Fine properties of sets of finite perimeter in doubling metric measure spaces

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1 Introduction

In the last years there has been a tremendous progress in the understanding of the Sobolev-space theory in the general setting of metric spaces, see for instance [28] and the references therein, [10], [29], [38] but the list is not exhaustive).

More recently, starting from the papers [20], [23], [11], [7] (see also [5] for the case of Finsler spaces and [4] for the case of weighted Sobolev spaces), some more progress has been made in [33] on the definition of BV functions and sets of finite perimeter in the general setting of metric measure spaces, i.e. metric spaces (X, d) endowed with a locally finite Borel measure μ. A basic assumption of the theory is that μ is a doubling measure, see (2.1).

The aim of this paper is to study the properties of the perimeter measure in this quite general setting. In particular, defining the essential boundary $\partial^* E$ of E as the set of points where neither the density of E nor the density of $X \setminus E$ is 0, we show that the perimeter measure is concentrated on $\partial^* E$ and is representable by an Hausdorff-type measure. As a consequence we show that $\partial^* E$ is not only μ-negligible (a well known consequence of the doubling property of μ) but it has a finite lower dimensional measure. Moreover, we show that the perimeter measure $P(E, \cdot)$ is almost everywhere asymptotically doubling, i.e.

$$\limsup_{\varrho \downarrow 0} \frac{P(E, B_{2\varrho}(x))}{P(E, B_\varrho(x))} < \infty \qquad \text{for } P(E, \cdot)\text{-a.e. } x \in X.$$

This information allows to differentiate with respect to the perimeter measure and therefore is useful in connection with blow-up methods, closer to the original ones adopted by De Giorgi in his pioneering work [14] on Euclidean sets of finite perimeter. In fact, the blow-up method provides stronger informations on sets of finite perimeter in the Heisenberg group, see [21], [22].

In [1] we obtained the above mentioned results on the perimeter measure under the following assumptions:

(i) X is Ahlfors-regular, i.e. there exist a dimension $s \geq 1$ and a constant $a > 0$ satisfying

$$a\varrho^s \leq \mu(B_\varrho(x)) \leq \frac{1}{a}\varrho^s \qquad \forall x \in X,\ \varrho \in (0, \operatorname{diam} X)\,. \tag{1.1}$$

(ii) The weak (1,1)-Poincaré inequality

$$\int_{B_\varrho(x)} |u(y) - u_{x,\varrho}|\, d\mu(y) \leq C_P \varrho \int_{B_{\lambda\varrho}(x)} g(y)\, d\mu(y)$$

holds. Here u is any locally Lipschitz function in X, $u_{x,\varrho}$ is the mean value of u on $B_\varrho(x)$ and g is an upper gradient of u according to Heinonen and Koskela, see [30].

(iii) The space BV and the class of sets of finite perimeter are built using the upper gradients.

After some time, and after the influence of the work [25], we realized that the assumption (i) could be removed and that the the proofs in [1] depend very little on the upper gradient structure but, rather, on some abstract properties that any reasonable "differentiation structure" must satisfy.

The way to remove assumption (i) is indeed very natural: while in the presence of (1.1) the space X has a definite dimension s, and therefore the perimeter measure can be represented by integration with respect to the Hausdorff measure $\mathcal{H}^{s-1}$, in the general doubling case the dimension is not constant, not even locally. Therefore, we have to consider the Hausdorff-type measure $\mathcal{H}^h$ built with the Carathéodory construction based on the function (see (2.3) for the precise definition)

$$h\left(\overline{B}_\varrho(x)\right) := \frac{\mu\left(\overline{B}_\varrho(x)\right)}{\operatorname{diam}\left(\overline{B}_\varrho(x)\right)}.$$

The function h, comparable to the $(s-1)$-th power the diameter of the ball under assumption (1.1), takes into account in a natural way of the local change of dimension of X. A nice example where this phenomenon occurs is the so-called Grušin plane: see the discussion in Bellaïche's contribution to [26].

Concerning assumption (iii), we relax it by defining BV functions and sets of finite perimeter relative to a given differentiation structure, following basically the relaxation approach in [33]. We assume that the differentiation structure is local (see §3 for details) and that it satisfies a suitable weak (1,1)-Poincaré inequality.

2 Notation and preliminary results

Throughout this paper we assume that (X,d) is a complete metric space. We denote the open ball $\{y \in X : d(x,y) < \varrho\}$ by $B_\varrho(x)$ and the closed ball $\{y \in X : d(x,y) \le \varrho\}$ by $\overline{B}_\varrho(x)$. We use the notation $B(X)$ for the collection of all closed balls of X and $\mathcal{B}(X)$ for the Borel σ-algebra of X.

We assume that (X,d) is endowed with a doubling measure μ, i.e. a σ-additive set function $\mu : \mathcal{B}(X) \to [0,\infty]$ finite on bounded sets and satisfying

$$\mu\left(B_{2\varrho}(x)\right) \le C_D \mu\left(B_\varrho(x)\right) \qquad \forall x \in X,\ \varrho > 0 \tag{2.1}$$

for some constant C_D. It is well known that the doubling property implies the density lower bound

$$\frac{\mu(B_\varrho(x))}{\mu(B_R(y))} \ge c\left(\frac{\varrho}{R}\right)^s, \qquad \forall 0 < \varrho \le R < \infty,\ x \in B_R(y) \tag{2.2}$$

for some constant $c = c(s, C_D)$, where s is any power greater than $\log_2 C_D$.

In the following, $\mathrm{Lip}_{\mathrm{loc}}(X)$ stands for the space of real valued Lipschiz functions on bounded subsets of X. More generally, whenever $A \subset X$ is an open set, by $u \in L^1_{\mathrm{loc}}(A)$ we mean that $u \in L^1(C)$ for any bounded and closed set $C \subset A$; the same convention applies to other functions spaces, convergence, and so on. Notice that, in view of the completeness and doubling assumptions, bounded and closed sets in X are compact.

Let $h : B(X) \to [0,\infty]$ be an increasing function. By the Carathéodory construction, h induces a Borel regular outer measure $\mathcal{H}^h$, defined for any set $B \subset X$ by $\sup_{\delta>0} \mathcal{H}^h_\delta(B)$, where

$$\mathcal{H}^h_\delta(B) := \inf\left\{\sum_{i\in I} h(B_i) : \ B_i \in B(X), \ \mathrm{diam}\,(B_i) < \delta, \ B \subset \bigcup_{i\in I} B_i\right\} \tag{2.3}$$

and we adopt the convention that $\mathcal{H}^h(\emptyset) = 0$. In the case when $h(B) = [\mathrm{diam}\,(B)]^\alpha$ the measure $\mathcal{H}^h$ reduces to $\mathcal{S}^\alpha$, the spherical *Hausdorff α-dimensional measure* (possibly up to a normalization factor).

For $E \subset X$, we denote by $E^c = X \setminus E$ the complement of E and by χ_E the characteristic function of E. For E, $F \subset X$ we denote by $E\Delta F$ the symmetric difference of E and F. If E is a Borel set, we denote the volume $\mu(E\cap B_\varrho(x))$ of E in $B_\varrho(x)$ by $m_E(x,\varrho)$. Moreover, $\partial^* E$ stands for the *essential boundary* of E, i.e. $x \in \partial^* E$ if

$$\text{neither } \lim_{\varrho\downarrow 0} \frac{m_E(x,\varrho)}{\mu(B_\varrho(x))} = 0 \qquad \text{nor } \lim_{\varrho\downarrow 0} \frac{m_{E^c}(x,\varrho)}{\mu(B_\varrho(x))} = 0.$$

We will use a classical covering theorem well adapted to the Hausdorff-type measures. The proof for the case $h(B) = [\mathrm{diam}\,(B)]^\alpha$ is given, for instance, in Theorem 1.10 of [16]; the same proof works if h is a doubling function, i.e. there exists a constant c_D such that

$$h(\overline{B}_{2\varrho}(x)) \le c_D h(\overline{B}_\varrho(x)) \qquad \text{whenever } x \in X, \ \varrho > 0. \tag{2.4}$$

Theorem 2.1 (Vitali covering theorem) *Let (X,d) be a metric space. Let $\mathcal{F}$ a family of closed balls and $K \subset X$ be such that:*

(i) for any $x \in K$ and any $\delta > 0$ the set

$$\left\{\varrho \in (0,\delta) : \ \overline{B}_\varrho(x) \in \mathcal{F}\right\}$$

is not empty;

(ii) there exist a doubling function h and a positive finite measure ν in $(X,\mathcal{B}(X))$ such that

$$\nu\left(\overline{B}_\varrho(x)\right) \ge h(\overline{B}_\varrho(x)) \qquad \forall\, \overline{B}_\varrho(x) \in \mathcal{F}.$$

Then, there exists a disjoint collection $\mathcal{G} \subset \mathcal{F}$ such that

$$\mathcal{H}^h\left(K \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0. \tag{2.5}$$

A simple and well known consequence of the above covering theorem (see for instance [17], 2.10.19) is the following implication

$$\limsup_{\varrho\downarrow 0} \frac{\nu(B_\varrho(x))}{h(\overline{B}_\varrho(x))} \ge t \quad \forall x \in B \qquad \Longrightarrow \qquad \nu(B) \ge t\mathcal{H}^h(B). \tag{2.6}$$

for any locally finite measure ν in X and any $B \in \mathcal{B}(X)$. Letting $t \uparrow \infty$ in (2.6) we obtain also

$$\limsup_{\varrho \downarrow 0} \frac{\nu(B_\varrho(x))}{h(\overline{B}_\varrho(x))} < \infty \qquad \text{for } \mathcal{H}^h\text{-a.e. } x \in X. \tag{2.7}$$

3 Differentiation structure

In this section, following basically (but not exactly) the presentation of [25], we introduce a list of axioms that the our set of pseudo-gradients must satisfy. Precisely, assume that we are given, for any $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$, a set $\mathrm{D}[u]$ of nonnegative Borel functions. Then, we consider the following 6 axioms:

Axiom 0. (Non triviality) $0 \in D[u]$ for any constant function u.

Axiom 1. (Upper linearity) If $g_1 \in \mathrm{D}[u_1]$, $g_2 \in \mathrm{D}[u_2]$ and $g \geq |\alpha| g_1 + |\beta| g_2$ μ-a.e., then $g \in \mathrm{D}[\alpha u_1 + \beta u_2]$.

Axiom 2. (Leibniz rule) If $g \in \mathrm{D}[u]$, then $\sup |\varphi| g + \mathrm{Lip}(\varphi)|u|$ belongs to $\mathrm{D}[\varphi u]$ whenever φ is a Lipschitz function.

Axiom 3. (Lattice property) If $g_1 \in \mathrm{D}[u_1]$, $g_2 \in \mathrm{D}[u_2]$ and $u = \min\{u_1, u_2\}$ then $\max\{g_1, g_2\} \in \mathrm{D}[u]$.

Axiom 4. (Locality) If $A \subset X$ is an open set and $g \in \mathrm{D}[u]$, then the function

$$g' = \begin{cases} g & \text{on } X \setminus A \\ h & \text{on } A \end{cases}$$

belongs to $\mathrm{D}(u)$ whenever $h \in \mathrm{D}[v]$ and $u \equiv v$ on A.

Axiom 5. (Weak Poincaré inequality) For any $g \in \mathrm{D}[u]$ and any ball $B_\varrho(x)$ we have

$$\int_{B_\varrho(x)} |u(y) - u_{x,\varrho}| \, d\mu(y) \leq C_P \varrho \int_{B_{\lambda\varrho}(x)} g(y) \, d\mu(y). \tag{3.1}$$

Here

$$u_{x,\varrho} := \frac{1}{\mu(B_\varrho(x))} \int_{B_\varrho(x)} u(y) \, d\mu(y)$$

is the mean value of u in $B_\varrho(x)$.

Notice that axiom A5 can also be stated in an apparently weaker form (more suitable for some applications)

$$\min_{m \in \mathbf{R}} \int_{B_\varrho(x)} |u(y) - m| \, d\mu(y) \leq C \varrho \int_{B_{\lambda\varrho}(x)} g(y) \, d\mu(y) \tag{3.2}$$

Indeed, it is easy to check that (3.2) implies (3.1) with $C_P = 2C$.

Definition 3.1 (D-structure) *We say that the set-valued map $u \mapsto \mathrm{D}[u]$ is a D-structure if axioms A0-A4 hold. We say that the D-structure satisfies the weak $(1,1)$-Poincaré inequality if axiom 5 holds. Finally, we say that the D-structure satisfies the $(1,1)$-Poincaré inequality if (3.1) holds with $\lambda = 1$.*

The paper [25] contains several examples of D-structures and of the induced Sobolev spaces (by a natural completion argument): weighted Sobolev spaces, Hajłasz–Sobolev spaces (in this case the locality axiom fails), combinatorial Sobolev spaces, upper gradients and infinitesimal stretch.

We need in the following an important consequence of the axioms A0-A5. The proof follows an argument of S.Semmes and is taken from [10] (see also [28]).

Theorem 3.2 (Quasi convexity) *For any pair of points $x, y \in X$ there exists a Lipschitz curve γ connecting x to y with length at most $Cd(x,y)$. Here C depends only on (C_D, λ, C_P).*

Proof We first prove that X is connected. Assume by contradition that $A \subset B_\varrho(x)$ is a nontrivial connected component of X. In view of the local compactness of X, the characteristic function of A is a locally Lipschitz function in X. Moreover, the locality axiom implies that $0 \in \mathrm{D}[\chi_A]$, and therefore the Poincaré inequality gives that χ_A is a.e. equal to a constant in any ball. Therefore, either A or $X \setminus A$ are negligible. Being A and $X \setminus A$ open, this contradicts the doubling property.

Let $x \in X$ and $\varepsilon > 0$ be fixed; since X is connected, any point $y \in X$ is ε-connected to x, namely there are finitely many points $x_1, \ldots, x_n$ with $x_1 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for $i = 1, \ldots, n-1$. Denoting by $u_\varepsilon(y)$ the infimum of

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

among all these families of points, it is clear that $u_\varepsilon(x) = 0$ and $\mathrm{Lip}(u_\varepsilon) \le 1$. Therefore, setting $u = \sup_{\varepsilon>0} u_\varepsilon$, the "telescope estimate" (see for instance [28], [10])

$$\int_{B_\varrho(x)} u_\varepsilon \, d\mu \le C(C_D, \lambda, C_P)\varrho\mu(B_\varrho(x))$$

and the dominated convergence theorem give

$$\int_{B_\varrho(x)} u(y)\, d\mu(y) \le C\varrho\mu(B_\varrho(x)). \tag{3.3}$$

By the local compactness of X, $u(y) < \infty$ implies the existence of a Lipschitz curve connecting x to y with length $u(y)$.

Let now $y_0 \in X$ and $\varrho = d(x, y_0)$. By (3.3) we obtain $y_1 \in B_{\varrho/2}(y_0)$ and a Lipschitz curve connecting x to y_1 with length at most $C\varrho$. Repeating this construction with y_1 in place of x, we obtain $y_2 \in B_{\varrho/4}(y_0)$ and a Lipschitz curve connecting y_1 to y_2 with length at most $C\varrho/2$. Iterating this argument and glueing all these curves the proof is achieved. □

4 Sets of finite perimeter

In this section we recall the main properties of sets of finite perimeter which will be useful in the sequel. According to [33] (see also [20], [23] and [11]) we define the class of sets of finite perimeter and the perimeter measure by a relaxation procedure, using as energy functional the L^1 norm of g, where g is any function in $\mathrm{D}[u]$. As a consequence, the class of sets of finite perimeter and the perimeter measure depend on the D-structure.

Definition 4.1 (Perimeter) *Let $E \in \mathcal{B}(X)$ and $A \subset X$ open. The perimeter of E in A, denoted by $P(E, A)$, is defined by*

$$P(E,A) := \inf \left\{ \liminf_{h\to\infty} \int_A g_h \, d\mu : \ (u_h) \subset \mathrm{Lip}_{\mathrm{loc}}(A), \ u_h \to \chi_E \ \textit{in} \ L^1_{\mathrm{loc}}(A), \ g_h \in \mathrm{D}[u_h] \right\}.$$

We say that E has finite perimeter in X if $P(E, X) < \infty$.

An analogous definition could be given for BV functions, defining

$$|Du|(A) := \inf \left\{ \liminf_{h\to\infty} \int_A g_h \, d\mu : \ (u_h) \subset \mathrm{Lip}_{\mathrm{loc}}(A), \ u_h \to u \ \text{in} \ L^1_{\mathrm{loc}}(A), \ g_h \in \mathrm{D}[u_h] \right\}$$

and saying that $u \in BV(X)$ if $u \in L^1(X)$ and $|Du|(X) < \infty$.

The basic example which motivated the present paper is the following:

Example 4.2 (Carnot–Carathéodory spaces) Let $\Omega \subset \mathbf{R}^n$ be an open connected set, let $Y_1, \ldots, Y_k$ be locally Lipschitz vector fields defined in Ω, and let ρ be the Carnot–Carathéodory distance induced by (Y_i). Then, assuming that $\rho(x, y) < \infty$ whenever x, $y \in \Omega$, the function

$$|Yu| := \left(|Y_1 u|^2 + \ldots + |Y_k u|^2\right)^{1/2}$$

is a minimal upper gradient of u whenever $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \rho)$ (see the discussion in §11.2 of [28]). Then, adopting as D-structure the one induced by upper gradients, the definitions of BV functions and sets of finite perimeter adopted in [20], [23] and [11] are equivalent to the one adopted here, with $X = \Omega$, $d = \varrho$ and μ equal to the Lebesgue measure and the D-structure induced by upper gradients, see the discussion in [33].

It turns also out that $u \in BV$ if and only if there exists measures $\mu_i = D_{Y_i} u$ such that

$$\int_X u Y_i^* \varphi \, d\mu = -\int_X \varphi \, d\mu_i \qquad \forall i = 1, \ldots, k, \ \varphi \in C^\infty_c(X)$$

(here $Y_i^* \varphi$ is the adjoint of $Y_i \varphi$, the partial derivative along the vector field Y_i) and the measure $|Du|$ is the total variation of the vector-valued measure $(\mu_1, \ldots, \mu_k)$.

Notice also that in general a Carnot–Carathéodory metric measure space (X, d, μ) is only doubling: the Ahlfors regularity condition holds, however, for Carnot groups.

The following properties easily follow from the definition of perimeter and the axioms A0-A4 (see [33] for a proof in the case of the upper gradients D-structure).

(a) (locality) $P(E,A)=P(F,A)$ whenever $(E\Delta F)\cap A$ is μ-negligible;

(b) (lower semicontinuity) $E\mapsto P(E,A)$ is lower semicontinuous with respect to the $L^1_{\rm loc}(A)$ topology;

(c) (subadditivity) $P(E\cup F,A)+P(E\cap F,A)\leq P(E,A)+P(F,A)$;

(d) (complementation) $P(E,A)=P(E^c,A)$.

By (c) and (d) it follows that

$$\max\{P(E\cup F,A),P(E\cap F,A),P(E\setminus F,A)\}\leq P(E,A)+P(F,A). \tag{4.1}$$

The proof of the following result, still obtained in [33], is not elementary: it uses the axioms A0-A4 and methods typical of the theory of Γ-convergence (see for instance [12]).

Theorem 4.3 *Let E be a set of finite perimeter in X. Then:*

(i) the set function $A\mapsto P(E,A)$ is the restriction to the open subsets of X of a finite Borel measure $P(E,\cdot)$ in X, defined by

$$P(E,B):=\inf\{P(E,A):\ A\supset B,\ A \textit{ open}\}\qquad\forall B\in\mathcal{B}(X);$$

(ii) if D supports the weak $(1,1)$-Poincaré inequality (3.1), the following relative isoperimetric inequality holds:

$$\min\{m_E(x,\varrho),m_{E^c}(x,\varrho)\}\leq C_I\left(\frac{\varrho^s}{\mu(B_\varrho(x))}\right)^{1/(s-1)}[P(E,B_{2\lambda\varrho}(x))]^{s/(s-1)} \tag{4.2}$$

where $s>1$ is any exponent satisfying (2.2);

(iii) for any $u\in{\rm Lip}_{\rm loc}(X)$ and any $g\in{\rm D}[u]$ the following coarea formula holds:

$$\int_{\mathbf{R}}P(\{u>t\},B)\,dt\leq\int_B g\,d\mu\qquad\forall B\in\mathcal{B}(X). \tag{4.3}$$

Proof. Properties (i), (ii), (iii) are proved in [33]. We repeat, for the reader's convenience, only the proof of (ii). By a well known result of Hajłasz and Koskela still true in our abstract setting (see [28], Theorem 5.1 and [25], Theorem 1.14), (3.1) and (2.2) imply a weak $(1^*,1)$-Poincaré inequality, i.e.

$$\left(\frac{1}{\mu(B_\varrho(x))}\int_{B_\varrho(x)}|u(y)-u_{x,\varrho}|^{s/(s-1)}\,d\mu(y)\right)^{(s-1)/s}\leq C\frac{\varrho}{\mu(B_\varrho(x))}\int_{B_{2\lambda\varrho}(x)}g\,d\mu$$

for any $u\in{\rm Lip}(B_{2\lambda\varrho}(x))$ and any $g\in{\rm D}[u]$. Taking into account the definition of $P(E,B_{2\lambda\varrho}(x))$, and noticing that by a truncation argument we need only to consider sequences (u_h) converging to χ_E in $L^1(B_{2\lambda\varrho}(x))$, we obtain (4.2). □

Remark 4.4 By a similar argument, we have

$$\min\{m_E(x,\varrho), m_{E^c}(x,\varrho)\} \le C_I \left(\frac{\varrho^s}{\mu(B_\varrho(x))}\right)^{1/(s-1)} [P(E, B_\varrho(x))]^{s/(s-1)} \tag{4.4}$$

whenever D supports a $(1,1)$-Poincaré inequality.

Finally, we will need the following canonical relation between perimeter and derivative of volume. The proof (see [1]) uses the lower semicontinuity of the variation and an approximation argument.

Lemma 4.5 (Localization) *Let E be a set of finite perimeter in X and $x \in X$. For a.e. $\varrho > 0$ the set $E \setminus B_\varrho(x)$ has finite perimeter in X and satisfies*

$$P(E \setminus B_\varrho(x), \partial B_\varrho(x)) \le \frac{d}{dr} m_E(x,r)\Big|_{r=\varrho}.$$

5 Representation of perimeter and doubling property

Throughout this section we assume that E is a set of finite perimeter in X. We define $h : B(X) \to [0,\infty)$ as

$$h(B) := \frac{\mu(B)}{\mathrm{diam}(B)}$$

and notice that h is a doubling function (i.e. it satisfies (2.4) with $c_D = C_D$), since μ is a doubling measure. We will consider the measure $\mathcal{H}^h$ built from h using the Carathéodory construction.

We recall an useful co-area inequality involving the $\mathcal{H}^h$-measure of level sets of Lipschitz functions; the proof of the inequality is given in Theorem 2.10.25 of [17] for the spherical Hausdorff measures $\mathcal{S}^\alpha$ (see also Proposition 3.1.2 of [2]) and works for $\mathcal{H}^h$ as well.

Proposition 5.1 *For any $u \in \mathrm{Lip}(X)$ and any $B \in \mathcal{B}(X)$ we have*

$$\int_{\mathbf{R}} \mathcal{H}^h\left(B \cap u^{-1}(t)\right)\, dt \le \mathrm{Lip}(u)\mu(B).$$

In particular, for any $x \in X$ the set $\partial B_\varrho(x)$ has finite $\mathcal{H}^h$-measure for a.e. $\varrho > 0$.

We begin our analysis of the perimeter measure by proving the absolute continuity of $P(E,\cdot)$ with respect to $\mathcal{H}^h$.

Lemma 5.2 (Absolute continuity) *We have $P(E,B) = 0$ whenever $B \in \mathcal{B}(X)$ is $\mathcal{H}^h$-negligible.*

PROOF. Assuming with no loss of generality that B is a compact set, for any $\epsilon > 0$ we can cover B by a finite number of open balls B_i^ϵ of radius $r_i^\epsilon < \epsilon$ and center x_i^ϵ such that $\sum_i h(\overline{B}_{r_i^\epsilon}(x_i^\epsilon)) < \epsilon$. By (4.3) with $u = d(\cdot, x_i^\epsilon)$ and $A = B_{2r_i^\epsilon}(x_i^\epsilon)$ we can find concentric open balls $\hat{B}_i^\epsilon \supset B_i^\epsilon$ with radius at most $2r_i^\epsilon$ such that

$$P(\hat{B}_i^\epsilon, X) \leq \frac{\mu(B_{2r_i^\epsilon}(x_i^\epsilon))}{r_i^\epsilon} \leq 2C_D h(\overline{B}_{r_i^\epsilon}(x_i^\epsilon)).$$

Denoting by $A_\epsilon \supset B$ the union of the balls $\hat{B}_i^\epsilon$, by locality and subadditivity of perimeter we get

$$P(E \cup A_\epsilon, X) = P(E \cup A_\epsilon, X \setminus B) \leq P(E, X \setminus B) + 2C_D\epsilon.$$

Since

$$\mu(A_\epsilon) \leq 2\epsilon \sum_i h(\overline{B}_{r_i^\epsilon}(x_i^\epsilon)) \leq 2C_D\epsilon^2 \to 0,$$

passing to the limit as $\epsilon \downarrow 0$ the lower semicontinuity of perimeter gives

$$P(E, X) \leq P(E, X \setminus B)$$

whence $P(E, B) = 0$. □

Now we prove that $P(E, \cdot)$ is representable by integration of $\mathcal{H}^h$ on $\partial^* E$; moreover, at $\mathcal{H}^h$-a.e. point of $\partial^* E$ we have lower bounds on $m_E(x, \varrho)$ and $m_{E^c}(x, \varrho)$ for *arbitrarily small* radii ϱ. The volume lower bounds will be improved in Theorem 5.4, showing that both fractions $m_E(x, \varrho)/\mu(B_\varrho(x))$ and $m_{E^c}(x, \varrho)/\mu(B_\varrho(x))$ are far from zero for *sufficiently small* radii $\varrho > 0$.

Theorem 5.3 (Hausdorff representation of perimeter) *The measure $P(E, \cdot)$ is concentrated on the set*

$$\Sigma_\gamma := \left\{ x : \limsup_{\varrho \downarrow 0} \min\{\frac{m_E(x, \varrho)}{\mu(B_\varrho(x))}, \frac{m_{E^c}(x, \varrho)}{\mu(B_\varrho(x))}\} \geq \gamma \right\} \subset \partial^* E$$

with $\gamma > 0$ depending only on (C_D, λ, C_I). Moreover $\partial^* E \setminus \Sigma_\gamma$ is $\mathcal{H}^h$-negligible, $\mathcal{H}^h(\partial^* E) < \infty$ and

$$P(E, B) = \int_{B \cap \partial^* E} \theta \, d\mathcal{H}^h \qquad \forall B \in \mathcal{B}(X)$$

for some Borel function $\theta : X \to [\gamma', \infty)$, with $\gamma' > 0$ depending only on (C_D, λ, C_I).

PROOF. We prove that $P(E, K) = 0$ for any compact set $K \subset X \setminus \Sigma_\gamma$. By Egorov theorem we can assume the existence of $r_0 > 0$ such that

$$\min\{m_E(x, \varrho), m_{E^c}(x, \varrho)\} < \gamma\mu(B_\varrho(x)) \qquad \forall x \in K, \ \varrho \in (0, r_0).$$

We define

$$\underline{m}_F(x, \varrho) := \frac{2}{\varrho} \int_{\varrho/2}^{\varrho} m_F(x, \tau) \, d\tau, \qquad \overline{\mu}(B_\varrho(x)) := \frac{1}{\varrho} \int_{\varrho}^{2\varrho} \mu(B_\tau(x)) \, d\tau,$$

and notice that $\underline{m}_F(x,\varrho) \le m_F(x,\varrho)$, $\mu(B_\varrho(x)) \le \overline{\mu}(B_\varrho(x))$ and that $\varrho \mapsto \underline{m}_F(x,\varrho)$, $\varrho \mapsto \overline{\mu}(B_\varrho(x))$ are continuous. By a continuity argument, either $\underline{m}_E(x,\varrho) < \gamma\overline{\mu}(B_\varrho(x))$ in $(0,r_0)$ or $\underline{m}_{E^c}(x,\varrho) < \gamma\overline{\mu}(B_\varrho(x))$ in $(0,r_0)$ and not both, provided $\gamma C_D^2 < 1$, because $\overline{\mu}(B_\varrho(x)) \le \mu(B_{2\varrho}(x))$ and $\underline{m}_E(x,\varrho) + \underline{m}_{E^c}(x,\varrho) \ge \mu(B_{\varrho/2}(x))$. Hence, possibly splitting K in two parts and replacing E by E^c, we can assume that $\underline{m}_E(x,\varrho) < \gamma C_D\mu(B_\varrho(x))$ in $(0,r_0)$ and therefore

$$m_E(x,\varrho) \le \underline{m}_E(x,2\varrho) \le \gamma C_D^2\mu(B_\varrho(x)) \qquad \forall\varrho \in (0,r_0/2). \tag{5.1}$$

Let $r \in (0,r_0/4)$ and let $x_1,\ldots,x_n \in K$ be recursively chosen in such a way that $d(x_i,x_j) \ge r$ for $i \ne j$ and $K \subset \cup_i B_r(x_i)$. We can find $\rho_i \in (r,2r)$ such that $\mathcal{H}^h(\partial B_{\rho_i}(x_i))$ is finite and

$$r\frac{d}{d\varrho}m_E(x,\rho)\bigg|_{\varrho=\varrho_i} \le m_E(x,2r) \le C_I\left(\frac{(2r)^s}{\mu(B_{2r}(x))}\right)^{1/(s-1)} [P(E,B_{2\lambda r}(x_i))]^{s/(s-1)},$$

where we have used (5.1) and we have assumed $\gamma C_D^2 < 1/2$. We can also choose recursively ρ_i in such a way that $\mathcal{H}^h(\partial B_{\rho_i}(x_i)\cap\partial B_{\rho_j}(x_j)) = 0$ whenever $i \ne j$. By Lemma 4.5 and (4.2) we get

$$\begin{aligned} P(E\setminus B_{\rho_i}(x_i),\partial B_{\rho_i}(x_i)) &\le \frac{1}{r}\left[m_E(x,2r)\right]^{1/s+(s-1)/s} \\ &\le 2(\gamma C_D^2)^{1/s}C_I^{(s-1)/s}P(E,B_{2\lambda r}(x_i)). \end{aligned}$$

Now we estimate the overlapping of the balls $B_{2\lambda r}(x_i)$. Let $x \in X$ be in all balls $B_{2\lambda r}(x_i)$, $i \in J$. Taking into account that $d(x_i,x_j) \ge r$, we obtain that the balls $B_{r/2}(x_i)$ are pairwise disjoint and, for $i \in J$, contained in $B_{(2\lambda+1)r}(x)$. Since

$$\mu(B_{(2\lambda+1)r}(x)) \le \mu(B_{(4\lambda+1)r}(x_i)) \le c(\lambda,C_D)\mu(B_{r/2}(x_i)),$$

adding with respect to $i \in J$ we obtain that the cardinality of J is at most $\xi = 1/c(\lambda,C_D)$.

Then, setting $A_r = \cup_i B_{\rho_i}(x_i)$, subadditivity and locality of perimeter give

$$\begin{aligned} P(E\setminus A_r,X) &= P(E\setminus A_r,X\setminus A_r) = P(E,X\setminus\overline{A}_r) + P(E\setminus A_r,\partial A_r) \\ &\le P(E,X\setminus K) + \sum_{i=1}^n P(E\setminus A_r,\partial B_{\rho_i}(x_i)) \\ &\le P(E,X\setminus K) + \sum_{i=1}^n P(E\setminus B_{\rho_i}(x_i),\partial B_{\rho_i}(x_i)) + P(\bigcup_{j\ne i} B_{\rho_j}(x_i),\partial B_{\rho_i}(x_i)) \\ &\le P(E,X\setminus K) + 2(\gamma C_D^2)^{1/s}C_I^{(s-1)/s}\xi P(E,\cup_i B_{2\lambda r}(x_i)). \end{aligned}$$

In the last line we have used Lemma 5.2 and the fact that $\partial B_{\rho_i}(x_i)\cap\partial B_{\rho_j}(x_j)$ is $\mathcal{H}^h$-negligible for $i \ne j$. Since A_r is contained in the $2r$-neighbourhood of K and

$$\mu(E\cap A_r) \le \sum_{i=1}^n m_E(x_i,2r) \le 2r(\gamma C_D^2)^{1/s}C_I^{(s-1)/s}\xi P(E,X) \to 0,$$

passing to the limit as $r \downarrow 0$ we obtain

$$P(E,X) \le P(E,X\setminus K) + 2(\gamma C_D^2)^{1/s} C_I^{(s-1)/s} \xi P(E,K).$$

Thus, $P(E,K)=0$, provided $2(\gamma C_D^2)^{1/s} C_I^{(s-1)/s}\xi < 1$ and $\gamma C_D^2 < 1/2$.

This proves that $P(E,\cdot)$ is concentrated on Σ_γ. Denoting by $c(\lambda, C_D)$ a constant such that $c(\lambda, C_D)\mu(\overline{B}_{2\lambda\varrho}(x)) \le \mu(\overline{B}_\varrho(x))$, by (4.2) we get

$$\limsup_{\varrho\downarrow 0} \frac{P(E, B_{2\lambda\varrho}(x))}{h(\overline{B}_{2\lambda\varrho}(x))} \ge (\gamma c(\lambda, C_D)/C_I)^{(s-1)/s} \qquad \forall x \in \Sigma_\gamma,$$

hence (2.6) gives $P(E,B) \ge \gamma' \mathcal{H}^h(B)$ for any Borel set $B \subset \Sigma_\gamma$ with γ' depending only on (C_D, λ, C_I). As a consequence Σ_γ is σ-finite with respect to $\mathcal{H}^h$ and Lemma 5.2 in conjunction with the Radon–Nikodým theorem gives

$$P(E,B) = \int_{B\cap\Sigma_\gamma} \theta \, d\mathcal{H}^h \qquad \forall B \in \mathcal{B}(X)$$

for some Borel function $\theta : X \to [\gamma', \infty)$.

It remains to prove that $\partial^* E \setminus \Sigma_\gamma$ is $\mathcal{H}^h$-negligible. Notice that, since X is connected, the diameter of any ball $\overline{B}_\varrho(x)$ is at least ϱ, provided $\varrho < \mathrm{diam}\,(X)/2$.

By (2.6) with $\nu = P(E,\cdot)$ we know that $P(E, B_\varrho(x)) = o(h(\overline{B}_\varrho(x)))$ $\mathcal{H}^h$-a.e. in $X\setminus\Sigma_\gamma$, because ν is concentrated on Σ_γ. The relative isoperimetric inequality, in conjunction with the lower bound on the diameter of balls, gives

$$\min\{\underline{m}_E(x,\varrho), \underline{m}_{E^c}(x,\varrho)\} = o\,(\overline{\mu}(B_\varrho(x))) \qquad \text{for } \mathcal{H}^h\text{-a.e. } x \in X\setminus\Sigma_\gamma$$

and, by a continuity argument again, either $\underline{m}_E(x,\varrho) = o\,(\overline{\mu}(B_\varrho(x)))$ or $\underline{m}_{E^c}(x,\varrho) = o\,(\overline{\mu}(B_\varrho(x)))$ (thus $x \notin \partial^* E$) for $\mathcal{H}^h$-a.e. $x \in X \setminus \Sigma_\gamma$. □

Now we prove a lower density estimate for both perimeter and area and, as a consequence, the asymptotic doubling property.

Theorem 5.4 *The measure $P(E,\cdot)$ satisfies*

$$\infty > \limsup_{\varrho\downarrow 0} \frac{P(E,B_\varrho(x))}{h(\overline{B}_\varrho(x))} \ge \liminf_{\varrho\downarrow 0} \frac{P(E,B_\varrho(x))}{h(\overline{B}_\varrho(x))} \ge \tau_1 \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X \tag{5.2}$$

$$\liminf_{\varrho\downarrow 0} \min\left\{\frac{m_E(x,\varrho)}{\mu(B_\varrho(x))}, \frac{m_{E^c}(x,\varrho)}{\mu(B_\varrho(x))}\right\} \ge \tau_2 \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X \tag{5.3}$$

with τ_1, $\tau_2 > 0$ depending only on (C_D, λ, C_I).

Proof. The upper estimate in (5.2) follows by (2.7) and Lemma 5.2.

In the proof of the lower estimate in (5.3) we can assume that (X,d) is a *length* space, i.e., that any pair of points x, $y \in X$ can be connected by a rectifiable curve of length

$d(x, y)$. Indeed, by Theorem 3.2 we know that any pair of points x, $y \in X$ can be connected by a Lipschitz curve with length at most $Cd(x, y)$. Hence, we can simply replace d by the geodesic metric

$$\tilde{d}(x, y) := \inf \left\{ \sum_{i=1}^{n-1} d(x_{i+1}, x_i) : \ x_1 = x, \ x_n = y \right\}$$

associated to d and, taking into account that the class of locally Lipschitz functions is invariant, define a new D-structure $\tilde{\mathrm{D}}$ on $(X, \tilde{d}, \mu)$ saying that $g \in \tilde{\mathrm{D}}[u]$ if and only if $g/C \in \mathrm{D}[u]$. It is easy to check that the axioms A0-A4 still hold, and that axiom A5 holds with $\tilde{\lambda} = C\lambda$ and $\tilde{C}_P = 2C_P$ (see (3.2)). Notice also that the perimeter measures induced by D and $\tilde{\mathrm{D}}$ are comparable.

Since any ball in a length space is a John domain, from Corollary 9.8 in [28] we obtain the $(1^*, 1)$-Poincaré inequality. In particular, by Remark 4.4, the relative isoperimetric inequality (4.4) follows. We will also use the fact (when applying Proposition 5.7) that the diameter of any ball $\overline{B}_\varrho(x)$ is at least ϱ for $\varrho < \mathrm{diam}\,(X)/2$.

We define

$$\alpha := \frac{1}{2sC_I^{(s-1)/s}}, \qquad m(\varrho) := [\mu(B_\varrho(x)]^{1/s}$$

and fix a strictly positive number $d < \min\{1/2, \gamma/C_D^{2s}, (\alpha/2)^s C_D^{-2s}\}$. Notice that m is a doubling function in $(0, \infty)$, with doubling constant $c_D = C_D^{1/s}$.

By applying Proposition 5.7 below both to E and to E^c, we need only to prove the lower estimate in (5.2) for any compact set $K \subset X$ where the following property holds: there exists $\varrho_0 > 0$ such that for any $x \in K$ and a.e. $\varrho \in (0, \varrho_0)$, the volume bounds $\mu(B_\varrho(x)) \geq m_E(x, \varrho) \geq d\mu(B_\varrho(x))$ imply

$$P(E, B_\varrho(x)) \leq 2P(E \setminus B_\varrho(x), \partial B_\varrho(x)) \tag{5.4}$$

and the volume bounds $\mu(B_\varrho(x)) \geq m_{E^c}(x, \varrho) \geq d\mu(B_\varrho(x))$ imply

$$P(E^c, B_\varrho(x)) \leq 2P(E^c \setminus B_\varrho(x), \partial B_\varrho(x)). \tag{5.5}$$

Let $v(\varrho) := [\min\{m_E(x, \varrho), m_{E^c}(x, \varrho)\}^{1/s}$; then, if $v(\varrho) \geq d^{1/s} m(\varrho)$, Lemma 4.5, (5.4) and the relative isoperimetric inequality give

$$\begin{aligned} sv'(\varrho) &\geq [m_E(x, \varrho)]^{(1-s)/s} P(E \setminus B_\varrho(x), \partial B_\varrho(x)) \geq \frac{1}{2}[m_E(x, \varrho)]^{(1-s)/s} P(E, B_\varrho(x)) \\ &\geq \alpha s \frac{m(\varrho)}{\varrho} \end{aligned} \tag{5.6}$$

for a.e. $\varrho \in (0, \varrho_0)$ such that $v(\varrho) = [m_E(x, \varrho)]^{1/s}$. Analogously, for a.e. $\varrho \in (0, \varrho_0)$ such that $v(\varrho) = [m_{E^c}(x, \varrho)]^{1/s}$ we obtain $v'(\varrho) \geq \alpha m(\varrho)/\varrho$. Summing up, we have checked that, for a.e. $\varrho \in (0, \varrho_0)$, $v(\varrho) \geq d^{1/s} m(\varrho)$ implies $v'(\varrho) \geq \alpha m(\varrho)/\varrho$.

By Theorem 5.3 we can also assume that

$$\limsup_{\varrho \downarrow 0} \frac{v(\varrho)}{m(\varrho)} \geq \gamma^{1/s}. \tag{5.7}$$

Let $\tilde{m}$ be given by Lemma 5.6 below and let $u := v - \beta\tilde{m}$, with $\beta \geq c_D d^{1/s}$, $\beta < \alpha/(2c_D)$ and $\beta < \gamma^{1/s}/c_D$ (this choice is possible, due to our choice of d). Notice that the negative part of Du is absolutely continuous with respect to $\mathcal{L}^1$ because v is nondecreasing and $\tilde{m}$ is an absolutely continuous function. Moreover $u(\varrho) \geq 0$ implies, by our choice of β, $v(\varrho) \geq d^{1/s} m(\varrho)$ and therefore

$$u'(\varrho) = v'(\varrho) - \beta\tilde{m}'(\varrho) \geq (\alpha - 2\beta c_D)\frac{m(\varrho)}{\varrho} \geq 0$$

again by our choice of β. In addition, by (5.7) we infer

$$\limsup_{\varrho\downarrow 0} \frac{u(\varrho)}{\tilde{m}(\varrho)} = \limsup_{\varrho\downarrow 0} \frac{v(\varrho)}{\tilde{m}(\varrho)} - \beta \geq \frac{\gamma^{1/s}}{c_D} - \beta > 0.$$

Then, the Calculus lemma below gives that u is strictly positive in $(0, \varrho_0)$, hence

$$v(\varrho) \geq \beta\tilde{m}(\varrho) > \frac{\beta}{c_D} m(\varrho) \qquad \forall \varrho \in (0, \varrho_0).$$

This proves (5.3) with $\tau_2 = (\beta/c_D)^s$.

Finally, using (5.3) in conjunction with the relative isoperimetric inequality (4.4) we obtain the lower bound in (5.2). $\square$

In the next proposition we list some elementary properties functions with finite pointwise variation needed in the proof of Theorem 5.4. We recall that the derivative in the sense of distributions of any function u with finite pointwise variation in an interval $I = (a, b)$ is representable by a finite measure Du; the fundamental theorem of calculus, in this setting, says that u coincides a.e. in I with the function $u(a+) + Du\left((a, t)\right)$ (see for instance [3], [36]).

Lemma 5.5 (Calculus Lemma) *Let $\varrho_0 > 0$ and let $u : (0, \varrho_0) \to \mathbf{R}$ be with finite pointwise variation. Let us assume that u is left continuous and that the negative part of Du is absolutely continuous with respect to $\mathcal{L}^1$. Then:*

(i) if $u' \geq 0$ a.e. in $(0, \varrho_0)$ then u is nondecreasing;

(ii) if

$$u' \geq 0 \qquad \text{a.e. in } \{u \geq 0\} \tag{5.8}$$

and $\limsup_{\varrho\downarrow 0} u(\varrho) > 0$, then u is strictly positive and nondecreasing in $(0, \varrho_0)$.

Proof. (i) It is well known that the absolutely continuous part of Du is given by $u'\mathcal{L}^1$. Then, the negative part of Du vanishes. Since, by the fundamental theorem of calculus in BV, u coincides a.e. with the nondecreasing function and left continuous function

$$\tilde{u}(t) := u(0+) + Du\left((0, t)\right),$$

the left continuity of u gives $u = \tilde{u}$ in $(0, \varrho_0)$.
(ii) It suffices to apply (i) to u^+, the positive part of u, to obtain that u^+ is nondecreasing. Since $u^+(\varrho) > 0$ for arbitrarily small $\varrho > 0$ we obtain that $u = u^+$ in $(0, \varrho_0)$. $\square$

We also need an elementary lemma, showing that for any doubling function m there is always a comparable function $\tilde{m}$ absolutely continuous (and doubling as well).

Lemma 5.6 *Let $\varrho_0 > 0$ and let $m : (0, \varrho_0) \to (0, \infty)$ be a nondecreasing doubling function. Then, there exists an absolutely continuous nondecreasing function $\tilde{m}$ satisfying*

$$\frac{1}{c_D} m \le \tilde{m} \le c_D m \quad \text{in } (0, \varrho_0), \qquad \tilde{m}' \le 2c_D \frac{m}{\varrho} \quad \text{a.e. in } (0, \varrho_0),$$

where c_D is the doubling constant of m.

Proof. Assume for simplicity $\varrho_0 = \infty$. We define $\tilde{m}(\varrho) = m(\varrho)$ if $\varrho = 2^i$ for some $i \in \mathbf{Z}$ and extend $\tilde{m}$ to $(0, \infty)$ by linear interpolation. For $\varrho > 0$, denoting by i the unique integer such that $2^{i-1} \le \varrho < 2^i$, we have

$$\tilde{m}(\varrho) \le \tilde{m}(2^i) = m(2^i) \le c_D m(2^{i-1}) \le c_D m(\varrho)$$

and

$$m(\varrho) \le m(2^i) \le c_D m(2^{i-1}) = c_D \tilde{m}(2^{i-1}) \le c_D \tilde{m}(\varrho).$$

Moreover, if $2^{i-1} < \varrho$ we have

$$\tilde{m}'(\varrho) = 2^{1-i} \left(m(2^i) - m(2^{i-1}) \right) \le 2c_D \frac{m(\varrho)}{\varrho}.$$

□

Proposition 5.7 (Asymptotic quasi-minimality) *Assume that the relative isoperimetric inequality (4.4) holds. Let $d \in (0, 1/2)$ and $M > 1$. Then, for $P(E, \cdot)$-a.e. $x \in X$ there exists $\varrho_x > 0$ such that, for a.e. $\varrho \in (0, \varrho_x)$, the volume bounds*

$$\frac{1}{2} \mu(B_\varrho(x)) \ge m_E(x, \varrho) \ge d\mu(B_\varrho(x))$$

imply

$$P(E, B_\varrho(x)) \le MP(E \setminus B_\varrho(x), \partial B_\varrho(x)).$$

Proof. Let $\mathcal{B}$ be the family of all balls $B_\varrho(x)$ of finite perimeter such that

(a) $P(E, \partial B_\varrho(x)) = 0$ and $\mu(\partial B_\varrho(x)) = 0$;

(b) $\frac{1}{2} \mu(B_\varrho(x)) \ge m_E(x, \varrho) \ge d\mu(B_\varrho(x))$;

(c) $P(E, B_\varrho(x)) > MP(E \setminus B_\varrho(x), \partial B_\varrho(x))$.

Notice that, for x given, the conditions in (a) are fulfilled with at most countably many exceptions and that, since X is connected, the diameter of any ball $B_\varrho(x)$ is at least ϱ whenever $\varrho < \operatorname{diam}(X)/2$. Let $B = \cap_h B_h$, where B_h is the set of points x such that

$$\left\{\varrho \in (0, 2^{-h}) : \text{ (b) and (c) hold } \right\}$$

has strictly positive measure. It is easy to check that

$$L_h := \left\{(x, \varrho) \in X \times (0, \infty) : \; \varrho < 2^{-h} \text{ and (b), (c) hold } \right\}$$

is a Borel subset of $X \times (0, \infty)$, and therefore

$$B = \bigcap_{h=1}^{\infty} \left\{x : \; \int_0^\infty \chi_{L_h}(x, \tau)\, d\tau > 0\right\}$$

is a Borel set as well.

We will prove that $P(E, K) = 0$ for any compact set $K \subset B$. To this aim, for any $\delta > 0$ we consider the family

$$\mathcal{F} = \left\{\overline{B}_\varrho(x) : \; x \in K, \; \varrho \in (0, \delta), \; B_\varrho(x) \in \mathcal{B}\right\}.$$

Notice that, for any ball $\overline{B}_\varrho(x) \in \mathcal{F}$, the second condition in (a), (b), the lower bound on the diameter of balls and the relative isoperimetric inequality (4.4) give $P(E, B_\varrho(x)) \geq (d/C_I)^{(s-1)/s} h(\overline{B}_\varrho(x))$. In particular, by the inclusion $K \subset B$, $\mathcal{F}$ fulfils the assumptions (i), (ii) of Theorem 2.1.

Hence, by applying Theorem 2.1 (with ν equal to a constant multiple of $P(E, \cdot)$), we can find a disjoint family of balls $(\overline{B}_{\varrho_i}(x_i))_{i \in I} \subset \mathcal{F}$ such that $\cup_i \overline{B}_{\varrho_i}(x_i)$ contains $\mathcal{H}^h$-almost all (hence $P(E, \cdot)$ almost all) of K. By the first condition in (a), the open set $A_\delta = \cup_i B_{\varrho_i}(x_i)$ satisfies $P(E, K \setminus A_\delta) = 0$.

Let $J \subset I$ be a finite family and let A_J be the union of the balls $B_{\varrho_i}(x_i)$, $i \in J$. By locality and subadditivity of perimeter we get

$$\begin{aligned}
P(E \setminus A_J, X) &= P(E \setminus A_J, X \setminus A_J) = P(E \setminus A_J, X \setminus \overline{A}_J) + P(E \setminus A_J, \partial A_J) \\
&= P(E, X \setminus \overline{A}_J) + P(E \setminus A_J, \partial A_J) \\
&\leq P(E, X \setminus A_J) + \sum_{i,\, j \in J} P(E \setminus B_{\varrho_i}(x_i), \partial B_{\varrho_j}(x_j)) \\
&= P(E, X \setminus A_J) + \sum_{i \in J} P(E \setminus B_{\varrho_i}(x_i), \partial B_{\varrho_i}(x_i)) \\
&\leq P(E, X \setminus A_J) + M^{-1} P(E, A_\delta).
\end{aligned}$$

Letting $J \uparrow I$ and using the lower semicontinuity of perimeter we infer

$$P(E \setminus A_\delta, X) \leq P(E, X \setminus A_\delta) + \frac{1}{M} P(E, A_\delta) \leq P(E, X \setminus K) + \frac{1}{M} P(E, A_\delta).$$

Since

$$\mu(A_\delta) \le 2 \sup_i \varrho_i \sum_i h(\overline{B}_{\varrho_i}(x_i)) \le 2\delta (C_I/d)^{(s-1)/s} P(E,X),$$

letting $\delta \downarrow 0$ and using again the lower semicontinuity of perimeter we obtain $P(E,X) \le P(E, X \setminus K) + P(E,K)/M$, hence $P(E,K) = 0$. $\square$

By (5.2) and the doubling property of h we obtain that the perimeter measure is a.e. asymptotically doubling.

Corollary 5.8 (Asymptotic doubling property) *The measure $P(E,\cdot)$ is a.e. asymptotically doubling, i.e.*

$$\limsup_{\varrho \downarrow 0} \frac{P(E, B_{2\varrho}(x))}{P(E, B_\varrho(x))} < \infty \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X.$$

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