Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities

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Abstract

We study the existence of positive solutions for a fourth order semilinear elliptic equation under Navier boundary conditions with positive, increasing and convex source term. Both bounded and unbounded solutions are considered. When compared with second order equations, several differences and difficulties arise. In order to overcome these difficulties new ideas are needed. But still, in some cases we are able to extend only partially the well-known results for second order equations. The theoretical and numerical study of radial solutions in the ball also reveal some new phenomena, not available for second order equations. These phenomena suggest a number of intriguing unsolved problems, which we quote in the final section.

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1 Introduction

In the last decades, positive solutions of the second order semilinear elliptic problem

$$\begin{cases} -\Delta u = \mu g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

have attracted a lot of interest, see e.g. [5, 6, 7, 9, 11, 13, 14, 16, 17, 18] and references therein. Here, Ω is a smooth bounded domain of $\mathbf{R}^n$ ($n \geq 2$), $\mu \geq 0$ and g is a positive, increasing and convex smooth function. By now, (1.1) is quite well understood. As a subsequent step, P.L. Lions [16, Section 4.2 (c)] suggests to study positive solutions to *systems* of semilinear elliptic equations, namely

$$\begin{cases} -\Delta u_i = \mu g_i(u_1, ..., u_m) & \text{in } \Omega \\ u_1 = ... = u_m = 0 & \text{on } \partial\Omega \end{cases} \qquad (i = 1, ..., m) \tag{1.2}$$

where $m \geq 2$ and the functions g_i are as just mentioned. In this paper we consider the case of two equations ($m = 2$) with $g_1(u_1, u_2) = u_2$ and $g_2(u_1, u_2) = g(u_1)$. Then, taking $\lambda = \mu^2$, system (1.2) reduces to the following semilinear biharmonic elliptic problem under Navier boundary conditions:

$$\begin{cases} \Delta^2 u = \lambda g(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{1.3}$$

We will essentially focus our attention on the cases where g is logarithmically convex, namely

$$g \in C^1(\mathbf{R}_+) \ , \qquad g(0) > 0 \ , \qquad s \mapsto \log g(s) \ \text{ is nonconstant increasing and convex} \ , \tag{1.4}$$

or g has a power-type behavior such as

$$g(s) = (1 + s)^p, \qquad p > 1 \ . \tag{1.5}$$

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Very little is known about (1.3) when g satisfies (1.4) or (1.5). As far as we are aware, only a couple of papers [3, 23] considering *Dirichlet* boundary conditions study this problem. But it is well-known that boundary conditions significantly change the nature of the problem and of the tools available in the proofs. For instance, under Navier boundary conditions one has maximum and comparison principles in any domain Ω. On the other hand, when dealing with Dirichlet boundary conditions one seeks solutions in $H^2_0(\Omega)$ and this allows one to extend solutions by 0 *outside* Ω; see, in particular, Problem 9.3 in Section 9.

The first purpose of the present paper is to extend to (1.3) some well-known results relative to (1.1). In Theorem 2.1 we assume that the source g satisfies (1.4) and we prove a full extension of the results available for (1.1). Although the results remain similar, the proof is completely different due to some technical difficulties, see Problem 9.1 in Section 9. We overcome this problem by generalizing a procedure developed in [3]. Then, we turn to the power-like case (1.5). When p is subcritical, namely $p \leq \frac{n+4}{n-4}$, by applying critical point techniques as in [2, 6, 9, 12] in Theorem 2.2 we completely extend the results relative to (1.1). But for supercritical p, namely $p > \frac{n+4}{n-4}$, we only have partial results, see Theorem 2.3.

The second (and perhaps most important) purpose of the present paper is to emphasize some striking differences between (1.1) and (1.3). These differences are not just the already mentioned technical difficulties in the proofs but also some unexpected and new behaviors of the solutions which are particularly evident in the radial setting, i.e. the case where $\Omega = B$, the unit ball. Let us mention a couple of these differences.

When $g(s) = e^s$ or $g(s) = (1+s)^p$ one can easily find explicit singular radial solutions of (1.1), see [7, 19]. For the same nonlinearities g, one can also find explicit singular solutions of the equation in (1.3) which satisfy the first boundary condition but *not* the second. Hence, apparently, these are "ghost" singular solutions which have nothing to do with problem (1.3). But in [3] it was shown that the "true" singular solutions have the same asymptotic blow up behavior as the ghost solutions. We have no explanation of this fact.

If g is critical, namely $g(s) = (1+s)^{\frac{n+2}{n-2}}$, problem (1.1) may be solved explicitly when $\Omega = B$, see [11, 14]. Up to rescaling and translations, the solutions are the restrictions to B of the positive entire solutions of the equation $-\Delta u = u^{\frac{n+2}{n-2}}$ over $\mathbf{R}^n$. For critical growth problems of fourth order, namely $g(s) = (1+s)^{\frac{n+4}{n-4}}$, the same result is not true. The reason is that Pohozaev identity *does not* ensure nonexistence of radial sign changing solutions of $\Delta^2 u = |u|^{\frac{8}{n-4}} u$ over $\mathbf{R}^n$, see Problem 9.4. With the aid of Mathematica we numerically show that the previous equation has both radial positive solutions which (for finite $|x|$) blow up towards $+\infty$ and solutions which change sign and (for finite $|x|$) blow up towards $-\infty$. Then, by a shooting method having the initial second derivative as parameter, in Theorem 4.2 we partially prove these numerical evidences.

These are just some differences between (1.1) and (1.3), for further differences see Section 4. These surprising results shed some light on semilinear fourth problems but still much work has to be done to reach a complete understanding of (1.3) and (1.2). This leads us to suggest some (difficult) unsolved problems in Section 9.

The paper is organized as follows. In next section we establish our main results for general domains Ω. In Section 3 we prove some analogies between (1.1) and (1.3) for a wide classs of nonlinearities g. In Section 4 we study the particular case where Ω is the unit ball and we emphasize some differences between (1.1) and (1.3). Sections 5-8 are devoted to the proofs of the results. Finally, in Section 9 we quote some open problems.

2 Main results

Throughout the paper we assume that Ω is a bounded smooth domain of $\mathbf{R}^n$ ($n \geq 5$) and $\lambda \geq 0$.

For $1 \leq p \leq \infty$ we denote by $|\cdot|_p$ the $L^p(\Omega)$ norm whereas, we denote by $\|\cdot\|$ the $H^2 \cap H^1_0(\Omega)$ norm, that is $\|u\|^2 = \int_\Omega |\Delta u|^2$. We fix some exponent q with $q > \frac{n}{4}$ and $q \geq 2$. The definitions and results below do not depend on the special choice of q. Consider the functional space

$$X(\Omega) = \left\{u \in W^{4,q}(\Omega) \mid u = \Delta u = 0 \text{ on } \partial\Omega\right\} \ .$$

Then, we say that a function $u \in L^2(\Omega)$, $u \geq 0$ is a **solution** of (1.3) if $g(u) \in L^1(\Omega)$ and

$$\int_\Omega u\Delta^2 v = \lambda \int_\Omega g(u)v \qquad \forall v \in X(\Omega).$$

A solution u of (1.3) is called **regular** (resp. **singular**) if $u \in L^\infty(\Omega)$ (resp. $u \notin L^\infty(\Omega)$). We also say that a solution u_λ of (1.3) is **minimal** if $u_\lambda \leq u$ a.e. in Ω for any further solution u of (1.3). Next, we define

$$\Lambda(g(u)) := \{\lambda \geq 0 : (1.3) \text{ admits a solution}\} \ , \qquad \lambda^*(g(u)) := \sup \Lambda(g(u)) \ . \tag{2.1}$$

When it is clear which g we are dealing with we will simply write Λ and λ^*. Clearly, $0 \in \Lambda$ so that $\Lambda \neq \emptyset$ and λ^* is well-defined. Finally, we call **extremal** a solution u^* of (1.3) with $\lambda = \lambda^*$.

Our first statement concerns the log-convex case (1.4). We set $f(s) := \log g(s)$, we assume that

$$f \in C^1(\mathbf{R}_+) \ , \quad s \mapsto f(s) \ \text{ is nonconstant increasing and convex} \tag{2.2}$$

so that (1.3) reads

$$\begin{cases} \Delta^2 u = \lambda e^{f(u)} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Then, we prove

Theorem 2.1 *Assume that f satisfies* (2.2)*. Then there exists $\lambda^* > 0$ such that:*
(i) for $0 < \lambda < \lambda^*$ problem (2.3) *admits a minimal regular solution;*
(ii) for $\lambda = \lambda^*$ problem (2.3) *admits at least a solution, not necessarily regular;*
(iii) for $\lambda > \lambda^*$ problem (2.3) *admits no solution.*

Next, we consider the power-type case:

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Our first result about (2.4) deals with the subcritical case. In such situation, critical point theory applies. We assume that the minimax variational characterization of mountain pass solutions given by Ambrosetti-Rabinowitz [2] is known to the reader. Then, we prove

Theorem 2.2 *Assume that $1 < p \leq \frac{n+4}{n-4}$. Then, any solution of problem* (2.4) *is regular and there exists $\lambda^* > 0$ such that:*
(i) for $0 < \lambda < \lambda^*$ problem (2.4) *admits at least two solutions: the minimal solution and a mountain pass solution;*
(ii) for $\lambda = \lambda^*$ problem (2.4) *admits a unique solution;*
(iii) for $\lambda > \lambda^*$ problem (2.4) *admits no solution.*

The supercritical case $p > \frac{n+4}{n-4}$ is more delicate and we only have partial results. Note that Theorem 2.1 defines $\lambda^*(e^{pu}) > 0$ for all $p > 1$. This number is in some sense "optimal" for the following statement:

Theorem 2.3 *Assume that* $p > \frac{n+4}{n-4}$. *Then there exists* $\lambda^* \geq \lambda^*(e^{pu})$ *such that:*
(i) *for* $0 < \lambda < \lambda^*$ *problem* (2.4) *admits a minimal solution which is regular whenever* $0 < \lambda < \lambda^*(e^{pu})$*;*
(ii) *for* $\lambda > \lambda^*$ *problem* (2.4) *admits no solutions.*

The upper bound $\lambda^*(e^{pu})$ for the regularity of minimal solutions is obtained by comparison arguments. The "optimal" choice of the function e^{pu} is a consequence of the fact that the function $u \mapsto pu$ is the smallest function f satisfying (2.2) and $e^{f(u)} \geq (1+u)^p$.

3 Some analogies between (1.1) and (1.3)

Throughout this section we deal with general nonlinearities g satisfying

$$g \in C^1(\mathbf{R}_+) \quad \text{is a nonconstant strictly positive, increasing and convex function.} \tag{3.1}$$

We collect here some results which will be useful in the sequel. We just give some hints of the proofs since they are essentially similar to previous works. We first establish some technical lemmas:

Lemma 3.1 *For all* $w \in L^1(\Omega)$ *such that* $w \geq 0$ *a.e. in* Ω *there exists a unique* $u \in L^1(\Omega)$ *such that* $u \geq 0$ *a.e. in* Ω *and which satisfies*

$$\int_\Omega u\Delta^2 v = \int_\Omega wv$$

for all $v \in C^4(\overline{\Omega}) \cap X(\Omega)$. *Moreover, there exists* $C > 0$ *(independent of* w*) such that* $|u|_1 \leq C\,|w|_1$.

Proof. It is similar to that of [5, Lemma 1] which makes use of a weak form of the maximum principle. This principle is proved in [3, Lemma 1] for polyharmonic equations in the ball under Dirichlet boundary conditions for which one can use Boggio's principle. Under Navier boundary conditions, Boggio's principle is replaced by the (strong) maximum principle for superharmonic functions and general domains Ω may be chosen. ∎

A weak form of the maximum principle reads as follows:

Lemma 3.2 *Assume that* $u \in L^1(\Omega)$ *satisfies*

$$\int_\Omega u\Delta^2 v \geq 0$$

for all $v \in C^4\left(\overline{\Omega}\right) \cap X(\Omega)$ *such that* $v \geq 0$ *in* Ω. *Then,* $u \geq 0$ *a.e. in* Ω.

Proof. This may be obtained using Lemma 3.1 and arguing as in [3, 5]. ∎

From Lemma 3.2 and arguing as for [3, Lemma 4], we obtain a weak form of the super-subsolution method:

Lemma 3.3 *Assume* (3.1). *Let* $\lambda > 0$, *assume that there exists* $\overline{u} \in L^2(\Omega)$, $\overline{u} \geq 0$ *such that* $g(\overline{u}) \in L^1(\Omega)$ *and*

$$\int_\Omega \overline{u}\Delta^2 v \geq \lambda \int_\Omega g(\overline{u})v \qquad \forall v \in X(\Omega)\ :\ v \geq 0 \text{ a.e. in } \Omega.$$

Then, there exists a solution u *of* (1.3) *which satisfies* $0 \leq u \leq \overline{u}$ *a.e. in* Ω.

By Lemma 3.3 we infer at once that the set Λ defined in (2.1) is an interval. We now show that it is bounded:

Lemma 3.4 *Assume* (3.1)*. Then,* $\alpha_g := \max\{\alpha > 0 :\ g(s) \geq \alpha s\ \forall s \geq 0\} > 0$ *and*

$$0 < \lambda^*(g(u)) < \frac{\lambda_1}{\alpha_g} \ , \tag{3.2}$$

where λ_1 *denotes the first eigenvalue of* Δ^2 *in* Ω *under Navier boundary conditions.*

Proof. A standard application of the Implicit Function Theorem implies $\lambda^* > 0$.
Let Φ_1 denote a positive eigenfunction corresponding to λ_1. Assume that $u \in L^2(\Omega)$ solves (1.3), then we have

$$\lambda_1 \int_\Omega u\Phi_1 = \int_\Omega u\Delta^2\Phi_1 = \lambda \int_\Omega g(u)\Phi_1 > \lambda\alpha_g \int_\Omega u\Phi_1$$

where the last inequality is strict since $g(u) > \alpha_g u$ for small u (recall that $g(0) > 0$). The upper bound for λ^* now follows at once. ■

We now show that minimal *regular* solutions of (1.3) are stable:

Proposition 3.5 *Assume* (3.1)*. Let* $\lambda \in (0, \lambda^*)$ *and suppose that the minimal solution* u_λ *of* (1.3) *is regular. Let* μ_1 *denote the least eigenvalue of the linearized operator* $\Delta^2 - \lambda g'(u_\lambda)$ *in* u_λ*; then* $\mu_1 > 0$*.*

Proof. Recall the variational characterization of μ_1:

$$\mu_1 = \mu_1(\lambda) = \inf_{w \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta w|^2 - \lambda \int_\Omega g'(u_\lambda)w^2}{\int_\Omega w^2} \ .$$

Clearly, the map $\lambda : (0, \lambda^*) \mapsto \mu_1(\lambda)$ is non increasing and, by Proposition 2 in [3], it is continuous from the left. For contradiction, suppose there exists $\lambda \in (0, \lambda^*)$ such that $\mu_1(\lambda) \leq 0$ and define $\lambda_0 := \sup\{\lambda \geq 0 : \mu_1(\lambda) > 0\}$. By the continuity from the left we have $\mu_1(\lambda_0) \geq 0$. If $\mu_1(\lambda_0) > 0$, by the second part of Proposition 2 in [3], we get $\mu_1(\lambda) > 0$ for some $\lambda > \lambda_0$, which contradicts the definition of λ_0. So it must be $\mu_1(\lambda_0) = 0$. Fix some $\gamma \in (\lambda_0, \lambda^*)$; then, u_γ is a strict supersolution of (1.3) with $\lambda = \lambda_0$; but Proposition 3 in [3] yields $u_{\lambda_0} = u_\gamma$ giving again a contradiction. ■

Next, we show that the interval Λ is closed, provided the minimal solution u_λ is regular for all λ and the nonlinearity g satisfies a growth condition which is verified by (1.4) and (1.5). Since by Lemma 3.3 the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing for all $x \in \Omega$, it makes sense to define

$$u^*(x) := \lim_{\lambda \to \lambda^*} u_\lambda(x) \qquad (x \in \Omega) \ . \tag{3.3}$$

The following statement tells us that u^* is the *extremal* solution:

Proposition 3.6 *Assume* (3.1) *and*

$$\lim_{s \to +\infty} \frac{g'(s)s}{g(s)} > 1 \ . \tag{3.4}$$

Assume that the minimal solution u_λ *of* (1.3) *is regular for all* $\lambda \in (0, \lambda^*)$ *and let* u^* *be as in* (3.3)*. Then,* $u^* \in H^2 \cap H_0^1(\Omega)$ *and* u^* *solves* (1.3) *for* $\lambda = \lambda^*$*. Moreover,* $u_\lambda \to u^*$ *in* $H^2 \cap H_0^1(\Omega)$ *as* $\lambda \to \lambda^*$*.*

Proof. Let u_λ be the minimal solution of (1.3), then:

$$\int\limits_\Omega u_\lambda \Delta^2 v = \lambda \int\limits_\Omega g(u_\lambda)v \qquad \forall v \in X(\Omega) \, , \tag{3.5}$$

and, by Proposition 3.5,

$$\lambda \int\limits_\Omega g'(u_\lambda)u_\lambda^2 \leq \int\limits_\Omega (\Delta u_\lambda)^2 = \lambda \int\limits_\Omega g(u_\lambda)u_\lambda. \tag{3.6}$$

From (3.4), it follows that for every $\varepsilon > 0$ there exists $C > 0$ such that $(1+\varepsilon)g(s)s \leq g'(s)s^2 + C$ for all $s \geq 0$. Arguing as in [7], and applying this last inequality and (3.6), we get:

$$\int\limits_\Omega (g'(u_\lambda)u_\lambda^2 + C) \geq (1+\varepsilon) \int\limits_\Omega g(u_\lambda)u_\lambda \geq (1+\varepsilon) \int\limits_\Omega g'(u_\lambda)u_\lambda^2,$$

which gives the existence of a constant $C_1 > 0$ such that:

$$\int\limits_\Omega g(u_\lambda)u_\lambda < C_1$$

and therefore

$$\|u_\lambda\|^2 = \int\limits_\Omega (\Delta u_\lambda)^2 < \lambda^* C_1. \tag{3.7}$$

If we let $\lambda \to \lambda^*$, by (3.7) and (3.3) we deduce that, up to a subsequence,

$$u_\lambda \rightharpoonup u^* \ \text{ in } H^2 \cap H_0^1(\Omega) \text{ as } \lambda \to \lambda^*. \tag{3.8}$$

Furthermore, (3.8) allows us to pass to the limit in (3.5) and to get that u^* solves (1.3) for $\lambda = \lambda^*$. Finally, by Lebesgue's Theorem, we have that:

$$\|u_\lambda\|^2 = \lambda \int\limits_\Omega g(u_\lambda)u_\lambda \to \lambda^* \int\limits_\Omega g(u^*)u^* = \|u^*\|^2 \ \textit{as } \lambda \to \lambda^*.$$

This, together with (3.8), shows that $u_\lambda \to u^*$ in $H^2 \cap H_0^1(\Omega)$ as $\lambda \to \lambda^*$. ■

If in addition $\{u_\lambda\}$ is *uniformly* bounded then we can improve Proposition 3.6 with the following:

Proposition 3.7 *Assume* (3.1). *Let u_λ denote the minimal solution of* (1.3) *and assume there exists $M > 0$ such that $|u_\lambda|_\infty < M$, for all $\lambda \in (0, \lambda^*)$. Then $u_\lambda \to u^*$ in $C^{4,\alpha}\left(\overline{\Omega}\right)$ for all $\alpha \in (0,1)$. Moreover, λ^* is a* ***turning point****, that is, there exists $\delta > 0$ such that the solutions (λ, u) of* (1.3)*, near (λ^*, u^*), form a differentiable curve $s \in (-\delta, +\delta) \mapsto (\lambda(s), u(s)) \in \mathbf{R}_+ \times C^{4,\alpha}\left(\overline{\Omega}\right) \cap X(\Omega)$, which satisfies: $u(0) = u^*$, $\lambda(0) = \lambda^*$, $\lambda'(0) = 0$ and $\lambda''(0) < 0$.*

Proof. We argue as in [9]. Since $\{u_\lambda\}$ is bounded in $L^\infty(\Omega)$, by elliptic regularity, we deduce the boundedness of $\{u_\lambda\}$ also in $W^{4,p}(\Omega)$, for every $p > 1$. Then, by Sobolev embedding, we get that, for every $0 < \alpha < 1$, $\{u_\lambda\}$ is bounded in $C^{3,\alpha}(\overline{\Omega})$ and so, again by elliptic regularity, $\{u_\lambda\}$ is also bounded in $C^{4,\alpha}(\overline{\Omega})$. Finally, from the compact embedding $C^{4,\alpha_1}(\overline{\Omega}) \subset C^{4,\alpha_2}(\overline{\Omega})$ (for every $\alpha_1 > \alpha_2$) we get the claimed convergence.
Let us now define the operator $F : (0, \lambda^*] \times C^{4,\alpha}\left(\overline{\Omega}\right) \cap X(\Omega) \to C^{0,\alpha}\left(\overline{\Omega}\right)$ by:

$$F(\lambda, u) := \Delta^2 u - \lambda g(u).$$

It is not difficult to verify that $F(\lambda, u)$ satisfies the hypotheses of Theorem 3.2 in [8], from which follows the existence of a curve of solutions, $(\lambda(s), u(s))$, such that $u(0) = u^*$and $\lambda(0) = \lambda^*$.

To show that $\lambda'(0) = 0$ and $\lambda''(0) < 0$, it is sufficient to differentiate $s \mapsto F(\lambda(s), u(s))$ twice with respect to s and evaluate these derivatives at $s = 0$. ■

A further step towards a better knowledge of the set of solutions of problem (1.3) is made by showing that this set is unbounded in $C^{4,\alpha}\left(\overline{\Omega}\right)$. Assume (3.1) and for every $u \in C^{0,\alpha}\left(\overline{\Omega}\right)$ let $v := G(\lambda, u) \in C^{0,\alpha}\left(\overline{\Omega}\right)$ be the unique solution of the problem:

$$\begin{cases} \Delta^2 v = \lambda g(u) & \text{in } \Omega \\ v = \Delta v = 0 & \text{on } \partial\Omega \ . \end{cases}$$

The solutions of (1.3) are fixed points of G. Furthermore, by elliptic regularity, we have that $v \in C^{4,\alpha}\left(\overline{\Omega}\right)$ and hence, from the compactness of the embedding $C^{4,\alpha}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega})$, we get that G is a compact operator from $C^{0,\alpha}\left(\overline{\Omega}\right)$ into $C^{0,\alpha}\left(\overline{\Omega}\right)$. So, if we call C_0 the component of the set

$$S := \left\{(\widetilde{\lambda}, u) \in (0, \lambda^*] \times C^{4,\alpha}\left(\overline{\Omega}\right) : u \text{ solves (1.3) with } \lambda = \widetilde{\lambda}\right\}$$

to which (0,0) belongs, we are in the framework of Theorem 6.2 in [22], from which it follows that:

Proposition 3.8 *Assume* (3.1)*. Then C_0 is unbounded in* $(0, \lambda^*] \times C^{4,\alpha}\left(\overline{\Omega}\right)$.

4 Some differences between (1.1) and (1.3): radial problems

In this section we assume that $\Omega = B$ (the unit ball). In this case, writing (1.3) in its original form of system (1.2), by [25, Theorem 1] we know that any regular solution of (1.3) is radially symmetric and radially decreasing. We discuss separately the exponential case (2.3) (when $f(u) \equiv u$) and the power case (2.4). For the latter, the critical case $p = \frac{n+4}{n-4}$ deserves particular attention. In radial coordinates $r = |x|$, (1.3) becomes

$$u^{iv}(r) + \frac{2\,(n-1)}{r} u'''(r) + \frac{(n-1)\,(n-3)}{r^2} u''(r) - \frac{(n-1)\,(n-3)}{r^3} u'(r) = \lambda g(u(r)) \qquad r \in [0, 1) \tag{4.1}$$

supported with Navier boundary conditions

$$u(1) = u''(1) + (n-1)u'(1) = 0 \ . \tag{4.2}$$

Moreover, regular solutions u are smooth and therefore $r \mapsto u(r)$ must be an even function of r, namely

$$u'(0) = u'''(0) = 0 \ . \tag{4.3}$$

The main purpose of the present section is to highlight several striking differences between (1.3) and the corresponding second order problem (1.1).
Another purpose of this section is to estimate λ^*. In order to give an upper bound for λ^* we use Lemma 3.4. The estimate (3.2) gives

$$\lambda^*(e^u) < \frac{\lambda_1}{e} \ , \qquad \lambda^*((1+u)^p) < \frac{(p-1)^{p-1}}{p^p} \lambda_1 \ . \tag{4.4}$$

It is well-known that $\lambda_1 = Z^4$, where Z is the first zero of the Bessel function $J_{\frac{n-2}{2}}$. According to [1] we have

n	5	6	7	8
λ_1	407.6653	695.6191	1103.3996	1657.0143

In order to give a lower bound for λ^*, we seek a supersolution for (1.3). For any $C > 0$ the function

$$U_C(r) = C\left(\frac{2n}{n+3}r^3 - 3\frac{n+1}{n+3}r^2 + 1\right) \tag{4.5}$$

belongs to $H^2 \cap H_0^1(\Omega)$ and satisfies the boundary conditions (4.2). We investigate for which C and λ we have $\Delta^2 U_C \geq \lambda g(U_C)$. The largest such λ gives a lower bound for λ^*. The choice of U_C in (4.5) as a supersolution is probably not optimal. Nevertheless, with Mathematica we could at least optimize the choice of the constant C and find the results listed in the tables in the following subsections.

The last purpose of this section is to determine the **ghost solutions** as mentioned in the introduction. More precisely, we determine solutions of (4.1) satisfying the first boundary condition in (4.2) but *not* the second. Of particular interest is the value of λ_g corresponding to the ghost solution. We will see that λ_g may be either larger or smaller than λ^*; apparently, the former case occurs for subcritical nonlinearities whereas the latter occurs for supercritical nonlinearities. However, this is not a rule, see the case of critical nonlinearities.

4.1 Exponential nonlinearity

When $f(u) = u$, (2.3) written in radial coordinates becomes

$$u^{iv}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)\,(n-3)}{r^3}u'(r) = \lambda e^{u(r)} \qquad r \in [0,1) \tag{4.6}$$

together with the boundary conditions (4.2). As may be checked by a simple calculation, for $\lambda = \lambda_e := 8(n-2)(n-4)$ the function $U(r) := -4\log r$ is a ghost solution, namely it solves (4.6) and the first boundary condition in (4.2) but not the second boundary condition. Contrary to what happens for the second order equation, the explicit form of a radial singular solution seems not simple to be determined, see also [3].

In dimensions $n = 5, 6, 7, 8$, the table below shows first for which values of C the function U_C defined in (4.5) is a supersolution of (4.6) and the corresponding lower bound for λ^*. We also give the upper bound obtained with (4.4). In the fifth column, we quote from [3] a lower bound for the extremal value $\lambda^*(D)$ of the corresponding *Dirichlet* problem; as for the eigenvalues, it is considerably larger than λ^*. Finally, in the last column, we quote λ_e, namely the value of λ corresponding to the ghost solution: it is considerably smaller than λ^*.

n	C	$\lambda^* \geq$	$\lambda^* <$	$\lambda^*(D) \geq$	λ_e
5	1.093	98.37	149.9716	235.89	24
6	1.132	158.48	255.9039	361.34	64
7	1.162	234.26	405.9180	523.16	120
8	1.185	325.76	609.5814	724.50	192

4.2 Power-type nonlinearity

In radial coordinates (2.4) reads

$$u^{iv}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)\,(n-3)}{r^3}u'(r) = \lambda(1+u(r))^p \qquad r \in [0,1) \tag{4.7}$$

together with the boundary conditions (4.2).

Let us first recall some results for the second order problem corresponding to (4.7), namely

$$-u''(r) - \frac{n-1}{r}u'(r) = \mu(1+u(r))^p\ , \qquad r \in [0,1)\ . \tag{4.8}$$

It is well-known [7] that the function $v_p(r) = r^{-2/(p-1)} - 1$ solves (4.8) (and satisfies the Dirichlet boundary condition $u(1) = 0$) if

$$\mu = \mu_p := \frac{2(np - n - 2p)}{(p-1)^2} \ .$$

Note that $\mu_p > 0$ if and only if $p > \frac{n}{n-2}$; note also that $\frac{n}{n-2}$ is the critical (largest) trace exponent q for which one has $H^1(\Omega) \subset L^{q+1}(\partial\Omega)$. Moreover, $u_p \in H^1_0(B)$ if and only if $p > \frac{n+2}{n-2}$, the critical Sobolev exponent.
For the fourth order problem, we consider the function

$$u_p(r) = r^{-4/(p-1)} - 1,$$

which solves (4.7) if

$$\lambda = \lambda_p := \frac{8(p+1)(2 + 2p - np + n)(4p - np + n)}{(p-1)^4} \ .$$

Note that

$$\lambda_p > 0 \iff p \in \left(1, \frac{n+2}{n-2}\right) \cup \left(\frac{n}{n-4}, \infty\right)$$

and that $u_p \in H^2 \cap H^1_0(B)$ if and only if $p > \frac{n+4}{n-4}$. The number $(n+2)/(n-2)$ is the critical exponent for the first order Sobolev inequality while $n/(n-4)$ is again the critical trace exponent q for the embedding $H^2(\Omega) \subset L^{q+1}(\partial\Omega)$. For $\lambda = \lambda_p$, the function u_p is a singular solution of equation (4.7) but u_p *does not* satisfy the second condition in (4.2); hence, it is not a solution of problem (1.3). The functions u_p are the ghost solutions. These facts suggest several problems which we quote in Section 9.

Also for (4.7) we used the function U_C in (4.5). In dimensions $n = 5, 6, 7, 8$, the tables below show both for which values of C the function U_C is a supersolution of (4.7) and the corresponding lower bound for λ^*. We also give the upper bound obtained with (4.4). The tables correspond, respectively, to the cases $p = 3/2$ (subcritical) and $p = 10$ (supercritical); in the first case we have $\lambda^* < \lambda_p$, whereas in the second we have $\lambda^* > \lambda_p$.

(p=3/2)

n	C	$\lambda^* \geq$	$\lambda^* <$	λ_p
5	0.801	72.09	156.91	2800
6	0.844	118.16	267.74	1920
7	0.878	177	424.69	1200
8	0.905	248.79	637.78	640

n	C	$\lambda^* \geq$	$\lambda^* <$	λ_p
5	0.111	9.99	15.79	1.542
6	0.115	16.1	26.94	6.001
7	0.118	23.79	42.74	12.648
8	1.121	33.26	64.19	21.46

(p=10)

4.3 The critical case

Of special interest is problem (2.4) in the critical case $p = \frac{n+4}{n-4}$. By Theorem 2.2 and [25, Theorem 1] we know that this problem admits at least two regular and radially symmetric solutions. Take any such solution u; then the function $v = \lambda^{\frac{n-4}{8}}(1+u)$ solves the problem

$$\begin{cases} \Delta^2 v = v^{\frac{n+4}{n-4}} & \text{in } B \\ v > \lambda^{(n-4)/8} & \text{in } B \\ v = \lambda^{(n-4)/8} & \text{on } \partial B \\ \Delta v = 0 & \text{on } \partial B \ . \end{cases} \tag{4.9}$$

Equivalently, $v = v(r)$ satisfies

$$v^{iv}(r) + \frac{2\,(n-1)}{r} v'''(r) + \frac{(n-1)\,(n-3)}{r^2} v''(r) - \frac{(n-1)\,(n-3)}{r^3} v'(r) - v(r)^{\frac{n+4}{n-4}} = 0 \tag{4.10}$$

with the boundary conditions

$$v(1) = \lambda^{(n-4)/8} \ , \qquad \Delta v(1) = v''(1) + (n-1)v'(1) = 0 \ , \tag{4.11}$$

and the regularity conditions $v'(0) = v'''(0) = 0$.
Consider now the critical problem over the whole space

$$\Delta^2 v = v^{\frac{n+4}{n-4}} \quad \text{in } \mathbf{R}^n \ . \tag{4.12}$$

By [15, Theorem 1.3], we know that (up to translations) any smooth positive solution of (4.12) has the form

$$v_d\,(x) = \frac{\left(n\left(n^2-4\right)(n-4)\,d^2\right)^{\frac{n-4}{8}}}{\left(1+d\,|x|^2\right)^{\frac{n-4}{2}}} \qquad (d>0)\,. \tag{4.13}$$

The main goal of this section is to describe the (smooth) continuation of solutions of (4.9) *outside* B. We obtain a new phenomenon, not available for the corresponding second order problem:

Proposition 4.1 *Let v be a (radial) solution of* (4.9)*; then it does not admit a positive radial extension to $\mathbf{R}^n$.*

Proof. By contradiction suppose there exists $\overline{v}$, positive radial extension of v to $\mathbf{R}^n$. Then, by [24, Theorem 4] we have that $\overline{v}$ coincides with one of the functions v_d in (4.13), for some $d > 0$. But this is impossible since for all d, the function v_d does not satisfy the second condition in (4.11). ∎

For the critical growth second order problem it is known (see e.g. [11, Theorem 7]) that the solutions of the equation in fact coincide in B with some of the functions v_d of the corresponding family (4.13) and it is so clear in which way they are continued. Proposition 4.1 tells us that fourth order problems behave differently: it is therefore natural to inquire in which way the solutions of (4.9) may be continued for $|x| > 1$.
To this end, we performed several numerical experiments with Mathematica. The next figures display the graphics of three solutions of

$$v^{iv}(r) + \frac{14}{r}v'''(r) + \frac{35}{r^2}v''(r) - \frac{35}{r^3}v'(r) - v(r)^3 = 0 \ . \tag{4.14}$$

All three solutions satisfy the initial conditions

$$v(0) = 4\sqrt{\frac{6}{5}} \approx 4.38178 \qquad v'(0) = v'''(0) = 0. \tag{4.15}$$

The distinction between the three solutions is made by the choice of the second derivative at $r = 0$: we take respectively

$$v''(0) = -\frac{8}{5}\sqrt{\frac{6}{5}} \approx -1.75271 \ , \qquad v''(0) = -\frac{8}{5}\sqrt{\frac{6}{5}} - 10^{-3} \ , \qquad v''(0) = -\frac{8}{5}\sqrt{\frac{6}{5}} + 10^{-3} \ . \tag{4.16}$$

Therefore, the first figure represents the function (4.13) for $n = 8$ and $d = 0.1$.
We performed further numerical experiments for other choices of n and d but the results were completely similar. Obviously, if one takes the "equilibrium" initial second derivative (the one of (4.13)), then the solution is precisely v_d. If one slightly increases this value, the corresponding solution has first a global minimum at positive level and then blows up towards $+\infty$. If one slightly decreases the equilibrium value, the corresponding solution vanishes, becomes negative and then blows up towards $-\infty$. These numerical results are partially confirmed by a rigorous proof. To be more precise, up to rescaling we may restrict our attention to the following problem

$$\begin{cases} u^{iv}(r) + \dfrac{2(n-1)}{r}u'''(r) + \dfrac{(n-1)(n-3)}{r^2}u''(r) - \dfrac{(n-1)(n-3)}{r^3}u'(r) = u^{\frac{n+4}{n-4}}(r) \quad r \in [0,\infty) \\[2ex] u(0) = 1 \ , \qquad u'(0) = u'''(0) = 0 \ , \qquad u''(0) = \gamma < 0 \ . \end{cases} \tag{4.17}$$

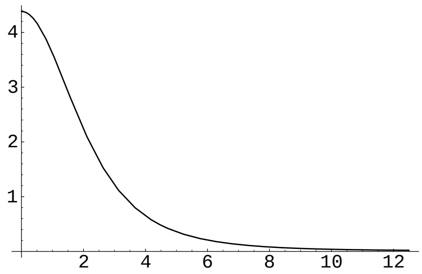

Figure 1: The plot of the solution of (4.14)-(4.15)-$(4.16)_1$

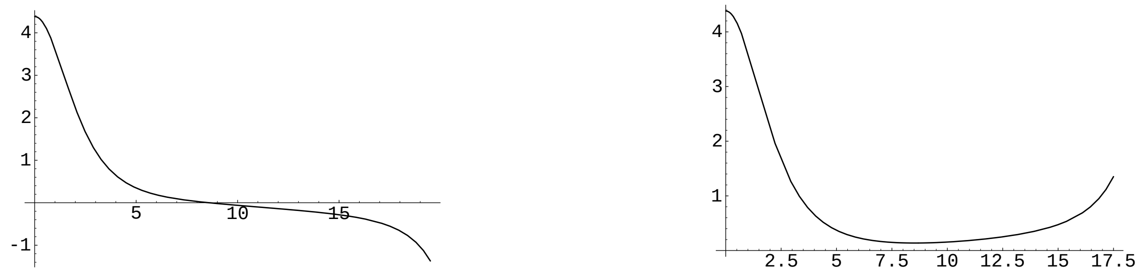

Figure 2: The plots of the solutions of (4.14)-(4.15) with $(4.16)_2$ and $(4.16)_3$

Here γ is the only free parameter while $u'(0) = u'''(0) = 0$ are the already mentioned regularity conditions. Existence and uniqueness of a local solution u_γ of (4.17) is quite standard. For a suitable choice of $\gamma < 0$, say $\gamma = \overline{\gamma}$, the unique solution $\overline{u} := u_{\overline{\gamma}}$ of (4.17) is in the family (4.13), namely

$$\overline{u}(r) = \frac{\left[n(n^2-4)(n-4)\right]^{\frac{n-4}{4}}}{\left(\sqrt{n(n^2-4)(n-4)} + r^2\right)^{\frac{n-4}{2}}} \ .$$

Then, we prove

Theorem 4.2 *For any $\gamma < 0$ let u_γ be the unique (local) solution of* (4.17). *Then:*
(i) if $\gamma < \overline{\gamma}$ there exists $R > 0$ such that $u_\gamma(R) = 0$, $u_\gamma(r) < \overline{u}(r)$ and $u'_\gamma(r) < 0$ for $r \in (0, R]$;
(ii) if $\gamma > \overline{\gamma}$ there exist $0 < R_1 < R_2 < \infty$ such that $u_\gamma(r) > \overline{u}(r)$ for $r \in (0, R_2)$, $u'_\gamma(r) < 0$ for $r \in (0, R_1)$, $u'_\gamma(R_1) = 0$, $u'_\gamma(r) > 0$ for $r \in (R_1, R_2)$ and $\lim_{r \to R_2} u_\gamma(r) = +\infty$.

Remark 4.3 The functions $u = u(r)$ displayed in the last plot of Figure 2 solve the following Dirichlet problem

$$\begin{cases} \Delta^2 u = u^{\frac{n+4}{n-4}} & \text{in } B_R \\ u = \gamma & \text{on } \partial B_R \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R \end{cases}$$

for some $\gamma, R > 0$. Then, the function $w(x) = \frac{u(Rx)}{\gamma} - 1$ satisfies

$$\begin{cases} \Delta^2 w = \lambda (1+w)^{\frac{n+4}{n-4}} & \text{in } B \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

for $\lambda = R^4 \gamma^{\frac{8}{n-4}}$, namely the Dirichlet problem for the equation in (2.4) in the unit ball.

We conclude this section with the table containing the value of $\lambda_{(n+4)/(n-4)}$ and the estimates of λ^* obtained with U_C in (4.5):

n	$(n+4)/(n-4)$	$\lambda_{(n+4)/(n-4)}$	C	$\lambda^* \geq$	$\lambda^* <$
5	9	25/16	0.123	11.07	17.65
6	5	9	0.235	32.9	56.98
7	11/3	441/16	0.335	67.54	128.72
8	3	64	0.425	116.84	245.48

5 Proof of Theorem 2.1

Note first that, up to rescaling λ, we may assume that

$$f(0) = 0 \ .$$

Then, we start with a "calculus" statement:

Lemma 5.1 *Assume that $\varphi \in C^1[0, +\infty)$ is a nonnegative, non-decreasing and convex function such that $\varphi(0) = 0$. Then for any $x \geq 0$ and any $\beta > 1$ we have $\varphi(\beta x) \geq \beta\varphi(x)$.*

Proof. If $\varphi(x) = 0$ (in particular if $x = 0$) the statement is trivial. For $x > 0$ such that $\varphi(x) > 0$, Lagrange's Theorem states that there exists $\xi_x \in (x, \beta x)$ such that

$$\varphi(\beta x) - \varphi(x) = (\beta - 1)x\varphi'(\xi_x), \tag{5.1}$$

and there exists $\alpha_x \in (0, x)$ such that

$$\varphi(x) = \varphi(x) - \varphi(0) = \varphi'(\alpha_x)\, x. \tag{5.2}$$

Divide (5.1) by $\varphi(x)$ so that, by (5.2) and using the convexity of φ, we obtain:

$$\frac{\varphi(\beta x)}{\varphi(x)} = 1 + \frac{\varphi'(\xi_x)(\beta - 1)\, x}{\varphi(x)} = 1 + \frac{(\beta - 1)\, x\varphi'(\xi_x)}{x\varphi'(\alpha_x)} \geq 1 + (\beta - 1) = \beta$$

which is the statement. ■

We now establish an improved version of [3, Lemma 5]:

Lemma 5.2 *Assume that for some $\mu > 0$ there exists a (possibly singular) solution u_0 of (2.3) with $\lambda = \mu$ Then, for all $0 < \lambda < \mu$ there exists a regular solution of (2.3).*

Proof. Let $0 < \lambda < \mu$, and consider the (unique) functions $u_1, u_2 \in L^2(\Omega)$ satisfying respectively

$$\int_\Omega u_1 \Delta^2 v = \lambda \int_\Omega e^{f(u_0)} v \qquad \text{and} \qquad \int_\Omega u_2 \Delta^2 v = \lambda \int_\Omega e^{f(u_1)} v \quad \forall v \in X(\Omega) \ ;$$

such functions exist by Lemma 3.1 and belong to $L^2(\Omega)$ since Lemma 3.2 entails

$$u_0 \geq \frac{\lambda}{\mu} u_0 = u_1 \geq u_2 \ \text{ a.e. in } \Omega.$$

We now need the following elementary statement:

$$\forall \vartheta > 1 \ \ \forall \alpha > 1 \ \ \exists \gamma > 0 \ \text{ s.t. } \quad e^{\vartheta f(s)} + \gamma - \alpha e^{f(s)} \geq 0 \quad \forall s \geq 0. \tag{5.3}$$

Fix $\vartheta := \frac{\mu}{\lambda} > 1$ and take $\alpha > \max\{\frac{n}{4}, 2\}$; then, (5.3) ensures that there exists $k > 0$ such that

$$e^{\frac{\mu}{\lambda}f(s)} + \frac{k}{\lambda} \geq \alpha e^{f(s)} \quad \forall s \geq 0. \tag{5.4}$$

Let $w \in X(\Omega)$ be the unique solution of the equation $\Delta^2 w = k$ in Ω; then $w \in L^\infty(\Omega)$ and $w > 0$ in Ω. Moreover, using Lemma 5.1 and (5.4) we get

$$\int_\Omega (u_1 + w)\Delta^2 v = \lambda \int_\Omega \left(e^{f(u_0)} + \frac{k}{\lambda}\right) v = \lambda \int_\Omega \left(e^{f(\frac{\mu}{\lambda}u_1)} + \frac{k}{\lambda}\right) v \geq$$

$$\geq \lambda \int_\Omega \left(e^{\frac{\mu}{\lambda}f(u_1)} + \frac{k}{\lambda}\right) v \geq \lambda\alpha \int_\Omega e^{f(u_1)} v = \alpha \int_\Omega u_2 \Delta^2 v$$

for all $v \in X(\Omega)$ such that $v \geq 0$ in Ω. Hence, by Lemma 3.2, we infer that $u_2 \leq \frac{u_1+w}{\alpha}$. Since $\alpha > 2$ and $w > 0$, this inequality, together with the monotonicity and convexity of f, implies that

$$f(u_2) \leq f\left(\frac{u_1}{\alpha} + \frac{w}{\alpha}\right) \leq f\left(\frac{1}{\alpha}u_1 + \left(1 - \frac{1}{\alpha}\right) w\right) \leq \frac{1}{\alpha}f(u_1) + \left(1 - \frac{1}{\alpha}\right) f(w).$$

In particular,

$$e^{f(u_2)} \leq e^{\frac{1}{\alpha}f(u_1)} e^{\left(1-\frac{1}{\alpha}\right)f(w)} ;$$

since $e^{\frac{1}{\alpha}f(u_1)} \in L^\alpha(\Omega)$ and $e^{\left(1-\frac{1}{\alpha}\right)f(w)} \in L^\infty(\Omega)$ we get at once that

$$e^{f(u_2)} \in L^\alpha(\Omega).$$

Finally, consider $u_3 \in L^2(\Omega)$ such that

$$\int_\Omega u_3 \Delta^2 v = \lambda \int_\Omega e^{f(u_2)} v \quad \forall v \in X(\Omega).$$

By elliptic regularity and the fact that $\alpha > \frac{n}{4}$, we deduce

$$u_3 \in W^{4,\alpha}(\Omega) \subset L^\infty(\Omega).$$

Moreover,

$$\int_\Omega (u_2 - u_3)\Delta^2 v = \lambda \int_\Omega \left(e^{f(u_1)} - e^{f(u_2)}\right) v \geq 0 \qquad \forall v \in X(\Omega) : v \geq 0 \text{ in } \Omega$$

so that by Lemma 3.2 we infer that $u_3 \leq u_2$. Hence,

$$\int_\Omega u_3 \Delta^2 v \geq \lambda \int_\Omega e^{f(u_3)} v \quad \forall v \in X(\Omega) : v \geq 0 \text{ in } \Omega.$$

Then u_3 is a weak bounded supersolution of (2.3) and the statement follows by Lemma 3.3. ■

Theorem 2.1 is now a straightforward consequence of (3.2) and of Lemma 5.2 and Proposition 3.6.

6 Proof of Theorem 2.2

The proof of Theorem 2.2 is obtained by combining some well-known results in [2, 6, 9, 12, 27]. Firstly, by applying the regularity results in [27], we prove

Proposition 6.1 *Assume that* $1 < p \leq \frac{n+4}{n-4}$ *and let* $u \in H^2 \cap H_0^1(\Omega)$ *be a solution of* (2.4)*. Then* u *is regular.*

Proof. If we show that $u \in L^q(\Omega)$ for every $q < \infty$, the statement will follow by elliptic regularity.
We first claim that for every $\varepsilon > 0$ there exist $q_\varepsilon \in L^{\frac{n}{4}}(\Omega)$ and $F_\varepsilon \in L^\infty(\Omega)$ such that:

$$(1 + u(x))^p = q_\varepsilon(x)u(x) + F_\varepsilon(x) \quad \text{and} \quad \|q_\varepsilon\|_{\frac{n}{4}} < \varepsilon \,. \tag{6.1}$$

Fix $M \geq 1$ and write

$$(1 + u)^p = \chi_{\{u \leq M\}}(1 + u)^p + \chi_{\{u>M\}}(1 + u)^p = \varphi(x) + \chi_{\{u>M\}}\frac{(1 + u)^p}{u}u \tag{6.2}$$

where $\chi_{\{.\}}$ is the characteristic function and $\varphi(x) = \chi_{\{u\leq M\}}(1 + u)^p \in L^\infty(\Omega)$. It is clear that $(1 + u)^p \leq (2u)^p$ whenever $u > M$. Moreover, using the embedding $H^2(\Omega) \subset L^{\frac{2n}{n-4}}(\Omega)$ and the fact that $p \leq \frac{n+4}{n-4}$, we have that $u^{p-1} \in L^{\frac{n}{4}}(\Omega)$, hence:

$$0 \leq a(x) := \chi_{\{u>M\}}\frac{(1 + u)^p}{u} \leq 2^p u^{p-1} \in L^{\frac{n}{4}}(\Omega)\,.$$

Therefore, we may write $(1 + u)^p = \varphi(x) + a(x)u$ with $\varphi \in L^\infty(\Omega)$ and $a \in L^{\frac{n}{4}}(\Omega)$. Applying Lemma B2 in [27], for every $\varepsilon > 0$ we obtain

$$a(x)u(x) = q_\varepsilon(x)u(x) + f_\varepsilon(x) \tag{6.3}$$

where q_ε and f_ε satisfy $\|q_\varepsilon\|_{\frac{n}{4}} < \varepsilon$ and $f_\varepsilon \in L^\infty(\Omega)$. Defining $F_\varepsilon(x) = f_\varepsilon(x) + \varphi(x)$, from (6.2) and (6.3) we obtain (6.1).
By (6.1), for every $\varepsilon > 0$, the equation in (2.4) can be rewritten as

$$\Delta^2 u = \lambda\Big(q_\varepsilon(x)u(x) + F_\varepsilon(x)\Big) \qquad \text{in } \Omega$$

so that the result follows by Steps 2 and 3 in [27]. ■

Consider the functional

$$J(u) = \int_\Omega |\Delta u|^2 - \frac{\lambda}{p + 1}\int_\Omega |1 + u|^{p+1}.$$

When $1 < p < \frac{n+4}{n-4}$, Proposition 3.5 enables us to argue as in the proof of Theorem 2.1 in [9] with minor changes; therefore, the existence of a (positive) mountain pass critical point for J follows.
When $p = \frac{n+4}{n-4}$, the embedding $H^2 \cap H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is not compact and the Palais-Smale condition for J does not hold at all levels. In order to find a mountain pass solution, we combine arguments from [6] and [12]. As in [6], we seek a second solution u of the form $u = u_\lambda + v$ with $v > 0$ in Ω so that v solves the problem

$$\begin{cases} \Delta^2 v = \lambda(1 + u_\lambda + v)^{\frac{n+4}{n-4}} - \lambda(1 + u_\lambda)^{\frac{n+4}{n-4}} & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

Setting $h(x,v)=\lambda(1+u_\lambda+v)^{\frac{n+4}{n-4}}-\lambda(1+u_\lambda)^{\frac{n+4}{n-4}}-\lambda v^{\frac{n+4}{n-4}}$, the previous problem reads

$$\begin{cases} \Delta^2 v=\lambda v^{\frac{n+4}{n-4}}+h(x,v) & \text{in } \Omega \\ v>0 & \text{in } \Omega \\ v=\Delta v=0 & \text{on } \partial\Omega \end{cases}$$

Finally, let $w=\lambda^{\frac{n-4}{8}}v$ and $f(x,w)=\lambda^{\frac{n-4}{8}}h(x,\lambda^{\frac{4-n}{8}}w)$, then w satisfies

$$\begin{cases} \Delta^2 w=w^{\frac{n+4}{n-4}}+f(x,w) & \text{in } \Omega \\ w>0 & \text{in } \Omega \\ w=\Delta w=0 & \text{on } \partial\Omega \end{cases} \tag{6.4}$$

The function f satisfies the hypotheses of Corollary 1 in [12]; therefore, we infer the existence of a positive solution of (6.4) or, equivalently, of a positive mountain pass solution for (2.4).
In order to conclude the proof of Theorem 2.2, we need to show that the extremal solution u^*, which exists by Proposition 3.6, is unique. To this end, recall that u^* is a classical solution in view of Proposition 6.1. Therefore, it suffices to argue as for Lemma 2.6 in [7].

7 Proof of Theorem 2.3

As $e^{ps}\geq(1+s)^p$ for all $s\geq 0$, arguing as in Theorem 8 in [11], we get that:

$$0<\lambda^*(e^{pu})\leq\lambda^*((1+u)^p). \tag{7.1}$$

By Lemma 5.2, for every $\lambda<\lambda^*(e^{pu})$ there exists a minimal regular solution u_λ of (2.3) with $f(u)=pu$. Such u_λ is also a bounded supersolution of (2.4), indeed

$$\int_\Omega u_\lambda\Delta^2 v=\lambda\int_\Omega e^{pu_\lambda}v\geq\lambda\int_\Omega(1+u_\lambda)^p v \qquad \forall v\in X(\Omega): v\geq 0 \text{ in } \Omega.$$

Then, by Lemma 3.3, for all $\lambda<\lambda^*(e^{pu})$ there exists a solution u_p of (2.4) such that $u_p\leq u_\lambda$.

8 Proof of Theorem 4.2

(i) Since $u_\gamma(0)=\overline{u}(0)$, $u_\gamma'(0)=\overline{u}'(0)$ and $u_\gamma''(0)<\overline{u}''(0)$, we have $u_\gamma(r)<\overline{u}(r)$ at least in a sufficiently small right neighborhood of $r=0$. For contradiction, assume that there exists (a first) $\rho>0$ such that

$$u_\gamma(\rho)=\overline{u}(\rho)\ , \qquad u_\gamma(r)<\overline{u}(r)<1 \qquad \forall r\in(0,\rho)\ . \tag{8.1}$$

Note that (4.17) may be rewritten as

$$\left\{r^{n-1}\left[\Delta u_\gamma(r)\right]'\right\}'=r^{n-1}u_\gamma^{\frac{n+4}{n-4}}(r)\ , \qquad \left\{r^{n-1}\left[\Delta\overline{u}(r)\right]'\right\}'=r^{n-1}\overline{u}^{\frac{n+4}{n-4}}(r) \qquad \forall r\in[0,\rho]\ . \tag{8.2}$$

By subtracting the two equations in (8.2) we readily obtain

$$\left\{r^{n-1}\left[\Delta u_\gamma(r)-\Delta\overline{u}(r)\right]'\right\}'=r^{n-1}[u_\gamma^{\frac{n+4}{n-4}}(r)-\overline{u}^{\frac{n+4}{n-4}}(r)] \qquad \forall r\in[0,\rho]\ . \tag{8.3}$$

Since both solutions u_γ and $\overline{u}$ are smooth, we have

$$\lim_{r\to 0}\left\{r^{n-1}\left[\Delta u_\gamma(r)-\Delta\overline{u}(r)\right]'\right\}=0\ ;$$

therefore, for any $r \in (0, \rho]$ we may integrate (8.3) over $[0, r]$ and obtain

$$r^{n-1} [\Delta u_\gamma(r) - \Delta\overline{u}(r)]' = \int_0^r t^{n-1}[u_\gamma^{\frac{n+4}{n-4}}(t) - \overline{u}^{\frac{n+4}{n-4}}(t)]\, dt < 0 \qquad \forall r \in (0, \rho] \ , \tag{8.4}$$

the last inequality being a consequence of (8.1). Note also that $\Delta u_\gamma(0) = n\gamma < n\overline{\gamma} = \Delta\overline{u}(0)$; this, combined with the strict decreasing of $r \mapsto \Delta[u_\gamma(r) - \overline{u}(r)]$ (see (8.4)) shows that

$$-\Delta(u_\gamma - \overline{u}) > 0 \qquad \text{in } B_\rho \ . \tag{8.5}$$

Moreover, (8.1) tells us that $(u_\gamma - \overline{u}) = 0$ on ∂B_ρ. This, together with (8.5) and the maximum principle shows that $u_\gamma > \overline{u}$ in B_ρ. This contradicts (8.1) and shows that $u_\gamma(r) < \overline{u}(r)$ as long as $u_\gamma(r)$ remains positive. The positivity interval for u_γ cannot be $(0, \infty)$, otherwise u_γ would be a positive solution of (4.12) which is not in the family (4.13), against [15, Theorem 1.3].
We have so far proved that there exists a finite $R > 0$ such that $u_\gamma(R) = 0$ and $u_\gamma(r) < \overline{u}(r)$ whenever $r \in (0, R]$. We now show that $u'_\gamma(r) < 0$ for all $r \in (0, R]$. If $u'_\gamma(R_\gamma) = 0$ for some $R_\gamma \le R$, then $\Delta u_\gamma(R_\gamma) \ge 0$; by integrating the first of (8.2) over $[0, r]$ for $r > R_\gamma$ and arguing as above we deduce that $\Delta u_\gamma(r) > 0$ for all $r > R_\gamma$ and, in turn, that $u'_\gamma(r) > 0$ for all $r > R_\gamma$. But then we would find $\rho > R_\gamma$ such that (8.1) holds, which we have just seen to be impossible. This contradiction shows that $u'_\gamma(r) < 0$ for all $r \in (0, R]$ and completes the proof of (i).

(ii) As in the proof of (i), it cannot be $u_\gamma(\rho) = \overline{u}(\rho)$ for some $\rho > 0$. Hence, for $r > 0$, $0 < \overline{u}(r) < u_\gamma(r)$ as long as the latter exists; if there exists no $R_1 > 0$ such that $u'_\gamma(R_1) = 0$, then $u'_\gamma(r) < 0$ for all $r > 0$ so that u_γ would be a positive global solution of (4.12) which is not in the family (4.13), against [15, Theorem 1.3]. So, let $R_1 > 0$ be the first solution of $u'_\gamma(R_1) = 0$; then, $\Delta u_\gamma(R_1) \ge 0$. By integrating the first of (8.2) over $[0, r]$ for $r > R_1$ we deduce that $\Delta u_\gamma(r) > 0$ for all $r > R_1$ and that $u'_\gamma(r) > 0$ for all $r > R_1$. Invoking once more [15, Theorem 1.3], we deduce that u_γ cannot exist globally; this proves the existence of R_2 and completes the proof of (ii).

9 Some unsolved problems

Problem 9.1 Prove Lemma 5.2 for general nonlinearities g.
For any strictly positive, increasing and convex function g, it is shown in [5] that (1.1) possesses a minimal *regular* solution for all $\mu < \mu^*$ (the extremal value). The proof takes advantage of the inequality $\Delta\Phi(u) \le \Phi'(u)\Delta u$ which holds for any smooth concave function Φ with bounded first derivative and such that $\Phi(0) = 0$. For the fourth order problem (1.3), this inequality seems out of reach and one should find other issues. On the other hand, the method used in Lemma 5.2 seems to apply only to functions g satisfying (1.4).

Problem 9.2 Find the critical dimensions.
Consider again the second order equation (1.1). For $g(u) = (1+u)^p$, it is shown in [18, Théorème 4] that if either $n \le 10$ or $n \ge 11$ and $p < \overline{p} := \frac{n-2\sqrt{n-1}}{n-4-2\sqrt{n-1}}$, then the extremal solution u^* (corresponding to μ^*) is bounded; on the other hand, it is shown in [7] that when Ω is a ball, if $n \ge 11$ and $p \ge \overline{p}$, then the extremal solution u^* is unbounded. Similarly, for $g(u) = e^u$, it is proved in [18, Théorème 3] that if $n \le 9$ then u^* is bounded, whereas from [7] we know that if Ω is a ball and $n \ge 10$, then u^* is unbounded. We call critical dimension $N(g(u))$ the largest dimension for which the semilinear equation with nonlinearity $g(u)$ admits a regular extremal solution in any domain Ω. Then, we just saw that for second order equations we have

$$N(e^u) = 9 \ , \qquad N((1+u)^p) = +\infty \quad \text{for } 1 < p < \overline{p} \ , \qquad N((1+u)^p) = 10 \quad \text{for } p \ge \overline{p} \ .$$

One is then interested in finding the critical dimensions also for fourth order problems. Two main difficulties arise. First, the counterpart of [7] fails due to the double boundary condition and no interpretation in terms of remainder terms for Hardy inequality is available, see [10]. Second, also the method in [18] fails since the very same arguments as in the proof of [18, Théorème 3] yield

$$\frac{\lambda a}{4}\int_\Omega [e^{(a+1)u_\lambda}-e^{u_\lambda}]+\frac{a^4}{16}\int_\Omega [e^{au_\lambda}|\nabla u_\lambda|^4]\geq \lambda\int_\Omega [e^{(a+1)u_\lambda}-2e^{(a+2)u_\lambda/2}+e^{u_\lambda}]\qquad \forall a>0$$

which allows no conclusion. If one assumes (with no motivation!) that the additional term $\int e^{au_\lambda}|\nabla u_\lambda|^4$ is a lower order term as $\lambda\to\lambda^*$, then we would have boundedness of the extremal solution for $n<20$. Nevertheless, as in [3], we believe that $N(e^u)=12$ for fourth order problems and that the critical dimension does not depend on the boundary condition (Navier or Dirichlet) considered.

Problem 9.3 Prove uniqueness for small λ.
If Ω is conformally contractible, then Reichel [23] proves that the equation in (1.3) under Dirichlet boundary conditions admits a unique smooth solution for small λ and suitable nonlinearities g. Conformally contractible domains are slightly more general than starshaped domains and allow to obtain uniqueness from a strict variational principle by means of a Pohozaev-type identity. Among other tools, the proof is based on a crucial extension argument (see Proposition 8 p.68 in [23]) which is not available under Navier boundary conditions. Is it possible to by-pass this difficulty and to obtain uniqueness for small λ also under Navier boundary conditions?

Problem 9.4 Nonexistence of entire nodal radial solutions of the critical growth equation.
The numerical results of Section 4.3 and Theorem 4.2 suggest the following conjecture: the equation

$$\Delta^2 u=|u|^{8/(n-4)}u\quad \text{in } \mathbf{R}^n \tag{9.1}$$

admits no radial sign changing solutions. Even if this result is well-known for the second order equation $-\Delta u=|u|^{4/(n-2)}u$, this conjecture appears hard to prove due to a lack of Lyapunov functional for (4.10). Let us also mention that (9.1) admits infinitely many (nonradial!) sign changing solutions, see [4].

Problem 9.5 Prove the missing part of Theorem 4.2.
In Theorem 4.2 we prove that there exists $R>0$ such that the problem

$$\begin{cases}\Delta^2 u=|u|^{8/(n-4)}u & \text{for } |x|<R\\ u=0 & \text{for } |x|=R\end{cases}\tag{9.2}$$

admits a positive radial solution. This problem is underdetemined as it lacks one boundary condition. It is well-known [20, 21, 26] that Pohozaev identity enables to exclude the existence of positive solutions of (9.2) complemented with a further boundary condition (either $\frac{\partial u}{\partial\nu}=0$ or $\Delta u=0$ for $|x|=R$). In view of the numerical results of Section 4.3, one should try to prove that the positive radial solution of (9.2) changes sign at $|x|=R$ and then blows up towards $-\infty$ at some finite $|x|>R$.

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