lipschitz boundary for all i.

Proof: in case the right-hand side of (2.4) is finite (otherwise there is nothing to prove), thanks to Remark 2.1 we first assume in addition that $A \in \mathcal{A}_0$ and $(A\cap B)\cap\Omega_i$ has Lipschitz boundary for every i. By definition of relaxation, there exist sequences of functions $\{u_j\}$ and $\{v_j\}$ in $L^1(\Omega)$, both converging to w in $L^1(\Omega)$, with $u_{j|A} \in C^{1,p(x)}(A)$ and $v_{j|B} \in C^{1,p(x)}(B)$ for every j, such that

$$\overline{F}(w,A) = \liminf_{j\to+\infty} F(u_j, A) < +\infty \qquad \text{and} \qquad \overline{F}(w,B) = \liminf_{j\to+\infty} F(v_j,B) < +\infty\,. \tag{2.5}$$

We take subsequences, which we relabel $\{u_j\}$ and $\{v_j\}$, such that the lower limits in (2.5) are limits and hence by (1.1)

$$\sup_{j\in\mathbf{N}}\int_A |Du_j|^{p(x)}\,dx+\sup_{j\in\mathbf{N}}\int_B |Dv_j|^{p(x)}\,dx<+\infty\,. \tag{2.6}$$

By the fundamental $L^{p(x)}$ estimate (Lemma 2.8) applied to u_j and v_j, for any $\sigma>0$ we can find $M_\sigma>0$ and a sequence $w_j:=\varphi_j u_j+(1-\varphi_j)v_j$, where the φ_j are cut-off functions between A' and A, such that

$$F(w_j,A'\cup B)\leq (1+\sigma)(F(u_j,A)+F(v_j,B))+M_\sigma\int_{A\cap B}|u_j-v_j|^{p(x)}\,dx+\sigma\,. \tag{2.7}$$

Then, if $C:=A\cap B\in\mathcal{A}_0$, since $C_i:=C\cap\Omega_i$ has Lipschitz boundary for every i, taking $q:=\sup_C p(x)<+\infty$, which is finite by (1.4), by Lemma 2.2 and (2.6)

$$\int_{A\cap B}|u_j-v_j|^{p(x)}\,dx\leq 2^{q-1}\int_C(|u_j-w|^{p(x)}+|v_j-w|^{p(x)})\,dx\to 0 \tag{2.8}$$

as $j\to+\infty$. Therefore, since $w_j\to w$ in $L^1(\Omega)$, by (2.7), (2.5) and (2.8)

$$\overline{F}(w,A'\cup B)\leq\liminf_{j\to+\infty}F(w_j,A'\cup B)\leq(1+\sigma)(\overline{F}(w,A)+\overline{F}(w,B))+\sigma$$

and hence (2.4) holds by the arbitrariness of $\sigma>0$. Finally, to prove (2.4) for any $A\in\mathcal{A}$, by Remark 2.1 it is enough to take $\widehat{A}\in\mathcal{A}_0$ such that $A'\subset\subset\widehat{A}\subset\subset A$ and $(\widehat{A}\cap B)\cap\Omega_i$ has Lipschitz boundary for every i. Then, by the monotonicity of $\overline{F}(w,\cdot)$

$$\overline{F}(w,A'\cup B)\leq\overline{F}(w,\widehat{A})+\overline{F}(w,B)\leq\overline{F}(w,A)+\overline{F}(w,B)$$

and the proof is complete. $\square$

3 Measure property of the relaxed functional

In this section we prove Theorem 1.8 showing that, for every function $u\in L^1(\Omega;\mathbf{R}^N)$, the relaxed functional $\overline{F}(u,\cdot)$ is the trace on $\mathcal{A}$ of a Borel measure on Ω. To this aim we will apply the De Giorgi-Letta criterion (Proposition 1.7) to the increasing set function $\overline{F}(u,\cdot)$, for every $u\in L^1(\Omega;\mathbf{R}^N)$, see Remark 1.6. As a consequence, we will then prove Theorem 1.9.

Proof of Theorem 1.8: we first consider the case $f(x,z):=|z|^{p(x)}$, so that in particular $f(x,\cdot)$ is convex in $\mathbf{R}^{nN}$ for a.e. $x\in\Omega$. Once obtained inner regularity (Proposition 3.1) and hence, by weak subadditivity, measure property (Proposition 3.3) for $f(x,z):=|z|^{p(x)}$, thanks to the fundamental $L^{p(x)}$ estimate, weak subadditivity and growth condition (1.1), we will then extend inner regularity (Proposition 3.4) and hence measure property to $\overline{F}(u,\cdot)$ for any choice of f in (1.5).

Step 1: the case $f(x,Du):=|Du|^{p(x)}$.
Define now $\Psi:L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ by

$$\Psi(u,A):=\begin{cases}\displaystyle\int_A|Du(x)|^{p(x)}\,dx & \text{if}\quad u\in C^{1,p(x)}(A;\mathbf{R}^N)\\ +\infty & \text{elsewhere on }\ L^1(\Omega;\mathbf{R}^N)\end{cases} \tag{3.1}$$

and let $\overline{\Psi}(\cdot,A)$ be the $L^1(\Omega)$-lower semicontinuous envelope of $\Psi(\cdot,A)$ for every $A\in\mathcal{A}$. Finally, let $\overline{\Psi}_-(u,\cdot)$ be the inner regular envelope of $\overline{\Psi}(u,\cdot)$, see (1.7), i.e, for every $u\in L^1(\Omega;\mathbf{R}^N)$

$$\overline{\Psi}_-(u,C):=\sup\{\overline{\Psi}(u,B)\mid B\in\mathcal{A}\,,\ B\subset\subset C\}\,,\qquad C\in\mathcal{A}\,. \tag{3.2}$$

Making use of the argument in [10, Lemma 2.3], we are able to prove the following

Proposition 3.1 (Inner regularity) *Let $f(x,z) := |z|^{p(x)}$ and $\Psi : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ be given by (3.1). Then for every $u\in L^1(\Omega;\mathbf{R}^N)$ the increasing set function $\overline{\Psi}(u,\cdot)$ is inner regular, i.e., for every $C\in\mathcal{A}$*

$$\overline{\Psi}(u,C)=\overline{\Psi}_-(u,C)\,, \tag{3.3}$$

where $\overline{\Psi}_-(u,C)$ is given by (3.2).

PROOF: by the monotonicity of $\overline{\Psi}(u,\cdot)$, it suffices to show that "$\leq$" holds in (3.3), in case $\overline{\Psi}_-(u,C)<+\infty$.

To this aim, for every $\varepsilon>0$ and $j\in\mathbf{N}_0:=\mathbf{N}\cup\{0\}$, let $A^j\in\mathcal{A}_0$ be such that $A^j\subset\subset A^{j+1}\subset\subset C$, $|\partial A^j|=0$ and

$$\overline{\Psi}_-(u,C)-\varepsilon\,2^{-j}\leq\overline{\Psi}(u,A^j)\leq\overline{\Psi}_-(u,C)\qquad\forall\,j\in\mathbf{N}_0\,. \tag{3.4}$$

Moreover, thanks to Remark 2.1 we can choose A^j so that $A^j_i:=A^j\cap\Omega_i$ has Lipschitz boundary for every i and j. In fact, once defined the sequence $\{A^j\}$ satisfying (3.4), it suffices to find $\widetilde{A}^j$, by induction on j, such that $A^j\subset\subset\widetilde{A}^j\subset\subset A^{j+1}$ and $\widetilde{A}^j\cap\Omega_i$ has Lipschitz boundary for every i.

For every $j\in\mathbf{N}_0$ let $\{u^j_h\}_h\subset L^1(\Omega)$, obviously depending also on ε, be such that

$$\lim_{h\to+\infty}\|u^j_h-u\|_{L^1(\Omega)}=0\quad\text{ and }\quad\overline{\Psi}(u,A^j)=\liminf_{h\to+\infty}\Psi(u^j_h,A^j)<+\infty\,. \tag{3.5}$$

Possibly passing to a subsequence, we can suppose that $u^j_h\to u$ a.e. on Ω, $\{u^j_{h|A^j}\}_h\subset C^{1,p(x)}(A^j)$, $\sup_h\int_{A^j}|Du^j_h|^{p(x)}\,dx<+\infty$ and that the lower limit in (3.5) is a limit. Then, since A^j_i has Lipschitz boundary for every i, by Lemma 2.2 and (3.5)

$$\lim_{h\to+\infty}\|u^j_h-u\|_{L^{p(x)}(A^j)}=0\qquad\forall\,j\in\mathbf{N}_0\,. \tag{3.6}$$

Set $A^{-1}:=\emptyset$ and let us consider a partition of unity $\{\phi_j\}_{j\in\mathbf{N}_0}$ relative to the open covering of C given by $\{A^{j+1}\setminus\overline{A}^{j-1}\}_{j\in\mathbf{N}_0}$. More precisely, for every $j\in\mathbf{N}_0$ we have that $\phi_j\in C^1_0(A^{j+1}\setminus\overline{A}^{j-1})$ and

$$0\leq\phi_j(x)\leq1\,,\quad\sum_{j=0}^{+\infty}\phi_j(x)=1\qquad\forall\,x\in C\,. \tag{3.7}$$

For every $j\in\mathbf{N}$, let $h(j)\in\mathbf{N}$ to be chosen later, set $v_j:=u^j_{h(j)}$ and

$$w_\varepsilon(x):=\sum_{j=1}^{+\infty}\phi_{j-1}(x)\,v_j(x)\,,\qquad x\in C\,. \tag{3.8}$$

Note that since $v_{j|A^j}\in C^{1,p(x)}(A^j)$, we have that $\phi_{j-1}(x)\,v_j(x)\in C^1_0(C)$ for every $j\in\mathbf{N}$. Moreover, since every x in C has a neighborhood contained at most in the union of three sets of the type $A^{j+1}\setminus\overline{A}^{j-1}$, for every $x\in C$ the infinite sum in the right-hand side of (3.8) reduces to a finite one, hence $w_\varepsilon\in C^1(C)$ for every $\varepsilon>0$. Taking $w_\varepsilon\equiv u$ in $\Omega\setminus C$, for every $t\in]0,1[$ the function tw_ε belongs to $L^1(\Omega)$ and by (3.7)

$$\begin{aligned}\|tw_\varepsilon-u\|_{L^1(\Omega)}&\leq t\,\|w_\varepsilon-u\|_{L^1(C)}+(1-t)\,\|u\|_{L^1(\Omega)}\\&=t\,\|\sum_{j=1}^{+\infty}\phi_{j-1}\,(v_j-u)\|_{L^1(C)}+(1-t)\,\|u\|_{L^1(\Omega)}\\&\leq t\sum_{j=1}^{+\infty}\int_{A^j}|u^j_{h(j)}-u|\,dx+(1-t)\,\|u\|_{L^1(\Omega)}\,.\end{aligned} \tag{3.9}$$

Moreover, since $f(x,\cdot)=|\cdot|^{p(x)}$ is convex, $0\leq\phi_{j-1}\leq1$ and the sum in (3.8) is locally finite, for $t\in]0,1[$

we have

$$
\begin{aligned}
\int_C f(x,tDw_\varepsilon)\,dx &= \int_C f\big(x,t\,(\sum_{j=1}^{+\infty}\phi_{j-1}Dv_j + D\phi_{j-1}v_j)\big)\,dx \\
&\le t\int_C f\big(x,\sum_{j=1}^{+\infty}\phi_{j-1}Dv_j\big)\,dx + (1-t)\int_C f\big(x,\frac{t}{1-t}\sum_{j=1}^{+\infty}D\phi_{j-1}v_j\big)\,dx \\
&\le \int_{A^1} f(x,Dv_1)\,dx + \sum_{j=2}^{+\infty}\int_{A^j\setminus A^{j-2}} f(x,Dv_j)\,dx + (1-t)\int_C f\big(x,\frac{t}{1-t}\sum_{k=1}^{+\infty}D\phi_{k-1}v_k\big)\,dx\,.
\end{aligned}
\tag{3.10}
$$

As for the last term in (3.10) we observe that $D\phi_{k-1}\equiv 0$ in A^0 for every $k\in\mathbf{N}$ and that $\sum_{k=1}^{+\infty}D\phi_{k-1}v_k = D\phi_{j-1}v_j + D\phi_j v_{j+1}$ in $A^j\setminus\overline{A}^{j-1}$. Hence, since $f(x,0)\equiv 0$, we have

$$
\int_C f\big(x,\frac{t}{1-t}\sum_{k=1}^{+\infty}D\phi_{k-1}v_k\big)\,dx = \sum_{j=1}^{+\infty}\int_{A^j\setminus A^{j-1}} f\big(x,\frac{t}{1-t}\,(D\phi_{j-1}v_j + D\phi_j v_{j+1})\big)\,dx\,. \tag{3.11}
$$

Now observe that since $A^{j-2}\subset A^j$ and $\{u_h^j\}\subset C^{1,p(x)}(A^{j-2})$, by (3.4) and (3.5)

$$
\overline{\Psi}_-(u,C) - \varepsilon\,2^{2-j} \le \overline{\Psi}(u,A^{j-2}) \le \liminf_{h\to+\infty}\int_{A^{j-2}} f(x,Du_h^j)\,dx \qquad \forall\, j=2,3,\dots \tag{3.12}
$$

Now, for every j, in $A^j\setminus\overline{A}^{j-1}$ we have $\sum_{k=1}^{+\infty}\phi_k = \phi_{j-1}+\phi_j\equiv 1$, hence $D\phi_{j-1}+D\phi_j\equiv 0$ and then, since $u_h^j\to u$ a.e. on Ω, we have $D\phi_{j-1}u_h^j + D\phi_j u_k^{j+1}\to 0$ a.e. on $A^j\setminus\overline{A}^{j-1}$ as $h,k\to+\infty$. Moreover, if $q_j := \sup_{A^j} p(x) < +\infty$, which si finite by (1.4) since $A^j\in\mathcal{A}_0$,

$$
f\left(x,\frac{t}{1-t}\,(D\phi_{j-1}u_h^j + D\phi_j u_k^{j+1})\right) \le \left(1+\left(\frac{t}{1-t}\right)^{q_j}\right) 2^{q_j-1}(\|D\phi_{j-1}\|_\infty^{q_j}\,|u_h^j|^{p(x)} + \|D\phi_j\|_\infty^{q_j}\,|u_k^{j+1}|^{p(x)})
$$

a.e. on $A^j\setminus\overline{A}^{j-1}$ whereas by (3.6) we have $\|u_h^j - u\|_{L^{p(x)}(A^j\setminus A^{j-1})} + \|u_k^{j+1}-u\|_{L^{p(x)}(A^j\setminus A^{j-1})}\to 0$ as $h,k\to+\infty$. Then, by the dominated convergence theorem, since $f(x,0)\equiv 0$ we infer

$$
\lim_{h\to+\infty}\lim_{k\to+\infty}\int_{A^j\setminus A^{j-1}} f\left(x,\frac{t}{1-t}\,(D\phi_{j-1}u_h^j + D\phi_j u_k^{j+1})\right)\,dx = 0 \tag{3.13}
$$

for every $j\in\mathbf{N}$. Hence by (3.5), (3.4), (3.12) and (3.13), since $\{u_h^j\}\subset C^{1,p(x)}(A^j)$, for every $j\in\mathbf{N}$ we deduce existence of $h(j)$ and $k(j)$ in $\mathbf{N}$ such that $h(j+1)\ge k(j)$ and

$$
\int_{A^j}|u_{h(j)}^j - u|\,dx \le \varepsilon\,2^{-j} \qquad \forall\, j\in\mathbf{N}\,, \tag{3.14}
$$

$$
\int_{A^j} f(x,Du_{h(j)}^j)\,dx \le \overline{\Psi}_-(u,C) + \varepsilon\,2^{-j} \qquad \forall\, j\in\mathbf{N}\,, \tag{3.15}
$$

$$
\int_{A^{j-2}} f(x,Du_{h(j)}^j)\,dx \ge \overline{\Psi}_-(u,C) - \frac{2\varepsilon}{2^{j-2}} \qquad \forall\, j=2,3,\dots\,, \tag{3.16}
$$

$$
\int_{A^j\setminus A^{j-1}} f\left(x,\frac{t}{1-t}\,(D\phi_{j-1}u_{h(j)}^j + D\phi_j u_k^{j+1})\right)\,dx \le \varepsilon\,2^{-j} \qquad \forall\, j\in\mathbf{N}\,,\ k\ge k(j)\,. \tag{3.17}
$$

In particular, by choosing $k=h(j+1)$ in (3.17) we obtain

$$
\int_{A^j\setminus A^{j-1}} f\left(x,\frac{t}{1-t}\,(D\phi_{j-1}u_{h(j)}^j + D\phi_j u_{h(j+1)}^{j+1})\right)\,dx \le \varepsilon\,2^{-j} \qquad \forall\, j\in\mathbf{N}\,. \tag{3.18}
$$

With this choiche of $\{h(j)\}_j$, by (3.9) and (3.14) we deduce that

$$tw_\varepsilon \to u \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0^+ \text{ and } t \to 1^- \tag{3.19}$$

and by (3.10), (3.15), (3.16), (3.11) and (3.18) that

$$\begin{aligned}\int_C f(x, tDw_\varepsilon)\,dx &\leq \overline{\Psi}_-(u,C) + \frac{\varepsilon}{2} + \sum_{j=2}^{+\infty}\left(\overline{\Psi}_-(u,C) + \frac{\varepsilon}{2^j} - \overline{\Psi}_-(u,C) + \frac{2\varepsilon}{2^{j-2}}\right) + (1-t)\sum_{j=1}^{+\infty}\frac{\varepsilon}{2^j} \\ &\leq \overline{\Psi}_-(u,C) + 5\varepsilon + (1-t)\,\varepsilon < +\infty\,.\end{aligned} \tag{3.20}$$

In particular we infer that $tw_\varepsilon \in C^{1,p(x)}(C)$ for every $\varepsilon > 0$ and $t \in]0,1[$ and hence, by (3.1), $\overline{\Psi}(tw_\varepsilon, C) = \int_C f(x, tDw_\varepsilon)\,dx$. Finally, as $\varepsilon \to 0^+$ and $t \to 1^-$, by (3.20) and (3.19) we obtain that $\overline{\Psi}(u,C) \leq \overline{\Psi}_-(u,C)$ and hence the assertion. □

Remark 3.2 If we define the localized functional $\Psi(u,A)$ by (1.16) with $f(x,z) = |z|^{p(x)}$, the argument of Proposition 3.1 may fail. In fact, the definition (3.8), with $v_j := u^j_{h(j)} \in C^{1,p(x)}(\Omega;\mathbf{R}^N)$, gives a function $w_\varepsilon \in C^{1,p(x)}(C;\mathbf{R}^n)$ which in general cannot be extended to a function in $C^{1,p(x)}(\Omega;\mathbf{R}^N)$ if the boundary of C is not smooth.

Now, since we have just proved (Proposition 3.1) that, in case $f(x,z) := |z|^{p(x)}$, the increasing set function $\overline{\Psi}(u,\cdot)$ is inner regular, and $\overline{\Psi}(u,\cdot)$ is superadditive, thanks to Proposition 1.7 we obtain measure property of $\overline{\Psi}(u,\cdot)$, for every $u \in L^1(\Omega;\mathbf{R}^N)$, if we show that $\overline{\Psi}(u,\cdot)$ is subadditive.

Proposition 3.3 (Subadditivity) *For every $w \in L^1(\Omega;\mathbf{R}^N)$ we have*

$$\overline{\Psi}(w, A\cup B) \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,B) \qquad \forall\, A,B \in \mathcal{A}\,. \tag{3.21}$$

PROOF: by inner regularity (Proposition 3.1), it is well known that weak subadditivity (Lemma 2.9 with $F = \Psi$) yields (3.21) for any $A,B \in \mathcal{A}$, provided $B \cap \Omega_i$ has Lipschitz boundary for every i. In fact, for any $C \in \mathcal{A}$ with $C \subset\subset A \cup B$, by enlarging a bit the subset $C \setminus \overline{B}$, we can find $A' \subset\subset A$ such that $C \subset A' \cup B$, which yields, by (2.4),

$$\overline{\Psi}(w,C) \leq \overline{\Psi}(w, A'\cup B) \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,B)$$

and hence (3.21), by inner regularity, letting $C \nearrow A \cup B$. Finally, to prove (3.21) for any $B \in \mathcal{A}$, in case $\overline{\Psi}(w, A\cup B) < +\infty$, for each small $\varepsilon > 0$ take $C \subset\subset A \cup B$ such that by inner regularity $\overline{\Psi}(w,C) \geq \overline{\Psi}(w, A\cup B) - \varepsilon$. By Remark 2.1, we can find an open set $\widetilde{B} \in \mathcal{A}$ with $C \setminus \overline{A} \subset\subset \widetilde{B} \subset\subset B$ and such that $\widetilde{B} \cap \Omega_i$ has Lipschitz boundary for every i. Then since $C \subset A \cup \widetilde{B}$

$$\overline{\Psi}(w,A\cup B) \leq \overline{\Psi}(w,C) + \varepsilon \leq \overline{\Psi}(w, A\cup\widetilde{B}) + \varepsilon \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,\widetilde{B}) + \varepsilon \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,B) + \varepsilon$$

and hence we obtain (3.21), letting $\varepsilon \to 0^+$. In case $\overline{\Psi}(w, A\cup B) = +\infty$, take $C \subset\subset A\cup B$ with $\overline{\Psi}(w,C) > 1/\varepsilon$, so that arguing as before

$$\frac{1}{\varepsilon} \leq \overline{\Psi}(w,C) \leq \overline{\Psi}(w, A\cup\widetilde{B}) \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,\widetilde{B}) \leq \overline{\Psi}(w,A) + \overline{\Psi}(w,B)$$

and hence (3.21) follows by letting again $\varepsilon \to 0^+$. □

Step 2: measure property of $\overline{F}(u,\cdot)$.
Consider now any Borel function f as in Theorem 1.8. We first prove the following

Proposition 3.4 (Inner regularity) *For every $w \in L^1(\Omega;\mathbf{R}^N)$, $\overline{F}(w,\cdot)$ is an inner regular set function.*

PROOF: since $\overline{F}(w,\cdot)$ is an increasing set function, if $\overline{F}_-(w,\cdot)$ is defined by (1.7), it suffices to prove that

$$\overline{F}(w,C) \leq \overline{F}_-(w,C) \tag{3.22}$$

for every fixed open set $C \in \mathcal{A}$ and every function $w \in L^1(\Omega)$ such that $\overline{F}_-(w,C) < +\infty$. To this aim note that growth condition (1.1) yields the estimate

$$\alpha\,\overline{\Psi}(w,A) \le \overline{F}(w,A) \le \int_A a(x)\,dx + \beta\,\overline{\Psi}(w,A) \tag{3.23}$$

for every $w \in L^1(\Omega)$ and $A \in \mathcal{A}$, where Ψ is given by (3.1), and the same estimate with $\overline{\Psi}_-$ and $\overline{F}_-$, respectively, instead of $\overline{\Psi}$ and $\overline{F}$ in (3.23). In particular, by the monotonicity and the inner regularity of $\overline{\Psi}(w,\cdot)$, see Proposition 3.1,

$$\overline{\Psi}(w,A) \le \overline{\Psi}(w,C) = \overline{\Psi}_-(w,C) \le \frac{1}{\alpha}\overline{F}_-(w,C) < +\infty \tag{3.24}$$

for every $A \in \mathcal{A}$ with $A \subset C$. For every $\varepsilon > 0$ we can choose an open set $A_\varepsilon \in \mathcal{A}$ with $A_\varepsilon \subset\subset C$ such that, by inner regularity of $\overline{\Psi}(w,\cdot)$ and absolute continuity of $a(x) \in L^1(\Omega)$,

$$\overline{\Psi}(w,C) \le \overline{\Psi}(w,A_\varepsilon) + \varepsilon \qquad \text{and} \qquad 0 \le \int_{C\setminus A_\varepsilon} a(x)\,dx \le \varepsilon\,. \tag{3.25}$$

Moreover, in order to apply Lemma 2.2, by Remark 2.1 we can suppose that $A_\varepsilon \cap \Omega_i$ has Lipschitz boundary for every i. Finally, let $B = B_\varepsilon := C \setminus \overline{A}_\varepsilon \in \mathcal{A}$, so that if $\widetilde{\Psi}(w,\cdot)$ is the Borel measure given by the extension of $\overline{\Psi}(w,\cdot)$ to Ω (see iii) in Proposition 1.7), by (3.25) we have

$$\overline{\Psi}(w,B) = \overline{\Psi}(w,C) - \widetilde{\Psi}(w,\overline{A}_\varepsilon) \le \overline{\Psi}(w,C) - \overline{\Psi}(w,A_\varepsilon) \le \varepsilon\,. \tag{3.26}$$

Since (3.24) yields $\overline{\Psi}(w,B) < +\infty$, there exists a sequence $\{v_j\} \subset L^1(\Omega)$, converging to w in $L^1(\Omega)$, such that $v_{j|B} \in C^{1,p(x)}(B)$ for every j and

$$\overline{\Psi}(w,B) = \lim_{j\to+\infty} \int_B |Dv_j|^{p(x)}\,dx < +\infty\,. \tag{3.27}$$

In particular, by (1.1), (3.25) and (3.27), since $B = C \setminus \overline{A}_\varepsilon$,

$$\liminf_{j\to+\infty} F(v_j,B) \le \int_{C\setminus A_\varepsilon} a(x)\,dx + \beta \lim_{j\to+\infty} \Psi(v_j,B) \le \varepsilon + \beta\,\overline{\Psi}(w,B)\,. \tag{3.28}$$

Choose now $A', A \in \mathcal{A}_0$ such that $A_\varepsilon \subset\subset A' \subset\subset A \subset\subset C$. In order to apply Lemma 2.2, by Remark 2.1 we can suppose again that $A \cap \Omega_i$ has Lipschitz boundary for every i. Since $\overline{F}(w,A) < +\infty$, there exists a sequence $\{u_j\} \subset L^1(\Omega)$, converging to w in $L^1(\Omega)$, such that $u_{j|A} \in C^{1,p(x)}(A)$ for every j and

$$\overline{F}(w,A) = \lim_{j\to+\infty} \int_A f(x,Du_j)\,dx < +\infty\,. \tag{3.29}$$

By the fundamental $L^{p(x)}$ estimate (Lemma 2.8) applied with u_j on A and v_j on B, for any $\sigma > 0$ we can find $M_\sigma > 0$ and a sequence $\{\varphi_j\}$ of smooth cut-off functions between A' and A such that

$$F(w_j, A' \cup B) \le (1+\sigma)(F(u_j,A) + F(v_j,B)) + M_\sigma \int_{A\cap B} |u_j - v_j|^{p(x)}\,dx + \sigma\,, \tag{3.30}$$

where $w_j := \varphi_j u_j + (1-\varphi_j)v_j$. By (3.27), (3.29) and (1.1) we have

$$\sup_j \int_{A\cap B} (|Du_j|^{p(x)} + |Dv_j|^{p(x)})\,dx < +\infty\,.$$

Moreover, $A \cap B = A \setminus \overline{A}_\varepsilon \in \mathcal{A}_0$ and, since $A_\varepsilon \subset\subset A$, then $A \cap B$ has boundary given by the disjoint union $\partial A \cup \partial A_\varepsilon$. Then $(A\cap B) \cap \Omega_i$ has Lipschitz boundary given by $\partial(A \cap \Omega_i) \cup \partial(A_\varepsilon \cap \Omega_i)$ for every i. By Lemma 2.2 we conclude that $\|u_j - v_j\|_{L^{p(x)}(A\cap B)} \to 0$ as $j \to +\infty$. Then, since $w_j = \varphi_j u_j + (1-\varphi_j)v_j \to w$ in $L^1(\Omega)$, by (3.30), (3.29) and (3.28) we obtain

$$\overline{F}(w, A' \cup B) \le \liminf_{j\to+\infty} F(w_j, A' \cup B) \le (1+\sigma)(\overline{F}(w,A) + \varepsilon + \beta\,\overline{\Psi}(w,B)) + \sigma\,. \tag{3.31}$$

Finally, since $B = C \setminus \overline{A}_\varepsilon$ yields $A' \cup B = C$, taking $\varepsilon > 0$ small so that $\varepsilon(1+\beta) \le \sigma$, by (3.26) and (3.31)

$$\overline{F}(w, C) \le (1+\sigma)(\overline{F}(w, A) + \sigma) + \sigma \tag{3.32}$$

and hence (3.22) holds by the arbitraryness of $\sigma > 0$. $\square$

Since we have just proved that $\overline{F}(w, \cdot)$ is inner regular for every $w \in L^1(\Omega; \mathbf{R}^N)$, arguing as in Proposition 3.3, by weak subadditivity (2.4) we obtain that $\overline{F}(w, \cdot)$ is subadditive. Since $\overline{F}(w, \cdot)$ is trivially superadditive, by Proposition 1.7 the proof of Theorem 1.8 is complete. $\square$

Proof of Theorem 1.9: for every i denote $\mathcal{A}_i := \mathcal{A}(\Omega_i)$ the family of open subsets of Ω_i. We remark that $\overline{F}$ is a local functional, i.e., $\overline{F}(u, A) = \overline{F}(v, A)$ if $u, v \in L^1(\Omega)$ and $u = v$ a.e. on A, for every $A \in \mathcal{A}$. This follows from measure and increasing property of $\overline{F}(u, \cdot)$, see e.g. [8, Lemma 4.4.2] for a similar proof. Hence it is well defined the local functional $F_i : L^1(\Omega_i; \mathbf{R}^N) \times \mathcal{A}_i \to [0, +\infty]$ given by $F_i(u, A) := F(\widetilde{u}, A)$, where $\widetilde{u} \in L^1(\Omega; \mathbf{R}^N)$ is any extension of u. Also, denote by $\overline{F}_i(\cdot, A)$ the $L^1(\Omega_i)$-lower semicontinuous envelope of $F_i(\cdot, A)$, so that $\overline{F}_i(u, A) = \overline{F}(\widetilde{u}, A)$ for every $u \in L^1(\Omega_i)$ and $A \in \mathcal{A}_i$, if $\widetilde{u} \in L^1(\Omega)$ is such that $\widetilde{u}_{|\Omega_i} = u$. Finally, denote in an analogous way by $\Psi_i : L^1(\Omega_i; \mathbf{R}^N) \times \mathcal{A}_i \to [0, +\infty]$ the local functional given by $\Psi_i(u, A) := \Psi(\widetilde{u}, A)$, where Ψ is given by (3.1), so that for every $A \in \mathcal{A}_i$

$$\Psi_i(u, A) := \begin{cases} \displaystyle\int_A |Du(x)|^{p_i}\, dx & \text{if } u \in C^{1,p_i}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega_i; \mathbf{R}^N) \end{cases}$$

where $p_i > 1$ is given by (1.3), and let $\overline{\Psi}_i(\cdot, A)$ denote its $L^1(\Omega_i)$-l.s.c. envelope.

If $A \in \mathcal{A}_i$ has Lipschitz boundary, it is easy to show that $\overline{\Psi}_i(u, A) = \int_A |Du|^{p_i}\, dx$ if $u \in W^{1,p_i}(A)$, and $+\infty$ elsewhere on $L^1(\Omega_i)$. Then, since by Proposition 3.1 the set function $\overline{\Psi}_i(u, \cdot)$ is inner regular on $\mathcal{A}_i$ for every $u \in L^1(\Omega_i)$, we obtain that for every $A \in \mathcal{A}_i$

$$\overline{\Psi}_i(u, A) = \begin{cases} \displaystyle\int_A |Du(x)|^{p_i}\, dx & \text{if } u \in W^{1,p_i}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega_i; \mathbf{R}^N)\,. \end{cases} \tag{3.33}$$

Finally, by (3.23) we also have

$$\alpha\, \overline{\Psi}_i(u, A) \le \overline{F}_i(u, A) \le \int_A a(x)\, dx + \beta\, \overline{\Psi}_i(u, A) \tag{3.34}$$

for every $u \in L^1(\Omega_i)$ and $A \in \mathcal{A}_i$.

These facts lead us to apply the classical integral representation theorem [9, Thm 1.1], see also [13, Thm. 20.1], to the relaxed functional $\overline{F}_i$ and write

$$\overline{F}_i(u, A) = \begin{cases} \displaystyle\int_A \varphi_i(x, Du(x))\, dx & \text{if } u \in W^{1,p_i}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega_i; \mathbf{R}^N) \end{cases} \tag{3.35}$$

for every $A \in \mathcal{A}_i$, where $\varphi_i : \Omega_i \times \mathbf{R}^{nN} \to [0, +\infty)$ is a quasi-convex function satisfying growth condition

$$\alpha |z|^{p_i} \le \varphi_i(x, z) \le a(x) + \beta |z|^{p_i} \tag{3.36}$$

for a.e. $x \in \Omega_i$ and all $z \in \mathbf{R}^{nN}$. In fact, Theorem 1.8 yields that $\overline{F}_i(u, \cdot)$ is a measure on $\mathcal{A}_i$ for every $u \in L^1(\Omega_i)$; for every $A \in \mathcal{A}_i$ the functional $\overline{F}_i(\cdot, A)$ is $L^1(\Omega_i)$-l.s.c., and hence $L^{p_i}(\Omega_i)$-l.s.c.; since $f(x, Du)$ does not depend on u, we infer that $\overline{F}_i(u + c, A) = \overline{F}_i(u, A)$ for every $u \in L^1(\Omega_i)$, $A \in \mathcal{A}_i$ and $c \in \mathbf{R}^N$; also $\overline{F}_i$ satisfies a standard p_i-growth condition, since by (3.34) and (3.33)

$$0 \le \overline{F}_i(u, A) \le \int_A (a(x) + \beta\, |Du(x)|^{p_i})\, dx$$

for every $u \in W^{1,p_i}(\Omega_i)$ and $A \in \mathcal{A}_i$; finally, by locality of F we infer that $\overline{F}_i$ is local, i.e., for every $A \in \mathcal{A}_i$ and all $u, v \in L^1(\Omega_i)$ such that $u = v$ a.e. on A, we have that $\overline{F}_i(u, A) = \overline{F}_i(v, A)$.

By [13, Thm. 20.1], we then obtain the integral representation (3.35) for every $A \in \mathcal{A}_i$ and $u \in L^1(\Omega_i)$ such that $u_{|A} \in W^{1,p_i}_{\rm loc}(A)$, where φ_i is a non negative quasi-convex function satisfying a p_i growth condition from above. Moreover, by (3.34) and (3.33) we obtain (3.35) for all u and A and growth condition (3.36).

Setting now $\varphi(x, \cdot) := \varphi_i(x, \cdot)$ if $x \in \Omega_i$ for some i, we then obtain the first part of the statement. In particular, (3.36) yields that φ satisfies growth condition (1.1) for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$. Moreover, for every $u \in L^1(\Omega)$, let $\widetilde{F}(u, \cdot)$ denote the Borel measure on Ω given by the extension of $\overline{F}(u, \cdot)$, see iii) in Proposition 1.7, and let $\mu(u, A) := \widetilde{F}(u, A \cap \Sigma)$ for $A \in \mathcal{A}$. If $u_{|A} \in W^{1,p(x)}_{\rm loc}(A)$, setting $A_i := A \cap \Omega_i$ we have

$$\begin{aligned}
\overline{F}(u, A) &= \sum_{i=1}^{+\infty} \overline{F}(u, A_i) + \widetilde{F}(u, A \cap \Sigma) = \sum_{i=1}^{+\infty} \overline{F}_i(u_{|\Omega_i}, A_i) + \widetilde{F}(u, A \cap \Sigma) \\
&= \sum_{i=1}^{+\infty} \int_{A_i} \varphi_i(x, Du)\, dx + \mu(u, A) = \int_A \varphi(x, Du)\, dx + \mu(u, A)
\end{aligned}$$

and hence the singular measure $\mu(u, \cdot)$ has support contained in the transition set Σ. On the other side, if $u_{|A} \notin W^{1,p(x)}_{\rm loc}(A)$, then $u_{|A_i} \notin W^{1,p_i}_{\rm loc}(A_i)$ for some i and hence, by (3.35), $\overline{F}(u, A) \geq \overline{F}_i(u_{|\Omega_i}, A_i) = +\infty$.

To prove the last assertion, let us fix $A \in \mathcal{A}$ and let $A' \in \mathcal{A}_0$ be such that $A' \subset\subset A$. If $q := \sup_{x \in A'} p(x) < +\infty$, see (1.4), by (1.1) we infer that f satisfies a q-growth condition on $A' \times \mathbf{R}^N$. Denote now by $H : L^1(A'; \mathbf{R}^N) \times \mathcal{A}(A') \to [0, +\infty]$ the local functional given by $H(u, B) := F(\widetilde{u}, B)$, where $\widetilde{u} \in L^1(\Omega)$ is any extension of u. Also, let $\overline{H}(\cdot, B)$ be the $L^1(A')$-l.s.c. envelope of $H(\cdot, B)$, for every $B \in \mathcal{A}(A')$. Then by an argument similar to the previous one

$$\overline{F}(v, A') = \overline{H}(v_{|A'}, A') = \int_{A'} \varphi(x, Dv(x))\, dx$$

holds for every function $v \in L^1(\Omega)$ with $v_{|A'} \in W^{1,q}_{\rm loc}(A')$. In particular, by repeating the argument for every $A' \subset\subset A$, we have

$$\overline{F}(v, A') = \int_{A'} \varphi(x, Dv(x))\, dx \qquad \forall\, v \in C^1(A; \mathbf{R}^N)\,,\ \forall\, A' \in \mathcal{A}_0\,,\ A' \subset\subset A\,. \tag{3.37}$$

Let now $u \in W^{1,p(x)}_{\rm loc}(A)$ and $\{u_k\} \subset C^1(A)$ be such that $u_k \to u$ in $W^{1,p(x)}_{\rm loc}(A)$. Moreover, for every $A' \in \mathcal{A}_0$, $A' \subset\subset A$, let $\{v_k\} \subset L^1(\Omega)$ be such that $v_{k|A} \in C^1(A)$, $v_{k|A'} = u_{k|A'}$ for every $k \in \mathbf{N}$ and $v_k \to u$ in $L^1(\Omega)$. Then, since φ is a Carathéodory function satisfying (1.1), by lower semicontinuity of $\overline{F}(\cdot, A')$, by (3.37) applied with $v = v_k$, by locality and by the dominated convergence theorem

$$\overline{F}(u, A') \leq \liminf_{k \to +\infty} \overline{F}(v_k, A') = \lim_{k \to +\infty} \int_{A'} \varphi(x, Du_k(x))\, dx = \int_{A'} \varphi(x, Du(x))\, dx\,.$$

Then, by measure property of $\overline{F}(u, \cdot)$, taking $A' \nearrow A$ one obtains $\overline{F}(u, A) \leq \int_A \varphi(x, Du(x))\, dx$. Finally, since $\mu(u, A) \geq 0$ in (1.8), the opposite inequality is trivial and hence $\mu(u, A) = 0$, as required. □

4 The case of piecewise continuous growth exponents

In this section we prove Theorems 1.13 and 1.14.

Proof of Theorem 1.13: the proof is a straightforward readaptation of Theorem 1.8, once we extend Lemma 2.2 by the following

Lemma 4.1 ([11, Lemma 3.8., Cor. 3.9]) *Let $p : \Omega \to [1, +\infty)$ be a regular piecewise continuous exponent. Let $A \in \mathcal{A}_0$ be such that $A_i := A \cap \Omega_i$ has Lipschitz boundary for every i. Moreover, let $\{u_j\} \subset W^{1,p(x)}_{\rm loc}(A; \mathbf{R}^N)$ and $u \in L^1(A; \mathbf{R}^N)$ be such that $u_j \to u$ in $L^1(A; \mathbf{R}^N)$ and*

$$\sup_{j \in \mathbf{N}} \int_A |Du_j(x)|^{p(x)}\, dx < +\infty\,.$$

Then

$$\lim_{j\to+\infty}\int_A |u_j-u|^{p(x)}\,dx=0\,.$$

In fact, as to Sec. 2, Remark 2.1 holds again, (2.1) follows from Lemma 4.1; Lemma 2.5 relies on (1.4) (see Remark 1.12), on Theorem 2.4 and on the convexity of $|\cdot|^{p(x)}$; Lemma 2.7 relies on (1.4) and on the convexity of $|\cdot|^{p(x)}$; Lemma 2.8 follows from Lemma 2.7 and (1.4); finally Lemma 2.9 depends on Lemma 2.8, Lemma 4.1, Remark 2.1 and (1.4).

Also, in Step 1 of Theorem 1.8, Proposition 3.1 relies on Remark 2.1, Lemma 4.1, (1.4) and on the convexity of $|\cdot|^{p(x)}$; of course, (3.6) and its consequences are to be read as

$$\lim_{h\to+\infty}\int_{A^j}|u_h^j-u|^{p(x)}=0 \qquad \forall\, j\in\mathbf{N}_0\,.$$

Also, Proposition 3.3 follows from Proposition 3.1, Lemma 2.9, Remark 2.1 and Lemma 4.1. Finally, in Step 2, Proposition 3.4 relies on Proposition 3.1, Remark 2.1, Lemma 4.1 and Lemma 2.8. □

PROOF OF THEOREM 1.14: let $\mathcal{A}_i$, F_i, $\overline{F}_i$, Ψ_i and $\overline{\Psi}_i$ be given as in the proof of Theorem 1.9, so that in particular, for every $A\in\mathcal{A}_i$,

$$\Psi_i(u,A):=\begin{cases}\displaystyle\int_A|Du(x)|^{p_i(x)}\,dx & \text{if } u\in C^{1,p_i(x)}(A;\mathbf{R}^N)\\ +\infty & \text{elsewhere on } L^1(\Omega_i;\mathbf{R}^N)\end{cases}$$

where $p_i:\Omega_i\to(1,+\infty)$ is given by Definition 1.11. We first show that for every $A\in\mathcal{A}_i$

$$\overline{\Psi}_i(u,A)=\begin{cases}\displaystyle\int_A|Du(x)|^{p_i(x)}\,dx & \text{if } u\in W^{1,p_i(x)}_{\rm loc}(A;\mathbf{R}^N)\\ +\infty & \text{elsewhere on } L^1(\Omega_i;\mathbf{R}^N)\,.\end{cases}\tag{4.1}$$

In fact, if $\overline{\Psi}_i(u,A)<+\infty$, then for every $A'\in\mathcal{A}_0$, with $A'\subset\subset A$, Lemma 2.5 yields $u\in W^{1,p_i(x)}_{\rm loc}(A';\mathbf{R}^N)$ and

$$\int_{A'}|Du|^{p_i(x)}\,dx\le\overline{\Psi}_i(u,A')\,.\tag{4.2}$$

To obtain equality in (4.2), it suffices to apply the following density result due to Zhikov [27], compare also [1, Lemma 4.2] or [11, Prop. 3.18] for a proof.

Proposition 4.2 *Let $p:\Omega\to[1,+\infty)$ be a continuous function satisfying* (1.10) *for every $A\subset\subset\Omega$. Then for every $u\in W^{1,p(x)}_{\rm loc}(\Omega;\mathbf{R}^N)$ there exists a sequence of smooth functions $\{u_j\}\subset C_0^\infty(\Omega;\mathbf{R}^N)$ such that $u_j\to u$ in $W^{1,p(x)}_{\rm loc}(\Omega;\mathbf{R}^N)$. If in addition $u\in W^{1,p(x)}(\Omega;\mathbf{R}^N)$, then $u_j\to u$ also in $L^1(\Omega;\mathbf{R}^N)$.*

Now, by inner regularity of $\overline{\Psi}_i(u,\cdot)$, letting $A'\nearrow A$ we obtain both $u\in W^{1,p_i(x)}_{\rm loc}(A;\mathbf{R}^N)$ and (4.1).

We now wish to apply the following integral representation result to the local functional $\mathcal{F}=\overline{F}_i$.

Theorem 4.3 ([11, Thm. 4.1]) *Let $p:\Omega\to[1,+\infty)$ be a continuous function satisfying* (1.10) *for every open set $A\subset\subset\Omega$. Let $\mathcal{F}:L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ satisfy the following conditions:*

i) *$\mathcal{F}$ is local, i.e., $\mathcal{F}(u,A)=\mathcal{F}(v,A)$ for every $A\in\mathcal{A}$ and $u,v\in L^1(\Omega;\mathbf{R}^N)$ with $u=v$ a.e. on A;*

ii) *for all $u\in L^1(\Omega;\mathbf{R}^N)$ the set function $\mathcal{F}(u,\cdot)$ is increasing, and is the trace on $\mathcal{A}$ of a Borel measure;*

iii) *there exist $\beta>0$ and $a(x)\in L^1_{\rm loc}(\Omega)$ such that*

$$0\le\mathcal{F}(u,A)\le\int_A(a(x)+\beta\,|Du|^{p(x)})\,dx$$

for all $u\in W^{1,p(x)}(\Omega;\mathbf{R}^N)$ and $A\in\mathcal{A}$;

iv) $\mathcal{F}(u+c,A)=\mathcal{F}(u,A)$ *for all* $u\in L^1(\Omega;\mathbf{R}^N)$, $A\in\mathcal{A}$, $c\in\mathbf{R}^N$;

v) $\mathcal{F}(\cdot,A)$ *is sequentially lower semicontinuous with respect to the strong convergence in* $L^1(\Omega;\mathbf{R}^N)$ *for all* $A\in\mathcal{A}$.

Then there exists a Carathéodory function $\varphi:\Omega\times\mathbf{R}^{nN}\to[0,+\infty)$ *such that*

$$\mathcal{F}(u,A)=\int_A\varphi(x,Du(x))\,dx$$

for every $A\in\mathcal{A}$ *and for every* $u\in L^1(\Omega;\mathbf{R}^N)$ *such that* $u_{|A}\in W^{1,p(x)}_{\rm loc}(A;\mathbf{R}^N)$*; in addition, the function* $\varphi(x,\cdot)$ *is quasi-convex on* $\mathbf{R}^{nN}$ *for a.e.* $x\in\Omega$ *and satisfies the growth condition*

$$0\le\varphi(x,z)\le a(x)+\beta\,|z|^{p(x)}$$

for a.e. $x\in\Omega$ *and all* $z\in\mathbf{R}^{nN}$.

Now $\overline{F}_i$ satisfies Theorem 4.3 on Ω_i, since (3.23) and (4.1) yield growth condition iii), whereas the other hypotheses are easily verified. Arguing similarly to Theorem 1.9, we then easily conclude with (1.8) and with the rest of the proof. □

5 An example with energy concentration

In this section we prove the statements contained in Example 1.15. More precisely, for every $A\in\mathcal{A}$ and $u\in W^{1,p(x)}_{\rm loc}(A)$, in Step 1 we first show that $\mu(u,A)=0$ in case $0_{\mathbf{R}^2}\notin A$. Secondly, in case $0_{\mathbf{R}^2}\in A$, in Step 2 we show that $\mu(u,A)=+\infty$ if $\lambda_1\ne\lambda_2$ in (1.13). Finally, in Step 3 we prove that $\mu(u,A)=0$ if $0_{\mathbf{R}^2}\in A$ but $\lambda_1=\lambda_2$. We first make the following

Remark 5.1 Under the hypotheses of Example 1.15, for every $u\in L^1(B_1)$ and $A\in\mathcal{A}$, we can easily find a sequence $\{u_k\}\in L^1(B_1)$ with $u_k\to u$ in $L^1(B_1)$ and $u_{k|A}\in W^{1,q}(A)$ for every $k\in\mathbf{N}$. Moreover, since $f(x,z):=|z|^{p(x)}\le 1+|z|^q$, then for every function $v\in W^{1,q}(A)$ there exists a sequence of smooth functions $\{v_k\}\subset C^{1,p(x)}(A)$ such that $v_k\to v$ in $L^1(A)$ and $\|v_k-v\|_{W^{1,p(x)}(A)}\to 0$ as $k\to+\infty$. Taking $v_k\equiv u$ on $B_1\setminus A$, this yields that for every $A\in\mathcal{A}$ and $u\in L^1(B_1)$

$$\overline{F}(u,A)=\inf\{\liminf_{k\to+\infty}\int_A|Du_k(x)|^{p(x)}\,dx\mid\{u_k\}\subset W^{1,q}(A)\,,\ u_k\to u\ \text{ in }\ L^1(A)\}\,. \tag{5.1}$$

Step 1: the case $0_{\mathbf{R}^2}\notin A$.
Following an argument by Zhikov, compare [29], we now show that there is no energy concentration on open sets which do not contain the origin.

Proposition 5.2 *Under the hypotheses of Example* 1.15*, if* $A\in\mathcal{A}$*,* $0_{\mathbf{R}^2}\notin A$ *and* $u_{|A}\in W^{1,p(x)}_{\rm loc}(A)$*, then* $\mu(u,A)=0$ *in* (1.8) *and hence* $\overline{F}(u,A)=\int_A|Du(x)|^{p(x)}\,dx$.

PROOF: it suffices to show that for every $A'\in\mathcal{A}_0$, with $A'\subset\subset A$, there exists a sequence of functions $\{u_j\}\subset W^{1,q}(A')$ such that $u_j\to u$ in $L^1(A')$ and

$$\lim_{j\to+\infty}\int_{A'}|Du_j(x)|^{p(x)}\,dx=\int_{A'}|Du(x)|^{p(x)}\,dx\,.$$

In fact, by (5.1), this yields $\overline{F}(u,A')\le\int_{A'}|Du(x)|^{p(x)}\,dx$ and hence, by inner regularity, letting $A'\nearrow A$ one obtains the assertion since $\mu(u,A)\ge 0$ and $\varphi(x,z)=f(x,z)=|z|^{p(x)}$ in (1.8).

For every $B\in\mathcal{A}$, we set $B_p:=B\cap\Omega_p$, $B_q:=B\cap\Omega_q$, where Ω_p and Ω_q are the subsets of Ω corresponding to the phases p and q of $p(x)$, see (1.11), i.e.,

$$\Omega_p:=\{x\in\Omega\mid x_1x_2>0\}\,,\qquad\Omega_q:=\{x\in\Omega\mid x_1x_2<0\}\,.$$

Suppose first in addition that A_q has Lipschitz boundary, and let $\widetilde{u} \in W^{1,q}(A)$ be a "smooth" extension to A of $u_{|A_q} \in W^{1,q}(A_q)$. Also, let $v(x) := \phi(x)(u(x) - \widetilde{u}(x))$, where $\phi \in C_0^1(A)$ is a smooth cut-off between A' and A. Since $v \in W_0^{1,p}(A_p)$, there exists a smooth sequence $\{v_j\} \subset C_0^1(A_p)$ such that $v_j \to v$ in $W_0^{1,p}(A_p)$. Setting now

$$u_j(x) := \begin{cases} u(x) & \text{if} \quad x \in A_q \\ \widetilde{u}(x) + v_j(x) & \text{if} \quad x \in A_p \end{cases}$$

we have that $\{u_j\} \in W^{1,q}(A)$, $u_j \to u$ in $L^1(A')$ and finally

$$\begin{aligned} \lim_{j\to+\infty} \int_{A'} |Du_j|^{p(x)}\,dx &= \lim_{j\to+\infty} \left(\int_{A'_q} |Du|^q\,dx + \int_{A'_p} |D\widetilde{u} + Dv_j|^p\,dx \right) \\ &= \int_{A'_q} |Du|^q\,dx + \int_{A'_p} |D\widetilde{u} + Dv|^p\,dx = \int_{A'} |Du|^{p(x)}\,dx\,. \end{aligned}$$

To conclude the proof, it suffices for every $A' \subset\subset A$ to take $A'' \in \mathcal{A}$, with $A' \subset\subset A'' \subset A$, such that A''_q has Lipschitz boundary, and repeat the previous argument with A'' instead of A. $\square$

Remark 5.3 As noticed by Zhikov, the proof of Proposition 5.2 does not hold if $0_{\mathbf{R}^2} \in A$, since we cannot find, in general, a $W^{1,q}(A)$-extension $\widetilde{u}$ of $u_{|A_q}$. However, since $q > 2$, $\widetilde{u}$ should be in particular continuous at $0_{\mathbf{R}^2}$, which cannot hold if $u = u_0$ is given by (1.12), since $u_{|A_q}$ takes distant values (1 and 0) on each neighborhood of the origin.

Step 2: the case $0_{\mathbf{R}^2} \in A$ *and* $\lambda_1 \neq \lambda_2$.
We prove that $\mu(u, A) = +\infty$, and hence (1.14) holds, if $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\rm loc}(A)$ with $\lambda_1 \neq \lambda_2$ in (1.13). To this aim, we first show that the relaxed functional can be described by imposing Dirichlet type conditions (Proposition 5.4). Compare [19, Lemma 2.1] for a similar result in case of functionals with standard p-growth. Secondly, we give a lower bound of the energy of a $W^{1,q}$-function with trace equal to the trace of u (Proposition 5.6). By (5.2), we then obtain a lower bound of the type $C \cdot R^{2-q}$, with $q > 2$, for the relaxed functional $\overline{F}(u, B_R)$, and hence the assertion as $R \to 0^+$.

If $1 < p \le q < 2p$ and $u \in W^{1,p(x)}_{\rm loc}(A)$, with $0_{\mathbf{R}^2} \in A \in \mathcal{A}$, then for a.e. $R > 0$ sufficiently small (so that in particular $B_R \subset\subset A$) the trace $u_{|\partial B_R}$ belongs to $W^{1,q}(\partial B_R)$ and hence there exists a function $v \in W^{1,q}(B_R)$ such that $v_{|\partial B_R} = u_{|\partial B_R}$. Then we have

Proposition 5.4 *Under the hypotheses of Example* 1.15*, if* $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ *and* $u \in W^{1,p(x)}_{\rm loc}(A)$*, then for a.e.* $R > 0$ *sufficiently small we have*

$$\begin{aligned} \overline{F}(u, B_R) = \inf\{ \liminf_{k\to+\infty} \int_{B_R} |Du_k(x)|^{p(x)}\,dx \mid \{u_k\} \subset W^{1,q}(B_R)\,, \\ u_k \to u \ \text{ in } \ L^1(B_R) \ \text{ and } \ u_{k|\partial B_R} = u_{|\partial B_R} \}\,. \end{aligned} \tag{5.2}$$

By the same argument we easily obtain the following

Corollary 5.5 *Under the hypotheses of Proposition* 5.4*, for a.e.* $R > 0$ *sufficiently small we also have*

$$\begin{aligned} \overline{F}(u, A \setminus \overline{B}_R) = \inf\{ \liminf_{k\to+\infty} \int_{A\setminus B_R} |Du_k(x)|^{p(x)}\,dx \mid \{u_k\} \subset W^{1,q}(A \setminus B_R)\,, \\ u_k \to u \ \text{ in } \ L^1(A \setminus B_R) \quad \text{and} \quad u_{k|\partial B_R} = u_{|\partial B_R} \}\,. \end{aligned}$$

Let now $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\rm loc}(A)$. We give a lower bound of the energy of any $W^{1,q}$-function with trace on ∂B_R equal to the trace of u, for a.e. $R > 0$ small.

Proposition 5.6 *Under the hypotheses of Example* 1.15*, let* $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ *and* $u \in W^{1,p(x)}_{\rm loc}(A)$ *be such that* $u_{|\partial B_R} \in W^{1,q}(\partial B_R)$ *for some* $R > 0$ *small (so that* $B_R \subset\subset A$*). Then for every function* $v \in W^{1,q}(B_R)$ *with* $v_{|\partial B_R} = u_{|\partial B_R}$*, we have*

$$\int_{B_R} |Dv(x)|^{p(x)}\,dx \ge \alpha(q)^{q-1} \cdot R^{2-q} \cdot \int_{\pi/2}^{\pi} (|\widetilde{u}(R,\theta) - \nu|^q + |\widetilde{u}(R, \theta + \pi) - \nu|^q)\,d\theta \tag{5.3}$$

where $\alpha(q) := \dfrac{q-2}{q-1} > 0$, $\nu := v(0_{\mathbf{R}^2})$ *and* $\widetilde{u}(\rho,\theta) := u(\rho\cos\theta, \rho\sin\theta)$.

Suppose now that $\lambda_1 \neq \lambda_2$ in (1.13). Since by the dominated convergence theorem

$$\lim_{R\to 0^+} \int_{\pi/2}^{\pi} (|\widetilde{u}(R,\theta) - \nu|^q + |\widetilde{u}(R,\theta+\pi) - \nu|^q)\, d\theta = \frac{\pi}{2}(|\lambda_1 - \nu|^q + |\lambda_2 - \nu|^q) > 0$$

and $R^{2-q} \to +\infty$ as $R \to 0^+$, by Propositions 5.4 and 5.6 we infer that there exists a sequence of radii $R_j \searrow 0$ such that $\overline{F}(u; B_{R_j}) \to +\infty$ as $j \to +\infty$. Then, since $B_R \subset A$ for $R > 0$ small, by monotonicity of $\overline{F}(u,\cdot)$ we obtain $\overline{F}(u,A) \geq \limsup_{R\to 0^+} \overline{F}(u, B_R) = +\infty$. As a consequence, we have $\mu(u,A) = +\infty$ and hence (1.14), if $\lambda_1 \neq \lambda_2$. We conclude Step 2 by proving Propositions 5.4 and 5.6.

Proof of Proposition 5.4: if $R > 0$ is such that $B_R \subset\subset A$ and $u_{|\partial B_R} \in W^{1,q}(\partial B_R)$, by (5.1) we immediately infer that "$\leq$" holds in (5.2). To obtain "$\geq$", and hence the assertion, it suffices to show that for any sequence $\{u_k\} \subset W^{1,q}(B_R)$ with $u_k \to u$ in $L^1(B_R)$, there exists another sequence $\{v_k\} \subset W^{1,q}(B_R)$ with $v_k \to u$ in $L^1(B_R)$, $v_{k|\partial B_R} = u_{|\partial B_R} \in W^{1,q}(B_R)$ for each k, such that

$$\liminf_{k\to+\infty} \int_{B_R} |Dv_k(x)|^{p(x)}\, dx \leq \liminf_{k\to+\infty} \int_{B_R} |Du_k(x)|^{p(x)}\, dx\,. \tag{5.4}$$

To this aim, we remark that since $u_{|A} \in W^{1,p(x)}_{\mathrm{loc}}(A)$, then for a.e. $R > 0$ small we have

$$\lim_{\rho\to R^-} \fint_{B_R\setminus B_\rho} |Du(x)|^{p(x)}\, dx = \int_{\partial B_R} |Du(x)|^{p(x)}\, d\mathcal{H}^1 < +\infty\,. \tag{5.5}$$

Of course, we can suppose the right-hand side of (5.4) to be finite and, to simplify the notation, take $R = 1$ in the sequel. More precisely, we reparametrize the domain Ω by a homothetic expansion so that $A \supset B_{R_0}$ for some $R_0 > 1$, (5.5) holds for $R = 1$ and $u_{|\partial B_1} \in W^{1,q}(\partial B_1)$. Possibly passing to subsequences we have $\sup_k \int_{B_1} |Du_k|^{p(x)}\, dx < +\infty$ and hence, by Rellich's theorem (compare Lemma 2.2), we infer that $\|u_k - u\|_{L^{p(x)}(B_1)} \to 0$ as $k \to +\infty$. We now set, for $0 < \rho < 1$,

$$g_k(\rho) := \int_{\partial B_\rho} |u_k - u|^{p(x)}\, d\mathcal{H}^1 \qquad \text{and} \qquad f_k(\rho) := \int_{\partial B_\rho} |D_\tau u_k|^{p(x)}\, d\mathcal{H}^1\,,$$

where τ is the unit vector tangent to ∂B_ρ. Passing again to a subsequence we can suppose that the lower limit in the right-hand side of (5.4) (with $R = 1$) is a finite limit and by the coarea formula [17, 3.2.11]

$$\int_0^1 g_k(\rho)\, d\rho = \int_{B_1} |u_k - u|^{p(x)}\, dx \leq 2^{-(1+q)k}\,, \qquad \int_0^1 f_k(\rho)\, d\rho \leq \int_{B_1} |Du_k|^{p(x)}\, dx \leq C < +\infty$$

for each k. Therefore, since

$$\mathrm{meas}(\{\rho \in (1 - 2^{1-k}, 1) \mid f_k(\rho) \leq 2^{k+1}\, C\}) \geq 3 \cdot 2^{-1-k}$$

and

$$\mathrm{meas}(\{\rho \in (0,1) \mid g_k(\rho) \leq 2^{-qk}\}) \geq 1 - 2^{-k}\,,$$

we can choose a sequence $\{r_k\}$ with

$$2^{-1-k} < r_k < 2^{1-k} \qquad \forall\, k \tag{5.6}$$

such that

$$\int_{\partial B_{1-r_k}} |u_k - u|^{p(x)}\, d\mathcal{H}^1 \leq 2^{-qk}\,, \qquad \int_{\partial B_{1-r_k}} |D_\tau u_k|^{p(x)}\, d\mathcal{H}^1 \leq 2^{k+1}\, C \tag{5.7}$$

for each k and by (5.5)

$$\fint_{B_1\setminus B_{1-r_k}} |Du(x)|^{p(x)}\, dx \leq C_0 < +\infty \qquad \forall\, k\,. \tag{5.8}$$

Taking now the exponent γ so that $1 < \gamma < q/(q-1)$ we define, in polar coordinates,

$$\widetilde{v}_k(\rho,\theta) := \begin{cases} \widetilde{u}_k(\Psi_k(\rho),\theta) & \text{if} \quad 0 \le \rho \le 1 - {r_k}^\gamma \\ \lambda_k(\rho)\,\widetilde{u}(1,\theta) + (1-\lambda_k(\rho))\,\widetilde{u}_k(1-r_k,\theta) & \text{if} \quad 1-{r_k}^\gamma < \rho \le 1 \end{cases}$$

for each $\theta \in [0,2\pi]$, where $\widetilde{w}(\rho,\theta) := w(\rho\cos\theta, \rho\sin\theta)$ for $w = u$, u_k, v_k,

$$\Psi_k(\rho) := \frac{1-r_k}{1-{r_k}^\gamma}\,\rho\,, \qquad 0 \le \rho \le 1-{r_k}^\gamma\,,$$

so that $\Psi_k(0) = 0$ and $\Psi_k(1-{r_k}^\gamma) = 1-r_k$, and

$$\lambda_k(\rho) := \frac{\rho - 1 + {r_k}^\gamma}{{r_k}^\gamma}\,, \qquad 1-{r_k}^\gamma \le \rho \le 1\,,$$

so that $\lambda_k(1-{r_k}^\gamma) = 0$ and $\lambda_k(1) = 1$. It is then easy to show that $\{v_k\} \subset W^{1,q}(B_1)$, $v_k \to u$ in $L^1(B_1)$, $v_{k|\partial B_1} = u_{|\partial B_1}$ for every $k \in \mathbf{N}$ and

$$\lim_{k\to+\infty} \int_{B_{1-{r_k}^\gamma}} |Dv_k(x)|^{p(x)}\,dx = \lim_{k\to+\infty} \int_{B_1} |Du_k(x)|^{p(x)}\,dx\,.$$

Therefore (5.4) (with $R=1$) holds if we show that

$$\lim_{k\to+\infty} \int_{B_1 \setminus B_{1-{r_k}^\gamma}} |Dv_k(x)|^{p(x)}\,dx = 0\,. \tag{5.9}$$

To prove (5.9), passing to polar coordinates we estimate

$$\begin{aligned} \int_{B_1\setminus B_{1-{r_k}^\gamma}} |Dv_k(x)|^{p(x)}\,dx \le 2^{q-1} \Bigg\{ & \int_{1-{r_k}^\gamma}^1 \rho\,d\rho \int_0^{2\pi} {r_k}^{-\gamma p(\theta)} |\widetilde{u}(1,\theta) - \widetilde{u}_k(1-r_k,\theta)|^{p(\theta)}\,d\theta \\ & + \int_{1-{r_k}^\gamma}^1 \rho\,d\rho \int_0^{2\pi} (|\rho^{-1}D_\theta\widetilde{u}(1,\theta)|^{p(\theta)} + |\rho^{-1}D_\theta\widetilde{u}_k(1-r_k,\theta)|^{p(\theta)})\,d\theta \Bigg\} \end{aligned} \tag{5.10}$$

where by (1.11)

$$p(\theta) := \begin{cases} p & \text{if} \quad \theta \in]0,\pi/2[\cup]\pi,3\pi/2[\\ q & \text{if} \quad \theta \in]\pi/2,\pi[\cup]3\pi/2,2\pi[\,. \end{cases}$$

Now we have $D_\tau u = \rho^{-1}D_\theta\widetilde{u}$; moreover, if $1-{r_k}^\gamma \le \rho \le 1$, then by (5.6) for k large we have $\rho^{-1} \le (1-{r_k}^\gamma)^{-1} \le 2$, so that

$$\int_{1-{r_k}^\gamma}^1 \rho\,d\rho \int_0^{2\pi} |\rho^{-1}D_\theta\widetilde{u}(1,\theta)|^{p(\theta)}\,d\theta \le 2^q\,{r_k}^\gamma \int_{\partial B_1} |D_\tau u|^{p(x)}\,d\mathcal{H}^1 \to 0$$

as $k \to +\infty$, since ${r_k}^\gamma \to 0$ and $u_{|\partial B_1} \in W^{1,q}(\partial B_1)$. Moreover, since $1-{r_k}^\gamma \le \rho \le 1$ and ${r_k}^\gamma \le r_k$ yield

$$\left(\frac{\rho}{1-r_k}\right)^{1-p(\theta)} \le \left(\frac{1-r_k}{1-{r_k}^\gamma}\right)^{p(\theta)-1} \le 1\,,$$

by (5.6) and (5.7) we have

$$\begin{aligned} & \int_{1-{r_k}^\gamma}^1 \rho\,d\rho \int_0^{2\pi} |\rho^{-1}D_\theta\widetilde{u}_k(1-r_k,\theta)|^{p(\theta)}\,d\theta = \\ & = \int_{1-{r_k}^\gamma}^1 \left(\frac{\rho}{1-r_k}\right)^{1-p(\theta)} d\rho \int_0^{2\pi} (1-r_k)^{1-p(\theta)}\,|D_\theta\widetilde{u}_k(1-r_k,\theta)|^{p(\theta)}\,d\theta \\ & \le {r_k}^\gamma \int_{\partial B_{1-r_k}} |D_\tau u_k|^{p(x)}\,d\mathcal{H}^1 \,\le\, (2^{1-k})^\gamma \cdot 2^{k+1} \cdot C \to 0 \end{aligned}$$

as $k \to +\infty$, since $\gamma > 1$. Hence the second integral in the right-hand side of (5.10) goes to 0 as $k \to +\infty$. As to the first integral, note that

$$|\widetilde{u}(1,\theta) - \widetilde{u}_k(1-r_k,\theta)|^{p(\theta)} \le 2^{q-1}\left(|\widetilde{u}(1,\theta) - \widetilde{u}(1-r_k,\theta)|^{p(\theta)} + |\widetilde{u}_k(1-r_k,\theta) - \widetilde{u}(1-r_k,\theta)|^{p(\theta)}\right)$$

and for a.e. $\theta \in [0,2\pi]$, by Jensen inequality,

$$|\widetilde{u}(1,\theta) - \widetilde{u}(1-r_k,\theta)|^{p(\theta)} \le {r_k}^{p(\theta)-1} \int_{1-r_k}^{1} |D_\rho \widetilde{u}(\sigma,\theta)|^{p(\theta)}\, d\sigma\,.$$

Now, for k large, $1/2 < 1-r_k \le \rho \le 1$ yields $1 \le \rho/(1-r_k) \le 2\rho$, whereas $2 \le 1-r_k \le \sigma \le 1$ yields $1 \le 2\sigma$. Moreover, since $\gamma > 1$ and $0 < r_k < 1$, then ${r_k}^{(\gamma-1)(1-p(\theta))} \le {r_k}^{(\gamma-1)(1-q)}$ and ${r_k}^{\gamma(1-p(\theta))} \le {r_k}^{\gamma(1-q)}$. As a consequence, we compute

$$\begin{aligned}
&2^{1-q}\int_{1-{r_k}^\gamma}^{1} \rho\, d\rho \int_0^{2\pi} {r_k}^{-\gamma p(\theta)} |\widetilde{u}(1,\theta) - \widetilde{u}_k(1-r_k,\theta)|^{p(\theta)}\, d\theta \le \\
&\le \int_0^{2\pi} {r_k}^{(\gamma-1)(1-p(\theta))} \int_{1-r_k}^{1} |D_\rho \widetilde{u}(\sigma,\theta)|^{p(\theta)}\, d\sigma\, d\theta \\
&+ \int_{1-{r_k}^\gamma}^{1} \rho\, d\rho \int_0^{2\pi} {r_k}^{-\gamma p(\theta)} |\widetilde{u}_k(1-r_k,\theta) - \widetilde{u}(1-r_k,\theta)|^{p(\theta)}\, d\theta \\
&\le {r_k}^{(\gamma-1)(1-q)} \int_0^{2\pi}\int_{1-r_k}^{1} 2\sigma |D_\rho \widetilde{u}(\sigma,\theta)|^{p(\theta)}\, d\sigma\, d\theta + {r_k}^{\gamma(1-q)} \frac{1}{1-r_k}\int_{\partial B_{1-r_k}} |u_k - u|^{p(x)}\, d\mathcal{H}^1 \\
&\le 2\,{r_k}^{(\gamma-1)(1-q)} \int_{B_1 \setminus B_{1-r_k}} |Du|^{p(x)}\, dx + 2\,{r_k}^{\gamma(1-q)} \int_{\partial B_{1-r_k}} |u_k - u|^{p(x)}\, d\mathcal{H}^1 := I_k + II_k\,.
\end{aligned}$$

By (5.6) and (5.7) we estimate $II_k \le 2\,(2^{-1-k})^{\gamma(1-q)}\, 2^{-qk} = 2^{\gamma(q-1)+1}(2^{-k})^{q+\gamma(1-q)} \to 0$ as $k \to +\infty$, since $\gamma < q/(q-1)$ yields $q + \gamma(1-q) > 0$. Also, by (5.6) and (5.8), since $|B_1 \setminus B_{1-r_k}| = \pi(2-r_k)r_k \le \pi\, 2^{2-k}$,

$$I_k \le 2\,(2^{-1-k})^{(\gamma-1)(1-q)}\,\pi\, 2^{2-k} \fint_{B_1 \setminus B_{1-r_k}} |Du|^{p(x)}\, dx \le \pi\, C_0\, 2^{(q-1)(\gamma-1)+3}\, 2^{-k(1+(\gamma-1)(1-q))} \to 0$$

as $k \to +\infty$, since $\gamma < q/(q-1)$ yields $1 + (\gamma-1)(1-q) > 0$. We finally obtain (5.9), which completes the proof. □

Proof of Corollary 5.5: it suffices to modify the proof of Proposition 5.4 by gluing the sequence $\{u_k\} \subset W^{1,q}(A \setminus B_R)$, $B_R \subset\subset A$, with the trace of u in ∂B_R, and make computations similar to the ones in Proposition 5.4. □

Proof of Proposition 5.6: by using polar coordinates ($\widetilde{v}(\rho,\theta) = v(\rho\cos\theta, \rho\sin\theta)$) one easily infers

$$\int_{B_R} |Dv(x)|^{p(x)}\, dx \ge \int_0^R \rho \int_{\pi/2}^{\pi} (|D_\rho \widetilde{v}(\rho,\theta)|^q + |D_\rho \widetilde{v}(\rho,\theta+\pi)|^q)\, d\theta\, d\rho\,.$$

Now, since $q > n = 2$, we have $W^{1,q}(B_R) \subset C^{0,1-2/q}(\overline{B}_R)$, whereas condition $v_{|\partial B_R} = u_{|\partial B_R}$ yields $\widetilde{v}(R,\theta) = \widetilde{u}(R,\theta)$ for a.e. $\theta \in [\pi/2,\pi] \cup [3\pi/2, 2\pi]$. Therefore, if $w_\theta(t)$ minimizes the variational problem

$$\min\left\{\int_0^R t\,|w'(t)|^q\, dt \mid w \in C^0([0,R])\,,\ t\,|w'(t)|^q \in L^1(0,R)\,,\ w(R) = \widetilde{u}(R,\theta)\,,\ w(0) = \nu\right\} \tag{5.11}$$

where $\nu = v(0_{\mathbf{R}^2}) \equiv \widetilde{v}(0,\theta)$, we have

$$\int_{B_R} |Dv(x)|^{p(x)}\, dx \ge \int_{\pi/2}^{\pi}\int_0^R t\,(|w_\theta'(t)|^q + |w_{\theta+\pi}'(t)|^q)\, dt\, d\theta\,. \tag{5.12}$$

By solving the Euler equation associated to (5.11),

$$\frac{d}{dt}(t \cdot |w'(t)|^{q-1}) = 0\,, \qquad 0 \le t \le R\,, \tag{5.13}$$

with Dirichlet data, we obtain

$$w_\theta(t) := (\widetilde{u}(R,\theta) - \nu)\left(\frac{t}{R}\right)^{\alpha(q)} + \nu\,, \qquad 0 \le t \le R\,, \qquad \alpha(q) := \frac{q-2}{q-1}. \tag{5.14}$$

Finally, by (5.12) and (5.14) it is easy to compute the lower bound (5.3). □

Step 3: the case $0_{\mathbf{R}^2} \in A$ and $\lambda_1 = \lambda_2$.
We prove that $\mu(u,A) = 0$, and hence (1.14) holds, if $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ with $\lambda_1 = \lambda_2$ in (1.13). To this aim, it suffices to prove the following

Proposition 5.7 *If $0_{\mathbf{R}^2} \in A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ is such that $\lambda_1 = \lambda_2$ in (1.13), then for a.e. $R > 0$ sufficiently small there exists a function $v_R \in W^{1,q}(B_R)$, with trace $v_{R|\partial B_R} = u_{|\partial B_R}$, such that*

$$\liminf_{R\to 0^+} \int_{B_R} |Dv_R(x)|^{p(x)}\,dx = 0\,. \tag{5.15}$$

In fact, by Corollary 5.5 we can find, for a.e. $R > 0$ small, a sequence $\{v_k^R\} \subset W^{1,q}(A \setminus B_R)$, with $v^R_{k|\partial B_R} = u_{|\partial B_R} \in W^{1,q}(\partial B_R)$, such that $v_k^R \to u$ in $L^1(A \setminus B_R)$ and

$$\lim_{k\to+\infty} \int_{A\setminus B_R} |Dv_k^R(x)|^{p(x)}\,dx = \overline{F}(u, A \setminus \overline{B}_R) = \int_{A\setminus B_R} |Du(x)|^{p(x)}\,dx\,, \tag{5.16}$$

the last equality being given by Proposition 5.2. Hence, taking $v_k^R(x) := v_R(x)$ if $x \in B_R$, by a diagonal procedure we obtain a sequence $\{w_k\} \subset W^{1,q}(A)$ such that $w_k \to u$ in $L^1(A)$ and by (5.15) and (5.16)

$$\lim_{k\to+\infty} \int_A |Dw_k(x)|^{p(x)}\,dx = \int_A |Du(x)|^{p(x)}\,dx\,. \tag{5.17}$$

Therefore, by (5.1) we obtain $\overline{F}(u,A) \le \int_A |Du|^{p(x)}\,dx$ and then, since $\mu(u,A) \ge 0$, by (1.8) we conclude that $\mu(u,A) = 0$ and hence (1.14) holds even in case $\lambda_1 = \lambda_2$. It finally remains the following

Proof of Proposition 5.7: if $u_{|\partial B_R} \in W^{1,q}(\partial B_R)$, we define in polar coordinates ($\widetilde{v}_R(\rho,\theta) := v_R(\rho\cos\theta, \rho\sin\theta)$)

$$\widetilde{v}_R(\rho,\theta) := (\widetilde{u}(R,\theta) - \lambda)\,\frac{\rho}{R} + \lambda\,, \qquad 0 \le \rho < R\,, \quad \theta \in [0, 2\pi]\,,$$

where $\lambda := \lambda_1 = \lambda_2$ is given by (1.13). Passing to polar coordinates we have

$$|Dv_R(x)|^2 = \frac{1}{R^2}\left(|\widetilde{u}(R,\theta) - \lambda|^2 + |D_\theta \widetilde{u}(R,\theta)|^2\right).$$

Since $u_{|\partial B_R} \in W^{1,q}(\partial B_R)$, if τ is the tangential direction to ∂B_R we estimate

$$\int_{B_R} |Dv_R|^q\,dx \le c(q)\left\{R^{1-q} \int_{\partial B_R} |u - \lambda|^q\,d\mathcal{H}^1 + R \int_{\partial B_R} |D_\tau u|^q\,d\mathcal{H}^1\right\} < +\infty$$

and hence $v_R \in W^{1,q}(B_R)$, whereas $v_{R|\partial B_R} = u_{|\partial B_R}$. Moreover, denoting by $I_p :=]0, \pi/2[\cup]\pi, 3\pi/2[$ and $I_q :=]\pi/2, \pi[\cup]3\pi/2, 2\pi[$ the sets of angles θ corresponding to $p(x) = p, q$, respectively, one has

$$\begin{aligned}\int_{B_R} |Dv_R(x)|^{p(x)}\,dx \le c\,(p,q)\,\Bigg\{ & R^{2-p} \int_{I_p} |\widetilde{u}(R,\theta) - \lambda|^p\,d\theta + R^{2-q} \int_{I_q} |\widetilde{u}(R,\theta) - \lambda|^q\,d\theta \\ & + R^{2-p} \int_{I_p} |D_\theta \widetilde{u}(R,\theta)|^p\,d\theta + R^{2-q} \int_{I_q} |D_\theta \widetilde{u}(R,\theta)|^q\,d\theta \Bigg\}\,.\end{aligned} \tag{5.18}$$

To show that the right-hand side of (5.18) goes to zero, as $R \to 0^+$, we make use of the following lemma (stated in any dimension), the proof of which is postponed.

Lemma 5.8 *If $u \in W^{1,p}(B_{R_0};\mathbf{R}^N)$ with $1 \le p < n$, $B_{R_0} \subset \mathbf{R}^n$ being the n-ball of radius R_0, then for a.e. $0 < R < R_0$ we have*

$$R^{1-p} \int_{\partial B_R} |u|^p \, d\mathcal{H}^{n-1} \le c\,(n,p) \left\{ \int_{B_R} |Du|^p \, dx + \left(\int_{B_R} |u|^{p^*} \, dx \right)^{p/p^*} \right\} \tag{5.19}$$

where $p^* := np/(n-p)$ is the Sobolev conjugate of p.

Now, by applying Lemma 5.8, with $n = 2$, to the function $u(\cdot) - \lambda$, which belongs to $W^{1,p}(B_{R_0})$ if $B_{R_0} \subset\subset A$, since $q < 2p$ yields $q < p^*$, by Sobolev embedding theorem we obtain

$$\begin{aligned} \liminf_{R\to 0^+} R^{2-p} \int_{I_p} |\widetilde{u}(R,\theta) - \lambda|^p \, d\theta &\le \liminf_{R\to 0^+} R^{1-p} \int_{\partial B_R} |u(x) - \lambda|^p \, d\mathcal{H}^1 \\ &\le \liminf_{R\to 0^+} c\,(2,p) \left\{ \int_{B_R} |Du|^p \, dx + \left(\int_{B_R} |u - \lambda|^{p^*} \, dx \right)^{p/p^*} \right\} = 0\,, \end{aligned}$$

by absolute continuity, hence the first integral in the right-hand side of (5.18) is small with R.

As to the second integral in (5.18), we recall that if $u \in W^{1,q}(A)$, with $A \subset \mathbf{R}^2$ bounded open set with Lipschitz boundary, and $q > 2$, then u is Hölder continuous in A and more precisely, by Morrey's theorem,

$$|u(x) - u(y)| \le c \, \|u\|_{W^{1,q}(A)} \, |x-y|^{1-2/q} \qquad \forall \, x, y \in A\,,$$

where $c > 0$ is an absolute constant, compare [5, Thm. 5.4]. Therefore, by (1.13) with $\lambda = \lambda_1 = \lambda_2$ we have

$$|\widetilde{u}(R,\theta) - \lambda| = |\widetilde{u}(R,\theta) - \widetilde{u}(0,\theta)| \le c \, \|u\|_{W^{1,q}(\widetilde{B}_R)} \, R^{1-2/q}$$

for every $\theta \in I_q$, where $\widetilde{B}_R := \{x \in B_R \mid x_1 x_2 < 0\}$. Since $u_{|\widetilde{B}_R} \in W^{1,q}(\widetilde{B}_R)$, we then obtain

$$R^{2-q} \int_{I_q} |\widetilde{u}(R,\theta) - \lambda|^q \, d\theta \le \pi \, c^q \, \|u\|^q_{W^{1,q}(\widetilde{B}_R)} \to 0$$

as $R \to 0^+$, by absolute continuity. Finally, as to the last two integrals in (5.18), since $u_{|B_{R_0}} \in W^{1,p(x)}(B_{R_0})$, setting

$$f(R) := R^{1-p} \int_{I_p} |D_\theta \widetilde{u}(R,\theta)|^p \, d\theta + R^{1-q} \int_{I_q} |D_\theta \widetilde{u}(R,\theta)|^q \, d\theta\,, \qquad 0 < R < R_0\,,$$

by the coarea formula, since $D_\tau u = R^{-1} D_\theta \widetilde{u}$ on ∂B_R (where τ is the tangent direction to ∂B_R), one has

$$\int_0^{R_0} f(R) \, dR = \int_0^{R_0} \int_{\partial B_R} |D_\tau u|^{p(x)} \, d\mathcal{H}^1 \, dR \le \int_{B_{R_0}} |Du|^{p(x)} < +\infty$$

and hence $f(R) \in L^1(0,R_0)$. Therefore, since $f(R) \ge 0$, we have $\liminf_{R\to 0^+} R\, f(R) = 0$, hence the last two integrals in (5.18) are small with R and finally (5.15) holds, as required. □

We conclude with the following

Proof of Lemma 5.8: setting $u_R(x) := u(Rx)$ for $x \in B_1$ and $0 < R < R_0$, we have that $u_R \in W^{1,p}(B_1)$ and for a.e. $0 < R < R_0$ the trace $u_{R|\partial B_1} \in W^{1,p}(\partial B_1)$. Now, by changing variable we have

$$\int_{\partial B_R} |u|^p \, d\mathcal{H}^{n-1} = R^{n-1} \cdot \int_{\partial B_1} |u_R|^p \, d\mathcal{H}^{n-1}\,. \tag{5.20}$$

Moreover, by [5, Thm. 5.22] we have the continuous immersion $W^{1,p}(B_1) \hookrightarrow L^r(\partial B_1)$ if $p < n$ and $p \le r \le (n-1)\,p/(n-p)$. In particular, for $r = p$ (which is good for any such p) we infer that for a.e. $0 < R < R_0$

$$\left(\int_{\partial B_1} |u_R|^p \, d\mathcal{H}^{n-1} \right)^{1/p} \le c \, \|u_R\|_{W^{1,p}(B_1)} \tag{5.21}$$

where $c > 0$ is an absolute constant. By changing variable $y = Rx$, we have

$$\|u_R\|^p_{W^{1,p}(B_1)} = R^{p-n} \int_{B_R} |Du|^p \, dx + R^{-n} \int_{B_R} |u|^p \, dx \tag{5.22}$$

whereas, since by Sobolev embedding theorem $u \in L^{p^*}(B_1)$, by Hölder inequality we have

$$\left(\int_{B_R} |u|^p \, dx \right)^{1/p} \leq \left(\int_{B_R} |u|^{p^*} \, dx \right)^{1/p^*} \cdot |B_R|^{1/p - 1/p^*}$$

and hence, since $|B_R|^{1/p-1/p^*} = c(n)\, R$,

$$R^{-n} \int_{B_R} |u|^p \, dx \leq c\,(n,p)\, R^{p-n} \cdot \left(\int_{B_R} |u|^{p^*} \, dx \right)^{p/p^*}. \tag{5.23}$$

Finally, by (5.20), (5.21), (5.22) and (5.23) we obtain (5.19), and the proof is complete. □

References

[1] ACERBI E., BOUCHITTÉ G., FONSECA I., Relaxation of convex functionals: the gap problem, *Preprint C. Mellon University.*

[2] ACERBI E., FUSCO N., A transmission problem in the calculus of variations, *Calc. Var.* **2** (1994), 1-16.

[3] ACERBI E., MINGIONE G., Regularity results for a class of functionals with non standard growth, *Arch. Rat. Mech. Anal.* **156** 2 (2001), 121-140.

[4] ACERBI E., MINGIONE G., Regularity results for a class of quasiconvex functionals with non standard growth, *Ann. S.N.S. Pisa Cl. Sci.* **30** (2001), 311-339.

[5] ADAMS R.A., *Sobolev spaces*, Academic Press, New York 1975.

[6] BOUCHITTÉ G., FONSECA I., MALÝ J., The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent, *Proc. Roy. Soc. Edinburg* **128A** (1998), 463-479.

[7] BRAIDES A., DEFRANCESCHI A., *Homogeneization of multiple integrals*, Oxford Lecture series in Maths. and its Appl. 12, Oxford University Press, Oxford 1998.

[8] BUTTAZZO G., *Semicontinuity, relaxation and integral representation in the calculus of variations,* Pitman Res. Notes in Math. 207, Longman, Harlow 1989.

[9] BUTTAZZO G., DAL MASO G., Integral representation and relaxation of local functionals, *Nonlinear Anal.* **9** (1985), 515-532.

[10] CORBO ESPOSITO A., DE ARCANGELIS R., Further remarks on some types of relaxation of variational integral functionals, *Nonlinear variational problems and partial differential equations*, Pitman Res. Notes in Math. 320, Longman, Harlow 1995, 157-181.

[11] COSCIA A., MUCCI D., Integral representation and Γ-convergence of variational integrals with $p(x)$-growth, *Preprint Università di Parma* n. 248 (2000).

[12] DACOROGNA B., *Direct methods in the calculus of variations*, Appl. Math. Sci. 78, Springer, Berlin, 1989.

[13] DAL MASO G., *An introduction to* Γ*-convergence*, Prog. in Nonl. Diff. Eq. and their Appl. 8, Birkhäuser, Boston 1993.

[14] DE GIORGI E., LETTA G., Une notion generale de convergence faible pour des fonctions croissantes d'ensemble, *Ann. S.N.S. Pisa Cl. Sci.* **4** (1977), 61-99.

[15] EDMUNDS D.E., RÁKOSNÍK J., Sobolev embeddings with variable exponents, *Studia Math.* **143** 3 (2000), 267-293.

[16] ESPOSITO L., LEONETTI F., MINGIONE G., Sharp regularity for functionals with (p, q)-growth. *Preprint Università di Parma* n. 266 (2001).

[17] FEDERER H., *Geometric measure theory,* Grundleheren math. wissen. 153, Springer, Berlin 1969.

[18] FONSECA I., MALÝ J., Relaxation of multiple integrals below the growth exponent, *Ann. Inst. H. Poincaré* **14** 3 (1997), 309-338.

[19] FUSCO N., On the convergence of integral functionals depending on vector-valued functions, *Ricerche Mat.* **32** 2 (1983), 321-329.

[20] IOFFE A.D., On lower semicontinuity of integral functionals I, *SIAM J. Control Optim.* **15** (1977), 521-538.

[21] KOVÁČIK O., RÁKOSNÍK J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41 (116)** 4 (1991), 592-618.

[22] MARCELLINI P., On the definition and the lower semicontinuity of certain quasiconvex integrals, *Ann. Inst. H. Poincaré* **3** (1986), 391-409.

[23] MARCELLINI P., Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), 1-30.

[24] MORREY C.B., Quasi-convexity and semicontinuity of multiple integrals, *Pacific J. Math.* **2** (1952), 25-53.

[25] RAJAGOPAL K.R., RŮŽIČKA M., Mathematical modelling of electrorheological fluids, *Cont. Mech. Therm.* **13** (2001), 59-78.

[26] ZHIKOV V.V., On the passage to the limit in nonlinear variational problems, *Russian Acad. Sci. Sb. Math.* **76** (2) (1993), 427-459.

[27] ZHIKOV V.V., On Lavrentiev's phenomenon, *Russian J. Math. Physics* **3** (1995), 249-269.

[28] ZHIKOV V.V., On some variational problems, *Russian J. Math. Physics* **5** (1997), 105-116.

[29] ZHIKOV V.V., KOZLOV S.M., OLIENIK O., *Homogeneization of differential operators and integral functionals*, Springer, Berlin 1994.

D. Mucci: Dipartimento di Matematica dell'Università di Parma, Via D'Azeglio 85/A, 43100 Parma (Italy), E-mail: domenico.mucci@unipr.it