A discontinuous finite element approach for the approximation of free discontinuity problems

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Abstract. We propose an approximation by Γ-convergence for free discontinuity problems by means of discrete functionals defined in spaces of discontinuous finite elements. The adaptivity of triangulation, which is required to have an isotropic surface energy, is parametrized in such a way that remeshing is not necessary. Some applications in image processing and fracture mechanics are considered.

AMS subject classification: 49J45, 65K10
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1 Introduction

Some recent models, coming e.g. from image processing, fracture mechanics, shape optimization, are often formulated as free discontinuity problems. Most of the energies, which are commonly employed, are defined in the space SBV of special functions with bounded variation [3], [18] and can be written in the general form

$$\int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u} g(x,u^+,u^-,\nu)\,d\mathcal{H}^{n-1}\,. \tag{1}$$

The finite element discretizations which have been studied up to now [5], [9], [11], [22] are always defined in the usual spaces of continuous elements. Roughly speaking, we could say that the underlying idea is the same: the bulk energies are treated by standard approximation arguments while the surface energies are represented in term of sharp gradients.

The aim of the present work is to propose an alternative approach, in some sense more natural for free discontinuity problems, based on discontinuous finite elements. Such a technique, to our knowledge, has never been used for this class of problems. More precisely, the discrete functionals are of the form

$$F_h(u) = \sum_{T\in \boldsymbol{T}_h} \int_T f(|\nabla u|)\,dx + \sum_{T\in \boldsymbol{T}_h} \int_{\partial T} s(b_h[u])\phi(\nu)\,d\mathcal{H}^1\,, \tag{2}$$

where $\boldsymbol{T}_h$ is an adaptive triangulation of the domain Ω and and u is a piecewise linear function on $\boldsymbol{T}_h$ which is not necessarily continuous across elements. Choosing properly the function s and the sequence b_h we prove that the functionals F_h Γ-converge in the strong topology of $L^1(\Omega)$ to the energy

$$F(u)=\int_\Omega f(|\nabla u|)\,dx+\int_{S_u}\phi(\nu_u)\,d\mathcal{H}^1\,. \tag{3}$$

Note that in this approach bulk and surface energies are clearly distinguished. In this way the energy balance, which is always crucial in this problems, is computed accurately, without introducing a threshold effect on the gradient which is typical when using continuous elements [24]. Besides, with minor changes in the proofs, the convergence result holds for more general energies of the form

$$\int_\Omega f(u,\nabla u)\,dx+\int_{S_u} g(u^+,u^-)\phi(\nu)\,d\mathcal{H}^{n-1}\,. \tag{4}$$

As usual, the main difficulty in the convergence result is the correct approximation of the surface energy, which in our case depends strongly on the choice of the triangulation $\boldsymbol{T}_h$. Consider for the sake of simplicity the Mumford-Shah functional

$$\int_\Omega |\nabla u|^2\,dx+\mathcal{H}^1(S_u)\,. \tag{5}$$

The adaptivity of the mesh $\boldsymbol{T}_h$ is a crucial condition to have convergence. Indeed, using a family of structured triangulations $\{\boldsymbol{T}_h\}$ it turns out that, despite of the easy numerical implementation, the Γ-limit is of the form

$$\int_\Omega |\nabla u|^2\,dx+\int_{S_u}\psi(\nu)\,d\mathcal{H}^1\,, \tag{6}$$

where the anisotropy function ψ is generated by the geometry of the mesh [22]. What's more, this effect cannot be eliminated using simply a family of unstructured triangulations because in this case the Γ-limit does not exist.

From this considerations it seems natural that for our functionals we can recover the correct Γ-limit only employing adaptive triangulations. In this direction a possible solution comes from [9]: let $\mathcal{T}_h$ be the set of all the regular triangulations $\boldsymbol{T}_h$ of Ω and let the corresponding finite element set $V_h(\Omega,\mathcal{T}_h)$ given by the union of the spaces $V_h(\Omega,\boldsymbol{T}_h)$ for $\boldsymbol{T}_h\in\mathcal{T}_h$. Then, following [9], it's not difficult to prove that the discrete energies F_h, defined in $V_h(\Omega,\mathcal{T}_h)$, Γ-converge to F with respect to the strong topology of $L^1(\Omega)$.

Similar discretizations has already been implemented in [6] for image segmentation and in [4] for a model in fracture mechanics. Nevertheless it is clear that the numerical minimization of (2) over the set $V_h(\Omega,\mathcal{T}_h)$ is not easy because it involves topological changes in the mesh which are not differentiable.

The approximation which is presented in this work is in some sense a compromise between the anisotropic one, defined on structured triangulations, and the previous one, defined on the whole family $\mathcal{T}_h$. Indeed we define in Section 2 a suitable family adaptive triangulations $\boldsymbol{T}_h$ which can be easily parametrized in $\mathbf{R}^n$ (with n depending on h) and which does not involve topological changes. In this way the discrete energy turns out to be differentiable with respect both to the values of the unknown u and to the position of the adaptive vertices. Then we prove in Sections 3 and 4 that this family of adaptive meshes is sufficiently general to ensure that F_h Γ-converges to F in the topology of $L^1(\Omega)$ and that sequences $\{u_{h_n}\}$ equibounded in energy are precompact.

Finally, some applications for image segmentation and fracture mechanics are presented in Section 5.

2 Preliminaries and statement of the convergence result

2.1 Spaces of functions with bounded variation

In this section we briefly recall some definitions and basic results about the spaces of functions with bounded variation which will be useful in the sequel, while we remand the reader to [3] for a deep treatise. Here the set $\Omega \subset \mathbf{R}^2$ is assumed to be open, bounded and with Lipschitz boundary.

Definition 2.1 *For $1 \le p < +\infty$ we define the spaces*

$$SBV^p(\Omega) = \{u \in SBV(\Omega) \,:\, \nabla u \in L^p(\Omega),\ \mathcal{H}^1(S_u) < +\infty\}\,.$$

$$GSBV^p(\Omega) = \{u \in L^1(\Omega) \,:\, ((u \wedge t) \vee -t) \in SBV^p(\Omega) \text{ for every } t \in \mathbf{R}^+\}\,.$$

Definition 2.2 *For $1 \le q \le +\infty$ and $m \ge 1$ let $\mathcal{W}^{m,q}(\Omega)$ be the set of functions $u \in SBV(\Omega)$ such that*

1. *S_u is essentially closed, i.e. $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$*
2. *$\overline{S_u}$ is the union of a finite number of segments*
3. *$u \in W^{m,q}(\Omega \setminus \overline{S_u})$.*

The following lemma, proved in [12] under more general hypothesis, shows the "density" property of the space $\mathcal{W}^{m,q}(\Omega)$ with respect to a large class of energies in $SBV(\Omega)$.

Lemma 2.3 *For $p > 1$, let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$, then there is a sequence $\{w_k\}$ such that $w_k \in \mathcal{W}^{m,\infty}(\Omega)$ for every $m \in \mathbf{N}$ and*

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega)\,, \tag{7}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbf{R}^2)\,, \tag{8}$$

$$\limsup_{k\to\infty} \|w_k\|_\infty \le \|u\|_\infty\,. \tag{9}$$

Moreover for every norm $\phi : \mathbf{R}^2 \to [0, +\infty)$ and for every open set $A \subset \Omega$ we have

$$\limsup_{k\to\infty} \int_{S_{w_k}\cap A} \phi(\nu)\, d\mathcal{H}^1 \le \int_{S_u \cap \bar{A}} \phi(\nu)\, d\mathcal{H}^1\,. \tag{10}$$

Under the additional assumption that $1 < p \le 2$ (using a p-capacity argument) we can assume that for every k the jump set S_{w_k} is a finite union of disjoint segments which do not intersect the boundary $\partial\Omega$.

Using a truncation argument it is easy to prove an analogous result for $u \in GSBV^p(\Omega)$.

2.2 Statement of the main convergence result

Let $\Omega \subset \mathbf{R}^2$ be an open bounded polygonal set. Let $f : \mathbf{R} \to [0, +\infty)$ be a convex function such that for $c_1, c_2 > 0$ and for $p > 1$

$$c_1|z|^p \le f(z) \le c_2(1+|z|^p) \quad \forall z \in \mathbf{R}. \tag{11}$$

Let $\phi : \mathbf{R}^2 \to [0, +\infty)$ be a norm. For $u \in GSBV^p(\Omega)$ we consider the functional

$$F(u) := \int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \phi(\nu_u)\,d\mathcal{H}^1. \tag{12}$$

For every $h > 0$ let $\boldsymbol{B}_h$ be a triangulation of $\mathbf{R}^2$. Let h_T and r_T be respectively the diameter and the internal radius of the triangle T. Assume that the family $\{\boldsymbol{B}_h\}$ is regular, i.e. for every $h > 0$ and for every triangle $T \in \boldsymbol{B}_h$ we have

$$h \le h_T \le C_1 h \qquad r_T \ge C_2 h, \tag{13}$$

for some constants $C_1 > 0$ and $C_2 > 0$ independent of h and T. Moreover, for the sake of simplicity, we assume that Ω can be represented by the union of the triangles $T \in \boldsymbol{B}_h$ contained in Ω. Finally we denote by K_h the set of vertices of the mesh $\boldsymbol{B}_h$.

Remark 2.4 *For every triangle T let ζ^i and θ^i (for $i = 1, \dots, 3$) be respectively the edges and the internal angles. It is well known that the regularity condition (13) implies a lower and an upper bound on the amplitude of the internal angles (the Zlamal's condition). Thus there exist $\theta_0 > 0$, independent of h, such that $\theta_0 \le \theta^i \le \pi - \theta_0$. Moreover it is not difficult to see that there exists a constant $C_3 > 0$ such that for every $T \in \boldsymbol{B}_h$ and for every edge ζ^i we have $C_3 h \le \mathcal{H}^1(\zeta^i)$.*

Let $a_h \le \frac{1}{2}$ be a positive infinitesimal sequence. For every $h > 0$ let $\mathcal{T}_h = \{\boldsymbol{T}_h\}$ the family of the triangulations nested in $\boldsymbol{B}_h$ and defined in the following way (see Figure 1): every element $T \in \boldsymbol{B}_h$ is divided into four sub-elements $T_1, \dots, T_4$, taking on every edge $[x_1x_2]$ of T a knot x' of $\boldsymbol{T}_h$ which satisfies the constraint

$$x' = tx_1 + (1-t)x_2 \qquad \text{for } a_h \le t \le 1 - a_h. \tag{14}$$

We will say that these vertices are adaptive, since for given h their position along the edge is variable and in practice it will be chosen in an optimal way. Finally, for every mesh $\boldsymbol{T}_h$, we denote by $\boldsymbol{E}_h$ the set of edges whose extrema are both adaptive vertices (the ones represented by dashed lines in Figure 1).

For every $h > 0$ and every $\boldsymbol{T}_h \in \mathcal{T}_h$, let $W_h(\Omega, \boldsymbol{T}_h)$ be the finite element space of discontinuous functions which are affine on every sub-element $T \in \boldsymbol{T}_h$ and which can have discontinuities only along the edges belonging to $\boldsymbol{E}_h$. Then our *finite element set* $V_h(\Omega)$ will be the union of the spaces $W_h(\Omega, \boldsymbol{T}_h)$ for $\boldsymbol{T}_h \in \mathcal{T}_h$.

For $u \in W_h(\Omega, \boldsymbol{T}_h)$ and $\zeta \in \boldsymbol{E}_h$, we denote by $[u]_\zeta$ the vector $(u_1^+ - u_1^-, u_2^+ - u_2^-)$, where $u_i^\pm$ are the values of u in the extrema of ζ, and we denote by $|[u]_\zeta|$ any norm applied to $[u]_\zeta$.

Theorem 2.5 *Let b_h be a positive sequence such that*

$$h a_h b_h \to +\infty \tag{15}$$

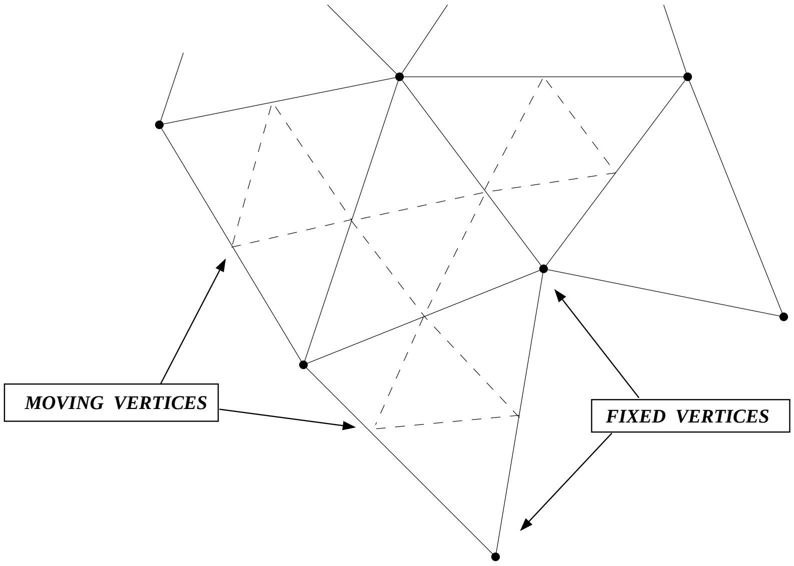

Figure 1: The nested triangulations $\boldsymbol{B}_h$ and $\boldsymbol{T}_h$.

and let $s : [0, +\infty) \to [0, 1]$ *be a non-decreasing continuous function such that*

$$s(0) = 0 \qquad \lim_{t \to +\infty} s(t) = 1\,. \tag{16}$$

For $u \in V_h(\Omega)$ *let*

$$F_h(u) = \sum_{T \in \boldsymbol{T}_h} \int_T f(|\nabla u|)\, dx + \sum_{\zeta \in \boldsymbol{E}_h} \int_\zeta s(b_h |[u]_\zeta|) \phi(\nu)\, d\mathcal{H}^1\,, \tag{17}$$

and let $F_h(u) = +\infty$ *otherwise in* $L^1(\Omega)$*. Then* F_h Γ*-converges, in the strong topology of* $L^1(\Omega)$*, to the functional*

$$F(u) = \int_\Omega f(|\nabla u|)\, dx + \int_{S_u} \phi(\nu_u)\, d\mathcal{H}^1\,, \tag{18}$$

if $u \in GSBV^p(\Omega) \cap L^1(\Omega)$ *and* $F(u) = +\infty$ *otherwise in* $L^1(\Omega)$*.*

The proof will follow from Proposition 3.7 and 4.2.

Remark 2.6 *Note that for numerical reasons it is better not to require any condition on the behavior of* $s(t)/t$ *for* $t \to 0$*, which are usually needed for compactness and lower semicontinuity. In some sense, as it will be clear later, this weaker choice of the hypothesis is possible thank to the discrete setting and thank to condition (15). Besides, when* $\lim_{t \to 0^+} s(t)/t = +\infty$ *the same convergence result holds true replacing (16) with* $b_h \to +\infty$*.*

Remark 2.7 *Note that the adaptive triangulations* $\boldsymbol{T}_h$ *are defined in such a way that there are no changes in its topology. In this way remeshing is not necessary and the implementation of the problems turns out to be easier and faster. Indeed, for h fixed, the minimization problem over the adaptive triangulations is in practice a minimization* *problem in $\mathbf{R}^m$ (where m is the number of edges in $\boldsymbol{E}_h$) because the coordinates of the adaptive vertices are easily parametrized in terms of the convex combination (14).*

Moreover note that the family $\boldsymbol{T}_h$ is not in general regular in the sense of (13). This fact does not affect convergence because, as a matter of fact, the bulk energies, which require regularity, are approximated by the mesh $\boldsymbol{T}_h$; indeed the motion of adaptive vertices, that may cause instabilities in the gradients, is concentrated only in a small neighborhood of the discontinuity lines whose measure is infinitesimal.

Proposition 2.8 *Let $u_h \in V_h(\Omega)$ such that $F_{h_n}(u_{h_n}) + \|u_{h_n}\|_\infty \leq C < \infty$, then u_h is strongly precompact in $L^1(\Omega)$.*

3 Γ-limsup inequality

Lemma 3.1 *Let $\{\boldsymbol{B}_h\}$ be a regular family of triangulations of $\mathbf{R}^2$. Let S be a segment with $\mathcal{H}^1(S) < +\infty$ and let $\mathbf{S}_h = \{T \in \boldsymbol{B}_h : T \cap S \neq \emptyset\}$, then there is a constant C such that, for h sufficiently small, $\#(\mathbf{S}_h) \leq C\mathcal{H}^1(S)/h$.*

Proof. By the regularity of the mesh, for every element T we have $|T| \geq \pi r_T^2 \geq \pi C h^2$. Let C_1 be the constant appearing in (13) and let $U(S, 2C_1 h) = \{x \in \Omega : d(x, S) < 2C_1 h\}$, then $U(S, 2C_1 h)$ contains all the elements $T \in \mathbf{S}_h$. Being S a segment

$$\frac{|U(S, 2C_1 h)|}{h} \leq 4C_1 \mathcal{H}^1(S) + O(h)\,.$$

By the lower bound on $|T|$, it follows that

$$\#(\mathbf{S}_h) \leq \frac{|U(S, 2C_1 h)|}{\pi C h^2} \leq C\frac{\mathcal{H}^1(S)}{h} + O(1)\,,$$

which gives the required estimate for h sufficiently small. ■

Let $S \subset \Omega$ be a segment with unit normal ν and let $\mathbf{S}_h$ be as in the previous lemma. For every positive h we define a discretization of S in $\mathcal{T}_h$ by means of a piecewise linear curve S_h which can be represented by the edges $\zeta \in \boldsymbol{E}_h$. To simplify the definitions we consider the line $\tilde{S} \subset \mathbf{R}^2$ which contains S and we assume that $\tilde{S}$ does not intersects any knot of the mesh $\boldsymbol{B}_h$. For every $T \in \mathbf{S}_h$ let ζ be an edge of ∂T such that $(\tilde{S} \cap \zeta) = \{p\}$. Then $p = x_h^i + t(x_h^j - x_h^i)$, where x_h^i and x_h^j are the extrema of ζ and $t \in (0,1)$. Let p_h be defined by

$$p_h = \begin{cases} x_h^i + a_h(x_h^j - x_h^i) & \text{if } 0 < t \leq a_h \\ p & \text{if } a_h \leq t \leq 1 - a_h \\ x_h^i + (1 - a_h)(x_h^j - x_h^i) & \text{if } 1 - a_h \leq t < 1. \end{cases} \tag{19}$$

Roughly speaking p_h is the projection of p on the segment $[x_h^i + a_h(x_h^j - x_h^i), x_h^i + (1 - a_h)(x_h^j - x_h^i)]$. In this way, as $\#(\tilde{S} \cap \partial T) = 2$, for every $T \in \mathbf{S}_h$ we have two points p_h and q_h (contained in ∂T) and then we can define $S_h \cap T$ as the segment $[p_h\, q_h]$. The set S_h will be clearly given by the union of the segments $S_h \cap T$ for $T \in \mathbf{S}_h$ (it's clear that with this definition S_h is a piecewise linear curve).

Lemma 3.2 *Let S be a segment, S_h (for every $h > 0$) its discretization in $\mathcal{T}_h$, $T \in \mathbf{S}_h$ such that $\#(\partial T \cap S) = 2$. Then*

$$|\mathcal{H}^1(S \cap T) - \mathcal{H}^1(S_h \cap T)| = O(a_h h)\,, \tag{20}$$

$$\Big|\int_{S\cap T}\phi(\nu)\,d\mathcal{H}^1-\int_{S_h\cap T}\phi(\nu_h)\,d\mathcal{H}^1\Big|=O(a_h h)\,. \tag{21}$$

Proof. Let p and q be the extrema of $S\cap T$, p_h and q_h be the ones of $S_h\cap T$. Note that $d(p,p_h)\le Ca_h h$, then from

$$\mathcal{H}^1(S_h\cap T)\le d(p_h,p)+\mathcal{H}^1(S\cap T)+d(q,q_h)\,,$$

$$\mathcal{H}^1(S\cap T)\le d(p,p_h)+\mathcal{H}^1(S_h\cap T)+d(q_h,q)\,,$$

it follows easily that

$$|\mathcal{H}^1(S_h\cap T)-\mathcal{H}^1(S\cap T)|\le Ca_h h=O(a_h h)\,.$$

Denote by s and s_h the vectors corresponding to $S\cap T$ and $S_h\cap T$ and note that

$$\|s\|_\phi=\int_{S\cap T}\phi(\nu)\,d\mathcal{H}^1$$

is a norm. Then by the equivalence of the norms in $\mathbf{R}^2$ it follows easily that

$$\Big|\int_{S_h\cap T}\phi(\nu_h)\,d\mathcal{H}^1-\int_{S\cap T}\phi(\nu)\,d\mathcal{H}^1\Big|\le\Big|\|s_s\|_\phi-\|s\|_\phi\Big|\le\|s_s-s\|_\phi\le C|s_h-s|\,,$$

which gives the second inequality. ■

Considering the estimate on the cardinality $\#(\mathbf{S}_h)$, the previous Lemma and the behavior in the extrema of each segment, the following global error estimates come easily.

Lemma 3.3 *Let S be a segment and let S_h (for every $h>0$) be its discretization in $\mathcal{T}_h$, then*

$$|\mathcal{H}^1(S)-\mathcal{H}^1(S_h)|=O(a_h+h)\,, \tag{22}$$

$$\Big|\int_S\phi(\nu)\,d\mathcal{H}^1-\int_{S_h}\phi(\nu_h)\,d\mathcal{H}^1\Big|=O(a_h+h)\,. \tag{23}$$

For the sake of clearness we state the following easy lemma, which we will use in the sequel.

Lemma 3.4 *Let $L:\mathbf{R}^2\to\mathbf{R}^2$ be an affine application and let ζ_i and ζ_j be unit vectors such that* $\arccos(\langle\zeta_i,\zeta_j\rangle)\ge\theta_0>0$*. Then*

$$|\nabla L|\le C\Big(|\nabla L\,\zeta_i|+|\nabla L\,\zeta_j|\Big)\,, \tag{24}$$

where C depends only on θ_0.

Lemma 3.5 *Let $S\subset\Omega$ be a segment. Let $T\in\boldsymbol{B}_h$ such that $\#(\partial T\cap S)=2$ and let S_h be the approximation of S in $\mathcal{T}_h$. Denote by L the piecewise affine map of T onto itself which keeps the natural correspondence between the subelements T_k and T_k' (as in Figure 2). Then $|\nabla L|\le C$, where C does not depend on h.*

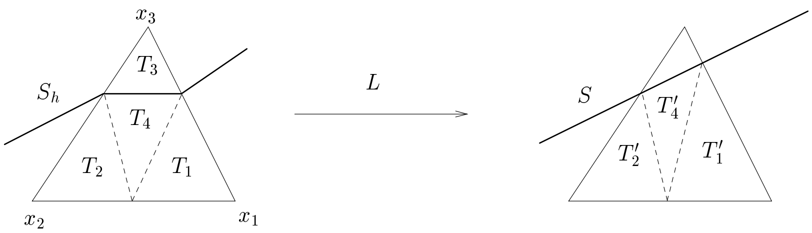

Figure 2: The piecewise affine map of Lemma 3.5.

Proof. Consider first the behavior of the triangles T_k for $k = 1, \dots, 3$ and take a system of coordinates with origin in x_k. We denote by ζ_k^i, for $i = 1, 2$, the unit vectors corresponding to the edges of T_k which contain x_k. Then $L_k = L_{|T_k}$ is a linear map with eigenvectors ζ_k^i and eigenvalues λ_k^i. Considering the three cases in (19) we get either $|\lambda_k^i| \le 1$ or

$$|\lambda_k^i| \le \frac{C_1 h}{C_3 h(1 - a_h)} \le C\,.$$

Since for $i = 1, 2$ we have $|\nabla L_k \zeta_k^i| = |L_k\, \zeta_k^i| = |\lambda_k^i|$ by Lemma 3.4 we have $|\nabla L_k| \le C$, where C depends only on θ_0.

Finally, since $L_4 = L_k$ on $\partial T_4 \cap \partial T_k$ (for $k = 1, 2, 3$), it is sufficient to choose a couple of edges of ∂T_4 which form an angle greater than or equal to θ_0. Then by Lemma 3.4 we have again $|\nabla L_4| \le C$. ∎

Proposition 3.6 *Let F and F_h be as in Theorem 2.5. Let $u \in \mathcal{W}^{2,\infty}(\Omega)$, then there exists $\{u_h\}$ such that $u_h \in V_h(\Omega)$ and*

$$u_h \longrightarrow u \text{ in } L^1(\Omega)\,, \tag{25}$$

$$\limsup_{h \to 0} F_h(u_h) \le \int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \phi(\nu_u)\,d\mathcal{H}^1 = F(u)\,. \tag{26}$$

Proof. Step 1. First of all we must take care of the fact that S_u may contain a knot of the mesh. Let $Q \subset \mathbf{R}^2$ be an open square such that $\Omega \subset\subset Q$. Since $\partial\Omega$ is polygonal and $u \in W^{2,\infty}(\Omega \setminus \overline{S_u})$ by the standard extension and embedding results for Sobolev spaces, see e.g. [25], there exists $v \in SBV(Q)$ such that $v = u$ in Ω, $\overline{S_v}$ is the union of segments S_i for $i = 1, \dots, m$ and $v \in W^{2,\infty}(Q \setminus \overline{S_v})$.

Let τ_i denote the unit vectors parallel to S_i. Let $\xi \in \mathbf{S}^1$ such that $\xi \ne \pm\tau_i$ for every $i = 1, \dots, m$. Denote by K_h the set of knots of the mesh $\boldsymbol{B}_h$ and define the set $Z_h = \{x_h \in K_h : x_h \notin \overline{S_v}\}$. Clearly $d(Z_h, \overline{S_v}) > 0$. Then for some $0 < t_h < C_1 h$ we have $(\overline{S_v} + t_h\xi) \cap K_h = \emptyset$. We will denote the function $v(x - t_h\xi)$ by $v^t(x)$, the set $\overline{S_{v^t}} = (\overline{S_v} + t_h\xi)$ by S^t and the sets $(S_i + t_h\xi)$ by S_i^t .

First of all, for every $h > 0$, we define a mesh $\boldsymbol{T}_h \in \mathcal{T}_h$ which fits to the discontinuity set S^t. Let $I = \{x \in \Omega : x = S_i^t \cap S_j^t\}$. For the sake of simplicity it will be useful to suppose (without loss of generality) that the segments intersect in their extrema. Moreover we assume that h is sufficiently small in such a way that we can localize the constructions.

Consider $p_k \in I$ and assume, for simplicity of notations, that it is the intersection point of the sets S_i^t for $1 \le i \le j$. It is easy to see that it is possible to choose $r_k > 0$

in such a way that for every $h > 0$ the sets $U(S_i^t, C_1 h) \setminus B(p_k, r_k h)$ are pairwise disjoint, for $1 \leq i \leq j$. We will denote by $B_{k,h}$ the ball $B(p_k, r_k h)$. Let $C_{k,h}$ be the union of elements $T \in \boldsymbol{B}_h$ such that $T \cap B_{k,h} \neq \emptyset$. Consider the segments $S_i^t \setminus C_{k,h}$ and let $S_{i,h}^t$ be their approximations in $\mathcal{T}_h$. Then it is possible to find a polygonal region $P_{k,h} \subset C_{k,h}$ (see Figure 3) in such a way that it contains p_k, $\partial P_{k,h}$ is representable in $\mathcal{T}_h$ and connects the sets $S_{1,h}^t, \ldots, S_{j,h}^t$.

This construction gives the position of some vertices of $\boldsymbol{T}_h$, all the others will be taken as $x' = \frac{1}{2}x^i + \frac{1}{2}x^j$ (if x' belongs to the edge $[x^i\, x^j]$).

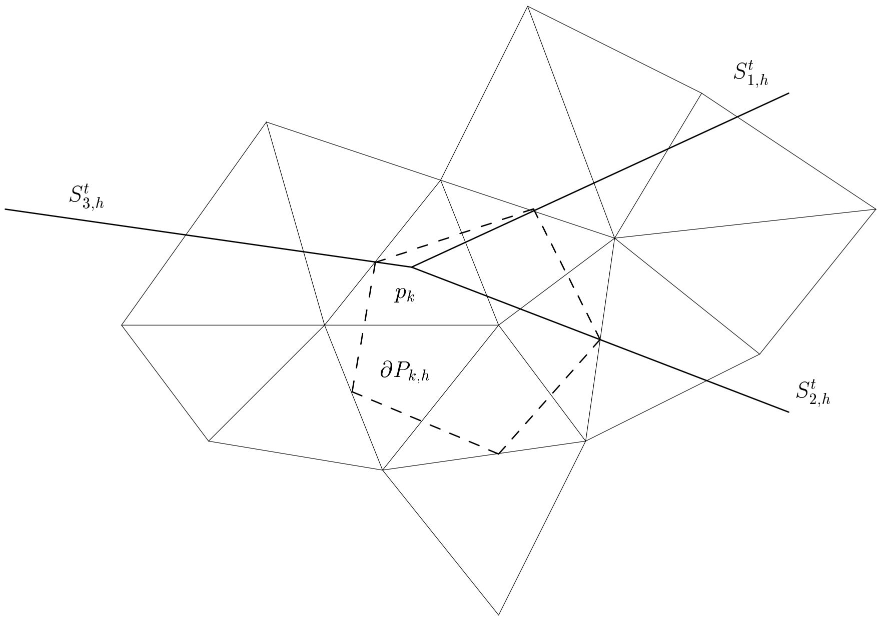

Figure 3: The construction around the point p_k.

Step 2. Now we can define the function u_h. Let Ω_h^s be the union of the elements $T \in \boldsymbol{B}_h$ such that $T \subset C_{k,h}$ or $T \cap S_{k,h}^t \neq \emptyset$ for some k. We denote the set $\Omega \setminus \Omega_h^s$ by Ω_h^c. Note that $|\Omega_h^s| \leq Ch\mathcal{H}^1(S_v) + \#(I)Ch^2$ and thus $|\Omega_h^s| \to 0$.

We consider first the set Ω_h^c. Since v^t is continuous in Ω_h^c we can define u_h as the Lagrange interpolation of v^t in $\boldsymbol{T}_h$. Note that, by its definition, the restriction of $\boldsymbol{T}_h$ to the set Ω_h^c is regular in the sense of (13). Hence, by a classical result in the theory of finite elements (see for instance [10] Theorem 3.1.6), we know that there exists a constant C, independent of h, such that, for every element $T \in \boldsymbol{T}_h$ with $T \subset \Omega_h^c$, and for every $p, q \in [1, \infty]$ we have

$$\|v^t - v_h\|_{m,q,T} \leq C|T|^{\frac{1}{q} - \frac{1}{p}} h^{2-m} |v^t|_{2,p,T} \,.$$

In particular for $p = \infty$, $q = 1$ and $m = 0$ we have

$$\|v^t - v_h\|_{1,T} \leq Ch^4 |v^t|_{2,\infty,T} \,, \tag{27}$$

while for $p = \infty$, $q = \infty$ and $m = 1$ we have

$$\|v^t - v_h\|_{1,\infty,T} \leq Ch |v^t|_{2,\infty,T} \,, \tag{28}$$

for a suitable constant C independent of h and T.

Consider now the set Ω_h^s. Let $T \in \boldsymbol{B}_h$ such that $T \cap S_{j,h}^t \neq \emptyset$. Consider the notations of Lemma 3.5 and Figure 2. For $k = 1, \dots, 4$, as $T_k' \subset (\Omega \setminus \overline{S^t})$ we have $v^t \in W^{2,\infty}(T_k')$. Thus we can define an auxiliary affine function $v_h : T_k' \to \mathbf{R}$ as the Lagrange interpolation of v^t in T_k'. It's easy to see that

$$\|v_h\|_{\infty,T_k'} \le \|v^t\|_{\infty,(\Omega\setminus\overline{S^t})},$$

$$\|\nabla v_h\|_{\infty,T_k'} \le \|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})}.$$

Then we define $u_h : T_k \to \mathbf{R}$ as $u_h(x) = v_h(L_k x)$. By Lemma 3.5 it follows that

$$\|\nabla u_h\|_{\infty,T_k} \le |\nabla L| \, \|\nabla v_h\|_{\infty,T_k'} \le C\|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})}.$$

Finally let $T \subset C_{k,h}$ and consider the sub-elements $T_k \in \boldsymbol{T}_h$. If $T_k \subset P_{k,h}$ we define $u_h \equiv 0$, while if $T_k \not\subset P_{k,h}$ we define u_h as the Lagrange interpolation of v^t in T_k. As before we have

$$\|\nabla u_h\|_{\infty,T_k'} \le \|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})}.$$

Step 3. Let us see that the sequence u_h verifies (25) and (26). It's clear that $v^t \to u$ strongly in $L^1(\Omega)$, moreover by (27) it follows that

$$\begin{aligned}
\int_\Omega |u_h - v^t| \, dx &\le \int_{\Omega_h^s} |u_h - v^t| \, dx + \int_{\Omega_h^c} |u_h - v^t| \, dx \\
&\le 2\,|\Omega_h^s| \, \|v^t\|_{\infty,\Omega} + \sum_{T \subset \Omega_h^c} \int_T |u_h - v^t| \, dx \\
&\le O(h) + \sum_{T \subset \Omega_h^c} C h^4 |v^t|_{2,\infty,T} \\
&\le O(h) + C h^2 |v|_{2,\infty,(\Omega\setminus\overline{S_v})},
\end{aligned}$$

which implies (25). The proof of the Γ-limsup inequality follows from

$$\limsup_{h\to 0} \int_\Omega f(|\nabla u_h|) \, dx = \int_\Omega f(|\nabla u|) \, dx, \tag{29}$$

$$\limsup_{h\to 0} \sum_{\zeta\in \boldsymbol{E}_h} \int_\zeta s(b_h|[u_h]_\zeta|)\phi(\nu) \, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu) \, d\mathcal{H}^1. \tag{30}$$

Let $1_{\Omega_h^c}$ be the indicator function of Ω_h^c. By the previous step, for every $T \in \boldsymbol{T}_h$ we have $\|\nabla u_h\|_{\infty,T} \le \|\nabla v^t\|_{\infty,T} < +\infty$ and thus by (11) the function $f(|\nabla u_h|)$ is bounded from above. Since $|\Omega_h^s| \to 0$, by (28) it follows that $f(|\nabla u_h|)$ converges pointwise to $f(|\nabla u|)$. Thus by dominated convergence we have

$$\lim_{h\to 0} \sum_{T \in \boldsymbol{T}_h} \int_T f(|\nabla u_h|) \, dx = \int_\Omega \lim_{h\to 0} f(|\nabla u_h|) \, dx = \int_\Omega f(|\nabla u|) \, dx. \tag{31}$$

Moreover, as $s(b_h[u_h]_\zeta) \le 1$ we have

$$\begin{aligned}
\sum_{\zeta \in \boldsymbol{E}_h} \int_\zeta s(b_h|[u_h]_\zeta|)\phi(\nu) \, d\mathcal{H}^1 &\le \sum_{j=1}^m \int_{S_{j,h}^t} \phi(\nu) \, d\mathcal{H}^1 + C \sum_{p_k\in I} \mathcal{H}^1(\partial P_{k,h}) \\
&\le \sum_{j=1}^m \int_{S_{j,h}} \phi(\nu_h) \, d\mathcal{H}^1 + O(h).
\end{aligned} \tag{32}$$

Then by Lemma 3.3

$$\limsup_{h\to 0}\sum_{\zeta\,\in\,\boldsymbol{E}_h}\int_\zeta s(b_h|[u_h]_\zeta|)\phi(\nu_\zeta)\,d\mathcal{H}^1 \le \sum_{j=1}^m \limsup_{h\to 0}\int_{S_{j,h}}\phi(\nu_h)\,d\mathcal{H}^1$$
$$\le \int_{S_u}\phi(\nu)\,d\mathcal{H}^1$$

and the proof is concluded. ■

Proposition 3.7 *For every $u \in GSBV^p(\Omega)$ we have $F''(u) \le F(u)$.*

Proof. By Lemma 2.3 we have a sequence $w_k \in \mathcal{W}^{2,\infty}(\Omega)$ such that

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega) \tag{33}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^p(\Omega,\mathbf{R}^2) \tag{34}$$

$$\limsup_{k\to 0}\int_{S_{w_k}}\phi(\nu)\,d\mathcal{H}^1 \le \int_{S_u}\phi(\nu)\,d\mathcal{H}^1\,. \tag{35}$$

Then by (34) and by a standard result on Nemitskii operators we have

$$\int_\Omega f(|\nabla w_k|)\,dx \longrightarrow \int_\Omega f(|\nabla u|)\,dx. \tag{36}$$

By Proposition 3.6, the lower-semicontinuity of $F''(u)$, (35) and (36) it follows that

$$F''(u) \le \liminf_{k\to+\infty} F''(w_k) \le \limsup_{k\to+\infty} F(w_k) \le F(u)\,,$$

which concludes the proof. ■

4 Γ-liminf inequality and compactness

Lemma 4.1 *Let $s : [0,+\infty) \to [0,1]$ be a non-decreasing continuous function such that*

$$s(0) = 0 \qquad \lim_{t\to+\infty} s(t) = 1\,, \tag{37}$$

then there exist a concave function $g : [0,+\infty) \to [0,1]$ verifying

$$\lim_{t\to 0^+}\frac{g(t)}{t} = +\infty \qquad \lim_{x\to+\infty} g(t) = 1\,, \tag{38}$$

and a constant $C > 0$ such that $g(t) \le s(t)$ for $t \ge C$.

Proof. For $k \ge 1$ let $z_k \in \mathbf{R}^+$ such that $s(z_k) \ge 1 - 1/(k+1)$ and $\lim_{k\to\infty} z_k = +\infty$. Let g_0 be a piecewise constant function defined by $g_0(t) = 1 - 1/(k+1)$ for $t \in (z_k, z_{k+1}]$ and $g_0(t) = 0$ for $t \in [0, z_1]$. Since $s(t)$ is non-decreasing we have easily $g_0 \le s$ for $t \ge 0$. Let g_1 be a piecewise linear function which for $k \ge 2$ interpolates the values $g_0(z_k) = 1 - 1/k$ in the points z_k, while in the interval $[0, z_2]$ it simply interpolates a value $0 < \eta < g_0(z_2)$ in zero and $g_0(z_2)$ in z_2. This function is increasing, it satisfies (38) and $g_1(t) \le s(t)$ for $t \ge z_2$ but in general it is not concave. Nevertheless it is easy to "stretch" it into a concave function, indeed it is sufficient to translate recursively the points z_k in such a way that the slope of g_1 decreases. It is simple to check that the resulting function has all the required properties. ■

Proposition 4.2 *Let h_n be a positive sequence such that $\lim_{n\to\infty} h_n = 0$. Let $u \in L^1(\Omega)$ and $u_{h_n} \in V_{h_n}(\Omega)$ such that $u_{h_n} \to u$ in $L^1(\Omega)$. Then*

$$F(u) \leq \liminf_{n\to\infty} F_{h_n}(u_{h_n}) \,. \tag{39}$$

Proof. For the sake of simplicity we abbreviate the subscripts h_n by n and we denote the set $S_{u_{h_n}}$ by S_n. It is not restrictive to assume that $|[u]_\zeta| = |u_1^+ - u_1^-| + |u_2^+ - u_2^-|$ and that $\liminf_{n\to\infty} F_n(u_n) < +\infty$. Then there exists a subsequence h_j of h_n, such that $\sup_j F_j(u_j) < +\infty$ and

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{j\to\infty} F_j(u_j) \,. \tag{40}$$

Moreover, up to a further subsequence (not relabelled), the limits

$$\lim_{j\to\infty} \int_\Omega f(|\nabla u_j|)\, dx \quad \text{and} \quad \lim_{j\to\infty} \int_{S_j} s(b_j|[u_j]|)\phi(\nu)\, d\mathcal{H}^1$$

exist and clearly (40) is still valid. In order to apply the compactness and lower semicontinuity theorem of Ambrosio we would like to have an estimate of the form

$$\int_\Omega f(|\nabla v_j|)\, dx + \int_{S_j} \psi(b_j|[v_j]|)\phi(\nu)\, d\mathcal{H}^1 \leq F_j(u_j) \,,$$

where ψ satisfies (38) and v_j is a sequence of function such that $v_j \to u$ in $L^1(\Omega)$. Since the function given by Lemma 4.1 satisfies $g(x) \leq s(x)$ only for $x \geq C$ it is not possible to choose simply $v_j = u_j$ and $\psi(t) = g(t)$. Thus, let $C_\delta > C$, where C is the constant appearing in Lemma 4.1, and let

$$\boldsymbol{E}_{h_j}^\sharp := \{\zeta \in \boldsymbol{E}_h \,:\, b_j|[u_j]_\zeta| > C_\delta\} \qquad \boldsymbol{E}_{h_j}^\flat := \{\zeta \in \boldsymbol{E}_h \,:\, b_j|[u_j]_\zeta| \leq C_\delta\}$$

be respectively the set of "essential" (sharp) and "removable" (small) discontinuities. For every adaptive knot x', denote by ζ^k (for $k = 1, \ldots, 4$) the adaptive edges containing the knot x' and by u_j^i for $i = 1, \ldots, 4$ the values of the function u_j in x'. The new sequence v_j will be defined modifying each function u_j in such a way that the small discontinuities will be removed. The precise definition will be split in four cases.
Case 1: all the edges ζ^k belong to $\boldsymbol{E}_{h_j}^\flat$. Let $v_j^i(x') = \frac{1}{4}(\sum_{l=1}^4 u_j^l)$ for every $i = 1, \ldots, 4$, in such a way that continuity is imposed in the knot x'. It's easy to see that for every index i we have

$$b_j|u_j^i - v_j^i| \leq C'_\delta \,, \tag{41}$$

where $C'_\delta = C_\delta$.
Case 2: only one edge belongs to $\boldsymbol{E}_{h_j}^\flat$. Then $v_j^i(x')$ will be defined as in the previous case, but this time (41) holds with $C'_\delta = 2C_\delta$.
Case 3: exactly two edges belong to $\boldsymbol{E}_{h_j}^\flat$. The edges $\zeta^k \in \boldsymbol{E}_{h_j}^\sharp$ divide the values u_j^i into two subsets, indexed by $1 \leq i \leq m$ and $m+1 \leq i \leq 4$, in such a way that for $1 \leq i_1 \leq i_2 \leq m$ or $m+1 \leq i_1 \leq i_2 \leq 4$ we have $b_j|u_j^{i_1} - u_j^{i_2}| \leq C_\delta$. Since the sharp discontinuities are not to be removed, we impose a onesided continuity condition as follows: $v_j^i(x') = \frac{1}{m}(\sum_{l=1}^m u_j^l)$ for $i = 1, \ldots, m$ and $v_j^i(x') = \frac{1}{4-m}(\sum_{l=m+1}^4 u_j^l)$ for $i = m+1, \ldots, 4$. Then (41) holds again for $C'_\delta = C_\delta$.
Case 4: since there is only edge belonging to $\boldsymbol{E}_{h_j}^\flat$ the values u_j^i are now divided into three subsets, reasoning as in the previous case we obtain a function v_j such that (41) holds again for $C'_\delta = C_\delta$.

(It's clear that the remaining case does not require any change in the function u_j).

Let us consider now the behavior of the discontinuities for v_j. If $\zeta \in \boldsymbol{E}_{h_j}^{\flat}$ by the previous construction it follows that $[v_j]_\zeta = 0$. If $\zeta \in \boldsymbol{E}_{h_j}^{\sharp}$ then $s(C_\delta) \leq s(b_j|[u_j]_\zeta|)$, thus for $Q_\delta = 1/s(C_\delta)$ we have $1 \leq Q_\delta s(C_\delta) \leq Q_\delta s(b_j|[u_j]_\zeta|)$. Then $g(b_j|[v_j]_\zeta|) \leq 1 \leq Q_\delta s(b_j|[u_j]_\zeta|)$ and thus

$$\frac{g(b_j|[v_j]_\zeta|)}{Q_\delta} \leq s(b_j|[u_j]_\zeta|)\,.$$

It follows that for $\psi(t) = s(C_\delta)g(t)$ we get

$$\sum_{\zeta \in \boldsymbol{E}_{h_j}} \int_\zeta \psi(b_j|[v_j]_\zeta|)\phi(\nu)\,d\mathcal{H}^1 \leq \sum_{\zeta \in \boldsymbol{E}_{h_j}} \int_\zeta s(b_j|[u_j]_\zeta|)\phi(\nu)\,d\mathcal{H}^1\,. \tag{42}$$

It remains to give some error estimates for the bulk energies. It easy to check that by (41) we have $\|u_j - v_j\|_\infty \leq C'_\delta/b_j$ for $C'_\delta = 2C_\delta$ and thus $v_j \to u$ in $L^1(\Omega)$.

For every $T \in \boldsymbol{B}_{h_j}$ let x^i be its vertices and denote by $\hat{u}_j^i$ the shape function relative to the knot x^i. If $x^i \in K_{h_j}$ we have

$$|\nabla \hat{u}_j^i| \leq \frac{1}{h_j a_j \cos\frac{\pi-\theta_0}{2}} \leq \frac{C_\delta^0}{h_j a_j}\,,$$

otherwise (i.e. if x^i is adaptive)

$$|\nabla \hat{u}_j^i| \leq \frac{1}{h_j a_j \sin\theta_0} \leq \frac{C_\delta^0}{h_j a_j}\,,$$

where C_δ^0 depends only on θ_0 and δ. Then for every $T \in \boldsymbol{B}_{h_j}$

$$|\nabla u_j - \nabla v_j| \leq \sum_{i=1}^{3} |u_j(x^i) - v_j(x^i)||\nabla \hat{u}_j^i|\,,$$

from the previous estimates and (15) we get

$$\lim_{j\to\infty} \|\nabla u_j - \nabla v_j\|_\infty \leq 3 \lim_{j\to\infty} \frac{C'_\delta}{b_j}\frac{C_\delta^0}{h_j a_j} \leq \lim_{j\to\infty} \frac{C''_\delta}{h_j a_j b_j} = 0\,.$$

In particular by (11) ∇v_j is bounded in $L^p(\Omega, \mathbf{R}^2)$ and we can write

$$\sum_{T \in \boldsymbol{B}_{h_j}} \int_T |\nabla v_j|^p\,dx + \sum_{\zeta \in \boldsymbol{E}_{h_j}} \int_\zeta \psi(b_j|[v_j]_\zeta|)\phi(\nu)\,d\mathcal{H}^1 \leq C < \infty\,.$$

For $m \in \mathbf{N}$ assume that j is sufficiently large, so that $b_j > m$. Since the function $\psi(t)$ is increasing, we can write

$$\int_{S_j} \psi(m|[v_j]|)\phi(\nu)\,d\mathcal{H}^1 \leq \int_{S_j} \psi(b_j|[v_j]|)\phi(\nu)\,d\mathcal{H}^1 \leq C\,. \tag{43}$$

Then by Ambrosio's theorem we have a subsequence v_k of v_j and a function $v \in GSBV(\Omega)$ such that v_k converges to v in measure, ∇v_k converges to ∇v weakly in $L^p(\Omega, \mathbf{R}^2)$ and

$$\int_{S_v} \psi(m|[v]|)\phi(\nu_v)\,d\mathcal{H}^1 \leq \liminf_{k\to\infty} \int_{S_k} \psi(m|[v_k]|)\phi(\nu)\,d\mathcal{H}^1\,.$$

As $u_n \to u$ in $L^1(\Omega)$, the functions v and u must coincide. Since $(\nabla v_k - \nabla u_k) \to 0$ in $L^\infty(\Omega, \mathbf{R}^2)$ we have $\nabla u_k \to \nabla u$ weakly in $L^p(\Omega, \mathbf{R}^2)$ and then

$$\int_\Omega f(|\nabla u|)\,dx \le \liminf_{k\to\infty} \int_\Omega f(|\nabla u_k|)\,dx = \lim_{j\to\infty} \int_\Omega f(|\nabla u_j|)\,dx\,. \tag{44}$$

Being

$$\lim_{m\to+\infty} \psi(m|t|) = \begin{cases} s(C_\delta) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0, \end{cases}$$

by monotone convergence we have

$$\begin{aligned} s(C_\delta)\int_{S_u} \phi(\nu)\,d\mathcal{H}^1 &\le \sup_m \int_{S_u} \psi(m|[u]|)\phi(\nu)\,d\mathcal{H}^1 \\ &\le \lim_{j\to\infty} \int_{S_j} \psi(b_j|[u_j]|)\phi(\nu)\,d\mathcal{H}^1\,. \end{aligned} \tag{45}$$

Passing to the limit for $C_\delta \to +\infty$ and joining the previous inequalities we get

$$\begin{aligned} G(u) &= \int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \phi(\nu)\,d\mathcal{H}^1 \\ &\le \lim_{j\to\infty} \int_\Omega f(|\nabla u_j|)\,dx + \lim_{j\to\infty} \int_{S_j} \psi(b_j|[u_j]|)\phi(\nu)\,d\mathcal{H}^1 \\ &= \liminf_{n\to\infty} F_n(u_n) \end{aligned}$$

and we conclude the proof. ■

Proof of Proposition 2.8. The sequence $\{v_n\}$ defined as in Proposition 4.2 satisfies $\|v_n - u_n\|_\infty \le C/b_n$ and

$$\int_\Omega f(|\nabla v_n|)\,dx + \int_{S_{v_n}} \psi(m|[v_n]|)\,d\mathcal{H}^1 \le C < \infty\,, \tag{46}$$

then by Ambrosio's theorem it is strongly precompact in $L^1(\Omega)$ and thus the same holds for u_n. ■

5 Applications

5.1 A variational model for image segmentation

The Mumford-Shah functional is one of the most studied free discontinuity problem. It was introduced, in a strong formulation, in [21] as a variational model for image segmentation. Here it is considered in the weak form

$$\min\left\{\beta\int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}^1(S_u) + \int_\Omega |u-g|^2\,dx\,,\ \text{for } u \in SBV^2(\Omega)\right\}\,, \tag{47}$$

where $\Omega \subset \mathbf{R}^2$ is usually an open square, α and β are positive parameters and $g \in L^\infty(\Omega)$ represents the gray level map of the original image.

Theorem 5.1 *Let b_h and $s(t)$ be as in Theorem 2.5. For $u \in L^2(\Omega) \cap V_h(\Omega)$ let*

$$MS_h(u) = \sum_{T \,\in\, \boldsymbol{T}_h} \beta \int_T |\nabla u|^2 \, dx + \alpha \sum_{\zeta \,\in\, \boldsymbol{E}_h} \int_\zeta s(b_h |[u]_\zeta|) \, d\mathcal{H}^1 + \sum_{T \,\in\, \boldsymbol{T}_h} \int_T |u - g|^2 \, dx \, ,$$

and let $F_h(u) = +\infty$ otherwise in $L^2(\Omega)$. Then F_h Γ-converges, in the strong topology of $L^2(\Omega)$, to the Mumford-Shah functional

$$MS(u) = \beta \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}^1(S_u) + \int_\Omega |u - g|^2 \, dx \, ,$$

if $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ and $F(u) = +\infty$ otherwise in $L^2(\Omega)$. Moreover for every sequence u_{h_n} of minimizers of F_{h_n} there is a subsequence u_{h_k} and a function $u \in SBV^2(\Omega)$, which minimizes F, such that $u_k \to u$ in $L^2(\Omega)$ and $\nabla u_{h_k} \to \nabla u$ in $L^2(\Omega, \mathbf{R}^2)$.

Proof. The proof of the convergence result follows easily from Theorem 2.5 using the continuity of the L^2 term. Let us consider compactness: by a truncation argument it is clear that the minimizers u_n must be equibounded in $L^\infty(\Omega)$. Then, arguing as in the proof of Proposition 2.8, there exist a subsequence u_k and a function $u \in SBV^2(\Omega)$ such that $u_k \to u$ in $L^2(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega, \mathbf{R}^2)$. Moreover by Γ-convergence we have $F_n(u_n) \to F(u)$ and then there exists a subsequence (not relabeled) such that

$$\int_\Omega |\nabla u|^2 \, dx = \lim_{k \to \infty} \int_\Omega |\nabla u_k|^2 \, ,$$

which implies the strong convergence in $L^2(\Omega, \mathbf{R}^2)$ of ∇u_k to ∇u. ∎

5.2 A boundary value problem in fracture mechanics

Another interesting field for the applications of free discontinuity problems is the propagation of fractures in brittle materials. A beautiful theory has been developed by Griffith (see [17] and [20]) using the following energy balance: let dl be the infinitesimal increment of crack length and let dE be the corresponding variation of elastic energy. Griffith suggested that the energy required to increase the crack should be γdl, where γ, sometimes denoted as critical dissipated energy, is a positive parameter depending on material toughness. Then Griffith postulated that the fracture evolves only when $-dE \geq \gamma dl$, which simply means that the release of elastic energy $-dE$ must be greater then the energy γdl spent to increase the fracture.

For the time being, in the contest of SBV spaces, a complete mathematical formulation is available only in the case of antiplanar displacements and Dirichlet boundary conditions [15] and recently also with our discontinuous finite element approach [16] (alternatively, in a different functional framework, it is possible to use also bidimensional displacements, see [14] and [8]).

For the sake of simplicity assume that $\Omega \subset \mathbf{R}^2$ is an open polygonal domain and let $\partial_D \Omega \subset \partial\Omega$ be made of finitely many connected components, which are relatively open in $\partial\Omega$. In order to impose a Dirichlet boundary condition on $\partial_D \Omega$, allowing in the same time the presence of fractures, we introduce an open polygonal domain $\widetilde{\Omega}$, which contains Ω, such that $\partial_D \Omega \subset \widetilde{\Omega}$ and $(\partial\Omega \setminus \partial_D \Omega) \subset \partial\widetilde{\Omega}$ (see Figure 4).

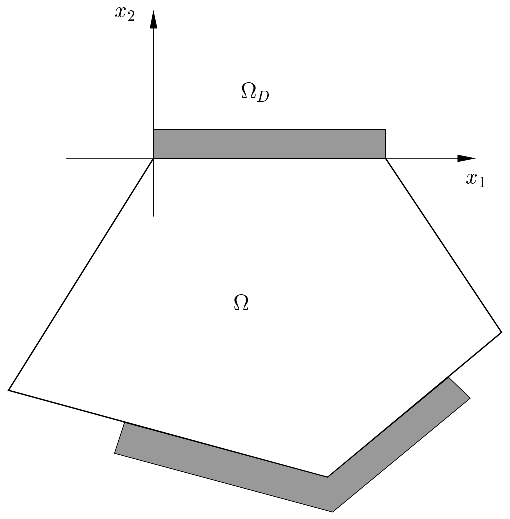

Figure 4: The set Ω, Ω_D (in gray) and the local system of coordinates used in the proof of Lemma 5.2.

Assume that there is a pre-existing crack $K \subset \Omega$ with $K \cap \partial\Omega_D = \emptyset$ such that K is the graph of a simple curve $\eta \in C^2([a,b])$. Let $\Omega_D = \widetilde{\Omega} \setminus \bar{\Omega}$ and consider the following minimization problem, which is the core of the incremental formulation in [15],

$$\min\left\{\int_\Omega |\nabla u|^2\,dx + \gamma\mathcal{H}^1(S_u \setminus K)\,,\ \text{for } u \in SBV^2(\widetilde{\Omega}) \text{ with } u = g \text{ in } \Omega_D\right\}. \tag{48}$$

For the proof of the convergence result we will need the following result, similar to Lemma 2.3, for boundary value problems.

Lemma 5.2 *Let $g \in W^{m,q}(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ and $u \in SBV^p(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ for $1 < p \leq 2$. Assume that $g \in W^{1,p}(\widetilde{\Omega})$ and $u = g$ in Ω_D. Then there is a sequence $\{w_k\}$ such that $w_k \in \mathcal{W}^{m,q}(\Omega)$, $w_k = g$ in Ω_D and*

$$w_k \to u \ \text{in } L^1(\widetilde{\Omega}), \tag{49}$$

$$\nabla w_k \to \nabla u \ \text{in } L^p(\widetilde{\Omega}, \mathbf{R}^2), \tag{50}$$

$$\limsup_{k\to+\infty} \|w_k\|_\infty \leq \|u\|_\infty\,. \tag{51}$$

Moreover, for every k, the discontinuity set S_{w_k} is the disjoint union of finitely many segments which do not intersect $\partial\Omega$ and for every open set $A \subset \widetilde{\Omega}$ we have

$$\limsup_{k\to+\infty} \int_{S_{w_k}\cap A} \phi(\nu)\,d\mathcal{H}^1 \leq \int_{S_u \cap \bar{A}} \phi(\nu)\,d\mathcal{H}^1\,. \tag{52}$$

Proof. Since $g \in W^{m,q}(\widetilde{\Omega})$ by the change of variable $u' = u - g$ it is not restrictive to assume that $g \equiv 0$. Consider first that $\partial_D\Omega$ is a segment, and let ν be its outer normal. To simplify the notations consider a system of coordinates (x_1, x_2) as in Figure

4 and assume that $\Omega_D = (0,\eta) \times (0,\gamma)$. In general this condition would be too restrictive but, considering that $g = 0$, it is not difficult to check that with minor details the same proof holds true. We denote by $\partial_D \Omega^t$ the section $(\partial_D \Omega + t\nu) \cap \Omega_D$ and by Ω_D^t the set $(\partial_D \Omega + s\nu) \cap \Omega_D$ for $s \in (t,\gamma)$.

Let u_n be the sequence given by Lemma 2.3 such that (7)-(10) hold in $\widetilde{\Omega}$. By (7)-(8) we have $u_n \to 0$ in $L^1(\Omega_D)$ and $\nabla u_n \to 0$ in $L^p(\Omega_D, \mathbf{R}^2)$. Moreover for every $\eta > 0$ (10) gives

$$\limsup_{n\to+\infty} \int_{S_{u_n}\cap\Omega_D^\eta} \phi(\nu)\, d\mathcal{H}^1 \leq \int_{S_g\cap\overline{\Omega_D^\eta}} \phi(\nu)\, d\mathcal{H}^1 = 0\,.$$

For every $\tau > 0$ there is $\delta \in (0,\tau)$ and a subsequence $\{u_i\}$ of $\{u_n\}$ such that for i sufficiently large $u_i \in W^{m,q}(\partial\Omega_D^\delta)$ and $u_i \to 0$ in $W^{1,p}(\partial\Omega_D^\delta)$. Now we can define a new sequence $\{v_i\}$ in the following way,

$$v_i(x_1,x_2) = \begin{cases} u_i(x_1,\delta) & \text{if } x \in \Omega_D^\tau \\ u_i(x_1,x_2) & \text{otherwise.} \end{cases}$$

Then $v_i \in \mathcal{W}^{m,q}(\widetilde{\Omega} \setminus \Omega_D^\delta)$, $v_i \in W^{m,q}(\Omega_D^\delta)$ and $v_i \to 0$ in $W^{1,p}(\Omega_D^\delta)$. Note that $\mathcal{H}^1(S_{v_i} \cap \Omega_D^\delta) = \emptyset$ and that in general it is not true that $v_i \in \mathcal{W}^{m,q}(\widetilde{\Omega})$. Let us check the convergence properties of $\{v_i\}$: since $v_i \to 0$ in $W^{1,p}(\Omega_D^\delta)$ we get

$$v_i \to u \text{ in } L^1(\widetilde{\Omega}), \qquad \nabla v_i \to \nabla u \text{ in } L^p(\widetilde{\Omega}, \mathbf{R}^2), \tag{53}$$

and it is clear that

$$\limsup_{i\to+\infty} \int_{S_{v_i}} \phi(\nu)\, d\mathcal{H}^1 \leq \limsup_{i\to+\infty} \int_{S_{u_i}} \phi(\nu)\, d\mathcal{H}^1 \leq \int_{S_u} \phi(\nu)\, d\mathcal{H}^1\,. \tag{54}$$

Denote by $\tilde{v}_i$ the extension to $\mathbf{R}^2$ of v_i restricted to Ω_D^δ (see [25]) thus $\tilde{v}_i \in W^{m,q}(\mathbf{R}^2)$ and $\tilde{v}_i \to 0$ in $W^{1,p}(\mathbf{R}^2)$. Then the sequence $\{w_i = v_i - \tilde{v}_i\}$ satisfies (53)-(54), has the same regularity of $\{v_i\}$ and $w_i = 0$ in Ω_D^τ.

It is not difficult to see that there exists a function $\psi \in C^\infty(\mathbf{R})$ such that $\psi(t) = t$ for $t \leq -1$, $\psi(t) = t+1$ for $t \geq 0$ and $1 \leq \psi'(t) \leq C$ for every t. Let $\psi^\delta(t) = \delta\psi(t/\delta)$ and $\Psi^\delta : \mathbf{R}^2 \to \mathbf{R}^2$ be the following map

$$\Psi^\delta(x_1,x_2) = \begin{cases} (\psi^\delta(x_1), x_2) & \text{if } x_1 \in (0,\eta) \\ (x_1,x_2) & \text{otherwise.} \end{cases}$$

Note that the functions Ψ^δ maps Ω_D into Ω_D^δ. It is clear that it is discontinuous in a set S_{Ψ^δ} such that $\mathcal{H}^1(S_{\Psi^\delta} \cap \widetilde{\Omega}) \leq 2\delta$, but elsewhere it is smooth and for $x \notin S_{\Psi^\delta}$ satisfies

$$|\Psi^\delta(x) - x|_\infty \leq \delta \qquad 1 \leq |\nabla\Psi_\delta(x)|_\infty \leq C\,. \tag{55}$$

Let $w_i^\delta(x) = w_i(\Psi^\delta(x))$ and for δ fixed let us check the convergence properties of this sequence to $u^\delta(x) = u(\Psi^\delta(x))$, as $i \to +\infty$. It is clear that $w_i^\delta \to u^\delta$ in $L^1(\widetilde{\Omega})$ and by (55) that $\nabla w_i^\delta \to \nabla u^\delta$ in $L^p(\widetilde{\Omega})$. Note that $w_i^\delta = 0$ in Ω_D and that $w_i^\delta \in W^{m,q}(\Omega \setminus S_{w_i^\delta})$ but the map Ψ^δ is not linear and thus we cannot say that $S_{w_i^\delta}$ is a union of segments. Thus consider a segment $S \subset S_{w_i}$ with normal ν, let S^+ and S^- be the half spaces "separated" by S and $w_i^\pm$ be the extension by reflection of w_i in $S^\pm$. For every $m \in \mathbf{N}$ let $-\delta = t^0 < t^1 < \ldots < t^m = 0$ a uniform subdivision of $[-\delta, 0]$ and let ψ_m^δ be the piecewise function with $\psi_m^\delta(t^i) = \psi^\delta(t^i)$. Denote by $\Psi_m^{-\delta}$ the inverse of the map Ψ_m^δ and by S_m^δ the set $\Psi_m^{-\delta}(S)$. Then S_m^δ is a piecewise linear curve and we prove that for every m

$$\int_{S_m^\delta} \phi(\nu)\, d\mathcal{H}^1 \le \int_S \phi(\nu)\, d\mathcal{H}^1 + C\delta\,. \tag{56}$$

By linearity it is sufficient to prove that

$$\int_{S_m^\delta} \phi(\nu)\, d\mathcal{H}^1 \le \int_S \phi(\nu)\, d\mathcal{H}^1 + C\frac{\delta}{m}\,,$$

assuming that the extrema of S are $(0,0)$ and (a,b), for $b \le \delta/m$, and the ones of S_m^δ are $(0,0)$ and (a,b'), where by (55) $b' = \psi_m^\delta(b)$. Then, considering that ϕ is 1-homogeneous we have

$$\int_S \phi(\nu)\, d\mathcal{H}^1 = \phi(-b,a) \qquad \int_{S_m^\delta} \phi(\nu)\, d\mathcal{H}^1 = \phi(-b',a)\,.$$

By (55) we have $|b'-b| \le \delta/m$, thus as ϕ is a norm we can write

$$\phi(-b,a) \ge \big|\phi(-b',a) - \phi(b'-b,0)\big| \ge \phi(-b',a) - \phi(\frac{\delta}{m},0) \ge \phi(-b',a) - C\frac{\delta}{m}\,,$$

which gives (56). Now we would like to modify the sequence $\{w_i^\delta\}$ in such a way that $S_{w_i^\delta} \subset S_m^\delta$. This can be done defining the function w_i^δ to be equal to $w_i^\pm(\psi^\delta)$ in the region delimited by the curves S_m^δ and $S_{w_i^\delta}$. Choosing for every i a value m_i sufficiently large we can redefine $\{w_i^\delta\}$ in order to satisfy the following properties: $S_{w_i^\delta}$ is piecewise linear, $w_i^\delta \to u^\delta$ and $\nabla w_i^\delta \to \nabla u^\delta$. Moreover by (56) it follows that

$$\limsup_{i\to+\infty} \mathcal{H}^1(S_{w_i^\delta}) \le \limsup_{i\to+\infty} \mathcal{H}^1(S_{w_i}) + 2\delta \le \mathcal{H}^1(S_u) + 2\delta\,.$$

Finally by a p-capacity argument we can assume that $S_{w_i^\delta}$ is the union of a finite number of segments.

Let now $\delta_k \searrow 0$ and for every k let u_k be an element $w_{i_k}^{\delta_k}$ of the sequence $\{w_i^{\delta_k}\}$ such that

$$\int_{S_{u_k}} \phi(\nu)\, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu)\, d\mathcal{H}^1 + 3\delta_k$$

and

$$\int_{\widetilde\Omega} |u_k - u^{\delta_k}|\, dx \le \delta_k\,, \qquad \int_{\widetilde\Omega} |\nabla u_k - \nabla u^{\delta_k}|^p\, dx \le \delta_k\,.$$

The resulting sequence $\{u_k\}$ satisfies the required properties, indeed

$$\limsup_{k\to+\infty} \int_{S_{u_k}} \phi(\nu)\, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu)\, d\mathcal{H}^1$$

and, since $|\psi^{\delta_k}(x) - x| \le \delta_k$ for $a.e.\, x \in \widetilde\Omega$, we have

$$\limsup_{k\to+\infty} \int_{\widetilde\Omega} |u_k - u|\, dx \le \limsup_{k\to+\infty} \int_{\widetilde\Omega} |u_k - u_k^\delta| + |u(\psi^{\delta_k}) - u|\, dx = 0$$

and finally by (55)

$$\limsup_{k\to+\infty}\int_{\widetilde\Omega}|\nabla u_k-\nabla u|^p\,dx \le C\limsup_{k\to+\infty}\int_{\widetilde\Omega}|\nabla u_k-\nabla u^{\delta_k}|^p+|\nabla u^{\delta_k}-\nabla u|^p\,dx$$
$$\le \limsup_{k\to+\infty}\int_{\widetilde\Omega}|\nabla\psi^{\delta_k}-I|^p|\nabla u|^p\,dx=0\,,$$

where the last limit follows by dominated convergence.

It remain to consider the general case, when $\partial_D\Omega$ is not a segment. Since $\partial_D\Omega$ is made of finitely many segments it possible to find a sequence δ_k in such a way that the previous construction works locally for every segment. Now we must glue the result, taking care of the difficulties which may come out only close to the points where two segments meets each other. As this intersections points are finite we can simply cut off the functions u_k by a capacitary potential. In this way all the regularity and convergence properties remain valid. ■

Remark 5.3 *Note that imposing the boundary condition we loose some regularity because in general w_k does not belong to $\mathcal{W}^{m,q}$ in the whole $\widetilde\Omega$.*

The following Corollary can be easily proved using the the regularity of the curve η and Lemma 5.2.

Corollary 5.4 *Let u and g as in Lemma 5.2 and let K as in (48). Then there exists a sequence $\{w_k\}$ such that $w_k\in\mathcal{W}^{m,q}(\Omega\setminus K)$, $w_k=g$ in Ω_D and (49)-(51) hold. Moreover, for every k, the discontinuity set $S_{w_k}\setminus K$ is the disjoint union of finitely many segments which do not intersect $\partial\Omega\cup K$ and for every open set $A\subset\widetilde\Omega$ we have*

$$\limsup_{k\to+\infty}\int_{(S_{w_k}\setminus K)\cap A}\phi(\nu)\,d\mathcal{H}^1\le\int_{(S_u\setminus K)\cap\bar A}\phi(\nu)\,d\mathcal{H}^1\,.$$

Let $\boldsymbol{B}_h$ be a regular family of triangulations in $\mathbf{R}^2$ and $\mathcal{T}_h$ be the family of subgrid meshes. Denote by $\boldsymbol{T}_h^r\in\mathcal{T}_h$ the one whose adaptive vertices are all defined by $x'=\frac{1}{2}(x_1+x_2)$ and by $\boldsymbol{E}_h^r$ the corresponding set of adaptive edges. Assume that $\partial\Omega$ and $\partial\Omega_D$ can be represented as (finite) unions of edges belonging to $\boldsymbol{E}_h^r$, in such a way that discontinuities are allowed on $\partial_D\Omega$. Let $g\in W^{2,\infty}(\widetilde\Omega)$ and let $g_h\in W_h(\mathbf{R}^2,\boldsymbol{T}_h^r)$ be its Lagrange interpolation.

Moreover, using the approximation argument of Section 3, it is not difficult to see that for every h there exists an approximation K_h of K in $\boldsymbol{E}_h$ such that $d(K_h,K)\to 0$ and

$$\int_{K_h}\langle\nu,\xi\rangle\,d\mathcal{H}^1\to\int_K\langle\nu,\xi\rangle\,d\mathcal{H}^1$$

for every $\xi\in S^1$. Finally we will denote by $\boldsymbol{K}_h$ the set of edges $\zeta\in\boldsymbol{E}_h$ contained in K_h.

Theorem 5.5 *Let b_h and $s(t)$ be as in Theorem 2.5. For $u\in L^1(\widetilde\Omega)$ let*

$$G_h(u)=\sum_{T\,\in\,\boldsymbol{T}_h}\int_T|\nabla u|^2\,dx+\gamma\sum_{\zeta\,\in\,\boldsymbol{E}_h\setminus\boldsymbol{K}_h}\int_\zeta s(b_h|[u]_\zeta|)\,d\mathcal{H}^1\,,$$

if $u \in V_h(\widetilde{\Omega})$ with $u = g_h$ on Ω_D and let $G_h(u) = +\infty$ otherwise in $L^1(\widetilde{\Omega})$. Then G_h Γ-converges, in the strong topology of $L^1(\widetilde{\Omega})$, to the Griffith energy

$$G(u) = \int_{\widetilde{\Omega}} |\nabla u|^2 \, dx + \gamma \mathcal{H}^1(S_u \setminus K) \, ,$$

if $u \in GSBV^2(\widetilde{\Omega}) \cap L^1(\widetilde{\Omega})$ with $u = g$ on Ω_D and $G(u) = +\infty$ otherwise in $L^1(\widetilde{\Omega})$. Moreover let $h_n \to 0$ and u_{h_n} a sequence of minimizers of G_{h_n}, if $\|u_{h_n}\|_\infty < \infty$ then there is a subsequence u_{h_k} and a function $u \in SBV^2(\widetilde{\Omega})$, which minimizes G, such that $u_k \to u$ in $L^2(\widetilde{\Omega})$ and $\nabla u_{h_k} \to \nabla u$ in $L^2(\widetilde{\Omega}, \mathbf{R}^2)$.

Proof. Γ-liminf inequality. Let $u_h \in V_h(\widetilde{\Omega})$ with $u_h \to u$ in $L^2(\widetilde{\Omega})$, we want to prove that

$$G(u) \leq \liminf_{h \to 0} G_h(u_h) \, .$$

It's not restrictive to assume that $u_h = g_h$ in Ω_D and that $\liminf_{h\to 0} G_h(u_h) < +\infty$. Let $v_h = g_h$ in Ω_D and let v_h be defined in Ω as in the proof of Proposition 4.2, in such a way that inequality (42) holds.

As the functional G_h is defined in $\widetilde{\Omega} \setminus K_h$ and G in $\widetilde{\Omega} \setminus K$ it is not possible to follow simply the proof of Proposition 4.2. By the slicing technique (see for instance [7]) it is sufficient to prove the Γ-liminf inequality for a one-dimensional section. Thus for $\xi \in S^1$ and $A \subset \mathbf{R}^2$ let A_ξ be the projection of A on the line $\langle y, \xi \rangle$ and for $y \in A_\xi$ let $A_\xi^y = \{t \in \mathbf{R} : (y + t\xi) \in A\}$. For simplicity we will identify any function in $\widetilde{\Omega}$ with its restriction to $\widetilde{\Omega}_\xi^y$. The sliced functionals are defined as

$$(G_h)_\xi^y(u_h) := \int_{\widetilde{\Omega}_\xi^y} |u_h'|^2 \, dt + s(b_h |[u_h](t)|) \mathcal{H}^0(t) \llcorner \left((\widetilde{\Omega} \setminus K_h)_\xi^y \right) \, ,$$

$$G_\xi^y(u) := \int_{\widetilde{\Omega}_\xi^y} |u'|^2 \, dt + \mathcal{H}^0 \left((S_u \setminus K)_\xi^y \right) \, .$$

We will prove that for every $\xi \in S^1$ and for a.e. $y \in \widetilde{\Omega}_\xi$ we have

$$G_\xi^y(u) \leq \liminf_{h \to 0} (G_h)_\xi^y(v_h) \, .$$

It is not difficult to see that for every $\xi \in S^1$ and for a.e. $y \in \Omega_\xi$ we have $\mathcal{H}^0\left((K_h)_\xi^y\right) = \mathcal{H}^0\left(K_\xi^y\right)$ and $d((K_h)_\xi^y, K_\xi^y) \to 0$. Then, modifying the functions v_h in a neighborhood of K_ξ^y, we can define a sequence $\tilde{v}_h$ in such a way that

$$(G_h)_\xi^y(v_h) \geq \int_{\widetilde{\Omega}_\xi^y} |\tilde{v}_h'|^2 \, dt + s(b_h |[\tilde{v}_h]|) \mathcal{H}^0(t) \llcorner \left((\widetilde{\Omega} \setminus K)_\xi^y \right) \, .$$

The rest follows as in the proof of Proposition 4.2 applying the compactness and lower-semicontinuity theorem in $(\Omega \setminus K)_\xi^y$.

Γ-limsup inequality. By Corollary 5.4 and by a standard diagonal argument it is sufficient to prove that for every function $w \in \mathcal{W}^{m,q}(\Omega \setminus K)$ and for every $\delta > 0$ there exists a sequence $w_h \in V_h(\widetilde{\Omega})$ such that

$$\limsup_{h \to 0} G_h(w_h) \leq G(w) + O(\delta) \, .$$

Let $\delta < d(S_w, K)$ and consider the map $T^\delta : [a,b] \times [-\delta, \delta] \to \Omega$ defined by $T^\delta(x,y) = \eta(x) + y\hat{n}(x)$, where $\hat{n}$ is the normal vector. It's easy to see that for δ sufficiently small T^δ is a C^1 diffeomorphism. We will denote by K^δ its image. As $d(K_h, K) \to 0$ we have $\mathcal{H}^1(K_h \cap K^\delta) \to 0$ and we can assume, without loss of generality, that for h sufficiently small K_h disconnects K^δ into $(K_h^\delta)^\pm$. Let $w^\pm$ be the extensions of w in $(K_h^\delta)^\pm$ such that $\|w^\pm\|_{m,q} \leq C\delta$, where C does not depend on δ. Such functions exists because the constants appearing in the extension estimates, which depend on δ, are uniformly bounded from above thanks to the special form of the maps T^δ. Now the discontinuities of w are now concentrated on K_h, thus, using the proof of Proposition 3.6, we can easily interpolate w and prove the required inequality.

Compactness. As $\mathcal{H}^1(K_h) \to \mathcal{H}^1(K)$ and $\mathcal{H}^1(K) < +\infty$, compactness follows by Theorem 5.1. ■

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