Università degli Studi di Pavia

APPROXIMATION RESULTS FOR FREE DISCONTINUITY FUNCTIONALS WITH LINEAR GROWTH

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Dedico questa tesi ai miei insegnanti di Scuola Media inferiore i quali,
forse per le mie origini contadine,
mi sconsigliarono vivamente di intraprendere
studi scientifici.

Preface

A number of variational problems recently under consideration involves integral functionals with "free discontinuities" (according to a terminology introduced in [22]): the variable function u is required to be smooth only outside a surface K, depending on u, and both u and K enter the structure of the functional. Hence, a typical form is:

$$F(u,K) = \int_{\Omega\setminus K} \phi(|\nabla u(x)|)\,dx + \int_{K\cap\Omega} f(|u^+(x) - u^-(x)|)\,d\mathcal{H}^{n-1}$$

where Ω is an open subset of $\mathbb{R}^n$, K is a $(n-1)$-dimensional compact set, $|u^+ - u^-|$ is the jump of u across K, while ϕ and f are given positive functions.

The natural weak formulation is obtained looking at K as the set of discontinuities of u, thus working in spaces of functions allowing hypersurfaces of discontinuities, such as the space $BV(\Omega)$ of functions of bounded variation.

The main difficulty in the actual minimization of F is the presence of the $(n-1)$-dimensional integral: the need of suitable approximations (leading to the convergence of minimum points) by means of more tractable functionals naturally arises. The method introduced in [10], when $\phi(t) = t^2$ and f is constant, makes use of integral functionals whose density depends on the average of the gradient on small balls. Here we apply this scheme to the case of ϕ with linear growth at infinity.

The aforementioned weak formulation of F in $BV(\Omega)$ takes the form:

$$F(u) = \int_{\Omega} \phi(|\nabla u(x)|)\,dx + \int_{S_u} f(|u^+(x) - u^-(x)|)\,d\mathcal{H}^{n-1} + c_0|D^c u|(\Omega) \tag{1.1}$$

where $Du = \nabla u\,d\mathcal{L}^n + |u^+ - u^-|\,d\mathcal{H}^{n-1} + D^c u$ is the decomposition of the measure derivative of u in its absolutely continuous, jump and Cantor part, respectively, and S_u denotes the set of discontinuity points of u. Assuming that ϕ is convex and f is concave, with

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c_0 = \lim_{t\to 0^+} \frac{f(t)}{t}\,,$$

it turns out that F is lower semicontinuous with respect to the L^1-topology. Notice that if ϕ has superlinear growth at infinity then $c_0 = +\infty$ and $F(u)$ is finite only if Du has no Cantor part (i.e. u belongs to the so called space of *special functions with bounded variation*). The well-known

Mumford-Shah functional falls within this case:

$$\int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^{n-1}(S_u)\,.$$

As pointed out in [10], it is not possible to obtain a variational approximation of F by usual integral functionals of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(\nabla u(x))\,dx$$

on Sobolev spaces; indeed, passing to the lower semicontinuous envelopes, this would lead to a convex limit, which contrasts with the non-convexity of F.

Heuristic arguments suggest that to get around the difficulty we have to prevent the consideration or the optimality of approximation gradients which are "too high" (with respect to $1/\varepsilon$), or to prevent that the effect of "high" gradients is concentrated on "small" regions. Several approximation methods (which are briefly presented in Chapter II) fit this requirements: see, e.g., the case where the functionals F_ε are restricted to finite elements spaces on regular triangulations of size ε ([7],[13],[26]); or the implicit constraint on the gradient through the addition of a higher order penalization ([1],[3],[25]); or the study of non-local models, where the effect of a "high" gradient is "spread" onto a set of size ε: this is the method which was first applied to the Mumford-Shah functional in [10] (see also [9],[14],[15],[17],[18]).

In Chapter III we investigate, in the one-dimensional case, the application of the method of the *average of the gradient* to the approximation of functionals with *linear growth* of the form (1.1); the extension to n dimensions (when $\phi(t) = t$) will be studied in Chapter IV.

Contents

Chapter I

Preliminaries

Let $n \geq 1$ be a fixed integer. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$ and the euclidean norm by $|x|$. The open ball with centre x and radius r is indicated by $B_r(x)$; the boundary of the unit ball $B_1(0)$ is denoted by S^{n-1}. The Lebesgue measure and the $(n-1)$ dimensional Hausdorff measure of a Borel set $B \subseteq \mathbb{R}^n$ are denoted by $|B|$ (or $\mathcal{L}^n(B)$) and $\mathcal{H}^{n-1}(B)$, respectively. Given Ω open subset of $\mathbb{R}^n$, $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω; $\mathcal{B}(\Omega)$ denotes the family of all Borel subsets of Ω. We use standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$.

1.1 Some basic tools in measure theory

The aim of this section is to recall some basic results of measure theory, as Radon-Nikodym's Theorem, Besicovitch's differentation Theorem, and approximate continuity of L^1-functions. Let's start with the well known Radon-Nikodym Theorem:

Theorem 1.1.1 (Radon-Nikodym) *Let μ be a σ-finite positive measure and ν a real or vector measure on a measure space $(X, \mathcal{E})$. Then there is a unique pair of $\mathbb{R}^m$-valued measures ν^a and ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in [L^1(X, \mu)]^m$ such that $\nu^a = f\mu$. The function f is called the density of ν with respect to μ and is denoted by ν/μ.*

The representation of the function f of the previous Theorem is given by the derivation Theorem:

Theorem 1.1.2 (Besicovitch) *Let μ be a positive Radon measure in an open set $\Omega \subseteq \mathbb{R}^n$, and ν an $\mathbb{R}^m$-valued Radon measure. Then, for μ-a.e. x in the support of μ the limit*

$$f(x) := \lim_{\varrho \to 0^+} \frac{\nu(B_\varrho(x))}{\mu(B_\varrho(x))}$$

exists in $\mathbb{R}^m$ and moreover the Radon-Nikodym decomposition of ν is given by $\nu = f\mu + \nu^s$, where $\nu^s = \nu \llcorner E$ and E is the μ-negligible set

$$E = (\Omega \setminus supp(\mu)) \cup \left\{ x \in supp(\mu) : \lim_{\varrho \to 0^+} \frac{|\nu|(B_\varrho(x))}{\mu(B_\varrho(x))} = +\infty \right\}.$$

A consequence of the derivation Theorem is the theory of approximate discontinuity points and approximate jump points of an L^1-function. If ν is a unit vector in $\mathbb{R}^n$, we split any ball $B_\varrho(x)$ into the two halves $B_\varrho^+(x,\nu) = \{y \in B_\varrho(x) : \langle y-x,\nu\rangle > 0\}$ and $B_\varrho^-(x,\nu) = \{y \in B_\varrho(x) : \langle y-x,\nu\rangle < 0\}$.

Let Ω be an open subset of $\mathbb{R}^n$.

Definition 1.1.3 *Let $u \in L^1_{loc}(\Omega)$ and $x \in \Omega$.*

We say that u has approximate limit *at x if there exists $z \in \mathbb{R}$ such that:*

$$\lim_{\varrho\to 0^+} \fint_{B_\varrho(x)} |u(y)-z|\,dy = 0.$$

The set S_u where this property fails is called approximate discontinuity set *of u. We say that x is an* approximate jump point *of u if there exist $a,b \in \mathbb{R}$ and $\nu \in \mathbb{R}^n$ with $|\nu| = 1$, such that $a \neq b$ and*

$$\lim_{\varrho\to 0^+} \fint_{B_\varrho^+(x,\nu)} |u(y)-a|\,dy = 0, \qquad \lim_{\varrho\to 0^+} \fint_{B_\varrho^-(x,\nu)} |u(y)-b|\,dy = 0. \tag{1.1}$$

The set of approximate jump points of u is denoted by J_u.

The vector z is uniquely determined for any point $x \in \Omega \setminus S_u$ and is called the *approximate limit* of u at x and denoted by $\tilde{u}(x)$. The triplet (a,b,ν), which turns out to be uniquely determined up to a permutation of a and b and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \tilde{u}$.

Proposition 1.1.4 *Let $u \in L^1_{loc}(\Omega)$; then S_u is a $\mathcal{L}^n$-negligible Borel set, J_u is a Borel subset of S_u and there exist Borel functions*

$$u^+(x), u^-(x), \nu_u(x)\colon J_u \to \mathbb{R}\times\mathbb{R}\times S^{n-1}$$

such that (1.1) holds for every $x \in J_u$.

Before concluding this section, we show a very useful tool, which, in the sequel, we shall apply several times. This is known as the Lemma of "sup of measures" (see, e.g., [2]).

Lemma 1.1.5 (sup of measures) *Let λ be a positive σ-finite Borel measure in Ω; let $\mu\colon \mathcal{A}(\Omega) \to \mathbb{R}$ be a superadditive set function and let $(\psi_i)_{i\in I}$ be a family of positive Borel functions. Suppose*

$$\mu(A) \geq \int_A \psi_i(x)d\lambda$$

for every $A \in \mathcal{A}(\Omega)$ and $i \in I$; then

$$\mu(A) \geq \int_A \sup_i \psi_i(x)d\lambda$$

for every $A \in \mathcal{A}(\Omega)$.

Proof. We have, by the regularity of the measures $\psi_i\lambda$,

$$\int_A \psi d\lambda = \sup\left\{\sum_{i=1}^k \int_{B_i} \psi_i d\lambda : \ (B_i) \quad \text{Borel partition of} \quad A, k \in \mathbb{N}\right\} =$$

$$\sup\left\{\sum_{i=1}^k \int_{K_i} \psi_i d\lambda : \ (K_i) \quad \text{disjoint compact subsets of} \quad A, k \in \mathbb{N}\right\} =$$

$$\sup\left\{\sum_{i=1}^k \int_{A_i} \psi_i d\lambda : \ (A_i) \quad \text{disjoint open subsets of} \quad A, k \in \mathbb{N}\right\} \leq \mu(A),$$

that is the thesis. ■

1.2 Functions of bounded variation

For a thorough treatment of BV functions we refer to [5].

Let Ω be an open subset of $\mathbb{R}^n$. We recall that the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure in Ω, i.e.,

$$\int_\Omega u(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx = -\int_\Omega \varphi(x)\,dD_i u, \qquad \text{for every } \varphi \in C_c^\infty(\Omega) \text{ and } i = 1, \dots, n$$

for some $Du = (D_1u, \dots, D_nu)$ Radon measure on Ω. The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$; indeed for any $u \in W^{1,1}(\Omega)$, the distributional derivative is given by the measure $\nabla u\mathcal{L}^n$. This inclusion is strict; for example the Heaviside's function $\chi_{(0,+\infty)}$ is a BV-function, with Du singular with respect to $\mathcal{L}^n$. The general structure of derivative of BV-functions is given by the Federer-Vol'pert Theorem:

Theorem 1.2.1 (Federer-Vol'pert) *For any $u \in BV(\Omega)$ the set S_u is countably $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover, $Du \llcorner J_u = (u^+ - u^-)\nu_u\mathcal{H}^{n-1} \llcorner J_u$, and $\nu_u(x)$ gives the approximate normal direction to J_u for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.*

In general, the singular part of Du with respect to $\mathcal{L}^n$ is not concentrated on J_u. Let's denote by D^cu the part of Du, singular with respect to $\mathcal{L}^n$, and concentrated on $\Omega\setminus S_u$; D^cu is the cantorian part of Du. It turns out that Du vanishes on $\mathcal{H}^{n-1}$-negligible sets; then, from Federer-Vol'pert Theorem

$$Du = \nabla u\mathcal{L}^n + (u^+ - u^-)\nu_u\mathcal{H}^{n-1} \llcorner J_u + D^cu.$$

The space $SBV(\Omega)$ of *special functions of bounded variation* can be defined as the space of the functions $u \in BV(\Omega)$ such that the singular part of their derivative with respect to the Lebesgue measure $\mathcal{L}^n$ is given by $(u^+ - u^-)\nu_u\mathcal{H}^{n-1} \llcorner J_u$. For such u, denoting by ∇u the density of the absolutely continuous part of Du, we have:

$$Du = \nabla u\mathcal{L}^n + (u^+ - u^-)\nu_u\mathcal{H}^{n-1} \llcorner J_u. \tag{2.1}$$

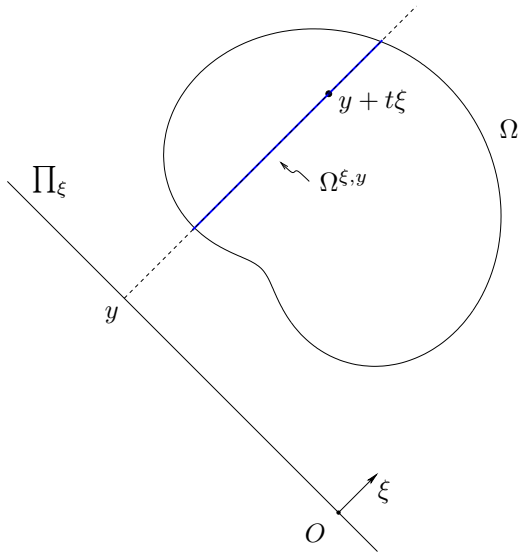
BV-functions are limits of bounded sequences of $W^{1,1}$-functions. Then it is natural to have the following compactness Theorem in BV-space:

Theorem 1.2.2 *Every sequence $(u_h) \subset BV(\Omega)$ satisfying*

$$\sup_{h\in\mathbb{N}} \left\{ \int_A |u_h(x)|dx + |Du_h|(A) \right\} < +\infty$$

for every $A \subset\subset \Omega$, admits a subsequence converging in $L^1(\Omega)$.

A very useful tool in the treatment of variational convergence of integral functionals, is the following slicing result by Ambrosio (see [4]). The basic idea is another way to look at derivatives of BV-functions, based on one-dimensional sections. We introduce, first, some notation. Let $\xi \in S^{n-1}$, and let $\prod_\xi := \{y \in \mathbb{R}^n : \ \langle y, \xi\rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \prod_\xi$ and $\Omega \subseteq \mathbb{R}^n$, we define $\Omega_{\xi,y} = \{t \in \mathbb{R} : \ y + t\xi \in \Omega\}$ and $\Omega^{\xi,y} = \{x \in \Omega : \ x = y + t\xi\}$. Moreover, if $u\colon \Omega \to \mathbb{R}$ we set $u_{\xi,y}\colon \Omega_{\xi,y} \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$.



Theorem 1.2.3 *Let $u \in BV(\Omega)$. Then, for all $\xi \in S^{n-1}$ the function $u_{\xi,y}$ belongs to $BV(\Omega_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.e. $y \in \prod_\xi$. For such y we have*

$$u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi\rangle, \quad \text{for a.e.} \quad t \in \Omega_{\xi,y}$$

$$S_{u_{\xi,y}} = \{t \in \mathbb{R} : \ y + t\xi \in S_u\}.$$

Moreover we have

$$\int_{\prod_\xi} |D^c u_{\xi,y}|(A_{\xi,y})d\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi\rangle|(A)$$

for all $A \in \mathcal{A}(\Omega)$, and for all Borel functions g

$$\int_{\prod_\xi} \sum_{t\in S_{u_{\xi,y}}} g(t)d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x)|\langle \nu_u, \xi\rangle|d\mathcal{H}^{n-1}.$$

Conversely, if $u \in L^1(\Omega)$ *and for all* $\xi \in (e_1, \dots, e_n)$ *and for a.e.* $y \in \prod_\xi$, $u_{\xi,y} \in BV(\Omega_{\xi,y})$ *and*

$$\int_{\prod_\xi} |Du_{\xi,y}|(\Omega_{\xi,y}) d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in BV(\Omega)$.

The slicing Theorem is a basic tool for the proof of the lower bound of the Γ-limit in the n-dimensional case, if it is known the lower bound in the one-dimensional case.

As to the proof of the upper bound, by definition of Γ-convergence (see section 1.4 below) it sufficies to compute the limit of the approximating functionals on a particular choice of approximating functions. Then it is necessary to have some density results. Here, we recall this useful density result, which can be found in [16].

Let Ω a bounded open set in $\mathbb{R}^n$; denote with $\mathcal{W}(\Omega)$ the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:
(i) $\mathcal{H}^{n-1}(\bar{S}_w \setminus S_w) = 0$
(ii)$\bar{S}_w$ is the intersection of Ω with the union of a finite member of pairwise disjoint $(n-1)$ dimensional simplexes;
(iii) $w \in W^{k,\infty}(\Omega \setminus \bar{S}_w)$ for every $k \in \mathbb{N}$.

Theorem 1.2.4 *Let* $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. *Then there exists a sequence* $w_j \in \mathcal{W}(\Omega)$ *such that* $w_j \to u$ *strongly in* $L^1(\Omega)$, $\nabla w_j \to \nabla u$ *strongly in* $L^2(\Omega, \mathbb{R}^n)$, $\limsup_h ||w_j||_\infty \leq ||u||_\infty$ *and*

$$\limsup_{j \to +\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j}) d\mathcal{H}^{n-1} \leq \int_{S_u} \phi(u^+, u^-, \nu_u) d\mathcal{H}^{n-1}$$

for every upper semicontinuous function ϕ *such that* $\phi(a, b, \nu) = \phi(b, a, -\nu)$ *for every* $a, b \in \mathbb{R}$ *and for every* $\nu \in S^{n-1}$.

Finally, we recall the fundamental properties of the space $GBV(\Omega)$ of *generalized functions of bounded variation.* This is the space of all functions $u \in L^1(\Omega)$ whose truncations $u^T := (u \wedge T) \vee (-T)$ are in $BV(\Omega)$, for every $T > 0$. For such function we can define

$$S_u := \bigcup_{T>0} S_{u^T},$$

and the approximate gradient and the traces $u^\pm$ as the limits of the corresponding quantities defined for u^T. Moreover, we define the measure

$$|D^c u|(B) := \sup_{T>0} |D^c u^T|(B) = \lim_{T \to +\infty} |D^c u^T|(B)$$

for every B Borel subset of Ω. It turns out that, even for GBV-functions, $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.

1.3 Functionals defined on BV

In this section we recall some results about a class of isotropic and translation invariant integral functionals defined on $BV(\Omega)$, of the form

$$F(u) = \int_\Omega \phi(|\nabla u(x)|)dx + \int_{S_u} f(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1} + c_0|D^c u|(\Omega)$$

Let us recall a lower semicontinuity result for F in the L^1-topology.

Definition 1.3.1 *We say that a sequence $(u_h) \in BV(\Omega)$ weakly* to $u \in BV(\Omega)$, if $u_h \to u$ in $L^1(\Omega)$, and $Du_h \to Du$ weakly*, i.e.*

$$\lim_{h\to+\infty} \int_\Omega \phi(x)dDu_h = \int_\Omega \phi(x)dDu, \quad \forall \phi \in C_c(\Omega).$$

Notice that u_h weakly* converge to u if and only if u_h converge to u in $L^1(\Omega)$ and u_h is bounded in $BV(\Omega)$ (see [5] Proposition 3.13).

Theorem 1.3.2 *Let $\phi\colon [0,+\infty) \to [0,+\infty]$ be an increasing, lower semicontinuous and convex function, let $f\colon (0,+\infty) \to [0,+\infty]$ be an increasing, lower semicontinuous and subadditive function and $c_0 \in [0,+\infty]$. If*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c_0 = \lim_{t\to 0^+} \frac{f(t)}{t}$$

then the functional F is sequentially weakly-lower semicontinuous in $BV(\Omega)$.*

Remark 1.3.3 In case f is concave, it is easy to deduce a lower semicontinuity result with respect to the L^1-topology. Indeed, if $u_h \to u$ in $L^1(\Omega)$ and $u_h^T = (u_h \wedge T) \vee (-T)$, then (u_h^T) is bounded in $BV(\Omega)$ if $(F(u_h))$ is a bounded sequence (just notice that $f(t) \geq c_M t$ on $[0, M]$ for a suitable $c_M > 0$).

Let us now quote an important relaxation result for functionals F as above, which can be easily obtained from result contained in [6] (see also [2]). Here $\bar{F}$ denote the largest l.s.c. functional smaller than F.

Theorem 1.3.4 *Let $\phi\colon \mathbb{R} \to [0,+\infty)$ be a convex lower semicontinuous function with $\phi(0) = 0$; let $f\colon \mathbb{R} \to [0,+\infty)$ be a subadditive and locally bounded function such that*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = \lim_{t\to 0} \frac{f(t)}{t} = c_0.$$

Consider the functional $F : BV(\Omega) \to [0,+\infty]$ defined by

$$F(u) = \begin{cases} \displaystyle\int_\Omega \phi(|\nabla u(x)|)dx + \int_{S_u} f(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1} & \\ & u \in SBV^2(\Omega) \cap L^\infty(\Omega), \mathcal{H}^{n-1}(S_u) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Then the relaxed functional $\bar{F}$ of F on $BV(\Omega)$ with respect to the L^1-topology is given by

$$\bar{F}(u) = \int_\Omega \phi(|\nabla u(x)|)dx + \int_{S_u} f(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1} + c_0|D^c u|(\Omega).$$

1.4 Γ-convergence

For the general theory see [19]. Let (X, d) be a metric space. Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that (F_j) Γ-*converges*, as $j \to +\infty$, to $F\colon X \to \overline{\mathbb{R}}$ if for all $u \in X$ we have:

i) (*lower semicontinuity inequality*) for every sequence (u_j) converging to u

$$F(u) \le \liminf_{j\to+\infty} F_j(u_j);$$

ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that:

$$F(u) \ge \limsup_{j\to+\infty} F_j(u_j);$$

The *lower* and *upper* Γ-*limits* of (F_j) are defined as

$$F'(u) = \inf\big\{\liminf_{j\to+\infty} F_j(u_j) : \ u_j \to u\big\}, \tag{4.1}$$

$$F''(u) = \inf\big\{\limsup_{j\to+\infty} F_j(u_j) : \ u_j \to u\big\}, \tag{4.2}$$

respectively.

We extend this definition of convergence to families depending on a real parameter. Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-*converges*, as $\varepsilon \to 0$, to $F\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence (ε_j) the sequence (F_{ε_j}) Γ-converges to F.

If we define the *lower* and *upper* Γ-*limits* of (F_ε) as

$$F'(u) = \inf\big\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\big\}, \tag{4.3}$$

$$F''(u) = \inf\big\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\big\}, \tag{4.4}$$

respectively, then (F_ε) Γ-converges to F if and only if

$$F'(u) = F''(u) = F(u) \qquad \text{for every } u \in X.$$

Both F' and F'' are lower semicontinuous on X. In the estimate of F' we shall use the following immediate consequence of the definition:

$$F'(u) = \inf\big\{\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j) : \ \varepsilon_j \to 0^+, \ u_j \to u\big\}. \tag{4.5}$$

It turns out that the infimum is attained.

The most important consequence of definition of Γ-convergence is represented by the Theorem of convergence of minima (see [19] Corollary 7.20):

Theorem 1.4.1 *Let $F_j\colon X \to \bar{\mathbb{R}}$ be a sequence of functions which Γ-converge to some $F : X \to \bar{\mathbb{R}}$; let $\inf_{y\in X} F_j(y) > -\infty$ for every $j \in \mathbb{N}$. Let $x_j \in X$ and let ε_j be a positive infinitesimal sequence, such that*

$$F_j(x_j) \le \inf_{y\in X} F_j(y) + \varepsilon_j;$$

(x_j is called an ε_j-minimizer for F_j). Assume that $x_j \to x$ for some $x \in X$. Then x is a minimum point of F, and

$$F(x) = \lim_{j \to +\infty} F_j(x_j).$$

Chapter II

Free discontinuity functionals and their variational approximation

2.1 Free discontinuity problems: some examples

Many variational problems can be formulated and studied in spaces of functions of bounded variation. The space BV appears as the natural setting to study variational models where both volume and surface energy densities have to be taken into account. The terminology "free discontinuity problems" was introduced by E. De Giorgi in [22] to indicate the class of variational problems that consist in the minimization of a functional, involving both a volume and a surface energy, depending on a closed set K and a function u usually smooth outside K. Notice that K is not fixed a priori. Here are some examples of free discontinuity problems.

Example 2.1.1 **Sets with prescribed mean curvature**.

Consider the problem

$$\min_{E\subseteq\mathbb{R}^n}\left\{\int_E g(x)dx+\mathcal{H}^{n-1}(\partial E)\right\}$$

where $g\in L^1(\mathbb{R}^n)$ is given. Let's compute the first variation; if g is continuous at a regular point $x\in\partial E$, and E minimizes the functional, then the equation

$$H(x)=g(x)\nu_E(x)$$

holds, where H is the mean curvature vector of ∂E and ν_E is the outer normal to E. This problem has been dealt with in the classical framework of sets of finite perimeter.

Example 2.1.2 **The Mumford-Shah image segmentation problem.**

The most famous example of free discontinuity problem is the minimization of the Mumford-Shah functional

$$MS(u,K)=\alpha\int_{\Omega\setminus K}|\nabla u(x)|^2dx+\beta\mathcal{H}^{n-1}(K\cap\Omega)+\gamma\int_{\Omega\setminus K}|u(x)-g(x)|^2dx$$

where $K \subseteq \Omega$ closed, $u \in C^1(\Omega \setminus K)$; here Ω is a bounded open set in $\mathbb{R}^n$, $\alpha, \beta, \gamma > 0$ are fixed parameters, and $g \in L^\infty(\Omega)$. This problem involves two classical objects: the Dirichlet integral and the area functional. Minimizing MS by the direct methods of the Calculus of Variations is not easy, because there is no topology on the closed sets which ensures compactness of minimizing sequences and lower semicontinuity of the Hausdorff measure. It is possible to give a meaningful weak formulation of the functional MS in SBV, setting, for $u \in SBV(\Omega)$:

$$F(u) = \int_{\Omega \setminus S_u} \left(|\nabla u(x)|^2 + \alpha |u(x) - g(x)|^2\right) dx + \beta \mathcal{H}^{n-1}(S_u).$$

The existence of minimizers of F in SBV does not lead immediately to a minimizing pair for MS, because in generale S_u is not closed, and its closure may be even the whole of Ω. Some work is needed to show that the closure of S_u for minimizers is not much larger than S_u. The key point of the theory developed in [20] is then to prove that if u is a minimizer, for any $x \in S_u$ and any ball $B_\varrho(x) \subseteq \Omega$ with ϱ small enough, the following density lower bound holds:

$$\mathcal{H}^{n-1}(S_u \cap B_\varrho(x)) \geq \vartheta_0 \varrho^{n-1} \tag{1.1}$$

where $\vartheta_0 = \vartheta_0(n)$ is a strictly positive dimensional constant. This estimate has a number of interesting consequences, but the information which can be deduced immediately is that if $u \in SBV(\Omega)$ and (1.1) holds, then

$$\mathcal{H}^{n-1}(\Omega \cap \bar{S_u} \setminus S_u) = 0.$$

At this point, it is not hard to show that u has a representative $\tilde{u} \in C^1(\Omega \setminus \bar{S_u})$ and that the pair $(\tilde{u}, \bar{S_u})$ is minimizing for the functional MS.

Example 2.1.3 **Connections with plasticity theory.**

Let $\phi\colon \mathbb{R} \to [0, +\infty)$ be a convex function, and $f\colon (0, +\infty) \to (0, +\infty)$ be a strictly subadditive function. The functional

$$F(u) = \int_0^l \phi(u'(t))dt + \sum_{t \in S_u} f(u^+(t) - u^-(y)),$$

for $u \in SBV(0, l)$, has recently been studied in [11] in connexion with the elastic properties of a bar.

Example 2.1.4 **Brittle fracture.**

Let $\Omega \subseteq \mathbb{R}^3$ be the reference configuration of an elastic body possibly subject to fracture, and $u\colon \Omega \to \mathbb{R}^3$ be the deformation. In fracture mechanics, one has to take into account both the bulk energy densities relative to the elastic deformation outside the fracture, and the energy necessary to produce the crack. If the material is hyperelastic and brittle, i.e. the elastic deformation outside the fracture can be modelled by an elastic energy density independent of the crack, it

is possible to study the existence of equilibria by minimizing a suitable functional subject to boundary conditions. Different models have been proposed; for example, in the isotropic case,

$$F(u,K) = \int_{\Omega\setminus K} W(\nabla u(x))dx + \int_K f(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1}(x)$$

with $c_1|z|^p \leq W(z) \leq c_2(1+|z|^p)$ for any 3×3 matrix z, for some $p > 1$, and $f(t) \to 0$ as $t \to 0$. F can be extended to SBV functions, giving rise to a functional of the form

$$\int_\Omega \phi(x, u(x), \nabla u(x))dx + \int_{J_u} f(u^+(x), u^-(x), \nu(x))d\mathcal{H}^{n-1}(x)$$

for $u \in SBV(\Omega)$.

2.2 Variational approximation

Even though a general existence theory is by now available, exact computation of solutions of free discontinuity problems can be very rarely performed. Hence, the computation of approximate solutions of free discontinuity problems is a crucial issue in the applications. Let us review both the approximation problem for free discontinuity functionals and one of the settings where linear functionals arise.

2.2.1 Approximation of the Mumford-Shah functional

The classical weak formulation of this functional is in a space of functions u allowing $(n-1)$-dimensional sets S_u of discontinuity (the free set K of discontinuity):

$$MS(u) = \alpha \int_\Omega |\nabla u(x)|^2 dx + \beta\mathcal{H}^{n-1}(S_u) + \gamma \int_\Omega |u(x) - g(x)|^2 dx$$

for $u \in SBV(\Omega)$. What makes the minimization of the Mumford-Shah functional very difficult is the presence of the surface term; thus, the need of a suitable approximation (leading to the convergence of minimum points) naturally arise. A first heuristic consideration suggests to use, as approximating functionals, energies of the form:

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(|\nabla u(x)|)dx + \int_\Omega |u(x) - g(x)|^2 dx$$

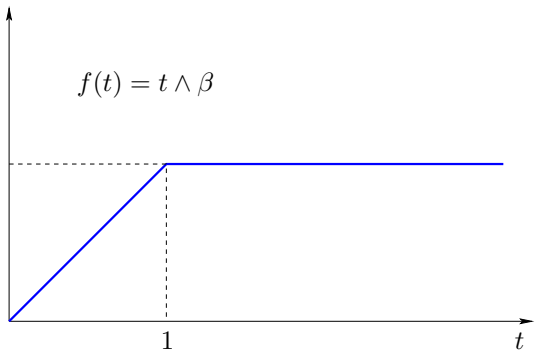
in Sobolev spaces, where $f_\varepsilon(t) \to t^2$ and $\varepsilon f_\varepsilon(t/\varepsilon) \to \beta$ (the energy of a linear function and the energy of a jump of unit length, respectively) as $\varepsilon \to 0$. One possible choice is

$$f_\varepsilon(t) = \frac{1}{\varepsilon} f(\varepsilon t^2),$$

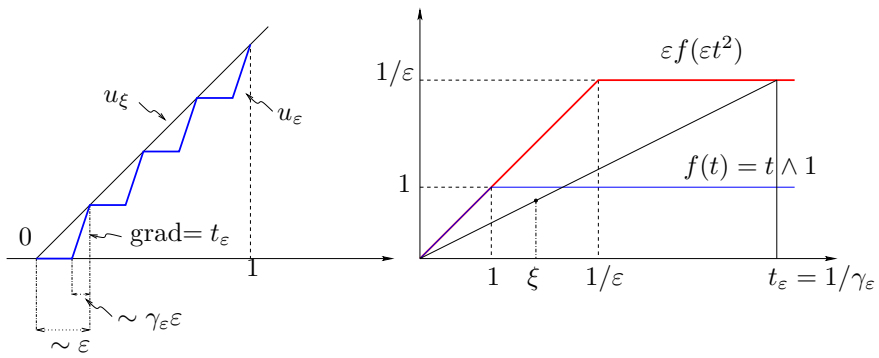
where f satisfy the conditions

$$\lim_{t\to 0} \frac{f(t)}{t} = 1 \quad \text{and} \quad \lim_{t\to+\infty} f(t) = \beta.$$

Take, for instance, $f(t) = t \wedge \beta$:



For the sequel, take, moreover, $\beta = 1$. In the one dimensional setting, the approximation of a linear function $u_\xi(x) = \xi x$ can be done by piecewise functions u_ε with gradient 0 or t_ε, with $t_\varepsilon \to +\infty$.



Hence, setting $\gamma_\varepsilon = 1/t_\varepsilon$, we obtain (in the assumption $\xi = 1$)

$$\int_0^1 f_\varepsilon(|u_\varepsilon'(x)|)dx = (1-\gamma_\varepsilon)\frac{1}{\varepsilon}f(0) + \gamma_\varepsilon \frac{1}{\varepsilon} f(\varepsilon t_\varepsilon^2) = \gamma_\varepsilon \frac{1}{\varepsilon} f\left(\frac{\varepsilon}{\gamma_\varepsilon^2}\right) \to 0$$

whenever $\gamma_\varepsilon = o(\varepsilon)$. Thus this type of approximation does not work, because the energy of linear function would be 0 (this, as the figure above suggests, is the value at ξ of the *convex envelope* of the integrand function). In fact it is *not possible* to obtain a variational approximation of MS, leading to the convergence of minimum points, by means of local integral functionals of the form:

$$\int_\Omega f_\varepsilon(|\nabla u(x)|)dx + \int_\Omega |(u(x) - g(x)|^2 dx$$

defined on the Sobolev space $W^{1,2}(\Omega)$. Indeed, if such an approximation existed, the functional MS would also be the variational limit of the convex functionals obtained by considering the *convex envelope* of f_ε, in contrast with the lack of convexity of MS.

Suitable modifications of the above setting (in order to obtain an approximation of the Mumford-Shah functional) aim to prevent the consideration or the optimality of gradients which are "too large" with respect to $1/\varepsilon$ or to prevent that the effect of "large" gradients is concentrated on "small" regions. The latter is the fundamental idea of the method of the average of the

gradients, applied by A.Braides and G.Dal Maso in [10]. They consider approximations of the form

$$E_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|^2 dy\right) dx$$

defined for $u \in W^{1,2}(\Omega)$, where f is a suitable continuous, non-decreasing and bounded function. These functionals are non-local in the sense that their energy density at a point $x \in \Omega$ depends on the behaviour of u in the whole set $B_\varepsilon(x) \cap \Omega$. In this case, the phenomenon previously considered of the convex envelope of the integrand function does not appear; indeed, the effect of high gradients is spread onto a region of size ε. If u is a linear function of gradient ξ, then

$$\frac{1}{\varepsilon}\int_0^1 f\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(0,1)} |u'(y)|^2 dy\right) dx \to f'(0)|\xi|^2.$$

Otherwise if we approximate a jump s with a piecewise affine function u_ε with gradient 0 or $t_\varepsilon = s/o(\varepsilon)$ around the jump point, then we obtain

$$\frac{1}{\varepsilon}\int_0^1 f\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(0,1)} |u'_\varepsilon(y)|^2 dy\right) dx \to 2f_\infty$$

where

$$f_\infty = \lim_{t\to+\infty} f(t).$$

This suggests that the Γ-limit must be of the Mumford-Shah type (the proof is given in [10]). Following a technique which is frequently used in Γ-convergence, they localize the problem and considering, for every open set $A \subseteq \Omega$, the functionals

$$F_\varepsilon(u, A) := \frac{1}{\varepsilon}\int_A f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|^2 dy\right) dx.$$

A crucial point is to prove that if ν_u is the unit normal to S_u, then, if F denote the Γ-limit,

$$F(u, A) \geq \int_A |\nabla u(x)|^2 dx,$$

$$F(u, A) \geq \int_{S_u\cap A} |\langle \nu_u(x), \xi\rangle| d\mathcal{H}^{n-1}.$$

for every $u \in SBV(\Omega) \cap L^\infty(\Omega)$, for every open set $A \subseteq \Omega$, and for every $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. From these estimates the inequality

$$F(u, A) \geq \int_\Omega |\nabla u(x)|^2 dx + \mathcal{H}^{n-1}(S_u)$$

follows exploiting the superadditivity of the set function $F(u, \cdot)$. The converse inequality is proved using a standard density argument based on the density in $SBV(\Omega)$ of the functions $u \in SBV(\Omega) \cap L^\infty(\Omega)$ with

$$\mathcal{H}^{n-1}(S_u) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} |\{x \in \mathbb{R}^n | d(x, S_u) < \varepsilon\}|.$$

More generally in [14] the author considers approximating functionals of the form

$$G_\varepsilon(u,A) = \frac{1}{\varepsilon}\int_A f_\varepsilon\left(\varepsilon g_\varepsilon(y,\nabla u(y))\psi_\varepsilon(x-y)dy\right)dx,$$

where f_ε are non decreasing concave functions with equibounded derivatives at 0, and g_ε satisfy standard growth conditions; while ψ_ε is a family of convolution kernels. The main result of this work is an integral representation theorem for a L^1-lower semicontinuous functional which are local measures and satisfies some growth conditions. This Theorem is then applied on the Γ-limit of the sequence F_ε, and furnishes an integral representation of the Γ-limit. Next, the author prove that the bulk energy density appearing in the expression of the represented Γ-limit can indeed be characterized under slightly stronger assumption; it turns out that this density depends, in some sense, on the behaviour of the sequence g_ε. The surface density of the limit depends, in general, not only on g_ε, but also on f_ε and ψ_ε. Non trivial examples which show the dependence of the surface density on ψ_ε can be deduced from the results in [18], where an explicit formula for the density is given under the assumption that the family (f_ε) contains one function only. For this general approach, other details can be found in [15] and [17].

Another type of approximation is given by a finite difference scheme. The idea is the same of the method of the average of the gradients previously considered, i.e. to prevent the effect of high gradients. But in this case, the average of the gradient is replaced by a finite difference; more precisely, in [24], the authors consider approximations of the form

$$F_\varepsilon(u) = \int_\Omega \varphi_\varepsilon\left(\frac{|u(x+\varepsilon)-u(x)|}{\varepsilon}\right)dx.$$

These are again non-local functionals. The convergence is first proved in the one-dimensional case, and is then extended, by a standard slicing argument, to the n-dimensional case. The main step is to show that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \bar{F}_{\varphi,\psi}(u)$$

for every $u \in L^1_{loc}(\mathbb{R})$, every $\varepsilon_n \to 0$, and every sequence $u_n \to u$ in $L^1_{loc}(\mathbb{R})$, where

$$F_{\varphi,\psi}(u) = \int_\Omega \varphi(|\nabla u(x)|)dx + \int_{S_u} \psi(|u^+(x)-u^-(x)|)d\mathcal{H}^{n-1},$$

φ is the lower Γ-limit of φ_ε, and ψ is the lower Γ-limit of $\varphi_\varepsilon(\cdot/\varepsilon)$. The result is very strong: indeed, in such way we obtain an approximation result for functionals with linear or superlinear growth in the gradient. For this type of approximation, see moreover [12] [21] [23] [24].

The methods of average of the gradients or finite difference are not the only methods to approximate the Mumford-Shah functional; as mentioned above methods were proposed to prevent the consideration or the optimality of approximation gradients which are "too high". For instance, a suitable penalization of the second derivative can produce the desired effect; in [1] the authors consider, in the one dimensional setting, functionals of the form

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'(t)|^2)dt + \varepsilon^3\int_I |u''(t)|^2 dt & u \in W^{2,2}(I) \\ +\infty & \text{otherwise} \end{cases}$$

where I is a bounded interval in $\mathbb{R}$ and f is a lower semicontinuous increasing function with finite derivative at 0 and bounded at $+\infty$.

The main result is given by the computation of the Γ-limit of the family F_ε. It turns out that (F_ε) Γ-converge to F in the L^1-topology, where

$$F(u) = \begin{cases} \alpha \displaystyle\int_I |u'(t)|^2 dt + m(\beta) \sum_{t \in S_u} \sqrt{|u^+(t) - u^-(t)|} & u \in SBV(I) \\ +\infty & \text{otherwise} \end{cases}$$

with α, β and $m(\beta)$ positive constant depends on f. The main step in the proof of this convergence result is to show that if u_ε is a bounded family in L^1 satisfying $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, then there exists a family (v_ε) in $SBV(I)$ which is near to u_ε in L^1, and such that

$$a \int_I |v'_\varepsilon(t)|^2 dt = \frac{1}{\varepsilon} \int_I f(\varepsilon |u'_\varepsilon(t)|^2) dt$$

and

$$\sum_{t \in S_{v_\varepsilon}} \sqrt{|v_\varepsilon^+(t) - v_\varepsilon^-(t)|} \leq \sup_\varepsilon F_\varepsilon(u_\varepsilon)/m(\beta).$$

The second one is the fundamental estimate which allows to obtain a lower bound for the lower Γ-limit, and it follows from an optimal profile problem, which has a non trivial solution 0 thanks to the term

$$\varepsilon^3 \int_I |u''_\varepsilon(t)|^2 dt.$$

By using a slight variation of this main result, the authors show that it is possible to approximate the Mumford-Shah functional, too.

In the paper [3], the authors generalize the previous result to the n-dimensional case, by using standard slicing technique to obtain the lower bound for the Γ-limit; a non trivial computation and classical density results furnish the upper bound and conclude the proof. A more general treatment of this arguments can be found in [25], where the author investigate the extension of the previous results for functional of the form

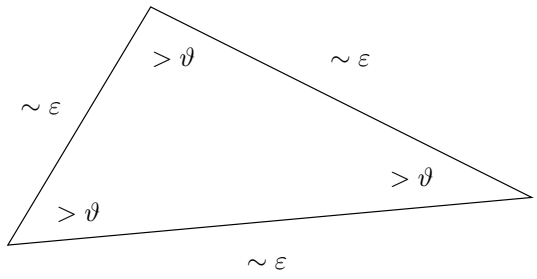
$$\frac{1}{\varepsilon} \int_\Omega f(\sqrt{\varepsilon} |\nabla u(x)|) dx + r(\varepsilon) \int_\Omega ||\nabla^2 u(x)||^2 dx$$

where $r(\varepsilon)$ is a function which vanishes as $\varepsilon \to 0^+$.

Another method for approximation is based on a restriction of the space of the possible approximating functions u_ε; an example is the use of finite elements. Consider the functional

$$\frac{1}{\varepsilon} \int_\Omega f(\varepsilon |\nabla u(x)|^2) dx + \int_\Omega |u(x) - g(x)|^2 dx$$

only on the space $V_\varepsilon(\Omega)$ which is the union of all the finite element spaces of continuous and piecewise affine functions on triangulations with the following property (for simplicity, in the case $n = 2$): *for each element T the length of the edges is of order ε and the amplitude of the internal angles is not less than a fixed value ϑ*:



In this conditions we have Γ-convergence to the Mumford-Shah functional. Results in this order of ideas are contained in [7] [13] [26] [27] [28].

2.2.2 Functionals with linear growth

Let us recall the well know Ambrosio-Tortorelli elliptic approximation for Mumford-Shah Functional:

$$AT(u,v) = \alpha \int_\Omega \left((1-v(x))^2 |\nabla u(x)|^2 + \frac{\beta}{2} \left(\varepsilon |\nabla v(x)|^2 + \frac{v(x)^2}{\varepsilon} \right) + \gamma |u(x) - g(x)|^2 \right) dx.$$

Then the term $\mathcal{H}^1(K \cap \Omega)$ in the Mumford-Shah functional is here replaced by a continuous variable v.

The key idea is that if K is a fixed curve in Ω and v_ε minimizes

$$G_\varepsilon(v) = \frac{1}{2} \int_{\Omega \setminus K} \left(\varepsilon |\nabla v(x)|^2 + \frac{v(x)^2}{\varepsilon} \right) dx$$

with $v = 1$ on K, then

$$G_\varepsilon(v_\varepsilon) \to \mathcal{H}^1(K).$$

Values of v_ε range from 0 to 1. In the framework of image segmentation v_ε may be viewed as a blurring of K with ε as "blurring radius".

A drawback of this method is the difficulty of recovering the actual boundaries from the edge-strength function v (indeed, global thresholding of v does not produce a satisfactory representation). The alternative method of *shape recovery by curve evolution* is based on the following idea: let Γ be a simple closed curve; in order to move Γ to where the image intensity gradient and hence v are high, we look for the stationary points of the functional

$$\int_\Gamma (1-v)^2 ds$$

where s denotes the arc length along Γ. In order to implement the evolution of Γ, assume that Γ is embedded in a surface $f_0 \colon \Omega \to \mathbb{R}$ as a level curve. Let $f(t,x,y)$ denote the evolving surface such that $f(0,x,y) = f_0(x,y)$. Then, in order to let all the level curves of f_0 evolve simultaneously, consider the functional

$$\int_{-\infty}^{+\infty} \int_{\Gamma_c} (1-v)^2 ds dc$$

where $\Gamma_c = \{(x,y) : \ f(t,x,y) = c\}$; by the coarea formula

$$\int_{-\infty}^{+\infty} \int_{\Gamma_c} (1-v)^2 ds dc = \int_\Omega (1-v)^2 |\nabla f(x,y)| dx dy.$$

Hence we arrive at the functional

$$E_\varepsilon(u,v) = \int_\Omega \left((1-v(x))^2 |\nabla u(x)| + \varepsilon |\nabla v(x)|^2 + \frac{v(x)^2}{\varepsilon} + \gamma |u(x) - g(x)|^2 \right) dx$$

proposed by J.Shah as a segmentation model (J. Shah, IEEE Conf. Comp. vision and Pattern Rec., 1996).

It can be proved ([2]) that E_ε converges to:

$$F(u) + \beta \int_\Omega |u(x) - g(x)| dx$$

where

$$F(u) = \int_\Omega |\nabla u(x)| dx + \int_{S_u} \frac{|u^+(x) - u^-(x)|}{1 + |u^+(x) - u^-(x)|} d\mathcal{H}^1 + |D^c u|(\Omega).$$

Then functionals with linear growth arise as segmentation models; the natural question is if, even in this case, there exists approximations by functionals defined in Sobolev spaces, of the same type than for Mumford-Shah.

Chapter III

The one-dimensional case

3.1 Setting of the problem and main results

Let (a,b) be an open interval of $\mathbb{R}$ and consider the functional $F : L^1(a,b) \to [0,+\infty]$ defined as follows:

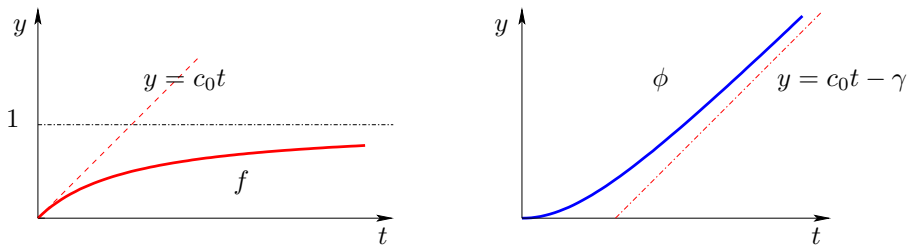
$$F(u) = \begin{cases} \displaystyle\int_a^b \phi(|u'(x)|)dx + \sum_{x\in S_u} f\left(|u^+(x) - u^-(x)|\right) + c_0|D^c u|(a,b) & \\ & \text{if } u \in GBV(a,b), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\phi, f : [0,+\infty) \to [0,+\infty)$ satisfy the following conditions:

(A0) ϕ is convex and f is concave, with $\phi(0) = f(0) = 0$ and there exists $c_0 \in \mathbb{R}$, with $c_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = \lim_{t\to 0} \frac{f(t)}{t} = c_0.$$

For example, as a choice for the applications:



By Theorem 5.4 in [5] (see also, e.g., [8] §2.4), F is sequentially lower semicontinuous in the L^1-topology.

We will prove an approximation result for F by means of a family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $L^1(a,b) \to [0,+\infty]$ of the form:

$$F_\varepsilon(u) = \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|dy\right) dx & \text{if } u \in W^{1,1}(a,b)\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1}$$

where f_ε is requested to satisfy the conditions *(A1)–(A3)* below.

(A1) For every $\varepsilon > 0$, $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ is a non-decreasing continuous function with $f_\varepsilon(0) = 0$; moreover, there exists $a_\varepsilon > 0$ such that $a_\varepsilon \to 0$ as $\varepsilon \to 0$ and f_ε is concave in $(a_\varepsilon, +\infty)$.

Simple heuristic considerations suggest that the volume term in the limit depends on the behaviour of f_ε near zero. Then we require that f_ε behaves as a suitable rescaling of ϕ in a neighborhood of zero; more precisely, we assume that:

(A2)
$$\lim_{(\varepsilon,t)\to(0,0)} \frac{f_\varepsilon(t)}{\varepsilon\phi\left(\frac{t}{\varepsilon}\right)} = 1$$

For reference convenience we point out that, in particular, *(A2)* implies:

$$\lim_{\varepsilon\to 0} \frac{f_\varepsilon(\varepsilon s)}{\varepsilon} = \phi(s) \quad \text{for every} \quad s \geq 0\,; \tag{1.2}$$

moreover, for every $\delta > 0$ there exist $t_\delta > 0$ and $\varepsilon_\delta > 0$ such that:

$$f_\varepsilon(t) \geq (1-\delta)\varepsilon\phi(t/\varepsilon)\,, \tag{1.3}$$

whenever $0 \leq t \leq t_\delta$ and $\varepsilon < \varepsilon_\delta$.

Analogously, we expect that the jump term in the limit depends on the pointwise behaviour of f_ε. Accordingly, we assume that:

(A3) $f_\varepsilon(t) \to f(t)$ uniformly on the compact subsets of $[0,+\infty)$.

Given f and ϕ as above, a possible choice for f_ε satisfying *(A1)–(A3)* is:

$$f_\varepsilon(t) = \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \leq t \leq t_\varepsilon, \\ f(t-t_\varepsilon) + \varepsilon\phi(\frac{t_\varepsilon}{\varepsilon}) & \text{if } t \geq t_\varepsilon, \end{cases}$$

where $t_\varepsilon \to 0$, and $t_\varepsilon/\varepsilon \to +\infty$. The only non-trivial assumption to verify is *(A2)*: since $(\varepsilon/t)\phi(\varepsilon/t) \to c_0$ as $(\varepsilon,t) \to (0,0)$, with $t \geq t_\varepsilon$, the check amounts to verify that:

$$\lim_{(\varepsilon,t)\to(0,0)} \frac{f(t-t_\varepsilon) + \varepsilon\phi(t_\varepsilon/\varepsilon)}{t} = 1\,.$$

This follows immediately from $f(t-t_\varepsilon)/(t-t_\varepsilon) \to c_0$ and $(\varepsilon/t_\varepsilon)\phi(t_\varepsilon/\varepsilon) \to c_0$ as $(\varepsilon,t) \to (0,0)$, with $t \geq t_\varepsilon$.

We point out two properties we shall need in the sequel.

Remark 3.1.1 It is easy to see that there exist sequences (c_h) and (d_h) of real numbers with

$$0 \leq c_h \leq c_0\,, \qquad \phi(t) = \sup_{h\in\mathbb{N}}(c_h t + d_h) \qquad \text{for every } t \geq 0. \tag{1.4}$$

REMARK 3.1.2 There exists $C > 0$ such that

$$f_\varepsilon(t) \le Ct \qquad \text{for every } t \ge 0 \text{ and } \varepsilon \text{ small enough.}$$

Proof. For every $\delta > 0$ Assumption (A2) yields the existence of $t_\delta, \varepsilon_\delta > 0$ such that

$$f_\varepsilon(t) \le (1+\delta)\varepsilon\phi(t/\varepsilon) \qquad \text{for every } 0 \le t \le t_\delta \text{ and } 0 < \varepsilon \le \varepsilon_\delta;$$

therefore, since $\phi(s) \le c_0 s$, we have:

$$f_\varepsilon(t) \le c_0(1+\delta)t \qquad \text{for every } 0 \le t \le t_\delta \text{ and } 0 < \varepsilon \le \varepsilon_\delta;$$

As to the interval $[t_\delta, +\infty)$, notice that we can suppose f_ε concave in $[t_\delta/2, +\infty)$ for every $0 < \varepsilon \le \varepsilon_\delta$; thus, if $m_\varepsilon t + q_\varepsilon$ is the affine function with coincide with f_ε in $t_\delta/2$ and t_δ, then $f_\varepsilon(t) \le m_\varepsilon t + q_\varepsilon$ if $t \ge t_\delta$. The pointwise convergence of f_ε gives the existence of m_0 and q_0 such that

$$f_\varepsilon(t) \le m_0 t + q_0 \qquad \text{for every } t \ge t_\delta \text{ and } 0 < \varepsilon \le \varepsilon_\delta.$$

The existence of a linear function majorizing f_ε on $[0, +\infty)$ uniformly in ε now easily follows. ■

Let us now state the main results of the paper.

Theorem 3.1.3 *Let $(F_\varepsilon)_{\varepsilon>0}$ be as in (1.1), with f_ε satisfying (A0),(A1), (A2) and (A3). Then (F_ε) Γ-converges, in $L^1(a,b)$ as $\varepsilon \to 0$, to $\mathcal{F}\colon L^1(a,b) \to [0,+\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_a^b \phi(|u'(x)|)dx + 2\sum_{x\in S_u} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) + c_0|D^c u|(a,b) & \\ & \text{if } u \in GBV(a,b)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.1.4 (Compactness) *Let u_j be a sequence in $L^1(a,b)$ such that*

$$||u_j||_\infty \le M, \quad F_{\varepsilon_j}(u_j) \le M$$

for a suitable constant M independent of j. Then there exists a subsequence (u_{j_k}) converging in $L^1(a,b)$ to a function $u \in BV(a,b)$.

As an example of application of these results (together with Theorem 1.4.1) we state the following corollary.

Corollary 3.1.5 *Let (ε_j) be a positive infinitesimal sequence and $g \in L^\infty(a,b)$. For every $u \in L^1(a,b)$, define:*

$$G_j(u) = F_{\varepsilon_j}(u) + \int_a^b |u(x) - g(x)|\,dx\,,$$

and

$$\mathcal{G}(u) = \mathcal{F}(u) + \int_a^b |u(x) - g(x)|\, dx\,.$$

For every j let x_j be an ε_j-minimizer of G_j in $L^1(a,b)$, i.e.

$$G_j(x_j) \le \inf_{L^1(a,b)} G_j + \varepsilon_j\,,$$

Then (x_j) converges, up to a subsequence, to a minimizer of G in $L^1(a,b)$.

Theorem 3.1.4 will be proved in the next section (Corollary 3.2.3) while Theorem 3.1.3 will be completed in § 3.5. In Section 3.6 we shall compute the relaxed functional of F_ε (which will be useful in the proof of the estimate for the Γ-lower limit):

Proposition 3.1.6 *For every $\varepsilon > 0$, the relaxed functional of F_ε in the L^1-topology is given by*

$$\bar{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(\frac{\varepsilon}{|(x-\varepsilon, x+\varepsilon)\cap(a,b)|}|Du|((x-\varepsilon,x+\varepsilon)\cap(a,b))\right) dx \tag{1.5}$$

for every $u \in BV(a,b)$.

Remark 3.1.7 Clearly, the functionals F_ε and $\mathcal{F}$ can be defined with (a,b) replaced by an arbitrary open subset A of $\mathbb{R}$; in this case we shall make the dependence on the set A explicit through the notation $F_\varepsilon(u,A)$ and $\mathcal{F}(u,A)$, respectively. Since every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals, it is not difficult to deduce, from Theorem 3.1.3, the convergence of $(F_\varepsilon(\cdot,A))$ to $\mathcal{F}(\cdot,A)$ for every open $A \subset \mathbb{R}$.

Remark 3.1.8 The Γ-convergence of (F_ε) easily implies the convergence of $(\tilde{F}_\varepsilon)$, where

$$\tilde{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\Big(\frac{1}{2}\int_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|\, dy\Big)\, dx$$

if $u \in W^{1,1}(a,b)$, and $\tilde{F}_\varepsilon(u) = +\infty$ otherwise in $L^1(a,b)$. Since $\tilde{F}_\varepsilon \le F_\varepsilon$, we have only to check the inequality

$$\liminf_{j\to+\infty} \tilde{F}_{\varepsilon_j}(u_j) \ge \mathcal{F}(u)$$

whenever $\varepsilon_j \to 0$ and $u_j \to u$ in $L^1(a,b)$. For any $\sigma > 0$ and j sufficiently large, it turns out that:

$$\tilde{F}_{\varepsilon_j}(u_j) \ge \frac{1}{\varepsilon_j}\int_{a+\sigma}^{b-\sigma} f_\varepsilon\Big(\frac{1}{2}\int_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(y)|\, dy\Big)\, dx \ge F_{\varepsilon_j}\big(u_j, (a+\sigma, b-\sigma)\big)\,;$$

hence $\liminf_{j\to+\infty} \tilde{F}_{\varepsilon_j}(u_j) \ge \mathcal{F}\big(u,(a+\sigma,b-\sigma)\big)$: let now σ tend to zero.

If not otherwise specified, ε_j will stand for a positive infinitesimal sequence.

3.2 Estimate from below of the volume and Cantor terms. Compactness

The main result of this section is Proposition 3.2.8. Moreover, the technical Lemma 3.2.2 will immediately imply the compactness result of Corollary 3.2.3.

Let us state a preliminary result (from [10], Lemma 4.2) whose proof follows by applying the mean value theorem for integrals to the function $\sum_{\alpha\in\mathbb{Z}} \psi(x+\alpha\eta)$.

Lemma 3.2.1 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. For every $\eta > 0$ there exists $x_0 \in (-\eta/2, \eta/2)$ such that*

$$\int_{\mathbb{R}} \psi(x)dx = \sum_{\alpha\in\mathbb{Z}} \eta\psi(x_0+\alpha\eta).$$

This allows to estimate $F_\varepsilon(u)$ by a useful integral sum (see (2.1) below).

Let us fix $u \in W^{1,1}(a,b)$, $\varepsilon > 0$ and φ_ε a cut-off function in $C_c^\infty(a,b)$ such that $0 \le \varphi_\varepsilon \le 1$ in (a,b) and $\varphi_\varepsilon = 1$ in $(a+\varepsilon, b-\varepsilon)$. Let ψ_ε be the continuous function defined by

$$\psi_\varepsilon(x) = \varphi_\varepsilon(x) f_\varepsilon\left(\varepsilon \fint_{(x-\varepsilon,x+\varepsilon)\cap(a,b)} |u'(t)|dt\right)$$

for $x \in (a,b)$, and 0 for $x \in \mathbb{R}\setminus(a,b)$. Then, taking Lemma 3.2.1 into account, with $\eta = 2\varepsilon$, we have

$$F_\varepsilon(u) \ge \frac{1}{\varepsilon}\int_a^b \psi_\varepsilon(x)dx = \frac{1}{\varepsilon}\int_{\mathbb{R}} \psi_\varepsilon(x)dx = \sum_{\alpha\in\mathbb{Z}} 2\psi_\varepsilon(x_\varepsilon + 2\varepsilon\alpha) \tag{2.1}$$

for a suitable $x_\varepsilon \in \mathbb{R}$. Therefore, if we set $x_\alpha = x_\alpha^\varepsilon = x_\varepsilon + 2\alpha\varepsilon$, $(\alpha \in \mathbb{Z})$, and

$$J_\varepsilon := \{\alpha \in \mathbb{Z} : \ x_\alpha \in (a+\varepsilon, b-\varepsilon)\}$$

we get ($\varphi_\varepsilon(x_\alpha) = 1$ if $\alpha \in J_\varepsilon$):

$$F_\varepsilon(u) \ge 2\sum_{\alpha\in J_\varepsilon} f_\varepsilon\left(\varepsilon \fint_{x_\alpha-\varepsilon}^{x_\alpha+\varepsilon} |u'(t)|dt\right). \tag{2.2}$$

Lemma 3.2.2 *Let $u \in W^{1,1}(a,b)$ and $\delta > 0$; let t_δ and ε_δ be as in (1.3). Fix $\varepsilon < \varepsilon_\delta$. With the notation above let*

$$\mathcal{P}_\varepsilon = \{(x_\alpha - \varepsilon, x_\alpha + \varepsilon) : \alpha \in J_\varepsilon\}.$$

Then we can select a subfamily $\mathcal{P}'_\varepsilon$ of $\mathcal{P}_\varepsilon$ such that

$$\textit{(i)} \quad \sharp(\mathcal{P}_\varepsilon \setminus \mathcal{P}'_\varepsilon) \le \frac{1}{2}F_\varepsilon(u)/f_\varepsilon(t_\delta);$$

$$\textit{(ii)} \quad F_\varepsilon(u) \ge (1-\delta)\int_{\bigcup \mathcal{P}'_\varepsilon} l(|u'(t)|)dt$$

whenever l is an affine function satisfying $l \le \phi$.

Since $f_\varepsilon(t_\delta) \to f(t_\delta) > 0$, the inequality (i) states that for any given $\delta > 0$ we can estimate $\sharp(\mathcal{P}_\varepsilon \setminus \mathcal{P}'_\varepsilon)$ by $F_\varepsilon(u)/f(t_\delta)$ for ε sufficiently small.

Proof. From (1.3) and (2.2), we get

$$F_\varepsilon(u) \geq 2(1-\delta) \sum_{\alpha \in H_\varepsilon} \varepsilon \phi \left(\fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)| dt \right),$$

where

$$H_\varepsilon = \left\{ \alpha \in J_\varepsilon : \ \varepsilon \fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)| dt < t_\delta \right\}.$$

Fix now an affine function $l : \mathbb{R} \to \mathbb{R}$ with $l \leq \phi$, and define

$$\mathcal{P}'_\varepsilon = \{(x_\alpha - \varepsilon, x_\alpha + \varepsilon) | \alpha \in H_\varepsilon\}.$$

Then

$$F_\varepsilon(u) \geq 2\varepsilon(1-\delta) \sum_{\alpha \in H_\varepsilon} l \left(\fint_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} |u'(t)| dt \right) =$$

$$= (1-\delta) \sum_{\alpha \in H_\varepsilon} \int_{x_\alpha - \varepsilon}^{x_\alpha + \varepsilon} l(|u'(t)|) dt = (1-\delta) \int_{\bigcup \mathcal{P}'_\varepsilon} l(|u'(t)|) dt.$$

Finally, from (2.2), and the monotonicity of f_ε, it follows that

$$F_\varepsilon(u) \geq 2 \sum_{\alpha \in J_\varepsilon \setminus H_\varepsilon} f_\varepsilon(t_\delta) = 2\sharp(J_\varepsilon \setminus H_\varepsilon) f_\varepsilon(t_\delta).$$

■

Corollary 3.2.3 (compactness) *Let u_j be a sequence in $L^1(a,b)$ such that*

$$||u_j||_\infty \leq M, \quad F_{\varepsilon_j}(u_j) \leq M$$

for a suitable constant M independent of j. Then there exists a subsequence (u_{j_k}) converging in $L^1(a,b)$ to a function $u \in BV(a,b)$.

Proof. Clearly, $u_j \in W^{1,1}(a,b)$ for every j. Since

$$\lim_{t \to +\infty} \frac{\phi(t)}{t} = c_0$$

there exists $c_1 \in \mathbb{R}$ such that $\phi(t) \geq c_0 t + c_1$. Fix $\delta > 0$ and apply Lemma 3.2.2 with $u = u_j$ and $l(t) = c_0 t + c_1$; we determine a family $\mathcal{P}_j$ of intervals of width $2\varepsilon_j$ covering a.e. $(a + \varepsilon_j, b - \varepsilon_j)$, and a subfamily $\mathcal{P}'_j$ such that $\sharp(\mathcal{P}_j \setminus \mathcal{P}'_j)$ is bounded independently of j, and

$$F_{\varepsilon_j}(u_j) \geq (1-\delta) \int_{\bigcup \mathcal{P}'_j} l(|u'_j(x)|) dx.$$

Then, if we define

$$v_j(x) = \begin{cases} u_j(x) & x \in \bigcup \mathcal{P}'_j \\ 0 & \text{otherwise} \end{cases}$$

it turns out that $v_j \in SBV(a,b)$, $||v_j||_\infty \leq M$, $\sharp S_{v_j}$ bounded independently of j, and

$$M \geq (1-\delta) \int_a^b l(|v'_j(x)|)dx = c'_0 \int_a^b |v'_j(x)|dx + c'_1$$

with $c'_0 > 0$. Therefore (v_j) is bounded in $BV(a,b)$ and we can extract a subsequence (v_{j_k}) converging in $L^1(a,b)$ to a function $u \in BV(a,b)$. Since

$$||v_j - u_j||_1 \leq 2\varepsilon_j M(\sharp(\mathcal{P}_j \setminus \mathcal{P}'_j) + 2)$$

we conclude that u_{j_k} converges to u in $L^1(a,b)$. ■

Corollary 3.2.4 *Let (u_j) be a converging sequence in $L^1(a,b)$. If $F_{\varepsilon_j}(u_j)$ is bounded, then the limit of (u_j) belongs to $GBV(a,b)$. In particular, if $F'(u) < +\infty$ then $u \in GBV(a,b)$.*

Proof. Let u be the L^1-limit of (u_j). For each $T > 0$ apply the previous Corollary to $u_j^T = (u_j \wedge T) \vee (-T)$: we get $(u \wedge T) \vee (-T) \in BV(a,b)$; hence $u \in GBV(a,b)$. ■

Lemma 3.2.5 *Let $u \in BV(a,b)$ and let (u_j) be a sequence in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and a.e. in (a,b). Suppose that there exists $\sigma \geq 0$ such that for every $x \in S_u$:*

$$|u^+(x) - u^-(x)| \leq \sigma.$$

Then, for every $j \in \mathbb{N}$, there exists $\tilde{u}_j \in W^{1,1}(a,b)$ such that

$$F_{\varepsilon_j}(\tilde{u}_j) \leq F_{\varepsilon_j}(u_j)$$

$\tilde{u}_j \to u$ in $L^1(a,b)$, and

$$\limsup_{j \to +\infty} ||\tilde{u}_j - u||_\infty \leq \sigma.$$

Proof. Let us denote by u a precise representative. It is not difficult to see that for every $\eta > 0$ there exists $\delta > 0$ such that whenever $x, y \in (a,b)$:

$$|x - y| < \delta \Rightarrow |u(x) - u(y)| < \sigma + \eta.$$

Indeed, suppose by contradiction that there exists $\eta_0 > 0$ such that for every $n \in \mathbb{N}$ we can find $x_n, y_n \in (a,b)$ satisfying $|x_n - y_n| < 1/n$ and

$$|u(x_n) - u(y_n)| \geq \sigma + \eta_0. \tag{2.3}$$

Up to a subsequence we can assume that x_n and y_n converge to a point $x_0 \in [a,b]$ and, moreover, that

$$x_n \to x_0^+ \quad \text{or} \quad x_n \to x_0^- \quad \text{and} \quad y_n \to x_0^+ \quad \text{or} \quad y_n \to x_0^-.$$

Then by taking the limit as $n \to +\infty$ in (2.3), we have a contradiction both if x_0 is a continuity or a jump point of u, and if x_0 is an end point of (a,b).

By means of this uniform control on the oscillation of u, we truncate now u_j. For any given $n \in \mathbb{N}$, let $\delta_n > 0$ be such that

$$|x-y| < \delta_n \Rightarrow |u(x)-u(y)| < \sigma + \frac{1}{n}$$

whenever $x, y \in (a,b)$. Consider a finite partition P_n of (a,b):

$$a = x_0 < x_1 < x_2 < \cdots < x_{k+1} = b$$

(we drop the dependence on n) with mesh size less than δ_n and such that each x_i (for every $i = 1, \ldots, k$) is a point of convergence for the sequence u_j. Let $j_n \in \mathbb{N}$ be such that if $j \geq j_n$

$$|u_j(x_i) - u(x_i)| \leq \frac{1}{n} \quad \text{for every } i = 1, \ldots, k.$$

We can choose j_n strictly increasing. Fix $j \in \mathbb{N}$ and let $j_n \leq j \leq j_{n+1}$. On the first and last interval of the partition P_n we define $\tilde{u}_j$ with constant value $u_j(x_1)$ and $u_j(x_k)$ respectively. Let now $[\xi, \eta]$ be any of the subintervals of $[x_i, x_{i+1}]$ for $i = 1, \ldots, k-1$. Without loss of generality we can assume $u_j(\xi) \leq u_j(\eta)$. Now define

$$\tilde{u}_j(x) = [u_j(x) \vee u_j(\xi)] \wedge u_j(\eta)$$

for every $x \in (\xi, \eta)$. Clearly, $\tilde{u}_j$ equals u_j at the endpoints ξ and η, hence $\tilde{u}_j \in W^{1,1}(a,b)$. Moreover, for every $x \in [\xi, \eta]$

$$u(\xi) - 1/n \leq u_j(\xi) \leq \tilde{u}_j(x) \leq \tilde{u}_j(\eta) \leq u(\eta) + 1/n.$$

Therefore:

$$|\tilde{u}_j(x) - u(x)| \leq \operatorname{osc}_{[\xi,\eta]} u + 2/n \leq \sigma + 3/n.$$

These inequalities hold in the first and last intervals of the decomposition, too. Hence we get the stated estimate about the upper limit of $||\tilde{u}_j - u||_\infty$. Since F_ε decreases by truncation, we have $F_{\varepsilon_j}(\tilde{u}_j) \leq F_{\varepsilon_j}(u_j)$. As to the L^1 convergence of the sequence $\tilde{u}_j$, consider, as above, the generic subinterval $[\xi, \eta]$ of the partition P_n; let $\tilde{u}'_j$ be the pointwise projection of u_j onto the interval $[u_j(\xi) - 1/n, u_j(\eta) + 1/n]$; since u takes values in this interval, we have:

$$|\tilde{u}_j - u| \leq |\tilde{u}_j - \tilde{u}'_j| + |\tilde{u}'_j - u| \leq 1/n + |u_j - u|, \quad \text{in } [\xi, \eta].$$

This implies the pointwise convergence of $\tilde{u}_j$ to u a.e. on the compact subsets of (a,b), hence on (a,b); the equiboundedness of the sequence gives the L^1 convergence. ■

REMARK 3.2.6 Given $u \in L^1(a,b)$, the set functions $A \mapsto F_\varepsilon(u,A)$ are increasing and superadditive. Consequently also the set function $A \mapsto F'(u,A)$ is increasing and superadditive, i.e.
(i) $F'(u,A_1) \le F'(u,A_2)$, whenever $A_1 \subseteq A_2 \subseteq (a,b)$;
(ii) $F'(u,A_1 \cup A_2) \ge F'(u,A_1) + F'(u,A_2)$, whenever $A_1 \cap A_2 = \emptyset$.

REMARK 3.2.7 Let $u \in BV(a,b)$ and λ be a positive Radon measure on (a,b); suppose

$$F'(u,I) \ge \int_I g(x)d\lambda$$

for every interval $I \subseteq (a,b)$, with g a non-negative Borel function. Then

$$F'(u,A) \ge \int_A g(x)d\lambda \tag{2.4}$$

for every A open subset of (a,b).

Proof. Let A be an open subset of (a,b); then A is a countable union of disjoint intervals I_h. From the monotonicity and superadditivity of F', for every $N \in \mathbb{N}$ we have

$$F'(u,A) \ge F'\left(u, \bigcup_{h=1}^N I_h\right) \ge \sum_{h=1}^N F'(u,I_h) \ge \int_{\bigcup_{h=1}^N I_h} g(x)d\lambda.$$

Passing to the limit as $N \to +\infty$ we obtain (2.4). ■

Proposition 3.2.8 *For every $u \in BV(a,b) \cap L^\infty(a,b)$*

$$F'(u) \ge \int_a^b \phi(|u'(x)|)dx, \qquad F'(u) \ge c_0|D^c u|(a,b).$$

Proof.

Step 1. We claim that if $u \in BV(a,b) \cap L^\infty(a,b)$ then: for every $\delta > 0$ and $l_h \le \phi$ affine function, with $l_h(t) = c_h t + d_h$ as in (1.4), the following inequality holds:

$$F'(u)\left(1 + \frac{6\sigma c_0}{f(t_\delta)}\right) \ge (1-\delta)\left(\int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)\right),$$

where t_δ is as in the assumption (1.3), and

$$\sigma = \sup_{x \in S_u} |u^+(x) - u^-(x)|.$$

Without loss of generality we can assume that there exists a sequence (u_j) in $W^{1,1}(a,b)$ such that $u_j \to u$ in $L^1(a,b)$ and a.e., and $F_{\varepsilon_j}(u_j) \to F'(u) < +\infty$. Let $\sigma = \sup_{x \in S_u} |u^+(x) - u^-(x)|$. Lemma 3.2.5 furnishes a sequence $\tilde{u}_j$ in $W^{1,1}(a,b)$, converging to u in $L^1(a,b)$, such that

$$F_{\varepsilon_j}(\tilde{u}_j) \le F_{\varepsilon_j}(u_j) \quad \text{and} \quad \limsup_{j \to +\infty} ||\tilde{u}_j - u||_\infty \le \sigma.$$

In particular $F_{\varepsilon_j}(\tilde{u}_j) \to F'(u)$. Let now $\eta > 0$ be fixed; we can suppose that $||\tilde{u}_j - u||_\infty \leq \sigma + \eta$ for every $j \in \mathbb{N}$. As shown in the first part of the proof of Lemma 3.2.5, there exists $\gamma > 0$ such that, if $J \subseteq (a, b)$, then

$$\text{diam} J < \gamma \Rightarrow \text{osc}_J u < \sigma + \eta;$$

therefore

$$\text{diam} J < \gamma \Rightarrow \text{osc}_J \tilde{u}_j < 3(\sigma + \eta). \tag{2.5}$$

For every $h \in \mathbb{N}$ let $l_h(t) = c_h t + d_h$ as in (1.4). Let $\delta > 0$ be fixed; apply Lemma 3.2.2 with $u = \tilde{u}_j$ and $l = l_h$. Thus, for j sufficiently large, we determine a uniform mesh $(x_\alpha)_{\alpha \in \mathbb{Z}}$ of $\mathbb{R}$ with size $2\varepsilon_j$ and a subfamily $\mathcal{P}'_j$ of

$$\mathcal{P}_j = \{(x_\alpha - \varepsilon_j, x_\alpha + \varepsilon_j) : \ x_\alpha \in (a + \varepsilon_j, b - \varepsilon_j)\}$$

with the following property

$$\sharp(\mathcal{P}_j \setminus \mathcal{P}'_j) \leq \frac{1}{2f(t_\delta)} F_{\varepsilon_j}(\tilde{u}_j), \qquad F_{\varepsilon_j}(\tilde{u}_j) \geq (1 - \delta) \int_{\bigcup \mathcal{P}'_j} l_h(|\tilde{u}'_j(x)|) dx. \tag{2.6}$$

Let $a_0 = \inf\left(\bigcup \mathcal{P}_j\right)$ and $b_0 = \sup\left(\bigcup \mathcal{P}_j\right)$; define (a.e.) v_j in (a_0, b_0) as follows

$$v_j(x) = \begin{cases} \tilde{u}_j(x) & x \in \bigcup \mathcal{P}'_j \\ \fint_J \tilde{u}_j(z) dz & x \in J \in \mathcal{P}_j \setminus \mathcal{P}'_j. \end{cases}$$

We consider v_j extended by continuity on (a, b) with values $v_j^+(a_0)$ and $v_j^-(b_0)$. Then $v_j \in SBV(a, b)$ and, by (2.5):

$$|v_j^+(x) - v_j^-(x)| \leq 3(\sigma + \eta)$$

for j sufficiently large (such that $2\varepsilon_j < \gamma$), for every $x \in S_{v_j}$. Moreover, by (2.6), $v_j \to u$ in $L^1(a, b)$ and

$$\sharp S_{v_j} \leq \frac{2}{f(t_\delta)} F_{\varepsilon_j}(\tilde{u}_j).$$

Therefore,

$$F_{\varepsilon_j}(\tilde{u}_j) \left(1 + \frac{6c_0}{f(t_\delta)}(\sigma + \eta)\right) \geq$$

$$\geq (1 - \delta) \left(\int_a^b l_h(|v'_j(x)|) dx + c_0 |D^s v_j|(a, b)\right) - (1 - \delta) d_h 2\varepsilon_j (\sharp(\mathcal{P}_j \setminus \mathcal{P}'_j) + 2) \geq$$

$$\geq (1 - \delta)(c_h |Dv_j|(a, b) + d_h(b - a)) - (1 - \delta) d_h 2\varepsilon_j (\sharp(\mathcal{P}_j \setminus \mathcal{P}'_j) + 2),$$

and, as $j \to +\infty$,

$$F'(u) \left(1 + \frac{6c_0}{f(t_\delta)}(\sigma + \eta)\right) \geq (1 - \delta)(c_h |Du|(a, b) + d_h(b - a)) \geq$$

$$\geq (1 - \delta) \left(\int_a^b l_h(|u'(x)|) dx + c_h |D^c u|(a, b)\right).$$

By the arbitrariness of η we conclude.

Step 2. Let $u \in BV(a,b) \cap L^\infty(a,b)$; let us fix $\sigma > 0$ and consider the finite set of points $\{x_1, \dots, x_{n-1}\} \subseteq S_u$ such that

$$|u^+(x_i) - u^-(x_i)| > \sigma$$

for $i = 1, \dots, n-1$. Let $a = x_0$ and $b = x_n$; then for every $i = 0, \dots, n-1$,

$$\sup_{x \in S_u \cap (x_i, x_{i+1})} |u^+(x) - u^-(x)| \leq \sigma.$$

By *Step 1* we have (recall Remark 3.1.7)

$$F'(u,(x_i,x_{i+1}))\left(1 + \frac{6c_0}{f(t_\delta)}\sigma\right) \geq (1-\delta)\left(\int_{x_i}^{x_{i+1}} l_h(|u'(x)|)dx + c_h|D^c u|(x_i,x_{i+1})\right)$$

for every $i = 0, \dots, n-1$. Then, by Remark 3.2.6

$$F'(u)\left(1 + \frac{6c_0}{f(t_\delta)}\sigma\right) \geq (1-\delta)\left(\int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)\right).$$

By arbitrariness of $\sigma > 0$ and $\delta > 0$, we obtain

$$F'(u) \geq \int_a^b l_h(|u'(x)|)dx + c_h|D^c u|(a,b)$$

and thus

$$F'(u) \geq \int_a^b l_h(|u'(x)|)dx \qquad \text{and} \quad F'(u) \geq c_h|D^c u|(a,b).$$

Step 3. Let $d_0 = \lim_{t\to+\infty} \phi(t) - c_0 t < 0$; without loss of generality, we can suppose $d_h > d_0$ and $c_h \geq 0$. Then, by Remark 3.2.7

$$F'(u,A) - d_0\lambda(A) \geq \int_A (l_h(|u'(x)|) - d_0)d\lambda$$

for every h and for every $A \in \mathcal{A}(\Omega)$. By Lemma 1.1.5,

$$F'(u,(a,b)) - d_0\lambda(a,b) \geq \int_a^b (\phi(|u'(x)|) - d_0)dx$$

and then

$$F'(u,(a,b)) \geq \int_a^b \phi(|u'(x)|)dx.$$

By taking the supremum on h in

$$F'(u) \geq c_h|D^c u|(a,b)$$

we have

$$F'(u) \geq c_0|D^c u|(a,b).$$

■

3.3 Estimate from below of the jump part

Lemma 3.3.1 *Let $u \in BV(a,b)$ and let (u_j) be a sequence in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$. Let $x \in S_u$; there exists $x_j^+, x_j^- \in (a,b)$ such that $x_j^\pm \to x^\pm$ and such that*

$$|u_j(x_j^+) - u^+(x)| \to 0, \qquad |u_j(x_j^-) - u^-(x)| \to 0.$$

Proof. By definition, for all $\sigma > 0$ there exists $\delta = \delta_\sigma > 0$ such that

$$\left| u^+(x) - \fint_x^{x+\delta} u(y)dy \right| < \frac{\sigma}{2};$$

clearly, we can assume that $\delta_\sigma < \sigma$. By the L^1-convergence of (u_j) there exists j_σ such that

$$\left| \fint_x^{x+\delta} u_j(y)dy - \fint_x^{x+\delta} u(y)dy \right| < \frac{\sigma}{2}$$

for all $j \geq j_\sigma$; by the mean value Theorem for integrals, for every $j \geq j_\sigma$ we can find $x_j^+ \in (x, x+\delta_\sigma)$ such that

$$|u_j(x_j^+) - u^+(x)| < \sigma.$$

For every $k \in \mathbb{N}$ let j_k be the integer j_σ corresponding to $\sigma = 1/k$. We can assume that j_k is strictly increasing. If we select the points x_j^+, defined above, with $j_k \leq j < j_{k+1}$, we get a sequence satisfying the required conditions. An analogous argument yields x_j^-. ■

Proposition 3.3.2 *For every $u \in BV(a,b)$*

$$F'(u) \geq 2 \sum_{x \in S_u} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right).$$

Proof.
Step 1. We claim that for any $\bar{x} \in S_u$

$$F'(u) \geq 2f\left(\frac{1}{2}|u^+(\bar{x}) - u^-(\bar{x})|\right).$$

Let (u_j) be a sequence in $W^{1,1}(a,b)$ converging to u in $L^1(a,b)$ and such that

$$F'(u) = \lim_{j \to +\infty} F_{\varepsilon_j}(u_j) < +\infty.$$

Fix $\bar{x} \in S_u$ and let $(x_j^\pm)$ be the sequences provided by Lemma 3.3.1 applied with $x = \bar{x}$. Assume $u^-(\bar{x}) < u^+(\bar{x})$; thus we suppose that $u_j(x_j^-) < u_j(x_j^+)$ for every j. Let $\tilde{u}_j$ be the continuous extension of u_j from (x_j^-, x_j^+) to (a,b) with the constant values $u_j(x_j^-)$ and $u_j(x_j^+)$. Clearly:

$$F_{\varepsilon_j}(u_j) \geq F_{\varepsilon_j}(\tilde{u}_j).$$

We can also assume that $\tilde{u}_j$ is non-decreasing, otherwise we replace $\tilde{u}_j$ by

$$\left(\tilde{u}_j(a) + \int_a^x (\tilde{u}_j'(t))^+ dt\right) \wedge \tilde{u}_j(b),$$

(this lower the value of F_{ε_j}). Apply now estimate (2.2) with $\varepsilon = \varepsilon_j$ and $u = \tilde{u}_j$; then

$$F_{\varepsilon_j}(\tilde{u}_j) \geq 2 \sum_{\alpha \in I_j} f_{\varepsilon_j}\left(\varepsilon_j \fint_{x_\alpha - \varepsilon_j}^{x_\alpha + \varepsilon_j} \tilde{u}_j'(t) dt\right),$$

where

$$I_j = \{\alpha \in J_{\varepsilon_j} : \ (x_\alpha - \varepsilon_j, x_\alpha + \varepsilon_j) \cap (x_j^-, x_j^+) \neq \emptyset\}.$$

The convergence $x_j^\pm \to \bar{x}$ yields that $(\sharp I_j)\varepsilon_j \to 0$ as $j \to +\infty$. For each $\alpha \in I_j$ let

$$\delta_j^\alpha = \varepsilon_j \fint_{x_\alpha - \varepsilon_j}^{x_\alpha + \varepsilon_j} \tilde{u}_j'(t) dt = \frac{1}{2}[u_j(x_\alpha + \varepsilon_j) - u_j(x_\alpha - \varepsilon_j)].$$

Given $\delta > 0$, we can find $t_\delta > 0$ and j_δ such that (see (1.3))

$$f_{\varepsilon_j}(t) \geq (1 - \delta)\varepsilon_j \phi(t/\varepsilon_j) \tag{3.1}$$

whenever $0 \leq t \leq t_\delta$ and $j \geq j_\delta$. Define

$$I_j' = \{\alpha \in I_j : \ \delta_j^\alpha \geq t_\delta\}, \qquad I_j'' = \{\alpha \in I_j : \ \delta_j^\alpha < t_\delta\}.$$

Then

$$F_{\varepsilon_j}(\tilde{u}_j) \geq 2\left(\sum_{\alpha \in I_j'} f_{\varepsilon_j}(\delta_j^\alpha) + \sum_{\alpha \in I_j''} f_{\varepsilon_j}(\delta_j^\alpha)\right).$$

From the subadditivity of f_{ε_j} in $(a_{\varepsilon_j}, +\infty)$ (see Assumption $(A1)$),

$$\sum_{\alpha \in I_j'} f_{\varepsilon_j}(\delta_j^\alpha) \geq f_{\varepsilon_j}\left(\sum_{\alpha \in I_j'} \delta_j^\alpha\right).$$

Let $\sharp I_j'' = N_j$; then, by the convexity of ϕ, and (3.1)

$$2 \sum_{\alpha \in I_j''} f_{\varepsilon_j}(\delta_j^\alpha) \geq 2(1 - \delta)\varepsilon_j N_j \sum_{\alpha \in I_j''} \frac{1}{N_j} \phi\left(\frac{\delta_j^\alpha}{\varepsilon_j}\right) \geq (1 - \delta)\phi\left(\frac{\sum_{\alpha \in I_j''} \delta_j^\alpha}{N_j \varepsilon_j}\right) 2N_j \varepsilon_j.$$

Since

$$2 \sum_{\alpha \in I_j} \delta_j^\alpha = u_j(x_j^+) - u_j(x_j^-) \to u^+(\bar{x}) - u^-(\bar{x}),$$

we can suppose that, up to a subsequence:

$$\sum_{\alpha \in I_j'} \delta_j^\alpha \to s_1 \quad \text{and} \quad \sum_{\alpha \in I_j''} \delta_j^\alpha \to s_2,$$

with $s_1 + s_2 = \frac{1}{2}|u^+(\bar{x}) - u^-(\bar{x})|$. Then, by the uniform convergence of f_{ε_j} to a concave function f, (see assumption (A3)) and taking into account the linear growth of ϕ (notice that $N_j\varepsilon_j \to 0$) we have:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(\tilde{u}_j) \geq 2(f(s_1) + (1-\delta)c_0 s_2) \geq 2[f(s_1) + (1-\delta)f(s_2)] \geq$$

$$\geq 2(1-\delta)f\left(\frac{|u^+(\bar{x}) - u^-(\bar{x})|}{2}\right),$$

where, in the last inequality, we have used the subadditivity of f. Finally, letting $\delta \to 0$:

$$F'(u) \geq \liminf_{j\to+\infty} F_{\varepsilon_j}(\tilde{u}_j) \geq 2f\left(\frac{|u^+(\bar{x}) - u^-(\bar{x})|}{2}\right).$$

Step 2. For any $N \in \mathbb{N}$ consider a finite subset $\{x_1, \ldots, x_N\}$ of S_u and let $I_1, \ldots, I_N$ be pairwise disjoint intervals such that $x_i \in I_i \subseteq (a,b)$ for every $i = 1, \ldots, N$. Apply the inequality of *Step 1* with (a,b) replaced by I_i and $\bar{x} = x_i$; then, by the subadditivity of $F'(u,\cdot)$ (Remark 3.2.6):

$$F'(u,(a,b)) \geq F'\left(u, \bigcup_{i=1}^N I_i\right) \geq \sum_{i=1}^N F'(u, I_i) \geq 2\sum_{i=1}^N f\left(\frac{1}{2}|u^+(x_i) - u^-(x_i)|\right).$$

Since N is arbitrary, we conclude. ■

3.4 Estimate from below of the lower Γ-limit

Theorem 3.4.1 *For every $u \in GBV(a,b)$ the inequality $F'(u) \geq \mathcal{F}(u)$ holds.*

Proof.

Step 1. Let us first consider the case $u \in BV(a,b) \cap L^\infty(a,b)$. Apply Propositions 3.2.8 and 3.3.2 with (a,b) replaced by an arbitrary open subinterval of (a,b): taking Remark 3.2.7 into account we get

$$F'(u,A) \geq \int_A \phi(|u'(x)|)dx, \qquad F'(u,A) \geq c_0|D^c u|(A)$$

and

$$F'(u,A) \geq 2\sum_{x\in S_u\cap A} f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right)$$

for every A open in (a,b). Let λ be the Borel measure defined by

$$\lambda(B) = \mathcal{L}^1(B) + \sharp(S_u \cap B) + c_0|D^c u|(B)$$

for every Borel subset B of (a,b). Let E be a Borel subset of $(a,b) \setminus S_u$ with $|E| = 0$ and on which $|D^c u|$ is concentrated, i.e. $|D^c u|((a,b) \setminus E) = 0$. Then

$$\mu(A) := F'(u,A) \geq \int_A \psi_i(x)d\lambda$$

for $i = 1, 2, 3$, and for every A open in (a, b), where

$$\psi_1(x) = \begin{cases} \phi(|u'(x)|) & x \in (a,b) \setminus (S_u \cup E) \\ 0 & x \in S_u \\ 0 & x \in E \end{cases}$$

$$\psi_2(x) = \begin{cases} 0 & x \in (a,b) \setminus (S_u \cup E) \\ 2f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) & x \in S_u \\ 0 & x \in E \end{cases}$$

$$\psi_3(x) = \begin{cases} 0 & x \in (a,b) \setminus (S_u \cup E) \\ 0 & x \in S_u \\ c_0 & x \in E\ . \end{cases}$$

Obviously

$$\psi(x) := \sup_i \psi_i(x) = \begin{cases} \phi(|u'(x)|) & x \in (a,b) \setminus (S_u \cup E) \\ 2f\left(\frac{1}{2}|u^+(x) - u^-(x)|\right) & x \in S_u \\ c_0 & x \in E \end{cases}$$

and then, from Lemma 1.1.5,

$$\mu(A) \geq \int_A \sup_i \psi_i(x) d\lambda = \int_A \psi(x) d\lambda = \mathcal{F}(u, A)$$

for every open subset A of (a, b).

Step 2. Let $u \in GBV(a, b)$ and let (u_j) be a sequence in $L^1(a, b)$ converging to u in $L^1(a, b)$ and such that

$$F'(u) = \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j).$$

Define

$$u_j^T = (u_j \wedge T) \vee (-T), \qquad u^T = (u \wedge T) \vee (-T).$$

Since $u_j^T \to u^T$ in $L^1(a, b)$, by *Step 1* we have

$$F'(u) = \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j) \geq \liminf_{j \to +\infty} F_{\varepsilon_j}(u_j^T) \geq$$

$$\geq \int_a^b \phi(|(u^T)'(x)| dx + 2 \sum_{x \in S_{u^T}} f\left(\frac{1}{2}|(u^T)^+(x) - (u^T)^-(x)|\right) + c_0 |D^c u^T|(a, b).$$

We conclude by taking the limit as $j \to +\infty$ and recalling that (see the definition of the space GBV in Section 1.2)

$$(u^T)' = \begin{cases} u' & \text{a.e. on} \quad \{|u| \leq T\} \\ 0 & \text{a.e. on} \quad \{|u| > T\} \end{cases}, \qquad |D^c(u^T)|(a, b) \to |D^c u|(a, b)$$

$$(u^T)^\pm(x) \to u^\pm(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_u.$$

■

3.5 Estimate from above of the upper Γ-limit

Lemma 3.5.1 *For every $u \in W^{1,1}(a,b)$ we have*

$$\lim_{j\to+\infty} F_{\varepsilon_j}(u) = \int_a^b \phi(|u'(x)|)dx.$$

Proof. Consider the functions

$$g_{\varepsilon_j}(x) = \fint_{(x-\varepsilon_j,x+\varepsilon_j)\cap(a,b)} |u'(y)|dy = \int_a^b \varphi_{\varepsilon_j}(x-y)|u'(y)|dy\,,$$

where

$$\varphi_{\varepsilon_j} = \frac{1}{2\varepsilon_j}\chi_{(-\varepsilon_j,\varepsilon_j)}.$$

Since $|u'| \in L^1(a,b)$, the sequence (g_{ε_j}) converges to $|u'|$ in $L^1(a,b)$ and a.e. in (a,b); by (1.2), and since f_{ε_j} is non-decreasing, it turns out that

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j} f_{\varepsilon_j}(\varepsilon_j g_{\varepsilon_j}(x)) = \phi(|u'(x)|)$$

for a.e. $x \in (a,b)$. Then, by Remark 3.1.2, we can pass to the limit under the integral; thus

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j}\int_a^b f_{\varepsilon_j}(\varepsilon_j g_{\varepsilon_j}(x))dx = \int_a^b \phi(|u'(x)|)dx.$$

■

Proposition 3.5.2 *For every $u \in SBV(a,b)$ the inequality $F''(u) \le \mathcal{F}(u)$ holds.*

Proof. Every $u \in SBV(a,b)$ is the L^1-limit of a sequence (u_h) such that

$$\sharp S_{u_h} < +\infty, \qquad \mathcal{F}(u_h) \to \mathcal{F}(u).$$

Indeed, if $S_u = \{x_i :\ i \in \mathbb{N}\}$, define, e.g.

$$u_h(x) = \int_a^x u'(t)dt + \sum_{x_i<x;\ i=1,\dots,h} (u^+(x_i) - u^-(x_i)).$$

Therefore, by the semicontinuity of F'', we can prove the stated inequality only in the case that $\sharp S_u$ is finite. We shall even suppose that S_u consists of a single point x_0, since the argument applied will be easily generalized to a finite number of jump points.

For any x let $I_j(x) = (x-\varepsilon_j, x+\varepsilon_j)$. By Proposition 3.1.6:

$$\overline{F}_{\varepsilon_j}(u) \le F_{\varepsilon_j}(u,(a,x_0)) + F_{\varepsilon_j}(u,(x_0,b)) + R_j, \tag{5.1}$$

where

$$R_j = \frac{1}{\varepsilon_j}\int_{I_j(x_0)} f_{\varepsilon_j}\Big(\frac{1}{2}|Du|(I_j(x))\Big)\,dx.$$

Notice that for $x \in I_j(x_0)$

$$|Du|(I_j(x)) \leq \int_{I_j(x)} |u'(t)|\,dt + |D^s u|(I_j(x)) \leq \int_{I_j(x)} |u'(t)|\,dt + |u^+(x_0) - u^-(x_0)|;$$

then for any $\sigma > 0$ there exists $j_\sigma \in \mathbb{N}$ such that for every $j \geq j_\sigma$ and $x \in I_j(x_0)$

$$|Du|\big(I_j(x)\big) \leq |u^+(x_0) - u^-(x_0)| + \sigma.$$

By $(A3)$ we immediately conclude that

$$\limsup_{j\to+\infty} R_j \leq 2f\Big(\frac{1}{2}|u^+(x_0) - u^-(x_0)|\Big).$$

Let us now pass to the limit in (5.1): taking Lemma 3.5.1 into account we get:

$$F''(u) \geq \limsup_{j\to+\infty} \overline{F}_{\varepsilon_j}(u) \leq \int_a^b \phi(|u'(x)|)\,dx + 2f\Big(\frac{1}{2}|u^+(x_0) - u^-(x_0)|\Big).$$

■

Theorem 3.5.3 *For every $u \in GBV(a,b)$ the inequality $F''(u) \leq \mathcal{F}(u)$ holds.*

Proof. By lower semicontinuity of F'' and by relaxation Theorem 1.3.4 we have $F''(u) \leq \mathcal{F}(u)$ for every $u \in BV(a,b)$. If $u \in GBV(a,b)$, it sufficies to pass to the limit as $T \to +\infty$ in $F''(u^T) \leq \mathcal{F}(u^T)$. ■

3.6 Relaxation and convergence of minima

Proof of Proposition 3.1.6

Denote by H_ε the functional on the right-hand side of (1.5). Let $I_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \cap (a,b)$ and $c_\varepsilon = \varepsilon/|I_\varepsilon(x)|$. It is easy to prove the L^1-l.s.c. of H_ε in BV. Indeed, if u_h is a sequence in BV converging to $u \in BV(a,b)$ in the L^1-topology, then, by Fatou's lemma and the lower semicontinuity of the total variation,

$$\liminf_{h\to+\infty} H_\varepsilon(u_h) = \liminf_{h\to+\infty} \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon |Du_h|(I_\varepsilon(x))\right)dx \geq$$

$$\geq \frac{1}{\varepsilon}\int_a^b \liminf_{h\to+\infty} f_\varepsilon\left(c_\varepsilon |Du_h|(I_\varepsilon(x))\right)dx = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon \liminf_{h\to+\infty} |Du_h|(I_\varepsilon(x))\right)dx \geq$$

$$\geq \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon |Du|(I_\varepsilon(x))\right)dx = H_\varepsilon(u).$$

Since $H_\varepsilon(u) \leq F_\varepsilon(u)$ for all $u \in BV(a,b)$, the relaxed functional $\bar{F}_\varepsilon$ is estimated from below by H_ε. Consider now the opposite inequality. Given $u \in BV(a,b)$, if (v_h) denotes the sequence

obtained from u (extended to a neighborhood of (a,b)) by standard mollification, then $v_h \to u$ in $L^1(a,b)$ and

$$|Dv_h|(I_\varepsilon(x)) \to |Du|(I_\varepsilon(x))$$

for a.e. $x \in (a,b)$ (see, e.g., [5] Proposition 3.7). Then by the dominated convergence theorem

$$\lim_{h\to+\infty} F_\varepsilon(v_h) = \frac{1}{\varepsilon}\int_a^b f_\varepsilon\left(c_\varepsilon|Du|(I_\varepsilon(x))\right)dx = H_\varepsilon(u).$$

This shows that $H_\varepsilon(u)$ is the relaxed functional of F_ε on $BV(a,b)$. ■

Proof of the corollary 3.1.5

By definition, $\{u_j\}$ is a sequence in $W^{1,1}(a,b)$ with

$$G_{\varepsilon_j}(u_j) \le \inf_{L^1(a,b)} G_{\varepsilon_j} + \varepsilon_j.$$

Since $g \in L^\infty(a,b)$ we can assume that (u_j) is equibounded. By Corollary 3.2.3 there exists $u_0 \in BV(a,b)$ such that $u_j \to u_0$ in $L^1(a,b)$. By Theorem 1.4.1, since G_{ε_j} Γ-converge to $\mathcal{G}$, u_0 is a minimum point of $\mathcal{G}$ on $L^1(a,b)$. ■

Chapter IV

The n-dimensional case

In this chapter we tackle the problem of the n-dimensional extension of the convergence result given for F_ε in the previous chapter. Unlike in the one-dimensional case, here we restrict the study to a fixed integrand function f, independent of ε.

4.1 Statement of the results

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $f : [0, +\infty) \to [0, +\infty)$ be a non-decreasing, strictly concave and C^2 function such that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 1\,.$$

We consider the functionals $F_\varepsilon : L^1(\Omega) \to \mathbb{R}$, with $\varepsilon > 0$, and $F\colon L^1(\Omega) \to \mathbb{R}$ defined by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)| dy\right) dx & u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)| dx + \int_{S_u} \theta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u \in GBV(\Omega) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

with

$$\theta(y) = 2 \int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n} y (\sqrt{1-t^2})^{n-1}\right) dt \qquad (y > 0), \tag{1.1}$$

where ω_n denotes the volume of the n-dimensional ball in $\mathbb{R}^n$ (with $\omega_0 = 1$).

The main result is the following theorem:

Theorem 4.1.1 *The family $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to $\mathcal{F}$ in $L^1(\Omega)$ as $\varepsilon \to 0$.*

As in the one-dimensional case, in the computation of the upper Γ limit it will be useful the following result (which will be proved in Section 4.5).

Proposition 4.1.2 *For every $\varepsilon > 0$, the relaxed functional of F_ε in the L^1-topology is given by*

$$\bar{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)dx \tag{1.2}$$

for every $u \in BV(\Omega)$.

Finally we conclude, as in the previous Chapter, with a corollary about the convergence of minima.

Corollary 4.1.3 *Let (ε_j) be a positive infinitesimal sequence and $g \in L^\infty(\Omega)$. For every $u \in L^1(\Omega)$, define:*

$$G_j(u) = F_{\varepsilon_j}(u) + \int_\Omega |u(x) - g(x)|dx,$$

and

$$G(u) = \mathcal{F}(u) + \int_\Omega |u(x) - g(x)|dx.$$

For every j let u_j be an ε_j-minimizer for G_j in $L^1(\Omega)$, i.e.

$$G_j(u_j) \le \inf_{L^1(\Omega)} G_j + \varepsilon_j;$$

then u_j converges, up to a subsequence, to a minimizer of G in $L^1(\Omega)$.

Remark 4.1.4 We shall need the following "localization" of the functional F_ε: for every open subset A of Ω, we set

$$F_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|dy\right)dx & u \in W^{1,1}(\Omega) \\ +\infty & u \in L^1(\Omega)\setminus W^{1,1}(\Omega). \end{cases}$$

Clearly, if $A \subset\subset \Omega$, the lower and upper Γ-limits of $(F_\varepsilon(\cdot, A))$ do not change by replacing Ω with any $\Omega' \supset\supset A$.

4.2 Compactness. Estimate from below of the volume and Cantor terms

The Lemma below is essentially Proposition 4.1 in [10] (the proof is the same, up to a minor modifications). This estimate of $F_\varepsilon(u)$ from below by the absolutely continuous part of the derivative is not enough to get a bound for the lower Γ-limit of F_ε (differently from what happens with the gradient squared which allows the application of the SBV compactness theorem). Though, we can deduce a compactness result for the family (F_ε). Denote by A_ϱ the following set

$$A_\varrho = \{x \in A : \ d(x, \partial A) > \varrho\}.$$

Lemma 4.2.1 *Let A an open subset of Ω, and let $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. For every $\varepsilon > 0$ and $\delta > 0$, there exists a function $v \in SBV(A) \cap L^\infty(A)$ such that:*

$$(1-\delta)\int_A |\nabla v(x)|dx \le F_\varepsilon(u, A),$$
$$\mathcal{H}^{n-1}(S_v \cap A_{6\varepsilon}) \le cF_\varepsilon(u, A),$$
$$||v||_{L^\infty(A)} \le ||u||_{L^\infty(A)},$$
$$||v-u||_{L^1(A_{6\varepsilon})} \le c\varepsilon F_\varepsilon(u, A)||u||_{L^\infty(A)},$$

where c is a constant depending only by n, δ and f.

Proposition 4.2.2 *Let $\{u_j\}$ be a sequence in $L^1(\Omega)$ such that*

$$||u_j||_\infty \le M, \quad F_{\varepsilon_j}(u_j) \le M$$

for some $M > 0$ independent on j; then there exists a subsequence (not relabelled) such that $u_j \to u_0$ for some $u_0 \in BV(\Omega)$.

Proof. Let $A \subset\subset \Omega$, with ∂A smooth. By Lemma 4.2.1 there exists a sequence (v_j) in $SBV(A)$ and a constant C independent of A such that

$$||v_j||_{BV(A)} \le C, \quad ||v_j||_{L^\infty(A)} \le M,$$

and

$$||v_j - u_j||_{L^1(A)} \to 0 \quad \text{as} \quad j \to +\infty.$$

Therefore there exists a subsequence (v_{j_k}) which converges to a function $u_0 \in BV(A)$, with

$$\|u_0\|_{BV(A)} \le C. \tag{2.1}$$

Clearly, also (u_{j_k}) converges to u_0 in $L^1(A)$. The arbitrariness of A and a diagonal argument allow to find a subsequence (u_{j_k}) which converges in $L^1_{\text{loc}}(\Omega)$ to a function $u_0 \in BV_{\text{loc}}(\Omega)$; actually, $u_0 \in BV(\Omega)$ by (2.1). Finally, the uniform bound of $\|u_j\|_{L^\infty(\Omega)}$ implies the $L^1(\Omega)$-convergence of (u_{j_k}) to u_0. ■

Corollary 4.2.3 *Let (u_j) be a converging sequence in $L^1(\Omega)$. If $F_{\varepsilon_j}(u_j)$ is bounded, then the limit of (u_j) belongs to $GBV(\Omega)$. In particular, if $F'(u) < +\infty$ then $u \in GBV(\Omega)$.*

Proof. For each $T > 0$ apply the previous Corollary to $u_j^T = (u_j \wedge T) \vee (-T)$: we get $(u \wedge T) \vee (-T) \in BV(\Omega)$; hence $u \in GBV(\Omega)$. ■

Given $y \in \mathbb{R}^{n-1}$ and $r > 0$, define

$$\tilde{Q}_r(y) = \{z \in \mathbb{R}^{n-1} : \ |z_i - y_i| < r, \ i = 1, \dots, n-1\}. \tag{2.2}$$

Lemma 4.2.4 *Let A be an open subset of $\mathbb{R}^{n-1}$, and $a, b \in \mathbb{R}$, with $a < b$. Let $u_j, u \in L^1\big((a,b)\times A\big)$ $(j \in \mathbb{N})$ and $u_j \to u$ in $L^1\big((a,b)\times A\big)$. For a.e. $x \in (a,b)$, for every $y \in A$ and $j \in \mathbb{N}$ with $\varepsilon_j < \frac{1}{\sqrt{n-1}} d(y, \partial A)$ we can define*

$$v_j^y(x) = \fint_{\tilde{Q}_{\varepsilon_j}(y)} u_j(x,s)\, ds.$$

Then there exists a subsequence of $(v_j^y)_j$ which converges to $u(\cdot, y)$ in $L^1(a,b)$ for a.e. $y \in A$.

Proof. There exists $N \subseteq (a,b)$, with $|N| = 0$ such that for every $x \in (a,b) \setminus N$

$$u_j(x,\cdot), u(x,\cdot) \in L^1(A).$$

In particular $v_j^y(x)$ is well-defined for every $x \in (a,b) \setminus N$ and $y \in A$, $\varepsilon_j < d(y, \partial A)/\sqrt{n-1}$. Let

$$\phi_j(s) = \frac{1}{|\tilde{Q}_{\varepsilon_j}(0)|} \chi_{\tilde{Q}_{\varepsilon_j}(0)}(s),$$

where $\chi_{\tilde{Q}_{\varepsilon_j}(0)}$ denotes the characteristic function of $\tilde{Q}_{\varepsilon_j}(0)$. We have

$$\int_A \left(\int_a^b |v_j^y(x) - u(x,y)| dx \right) dy = \int_A dy \int_a^b \left| \int_{\mathbb{R}^{n-1}} (u_j(x,s)\phi_j(s-y) - u(x,y)) ds \right| dx \leq$$

$$\leq \int_A dy \int_a^b dx \int_{\mathbb{R}^{n-1}} |u_j(x,s) - u(x,s)| \phi_j(s-y) ds +$$

$$+ \int_A dy \int_a^b \left| \int_{\mathbb{R}^{n-1}} u(x,s)\phi_j(s-y) ds - u(x,y) \right| dx.$$

Let I_j' and I_j'' be the two integrals terms on the right-hand side of the inequality above. By standard properties of L^p-functions

$$\lim_{|\xi| \to 0} ||u(\cdot - \xi) - u||_{L^1(A\times(a,b))} = 0 \tag{2.3}$$

if $u \in L^1(A \times (a,b))$; then

$$I_j'' \leq \int_a^b dx \int_A dy \int_{\mathbb{R}^{n-1}} |u(x,s) - u(x,y)| \phi_j(s-y) ds = \int_a^b dx \int_A dy \int_{\mathbb{R}^{n-1}} |u(x,y+z) - u(x,y)| \phi_j(z) dz =$$

$$= \int_{\mathbb{R}^{n-1}} \phi_j(z) ||u(\cdot, \cdot + z) - u||_{L^1(A\times(a,b))} dz \leq \sup_{|z| \leq c\varepsilon_j} ||u(\cdot, \cdot + z) - u||_{L^1(A\times(a,b))}$$

for some c positive constant. By (2.3) I_j'' tends to 0 as $j \to +\infty$. Let us now consider I_j';

$$I_j' = \int_A \left(\int_a^b (|u_j - u|(x,\cdot) * \phi_j)(y) dx \right) dy \leq$$

$$\leq \int_a^b ||(u_j - u)(x,\cdot)||_{L^1(A)} ||\phi_j||_{L^1(\mathbb{R}^{n-1})} dx = ||u_j - u||_{L^1((a,b)\times A)},$$

which tends to 0 as $j \to +\infty$. Thus we conclude that for any A open subset of $\mathbb{R}^{n-1}$

$$\lim_{j \to +\infty} \int_A \left(\int_a^b |v_j^y(x) - u(x,y)| dx \right) dy = 0.$$

In particular there exists a subsequence $(v^y_{j_k})_k$ which converges to $u(x,\cdot)$ in $L^1(a,b)$ for a.e. $y \in A$.
■

REMARK 4.2.5 The result of the previous lemma can be immediately generalized to the case where (a,b) is replaced by any open subset of $\mathbb{R}$.

Let $\xi \in S^{n-1}$ and let $\nu_1, \dots, \nu_n$ be an orthonormal basis of $\mathbb{R}^n$, with $\nu_1 = \xi$. For every $r > 0$ and $x \in \mathbb{R}^n$, define:

$$Q^\xi_r(x) = \{y \in \mathbb{R}^n : \ \langle y - x, \nu_i \rangle| < r, \quad i = 1, \dots, n\}.$$

Lemma 4.2.6 *There exists a sequence (c_h) of positive real numbers, with $\lim_{h\to+\infty} c_h = 1$, such that, for every $u \in W^{1,1}(\Omega)$, $\xi \in \mathbb{R}^n$ and A, A' open subsets of Ω, with $A' \subset\subset A$, and for every $\varepsilon < d(A', \partial A)/2$, the following inequality holds:*

$$F_\varepsilon(u, A) \geq \frac{1}{h\sigma^h_\varepsilon} \int_{A'} f^h \left(\sigma^h_\varepsilon \fint_{Q^\xi_{\sigma^h_\varepsilon}(z)} |\nabla u(y)| dy \right) dz,$$

where $f^h(t) = f(c_h h t)$ and $\sigma^h_\varepsilon = \varepsilon/h$.

Proof. For ease of notation we shall drop the superscript ξ in $Q^\xi_r(z)$ ($\xi \in S^{n-1}$ fixed). For every $h > 0$ let

$$Z_h = \{\alpha \in \mathbb{Z}^n : \ Q_{1/h}(2\alpha/h) \subseteq B_1(0)\}, \quad N_h = \sharp Z_h.$$

Clearly

$$c_h := \frac{2^n N_h}{\omega_n h^n} \to 1 \quad h \to +\infty,$$

and, for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$Z_h = \{\alpha \in \mathbb{Z}^n : \ Q^h_\varepsilon(x, \alpha) := x + Q_{\varepsilon/h}(2\alpha\varepsilon/h) \subseteq B_\varepsilon(x)\}.$$

Fix now $u \in W^{1,1}(\Omega)$; then for every $\varepsilon > 0$ and $x \in \Omega$, with $d(x, \partial\Omega) > \varepsilon$:

$$\fint_{B_\varepsilon(x)} |\nabla u(y)| dy \geq \sum_{\alpha \in Z_h} \frac{\left(\frac{2\varepsilon}{h}\right)^n}{\omega_n \varepsilon^n} \fint_{Q^h_\varepsilon(x,\alpha)} |\nabla u(y)| dy = \sum_{\alpha \in Z_h} \frac{c_h}{N_h} \fint_{Q^h_\varepsilon(x,\alpha)} |\nabla u(y)| dy.$$

By the concavity of f,

$$f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u(y)| dy \right) \geq \sum_{\alpha \in Z_h} \frac{1}{N_h} f\left(\varepsilon c_h \fint_{Q^h_\varepsilon(x,\alpha)} |\nabla u(y)| dy \right).$$

Let $A' \subset\subset A$, and $2\varepsilon < d(A', \partial A)$; we can find an open subset A'' of A such that

$$A' \subset\subset A'' \subset\subset A, \quad d(A', \partial A'') > \varepsilon, \quad d(A'', \partial A) > \varepsilon.$$

Thus, if we set $\sigma_\varepsilon^h = \varepsilon/h$, we get:

$$F_\varepsilon(u, A'') = \frac{1}{\varepsilon}\int_{A''} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u(y)|dy\right) dx \geq$$

$$\geq \sum_{\alpha\in Z_h} \frac{1}{N_h}\left(\frac{1}{h\sigma_\varepsilon^h}\int_{A''} f^h\left(\sigma_\varepsilon^h \fint_{Q_\varepsilon^h(x,\alpha)} |\nabla u(y)|dy\right) dx\right)$$

with f^h as in the statement of the lemma. The change of variables $z = x + 2\sigma_\varepsilon^h\alpha$ now yields:

$$F_\varepsilon(u, A) \geq \sum_{\alpha\in Z_h} \frac{1}{N_h}\left(\frac{1}{h\sigma_\varepsilon^h}\int_{A'} f^h\left(\sigma_\varepsilon^h \fint_{Q_{\sigma_\varepsilon^h}(z)} |\nabla u(y)|dy\right) dz\right).$$

Since the N_h terms of the sum do not depend on α, we conclude. ■

Remark 4.2.7 It is easy to see that Remark 3.2.6 holds unchanged in the n-dimensional setting; then, given $u \in L^1(\Omega)$, the set function $A \mapsto F'(u, A)$ is increasing and superadditive, i.e.
(i) $F'(u, A_1) \leq F'(u, A_2)$, whenever $A_1 \subseteq A_2 \subseteq (a, b)$;
(ii) $F'(u, A_1 \cup A_2) \geq F'(u, A_1) + F'(u, A_2)$, whenever $A_1 \cap A_2 = \emptyset$.

Proposition 4.2.8 *For every $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$F'(u, A) \geq \int_A |\nabla u(x)|dx + |D^c u|(A).$$

Proof. Let (ε_j) be a positive infinitesimal sequence and let (u_j) be a sequence in $W^{1,1}(\Omega)$ converging to $u \in L^1(\Omega)$ and such that $F_{\varepsilon_j}(u_j, A) \to F'(u, A)$ as $j \to +\infty$. Let $\xi \in S^{n-1}$. By Lemma 4.2.6, applied with $\varepsilon = \varepsilon_j$ and $u = u_j$, and by Fubini's Theorem, if A and A' are open subsets of Ω, with $A' \subset\subset A$, and $2\varepsilon_j < d(A', \partial A)$, we have (here $\sigma_j^h = \varepsilon_j/h$):

$$F_{\varepsilon_j}(u_j, A) \geq \frac{1}{h\sigma_j^h}\int_{A'} f^h\left(\sigma_j^h \fint_{Q^\xi_{\sigma_j^h}(z)} |\nabla u_j(x)|dx\right) dz =$$

$$= \int_{\prod_\xi}\left(\frac{1}{h\sigma_j^h}\int_{A'_{\xi,y}} f^h\left(\sigma_j^h \fint_{Q^\xi_{\sigma_j^h}(y+t\xi)} |\nabla u_j(x)|dx\right) dt\right) d\mathcal{H}^{n-1}(y)$$

where $\prod_\xi$ and $A'_{\xi,y}$ stand for the subspace orthogonal to ξ and for the one dimensional section of A' in the direction ξ, as in Theorem 1.2.3. It is not restrictive to assume $\xi = e_1$ (and we shall drop the superscript ξ); let $y = (t, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ denote the generic element of $\mathbb{R}^n$, and denote $A'_{e_1,y}$ as $A'_{\tilde{y}}$. Then, recalling the definition of $\tilde{Q}$

$$\fint_{Q_{\sigma_j^h}(y+te_1)} |\nabla u_j(x)|dx \geq \fint_{Q_{\sigma_j^h}(t,\tilde{y})} \left|\frac{\partial u_j}{\partial x_1}(x)\right| dx \geq \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left|\fint_{\tilde{Q}_{\sigma_j^h}(\tilde{y})} \frac{\partial u_j}{\partial s}(s, \tilde{x})d\tilde{x}\right| ds,$$

hence

$$\fint_{Q_{\sigma_j^h}(y+te_1)} |\nabla u_j(x)|dx \geq \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left| \frac{d}{ds} \fint_{\tilde{Q}_{\sigma_j^h}(\tilde{y})} u_j(s,\tilde{x})d\tilde{x} \right| ds.$$

For any fixed $A' \subset\subset A$, we can find an open set P with $A' \subset\subset P \subset\subset A$ and which is a finite union of sets of the form $E \times G$, with $E \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}^{n-1}$ open sets. Then, by Lemma 4.2.4 and the subsequent remark, the sequence $(v_j^{\tilde{y}})$ defined by

$$v_j^{\tilde{y}}(s) = \fint_{\tilde{Q}_{\sigma_j^h}(\tilde{y})} u_j(s,\tilde{x})d\tilde{x}, \quad s \in A'_{\tilde{y}}$$

converges (up to a subsequence, which does not affect the rest of the proof) to $u^{\tilde{y}} := u(\cdot,\tilde{y})$ in $L^1(A'_{\tilde{y}})$ for a.e. $\tilde{y}$ in the set $\tilde{A}' = \{\tilde{y} \in \mathbb{R}^{n-1} : \ A'_{\tilde{y}} \neq \emptyset\}$. By Fatou's Lemma:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq \int_{\prod_{e_1}} \frac{1}{h} \left(\liminf_{j\to+\infty} \frac{1}{\sigma_j^h} \int_{A'_{\tilde{y}}} f^h \left(\sigma_j^h \fint_{t-\sigma_j^h}^{t+\sigma_j^h} \left| \frac{dv_j^{\tilde{y}}}{ds}(s) \right| ds \right) dt \right) d\tilde{y};$$

thus, taking the one-dimensional Γ-convergence result into account:

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq \int_{\prod_{e_1}} \frac{1}{h} \left(\int_{A'_{\tilde{y}}} c_h h|(u^{\tilde{y}})'(s)|ds + c_h h|D^c u^{\tilde{y}}|(A'_{\tilde{y}}) \right) d\tilde{y}.$$

By the slicing Theorem 1.2.3 we deduce that

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) \geq c_h \left(\int_{A'} |\langle \nabla u(x), e_1\rangle|dx + |\langle D^c u, e_1\rangle|(A') \right).$$

As mentioned above, this result holds with any ξ in place of e_1; therefore, since $c_h \to 1$ as $h \to +\infty$ and $A' \subset\subset A$ is arbitrary, we get

$$F'(u, A) \geq \int_A |\langle \nabla u(x), \xi\rangle|dx + |\langle D^c u, \xi\rangle|(A)$$

for every $\xi \in S^{n-1}$. Superadditivity of F' and Lemma 1.1.5 conclude the proof. ■

4.3 Estimate from below of the surface term

In this section for any sequence (F_{ε_j}) and any given function $u \in BV$ we shall apply Besicovitch's differentiation Theorem, with respect to $\mathcal{H}^{n-1} \llcorner S_u$, to the lower Γ-limit considered as a set function (Proposition 4.3.1). Through a rescaling argument the density of this bound will be estimated in terms of the lower Γ-limit on the functions u_0 obtained by "blow-up" (Proposition 4.3.2 and subsequent Corollary). Let us notice that, when considering F' for u_0 on a unit ball B_1 (or on a cilinder C_1 of unit size, as in Proposition 4.3.5), we shall assume as Ω any set strictly containing B_1 (or C_1): see Remark 4.1.4. The passage to the evaluation of F' on a cilinder with axis normal to the jump set (Proposition 4.3.5) allows an explicit computation of the lower bound (see Proposition 4.3.6 together with 4.3.7).

Proposition 4.3.1 *Let (ε_j) be a positive infinitesimal sequence, $A \in \mathcal{A}(\Omega)$ and let $\Phi'(\cdot, A)$ be the lower Γ-limit of $\big(F_{\varepsilon_j}(\cdot, A)\big)$. Then $\Phi'(u, \cdot)\colon \mathcal{A}(\Omega) \to \mathbb{R}$, is the trace on $\mathcal{A}(\Omega)$ of a Borel measure on Ω; moreover, for every $u \in BV(\Omega)$*

$$\Phi'(u, A) \geq \int_{S_u \cap A} h(x) d\mathcal{H}^{n-1}$$

where, for $\mathcal{H}^{n-1}$ a.e. $x \in S_u$:

$$h(x) = \lim_{\varrho \to 0} \frac{\Phi'(u, B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}}.$$

Proof. The proof of the first part of the Lemma ($\Phi'(u, \cdot)$ trace of a Borel measure) can be obtained by Proposition 4.3 and Theorem 4.6 of [14] (these results are shown in the case $p > 1$, but the same proof works in the case $p = 1$).

Given $u \in BV(\Omega)$, for every $k \in \mathbb{N}$ let

$$J_k = \{x \in J_u : \ |u^+(x) - u^-(x)| > 1/k\}.$$

Clearly, $\mathcal{H}^{n-1}(J_k) < +\infty$; let

$$\nu_k = \mathcal{H}^{n-1} \llcorner J_k,$$

and denote by μ the Borel measure which extends $\Phi'(u, \cdot)$; we can assume $\Phi'(u, \Omega) < +\infty$. By Besicovitch's differentiation theorem, for ν_k-a.e. $x \in \Omega$ the limit

$$g(x) = \lim_{\varrho \to 0} \frac{\mu(B_\varrho(x))}{\nu_k(B_\varrho(x))}$$

exists and is finite; moreover, the Radon-Nikodym decomposition of μ is given by

$$\mu = g\nu_k + \mu^s, \qquad \mu^s \perp \nu^k.$$

Since J_k is $\mathcal{H}^{n-1}$-rectifiable, for $\mathcal{H}^{n-1}$-a.e. $x \in J_k$ we have

$$\frac{\nu_k(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = \frac{\mathcal{H}^{n-1}(B_\varrho(x) \cap J_k)}{\omega_{n-1}\varrho^{n-1}} \to 1, \qquad \varrho \to 0$$

(see, e.g. , [5] Theorem 2.63). Thus, for $\mathcal{H}^{n-1}$-e.e. $x \in J_k$

$$g(x) = \lim_{\varrho \to 0} \frac{\mu(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = \lim_{\varrho \to 0} \frac{\Phi'(u, B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = h(x).$$

Taking into account that μ^s is non-negative, we deduce that for every $A \in \mathcal{A}(\Omega)$

$$\Phi'(u, A) \geq \int_\Omega h(x) d\nu_k = \int_{J_k \cap A} h(x) d\mathcal{H}^{n-1}.$$

The conclusion follows considering the supremum for $k \in \mathbb{N}$. ■

Proposition 4.3.2 *Let Φ' be as in Proposition 4.3.1; then for every $u \in BV(\Omega)$*

$$\liminf_{\varrho\to 0} \frac{\Phi'(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq F'(u_0, B_1(x_0)), \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e.} \quad x_0 \in S_u,$$

where u_0 is the function given by

$$u_0(x) = \begin{cases} u^+(x_0) & \langle x - x_0, \nu\rangle \geq 0 \\ u^-(x_0) & \langle x - x_0, \nu\rangle < 0 \end{cases}$$

with $\nu = \nu_u(x_0)$.

Proof. We can assume $x_0 = 0$. Let ϱ_k be a decreasing infinitesimal sequence; for every $k \in \mathbb{N}$ there exists $w_j \in W^{1,1}(\Omega)$ such that $w_j \to u$ in $L^1(\Omega)$ and

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(w_j, B_{\varrho_k}(0)) \leq \Phi'(u, B_{\varrho_k}(0)) + \frac{\varrho_k^{n-1}}{k}.$$

Let $\bar{j} = j(k)$ be such that $\varepsilon_{\bar{j}}/\varrho_k \leq 1/k$ and

$$F_{\varepsilon_{\bar{j}}}(w_{\bar{j}}, B_{\varrho_k}(0)) \leq \Phi'(u, B_{\varrho_k}(0)) + \frac{\varrho_k^{n-1}}{k},$$

$$||w_{\bar{j}} - u||_{L^1(\Omega)} \leq \frac{1}{k},$$

$$\int_{B_2(0)} |w_{\bar{j}}(\varrho_k x) - u(\varrho_k x)|dx \leq \frac{1}{k}.$$

Let $u_k = w_{j(k)}$. We can suppose that the sequence $j(k)$ is increasing, and then we set $\sigma_k = \varepsilon_{j(k)}$; then $u_k \to u$ in $L^1(\Omega)$ and

$$\int_{B_2(0)} |u_k(\varrho_k x) - u(\varrho_k x)|dx \leq \frac{1}{k},$$

$$F_{\sigma_k}(u_k, B_{\varrho_k}(0)) \leq \Phi'(u, B_{\varrho_k}(0)) + \frac{\varrho_k^{n-1}}{k}.$$

Since

$$\int_{B_2(0)} |u_k(\varrho_k x) - u_0(\varrho_k x)|dx \leq \frac{1}{k} + \int_{B_2(0)} |u(\varrho_k x) - u_0(\varrho_k x)|dx \to 0$$

as $k \to +\infty$,

$$\liminf_{k\to+\infty} \frac{\Phi'(u, B_{\varrho_k}(0))}{\varrho_k^{n-1}} \geq \liminf_{k\to+\infty} \frac{F_{\sigma_k}(u_k, B_{\varrho_k}(0))}{\varrho_k^{n-1}}.$$

Let $v_k(t) = u_k(\varrho_k t)$; then

$$\fint_{B_{\sigma_k}(x)} |\nabla u_k(y)|dy = \frac{\varrho_k^{n-1}}{\omega_n \sigma_k^n} \int_{B_{\sigma_k/\varrho_k}(x/\varrho_k)} |\nabla v_k(\eta)|d\eta.$$

Finally, setting $x/\varrho_k = z$,

$$\frac{F_{\sigma_k}(u_k, B_{\varrho_k}(0))}{\varrho_k^{n-1}} = \frac{1}{\sigma_k/\varrho_k} \int_{B_1(0)} f\left(\frac{\sigma_k}{\varrho_k} \fint_{B_{\sigma_k/\varrho_k}(x)} |\nabla v_k(y)|dy\right) dx\,.$$

Since $\gamma_k := \sigma_k/\varrho_k \to 0$ as $k \to +\infty$, and $v_k \to u_0$ in $L^1(B_2(0))$, we conclude by the arbitrariness of ϱ_k and the definition of F'. ■

From the two propositions just proved the following result immediately follows:

Corollary 4.3.3 *Let $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$F'(u, A) \geq \int_{S_u \cap A} \sigma(x)\, d\mathcal{H}^{n-1},$$

where

$$\sigma(x_0) = \omega_{n-1}^{-1} F'\big(u_0, B_1(x_0)\big), \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S_u,$$

with ($\nu = \nu_u(x_0)$)

$$u_0(x) = \begin{cases} u^+(x_0) & \langle x - x_0, \nu\rangle \geq 0 \\ u^-(x_0) & \langle x - x_0, \nu\rangle < 0 \ . \end{cases}$$

Let $\nu \in S^{n-1}$. For any $y \in \mathbb{R}^n$ denote by y_ν and $y_{\nu^\perp}$ the projections onto the subspaces $V = \{t\nu : t \in \mathbb{R}\}$ and $V^\perp$, respectively. For $\varrho > 0$ and $x \in \mathbb{R}^n$ define

$$C_\varrho^\nu(0) = \{y \in \mathbb{R}^n : \ |y_\nu| < \varrho, \ |y_{\nu^\perp}| < \varrho\}, \qquad C_\varrho^\nu(x) = x + C_\varrho^\nu(0).$$

The next lemma proves that the "'transition set" between two constant values shrinks onto the interface.

Lemma 4.3.4 *Let $\nu \in S^{n-1}$ and*

$$u_0(x) = \begin{cases} a & \langle x, \nu\rangle \geq 0 \\ b & \langle x, \nu\rangle < 0 \ . \end{cases}$$

Let $\Omega' \supset\supset C_1^\nu(0)$. For any A open subset of $C_1^\nu(0)$, there exist a positive infinitesimal sequence (ε_j) and a sequence (u_j) in $W^{1,1}(\Omega')$ converging to u_0 in $L^1(\Omega')$ and such that

$$\lim_{j \to +\infty} F_{\varepsilon_j}(u_j, A) = F'(u_0, A)$$

$$u_j(x) = a \quad \langle x, \nu\rangle \geq a_j; \qquad u_j(x) = b \quad \langle x, \nu\rangle \leq -b_j,$$

where (a_j) and (b_j) are suitable positive infinitesimal sequences.

Proof. It is not restrictive to assume $\nu = e_1$. Fix A open subset of $C_1^\nu(0)$.
Step 1. Let $0 < \varepsilon < d(C_1^\nu(0), \partial\Omega')$, $\sigma > 0$ and $u_1, u_2, \varphi \in W^{1,1}(\Omega')$, with φ given by

$$\varphi(x) = \begin{cases} 0 & x \in A_\varepsilon \\ \text{affine} & x \in S_\varepsilon \\ 1 & x \in B_\varepsilon \end{cases}$$

where $A_\varepsilon = \{x \in \Omega' : \ x_1 \leq -2\varepsilon - \sigma\}$, $B_\varepsilon = \{x \in \Omega' : \ x_1 \geq -2\varepsilon\}$, and $S_\varepsilon = \{x \in \Omega' : \ -2\varepsilon - \sigma < x_1 < -2\varepsilon\}$. In particular, $|\nabla\varphi| \leq 1/\sigma$. Then, denoting by $v = \varphi u_1 + (1 - \varphi)u_2$, we have

$$\varepsilon F_\varepsilon(v, A) = \int_{A \cap \tilde{A}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u_2(y)| dy\right) dx + \int_{A \cap \tilde{B}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u_1(y)| dy\right) dx +$$

$$+ \int_{A \cap \tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla v(y)| dy\right) dx,$$

where $\tilde{A}_\varepsilon = \{x \in \Omega' : \ x_1 \le -3\varepsilon - \sigma\}$, $\tilde{B}_\varepsilon = \{x \in \Omega' : \ x_1 - \varepsilon\}$, and $\tilde{S}_\varepsilon = \{x \in \Omega' : \ -3\varepsilon - \sigma < x_1 < -\varepsilon\}$. By the subadditivity of f,

$$\int_{A\cap\tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla v(y)|dy\right) \le$$

$$\le \int_{A\cap\tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} (\varphi(y)|\nabla u_1(y)| + (1-\varphi(y))|\nabla u_2(y)| + |u_1(y)-u_2(y)||\nabla\varphi(y)|)dy\right) dx \le$$

$$\le \int_{A\cap\tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} \varphi(y)|\nabla u_1(y)|dy\right) dx + \int_{A\cap\tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} (1-\varphi(y))|\nabla u_2(y)|dy\right) dx +$$

$$+ \int_{A\cap\tilde{S}_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |u_1(y)-u_2(y)||\nabla\varphi(y)|dy\right) dx.$$

Then (since $f(t) \ge t$)

$$F_\varepsilon(v, A) \le F_\varepsilon(u_1, A \cap (\tilde{B}_\varepsilon \cup \tilde{S}_\varepsilon)) + F_\varepsilon(u_2, A \cap (\tilde{A}_\varepsilon \cup \tilde{S}_\varepsilon)) + I_\varepsilon \tag{3.1}$$

with

$$I_\varepsilon \le \frac{1}{\sigma} \int_{\tilde{S}_\varepsilon \cap C_1^\nu(0)} \fint_{B_\varepsilon(x)} |u_1(y) - u_2(y)|dydx.$$

A simple application of Fubini's theorem now gives

$$I_\varepsilon \le \frac{1}{\sigma} \int_{\Omega'_-} |u_1(y) - u_2(y)|\, dy,$$

where $\Omega'_- = \{x \in \Omega : \ x_1 \le 0\}$. *Step 2.* Let $\varepsilon_j \to 0$ and (u_j) in $W^{1,1}(\Omega')$ converging to u_0 in $L^1(\Omega')$ such that

$$F_{\varepsilon_j}(u_j, A) \to F'(u_0, A).$$

For every $j \in \mathbb{N}$ apply estimate (3.1) with $\varepsilon = \varepsilon_j$, $u_1 = u_j$ and $u_2 = b$. If $v_j = \varphi u_j + (1-\varphi)b$, since $F_{\varepsilon_j}(b, A) = 0$, we have:

$$F_{\varepsilon_j}(v_j, A) \le F_{\varepsilon_j}(u_j, A) + \frac{1}{\sigma} \int_{\Omega'_-} |u_j(y) - b|dy.$$

Notice that

$$\int_{\Omega'_-} |u_j(y) - b|dy = \int_{\Omega'_-} |u_j(y) - u_0(y)|dy \to 0$$

as $j \to +\infty$. Therefore, if (σ_h) is a positive infinitesimal sequence, we can find a strictly increasing sequence (j_h) in $\mathbb{N}$ such that

$$\frac{1}{\sigma_h} \int_{\Omega'_-} |u_{j_h} - b|dy \to 0 \qquad \text{as } h \to +\infty.$$

Then

$$\liminf_{h\to+\infty} F_{\varepsilon_{j_h}}(v_{j_h}, A) \le \liminf_{h\to+\infty} F_{\varepsilon_{j_h}}(u_{j_h}, A) = F'(u_0, A).$$

Moreover,

$$v_{j_h}(x) = \begin{cases} b & x_1 \le -2\varepsilon_{j_h} - \sigma_h \\ u_{j_h} & x_1 \ge 0\ . \end{cases}$$

Clearly $v_{j_h} \to u_0$ in $L^1(\Omega')$, and $v_{j_h}(x) = b$ when x_1 is negative and outside a left neighborhood of 0, shrinking to 0 as $h \to +\infty$. An analogous procedure allows to modify v_{j_h} so that, while mantaining the convergence of the functionals to $F'(u_0, A)$, it takes the value a at the points x with x_1 positive and outside a neighborhood of 0, shrinking to 0. ■

Proposition 4.3.5 *For $\mathcal{H}^{n-1}$ a.e. $x_0 \in S_u$*

$$F'(u_0, B_1(x_0)) = F'(u_0, C_1^\nu(x_0))$$

where $\nu = \nu_u(x_0)$ and u_0 is as in Corollary 4.3.3.

Proof. Clearly, the relevant inequality is $F'(u_0, B_1(x_0)) \ge F'(u_0, C_1^\nu(x_0))$. It is not restrictive to assume $x_0 = 0$ and $\nu = e_1$. Let $0 < \delta < 1$ and $\delta < \tau < 1$; let u_j be a sequence converging to u in L^1. Let $S_j = (-b_j, a_j) \times \mathbb{R}^{n-1}$, where a_j and b_j are given by previous Lemma. Then $S_j \cap C_1^\tau(0) \subset\subset B_1(0)$ and, always by previous Lemma, we can suppose $u_j = a$ if $x_1 \ge a_j$ and $u_j = b$ if $x_1 \le -b_j$. Now, we extend the definition of u_j at all $C_1^\tau(0)$, simply by setting $\tilde{u}_j = u_j$ in $B_1(0) \cap C_1^\tau(0)$, $\tilde{u}_j = a$ if $x_1 \ge a_j$ and $\tilde{u}_j = b$ if $x_1 \le -b_j$. Then

$$F_{\varepsilon_j}(u_j, B_1(0)) \ge F_{\varepsilon_j}(u_j, B_1(0) \cap C_\delta^\nu(0)) = F_{\varepsilon_j}(\tilde{u}_j, C_\delta^\nu(0))$$

By taking the infimum on all sequences we have

$$F'(u_0, B_1(0)) \ge F'(u_0, C_\delta^\nu(0))$$

and the conclusion follows by taking the supremum on $\delta \in (0, 1)$. ■

Proposition 4.3.6 *Let $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ be given by*

$$u_0(x) = \begin{cases} a & x_1 \ge 0 \\ b & x_1 < 0\,, \end{cases}$$

with $a, b \in \mathbb{R}$ and $a \ne b$; then

$$F'(u_0, C_1^{e_1}(0)) \ge \omega_{n-1} \inf_X G\,,$$

where

$$G(v) = \int_{\mathbb{R}} f\left(\frac{(n-1)\omega_{n-1}}{\omega_n} \int_0^1 [v(\xi + \sqrt{1-\eta^2}, \eta) - v(\xi - \sqrt{1-\eta^2}, \eta)]\eta^{n-2} d\eta\right) d\xi\,,$$

and X is the space of all functions $v \in W^{1,1}_{loc}(\mathbb{R} \times (0,1))$ non-decreasing in the first variable, such that there exists $\xi_0 < \xi_1$ (depending on v) with $v = b$ if $\xi \le \xi_0$, and $v = a$ if $\xi \ge \xi_1$.

Proof. Denote $C_1^{e_1}(0)$ by C; let (u_j) be a sequence in $W^{1,1}(C_2^{e_1}(0))$ converging to u_0 in $L^1(C_2^{e_1}(0))$ and such that

$$\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, C) = \lim_{j\to+\infty} F_{\varepsilon_j}(u_j, C) = F'(u_0, C).$$

We can suppose $a > 0$ and $b = 0$. Moreover, by Lemma 4.3.4 (with $A = C$), we can assume that $u_j(x) = 0$ if $x_1 \le -b_j$, and $u_j(x) = a$ if $x_1 \ge a_j$, for some positive $a_j, b_j \to 0$.

Let

$$\alpha_j(x_1, x_2, \dots, x_n) = \int_{-1}^{x_1} \left(\frac{\partial}{\partial t} u_j(t, x_2, \dots, x_n)\right)^+ dt\,,$$

and $v_j = \alpha_j \wedge a$. Then $v_j \in W^{1,1}(C_2^{e_1}(0))$, $v_j(x) = u_0(x)$ if $x_1 \notin (-b_j, a_j)$, and v_j is non-decreasing in the first variable. Moreover

$$|\nabla u_j| \ge \left|\frac{\partial u_j}{\partial x_1}\right| \ge \frac{\partial v_j}{\partial x_1} \ge 0.$$

Then

$$F_{\varepsilon_j}(u_j, C) \ge \frac{1}{\varepsilon_j} \int_C f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x)} \frac{\partial v_j}{\partial s_1}(s)ds\right) dx.$$

Let B_ϱ^{n-1} be the $(n-1)$-dimensional ball of center 0 and radius ϱ. Since $C = (-1,1) \times B_1^{n-1}$, we have

$$F_{\varepsilon_j}(u_j, C) \ge \frac{1}{\varepsilon_j} \int_{B_1^{n-1}} dx_2 \cdots dx_n \int_{-1}^{1} f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x)} \frac{\partial v_j}{\partial s_1}(s)ds\right) dx_1.$$

Let $(\bar{x}_2, \cdots, \bar{x}_n) \in B_1^{n-1}$ be such that

$$\int_{-1}^{1} f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x_1, \bar{x}_2, \cdots, \bar{x}_n)} \frac{\partial v_j}{\partial s_1}(s)ds\right) dx_1 =$$

$$= \min_{(x_2, \cdots, x_n) \in B_1^{n-1}} \int_{-1}^{1} f\left(\varepsilon_j \fint_{B_{\varepsilon_j}(x_1, \cdots, x_n)} \frac{\partial v_j}{\partial s_1}(s)ds\right) dx_1\,.$$

We can assume, up to a translation, that for all i, $\bar{x}_i = 0$. Then

$$F_{\varepsilon_j}(u_j, C) \ge \frac{\omega_{n-1}}{\varepsilon_j} \int_{-1}^{1} f\left(\frac{1}{\omega_n \varepsilon_j^{n-1}} \int_{B_{\varepsilon_j}(x_1, 0)} \frac{\partial v_j}{\partial s_1}(s)ds\right) dx_1\,.$$

By Fubini's Theorem,

$$\int_{B_{\varepsilon_j}(x_1,0)} \frac{\partial v_j}{\partial s_1}(s)ds = \int_{B_{\varepsilon_j}^{n-1}} \left[v_j\big(x_1 + \sqrt{\varepsilon_j^2 - |y|^2}, y\big) - v_j\big(x_1 - \sqrt{\varepsilon_j^2 - |y|^2}, y\big)\right] dy\,.$$

Define

$$\hat{v}_j(t, \varrho) = \fint_{\partial B_\varrho^{n-1}} v_j(t, y) d\mathcal{H}^{n-2}(y), \quad \varrho \in (0, \varepsilon_j)\,.$$

Then

$$\int_{B_{\varepsilon_j}(x_1,0)} \frac{\partial v_j}{\partial s_1}(s)ds = \int_0^{\varepsilon_j} d\varrho \int_{\partial B_\varrho^{n-1}} \left[v_j\big(x_1 + \sqrt{\varepsilon_j^2 - |y|^2}, y\big) - v_j\big(x_1 - \sqrt{\varepsilon_j^2 - |y|^2}, y\big)\right] d\mathcal{H}^{n-2}(y)$$

$$= \int_0^{\varepsilon_j} \mathcal{H}^{n-2}(\partial B_\varrho^{n-1})[\hat{v}_j(x_1 + \sqrt{\varepsilon_j^2 - \varrho^2}, \varrho) - \hat{v}_j(x_1 - \sqrt{\varepsilon_j^2 - \varrho^2}, \varrho)]d\varrho\,.$$

By the change of variables $x_1/\varepsilon_j = \xi$ and $\varrho = \varepsilon_j \eta$, we obtain, for j sufficiently large:

$$F_{\varepsilon_j}(u_j, C) \geq \omega_{n-1} \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\frac{(n-1)\omega_{n-1}}{\omega_n} \int_0^1 [\hat{v}_j(\varepsilon_j \xi + \varepsilon_j \sqrt{1-\eta^2}, \varepsilon_j \eta) + \right.$$

$$\left. -\hat{v}_j(\varepsilon_j \xi - \varepsilon_j \sqrt{1-\eta^2}, \varepsilon_j \eta)] \eta^{n-2} d\eta \right) d\xi \,.$$

Let $w_j(x) = \hat{v}_j(\varepsilon_j x)$. Clearly w_j is non-decreasing in the first variable, and there exist $\xi_0 < \xi_1$ such that $w_j(x) = a$ if $x \geq \xi_1$ and $w_j(x) = 0$ if $x \leq \xi_0$. Then w_j can be extended to all $\mathbb{R} \times (0,1)$ (with values 0 and a) and thus $w_j \in X$. Hence $F'(u, C) \geq \omega_{n-1} G(w_j)$. ■

Proposition 4.3.7 *With the notation of Proposition 4.3.6 we have*

$$\inf_X G \geq \vartheta(|a-b|) \,.$$

Proof. We can suppose $a > 0$ and $b = 0$. Recall that

$$G(v) = \int_{\mathbb{R}} f\left(a_n \int_0^1 \Big[v(\xi + \sqrt{1-\eta^2}, \eta) - v(\xi - \sqrt{1-\eta^2}, \eta) \Big] \eta^{n-2} \, d\eta \right) d\xi$$

where $a_n = (n-1)\omega_{n-1}/\omega_n$. For each $k \in \mathbb{N}$ we now consider a discrete version of G defined on the space of the functions on $S = \mathbb{R} \times [0,1]$ which are constant on each of the squares determined by a coordinate partition of S and with sides of length $1/k$. We also require the monotonicity in the first variable and the constant value 0 and a on the left and right of $[\xi_0, \xi_1] \times [0,1]$, respectively, for some $\xi_0 < \xi_1$.

Clearly, we can deal only with the values on the nodes; thus, for any $N \in \mathbb{N}$, define Y_k^N as the set of functions

$$v = (v^{i,j})_{i,j} \colon \mathbb{Z} \times \{1, \ldots, k-1\} \to [0, a],$$

such that:

a) for every j the function $i \mapsto v^{i,j}$ is increasing;

b) $v^{i,j} = 0$ if $i < -Nk$ and $v^{i,j} = a$ if $i \geq Nk$.

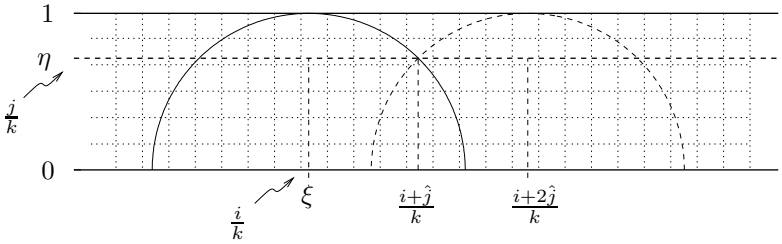
Let $Y_k = \bigcup_{N \in \mathbb{N}} Y_k^N$, and let $G_k \colon Y_k \to \mathbb{R}$ be defined by:

$$G_k(v) = \sum_{i=-(N+1)k}^{(N+1)k} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k} [v]_{i,j} \right), \qquad \text{on } Y_k^N,$$

where

$$[v]_{i,j} = \big(v^{i+\hat{j},j} - v^{i-\hat{j},j} \big)(j/k)^{n-2},$$

with $\hat{j}$ denoting the integer part of $\sqrt{k^2 - j^2}$ (see a sketch in the next figure).



Step 1. Each minimizer of G_k on Y_k^N takes only the values 0 and a.

Let v be a minimizer of G_k on Y_k^N. Suppose, by contradiction, that there exists i_0, j_0 with $v^{i_0,j_0} = c \in (0,a)$. We can assume that for a suitable $s \in \mathbb{N}$:

$$v^{i_0-1,j_0} < c\,, \qquad c = v^{i_0,j_0} = v^{i_0+1,j_0} = \cdots = v^{i_0+s,j_0}\,, \qquad v^{i_0+s+1,j_0} > c\,.$$

Then, given $t \in \mathbb{R}$, we define $v_t = (v_t^{i,j})_{i,j}$ as $v_t^{i_0+l,j_0} = c + t$, if $0 \le l \le s$, and $v_t = v$ otherwise. For $|t|$ sufficiently small, $v_t \in Y_k^N$. Let

$$I_1 = \{i \in \mathbb{Z} :\ i + \hat{j}_0 \in [i_0, i_0+s]\}\,, \quad I_2 = \{i \in \mathbb{Z} :\ i - \hat{j}_0 \in [i_0, i_0+s]\}\,.$$

Notice that if $i \notin I_1 \Delta I_2$ then $[v_t]_{i,j} = [v]_{i,j}$. Therefore:

$$G_k(v_t) = \sum_{i \in I_1 \setminus I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v_t]_{i,j}\right) + \sum_{i \in I_2 \setminus I_1} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v_t]_{i,j}\right) +$$

$$+ \sum_{i \notin I_1 \Delta I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right) =$$

$$= \sum_{i \in I_1 \setminus I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j} + \frac{a_n}{k}\left(\frac{j_0}{k}\right)^{n-2} t\right) + \sum_{i \in I_2 \setminus I_1} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j} - \frac{a_n}{k}\left(\frac{j_0}{k}\right)^{n-2} t\right)$$

$$+ \sum_{i \notin I_1 \Delta I_2} \frac{1}{k} f\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i,j}\right).$$

The function $t \mapsto G_k(v_t)$ is twice continuously differentiable in $t = 0$ (due to the smoothness of f), and:

$$\frac{d^2}{dt^2} G_k(v_t)\big|_{t=0} =$$

$$= \frac{a_n^2}{k^3}\left(\frac{j_0}{k}\right)^{2(n-2)} \left[\sum_{i \in I_1 \setminus I_2} f''\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i_1,j}\right) + \sum_{i \in I_2 \setminus I_1} f''\left(a_n \sum_{j=1}^{k-1} \frac{1}{k}[v]_{i_2,j}\right)\right] < 0\,.$$

by strict concavity of f; this is a contradiction, since v is a minimizer for G_k. *Step 2.* We claim that if $v \in Y_k^N$ takes only the values 0 and a, then

$$G_k(v) \ge G_k(\overline{v}),$$

where

$$\overline{v}^{i,j} = \begin{cases} 0 & i < Nk, \\ a & i \geq Nk \end{cases}$$

Indeed, assume that

$$E_v := \{i \in \mathbb{Z} : \ \exists j \ \ v^{i+\hat{j},j} = a, \ i + \hat{j} < Nk\} \neq \emptyset$$

(otherwise $v = \overline{v}$). Let $i_0 = \min E_v$ and

$$j_M = \max\{j : \ \ v^{i_0+\hat{j},j} = a, \ i_0 + \hat{j} < Nk\},$$
$$J_M = \{j : \ \hat{j} = \hat{j}_M, \ v^{i_0+\hat{j},j} = a\}, \qquad j_0 = \min J_M.$$

Then $v^{i_0+\hat{j}_0,j_0} = a$ and

$$j < j_0 \quad \Rightarrow \quad v^{i_0+\hat{j}_0,j} = 0. \tag{3.2}$$

Indeed, if we had $j < j_0$ with the property $v^{i_0+\hat{j}_0,j} = a$, then we would have $\hat{j} > \hat{j}_0 = \hat{j}_M$, and

$$Nk > i_0 + \hat{j}_0 = i_0 - l + \hat{j}, \quad \text{with } l = \hat{j} - \hat{j}_0 > 0;$$

therefore

$$v^{i_0-l+\hat{j},j} = a,$$

and $i_0 - l \in E$, which contrasts with the definition of i_0. Denote by w the function obtained by modifying v in $(i_0 + \hat{j}, j)$ for $j \in J_M$:

$$w^{i_0+\hat{j},j} = 0 \quad \text{for } j \in J_M, \qquad w^{i,j} = v^{i,j} \quad \text{otherwise.}$$

We want to show that

$$G_k(v) \geq G_k(w). \tag{3.3}$$

Notice that in the sum over i defining $G_k(v)$ the terms $v^{i_0+\hat{j},j}$ $(j \in J_M)$ appear only if $i = i_0$ or $i = i_0 + 2\hat{j}_0$. Accordingly, let us write $kG_k(v)$ as:

$$kG_k(v) = f(q+\delta) + f(p) + \sum_{i \notin \{i_0, i_0+2\hat{j}_0\}} f\left(\sum_j \frac{a_n}{k}[v]_{i,j}\right), \tag{3.4}$$

where:

$$q = \sum_{j \notin J_M} \frac{a_n}{k}[v]_{i_0,j}, \qquad \delta = \sum_{j \in J_M} \frac{a_n a}{k}\left(\frac{j}{k}\right)^{n-2}, \qquad p = \sum_{j=0}^{k-1} \frac{a_n}{k}[v]_{i_0+2\hat{j}_0,j}.$$

An analogous splitting can be written for $kG_k(w)$: clearly, the last term is the same as in (3.4). Thus:

$$kG_k(v) - kG_k(w) = f(q+\delta) - f(q) - [f(p+\delta) - f(p)]. \tag{3.5}$$

By the definition of j_0 it turns out that:

$$q = \sum_{j<j_0} \frac{a_n}{k}[v]_{i_0,j}.$$

Moreover, if $j < j_0$ then $\hat{j} \geq \hat{j}_0$; the monotonicity in the first variable and (3.2) yield:

$$v^{i_0+2\hat{j}_0-\hat{j},j} \leq v^{i_0+\hat{j}_0,j} = 0.$$

Hence:

$$j < j_0 \quad \Rightarrow \quad v^{i_0+2\hat{j}_0-\hat{j},j} = 0,$$

so that

$$j < j_0 \quad \Rightarrow \quad [v]_{i_0+2\hat{j}_0,j} = v^{i_0+2\hat{j}_0+\hat{j},j} - v^{i_0+2\hat{j}_0-\hat{j},j} \geq v^{i_0+\hat{j},j} = [v]_{i_0,j}.$$

Therefore

$$p \geq \sum_{j<j_0} \frac{a_n}{k}[v]_{i_0+2\hat{j}_0,j} \geq q.$$

Notice now that the strict concavity of f implies:

$$q \leq p \quad \Rightarrow \quad f(q+\delta) - f(q) \geq f(p+\delta) - f(p).$$

This, together with (3.5), proves (3.3). If E_w is not empty, it has a minimum strictly greater than $i_0 = \min E_v$. A finite iteration of the above argument proves the claim.

Step 3. We have shown that for every $N > 0$, $\inf_{Y_k^N} G_k = G_k(\bar{v})$, where $\bar{v}$ does not depend on N. Then $\inf_{Y_k} G_k = \inf_{N>0} \inf_{Y_k^N} G_k = G_k(\bar{v})$. Let's compute

$$\lim_{k\to+\infty} G_k(\bar{v})$$

where, up to a translation, we can suppose $\bar{v}$ given by

$$\bar{v}^{i,j} = \begin{cases} 0 & i < 0 \\ a & i \geq 0. \end{cases}$$

We have

$$G_k(\bar{v}) = 2\sum_{i=-k}^{0} \frac{1}{k} f\left(a_n \sum_{j=0, i+\hat{j}>0}^{k-1} \frac{a}{k}\left(\frac{j}{k}\right)^{n-2}\right) = 2\sum_{i=-k}^{0} \frac{1}{k} f\left(\frac{a_n a}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2}\right)$$

where $\hat{i}$ denotes the integer part of $\sqrt{k^2 - i^2}$. Obviously

$$\frac{1}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2} \leq \int_0^{\frac{\hat{i}}{k}} t^{n-2}dt.$$

Moreover

$$\int_0^{\frac{\hat{i}}{k}} t^{n-2}dt \leq \frac{1}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j+1}{k}\right)^{n-2} = \frac{1}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2} + b_k$$

where

$$b_k = \frac{1}{k}\left(\frac{\hat{i}+1}{k}\right)^{n-2} \to 0, \quad k \to +\infty.$$

Then

$$\int_0^{\frac{\hat{i}}{k}} t^{n-2}dt - b_k \le \frac{1}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2} \le \int_0^{\frac{\hat{i}}{k}} t^{n-2}dt.$$

By the monotonicity of f,

$$\sum_{i=-k}^{0}\frac{1}{k}f\left(\int_0^{\frac{\hat{i}}{k}} t^{n-2}dt - b_k\right) \le \sum_{i=-k}^{0}\frac{1}{k}f\left(\frac{1}{k}\sum_{j=0}^{\hat{i}}\left(\frac{j}{k}\right)^{n-2}\right) \le$$

$$\le \sum_{i=-k}^{0}\frac{1}{k}f\left(\int_0^{\frac{\hat{i}}{k}} t^{n-2}dt\right).$$

Thus, by the definition of the Riemann integral as the limit of the Riemann sums, we have, by taking $k \to +\infty$,

$$\inf_{Y_k} G_k \to 2\int_0^1 f\left(\frac{\omega_{n-1}}{\omega_n}a(\sqrt{1-t^2})^{n-1}\right)dt$$

Step 4. Let us show that a function $v \in X$ can be approximated by a sequence of continuous functions belongs to X.

Indeed, fix $\tau > 0$ and let ϱ_ε be a convolution kernel. Consider the function $v_\tau \colon \mathbb{R}^n \to [0,a]$ given by

$$v_\tau(x) = \begin{cases} a & x_1 \ge \xi_1, -\tau \le x_2 \le 1+\tau \\ v & x \in S \\ 0 & \text{otherwise} \end{cases}$$

where ξ_1 is such that $v = a$ for every $x_1 \ge \xi_1$. Then $v_\tau * \varrho_{\varepsilon|S} \in X$ is a continuous function, and $v_\tau * \varrho_{\varepsilon|S} \to v$ in $L^1(S)$ and a.e. $x \in S$, as $\varepsilon \to 0$. *Step 5.* We conclude the proof. Let $\sigma > 0$; then there exists $v_\sigma \in X$ such that

$$\inf_X G \ge G(v_\sigma) - \sigma. \tag{3.6}$$

By *Step 4* we can suppose v_σ continuous, hence uniformly continuous. Hence there exists $\delta > 0$ such that

$$|(x,y) - (x',y')|_1 < \delta \Rightarrow |v_\sigma(x,y) - v_\sigma(x',y')| < \sigma\,.$$

Fix $k \in \mathbb{N}$, and consider the function $w^{i,j} \in Y_k$ given by

$$w^{i,j}(Q_k^{i,j}) = \fint_{Q_k^{i,j}} v_\sigma dx$$

where $Q_k^{i,j} = [i/k,(i+1)/k) \times [j/k,(j+1)/k)$. Let $\xi \in (i/k,(i+1)/k)$ and $\eta \in (j/k,(j+1)/k)$; then

$$\frac{i+(j+1)^\wedge}{k} \le \xi + \sqrt{1-\eta^2} \le \frac{i+\hat{j}+1}{k}$$

Thus, for k sufficiently large (say $k \geq \bar{k}$)and for every $(x', y') \in Q_k^{i+\hat{j},j}$, $|(\xi + \sqrt{1-\eta^2}, \eta) - (x', y')| < \delta$; analougly, one can be obtain $|(\xi - \sqrt{1-\eta^2}, \eta) - (x', y')| < \delta$. Then for every $k \geq \bar{k}$ and for every $\xi \in (i/k, (i+1)/k)$ and $\eta \in (j/k, (j+1)/k)$,

$$(v_\sigma(\xi + \sqrt{1-\eta^2}, \eta) - v_\sigma(\xi - \sqrt{1-\eta^2}, \eta))\eta^{n-2} \geq [w]_{i,j} - \sigma\,. \tag{3.7}$$

This implies

$$\int_0^1 v_\sigma(\xi + \sqrt{1-\eta^2}, \eta) - v_\sigma(\xi - \sqrt{1-\eta^2}, \eta))\eta^{n-2} d\eta \geq \frac{1}{k}\sum_{j=0}^{k-1}[w]_{i,j} - \sigma.$$

By the mean value Theorem for integrals, we have

$$G(v_\sigma) = \sum_{i=-k}^{k} \frac{1}{k} f\left(\int_0^1 v_\sigma(\xi(i) + \sqrt{1-\eta^2}, \eta) - v_\sigma(\xi(i) - \sqrt{1-\eta^2}, \eta)\eta^{n-2} d\eta\right)$$

where $\xi(i) \in (i/k, (i+1)/k)$. By (3.6) and (3.7), and by the arbitrariness of σ, we obtain

$$\inf_X G \geq \lim_{k\to+\infty} \inf_{Y_k} G_k$$

and this, by *Step 3*, concludes the proof. ■

4.4 Estimate from below of the lower Γ-limit

Theorem 4.4.1 *For every $u \in GBV(\Omega)$*

$$F'(u) \geq \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

Proof. Let $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$. From Proposition 4.2.8 we have

$$F'(u, A) \geq \int_A |\nabla u(x)| dx + |D^c u|(A);$$

moreover, Corollary 4.3.3 and Propositions 4.3.5, 4.3.6 and 4.3.7 give

$$F'(u, A) \geq \int_{S_u \cap A} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1}\,.$$

Let $\lambda = \mathcal{L}^n + \mathcal{H}^{n-1} \llcorner S_u + |D^c u|$. Let E be a Borel subset of Ω with $|E| = 0$ and such that $|D^c u|$ is concentrated on E, i.e. $|D^c u|(\Omega \setminus E) = 0$. Then

$$\mu(A) := F'(u, A) \geq \int_A \psi_i(x) d\lambda$$

for $i = 1, 2$, where

$$\psi_1(x) = \begin{cases} |\nabla u(x)| & x \in \Omega \setminus (S_u \cup E) \\ 0 & x \in S_u \\ 1 & x \in E \end{cases}$$

$$\psi_2(x) = \begin{cases} 0 & x \in \Omega \setminus (S_u \cup E) \\ \vartheta(|u^+(x) - u^-(x)|) & x \in S_u \\ 0 & x \in E. \end{cases}$$

Obviously

$$\psi(x) := \sup_i \psi_i(x) = \begin{cases} |\nabla u(x)| & x \in \Omega \setminus (S_u \cup E) \\ \vartheta(|u^+(x) - u^-(x)|) & x \in S_u \\ 1 & x \in E \end{cases}$$

and then, from Lemma 1.1.5,

$$\mu(A) \geq \int_A \sup_i \psi_i(x) d\lambda = \int_A \psi(x) d\lambda = \mathcal{F}(u, A)\,.$$

In particular $F'(u) \geq \mathcal{F}(u)$ for every $u \in BV(\Omega)$. The extension of this inequality to the whole $GBV(\Omega)$ is the same as in the one-dimensional case (see *Step 2* in the proof of Theorem 3.4.1). ■

4.5 Estimate from above of the upper Γ-limit

Lemma 4.5.1 *Let $u \in W^{1,1}(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\lim_{j \to +\infty} F_{\varepsilon_j}(u, A) = \int_A |\nabla u(x)| dx.$$

Proof. Consider the sequence

$$g_j(x) = \fint_{B_{\varepsilon_j}(x) \cap \Omega} |\nabla u(y)| dy.$$

Since $g \in L^1(\Omega)$, from Lebesgue differentation Theorem g_j converge to $|\nabla u|$ in $L^1(\Omega)$ and a.e. in Ω; by hypothesis on f, there exists $b \geq 1$ such that $f(t) \leq bt$ for all $t \geq 0$. Then

$$\frac{1}{\varepsilon_j} f(\varepsilon_j g_j(x)) \leq b g_j(x)$$

for every $x \in \Omega$ and for every $j \in \mathbb{N}$. Since

$$\lim_{j \to +\infty} \frac{1}{\varepsilon_j} f(\varepsilon_j g_j(x)) = |\nabla u(x)|,$$

then, by the dominated convergence Theorem, we obtain

$$\lim_{j \to +\infty} \frac{1}{\varepsilon_j} \int_A f(\varepsilon_j g_j(x)) dx = \int_A |\nabla u(x)| dx.$$

■

Proposition 4.5.2 *Let $u \in GBV(\Omega)$; then*

$$F''(u) \leq \int_\Omega |\nabla u(x)| dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|) d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

Proof.

Step 1. Assume first that u is in the space $\mathcal{W}(\Omega)$ provided by the approximation result of Theorem 1.2.4; it is easy to see that we can reduce to the case in which

$$\bar{S}_u \subseteq K \subseteq \Omega,$$

with K a $(n-1)$-dimensional simplex. It is not restrictive to assume that K is contained in the hyperplane $\{x_1 = 0\}$.

Let $\bar{F}_\varepsilon$ be the relaxed functional of F_ε in the L^1-topology; then (as shown in Proposition 4.1.2)

$$\bar{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)dx$$

for every $u \in BV(\Omega)$. Since

$$F'' \geq \limsup_{\varepsilon\to 0} \bar{F}_\varepsilon\,,$$

we shall obtain a bound for $F''(u)$ if we estimate $\bar{F}_\varepsilon(u)$.

Set

$$h(x) = u^+(x) - u^-(x), \quad x \in \Omega \cap \{x_1 = 0\}.$$

Since $u \in \mathcal{W}(\Omega)$, the function h is continuous. Assume $2\varepsilon < d(\bar{S}_u, \partial\Omega)$, and let $(S_u)_\varepsilon = \{x \in \Omega : d(x, \bar{S}_u) < \varepsilon\}$; then

$$\begin{aligned}
\bar{F}_\varepsilon(u) &= \frac{1}{\varepsilon}\int_\Omega f\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)dx \leq \\
&\leq \frac{1}{\varepsilon}\int_{\Omega\setminus(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|dy\right)dx + \\
&+ \frac{1}{\varepsilon}\int_{(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)}|\nabla u(y)|dy + \frac{1}{\omega_n\varepsilon^{n-1}}|Du|^s(B_\varepsilon(x))\right)dx,
\end{aligned}$$

In view of Lemma 4.5.1 it is easy to see that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega\setminus(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|dy\right)dx = \int_\Omega |\nabla u(x)|dx.$$

By the concavity of f,

$$\frac{1}{\varepsilon}\int_{(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)}|\nabla u(y)|dy + \frac{1}{\omega_n\varepsilon^{n-1}}|Du|^s(B_\varepsilon(x))\right)dx \leq$$

$$\leq \frac{1}{\varepsilon}\int_{(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)}|\nabla u(y)|dy\right)dx + R_\varepsilon,$$

where

$$R_\varepsilon = \frac{1}{\varepsilon}\int_{(S_u)_\varepsilon} f\left(\frac{1}{\omega_n\varepsilon^{n-1}}|Du|^s(B_\varepsilon(x))\right)dx.$$

Since $|(S_u)_\varepsilon| \to 0$, then

$$\frac{1}{\varepsilon}\int_{(S_u)_\varepsilon} f\left(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u(y)|dy\right) dx \to 0.$$

Up to now we have proved that

$$\limsup_{\varepsilon\to 0} \bar{F}_\varepsilon(u) \le \int_\Omega |\nabla u(x)|dx + \limsup_{\varepsilon\to 0} R_\varepsilon.$$

Let $(S_u)^0_\varepsilon = \{y \in \mathbb{R}^{n-1} :\ (0,y) \in (S_u)_\varepsilon\}$; by Fubini's theorem we have

$$R_\varepsilon \le \frac{1}{\varepsilon}\int_{(S_u)^0_\varepsilon} \left(\int_{-\varepsilon}^{\varepsilon} f\Big(\frac{1}{\omega_n \varepsilon^{n-1}}|Du|^s(B_\varepsilon(s,y))\Big)\, ds\right) dy.$$

Fix $\sigma > 0$; since h is uniformly continuous on any fixed neighborhood of $\bar{S}_u$, for ε sufficiently small and for every $(s,y) \in (-\varepsilon,\varepsilon) \times (S_u)^0_\varepsilon$ we have

$$|Du|^s(B_\varepsilon(t,y)) \le \omega_{n-1}(\sqrt{\varepsilon^2 - t^2})^{n-1}(|h(0,y)| + \sigma).$$

Therefore

$$R_\varepsilon \le \frac{1}{\varepsilon}\int_{(S_u)^0_\varepsilon} \left(\int_{-\varepsilon}^{\varepsilon} f\Big(\frac{\omega_{n-1}}{\omega_n}\big(\sqrt{1-(t/\varepsilon)^2}\big)^{n-1}(|h(0,y)| + \sigma)\Big)dt\right) dy.$$

By the change of variable $s = t/\varepsilon$ we deduce that

$$\limsup_{\varepsilon\to 0} R_\varepsilon \le \int_{\bar{S}_u} \left(\int_{-1}^{1} f\Big(\frac{\omega_{n-1}}{\omega_n}(\sqrt{1-t^2})^{n-1}(|h(z)| + \sigma)\Big)dt\right) d\mathcal{H}^{n-1}(z).$$

By taking the limit as $\sigma \to 0$, we conclude.

Step 2. In the case $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, we can apply Theorem 1.2.4, with $\phi(a,b,\nu) = \vartheta(|a-b|)$. Then there exists a sequence $w_j \to u$ in $L^1(\Omega)$, with $w_j \in \mathcal{W}(\Omega)$, such that $\nabla w_j \to \nabla u$ in $L^2(\Omega,\mathbb{R}^n)$ and

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \vartheta(|w_j^+(x) - w_j^-(x)|)d\mathcal{H}^{n-1} \le \int_{S_u} \vartheta(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1};$$

thus, by the lower semicontinuity of F'' and the *Step 1*,

$$F''(u) \le \liminf_{j\to+\infty} F''(w_j) \le \int_\Omega |\nabla u(x)|dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1}.$$

Using now the relaxation Theorem 1.3.4, we have

$$F''(u) \le \int_\Omega |\nabla u(x)|dx + \int_{S_u} \vartheta(|u^+(x) - u^-(x)|)d\mathcal{H}^{n-1} + |D^c u|(\Omega)$$

for every $u \in BV(\Omega)$. Finally, by a truncation argument and again the lower semicontinuity of F'' we obtain the desired inequality in $GBV(\Omega)$. ■

4.6 Relaxation and convergence of minima

Proof of Proposition 4.1.2

Denote by H_ε the functional on the right-hand side of (1.2). Let $I_\varepsilon(x) = B_\varepsilon(x) \cap (a,b)$ and $c_\varepsilon = \varepsilon/|I_\varepsilon(x)|$. It is easy to prove the L^1-l.s.c. of H_ε in BV. Indeed, if u_h is a sequence in BV converging to $u \in BV(\Omega)$ in the L^1-topology, then, by Fatou's lemma and the lower semicontinuity of total variation,

$$\liminf_{h\to+\infty} H_\varepsilon(u_h) = \liminf_{h\to+\infty} \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(c_\varepsilon|Du_h|(I_\varepsilon(x))\right)dx \geq$$

$$\geq \frac{1}{\varepsilon}\int_\Omega \liminf_{h\to+\infty} f_\varepsilon\left(c_\varepsilon|Du_h|(I_\varepsilon(x))\right)dx = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(c_\varepsilon \liminf_{h\to+\infty}|Du_h|(I_\varepsilon(x))\right)dx \geq$$

$$\geq \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(c_\varepsilon|Du|(I_\varepsilon(x))\right)dx = H_\varepsilon(u).$$

Since $H_\varepsilon(u) \leq F_\varepsilon(u)$ for all $u \in BV(\Omega)$, the relaxed functional $\bar{F}_\varepsilon$ is estimated from below by H_ε. Consider now the opposite inequality. Given $u \in BV(\Omega)$, if (v_h) denotes the sequence obtained from u (extended to a neighborhood of Ω) by standard mollification, then $v_h \to u$ in $L^1(\Omega)$ and

$$|Dv_h|(I_\varepsilon(x)) \to |Du|(I_\varepsilon(x))$$

for a.e. $x \in \Omega$ (see [5] Proposition 3.7). Then by the dominated convergence theorem

$$\lim_{h\to+\infty} F_\varepsilon(v_h) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(c_\varepsilon|Du|(I_\varepsilon(x))\right)dx = H_\varepsilon(u).$$

This show that $H_\varepsilon(u)$ is the relaxed functional of F_ε on $BV(\Omega)$. ■

Proof of the corollary 4.1.3

By definition, $\{u_j\}$ is a sequence in $W^{1,1}(\Omega)$ with

$$G_{\varepsilon_j}(u_j) \leq \inf_{L^1(\Omega)} G_{\varepsilon_j} + \varepsilon_j.$$

Since $g \in L^\infty(a,b)$ we can assume that (u_j) is equibounded. By Proposition 4.2.2 there exists $u_0 \in BV(\Omega)$ such that $u_j \to u_0$ in $L^1(\Omega)$. By Theorem 1.4.1, since G_{ε_j} Γ-converge to $\mathcal{G}$, u_0 is a minimum point of $\mathcal{G}$ on $L^1(\Omega)$. ■

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