Boundary regularity results for weak solutions of subquadratic elliptic systems

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Zusammenfassung:

Die vorliegende Arbeit liefert einen Beitrag zur Regularitätstheorie für nichtlineare elliptische Systeme partieller Differentialgleichungen zweiter Ordnung. Wir betrachten schwache Lösungen $u \in g + W_0^{1,p}(\Omega, \mathbb{R}^N)$ mit vorgeschriebenen Randwerten $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ des inhomogenen elliptischen Systems

$$-\operatorname{div} a(\,\cdot\,, u, Du) \,=\, b(\,\cdot\,, u, Du) \qquad \text{in } \Omega$$

für ein beschränktes C^1-Gebiet $\Omega \subset \mathbb{R}^n$ und Koeffizienten $a(\cdot,\cdot,\cdot)$, die den üblichen Bedingungen bzgl. Stetigkeit, Wachstum und Elliptizität genügen. Die Inhomogenität $b(\cdot,\cdot,\cdot)$ sei eine Carathéodory-Funktion, die entweder eine kontrollierbare oder eine natürliche Wachstumsbedingung erfüllt. Unter diesen Voraussetzungen werden vor allem für den subquadratischen Fall $1 < p < 2$ höhere Integrierbarkeits- bzw. Regularitätsaussagen der folgenden Art (bis zum Rand von Ω) erzielt:

Sind Ω sowie die Randdaten g von der Klasse $C^{1,\alpha}$, $\alpha \in (0,1)$, und sind die Koeffizienten Hölder-stetig mit Exponent α in den ersten beiden Variablen, so geben wir mithilfe der Methode der $\mathcal{A}$-harmonischen Approximation eine *Charakterisierung der regulären Punkte* von Du bis zum Rand. Der Beweis führt direkt zur optimalen höheren Regularität auf der regulären Menge (d. h. lokale Hölder-Stetigkeit von Du zum Exponenten α).

Für C^1-Randwerte g sowie gleichmäßig stetige Koeffizienten zeigen wir *Calderón-Zygmund-Abschätzungen*, ein höheres Integrabilitätsresultat, bei dem im Unterschied zu klassischen Resultaten nach Gehring der Gewinn an Integrierbarkeit in quantifizierter Weise bestimmt wird. Hängen die Koeffizienten nicht explizit von u ab und liegt die Inhomogenität $b(x,u,z) \equiv b(x)$ in $L^{p/(p-1)}$, so gilt: $b \in L^{q/(p-1)}(\Omega, \mathbb{R}^N)$ und $g \in W^{1,q}(\Omega, \mathbb{R}^N)$ garantieren $Du \in L^q(\Omega, \mathbb{R}^{nN})$ für $q \in [p, \frac{np}{n-2} + \delta_1]$ (bzw. q beliebig, falls $n = 2$).

In *niedrigen Dimensionen* $n \in (p, p+2]$ beweisen wir außerdem mit der direkten Methode und Morrey-Abschätzungen: u ist lokal Hölder-stetig zu jedem Exponenten $\lambda \in (0, 1 - \frac{n-2}{p})$ außerhalb einer singulären Menge, deren Hausdorffdimension kleiner als $n - p$ ist. Dieses Resultat gilt sowohl für nicht-degenerierte als auch für degenerierte Systeme.

Im letzten Teil der Arbeit beschäftigen wir uns mit Techniken, die eine *Abschätzung der Hausdorffdimension der singulären Menge* von Du in $\overline{\Omega}$ erlauben. Dabei finden alle bisher erzielten Resultate ihre Anwendung. Sind Ω und g von der Klasse $C^{1,\alpha}$ für ein $\alpha \in (0,1)$ und die Koeffizienten Hölder-stetig mit Exponent α in den ersten beiden Variablen, so stellt sich heraus, dass die Hausdorff-Dimension der singulären Menge von Du höchstens $\min\{n-p, n-2\alpha\}$ ist, falls $n \in (p, p+2]$ erfüllt ist. Somit ist insbesondere für $\alpha > \frac{1}{2}$ *fast jeder Randpunkt regulär* (für eine natürliche Wachstumsbedingung an die Inhomogenität wird dies nur für den Fall $p = 2$ gezeigt). Ferner gilt dieselbe Aussage für Koeffizienten der Form $a(x,u,z) \equiv a(x,z)$ unter einer kontrollierbaren Wachstumsbedingung ohne Einschränkung an die Dimension n. Der Beweis basiert auf endlichen Differenzen-Operatoren, Interpolationstechniken und gebrochenen Sobolev-Räumen. Um dieser Strategie auch am Rand folgen zu können, stellen wir zwei unterschiedliche Methoden vor: für kontrollierbares Wachstum gehen wir indirekt vor und nutzen eine Familie von Vergleichsabbildung, die Lösungen eines regularisierten Systems sind, sowie Calderón-Zygmund-Abschätzungen. Für natürliches Wachstum hingegen argumentieren wir direkt und verwenden die Tatsache, dass schichtweise gemittelte Koeffizienten in normaler Richtung schwach differenzierbar sind.

Abstract:

The current thesis makes a contribution to the field of regularity theory of second-order nonlinear elliptic systems. We consider weak solutions $u \in g+W_0^{1,p}(\Omega,\mathbb{R}^N)$ of the inhomogeneous elliptic system

$$-\operatorname{div} a(\,\cdot\,,u,Du) \;=\; b(\,\cdot\,,u,Du) \qquad \text{in } \Omega$$

with prescribed boundary data $g \in W^{1,p}(\Omega,\mathbb{R}^N)$, a bounded domain $\Omega \subset \mathbb{R}^n$ of class C^1 and a vector field $a(\cdot,\cdot,\cdot)$ which satisfies standard continuity, ellipticity and growth conditions. The inhomogeneity $b : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ is assumed to be a Carathéodory function obeying either a controllable or a natural growth condition. Under these assumptions, the following higher integrability and regularity results (up to the boundary of Ω) are achieved, mainly for the subquadratic case $1 < p < 2$:

We first require that Ω and g are of class $C^{1,\alpha}$, $\alpha \in (0,1)$, and that the coefficients are Hölder continuous with exponent α with respect to the first and second variable. Via the method of $\mathcal{A}$-harmonic approximation we give a *characterization of regular points* for Du up to the boundary which extends known results to the inhomogeneous case. The proof yields directly the optimal higher regularity on the regular set (i. e., local Hölder continuity of Du with exponent α).

Provided that the boundary data g is of class C^1 and that the coefficients are uniformly continuous we then show *Calderón-Zygmund estimates*, a higher integrability result that yields, in contrast to classical higher integrability obtained from the application of Gehring's Lemma, a quantified gain in the higher integrability exponent. If the coefficients do not depend explicitly on u and if the inhomogeneity $b(x,u,z) \equiv b(x)$ belongs to $L^{p/(p-1)}$, then there holds: $b \in L^{q/(p-1)}(\Omega,\mathbb{R}^N)$ and $g \in W^{1,q}(\Omega,\mathbb{R}^N)$ imply $Du \in L^q(\Omega,\mathbb{R}^{nN})$ for $q \in [p, \frac{np}{n-2} + \delta_1]$ (or q arbitrary if $n = 2$).

Moreover, in *low dimensions* $n \in (p, p+2]$, we prove via the direct method and Morrey-type estimates: u is locally Hölder continuous with every exponent $\lambda \in (0, 1 - \frac{n-2}{p})$ outside a singular set of Hausdorff dimension less than $n - p$. This result holds true both for non-degenerate and degenerate systems.

The last part of the thesis is devoted to techniques which allow us to *estimate the Hausdorff dimension of the singular set* of Du in $\overline{\Omega}$. Here, all the result achieved so far are of importance. Assuming that Ω and g are of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and that the coefficients are Hölder continuous with exponent α with respect to the first and second variable, we find: The Hausdorff dimension of the singular set of Du does not exceed $\min\{n-p, n-2\alpha\}$ whenever $n \in (p, p+2]$. In particular, for $\alpha > \frac{1}{2}$ this implies that *almost every boundary point is in fact a regular one* (for a natural growth condition this is proved only for $p = 2$). Furthermore, this conclusion remains valid for coefficients of the form $a(x,u,z) \equiv a(x,z)$ and inhomogeneities of controllable growth without any restriction on the dimension n. The proof is based on finite difference operators, interpolation techniques and fractional Sobolev spaces. To extend this strategy up to the boundary, we present two different methods: for controllable growth we proceed directly and use a family of comparison maps (which are solutions of some regularized system) as well as Calderón-Zygmund estimates. For natural growth, however, we argue in a direct way and employ the fact that slicewise mean values of the coefficients are weakly differentiable in the normal direction.

Contents

Chapter 1

Introduction

Partial differential equations are often motivated by problems from science and serve as simplified models of physical phenomena. In general, we investigate the existence of a solution, and furthermore, its qualitative properties like regularity and differentiability. An intuitive example is the solution to the minimal surface equation – such as a soap film realizing the least surface area amongst all surfaces spanned by a wire. This equation like many other partial differential equations in science arises from the universal principle that nature favours states of minimal type or energy. For this reason, partial differential equations have been of substantial interest for a long time, and they have finally been studied in a systematic way – independent of practical applications – since the end of the 19th century. One of the crucial moments was the year 1900 when David Hilbert formulated 23 unsolved mathematical problems in his famous lecture at the International Congress of Mathematicians in Paris, one of them being

Are the solutions of regular problems in the calculus of variations necessarily analytic?

In general, this question was answered in the negative, which in turn raised new questions when trying to obtain regularity results in some weaker sense. One discarded the strategy to search for classical solutions (i. e., solutions which are sufficiently smooth). Instead, even in the cases where the previous question is answered in the affirmative, one first looks for "weak" solutions in suitable Sobolev spaces solving the equation in an integrated form. This allows to infer the existence of weak solutions via methods from functional analysis like Galerkin's method for nonlinear monotone operators. However, in the following we will only briefly touch existence problems.

Then, in a second step, one is concerned with the regularity properties of these solutions. Starting from the famous papers of De Giorgi, Nash and Moser [DG57, Nas58, Mos60] the theory of (scalar-valued) solutions to single equations is by now well-understood. In particular, it has been shown, under quite general assumptions on the coefficients of the equation, that solutions are in fact smooth. On the other hand, in the vectorial case counterexamples of De Giorgi [DG68] and of Giusti and Miranda [GM68b] dating from 1968 have revealed that solutions to elliptic systems (as well as minima of variational integrals) may develop singularities even if the coefficients are analytic. Hence, in contrast to equations, we can only expect partial regularity results for general nonlinear systems, which means that the solution is regular outside a *singular* set. Having to abandon full regularity, we are then interested in estimating the size of the singular set. This will be the main objective of this thesis, focusing on estimates up to the boundary and the subquadratic setting.

The different chapters of this work are mostly self-contained. Thus, we do not provide an extensive discussion of the historical background of the results in this introduction and postpone it to the following chapters. For a broader discussion, we refer to Giaquinta's monograph [Gia83] and Mingione's recent survey article [Min06]. Here, we rather concentrate on giving a rough overview of the results achieved in the current work and how they fit in the framework of dimension reduction of the singular set. We also give a brief explanation of some features of the proofs. We will now begin by describing the system under consideration:

Let $n, N \in \mathbb{N}$, $n \geq 2$, $p \in (1,2)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1. We consider weak solutions $u \in g + W_0^{1,p}(\Omega, \mathbb{R}^N)$ of the inhomogeneous elliptic system

$$-\operatorname{div} a(\,\cdot\,, u, Du) \;=\; b(\,\cdot\,, u, Du) \qquad \text{in } \Omega \tag{1.1}$$

with prescribed boundary values $g \in W^{1,p}(\Omega, \mathbb{R}^N)$. The vector field $a\colon \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is supposed to be of class C^1 with respect to the last variable (possibly apart from the origin) and to satisfy standard ellipticity and growth conditions

$$\begin{cases} |a(x,u,z)| \;\leq\; L\left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}}, \\ \nu\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \;\leq\; D_z a(x,u,z)\,\lambda \cdot \lambda \;\leq\; L\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,u,z) - a(\bar{x},\bar{u},z)| \;\leq\; L\left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}}\omega\big(|x-\bar{x}| + |u-\bar{u}|\big) \end{cases}$$

for all $x, \bar{x} \in \Omega$, $u, \bar{u} \in \mathbb{R}^N$ and $z, \lambda \in \mathbb{R}^{nN}$, where $0 < \nu \leq L$ and $\mu \in [0,1]$ are arbitrary constants and $\omega\colon \mathbb{R}^+ \to (0,1]$ is a modulus of continuity. The inhomogeneity $b : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ is assumed to be a Carathéodory function obeying either a controllable or a natural growth condition, i. e.,

$$|b(x,u,z)| \;\leq\; L_1\,(1 + |z|^2)^{\frac{p-1}{2}} \qquad \text{or} \qquad |b(x,u,z)| \;\leq\; L_2\,(1 + |z|^2)^{\frac{p}{2}}\,.$$

We want to comment briefly on the weak formulation of the Dirichlet problem (1.1) and a suitable space for weak solutions depending on which growth condition on the inhomogeneity is assumed: Here the term weak solution signifies that u solves (1.1) in integrated form, i. e., there holds $\int_\Omega a(\cdot, u, Du) \cdot D\varphi\,dx = \int_\Omega b(\,\cdot\,, u, Du) \cdot \varphi\,dx$ for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$. The boundary condition $u = g$ on $\partial\Omega$ is to be understood in the sense of traces. In particular, the existence of second derivatives of u is not required for the weak formulation of (1.1). In general, we shall consider weak solutions in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$. Then, taking into account the growth condition on the coefficients and on the inhomogeneity, we note that the integrals arising in the weak formulation are well-defined and finite. In case of a natural growth condition, however, we restrict our attention to *bounded* weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$. To justify this restriction, we recall the following example from [Hil82, Section 2]: Considering the equation $\triangle u = |Du|^2$ in $B_{1/2} \subset \mathbb{R}^2$, we observe that the functions $u_1 \equiv 0$ and $u_2 = \log\log(1/|x|) - \log\log 2$ are two distinct solutions in $W^{1,2}(B_{1/2})$ both vanishing on the boundary $\partial B_{1/2}$. A straightforward adaption of this example also applies in the subquadratic setting. Hence, taking $W^{1,p}(\Omega, \mathbb{R}^N)$ to be the class of admissible weak solutions may result in a violation of the "principle of local uniqueness" which in turn is related to the occurence of irregular weak solutions even in the case of equations, see also [LU68, Section 1.2]. Apart from boundedness, we will have to assume an additional smallness condition on the weak solution u. More precisely, we will assume for the remainder of this introduction that one of the following two conditions holds:

(1) the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition,

(2) the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a natural growth condition and $u \in L^\infty(\Omega,\mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \le M$ and $2L_2M < \nu$.

Keeping in mind these assumptions, we are concerned with the following topics related to higher integrability and regularity (up to the boundary of Ω):

Partial regularity of Du

We now consider non-degenerate systems ($\mu > 0$) under the assumption of Hölder continuous coefficients, that is $\omega(t) = \min\{1, t^\alpha\}$ for some $\alpha > 0$. As mentioned above, passing from equations to systems (i. e., from $N = 1$ to $N > 1$), weak solutions may develop singularities. Consequently, in a first step, one is interested in proving a partial regularity result, namely that Du is locally Hölder continuous outside a set of $\mathcal{L}^n$-measure zero. For this purpose we introduce the set of regular points

$$\mathrm{Reg}_{Du}(\overline{\Omega}) := \big\{x \in \overline{\Omega} : Du \in C^0(U \cap \overline{\Omega}, \mathbb{R}^{nN}) \text{ for a neighbourhood } U \text{ of } x\big\}$$

and the set of singular points $\mathrm{Sing}_{Du}(\overline{\Omega}) := \overline{\Omega} \setminus \mathrm{Reg}_{Du}(\overline{\Omega})$ of the gradient Du. The proof of partial regularity results for nonlinear systems usually relies on a linearization technique which involves the frozen (linearized) system. Since solutions to linear systems enjoy good a priori regularity estimates, a comparison principle yields a decay estimate for Du which is the crucial step in order to control its local behaviour at a given point in $\overline{\Omega}$. Actually, there are different proofs of partial regularity, which mainly differ in the implementation of the linearization described above. By now, these techniques are the indirect approach via the blow-up technique, the direct approach, and the method of $\mathcal{A}$-harmonic approximation. Partial regularity results using these methods were first achieved in the interior (in the quadratic case) by Morrey, Giusti and Miranda [Mor68, GM68a], Giaquinta, Modica and Ivert [GM79, Ive79], and Duzaar and Grotowski [DG00], respectively. Furthermore, Grotowski and Hamburger [Gro00, Ham07] succeeded in extending these techniques up to the boundary in the (super-)quadratic case and gave a characterization of regular boundary points (see also [Kro05] for the analogous results concerning almost minimizers of quasiconvex variational integrals).

Various subsequent papers were concerned with regularity results for more general nonlinear systems. We only mention the role of the modulus of continuity $\omega(\cdot)$: The assumption of Hölder continuity was weakened by Duzaar, Gastel and by Wolf to Dini-continuous coefficients requiring merely $\int_0^r \frac{\omega(\rho)}{\rho} d\rho < \infty$ for some $r > 0$, which still allows to conclude a partial regularity result for Du, see [DG02, Wol01a]. Assuming merely continuity of the coefficients, Foss and Mingione [FM08] recently gave a positive answer to the question of low order partial regularity.

Our first result in this paper is a partial regularity result for inhomogeneous systems with sublinear growth, stating that Du is in fact not only continuous but Hölder continuous with optimal exponent on the set of regular points $\mathrm{Reg}_{Du}(\overline{\Omega})$, and a characterization of $\mathrm{Reg}_{Du}(\overline{\Omega})$ (see Theorem 3.1 and Theorem 3.2):

Theorem 1.1: *Consider $p \in (1,2)$, $\alpha \in (0,1)$, a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^{1,\alpha}$ and $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. Let $u \in g + W_0^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.1) under the assumptions stated above with $\omega(t) = \min\{1, t^\alpha\}$. Then, for $y \in \mathrm{Reg}_{Du}(\overline{\Omega})$ there holds: Du is Hölder continuous with exponent α in a neighbourhood of y in $\overline{\Omega}$, and the set of singular boundary points is contained in $\Sigma_1 \cup \Sigma_2$ with*

$$\Sigma_1 = \Big\{ y \in \overline{\Omega} \colon \liminf_{\rho \to 0^+} \fint_{\Omega \cap B_\rho(y)} \big| V(Du) - \big(V(Du)\big)_{\Omega \cap B_\rho(y)} \big|^2 \, dx > 0 \Big\},$$

$$\Sigma_2 = \Big\{ y \in \overline{\Omega} \colon \limsup_{\rho \to 0^+} \big| \big(V(Du)\big)_{\Omega \cap B_\rho(y)} \big| = \infty \Big\},$$

where $V \colon \mathbb{R}^N \to \mathbb{R}^N$ is given by $V(\xi) = (1 + |\xi|^2)^{(p-2)/4} \xi$ for all $\xi \in \mathbb{R}^N$. In particular, we have $\mathcal{L}^n(\mathrm{Sing}_{Du}(\overline{\Omega})) = 0$.

The homogeneous case was treated in [Bec05]. Moreover, Wolf [Wol01b] already achieved some regularity results for the subquadratic situation. Here we follow ideas of Grotowski [Gro00, Gro02b] for the characterization of regular boundary points in the quadratic case $p = 2$ and from Duzaar, Grotowski and Kronz [DGK05] for the subquadratic situation: Our proof of this partial regularity result is based on the method of $\mathcal{A}$-harmonic approximation introduced by Duzaar and Steffen [DS02]: using good a priori estimates up to the boundary for solutions of linear systems with constant coefficients and an adequate Caccioppoli inequality, this method allows us to derive an excess-decay estimate for the gradient of the weak solution u of the nonlinear system (1.1). The presence of an inhomogeneity, in particular in case of a natural growth condition, demands technical modifications, e. g. the derivation of the Caccioppoli inequality becomes considerably more involved compared to the homogeneous situation. From Campanato's integral characterization of Hölder continuous functions we finally conclude the desired local Hölder continuity of Du.

Since the boundary $\partial\Omega$ itself is of Lebesgue measure zero, Theorem 1.1 does not yield the existence of a single regular boundary point, whereas due to a counterexample of Giaquinta [Gia78] the existence of irregular boundary points has been known for a while. In order to close this gap, the remaining part of the thesis is devoted to finding conditions which guarantee that the sets Σ_1 and Σ_2 defined above are not only $\mathcal{L}^n$-negligible sets, but even allow a suitable upper bound on their Hausdorff dimension. To this end, we first observe that a measure density result due to Giusti allows us to gain control of the Hausdorff dimension of Σ_1 and Σ_2, provided that Du belongs to some "better" space. For example, if the coefficients do not depend on (x, u), then standard difference quotients reveal $Du \in W^{1,p}(\Omega, \mathbb{R}^{nN})$ which in turn implies that the Hausdorff dimension of Σ_1 and Σ_2 does not exceed $n-p$. Thus, higher integrability or higher differentiability of Du will be of central interest. These considerations naturally lead to the investigation of Calderón-Zygmund estimates, a technique which will enable us to carry higher integrability of the right-hand side and the boundary values over to the weak solution.

Calderón-Zygmund estimates

In Chapter 5 we focus on weak solutions $u \in g + W_0^{1,p}(\Omega, \mathbb{R}^N)$ of the Dirichlet problem (1.1) in the special situation where the coefficients do not depend explicitly on u, i. e. $a(x, u, z) \equiv a(x, z)$, and where the inhomogeneity $b(x, u, z) \equiv b(x)$ belongs to $L^{p/(p-1)}$. We study higher integrability results for an arbitrary modulus of continuity $\omega(\cdot)$ and both the non-degenerate ($\mu > 0$) and the degenerate ($\mu = 0$) case. Roughly speaking, we are concerned with the question to what extent higher integrability of the inhomogeneity b and of the boundary values Dg is inherited by Du.

For the case of equations ($N = 1$), Caffarelli and Peral [CP98] introduced a method based on Calderón-Zygmund type covering arguments which allows to prove

$$b \in L^{\frac{q}{p-1}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \quad \Rightarrow \quad Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN}) \tag{1.2}$$

without any restriction on q. The crucial point here is that one obtains L^∞-estimates for the gradient of the weak solution to a suitable comparison problem. Since an analogous L^∞-estimate is available for systems exhibiting a special structure such as the p-Laplacean, the latter assertion also holds in this situation, see [Iwa83]. We mention that both results were extended later by Acerbi and Mingione to non-standard $p(x)$-growth. In contrast, for general nonlinear systems a corresponding comparison estimate can no longer be expected. In the superquadratic case Kristensen and Mingione proved in [KM06] that for $q \leq \frac{np}{n-2}$ higher integrability in the sense of (1.2) is still obtained. Moreover, if the boundary data is assumed to satisfy $g \in W^{1,q}(\Omega, \mathbb{R}^N)$, the higher integrability estimate is achieved for the whole domain Ω. Arguing similarly to [KM06], we will prove the analogous result in the subquadratic case (see Theorem 5.1):

Theorem 1.2: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 and let $u \in g + W^{1,p}_0(\Omega, \mathbb{R}^N)$ be a weak solution of (1.1) with coefficients $a(x,u,z) \equiv a(x,z)$ and inhomogeneity $b(x,u,z) \equiv b(x)$. Assume that $g \in W^{1,q}(\Omega, \mathbb{R}^N), b \in L^{q/(p-1)}$ with $q \in [p, s_1]$ and*

$$s_1 \in (p, \infty) \quad \text{if } n = 2, \qquad \text{and} \qquad s_1 = \frac{np}{n-2} \quad \text{if } n > 2$$

Then, there holds $Du \in L^q(\Omega, \mathbb{R}^{nN})$ with

$$\int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{q}{2}}\, dx \leq c \int_\Omega \big(\mu^2 + |Dg|^2 + |b|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx$$

for a constant c depending only on the structure constants and Ω.

As a main feature of this Calderón-Zygmund result we find a quantified gain in the higher integrability exponent – in contrast to classical higher integrability obtained from the application of Gehring's Lemma. In the first step of the proof we deduce in Chapter 4 that the solution to a suitable frozen comparison problem belongs to $W^{2,p}$. This is achieved by the use of standard difference quotient techniques. However, some difficulties arise from the facts that we need the higher differentiability result up to the boundary and that our strategy immediately covers degenerate systems with $\mu = 0$. The Sobolev-Poincaré inequality implies a $W^{1,np/(n-2)}$-estimate (respectively $W^{1,\infty}$ if $n = 2$) for the comparison solution. The proof of Theorem 5.1 is then based on a local comparison principle, basic properties of the Hardy-Littlewood maximal function and Calderón-Zygmund coverings, applied to the super-level sets of the maximal functions of $|Du|^p$ and $|Dg|^p + |b|^{p/(p-1)}$, respectively.

Having solved the problem of higher integrability, we are now in a position to deal with the second obstacle to proving an upper bound for the Hausdorff dimension of the singular set $\mathrm{Sing}_{Du}(\overline{\Omega})$, namely with the fact that the coefficients may depend explicitly on u. Let us explain why this is a critical point in our situation: considering coefficients of the form $a(x,z)$ which are Lipschitz-continuous with respect to the x-variable, it is well-known that the Hausdorff dimension of $\mathrm{Sing}_{Du}(\overline{\Omega})$ does not exceed $n-2$. Contrarily imposing only Hölder continuity with an arbitrarily small exponent we trivially have $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\overline{\Omega})) \leq n$. This suggests that the degree of Hölder continuity of the coefficients is related not only to the regularity of the solution, but also to the size of the singular set. Starting from this observation, Mingione [Min03b] accomplished in some sense an interpolation between Lipschitz continuity on the one hand and Hölder continuity on the other, and obtained $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Omega)) \leq n-2\alpha$ in the interior, provided that the coefficients are Hölder continuous in x with exponent $\alpha \in (0,1)$. We now pass to coefficients of the form $a(x,u,z)$. Following the above philosophy, we need to investigate the regularity of the map $x \mapsto (x,u(x))$. However, recalling that the weak solution u to (1.1) might develop singularities, this map need not to be Hölder continuous. Anyway, at this stage we may exploit the fact that u is actually a weak solution, and therefore, we next study a situation where u is locally Hölder continuous at least outside "irrelevant" sets (i. e., sets which are negligible with respect to the $\mathcal{H}^{n-1}$-measure since our final aim is to prove the existence of regular boundary points).

Partial regularity of u in low dimensions

In Chapter 6, we return to the case where the prescribed boundary data g is of class C^1 and where no further assumption on the modulus of continuity $\omega(\cdot)$ is made. We study partial regularity of u in low dimensions $n \in (p, p+2]$. Several results in slightly different situations were established by Campanato, e. g. in [Cam82b, Cam87a, Cam87b], mostly in the superquadratic case. He observed that the assumption $n \leq p+2$ allows to prove that the weak solution u is locally Hölder continuous outside a singular set of Hausdorff dimension less than $n-p$. In particular, almost every boundary point is a regular one for u (but not yet for its gradient Du). Some extensions concerning u-dependence and inhomogeneities were given later by Arkhipova [Ark97, Ark03] and by Idone [Ido04a, Ido04b]. In Theorem 6.1 we provide the corresponding up-to-the-boundary result for subquadratic systems with inhomogeneities:

Theorem 1.3: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 and $g \in C^1(\overline{\Omega}, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.1) under the assumptions stated above. Then there exists a constant $\delta > 0$ such that for $n > p > n-2-\delta$ there holds*

$$\dim_{\mathcal{H}}\big(\overline{\Omega} \setminus \mathrm{Reg}_u(\overline{\Omega})\big) < n-p \quad \text{and} \qquad u \in C^{0,\lambda}_{\mathrm{loc}}\big(\mathrm{Reg}_u(\overline{\Omega}), \mathbb{R}^N\big)$$

for all $\lambda \in \big(0, \min\{1 - \frac{n-2-\delta}{p}, 1\}\big)$.

It is worth mentioning that this result applies to both the non-degenerate and the degenerate case. The main difficulty lies, once more, in the derivation of a suitable comparison estimate which has already been exploited in a weaker version in the Calderón-Zygmund estimates. The proof of Theorem 1.3 is then obtained by the direct method and relies on certain Morrey-type estimates from Campanato's papers. Dealing with inhomogeneities obeying a natural growth condition requires some technical modifications which are adapted from Arkhipova's work [Ark03].

Existence of regular boundary points

The last part of the thesis is devoted to estimates of the Hausdorff dimension of the singular set of Du in $\overline{\Omega}$. In particular, in some cases we will prove that the dimension is less than $n-1$, thus coming up with the existence of regular boundary points. Here we consider the non-degenerate case $\mu = 1$ and $\omega(t) = \min\{1, t^\alpha\}$ for some $\alpha \in (0,1)$. For a long time suitable upper bounds for the Hausdorff dimension of the singular set $\mathrm{Sing}_u(\overline{\Omega})$ of u were known only for special situations – such as elliptic equations, quasilinear systems, see [Wie76, HW75, Gro00, Gro02a], or low dimensions. For the general situation, it was a long-standing open question to find conditions which allow to infer an analogous estimate for $\mathrm{Sing}_{Du}(\overline{\Omega})$ of the gradient Du. As mentioned above, the problem concerning the dimension reduction in the interior of Ω was first tackled by Mingione in [Min03b, Min03a] where he succeeded in showing that the Hausdorff dimension of $\mathrm{Sing}_{Du}(\Omega)$ is not larger than $n-2\alpha$, provided that the coefficients do not depend explicitly on u or that the assumption of low dimension is satisfied. Also inhomogeneities with natural growth were included in these interior estimates. Assuming that the bounded domain Ω and the prescribed boundary data g are of class $C^{1,\alpha}$, Duzaar, Kristensen and Mingione [DKM07] eventually obtained the essential estimate $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\overline{\Omega})) \leq n - 2\alpha$ up to the boundary (for $p \in (1,2)$ for homogeneous systems, for $p \geq 2$ for inhomogeneous systems with a controllable growth condition). In particular, for $\alpha > \frac{1}{2}$ this implies that almost every boundary point is in fact a regular one. Our first result in this context is given in Theorems 7.1, 7.2 and extends [DKM07] to subquadratic systems with inhomogeneities of controllable growth:

Theorem 1.4: *Let Ω be a domain of class $C^{1,\alpha}$ and $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for an exponent $\alpha > 1/2$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of (1.1) under the assumptions stated above and a controllable growth condition on $b(\cdot,\cdot,\cdot)$. Furthermore, let one of the following assumptions be fulfilled:*

(i) the vector field $a(\cdot,\cdot,\cdot)$ is independent of u, i. e., $a(x,u,z) \equiv a(x,z)$,

(ii) the assumption $p > n-2$ of low dimension holds.

Then $\mathcal{H}^{n-1}$-almost every boundary point is a regular point for Du.

The proof of the results in [Min03b, Min03a] is based on finite difference operators, interpolation techniques and fractional Sobolev spaces, combined in a delicate iteration scheme. The main difficulty is to find estimates which are known as Nikolski-type estimates and which bound the integral of finite differences of Du in terms of the step-size. Extending this strategy up to the boundary, we initially get the corresponding estimates from testing the system with classical differences only in tangential direction, but the missing normal direction cannot be immediately obtained by exploiting the system. To overcome this problem, we follow the arguments of Duzaar, Kristensen and Mingione [DKM07] and construct a family of comparison maps which are solutions of some regularized system and for which the existence of second-order derivatives is known. Then, in every step of the iteration, we gain via the Calderón-Zygmund theory some higher integrability of the gradient of the comparison map, which in turn is used to improve the integrability of Du. Hence, for the situations (i) and (ii) in Theorem 1.4, we find a suitable fractional Sobolev estimate for Du, which ensures that the Hausdorff dimension of the singular set $\mathrm{Sing}_{Du}(\overline{\Omega})$ does not exceed

$n - 2\alpha$ and $\min\{n-p, n-2\alpha\}$, respectively. This immediately implies the statement of the Theorem.

Moreover, extending Mingione's strategy up to the boundary we present a second approach (implemented only in the quadratic case $p = 2$) which applies to systems with inhomogeneities of natural growth. In the low dimensional case we obtain (see Theorem 8.1):

Theorem 1.5: *Consider* $n \in \{2,3,4\}$ *and* $\alpha > 1/2$. *Let* Ω *be a domain of class* $C^{1,\alpha}$ *and* $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. *Let* $u \in W^{1,2}(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ *be a weak solution of (1.1) under the assumptions stated above and a natural growth condition on* $b(\cdot,\cdot,\cdot)$ *(with* $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq M$ *for some* $M > 0$ *such that* $2L_2M < \nu$*). Then* $\mathcal{H}^{n-1}$*-almost every boundary point is a regular point for* Du.

In contrast to the previous proof, we make use of the system in a direct way and employ an observation of Kronz [Kro], namely that slicewise mean values of the coefficients are weakly differentiable in the normal direction which is essential for the up-to-the-boundary estimates. This enables us to find in every step of the iteration the desired Nikolski-type estimates and to end up with a fractional Sobolev estimate for Du analogous to above.

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Chapter 2

Preliminaries

2.1 Notation

We start with some remarks on the notation used throughout the whole work: we write

$$B_\rho(x_0) = \big\{x \in \mathbb{R}^n : |x - x_0| < \rho\big\}\,,$$
$$B_\rho^+(x_0) = \big\{x \in \mathbb{R}^n : x_n > 0, |x - x_0| < \rho\big\}$$

for an open ball, respectively the intersection of an open ball with the upper half-space $\mathbb{R}^{n-1} \times \mathbb{R}^+$, centred at a point $x_0 \in \mathbb{R}^n$ (respectively $\in \mathbb{R}^{n-1} \times \mathbb{R}^+$ in the latter case) with radius $\rho > 0$. Be careful with this notation: the centre x_0 is not assumed to be located in general on the plane $\mathbb{R}^{n-1} \times \{0\}$. For ease of notation it might even occur the case $B_\rho(x_0) \equiv B_\rho^+(x_0)$ when $B_\rho(x_0) \subset \mathbb{R}^{n-1} \times \mathbb{R}^+$. Sometimes it will be convenient to treat the n-th component of $x \in \mathbb{R}^n$ separately; therefore, we set $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Furthermore, we write

$$\Gamma_\rho(x_0) \;=\; \big\{x \in \mathbb{R}^n : |x - x_0| < \rho,\, x_n = 0\big\}$$

if $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. In the case $x_0 = 0$ (respectively $\rho = 1$) we will use the short hand notations $B_\rho := B_\rho(0)$, $B := B_1$ as well as $B_\rho^+ := B_\rho^+(0)$, $B^+ := B_1^+$, $\Gamma_\rho := \Gamma_\rho(0)$, and $\Gamma = \Gamma_1(0)$. Accordingly,

$$Q_\rho(x_0) \;=\; \big\{x \in \mathbb{R}^n : |x_i - (x_0)_i| < \rho,\ \text{for all}\ 1 \le i \le n\big\}$$

denotes the open cube centred at x_0 with side length $l(Q_\rho(x_0)) = 2\rho$, and $Q_\rho^+(x_0)$ denotes the cube intersected with the upper half-plane. The boundary part $Q_\rho(x_0) \cap \{x_n = 0\}$ will as well be denoted by $\Gamma_\rho(x_0)$, but the precise meaning of $\Gamma_\rho(x_0)$ will always be clear from the context.

The function spaces considered below are mainly Hölder spaces $C^{k,\alpha}$, Lebesgue spaces L^p and Sobolev spaces $W^{k,p}$ for $k \in \mathbb{N}_0$, $\alpha \in (0, 1]$ and $p \in [1, \infty]$ on bounded domains $\Omega \subset \mathbb{R}^n$.

Also fractional Sobolev spaces, Morrey spaces and Campanato spaces will play a crucial role in the sequel; the definitions and some important properties will be introduced and discussed later in more detail.

A function $u\colon \Omega \to \mathbb{R}^N$ is called Hölder continuous with exponent α on Ω if there exists a constant $0 < c < \infty$ such that for all points $x, y \in \Omega$ the estimate $|u(x) - u(y)| \leq c|x-y|^\alpha$ is satisfied (analogously for the closure $\overline{\Omega}$). Then the Hölder seminorm of u is defined as

$$[u]_{C^{0,\alpha}(\Omega,\mathbb{R}^N)} := \sup_{x\neq y\in\Omega} \Big\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} \Big\}.$$

The Hölder space $C^{k,\alpha}(\Omega,\mathbb{R}^N)$ consists of all functions $u \in C^k(\Omega,\mathbb{R}^N)$, i. e., k times continuously differentiable, for which the norm

$$\|u\|_{C^{k,\alpha}(\Omega,\mathbb{R}^N)} := \sum_{|\beta|\leq k} \sup_{x\in\Omega} |D^\beta u(x)| + \sum_{|\beta|=k} [D^\beta u]_{C^{0,\alpha}(\Omega,\mathbb{R}^N)}$$

is finite. Here, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ denotes a multi-index of length $|\beta| := \beta_1 + \ldots + \beta_n$ and $D^\beta u := D_1^{\beta_1} \ldots D_n^{\beta_n} u$.

The space L^p is defined as

$$L^p(\Omega,\mathbb{R}^N) := \big\{ u : \Omega \to \mathbb{R}^N : u \text{ is Lebesgue-measurable}, \|u\|_{L^p(\Omega,\mathbb{R}^N)} < \infty \big\}$$

equipped with the norm

$$\|u\|_{L^p(\Omega,\mathbb{R}^N)} = \begin{cases} \Big(\int_\Omega |u|^p \, dx \Big)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \operatorname{ess\,sup}_\Omega |u| & (p = \infty)\,, \end{cases}$$

where we consider classes of function which differ only on a set of Lebesgue measure zero. Endowed with this norm $L^p(\Omega,\mathbb{R}^N)$ is a Banach space (and in the case $p = 2$ even a Hilbert space).

The Sobolev space $W^{k,p}$ is defined as

$$W^{k,p}(\Omega,\mathbb{R}^N) := \big\{ u \in L^p(\Omega,\mathbb{R}^N) : D^\beta u \in L^p(\Omega,\mathbb{R}^N) \quad \forall\, |\beta| \leq k \big\}\,,$$

where $D^\beta u$ denotes the weak derivative of u. $W^{k,p}(\Omega,\mathbb{R}^N)$ is also a Banach space, endowed with the norm

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R}^N)} = \begin{cases} \Big(\sum_{|\beta|\leq k} \int_\Omega |D^\beta u|^p \, dx \Big)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \sum_{|\beta|\leq k} \operatorname{ess\,sup}_\Omega |D^\beta u| & (p = \infty)\,. \end{cases}$$

Furthermore, we denote by $W_0^{k,p}(\Omega,\mathbb{R}^n)$ the closure of $C_0^\infty(\Omega,\mathbb{R}^N)$ in the space $W^{k,p}(\Omega,\mathbb{R}^N)$. Here, we also introduce the following notation for $W^{1,p}$-functions defined on some intersected ball $B_\rho^+(x_0)$ or cube $Q_\rho^+(x_0)$ and which vanish on the flat part of the boundary:

$$\begin{aligned} W_\Gamma^{1,p}(B_\rho^+(x_0),\mathbb{R}^N) &:= \big\{ u \in W^{1,p}(B_\rho^+(x_0),\mathbb{R}^N) : u = 0 \text{ on } \Gamma_{\sqrt{\rho^2-(x_0)_n^2}}(x_0'') \big\}\,, \\ W_\Gamma^{1,p}(Q_\rho^+(x_0),\mathbb{R}^N) &:= \big\{ u \in W^{1,p}(Q_\rho^+(x_0),\mathbb{R}^N) : u = 0 \text{ on } \Gamma_\rho(x_0'') \big\}\,, \end{aligned}$$

where $(x_0)_n < \rho$ is satisfied and where $x_0'' := \big((x_0)', 0\big)$ is the projection of x_0 onto $\mathbb{R}^{n-1} \times \{0\}$.

For a given set $X \subset \mathbb{R}^n$ we denote by $\mathcal{L}^n(X) = |X|$ and $\mathcal{H}^k(X)$ its n-dimensional Lebesgue-measure and k-dimensional Hausdorff measure, respectively. Furthermore, if $h \in L^1(X, \mathbb{R}^N)$ and $0 < |X| < \infty$, we denote the average of h by

$$(h)_X := \fint_X h\,dx := \frac{1}{|X|}\int_X h\,dx\,.$$

On balls and cubes we use from time to time the ambiguous abbreviation $(h)_{z,\rho}$ instead of $(h)_{B_\rho(z)}$ and $(h)_{Q_\rho(z)}$, respectively.

We will often provide up-to-the-boundary estimates. For this purpose we introduce bounded domains Ω in $\mathbb{R}^n$, for some $n \geq 2$, obeying a certain boundary regularity condition: the boundary of Ω is said to be of class $C^{k,\tau}$ for $k \in \mathbb{N}_0$ and some $\tau \in (0,1)$ if for every boundary point $x_0 \in \partial\Omega$ there exist a radius $r > 0$ and a function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ of class $C^{k,\tau}$ such that (up to an isometry) Ω is locally represented by $\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > h(x')\}$. Thus we can locally straighten the boundary $\partial\Omega$ via a $C^{k,\tau}$-transformation defined by $(x', x_n) \mapsto (x', x_n - h(x'))$.

The constants c appearing in the different estimates will all be chosen greater than or equal to 1, and they may vary from line to line. The dependencies of the constants are usually indicated, and constants that are referred to will be signed in a unique way.

2.2 Morrey and Campanato spaces

We will also use the Morrey spaces $L^{p,\varsigma}(\Omega, \mathbb{R}^N)$ and the Campanato spaces $\mathcal{L}^{p,\varsigma}(\Omega, \mathbb{R}^N)$. For more details, the proofs of the Theorems below and an elaborate overview of the fundamental properties of these spaces, we refer to the original papers of Campanato [Cam63, Cam64, Cam65] and Meyers [Mey64], and to the monographs of Giusti, [Giu03, Chapter 2.3], or of Giaquinta, [Gia83, Chapter 3]. In the sequel we shall use the following definitions:

Definition: *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $1 \leq p < \infty$. By $L^{p,\varsigma}(\Omega, \mathbb{R}^N)$, $\varsigma \geq 0$, we denote the linear (Morrey) space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that*

$$\|u\|^p_{L^{p,\varsigma}(\Omega,\mathbb{R}^N)} := \sup_{y\in\Omega, 0<\rho\leq \operatorname{diam}\Omega} \rho^{-\varsigma} \int_{B_\rho(y)\cap\Omega} |u|^p\,dx < \infty\,.$$

By $\mathcal{L}^{p,\varsigma}(\Omega, \mathbb{R}^N)$, $0 \leq \varsigma \leq n+p$, we denote the linear (Campanato) space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$[u]^p_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)} := \sup_{y\in\Omega, 0<\rho\leq \operatorname{diam}\Omega} \rho^{-\varsigma} \int_{B_\rho(y)\cap\Omega} \big|u - (u)_{B_\rho(y)\cap\Omega}\big|^p\,dx < \infty\,.$$

In fact, both conditions stated above depend only on the behaviour of u for radii $\rho \to 0$. The Morrey space $L^{p,\varsigma}(\Omega, \mathbb{R}^N)$ is a Banach space with the norm $\|\cdot\|_{L^{p,\varsigma}(\Omega,\mathbb{R}^N)}$ defined above. We mention that the Morrey spaces $L^{p,\varsigma}(\Omega, \mathbb{R}^N)$ reduce to zero for $\varsigma > n$ in view of Lebesgue's differentiation theorem. Furthermore, in the definition of the Campanato spaces, it is obvious that by $[\,\cdot\,]_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)}$ only a seminorm is given, but $\mathcal{L}^{p,\varsigma}(\Omega, \mathbb{R}^N)$ is also a Banach space, endowed with the norm $\|\cdot\|_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)} := [\,\cdot\,]_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)} + \|\cdot\|_{L^p(\Omega,\mathbb{R}^N)}$.

We next consider domains $\Omega \subset \mathbb{R}^n$ satisfying a so-called Ahlfors regularity condition, i. e., there exists a positive constant k_Ω such that

$$(K_\Omega) \qquad |B_\rho(x_0) \cap \Omega| \geq k_\Omega\, \rho^n \qquad \text{for all points } x_0 \in \overline{\Omega} \text{ and every radius } \rho \leq \operatorname{diam}(\Omega)\,,$$

which means that the domains have no external cusps. The constant k_Ω depends only on n and the domain Ω, precisely only on the similarity class of Ω, i. e., $k_{t\Omega} = k_\Omega$ for any $t > 0$. The latter condition is for example satisfied by the large class of domains with Lipschitz-continuous boundary. Now we can deduce important equivalent formulations for Morrey and Campanato spaces, namely an isomorphy between Morrey and Campanato spaces and an integral characterization of Hölder continuous maps:

Theorem 2.1 ([Giu03], Proposition 2.3 and Theorem 2.9): *Consider $p \in [1,\infty)$. If Ω is a bounded open set satisfying the condition (K_Ω) and if $0 \leq \varsigma < n$, then $L^{p,\varsigma}(\Omega,\mathbb{R}^N)$ is isomorphic to $\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)$. Furthermore, if Ω is a bounded open set without internal cusps and if $n < \varsigma \leq n+p$, then $\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)$ is isomorphic to the space of Hölder continuous functions $C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$ with exponent $\lambda = \frac{\varsigma-n}{p}$, and the following estimates hold true:*

$$[u]_{C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)} \leq c\,[u]_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)} \qquad \text{and} \qquad \|u\|_{C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)} \leq c\,\|u\|_{\mathcal{L}^{p,\varsigma}(\Omega,\mathbb{R}^N)}$$

with a constant c depending only on n, p, ς and Ω.

Remark 2.2: We still want to comment on the remaining case $\varsigma = n$: the Morrey space $L^{p,n}(\Omega,\mathbb{R}^N)$ is isomorphic to $L^\infty(\Omega,\mathbb{R}^N)$ with the identity

$$\|u\|_{L^{p,n}(\Omega,\mathbb{R}^N)} = 2^{\frac{n}{p}}\, \|u\|_{L^\infty(\Omega,\mathbb{R}^N)}$$

(see [Giu03, Proposition 2.2]), whereas the Campanato space $\mathcal{L}^{p,n}(\Omega,\mathbb{R}^N)$ is also called the BMO-space, i. e., the space of all functions with bounded mean oscillation.

We will use the isomorphy stated in the latter theorem in the following form:

Theorem 2.3 ([KM06], Theorem 2.2): *Let $B_r \subset \mathbb{R}^n$ be a ball, $p \in (1,n]$ and $\varsigma \in (n-p,n]$. If $u \in W^{1,p}(B_r,\mathbb{R}^N)$ and $Du \in L^{p,\varsigma}(B_r,\mathbb{R}^{nN})$ then $u \in C^{0,\lambda}(\overline{B}_r,\mathbb{R}^N) \cap \mathcal{L}^{p,\varsigma+p}(B_r,\mathbb{R}^N)$, where $\lambda := 1-(n-\varsigma)/p$. Moreover, there exists a constant c depending only on n, p (but independent of the radius r) such that*

$$[u]_{C^{0,\lambda}(\overline{B_r},\mathbb{R}^N)} \leq c\,[u]_{\mathcal{L}^{p,\varsigma+p}(B_r,\mathbb{R}^N)} \leq c\,\|Du\|_{L^{p,\varsigma}(B_r,\mathbb{R}^N)}\,.$$

The same result holds true if B_r is replaced by a bounded Lipschitz domain Ω. In this case, the constant c also depends on the Lipschitz constant of $\partial\Omega$.

2.3 Fractional Sobolev spaces and interpolation

We now extend the notion of the previously defined Sobolev spaces $W^{k,p}$ by allowing also noninteger values $k \notin \mathbb{N}_0$, i. e., by introducing fractional Sobolev spaces; in the sequel we will use the notation of [Ada75] (cf. also the papers [KM06, DKM07]). For a bounded open set $A \subset \mathbb{R}^n$, parameters $\theta \in (0,1)$ and $q \in [1,\infty)$ we write $u \in W^{\theta,q}(A,\mathbb{R}^N)$ provided that $u \in L^q(A,\mathbb{R}^N)$ and the following Gagliardo-type norm of u defined as

$$\|u\|_{W^{\theta,q}(A)} := \Big(\int_A |u(x)|^q\,dx\Big)^{\frac{1}{q}} + \Big(\int_A\int_A \frac{|u(x)-u(y)|^q}{|x-y|^{n+q\theta}}\,dx\,dy\Big)^{\frac{1}{q}}$$

is finite.

In order to formulate a general criterion for a function to belong to a fractional Sobolev space we introduce the finite difference operator $\tau_{e,h}$ via

$$\tau_{e,h}G(x) \equiv \tau_{e,h}(G)(x) := G(x+he) - G(x)$$

for a vector valued function $G : A \to \mathbb{R}^N$, a vector $e \in B_1 \subset \mathbb{R}^n$ and a real number $h \in \mathbb{R}$. This makes sense whenever $x, x+he \in A$ which will always hold in the following when using $\tau_{e,h}$. If $e = e_s$, $s \in \{1,\dots,n\}$, is a standard basis vector, we use the abbreviation $\tau_{s,h}$ instead of $\tau_{e_s,h}$. These finite differences are related to the fractional Sobolev spaces (in the interior as well as in an up-to-the-boundary version) via the next lemma:

Lemma 2.4 ([KM05], Lemma 2.5 and [DKM07], Lemma 2.2): *Let $G \in L^q(Q_R^+,\mathbb{R}^N)$, $q \geq 1$, and assume that for $\theta \in (0,1]$, $M > 0$ and some $0 < r < R$ we have*

$$\sum_{s=1}^n \int_{Q_r^+} |\tau_{s,h}G|^q\,dx \leq M^q\,|h|^{q\theta}$$

for every $h \in \mathbb{R}$ satisfying $0 < |h| \leq d$ where $0 < d < \min\{1, R-r\}$ is a fixed number. In the case $s = n$ we only allow positive values of h. Then $G \in W^{b,q}(Q_\rho^+,\mathbb{R}^N)$ for every $b \in (0,\theta)$ and $\rho < r$. Moreover, there exists a constant $c = c(n,q)$ (in particular, independent of M and G) such that the following inequality holds true:

$$\int_{Q_\rho^+}\int_{Q_\rho^+} \frac{|G(x)-G(y)|^q}{|x-y|^{n+bq}}\,dx\,dy \leq c\Big(\frac{M^q\varepsilon^{q(\theta-b)}}{\theta-b} + \frac{|Q_R^+|}{\varepsilon^{n+bq}}\int_{Q_R^+}|G|^q\,dx\Big),$$

where $\varepsilon := \min\{r-\rho, d\}$. This result also holds true in the interior without any constraint on the sign of h with respect to the direction of the differences $\tau_{s,h}$; moreover we can also consider (half-)balls instead of cubes.

Proof: This proof is an adapted version of the proof of [KM05, Lemma 2.5] where the interior situation was considered. Therefore, we only present the calculations for the boundary situation. For a vector $v = \sum_{s=1}^n v_s e_s \in \mathbb{R}^n$ we write $v^{(k)} = \sum_{s=1}^k v_s e_s$ for $k = 1,\dots,n$ with $v^{(0)} = 0$. Then, we have

$$\big|G(x+v) - G(x)\big| = \Big|\sum_{s=1}^n \tau_{s,v_s}G(x+v^{(s-1)})\Big| \leq \sum_{s=1}^n \big|\tau_{s,v_s}G(x+v^{(s-1)})\big|$$

whenever $x + v^{(s-1)} \in Q_R^+$. We next calculate for $\varepsilon = \min\{r-\rho, d\}$ defined as above:

$$\begin{aligned}\int_{Q_\rho^+}\big|G(x+v)-G(x)\big|^q\,dx &\leq \int_{Q_\rho^+}\Big(\sum_{s=1}^n \big|\tau_{s,v_s}G(x+v^{(s-1)})\big|\Big)^q\,dx \\ &\leq n^{q-1}\int_{Q_\rho^+}\sum_{s=1}^n \big|\tau_{s,v_s}G(x+v^{(s-1)})\big|^q\,dx \leq n^q M^q |v|^{q\theta}\end{aligned}$$

by assumption for all $v \in \mathbb{R}^n$ with $|v| \le \varepsilon$ and $v_n \ge 0$. Hence, we obtain for each $b \in (0,\theta)$:

$$\int_{\{0<|v|\le\varepsilon, v_n\ge 0\}} \int_{Q_\rho^+} \frac{|G(x+v)-G(x)|^q}{|v|^{n+qb}}\,dx\,dv \le n^q M^q \int_{\{0<|v|\le\varepsilon, v_n\ge 0\}} |v|^{-n+q(\theta-b)}\,dv \le c(n,q)\frac{M^q \varepsilon^{q(\theta-b)}}{\theta-b}\,.$$

Taking into account the symmetry with respect to x and y we thus infer an estimate for points $(x,y)\in Q_\rho^+\times Q_\rho^+$ satisfying $|x-y|\le\varepsilon$:

$$\int_{\{(x,y)\in Q_\rho^+\times Q_\rho^+ : |x-y|\le\varepsilon\}} \frac{|G(y)-G(x)|^q}{|y-x|^{n+qb}}\,dx\,dy \le 2\,c(n,q)\frac{M^q\varepsilon^{q(\theta-b)}}{\theta-b}\,.$$

Otherwise if we consider points $(x,y)\in Q_\rho^+\times Q_\rho^+$ satisfying $|x-y|>\varepsilon$, we use the L^q-estimate

$$\int_{\{(x,y)\in Q_\rho^+\times Q_\rho^+ : |x-y|>\varepsilon\}} \frac{|G(y)-G(x)|^q}{|y-x|^{n+qb}}\,dx\,dy \le 2^q\varepsilon^{-n-bq}\big|Q_\rho^+\big|\int_{Q_\rho^+}|G|^q\,dx\,.$$

Combining the last two inequalities we arrive at the desired estimate. □

In the case where G is the weak derivative of a $W^{1,q}$ function v and where an estimate for finite differences only in tangential direction is known, we are still in a position to state a fractional differentiability result which is limited to the tangential derivative of v:

Lemma 2.5: *Let $v\in W^{1,q}(Q_R^+,\mathbb{R}^N)$, $q\ge 1$, and assume that for $\theta\in(0,1]$, $M>0$ and some $0<r<R$ we have*

$$\sum_{s=1}^{n-1}\int_{Q_r^+}|\tau_{s,h}Dv|^q\,dx \le M^q\,|h|^{q\theta} \tag{2.1}$$

for every $h\in\mathbb{R}$ satisfying $0<|h|\le d$ where $0<d<\min\{1,R-r\}$ is a fixed number. Then $D'v=(D_1v,\dots,D_{n-1}v)\in W^{b,q}(Q_\rho^+,\mathbb{R}^{(n-1)N})$ for every $b\in(0,\theta)$ and $\rho<r$.

Proof: We first fix $b\in(0,\theta)$ and $\rho\in(0,r)$. Now we consider arbitrary numbers $h'\in\mathbb{R}^+$ and $h\in\mathbb{R}$ satisfying $0<|h|,|h'|<\min\{d,\frac{r-\rho}{3}\}$. Then, using Young's inequality, standard properties of the difference operator and the assumption (2.1) on finite differences in tangential direction, we conclude for every $\varepsilon\in(0,\theta)$ and $s\in\{1,\dots,n-1\}$:

$$\begin{aligned}
&|h'|^{-(\theta-\varepsilon)q+}\,|h|^{-(1+\varepsilon)q}\int_{Q_{r-2d}^+}|\tau_{n,h'}\tau_{s,h}\tau_{s,-h}v|^q\,dx\\
&\le \big(|h'|^{-q}\,|h|^{-\theta q}+|h|^{-q-\theta q}\big)\int_{Q_{r-2d}^+}|\tau_{n,h'}\tau_{s,h}\tau_{s,-h}v|^q\,dx\\
&\le 2\,|h'|^{-q}\,|h|^{-\theta q}\int_{Q_{r-d}^+}|\tau_{s,h}\tau_{n,h'}v|^q\,dx+2\,|h|^{-q-\theta q}\int_{Q_{r-d}^+}|\tau_{s,h}\tau_{s,-h}v|^q\,dx\\
&\le 2\,|h|^{-\theta q}\int_{Q_r^+}|\tau_{s,h}D_nv|^q\,dx+2\,|h|^{-\theta q}\int_{Q_r^+}|\tau_{s,h}D_sv|^q\,dx \le 4\,M^q
\end{aligned}$$

uniformly in h, h'. From [Dom04, Lemma 2.2.1] we infer (for possibly smaller values of $|h|$)

$$|h'|^{-(\theta-\varepsilon)q}\,|h|^{-q}\int_{Q_{r-2d}^+}|\tau_{n,h'}\tau_{s,h}v|^q\,dx \le c\Big(\int_{Q_R^+}|Du|^q\,dx+M^q\Big),$$

and the constant c depends only on $\theta, q, \varepsilon, d$ and $r-\rho$. Considering the limit $h \to 0$, we hence end up with

$$|h'|^{-(\theta-\varepsilon)q} \int_{Q^+_{r-2d}} |\tau_{n,h'} D_s v|^q \, dx \,\le\, c \Big(\int_{Q^+_R} |Du|^q \, dx + M^q \Big) .$$

Keeping in mind that the index $s \in \{1, \ldots, n-1\}$ is arbitrary, we may combine the latter inequality with (2.1) to find

$$\sum_{s=1}^{n} \int_{Q^+_{r-2d}} |\tau_{s,h} D' v|^q \, dx \,\le\, c\, |h|^{(\theta-\varepsilon)q} \Big(\int_{Q^+_R} |Du|^q \, dx + M^q \Big)$$

for every $h \in \mathbb{R}$ satisfying $0 < |h| \le \min\{d, \frac{r-\rho}{3}\}$ where we only allow positive values of h if $s = n$. Setting $\varepsilon = (\theta - b)/2$ the application of the previous Lemma 2.4 with θ, r replaced by $\theta - \varepsilon, r - 2d$ yields the desired result. □

The following lemma makes it possible to switch easily from a given decay estimate for finite differences of $V(G)$ (where $V(\xi) = (1 + |\xi|^2)^{(p-2)/4}\xi$ for all $\xi \in \mathbb{R}^k$, see Appendix A.1) to the corresponding decay estimate for the finite differences of G:

Lemma 2.6 ([DKM07], Lemma 2.3): *Let $G \in L^p(Q^+_R, \mathbb{R}^N)$, $1 < p < 2$, $s \in \{1, \ldots, n\}$, and assume that for $\theta \in (0, 1]$, $M > 0$ and $0 < r < R$ we have*

$$\int_{Q^+_r} \big|\tau_{s,h}(V(G))\big|^2 \, dx \,\le\, M^2 |h|^{2\theta} ,$$

for every $0 < |h| \le \min\{d, R - r\}$, where $0 < d \le \min\{1, R - r\}$ is a fixed number. In the case $s = n$ we only allow positive values of h. Then, we have

$$\int_{Q^+_r} \big|\tau_{s,h} G\big|^p \, dx \,\le\, c(n, N, p)\, \|1 + G\|^{\frac{(2-p)p}{2}}_{L^p(Q^+_R)} M^p |h|^{p\theta} .$$

This result also holds true in the interior without any constraint on the sign of h with respect to the direction of the differences $\tau_{s,h}$; moreover we can also consider (half-)balls instead of cubes.

Proof: From Hölder's inequality and Lemma A.3 (i) we obtain

$$\begin{aligned}\int_{Q^+_r} |\tau_{s,h} G|^p \, dx &\le \Big(\int_{Q^+_r} \big(1 + |G(x)|^2 + |G(x + he_s)|^2\big)^{\frac{p}{2}} \, dx \Big)^{\frac{2-p}{2}} \\ &\quad \cdot \Big(\int_{Q^+_r} \big(1 + |G(x)|^2 + |G(x + he_s)|^2\big)^{\frac{p-2}{2}} |\tau_{s,h} G|^2 \, dx \Big)^{\frac{p}{2}} \\ &\le\, c(n, N, p)\, \|1 + G\|^{\frac{(2-p)p}{2}}_{L^p(Q^+_R)} \Big(\int_{Q^+_r} \big|\tau_{s,h}(V(G))\big|^2 \, dx \Big)^{\frac{p}{2}} ,\end{aligned}$$

and the conclusion is an immediate consequence of the assumption concerning the L^2 norm of $|\tau_{s,h}(V(G))|$. □

The following interpolation inequality can be found in [Cam82a], Lemma 2.V., and is essentially based on the inequality in [CC81], Theorem 2.I, for the case $p = 2$.

Theorem 2.7: *Let $\lambda \in (0,1]$, $\theta \in (0,1]$, $p \in (1,\infty)$ and $u \in C^{0,\lambda}(\overline{Q},\mathbb{R}^N)$ such that $Du \in W^{\theta,p}(Q,\mathbb{R}^{nN})$ with $p\theta < n$, where $Q \subset \mathbb{R}^N$ is an (upper) cube. Then*

$$Du \in L^s(Q,\mathbb{R}^{nN}) \qquad \text{for all } s < \frac{np(1+\theta)}{n-p\theta\lambda}\,.$$

Moreover,

$$\int_Q |Du|^s\,dx \;\leq\; c\big(n,N,p,\theta,\lambda,s,|Q|,\|u\|_{W^{1+\theta,p}(Q,\mathbb{R}^N)},[u]_{C^{0,\lambda}(\overline{Q},\mathbb{R}^N)}\big)\,.$$

A different definition for fractional Sobolev spaces, based on pointwise inequalities, can be derived as follows: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \geq 1$ and $\theta \in (0,1]$. Following the approach of Hajłasz in [Haj96], we set

$$D^{\theta,p}(\Omega;f) := \big\{g \in L^p(\Omega)\colon \exists\, E \subset \Omega,\ |E| = 0 \text{ such that } |f(x)-f(y)| \leq |x-y|^\theta(g(x)+g(y)) \text{ for all } x,y \in \Omega\setminus E\big\}\,,$$

and we define the fractional Sobolev space via

$$M^{\theta,p}(\Omega,\mathbb{R}^N) \;:=\; \big\{f \in L^p(\Omega,\mathbb{R}^N)\colon D^{\theta,p}(\Omega;f) \neq \emptyset\big\}\,.$$

We highlight that this definition has its origin in the definition of Sobolev spaces in the context of arbitrary metric spaces (replacing $|x-y|$ by $\mathrm{dist}(x,y)$) and that it does not make use of the notion of derivatives (for a more detailed discussion of the metric setting we refer to [HK00]). Employing the Hardy-Littlewood maximal function we have in fact that for the integer order $\theta = 1$ and sufficiently regular domains (e.g. with Lipschitz boundary) this "metric" Sobolev space coincides with the classical Sobolev space; more precisely, provided that $p > 1$, there holds

$$M^{1,p}(\Omega,\mathbb{R}^N) \;=\; W^{1,p}(\Omega,\mathbb{R}^N)$$

for all bounded domains Ω with the so-called extension property, meaning that there exists a bounded linear operator $E : W^{1,p}(\Omega,\mathbb{R}^N) \to W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ such that for every $f \in W^{1,p}(\Omega,\mathbb{R}^N)$ there holds $Ef = f$ almost everywhere in Ω. We note that the equivalence fails if $p = 1$, see [Haj95]. Furthermore, the definitions of the classical and the metric fractional Sobolev spaces immediately yield for all bounded domains Ω, fractional orders $\theta \in (0,1)$ and $p \in [1,\infty)$ the following inclusion:

$$M^{\theta,p}(\Omega,\mathbb{R}^N) \subseteq W^{\theta',p}(\Omega,\mathbb{R}^N) \qquad \text{for all } \theta' \in (0,\theta).$$

$M^{\theta,p}(\Omega,\mathbb{R}^N)$ is equipped with the norm

$$\|f\|_{M^{\theta,p}(\Omega,\mathbb{R}^N)} \;:=\; \|f\|_{L^p(\Omega,\mathbb{R}^N)} + \inf_{g\in D^{\theta,p}(\Omega;f)} \|g\|_{L^p(\Omega)}\,.$$

We observe that if $p \in (1,\infty)$ then, due to the convexity of L^p, to every $f \in M^{\theta,p}(\Omega,\mathbb{R}^N)$ there exists a unique function $g \in L^p(\Omega)$ which minimizes the $L^p(\Omega)$-norm amongst all functions in $D^{\theta,p}(\Omega;f)$.

The following lemma provides an integral characterization of fractional Sobolev spaces:

Lemma 2.8: *Let $\Omega \subset \mathbb{R}^n$ be a domain which fulfills an Ahlfors condition (K_Ω), $\theta \in (0,1]$, $p \in (1,\infty)$. Then the following two statements are equivalent:*

(i) $f \in M^{\theta,p}(\Omega,\mathbb{R}^N)$

(ii) $f \in L^1(\Omega,\mathbb{R}^N)$ *and there exists a function* $h \in L^p(\Omega)$ *and a radius* $R_0 > 0$ *such that*

$$\fint_{B_\rho(x_0)\cap\Omega} |f - (f)_{B_\rho(x_0)\cap\Omega}|\,dx \;\le\; \rho^\theta\, h(x_0) \tag{2.2}$$

for almost all $x_0 \in \overline{\Omega}$ *and* $\rho \le R_0$.

Proof: The implication (i) ⇒ (ii) follows by standard properties of the Hardy-Littlewood maximal function for the choice $h = 4M(g)$ with $g \in D^{\theta,p}(\Omega; f)$.

For the reverse implication (ii) ⇒ (i) we follow the proof of Campanato's integral characterization of Hölder continuous functions, see e. g. [Sim96, Chapt. 1.1, Lemma 1]. We first define the exceptional set

$$E \;:=\; \big\{x_0 \in \overline{\Omega} : h(x_0) = \infty \;\text{ or }\; (2.2) \text{ is not fulfilled } \;\text{or}\; x_0 \text{ is not a Lebesgue point of } f\big\}$$

which is in view of Lebesgue's Lemma of Lebesgue measure zero. Then we consider $\rho \in (0, R_0]$ and $y \in \overline{\Omega} \setminus E$. By assumption (ii) and the Ahlfors condition on Ω we observe

$$\fint_{B_{\rho/2}(y)\cap\Omega} |f - (f)_{B_\rho(y)\cap\Omega}|\,dx \;\le\; c(n,k_\Omega)\,\rho^\theta\, h(y)\,.$$

Moreover, using the given integral inequality on $B_{\rho/2}(y)\cap\Omega$ in place of $B_\rho(y)\cap\Omega$ we infer

$$\fint_{B_{\rho/2}(y)\cap\Omega} |f - (f)_{B_{\rho/2}(y)\cap\Omega}|\,dx \;\le\; \rho^\theta\, h(y)\,.$$

Combining these estimates we obtain $\big|(f)_{B_{\rho/2}(y)\cap\Omega} - (f)_{B_\rho(y)\cap\Omega}\big| \;\le\; c(n,k_\Omega)\,\rho^\theta\, h(y)$ for the mean values of f. Now, for every $k \in \mathbb{N}_0$ we can choose $\rho = 2^{-k}R_0$; consequently, we have

$$\big|(f)_{B_{2^{-k}R_0}(y)\cap\Omega} - (f)_{B_{2^{-k-1}R_0}(y)\cap\Omega}\big| \;\le\; c(n,k_\Omega)\,R_0^\theta\, h(y)\, 2^{-k\theta}\,.$$

Due to the fact that $2^{-k\theta}$ is the k-th term of a convergent geometric series, the sequence of the mean values $\{(f)_{B_{2^{-k}R_0}(y)\cap\Omega}\}_{k\in\mathbb{N}}$ is convergent. Keeping in mind $y \notin E$, we may use Lebesgue's Lemma and we note that its limit is in fact $f(y)$. Moreover, we see

$$\begin{aligned}\big|(f)_{B_{2^{-k}R_0}(y)\cap\Omega} - f(y)\big| &\le \sum_{j=k}^{\infty} \big|(f)_{B_{2^{-j}R_0}(y)\cap\Omega} - (f)_{B_{2^{-j-1}R_0}(y)\cap\Omega}\big| \\ &\le\; c\,R_0^\theta\, h(y) \sum_{j=k}^{\infty} 2^{-j\theta} \;\le\; c\,R_0^\theta\, h(y)\, 2^{-k\theta}\end{aligned}$$

for a constant c which depends only on n, k_Ω and θ. Furthermore, from the assumption in (ii) with the choice $\rho = 2^{-k}R_0$ and the latter inequality we infer

$$\fint_{B_\rho(y)\cap\Omega} |f - f(y)|\,dx \;\le\; c(n,k_\Omega,\theta)\, h(y)\, \rho^\theta \tag{2.3}$$

for all radii $\rho = 2^{-k}R_0$, $k \in \mathbb{N}_0$. In fact, the previous inequality holds for any radius $\rho \in (0, R_0]$ since for any such ρ there exists $k \in \mathbb{N}_0$ such that $2^{-k-1}R_0 < \rho \le 2^{-k}R_0$ and $B_\rho(y) \cap \Omega \subseteq B_{2^{-k}R_0}(y) \cap \Omega$.

We next choose two arbitrary points $y, z \in \overline{\Omega} \setminus E$ such that $|y - z| \le \frac{R_0}{2}$. Then we apply (2.3) on balls with radius $\rho = 2|y - z|$ with centres y and z, respectively. Since we have the inclusions

$$B_{\rho/2}\big(\tfrac{1}{2}(y+z)\big) \cap \Omega \subset B_\rho(y) \cap B_\rho(z) \cap \Omega$$

this gives

$$\begin{aligned}
&\fint_{B_{\rho/2}(\frac{1}{2}(y+z))\cap\Omega} \big(|f - f(y)| + |f - f(z)|\big)\, dx \\
&\quad \le c\Big(\fint_{B_\rho(y)\cap\Omega} |f - f(y)|\, dx + \fint_{B_\rho(z)\cap\Omega} |f - f(z)|\, dx \Big) \\
&\quad \le c\, \rho^\theta \big(h(y) + h(z)\big)
\end{aligned}$$

for a constant c depending only on n, k_Ω and θ. This implies $|f(y) - f(z)| \le c\, \rho^\theta (h(y) + h(z))$, and thus yields the desired pointwise inequality, provided that the distance of the points is less or equal than $\frac{R_0}{2}$. Therefore, in view of the boundedness of Ω and $h \in L^p(\Omega)$, a standard covering argument reveals $f \in L^p(\Omega, \mathbb{R}^N)$. Taking into account

$$|f(y) - f(z)| \le c(\theta, R_0)\, |x - y|^\theta \,\big(|f(x)| + |f(y)|\big)$$

if $|x - y| > \frac{R_0}{2}$, we observe that the function $g = c\,(h + |f|)$ belongs to $D^{\theta,p}(\Omega; f)$ for a constant c depending only on n, k_Ω, θ and R_0. This completes the proof of the lemma. □

Remarks 2.9: In fact, we have proved the following local version of the integral characterization: let $x_0 \in \overline{\Omega}$ and $R > 0$ such that

$$\fint_{B_r(z)\cap\Omega} |f - (f)_{B_r(z)\cap\Omega}|\, dx \;\le\; r^\theta\, h(z)$$

for almost all $z \in \overline{\Omega}$ and $B_r(z) \subset B_R(x_0)$. Then there holds $f \in M^{\theta,p}(B_{R/2}(x_0) \cap \Omega, \mathbb{R}^N)$ with

$$|f(x) - f(y)| \;\le\; c(n, k_\Omega, \theta)\, |x - y|^\theta \,\big(h(x) + h(y)\big)$$

for almost all $x, y \in B_{R/2}(x_0) \cap \Omega$. Furthermore, we mention that using Jensen's inequality and the fact that the Hardy Littlewood maximal operator is a bounded map from L^p to itself this characterization allows us to infer the inclusion

$$W^{\theta,p}(\Omega, \mathbb{R}^N) \subseteq M^{\theta,p}(\Omega, \mathbb{R}^N)$$

whenever Ω satisfies an Ahlfors condition (K_Ω), $\theta \in (0,1)$ and $p \in (1, \infty)$.

Moreover, we note that (i) implies indeed the following statement: there exists a function $h \in L^p(\Omega)$ and a radius $R_0 > 0$ such that

$$\Big(\fint_{B_\rho(x_0)\cap\Omega} |f - (f)_{B_\rho(x_0)\cap\Omega}|^q\, dx \Big)^{\frac{1}{q}} \;\le\; \rho^\theta\, h(x_0)$$

for all $q < p$ and almost all $x_0 \in \overline{\Omega}$ and $\rho \le R_0$.

Chapter 3

Partial regularity for inhomogeneous systems

In the sequel we consider weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1,2)$ of the following nonlinear, inhomogeneous elliptic systems of partial differential equations of second order

$$-\operatorname{div} a(\,\cdot\,, u, Du) \;=\; b(\,\cdot\,, u, Du) \qquad \text{in } \Omega\,. \tag{3.1}$$

Here Ω denotes a bounded domain in $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. For the right-hand side, we are going to investigate the controllable and the natural growth condition, which will be explained in the following. In the second case, we will have to restrict ourselves to *bounded* weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

We shall now prove partial regularity for the gradient Du and, in particular, how the set of regular points of Du is characterized both in the interior and at the boundary (under additional assumptions concerning the regularity of the boundary values on $\partial\Omega$). This extends the results in [Bec05] where the homogeneous situation $b \equiv 0$ was studied.

We first give a short overview of partial regularity results in the interior and at the boundary. For a broader discussion we refer to [Gia83, Gro00, Min06], where examples and motivations can be found explaining the development of regularity theory and the idea of partial regularity throughout the last century. In 1968, De Giorgi demonstrated in [DG68] that, in contrast to equations, we cannot in general expect a weak solution to a nonlinear system to be a classical one (i. e., of class C^2). The best we can hope for is partial regularity, in other words that there exists a set $\Omega_0 \subset \Omega$ such that $\Omega \setminus \Omega_0$ is small in a certain sense, for instance of Lebesgue measure zero, and that u or even Du is locally regular (e. g. Hölder continuous) in Ω_0. There are different approaches to prove partial regularity: in the interior, Giaquinta, Modica and Ivert [GM79, Ive79] were the first to utilize the direct method; the blow-up technique was earlier applied in the setting of elliptic systems by Giusti and Miranda [GM68a]; furthermore, Duzaar and Steffen [DS02] introduced the method of $\mathcal{A}$-harmonic approximation, which is inspired by Simon's proof of the regularity theorem of Allard and which extends the method of harmonic approximation (i. e., approximating with functions solving the Laplace equation) in a natural way to bounded elliptic operators with constant coefficients. Based on the latter approach, Duzaar and Grotowski [DG00] gave a new proof of the partial regularity of Du.

In the situation considered in this paper we will use a version of the latter technique which has been applied to various situations concerning regularity in the past few years. The idea of $\mathcal{A}$-harmonic approximation is the following: Given a linear system $\operatorname{div}(\mathcal{A}\, Dv)$ with constant coefficients, we know that a function w which is approximately $\mathcal{A}$-harmonic, i. e., for which $⨍_{B_R(x_0)} \mathcal{A}\, Dw \cdot D\varphi \, dx$ is sufficiently small for all test functions $\varphi \in C^1_0(B_R(x_0), \mathbb{R}^N)$, is close to an $\mathcal{A}$-harmonic function h in the L^p-sense. Therefore, we consider an appropriate freezing of the original nonlinear system denoted by $\mathcal{A}$ and we apply a comparison argument involving the solution u of the original system and the L^p-close $\mathcal{A}$-harmonic approximation h. Using good a priori estimates for h and a Caccioppoli-type inequality, we then find an excess-decay estimate in points where certain smallness assumptions (see below) are satisfied, i. e., where the so-called regularity criterion applies. Finally, by Campanato's characterization of Hölder continuous functions, we conclude the desired partial regularity result.

Apart from the $\mathcal{A}$-harmonic approximation lemma all proofs are direct. This gives a good control on the dependencies on the structure conditions and enables us to directly establish the optimal regularity result. It is optimal in the following sense: if $(1 + |z|^2)^{\frac{1-p}{2}} a(x, u, z)$ is uniformly Hölder continuous in x and u with exponent α, then Du is partially Hölder continuous with *the same* exponent α.

In the subquadratic case, where $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $p \in (1, 2)$ and where the coefficients $a(\cdot, \cdot, \cdot)$ satisfy a corresponding $(p-1)$-growth condition, only few partial regularity results are known. We first concentrate on the **interior situation**: In [Pep71], Pepe applied the blow-up technique to a special quasi-linear system and showed partial Hölder continuity of u, and, due to the quasi-linearity of the system, the singular set is of $(n-p)$ dimensional Hausdorff measure zero. In [Wol01b], Wolf considered solutions of nonlinear systems for both homogeneous and inhomogeneous systems, but the regularity result is not optimal in the above sense. Theorem 1.1 in [Bec05] and Theorem 3.1 below close this gap since there, for homogeneous and inhomogeneous systems, respectively, we have shown an (optimal) result analogous to that of the quadratic case (see e. g. [GM68a] combined with [Ham95], or [DG00]) and to that of the superquadratic case $p \geq 2$ (see [Ham07]). Furthermore, we give a characterization of regular points, similar to that of the (super-)quadratic case, where the set of regular interior points is defined by

$$\operatorname{Reg}_{Du}(\Omega) := \left\{ x \in \Omega : Du \in C^0(U, \mathbb{R}^{nN}) \text{ for some neighbourhood } U \text{ of } x \right\}.$$

More precisely, we obtain – exactly as for homogeneous systems – that under the structure conditions introduced in Section 3.1 we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\operatorname{Reg}_{Du}(\Omega), \mathbb{R}^N)$, and the set of singular points $\mathrm{Sing}_{Du}(\Omega) := \Omega \setminus \mathrm{Reg}_{Du}(\Omega) \subset \Pi_1 \cup \Pi_2$ is of Lebesgue measure zero, with

$$\begin{aligned} \Pi_1 &= \Big\{x_0 \in \Omega \,:\, \liminf_{\rho\to 0^+} \fint_{B_\rho(x_0)} \big|V(Du) - \big(V(Du)\big)_{x_0,\rho}\big|^2\,dx > 0\Big\}, \\ \Pi_2 &= \Big\{x_0 \in \Omega \,:\, \limsup_{\rho\to 0^+} \big(\big|(u)_{x_0,\rho}\big| + \big|\big(V(Du)\big)_{x_0,\rho}\big|\big) = \infty\Big\}, \end{aligned}$$

where $V(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}}\xi$ for $\xi \in \mathbb{R}^k$. We note that Carozza, Fusco and Mingione [CFM98, CM01] studied the problem of partial regularity for minimizers of quasiconvex integrals in the subquadratic setting, obtaining the same characterization for the set of regular points.

We now focus on the **boundary situation**: in the early 70's, Colombini [Col71] considered the case of quasi-linear systems and showed partial Hölder continuity of weak solutions outside a singular set of Hausdorff dimension not greater than $n-p$ (with $p \geq 2$). Furthermore, there are some papers, in particular by Campanato [Cam87b] and recently by Arkhipova [Ark03], in which the authors obtain partial regularity up to the boundary in low dimensions. In the case of general systems and arbitrary dimensions, regularity up to the boundary was lately studied for the first time by Grotowski [Gro00] via the $\mathcal{A}$-harmonic approximation method in the case $p = 2$, and by Hamburger [Ham07] using a version of the blow-up technique for the superquadratic case. For the analogous characterization of regular boundary points for almost minimizers of quasiconvex variational integrals we refer to [Kro05]. In what follows we will proceed analogously to Grotowski and provide a similar characterization of the set of regular boundary points which is defined by

$$\mathrm{Reg}_{Du}(\partial\Omega) := \big\{x \in \partial\Omega : Du \in C^0(U \cap \overline{\Omega}, \mathbb{R}^{nN}) \text{ for a neighbourhood } U \text{ of } x\big\}.$$

Here we assume the boundary $\partial\Omega$ to be of class $C^{1,\alpha}$ and further $u = g$ on $\partial\Omega$ for a function $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

To this end, we first state a priori estimates valid up to the boundary for weak solutions $u \in W^{1,p}$ of homogeneous linear systems with constant coefficients. This allows us to derive an excess-decay estimate at the boundary. Combined with the excess-decay estimate in the interior, we show that Du is locally Hölder continuous with exponent α in a boundary neighbourhood of every point $y \in \mathrm{Reg}_{Du}(\partial\Omega)$, and that the set of singular boundary points $\mathrm{Sing}_{Du}(\partial\Omega) := \partial\Omega \setminus \mathrm{Reg}_{Du}(\partial\Omega)$ satisfies the inclusion $\mathrm{Sing}_{Du}(\partial\Omega) \subset \widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2$, where

$$\begin{aligned} \widetilde{\Sigma}_1 &= \Big\{y \in \partial\Omega \,:\, \liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(y)} \big|V(D_{\nu_{\partial\Omega}(y)}u) - \big(V(D_{\nu_{\partial\Omega}(y)}u)\big)_{\Omega\cap B_\rho(y)}\big|^2\,dx > 0\Big\}, \\ \widetilde{\Sigma}_2 &= \Big\{y \in \partial\Omega \,:\, \limsup_{\rho\to 0^+} \big|\big(V(D_{\nu_{\partial\Omega}(y)}u)\big)_{\Omega\cap B_\rho(y)}\big| = \infty\Big\}; \end{aligned}$$

here $\nu_{\partial\Omega}(y)$ denotes the inward-pointing unit normal to $\partial\Omega$ at y. This means that for the regularity criterion at the boundary, only the normal derivative is of importance. We emphasize that since the boundary $\partial\Omega$ itself is of Lebesgue measure zero, this does not yield the existence of a single regular boundary point (whereas it was known for a while that singularities may occur at the boundary even if the boundary data is smooth, see the example in [Gia78]). For the existence we refer to the recent paper [DKM07] by Duzaar, Kristensen and Mingione and to Chapters 7, 8 further below.

In what follows we set the main focus on the treatment of the inhomogeneity: the proof is quite similar to the proof in the homogeneous situation, therefore we will not perform all the estimates in the proof, but rather concentrate on the modifications necessary to adapt it to the inhomogeneous case. For these modifications we mainly refer on the one hand to [DG00], where optimal interior regularity was considered in the quadratic case and where also inhomogeneities under a natural growth condition were taken into account, and on the other hand to [Gro00, Gro02b], respectively, where useful tools and techniques for the treatment of the boundary situation are provided.

3.1 Structure conditions and results

We impose on the coefficients $a\colon \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ of the inhomogeneous system (3.1) – exactly as in the homogeneous case – standard boundedness, differentiability, growth and ellipticity conditions: the functions $(x,u,z) \mapsto a(x,u,z)$ and $(x,u,z) \mapsto D_z a(x,u,z)$ are continuous, and for fixed $L > 0$, $\alpha \in (0,1)$ and all triples $(x,u,z), (\bar{x},\bar{u},z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ there holds that:

(H1) a has polynomial growth:
$$\big|a(x,u,z)\big| \le L\left(1+|z|^{p-1}\right),$$

(H2) a is differentiable with respect to z with bounded and continuous derivatives:
$$\big|D_z a(x,u,z)\big| \le L\,,$$

(H3) a is uniformly strongly elliptic:
$$D_z a(x,u,z)\,\lambda\cdot\lambda \ge \left(1+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\,\lambda \in \mathbb{R}^{nN}\,,$$

(H4) There exists a modulus of continuity ω with $\omega(t) \le \min(1,t^{\alpha})$ and $K : [0,\infty) \to [1,\infty)$ monotone nondecreasing such that
$$\big|a(x,u,z)-a(\bar{x},\bar{u},z)\big| \le L\,K(|u|)\left(1+|z|^2\right)^{\frac{p-1}{2}}\omega\big(|x-\bar{x}|+|u-\bar{u}|\big)\,.$$

Finally, we assume the following boundary condition:

(H5) g is in $C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$.

In (H4), which describes the Hölder continuity in the first two arguments, we have taken without loss of generality $K \ge 1$. Furthermore, the function g will specify the values of the weak solution u on the boundary $\partial\Omega$. We mention that the exponents are all chosen equal to $\alpha \in (0,1)$, i. e., the exponent of the regularity class of the domain Ω, the modulus of continuity ω and the Hölder exponent of Dg. This is no restriction to the general situation with different exponents since we have seen in [Bec05] for the homogeneous case that only the minimal exponent determines the class of regularity for the gradient Du. Moreover, multiplying (3.1) by $\bar{\nu} > 0$ we may consider any ellipticity constant $\bar{\nu}$ instead of 1.

In the proof of the characterization of regular boundary points (see Theorem 3.2 below) we will transform our system to the model situation on a half-ball with Dirichlet boundary values equal to zero on Γ. Performing this transformation we will end up with a modified ellipticity constant, i. e., (H3) will be transformed into the following condition: there exists a number $\nu \in (0,1)$ such that

$$\textbf{(H3)*} \qquad D_z a(x,u,z)\,\lambda\cdot\lambda \,\geq\, \big(\nu^{-2}+|z|^2\big)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\,\lambda\in\mathbb{R}^{nN}$$

is fulfilled. This means that ν plays the role of an ellipticity constant. In Chapter 3.2 we will see that ν depends only on the boundary values g, and if we assume natural growth for the inhomogeneity then it depends additionally on the smallness condition (3.3) on $\|u\|_{L^\infty(\Omega)}$ further below. Moreover, since the gradient $D_z a(x,u,z)$ is continuous, we may conclude the existence of a modulus of continuity on compact subsets of $\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$, i. e., there exists a function $\chi:[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfying

$$\begin{array}{ll} \chi(t,0)=0 & \text{for all } t\geq 0\\ t\mapsto\chi(t,s) & \text{is monotone nondecreasing for fixed } s\\ s\mapsto\chi^2(t,s) & \text{is concave and monotone nondecreasing for fixed } t \end{array}$$

such that for all $(x,u,z),(\bar{x},\bar{u},\bar{z})\in\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$ with $|u|+|z|+|u-\bar{u}|+|z-\bar{z}|\leq M_0+1$ we have

$$\begin{aligned} \big|D_z a(x,u,z)-D_z a(\bar{x},\bar{u},\bar{z})\big| \,&\leq\, L\,\chi\big(M_0,|x-\bar{x}|^2+|u-\bar{u}|^2+|z-\bar{z}|^2\big)\\ &=:\, L\,\chi_{M_0}\big(|x-\bar{x}|^2+|u-\bar{u}|^2+|z-\bar{z}|^2\big)\,. \end{aligned} \tag{3.2}$$

The right-hand side $b:\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^N$ is a Carathéodory function, i.e. measurable with respect to x and continuous with respect to (u,z), and fulfills one of the following growth conditions:

(B1) controllable growth:

$$|b(x,u,z)|\leq L\big(1+|z|^2\big)^{\frac{p-1}{2}}$$

for all $(x,u,z)\in\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$

(B2) natural growth: there exist L_1,L_2 (possibly depending on $M>0$), such that

$$|b(x,u,z)|\leq L_1(M)\,|z|^p+L_2(M)$$

for all $(x,u,z)\in\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$ with $|u|\leq M$.

Assuming the latter condition we have to additionally require: the solution u of the inhomogeneous system (3.1) to be bounded with $|u|\leq M$ for some $M>0$ satisfying

$$2\,L_1(M)\,M\,<\,1\,. \tag{3.3}$$

A discussion about the need of such a smallness condition appears in Giaquinta's monograph [Gia83, Chapter 6] and in [Hil82, Section 2].

In this context we now specify the notion *weak solution*:

Definition: $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ $(\cap L^\infty(\Omega,\mathbb{R}^N))$ *is called a* (bounded) weak solution *of the Dirichlet problem*

$$\left\{\begin{aligned} -\operatorname{div} a(x,u,Du) \,&=\, b(x,u,Du) \qquad && \textit{in } \Omega\,,\\ u\,&=\,g && \textit{on } \partial\Omega\,, \end{aligned}\right. \tag{3.4}$$

if there holds

$$\begin{cases} \displaystyle\int_\Omega a(\,\cdot\,,u,Du)\cdot D\varphi\,dx \,=\, \int_\Omega b(\,\cdot\,,u,Du)\cdot\varphi\,dx & \forall\,\varphi\in C_0^\infty(\Omega,\mathbb{R}^N) \\ u \,=\, g & \text{on } \partial\Omega\,, \end{cases} \tag{3.5}$$

where the latter equality is to be understood in the sense of traces.

By approximation and the growth assumption on $a(\cdot,\cdot,\cdot)$ with respect to the last variable, we see that the identity (3.5) holds for a larger class of test functions, taking into account the different growth conditions of $b(\cdot,\cdot,\cdot)$: if we assume (B1) then all functions $\varphi\in W_0^{1,p}(\Omega,\mathbb{R}^N)$ are admissible test functions, whereas for (B2) we additionally have to demand boundedness, i. e., $\varphi\in W_0^{1,p}(\Omega,\mathbb{R}^N)\cap L^\infty(\Omega,\mathbb{R}^N)$.

In what follows we consider weak solutions u of (3.1) which coincide on the boundary of the domain Ω with g (introduced in (H5)). On the one hand we will study the regularity of u in the interior of Ω: here, the regularity of the boundary data (g and $\partial\Omega$) is not involved in the estimates so that we obtain Hölder continuity with exponent α (see (H4)) of the first derivative of the solution u outside a negligible closed subset which is analogous to the results in the quadratic and superquadratic case (cf. [DG00, Ham07]):

Theorem 3.1: *Consider $p\in(1,2)$, Ω a bounded domain in $\mathbb{R}^n$, $n\geq 2$, and $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ a weak solution of the inhomogeneous system (3.1), where the coefficients $a\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ fulfill the assumptions (H1)-(H4). Furthermore, we assume one of the following structure conditions on the inhomogeneity:*

1. *the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition (B1),*
2. *the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2) and $u\in L^\infty(\Omega,\mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)}\leq M$ and $2L_1(M)M<1$.*

Then there holds $u\in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{Reg}_{Du}(\Omega),\mathbb{R}^N)$ and $\mathrm{Sing}_{Du}(\Omega)\subseteq\Pi_1\cup\Pi_2$ with

$$\Pi_1=\Big\{x_0\in\Omega\,:\,\liminf_{\rho\to0^+}\fint_{B_\rho(x_0)}\big|V(Du)-\big(V(Du)\big)_{x_0,\rho}\big|^2\,dx\,>\,0\Big\}\,,$$
$$\Pi_2=\Big\{x_0\in\Omega\,:\,\limsup_{\rho\to0^+}\big(\big|(u)_{x_0,\rho}\big|+\big|\big(V(Du)\big)_{x_0,\rho}\big|\big)\,=\,\infty\Big\}\,.$$

In particular, we have $\mathcal{L}^n(\mathrm{Sing}_{Du}(\Omega))=0$.

Remark: In the second case when $b(\cdot,\cdot,\cdot)$ fulfills a natural growth condition we have a better inclusion. Due to the fact that the solution u is a priori assumed to be bounded, the condition in the definition of the set Π_2 on the mean value of u, i. e., $\limsup_{\rho\to0^+}|(u)_{x_0,\rho}|=\infty$, is unnecessary.

Furthermore, we obtain the following characterization of regular boundary points analogous to the results in the quadratic and superquadratic case (cf. [Gro00, Gro02b] and [Ham07]):

Theorem 3.2: *Consider $p\in(1,2)$, Ω a bounded domain of class $C^{1,\alpha}$ in $\mathbb{R}^n$, $n\geq2$, and some $\alpha\in(0,1)$. Let $u\in g+W_0^{1,p}(\Omega,\mathbb{R}^N)$ be a weak solution of the inhomogeneous system (3.1), where the coefficients $a\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ fulfill the assumptions (H1)-(H4). Furthermore, we assume (H5) and one of the following structure conditions on the inhomogeneity:*

1. *the inhomogeneity* $b(\cdot,\cdot,\cdot)$ *obeys a controllable growth condition (B1),*
2. *the inhomogeneity* $b(\cdot,\cdot,\cdot)$ *obeys a natural growth condition (B2) and* $u \in L^\infty(\Omega,\mathbb{R}^N)$ *with* $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \le M$ *and* $2L_1(M)M < 1$.

Then, for $y \in \mathrm{Reg}_{Du}(\partial\Omega)$ *there holds: Du is Hölder continuous with exponent* α *in a neighbourhood of* y *in* $\overline{\Omega}$*, and the set of singular boundary points is contained in* $\widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2$ *with*

$$\widetilde{\Sigma}_1 = \Big\{ y \in \partial\Omega \,:\, \liminf_{\rho\to 0^+} ⨍_{\Omega\cap B_\rho(y)} \big| V(D_{\nu_{\partial\Omega}(y)}u) - \big(V(D_{\nu_{\partial\Omega}(y)}u)\big)_{\Omega\cap B_\rho(y)}\big|^2\, dx \,>\, 0 \Big\}\,,$$
$$\widetilde{\Sigma}_2 = \Big\{ y \in \partial\Omega \,:\, \limsup_{\rho\to 0^+} \big|\big(V(D_{\nu_{\partial\Omega}(y)}u)\big)_{\Omega\cap B_\rho(y)}\big| \,=\, \infty \Big\}\,,$$

where $\nu_{\partial\Omega}(y)$ *denotes the inward-pointing unit normal to* $\partial\Omega$ *at* y.

Remark: Since the solution u coincides with g on $\partial\Omega$ we do not need the assumption on the mean values of u at the boundary even in the case of a controllable growth condition.

Remark 3.3: We mention here that if we consider bounded weak solutions of (3.1) and if the inhomogeneity satisfies only an "almost" natural growth condition of the form

(B3) there exist L_1, L_2 (possibly depending on $M > 0$), such that:
$$|b(x,u,z)| \le L_1(M)\,|z|^{\tilde p} + L_2(M)$$
for all $(x,u,z) \in \overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$ with $|u| \le M$ and some $\tilde p < p$,

then the conclusions of our main Theorems 3.1 and 3.2 follow without any assumption of the form (3.3). The only point, where we actually use condition (3.3) is the proof of the Caccioppoli inequality, where we will point out the necessary modifications (cf. e. g. [CW98, Chapt. 12, Remark 4.2]) for dealing with the "almost" natural growth condition (B3).

3.2 The transformed system

In this section, we will explicitly transform the system to the model situation of the upper half-sphere, and prove for the transformed system (for some $r > 0$)

$$\begin{cases} -\operatorname{div}\tilde a(x,v,Dv) \,=\, \tilde b(x,v,Dv) & \text{in } B_r^+\,,\\ v \,=\, 0 & \text{on } \Gamma_r \end{cases}$$

that the new coefficients and the new inhomogeneity defined in (3.9), (3.10) below still satisfy similar structure conditions as introduced in the previous section.

We now consider the transformation to the model situation: Let $z \in \partial\Omega$ be a boundary point. After an affine transformation and a rotation we may assume $z = 0$ and $\nu_{\partial\Omega}(z) = e_n$, where $\nu_{\partial\Omega}(z)$ denotes the inner unit normal vector in z to the boundary $\partial\Omega$. Our assumptions on the regularity of the boundary of Ω ensure the existence of a $C^{1,\alpha}$-function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ with $h(0) = 0$, $\nabla h(0) = 0$, such that for some $r > 0$ there holds

$$\Omega \cap B_r(0) = \big\{ x \in B_r(0) : x_n > h(x') \big\}\,;$$

we here recall $x' = (x_1, \dots, x_{n-1})$. We choose the radius r sufficiently small such that for all $x' \in \mathbb{R}^{n-1}$ with $|x'| < \sqrt{2}r$ there holds:

$$|\nabla h(x')| < \tfrac{1}{k} \tag{3.6}$$

for some number $k \geq 2$ to be defined later. Since the boundary $\partial\Omega$ is compact, we can choose a common r that is suitable for all boundary points. We define the mappings $\mathcal{T}$, $\mathcal{T}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ of class $C^{1,\alpha}$ by

$$\begin{aligned} \mathcal{T}(x) &= \big(x', x_n - h(x')\big)\,, \\ \mathcal{T}^{-1}(y) &= \big(y', y_n + h(y')\big)\,. \end{aligned}$$

We see that $\mathcal{T}$ locally flattens the boundary (meaning that $\mathcal{T}(B_r \cap \partial\Omega) \subset \Gamma_r$) and $\mathcal{T}^{-1}$ is its inverse. In particular, we have $\mathcal{T}(0) = 0$ by the assumptions above. The Jacobian $D\mathcal{T}(x)$ is given by:

$$D\mathcal{T}(x) = \left(\begin{array}{ccc|c} & & & 0 \\ & Id_{n-1} & & \vdots \\ & & & 0 \\ \hline -D_1 h(x') & \cdots & -D_{n-1}h(x') & 1 \end{array}\right).$$

We have $\det D\mathcal{T} = \det D\mathcal{T}^{-1} = 1$. Condition (3.6) implies for every $x \in B_{\sqrt{2}r}$ and all vectors $w \in \mathbb{R}^n$:

$$\sqrt{1 - \tfrac{1}{k}}\,|w| \leq |D\mathcal{T}(x)w| \leq \sqrt{1 + \tfrac{1}{k}}\,|w| \tag{3.7}$$

and therefore, the corresponding estimates for $|w^T D\mathcal{T}(x)|$ and multiplications by matrices. Since $\mathcal{T}^{-1}$ has the same structure as $\mathcal{T}$, we infer (3.7) also for $D\mathcal{T}^{-1}$ instead of $D\mathcal{T}$, and hence we observe that $\mathcal{T}$ and $\mathcal{T}^{-1}$ have for any choice of $k \geq 2$ Lipschitz constants between $1/\sqrt{2}$ and $\sqrt{2}$. Keeping in mind the structure of $\mathcal{T}$ we find for every $\rho \leq \sqrt{2}r$ the inclusions

$$B^+_{\rho/\sqrt{2}} \subset \mathcal{T}(\Omega \cap B_\rho) \subset B^+_{\sqrt{2}\rho}. \tag{3.8}$$

Using the change of variables formula we observe (note $\det D\mathcal{T}^{-1} = 1$) for the $\mathcal{L}^n$-measure of these sets

$$|B^+_{\rho/\sqrt{2}}| \leq |\Omega \cap B_\rho| \leq |B^+_{\sqrt{2}\rho}|.$$

We now consider a solution u of the Dirichlet problem (3.4) under the assumptions (H1)-(H4) on the coefficients $a(\cdot,\cdot,\cdot)$, (H5) on the boundary data g as well as (B1) and (B2), respectively on the inhomogeneity $b(\cdot,\cdot,\cdot)$. We next show that the function

$$\tilde{v}(y) := \tilde{u}(y) - \tilde{g}(y) := u \circ \mathcal{T}^{-1}(y) - g \circ \mathcal{T}^{-1}(y)$$

(which is the transformation to the half-ball) is a weak solution to a system having the same structure as (3.4), on the domain B_r^+, for some $r > 0$. We will further show that the coefficients of the new system satisfy structure conditions analogous to (H1)-(H4) and (B1) and (B2), respectively (for some different constants and exponents given in terms of the boundary data and of the structure constants of the original system). By definition we see $\tilde{v} \in W^{1,p}_\Gamma(B_r^+, \mathbb{R}^N)$, hence, we have reduced our problem to the model situation of a half-ball with zero boundary values on Γ_r.

The transformed system: We begin by considering a test function $\varphi \in C_0^\infty(\Omega \cap B_{r/\sqrt{2}}, \mathbb{R}^N)$ and define $\tilde{\varphi} = \varphi \circ \mathcal{T}^{-1} \in C_0^{1,\alpha}(B_r^+, \mathbb{R}^N)$. Using the identity $\det D\mathcal{T}^{-1} = 1$ and the inclusion (3.8), the system (3.1) may now be transformed to a half-ball:

$$\begin{aligned}
0 &= \int_\Omega a\big(x, u(x), Du(x)\big) \cdot D\varphi(x)\, dx - \int_\Omega b\big(x, u(x), Du(x)\big) \cdot \varphi(x)\, dx \\
&= \int_{B_r^+} a\Big(\mathcal{T}^{-1}(y), u\big(\mathcal{T}^{-1}(y)\big), Du\big(\mathcal{T}^{-1}(y)\big)\Big) \cdot D\varphi\big(\mathcal{T}^{-1}(y)\big)\, dy \\
&\quad - \int_{B_r^+} b\Big(\mathcal{T}^{-1}(y), u\big(\mathcal{T}^{-1}(y)\big), Du\big(\mathcal{T}^{-1}(y)\big)\Big) \cdot \varphi\big(\mathcal{T}^{-1}(y)\big)\, dy \\
&= \int_{B_r^+} a\Big(\mathcal{T}^{-1}(y), \big(\tilde{v} + \tilde{g}\big)(y), \big(D\tilde{v} + D\tilde{g}\big)(y) \left(D\mathcal{T}^{-1}(y)\right)^{-1}\Big) \cdot D\tilde{\varphi}(y) \left(D\mathcal{T}^{-1}(y)\right)^{-1} dy \\
&\quad - \int_{B_r^+} b\Big(\mathcal{T}^{-1}(y), \big(\tilde{v} + \tilde{g}\big)(y), \big(D\tilde{v} + D\tilde{g}\big)(y) \left(D\mathcal{T}^{-1}(y)\right)^{-1}\Big) \cdot \tilde{\varphi}(y)\, dy\,,
\end{aligned}$$

where we have employed $\tilde{u} = \tilde{v} + \tilde{g}$ in the last equality. Due to $\left(D\mathcal{T}^{-1}(y)\right)^{-1} = D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)$ this can be rewritten as:

$$\int_{B_r^+} \tilde{a}\big(y, \tilde{v}(y), D\tilde{v}(y)\big) \cdot D\tilde{\varphi}(y)\, dy \,=\, \int_{B_r^+} \tilde{b}\big(y, \tilde{v}(y), D\tilde{v}(y)\big) \cdot \tilde{\varphi}(y)\, dy\,,$$

where the coefficients $\tilde{a}(y, v, z) \in \mathbb{R}^{nN}$ and the inhomogeneity $\tilde{b}(y, v, z) \in \mathbb{R}^N$ are given by

$$\tilde{a}(y, v, z) := a\Big(\mathcal{T}^{-1}(y), v + \tilde{g}(y), \big(z + D\tilde{g}(y)\big)\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big)\, D\mathcal{T}^t\big(\mathcal{T}^{-1}(y)\big)\,, \tag{3.9}$$

$$\tilde{b}(y, v, z) := b\Big(\mathcal{T}^{-1}(y), v + \tilde{g}(y), \big(z + D\tilde{g}(y)\big)\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big)\,, \tag{3.10}$$

with $D\mathcal{T}^t$ denoting the transpose of $D\mathcal{T}$. For arbitrary functions $\tilde{\varphi} \in C_0^\infty(B_r^+, \mathbb{R}^N)$ we may invert the calculations above (i. e., test the original system (3.1) by $\tilde{\varphi} \circ \mathcal{T}$) and conclude that $\tilde{v}$ is a weak solution of the (partial) Dirichlet problem

$$\begin{cases} \operatorname{div} \tilde{a}(\,\cdot\,, \tilde{v}, D\tilde{v}) \,=\, \tilde{b}(\,\cdot\,, \tilde{v}, D\tilde{v}) & \text{in } B_r^+\,, \\ \tilde{v} \,=\, 0 & \text{on } \Gamma_r\,. \end{cases} \tag{3.11}$$

In the sequel we show how the structure conditions on $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$, respectively, are transferred to the new coefficients $\tilde{a}(\cdot,\cdot,\cdot)$ and the new inhomogeneity $\tilde{b}(\cdot,\cdot,\cdot)$. For ease of notation we will omit the specification of Ω and B_r^+ in the norms appearing below. In what follows we consider y, $\bar{y} \in B_r^+$, v, $\bar{v} \in \mathbb{R}^N$ and z, C, $\overline{C} \in \mathbb{R}^{nN}$. Hence, $\mathcal{T}^{-1}(y)$ and $\mathcal{T}^{-1}(\bar{y})$ belong to the ball with the same radius increased by the factor $\sqrt{2}$, and for the latter ball inequality (3.6) is fulfilled. In particular we have (3.7) with x replaced by $\mathcal{T}^{-1}(y)$ (or by $\mathcal{T}^{-1}(\bar{y})$), i. e., there holds

$$\sqrt{1 - \tfrac{1}{k}}\, |C| \,\leq\, \big|D\mathcal{T}(\mathcal{T}^{-1}(y))\, C\big| \,\leq\, \sqrt{1 + \tfrac{1}{k}}\, |C|\,.$$

Here we choose the factor $k \geq 2$ according to the growth condition of the inhomogeneity in such a way that

$$\begin{cases} k \;= 2 & \text{if condition (B1) is considered,} \\ (1 + \frac{1}{k})^2 (1 - \frac{1}{k})^{-1} \;\leq 1 + \frac{1 - 2L_1(M)M}{4L_1(M)M} & \text{if condition (B2) is considered.} \end{cases} \tag{3.12}$$

Moreover, we will use the following identity (see the definition of $\tilde g$):

$$D\tilde g(y)\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big) \;=\; Dg\big(\mathcal{T}^{-1}(y)\big)\, D\mathcal{T}^{-1}(y)\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big) \;=\; Dg\big(\mathcal{T}^{-1}(y)\big)\,.$$

The coefficients $\tilde a(\cdot,\cdot,\cdot)$: For detailed calculations (in particular concerning the Hölder continuity of $\tilde a(\cdot,\cdot,\cdot)$ with respect to the first and second variable) we refer to [Bec05, Chapter 2.3]; here we only recapitulate the results and note the modifications necessary for the treatment of the inhomogeneity. We mention that – in contrast to [Gro00] – the original Dirichlet problem is reduced to zero boundary values on Γ_r. On the one hand, in particular in the inhomogeneous situation, we have to pay more attention to the transformation arguments: here we have to accept more restrictions on the radius and we do not obtain the same structure conditions (i. e., we will only deduce (H3)* instead of (H3)); moreover, the estimates in the Caccioppoli inequality will become slightly more technical. But on the other hand this reduction facilitates the notation within the deduction of the excess-decay estimates, because the function g does no longer appear in the model case; furthermore, we can proceed analogously to [Bec05] and cite some parts of the regularity proof.

Firstly we see that $D_z\tilde a(\cdot,\cdot,\cdot)$ (as well as $D_z a(\cdot,\cdot,\cdot)$) may be written as a bilinear form on $\mathbb{R}^{nN}$:

$$\begin{aligned}
&D_z\tilde a(y,v,z)\,(C,\overline{C})\\
&\quad= D_z a\Big(\mathcal{T}^{-1}(y), v+\tilde g(y), \big(z+D\tilde g(y)\big)\, D\mathcal{T}(\mathcal{T}^{-1}(y))\Big)\,\Big(C\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big), \overline{C}\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big)\,.
\end{aligned}$$

Taking into account the assumption (H1) we infer

$$\begin{aligned}
\big|\tilde a(y,v,z)\cdot C\big| \;&=\; \Big|a\Big(\mathcal{T}^{-1}(y), v+\tilde g(y), \big(z+D\tilde g(y)\big)\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big)\cdot C\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big|\\
&\le\; \sqrt{2}\, L\,\Big(1+\big|z\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)+Dg\big(\mathcal{T}^{-1}(y)\big)\big|^{p-1}\Big)\,|C|\\
&\le\; 2\sqrt{2}\, L\,\big(1+\|Dg\|_\infty\big)\,\big(1+|z|^{p-1}\big)\,|C|\,.
\end{aligned}$$

From condition (H2) and the regularity assumptions on $\mathcal{T}$ we further deduce that $\tilde a(y,v,z)$ is differentiable with respect to z with continuous derivative. Moreover, for the growth of $D_z\tilde a$ we see via the representation above:

$$\big|D_z\tilde a(y,v,z)\,(C,\overline{C})\big| \;\le\; 2\,L\,|C|\,|\overline{C}|\,,$$

i. e., a condition as in (H2). Next we turn our attention to the ellipticity condition: here we apply Young's inequality, (H3) and the fact that $p<2$ in order to achieve

$$\begin{aligned}
D_z\tilde a(y,v,z)\,(C,C) \;&\ge\; \Big(1+\big|z\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)+Dg\big(\mathcal{T}^{-1}(y)\big)\big|^2\Big)^{\frac{p-2}{2}}\,\big|C\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\big|^2\\
&\ge\; (1-\tfrac{1}{k})\,\Big(1+(1+k)\,\|Dg\|_\infty^2+(1+\tfrac{1}{k})\,\big|z\, D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\big|^2\Big)^{\frac{p-2}{2}}\,|C|^2\\
&\ge\; (1-\tfrac{1}{k})\,(1+\tfrac{1}{k})^{p-2}\,\Big(\big(1+(1+k)\|Dg\|_\infty\big)^2+|z|^2\Big)^{\frac{p-2}{2}}\,|C|^2\,, \qquad (3.13)
\end{aligned}$$

or, for the choice $k=2$, the less complicated representation

$$D_z\tilde a(y,v,z)\,(C,C) \;\ge\; 2^{p-3}\,\big(1+\|Dg\|_\infty^2+|z|^2\big)^{\frac{p-2}{2}}\,|C|^2\,.$$

To infer Hölder continuity of the map $(x,u) \to \tilde{a}(x,u,z)\,\big(1+|z|^2\big)^{\frac{p-1}{2}}$ we need, apart from the inequalities (3.7), a further restriction on the radius: we choose r sufficiently small such that

$$\|\mathcal{T}\|_{C^{1,\alpha}}\,\|\mathcal{T}^{-1}\|_{C^{1,\alpha}}\,(2r)^{\alpha} \;\le\; \tfrac{1}{2\sqrt{2}}\,. \tag{3.14}$$

This ensures in particular $r < \frac{1}{2}$; moreover, this choice is always possible due to the fact that the $C^{1,\alpha}$ norms of $\mathcal{T}$ and of $\mathcal{T}^{-1}$, respectively, are bounded and therefore the left-hand side in (3.14) converges to 0 as $r \to 0$. Hence, applying (H2) we obtain (cf. [Bec05] for detailed calculations)

$$\big|\big(\tilde{a}(y,v,z)-\tilde{a}(\bar{y},\bar{v},z)\big)\big| \;\le\; L\,c\,\widetilde{K}(|v|)\,\big(1+|z|^2\big)^{\frac{p-1}{2}}\,\omega\big(|y-\bar{y}|\,+\,|v-\bar{v}|\big)$$

for $\widetilde{K}(|v|) := K(|v|+\|g\|_\infty)$ and a constant c depending only on $\|g\|_{C^{1,\alpha}}, \|\mathcal{T}\|_{C^{1,\alpha}}$ and $\|\mathcal{T}^{-1}\|_{C^{1,\alpha}}$.

The inhomogeneity $\tilde{b}(\cdot,\cdot,\cdot)$: For the assumption of controllable growth (B1) we easily derive

$$\begin{aligned}
\big|\tilde{b}(y,v,z)\big| \;&=\; \Big|b\Big(\mathcal{T}^{-1}(y), v+\tilde{g}(y), \big(z+D\tilde{g}(y)\big)\,D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)\Big)\Big| \\
&\le\; L\,\Big(1+\big|z\,D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)+Dg\big(\mathcal{T}^{-1}(y)\big)\big|^2\Big)^{\frac{p-1}{2}} \\
&\le\; 2\,L\,\big(1+\|Dg\|_\infty\big)\,\big(1+|z|^2\big)^{\frac{p-1}{2}}\,.
\end{aligned}$$

If, in contrast, we assume natural growth (B2) and $|v+\tilde{g}(y)| \le M$, we obtain

$$\begin{aligned}
\big|\tilde{b}(y,v,z)\big| \;&=\; \Big|b\Big(\mathcal{T}^{-1}(y), v+\tilde{g}(y), z\,D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)+Dg\big(\mathcal{T}^{-1}(y)\big)\Big)\Big| \\
&\le\; L_1(M)\,\big|z\,D\mathcal{T}\big(\mathcal{T}^{-1}(y)\big)+Dg\big(\mathcal{T}^{-1}(y)\big)\big|^p + L_2(M) \\
&\le\; L_1(M)\,\Big((1+\tfrac{1}{k})^2|z|^2+(1+k)\|Dg\|_\infty^2\Big)^{\frac{p}{2}} + L_2(M) \\
&\le\; (1+\tfrac{1}{k})^p L_1(M)\,|z|^p + (1+k)\,L_1(M)\,\|Dg\|_\infty^p + L_2(M)\,.
\end{aligned}$$

Conclusion: We now rescale the transformed system (3.11) by the factor $(1-\frac{1}{k})\,(1+\frac{1}{k})^{p-2}$, i. e., by the factor appearing in the ellipticity condition in (3.13), meaning that we define the new coefficients and the new right-hand side by

$$\begin{aligned}
\hat{a}(\,\cdot\,,\,\cdot\,,\,\cdot\,) &:= (1-\tfrac{1}{k})^{-1}\,(1+\tfrac{1}{k})^{2-p}\,\tilde{a}(\,\cdot\,,\,\cdot\,,\,\cdot\,)\,, \\
\hat{b}(\,\cdot\,,\,\cdot\,,\,\cdot\,) &:= (1-\tfrac{1}{k})^{-1}\,(1+\tfrac{1}{k})^{2-p}\,\tilde{b}(\,\cdot\,,\,\cdot\,,\,\cdot\,)\,.
\end{aligned}$$

Then we see: $\tilde{v}$ is a weak solution of

$$\begin{cases} -\operatorname{div}\hat{a}(\,\cdot\,,\tilde{v},D\tilde{v}) \,=\, \hat{b}(\,\cdot\,,\tilde{v},D\tilde{v}) & \text{in } B_r^+\,, \\ \qquad\qquad \tilde{v} \,=\, 0 & \text{on } \Gamma_r \end{cases} \tag{3.15}$$

for r sufficiently small. Assuming a controllable growth condition on the inhomogeneity $b(\cdot,\cdot,\cdot)$, we come to the conclusion that for the new system there hold conditions analogous to (H1), (H2), (H3)*, (H4) and (B1) with constants

$$\begin{aligned}
\widehat{L} \;&=\; L\,c_L\big(\|g\|_{C^{1,\alpha}}, \|\mathcal{T}\|_{C^{1,\alpha}}, \|\mathcal{T}^{-1}\|_{C^{1,\alpha}},\big)\,, \\
\widehat{K}(\cdot) \;&=\; K\big(\,\cdot+\|g\|_\infty\big)\,, \\
\nu \;&=\; \nu\big(\|Dg\|_\infty\big)\,,
\end{aligned}$$

i. e., the new structural constants depend only on the boundary $\partial\Omega$ and the boundary data g. Otherwise, if we assume a natural growth condition we first note that the number k introduced in (3.12) depends only on M and $L_1(M)$. Thus, keeping in mind the normalization of the coefficients $\hat{a}(\cdot,\cdot,\cdot)$ by the factor $(1-\frac{1}{k})(1+\frac{1}{k})^{p-2}$ for an appropriate number $k = k(M, L_1(M))$, we infer conditions analogous to (H1), (H2), (H3)* and (H4) with constants

$$\begin{aligned} \widehat{L} &= L\, c_L\big(M, L_1(M), \|g\|_{C^{1,\alpha}}, \|\mathcal{T}\|_{C^{1,\alpha}}, \|\mathcal{T}^{-1}\|_{C^{1,\alpha}}, \big)\,, \\ \widehat{K}(\cdot) &= K\big(\,\cdot + \|g\|_\infty\big)\,, \\ \nu &= \nu\big(M, L_1(M), \|Dg\|_\infty\big)\,, \end{aligned}$$

i. e., the new structural constants depend here additionally on the constants appearing in the smallness condition (3.3). We briefly comment on the dependence of the ellipticity constant ν upon the parameters M and $L_1(M)$: when $2L_1(M)M \nearrow 1$, then $k \to \infty$ and consequently $\nu \searrow 0$. This takes no effect on the ellipticity of the transformed system because (3.3) is a global condition on $\overline{\Omega}$ and hence, ν is bounded from below uniformly for every transformed system. Moreover, (B2) transforms to the following condition: whenever we consider $v \in \mathbb{R}^N$ such that $|v + \tilde{g}(y)| \le M$ we obtain: $|\hat{b}(y, v, z)| \le \widehat{L}_1(M)|z|^p + \widehat{L}_2(M)$ where

$$\begin{aligned} \widehat{L}_1(M) &= \big(1+\tfrac{1}{k}\big)^2 \big(1-\tfrac{1}{k}\big)^{-1} L_1(M) \,=\, c_{L_1}\big(M, L_1(M)\big)\,, \\ \widehat{L}_2(M) &= c_{L_2}\big(M, L_1(M), L_2(M), \|Dg\|_\infty\big)\,. \end{aligned}$$

The condition $|v + \tilde{g}(y)| \le M$ required here is indeed natural: later we will apply condition (B2) for the transformed solution $\tilde{v}(y) = \tilde{u}(y) - \tilde{g}(y)$ instead of v such that equivalently (by definition of $\tilde{v}$) the condition $|\tilde{u}(y)| = |u(\mathcal{T}^{-1}(y)| \le M$ is required. Finally we calculate using the explicit representation of $\widehat{L}_1(M)$ and the definition (3.12) of k (note the assumption (3.3) on the quantity $L_1(M)M$):

$$\begin{aligned} 2\widehat{L}_1(M)\, M &= 2\big(1+\tfrac{1}{k}\big)^2 \big(1-\tfrac{1}{k}\big)^{-1} L_1(M)\, M \\ &\le 2\,\big(1 + \tfrac{1-2L_1(M)M}{4L_1(M)M}\big) L_1(M)\, M \\ &= L_1(M)\, M + \tfrac{1}{2} \,<\, 1\,. \end{aligned}$$

Consequently, there holds a condition analogous to $2L_1(M)M < 1$ for the transformed problem. Altogether, this means we have proved that the transformation to the model situation of a half-ball preserves all the structure conditions, both in the case of a controllable and of a natural growth condition on the inhomogeneity $b(\cdot,\cdot,\cdot)$.

In summary, in the sequel we shall consider weak solutions $u \in W^{1,p}(B^+, \mathbb{R}^N)$ of the elliptic system

$$\begin{cases} \displaystyle\int_{B^+} a(\,\cdot\,, u, Du)\cdot D\varphi\, dx = \int_{B^+} b(\,\cdot\,, u, Du)\cdot \varphi\, dx & \forall\, \varphi \in C_0^\infty(B^+, \mathbb{R}^N) \\ u = 0 & \text{on } \Gamma\,. \end{cases} \tag{3.16}$$

in the model case of a half-ball, where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the assumptions (H1), (H2), (H3)* and (H4). Furthermore, with respect to the inhomogeneity $b(\cdot,\cdot,\cdot)$ either a controllable growth condition (B1) or a (transformed) natural growth condition (B2)* is assumed, meaning that

(B1) $|b(x,u,z)| \le L\big(1+|z|^2\big)^{\frac{p-1}{2}}$ for all $(x,u,z) \in B^+ \cup \Gamma \times \mathbb{R}^N \times \mathbb{R}^{nN}$,

or

(B2)* $|b(x,u,z)| \le L_1(M)\,|z|^p + L_2(M)$ for all $(x,u,z) \in B^+ \cup \Gamma \times \mathbb{R}^N \times \mathbb{R}^{nN}$ with $|u+g| \le M$

for some constants $L_1(M)$, $L_2(M)$. In the latter case we will further require

$$u \in L^\infty(B^+,\mathbb{R}^N) \quad \text{with } \|u+g\|_{L^\infty(B^+,\mathbb{R}^N)} \le M \text{ and } 2L_1(M)M < 1\,. \tag{3.17}$$

Then, our objective is to infer a characterization of regular boundary points for the model problem under these assumptions. As we will see in Section 3.6.3, this suffices to obtain the desired characterization stated in Theorem 3.2 for the general situation.

3.3 Linear theory

In this section we first provide an *a priori* estimate for solutions of linear elliptic systems of second order with constant coefficients in the subquadratic case. In the corresponding quadratic situation it is well known that $W^{1,2}$-solutions are smooth up to the boundary. Using different techniques with the L^p-theory in a global version as an essential tool it is possible (see [Bec05, Chapter 4]) to overcome the difficulties arising from the fact that we treat the case $1 < p < 2$ in order to obtain regularity up to the boundary also in this case. Secondly, we present a suitable $\mathcal{A}$-harmonic approximation lemma.

We now consider the (partial) Dirichlet problems

$$\begin{cases} \operatorname{div}(A\,Du) = 0 & \text{in } B_\rho^+(x_0)\,, \\ u = 0 & \text{on } \Gamma_\rho(x_0)\,. \end{cases} \tag{3.18}$$

for $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ in order to prove C^∞-regularity up to the boundary, and

$$\operatorname{div}(A\,Du) = 0 \qquad \text{in } B_\rho(x_0) \tag{3.19}$$

for some $x_0 \in \mathbb{R}^n$ in order to derive the corresponding estimates in the interior. Here, we assume the coefficients $A \in \mathbb{R}^{nN}$ to be bounded and elliptic in the sense of Legendre-Hadamard, i. e., that we have for some $0 < \nu \le L$

(A1) $|A(C,\overline{C})| \le \Lambda\,|C|\,|\overline{C}|$ $\forall\, C,\overline{C} \in \mathbb{R}^{nN}$

(A2) $A(\xi\otimes\eta,\xi\otimes\eta) \ge \nu\,|\xi|^2\,|\eta|^2$ $\forall\, \xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n\,.$

Theorem 3.4 ([Bec05], Satz 4.5; [DGK05], Lemma 5): *Let $p \in (1,2)$ and let A be constant coefficients which satisfy conditions (A1) and (A2). There holds:*

(i) Assume $u \in W^{1,p}_\Gamma(B_\rho^+(x_0),\mathbb{R}^N)$ to be weak solutions of the system (3.18). Then $u \in C^\infty(B_\rho^+(x_0)\cup\Gamma_\rho(x_0),\mathbb{R}^N)$, and

$$\sup_{B^+_{\rho/2}(x_0)} \big(|Du| + \rho\,|D^2u|\big) \le c\left(\fint_{B_\rho^+(x_0)} |Du|^p\,dx\right)^{\frac1p}.$$

(ii) Assume $u \in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$ *to be weak solutions of the system (3.19). Then* $u \in C^\infty(B_\rho(x_0),\mathbb{R}^N)$*, and*

$$\sup_{B_{\rho/2}(x_0)} \big(|Du| + \rho\,|D^2u|\big) \le c\Big(\fint_{B_\rho(x_0)} |Du|\,dx\Big).$$

In both situations the constant c *depends only on* n, N, p *and* $\frac{\Lambda}{\nu}$.

Furthermore, we state the following results concerning $\mathcal{A}$-harmonic approximation:

Lemma 3.5 ($\mathcal{A}$-harm. approximation; [Bec05], Lemma 4.7, [DGK05], Lemma 6): *Let* λ, Λ *be positive constants. Then for every* $\varepsilon > 0$ *there exists* $\delta = \delta(n, N, p, \frac{\nu}{\Lambda}, \varepsilon)$ *with the following property:*

(i) For every bilinear form $\mathcal{A}$ *on* $\mathbb{R}^{nN}$ *which is elliptic in the sense of Legendre-Hadamard with ellipticity constant* ν *and upper bound* Λ *and for every* $u \in W^{1,p}_\Gamma(B^+_\rho(x_0),\mathbb{R}^N)$ *(with some* $\rho > 0$, $x_0 \in \mathbb{R}^{n-1}\times\{0\}$*) satisfying:*

$$\fint_{B^+_\rho(x_0)} |V(Du)|^2\,dx \le \gamma^2 \le 1,$$

$$\Big|\fint_{B^+_\rho(x_0)} \mathcal{A}(Du, D\varphi)\,dx\Big| \le \delta\,\gamma \sup_{B^+_\rho(x_0)} |D\varphi| \qquad \forall\,\varphi \in C^1_0(B^+_\rho(x_0),\mathbb{R}^N),$$

there exists an $\mathcal{A}$*-harmonic function* $h \in W^{1,p}_\Gamma(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ *(meaning that for all* $\varphi \in C^1_0(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ *there holds* $\int_{B^+_{\rho/2}(x_0)} \mathcal{A}(Dh, D\varphi)\,dx = 0$*), which satisfies*

$$\fint_{B^+_{\rho/2}(x_0)} \Big|V\Big(\frac{u-\gamma h}{\rho}\Big)\Big|^2\,dx \le \gamma^2\varepsilon \quad \text{and} \quad \fint_{B^+_{\rho/2}(x_0)} \big|V(Dh)\big|^2\,dx \le 2^{n+3}.$$

(ii) For every bilinear form $\mathcal{A}$ *on* $\mathbb{R}^{nN}$ *which is elliptic in the sense of Legendre-Hadamard with ellipticity constant* ν *and upper bound* Λ *and for every* $u \in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$ *satisfying:*

$$\fint_{B_\rho(x_0)} |V(Du)|^2\,dx \le \gamma^2 \le 1,$$

$$\Big|\fint_{B_\rho(x_0)} \mathcal{A}(Du, D\varphi)\,dx\Big| \le \delta\,\gamma \sup_{B_\rho(x_0)} |D\varphi| \qquad \forall\,\varphi \in C^1_0(B_\rho(x_0),\mathbb{R}^N),$$

there exists an $\mathcal{A}$*-harmonic function* $h \in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$ *which satisfies*

$$\fint_{B_\rho(x_0)} \Big|V\Big(\frac{u-\gamma h}{\rho}\Big)\Big|^2\,dx \le \gamma^2\varepsilon \quad \text{and} \quad \fint_{B_\rho(x_0)} \big|V(Dh)\big|^2\,dx \le 2.$$

3.4 A Caccioppoli inequality

As usual the first step in proving a regularity theorem for solutions u of elliptic systems is to establish a suitable reverse-Poincaré or Caccioppoli inequality. This means that a certain integral of Du (here the L^2-norm of $V(Du)$ on a half-ball) is essentially controlled in terms of the solution u itself on a larger domain (or, in our model situation, on a larger half-ball). In the first step we will study the boundary situation.

Lemma 3.6 (Caccioppoli inequality at the boundary): *Let $M_0 > 0$, $\xi \in \mathbb{R}^N$ with $|\xi| \leq M_0$ and let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a weak solution of (3.16) with coefficients $a(\cdot,\cdot,\cdot)$ satisfying the assumptions (H1), (H2), (H3)* and (H4). Furthermore, assume that one of the following conditions holds:*

1. *the inhomogeneity fulfills a controllable growth condition (B1),*
2. *the inhomogeneity fulfills a natural growth condition (B2)*, and (3.17) is satisfied.*

Then for all $x_0 \in \Gamma$ and $\rho < \rho_{cacc} \leq 1 - |x_0|$ there holds

$$\fint_{B^+_{\rho/2}(x_0)} |V(Du - \xi \otimes e_n)|^2 \, dx \, \leq \, c_{cacc} \Big(\fint_{B^+_\rho(x_0)} \Big| V\Big(\frac{u - \xi\, x_n}{\rho}\Big)\Big|^2 \, dx + \rho^{2\alpha}\Big) \, .$$

The constant c_{cacc} depends in the first case only on n, p, ν, L, M_0 and $K(M_0)$, whereas in the second case it depends additionally on $M, L_1(M)$ and $L_2(M)$; the radius ρ_{cacc} is given in the first case by $1 - |x_0|$, and in the second case it depends only on $M_0, M, L_1(M)$ and $\|Dg\|_{L^\infty}$.

Proof: Without loss of generality we may assume $M > 0$. We consider a cut-off function $\eta \in C^\infty_0(B_{\rho/2}(x_0), [0,1])$ satisfying $\eta = 1$ on $B_{\rho/2}(x_0)$ and $|\nabla\eta| \leq \frac{4}{\rho}$. Both in the case of controllable and natural growth (keep in mind that in the latter case u is bounded) the function $\varphi = \eta^2(u - \xi\, x_n)$ can be taken as a test function in (3.16). Hence, using the abbreviation $X := \xi \otimes e_n$ we obtain

$$\begin{aligned}
\fint_{B^+_\rho(x_0)} b(\,\cdot\,, u, Du) \cdot \varphi \, dx &= \fint_{B^+_\rho(x_0)} a(\,\cdot\,, u, Du) \cdot D\varphi \, dx \\
&= \fint_{B^+_\rho(x_0)} a(\,\cdot\,, u, Du) \cdot (Du - X)\, \eta^2 \, dx \\
&\quad + \fint_{B^+_\rho(x_0)} a(\,\cdot\,, u, Du) \cdot \big((u - \xi\, x_n) \otimes \nabla\eta\big)\, 2\eta \, dx \, .
\end{aligned}$$

Since $a(x_0, 0, X)$ is constant, the integral $\fint_{B^+_\rho(x_0)} a(x_0,0,X) \cdot D\varphi \, dx$ vanishes and thus we conclude

$$\begin{aligned}
&\fint_{B^+_\rho(x_0)} \big[a(\,\cdot\,, u, Du) - a(\,\cdot\,, u, X)\big] \cdot (Du - X)\, \eta^2 \, dx \\
&\quad = -2 \fint_{B^+_\rho(x_0)} \big[a(\,\cdot\,, u, Du) - a(\,\cdot\,, u, X)\big] \cdot \big((u - \xi\, x_n) \otimes \nabla\eta\big)\, \eta \, dx \\
&\qquad - \fint_{B^+_\rho(x_0)} \big[a(\,\cdot\,, u, X) - a(\,\cdot\,, \xi\, x_n, X)\big] \cdot D\varphi \, dx \\
&\qquad - \fint_{B^+_\rho(x_0)} \big[a(\,\cdot\,, \xi\, x_n, X) - a(x_0, 0, X)\big] \cdot D\varphi \, dx + \fint_{B^+_\rho(x_0)} b(\,\cdot\,, u, Du) \cdot \varphi \, dx \\
&\quad =: - \fint_{B^+_\rho(x_0)} (2\, I + II + III - IV)\, dx
\end{aligned} \tag{3.20}$$

with the obvious labelling. To estimate the terms I, II and III we decompose the half-ball into sets of the form

$$B_{(\leq)(>)} := B^+_\rho(x_0) \cap \big\{x : |Du(x) - X| \leq 1\big\} \cap \big\{x : \big|\tfrac{u(x) - \xi\, x_n}{\rho}\big| > 1\big\},$$

where other combinations involving $>$ and $\leq$ are defined analogously. If we do not restrict one of the two expressions $|Du(x)-X|$ respectively $|\frac{u(x)-\xi\, x_n}{\rho}|$ in certain computations, we replace the sign by a dot, for instance $B_{(\cdot)(\leq)} = B_{(\leq)(\leq)} \cup B_{(>)(\leq)}$. On these sets we can use Lemma A.1 (i) because for every $\zeta \in \mathbb{R}^k$ there holds:

$$\begin{aligned} \text{if } |\zeta|\leq 1: &\quad \min\{|\zeta|^2, |\zeta|^p\} = |\zeta|^2 \leq \sqrt{2}\,|V(\zeta)|^2 \\ \text{if } |\zeta|> 1: &\quad \min\{|\zeta|^2, |\zeta|^p\} = |\zeta|^p \leq \sqrt{2}\,|V(\zeta)|^2 . \end{aligned}$$

Hence, we will use in nearly every calculation Lemma A.1 and Young's inequality in its general form, i. e., that for all $a, b \geq 0$, $\varepsilon > 0$ and $q > 1$ there holds

$$a\cdot b \leq \tfrac{q-1}{q}\,\varepsilon\, a^{\frac{q}{q-1}} + \tfrac{1}{q}\,\varepsilon^{1-q}\, b^q .$$

Keeping in mind $\rho < 1$, we use the assumptions on the coefficients $a(\cdot,\cdot,\cdot)$ to estimate the various terms: for term I we apply condition (H2) if $x \in B_{(\leq)(\cdot)}$ and condition (H1) if $x \in B_{(>)(\cdot)}$ to conclude completely similarly to the derivation of the estimate (5.4) in [Bec05]

$$2\,I \leq \varepsilon\,|V(Du-X)|^2\,\eta^2 + c(p,M_0)\left(L^2\,\varepsilon^{-1} + L^p\,\varepsilon^{1-p}\right)\left|V\left(\tfrac{u-\xi\,x_n}{\rho}\right)\right|^2 . \tag{3.21}$$

Using condition (H4) and the fact that we have $|u-\xi\,x_n| \leq \rho$ in the set $B_{(\cdot)(\leq)}$, we find for the remaining terms II and III in a standard way (see (5.5) in [Bec05] for detailed computations)

$$\begin{aligned} II + III \leq\; & 2\,\varepsilon\,|V(Du-X)|^2\,\eta^2 + c(p,M_0)\,K(M_0)^{\frac{p}{p-1}}\left(L + L^{\frac{p}{p-1}}\,\varepsilon^{\frac{1}{1-p}}\right)\left|V\left(\tfrac{u-\xi\,x_n}{\rho}\right)\right|^2 \\ & + c(p,M_0)\,K(M_0)^{\frac{p}{p-1}}\left(L + L^{\frac{p}{p-1}}\,\varepsilon^{\frac{1}{1-p}} + L^2\,\varepsilon^{-1}\right)\rho^{2\alpha} . \end{aligned} \tag{3.22}$$

In the next step we consider the remaining integrals in (3.20), i. e., the left-hand side and the integral involving IV. Taking into consideration the ellipticity of $D_z a$ in (H3)* we find for the left-hand side of (3.20) via Young's inequality for all $\delta \in (0,1]$:

$$\begin{aligned} & \fint_{B_\rho^+(x_0)} \left[a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,X)\right]\cdot(Du-X)\,\eta^2\,dx \\ & \quad = \fint_{B_\rho^+(x_0)} \int_0^1 D_z a\big(\,\cdot\,,u,X+t(Du-X)\big)\big(Du-X,Du-X\big)\,\eta^2\,dt\,dx \\ & \quad \geq \fint_{B_\rho^+(x_0)} \int_0^1 \left(\nu^{-2} + |X+t(Du-X)|^2\right)^{\frac{p-2}{2}} |Du-X|^2\eta^2\,dt\,dx \\ & \quad \geq \fint_{B_\rho^+(x_0)} \left(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\right)^{\frac{p-2}{2}} |Du-X|^2\eta^2\,dx . \end{aligned} \tag{3.23}$$

We now have to distinguish the growth conditions for the inhomogeneity $b(\cdot,\cdot,\cdot)$ in order to bound the integrand IV:

Controllable growth condition (B1): First we see

$$IV = b(\,\cdot\,,u,Du)\cdot\varphi \leq L\,(1+|Du|^2)^{\frac{p-1}{2}}\,|u-\xi\,x_n|\,\eta^2,$$

we then estimate the integrand completely analogously to the terms I, II and III above on the different sets $B_{(\cdot)(\cdot)}$ and we obtain

$$IV \;\le\; \begin{cases} c(p,M_0)\,L\big(\big|\frac{u-\xi\,x_n}{\rho}\big|^2+\rho^2\big) & \text{on } B_{(\le)(\le)} \\ c(p,M_0)\,L\,\big|\frac{u-\xi\,x_n}{\rho}\big|^p & \text{on } B_{(\le)(>)} \\ \frac{\varepsilon}{\sqrt{2}}\,|Du-X|^p\,\eta^2+c(p,M_0)\,L^2\,\varepsilon^{-1}\,\big|\frac{u-\xi\,x_n}{\rho}\big|^2 & \text{on } B_{(>)(\le)} \\ \frac{\varepsilon}{\sqrt{2}}\,|Du-X|^p\,\eta^2+c(p,M_0)\,L^p\,\varepsilon^{1-p}\,\big|\frac{u-\xi\,x_n}{\rho}\big|^p & \text{on } B_{(>)(>)}\,. \end{cases}$$

By Lemma A.1, (i) this yields

$$IV \;\le\; \varepsilon\,|V(Du-X)|^2\,\eta^2+c(p,M_0)\,\big(L^2\,\varepsilon^{-1}+L^p\,\varepsilon^{1-p}\big)\,\Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2+c(p,M_0)\,L\,\rho^{2\alpha}\,. \tag{3.24}$$

Making the choice $\delta=1$ we obtain in (3.23)

$$\begin{aligned} &\fint_{B_\rho^+(x_0)}\big[a(\,\cdot\,,u,Du)-a(\,\cdot\,,u,X)\big]\cdot(Du-X)\,\eta^2\,dx \\ &\qquad\ge\;(2+2\,M_0^2)^{\frac{p-2}{2}}\fint_{B_\rho^+(x_0)}\big(\nu^{-2}+|Du-X|^2\big)^{\frac{p-2}{2}}\,|Du-X|^2\,\eta^2\,dx \\ &\qquad\ge\;c_1^{-1}(p,M_0,\nu)\fint_{B_\rho^+(x_0)}|V(Du-X)|^2\,\eta^2\,dx\,. \end{aligned}$$

Combining this with (3.20)-(3.22) and (3.24), i. e., the estimates for each of the integrands, we obtain

$$\begin{aligned} &\big(c_1^{-1}(p,\nu,M_0)-4\varepsilon\big)\fint_{B_\rho^+(x_0)}|V(Du-X)|^2\,\eta^2\,dx \\ &\quad\le\;c(p,M_0)\,K(M_0)^{\frac{p}{p-1}}\,\big(L^2\,\varepsilon^{-1}+L^p\,\varepsilon^{1-p}+L+L^{\frac{p}{p-1}}\,\varepsilon^{\frac{1}{1-p}}\big)\fint_{B_\rho^+(x_0)}\Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2\,dx \\ &\qquad+c(p,M_0)\,K(M_0)^{\frac{p}{p-1}}\,\big(L+L^{\frac{p}{p-1}}\,\varepsilon^{\frac{1}{1-p}}+L^2\,\varepsilon^{-1}\big)\,\rho^{2\alpha}\,. \end{aligned}$$

Choosing $\varepsilon=\frac{1}{8}\,c_1^{-1}(p,\nu,M_0)$ and dividing this by $\frac{1}{2}\,c_1^{-1}(p,\nu,M_0)$ finally yields

$$\begin{aligned} \fint_{B_{\rho/2}^+(x_0)}|V(Du-X)|^2\,dx\;&\le\;2^n\fint_{B_\rho^+(x_0)}|V(Du-X)|^2\,\eta^2\,dx \\ &\le\;c(n,p,L,\nu,M_0)\,K(M_0)^{\frac{p}{p-1}}\Big[\fint_{B_\rho^+(x_0)}\Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2\,dx\;+\;\rho^{2\alpha}\Big]\,, \end{aligned}$$

meaning that we have established the desired result if the inhomogeneity obeys a controllable growth condition.

Natural growth condition (B2)*: In this case we first mention that the solution u vanishes on Γ by assumption and hence, in view of (3.17) g is bounded on Γ by M. This enables us to calculate

$$|u-\xi x_n|\;\le\;|u+g|+|g(x_0)|+|g-g(x_0)|+|\xi x_n|\;\le\;2M+(\|Dg\|_{L^\infty}+|\xi|)\,\rho\,.$$

Therefore, we estimate the inhomogeneity utilizing the growth condition (B2)*, Young's inequality, Lemma A.1, (i) and $|\xi| \le M_0$ to infer for every $\delta > 0$ that

$$\begin{aligned}
IV &\le \big[L_1(M)\,|Du|^p + L_2(M)\big]\,|u-\xi x_n|\,\eta^2 \\
&\le \big[L_1(M)\big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p}{2}} + L_2(M)\big]\,|u-\xi x_n|\,\eta^2 \\
&= L_1(M)\,(1+\delta)\,|Du-X|^2\,\big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p-2}{2}}|u-\xi x_n|\,\eta^2 \\
&\quad + \big[L_1(M)\,\big(\nu^{-2} + (1+\delta^{-1})\,|X|^2\big)\,\big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p-2}{2}} \\
&\qquad + L_2(M)\big]\,|u-\xi x_n|\,\eta^2 \\
&\le L_1(M)\,(1+\delta)\,\big(2M + (\|Dg\|_{L^\infty} + |\xi|)\,\rho\big) \\
&\qquad \times \big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p-2}{2}}\,|Du-X|^2\,\eta^2 \\
&\quad + \big[L_1(M)\,\big(\nu^{-2} + (1+\delta^{-1})\,|X|^2\big) + L_2(M)\big]|u-\xi x_n|\,\eta^2 \\
&\le L_1(M)\,(1+\delta)\,\big(2M + (\|Dg\|_{L^\infty} + M_0)\,\rho\big) \\
&\qquad \times \big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p-2}{2}}|Du-X|^2\,\eta^2 \\
&\quad + \sqrt{2}\,\big[L_1(M)\,\big(\nu^{-2} + (1+\delta^{-1})\,M_0^2\big) + L_2(M)\big]\Big(\Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2 + \rho^{2\alpha}\Big) \\
&=: (IV_a + IV_b)\,.
\end{aligned}$$

The first term on the right-hand side of the last inequality will now be absorbed in (3.23) by employing the smallness condition $2L_1(M)\,M < 1$. For this purpose we choose

$$\delta = \frac{1-2L_1(M)M}{4L_1(M)M}$$

(implying that $(1+\delta)2L_1(M)M$ is the arithmetic mean of $2L_1(M)M$ and 1). We continue by setting

$$\rho_0 \;:=\; \min\Big\{1-|x_0|\,,\; \frac{M-2L_1(M)M^2}{(1+2L_1(M)M)(\|Dg\|_{L^\infty}+M_0)}\Big\}\,,$$

meaning that we take once again the arithmetic mean if necessary; therefore, ρ_0 is a quantity depending only on $M_0, M, L_1(M)$ and $\|Dg\|_{L^\infty}$. In particular, the choices for δ and ρ_0 allow us to compute

$$\begin{aligned}
1 - L_1(M)\,&(1+\delta)\,\big(2M + (\|Dg\|_{L^\infty} + M_0)\,\rho_0\big) \\
&\ge\; 1 - L_1(M)\,\frac{1+2L_1(M)M}{4L_1(M)M}\,\Big(2M + \frac{M-2L_1(M)M^2}{1+2L_1(M)M}\Big) \\
&=\; 1 - \frac{1+2L_1(M)M}{4M}\,\frac{3M+2L_1(M)M^2}{1+2L_1(M)M} \\
&=\; \frac{1-2L_1(M)M}{4} \;=\; c^{-1}\big(M, L_1(M)\big) \;>\; 0\,,
\end{aligned}$$

where we have employed that $2L_1(M)M < 1$ by assumption. Furthermore, in view of the choice of δ, we find $(1+\delta)^{(p-2)/2} \ge (1+\delta)^{-1} = c^{-1}(M, L_1(M))$ as well as $(1+\delta^{-1})^{(p-2)/2} \ge c^{-1}(M, L_1(M))$. Hence, we use (3.23) and $|X| \le M_0$ to obtain for all radii $\rho \in (0, \rho_0)$:

$$⨍_{B_\rho^+(x_0)} \big[a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,X)\big]\cdot(Du-X)\,\eta^2\,dx - ⨍_{B_\rho^+(x_0)} IV_a\,dx$$

$$
\begin{aligned}
&\geq c^{-1}\big(M, L_1(M)\big) \fint_{B_\rho^+(x_0)} \big(\nu^{-2} + (1+\delta^{-1})\,|X|^2 + (1+\delta)\,|Du-X|^2\big)^{\frac{p-2}{2}} |Du-X|^2 \eta^2\,dx \\
&\geq c^{-1}\big(M, L_1(M)\big) \fint_{B_\rho^+(x_0)} \big(\nu^{-2} + |X|^2 + |Du-X|^2\big)^{\frac{p-2}{2}} |Du-X|^2 \eta^2\,dx \\
&\geq c_2^{-1}\big(p,\nu,M_0,M,L_1(M)\big) \fint_{B_\rho^+(x_0)} |V(Du-X)|^2 \eta^2\,dx\,.
\end{aligned}
$$

We note that the constant c_2^{-1} approaches 0 as $2L_1(M)M \nearrow 1$ in condition (3.17). Combining (3.20)-(3.22) and the definition of IV_b, we infer similarly to the case of a controllable growth condition that

$$
\begin{aligned}
&\big(c_2^{-1}\big(p,\nu,M_0,M,L_1(M)\big) - 3\varepsilon\big) \fint_{B_\rho^+(x_0)} |V(Du-X)|^2\,\eta^2\,dx \\
&\leq c(p,L,M_0,M,L_1(M),L_2(M),\varepsilon)\,K(M_0)^{\frac{p}{p-1}} \Big(\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2 dx + \rho^{2\alpha}\Big)\,.
\end{aligned}
$$

Choosing $\varepsilon = \frac{1}{6}\,c_2^{-1}$ and dividing this by $\frac{1}{2}\,c_2^{-1}$ then yields for all $\rho \in (0,\rho_0)$

$$
\begin{aligned}
\fint_{B_{\rho/2}^+(x_0)} |V(Du-X)|^2\,dx &\leq 2^n \fint_{B_\rho^+(x_0)} |V(Du-X)|^2\,\eta^2\,dx \\
&\leq c\,K(M_0)^{\frac{p}{p-1}} \Big[\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2 dx \,+\, \rho^{2\alpha}\Big]
\end{aligned}
$$

with a constant c which depends only on $n,p,L,\nu,M_0,K(M_0),M,L_1(M)$ and $L_2(M)$, and the proof of the lemma is complete. □

Remark: Actually, we only have to confine ourselves to small radii if $\|Dg\|_{L^\infty} \neq 0$, whereas we have $\rho_{cacc} = 1 - |x_0|$ if $\|Dg\|_{L^\infty} = 0$. To prove this assertion it remains to consider radii $\rho \geq \rho_0$ where the radius ρ_0 is defined in the proof above. We apply Lemma A.1 and the Lemma with $\xi = 0$ (then it is easy to see that we do not require any smallness assumption on the radius). Hence, we find the following estimate

$$
\begin{aligned}
\fint_{B_{\rho/2}^+(x_0)} |V(Du-X)|^2\,dx \;&\leq\; c(p)\Big(\fint_{B_{\rho/2}^+(x_0)} |V(Du)|^2\,dx + M_0^2\Big) \\
&\leq c\,\Big[\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u}{\rho}\Big)\Big|^2 dx \,+\, \rho^{2\alpha} + M_0^2\Big] \\
&\leq c\,\Big[\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2 dx \,+\, M_0^2 \,+\, \rho^{2\alpha}\Big] \\
&\leq c\big(n,p,L,\nu,M_0,K(M_0),M,L_1(M),L_2(M)\big)\Big[\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u-\xi\,x_n}{\rho}\Big)\Big|^2 dx \,+\, \rho^{2\alpha}\Big]
\end{aligned}
$$

where in the last inequality we have used the fact that for all radii $\rho \geq \rho_0$ under consideration there holds:

$$
M_0^2 \;\leq\; M_0^2\,\rho_0^{-2\alpha}\rho^{2\alpha} \;=\; c(M_0,M,L_1(M))\,\rho^{2\alpha}\,.
$$

In the interior we find an analogous result. For later application the following form of the Caccioppoli inequality will be convenient:

Lemma 3.7 (Caccioppoli inequality in the interior): *Consider $\bar{\mu} \in \mathbb{R}^N$, $\Upsilon \in \mathbb{R}^{nN}$ with $|\bar{\mu}|, |\Upsilon| \leq M_0$ and let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of under the assumptions (H1)-(H4). Furthermore, assume that one of the following conditions holds:*

1. *the inhomogeneity fulfills a controllable growth condition (B1),*
2. *the inhomogeneity fulfills a natural growth condition (B2); additionally we suppose: $u \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|u\|_{L^\infty} \leq M$ and $2L_1(M)M < 1$.*

Then for every $x_0 \in \Omega$, $\rho \in (0,1)$ such that $B_\rho(x_0) \Subset \Omega$ there holds

$$\fint_{B_{\rho/2}(x_0)} |V(Du) - V(\Upsilon)|^2 \, dx \leq \widehat{c}_{cacc} \Big(\fint_{B_\rho(x_0)} \Big| V\Big(\frac{u(x) - \bar{\mu} - \Upsilon(x - x_0)}{\rho} \Big) \Big|^2 dx + \rho^{2\alpha} \Big),$$

The constant $\widehat{c}_{cacc}$ depends in the first case only on n, N, p, L, M_0 and $K(2M_0)$, whereas in the second case it depends additionally on $M, L_1(M)$ and $L_2(M)$.

Proof: We proceed similarly to the proof of the Caccioppoli inequality at the boundary, however, the occurrence of $\bar{\mu}$ necessitates some modifications in the choices of the quantities appearing within the proof. Instead of $u - \xi x_n$ we will consider the map $v(x) := u(x) - \bar{\mu} - \Upsilon(x - x_0)$. Let $\eta \in C_0^\infty(B_{\rho/2}(x_0), [0,1])$ be a cut-off function satisfying $\eta = 1$ in $B_{\rho/2}(x_0)$ and $|\nabla \eta| \leq \frac{4}{\rho}$. We now test the system (3.1) with the function $\varphi = \eta^2 v$ and analogously to (3.20) we obtain

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \big[a(\,\cdot\,, u, Du) - a(\,\cdot\,, u, \Upsilon)\big] \cdot (Du - \Upsilon)\, \eta^2 \, dx \\
&= -2 \fint_{B_\rho(x_0)} \big[a(\,\cdot\,, u, Du) - a(\,\cdot\,, u, \Upsilon)\big] \cdot \big(v \otimes \nabla \eta\big)\, \eta \, dx \\
&\quad - \fint_{B_\rho(x_0)} \big[a(\,\cdot\,, u, \Upsilon) - a(\,\cdot\,, \bar{\mu} + \Upsilon(x - x_0), \Upsilon)\big] \cdot D\varphi \, dx \\
&\quad - \fint_{B_\rho(x_0)} \big[a(\,\cdot\,, \bar{\mu} + \Upsilon(x - x_0), \Upsilon) - a(x_0, \bar{\mu}, \Upsilon)\big] \cdot D\varphi \, dx + \fint_{B_\rho(x_0)} b(\,\cdot\,, u, Du) \cdot \varphi \, dx \\
&=: -\fint_{B_\rho(x_0)} \big(2\, I + II + III - IV\big)\, dx
\end{aligned} \tag{3.25}$$

with the obvious abbreviations of the integrands. This time we decompose $B_\rho(x_0)$ in sets of the form

$$B_{(\leq)(>)} := B_\rho(x_0) \cap \big\{x : |Du(x) - \Upsilon| \leq 1\big\} \cap \big\{x : \big|\tfrac{v(x)}{\rho}\big| > 1\big\},$$

with analogous definitions for other combinations involving $>$ and $\leq$. For the estimates for I - III we now refer to Lemma 3.6: the only difference is the application of the Hölder continuity in (H4) since K depends on the second argument of $a(\cdot, \cdot, \cdot)$ and hence, on $|\bar{\mu} + \Upsilon(x - x_0)| \leq 2\, M_0$. Therefore, in the Caccioppoli inequality in the interior we have $K(2M_0)$ instead of $K(M_0)$. Combining these estimates we find (cf. (3.21) and (3.22)):

$$\begin{aligned}
I &\leq \varepsilon\, |V(Du - \Upsilon)|^2\, \eta^2 + c(p, M_0)\, \big(L^2\, \varepsilon^{-1} + L^p\, \varepsilon^{1-p}\big)\, \big|V\big(\tfrac{v}{\rho}\big)\big|^2 \\
II + III &\leq 2\,\varepsilon\, |V(Du - \Upsilon)|^2\, \eta^2 + c(p, M_0)\, K(2M_0)^{\frac{p}{p-1}}\, \big(L + L^{\frac{p}{p-1}}\, \varepsilon^{\frac{1}{1-p}}\big)\, \big|V\big(\tfrac{v}{\rho}\big)\big|^2 \\
&\quad + \big(L + L^{\frac{p}{p-1}}\, \varepsilon^{\frac{1}{1-p}} + L^2\, \varepsilon^{-1}\big)\, K(2M_0)^{\frac{p}{p-1}}\, c(p, M)\, \rho^{2\alpha}\,.
\end{aligned}$$

As previously we now distinguish the different growth conditions on the inhomogeneity:

Controllable growth condition (B1): for the remaining integrand IV we find (cf. (3.24)):

$$IV \;\leq\; \varepsilon\,|V(Du-\Upsilon)|^2\,\eta^2 + c(p,M_0)\,\big(L^2\,\varepsilon^{-1} + L^p\,\varepsilon^{1-p}\big)\,\big|V\big(\tfrac{v}{\rho}\big)\big|^2 + c(p,M_0)\,L\,\rho^{2\alpha}\,.$$

Utilizing the ellipticity of $D_z a$ in (H3), we may estimate the integral on the left-hand side of (3.25) from below by

$$\fint_{B_\rho(x_0)} \big[a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,\Upsilon)\big]\cdot(Du-\Upsilon)\,\eta^2\,dx \;\geq\; c_1^{-1}(p,M_0)\fint_{B_\rho(x_0)} |V(Du-\Upsilon)|^2\eta^2\,dx\,.$$

These estimates allow us to deduce from (3.25):

$$\begin{aligned}
&\big(c_1^{-1} - 4\,\varepsilon\big)\fint_{B_\rho(x_0)} |V(Du-\Upsilon)|^2\,\eta^2\,dx \\
&\quad\leq\; c(p,L,M_0,\varepsilon)\,K(2M_0)^{\frac{p}{p-1}}\fint_{B_\rho(x_0)} \Big|V\Big(\frac{v}{\rho}\Big)\Big|^2\,dx \;+\; c(p,L,M_0,\varepsilon)\,K(2M_0)^{\frac{p}{p-1}}\,\rho^{2\alpha}\,.
\end{aligned}$$

With the choice $\varepsilon := \frac{1}{8}\,c_1^{-1}$ and taking into account Lemma A.1 (v) and the definition of v, we immediately obtain the assertion of the lemma:

$$\begin{aligned}
\fint_{B_{\rho/2}(x_0)} |V(Du) - V(\Upsilon)|^2\,dx &\leq 2^n \fint_{B_\rho(x_0)} |V(Du) - V(\Upsilon)|^2\,\eta^2\,dx \\
&\leq\; c(n,N,p)\fint_{B_\rho(x_0)} |V(Du-\Upsilon)|^2\,\eta^2\,dx \\
&\leq\; c(n,N,p,L,M_0)\,K(2M_0)^{\frac{p}{p-1}}\Big(\fint_{B_\rho(x_0)} \Big|V\Big(\frac{v}{\rho}\Big)\Big|^2\,dx + \rho^{2\alpha}\Big)\,.
\end{aligned}$$

Natural growth condition (B2): Here, we proceed exactly as in the boundary situation: the ellipticity of $D_z a$ yields for every $\delta \in (0,1]$:

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \big[a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,\Upsilon)\big]\cdot(Du-\Upsilon)\,\eta^2\,dx \\
&\qquad\geq \fint_{B_\rho(x_0)} \big(1 + (1+\delta^{-1})\,|\Upsilon|^2 + (1+\delta)\,|Du-\Upsilon|^2\big)^{\frac{p-2}{2}}|Du-\Upsilon|^2\eta^2\,dx\,,
\end{aligned}$$

and for the inhomogeneity we obtain via the growth condition (B2):

$$\begin{aligned}
&\fint_{B_\rho(x_0)} IV\,dx \;\leq\; \fint_{B_\rho(x_0)} \big(L_1(M)\,|Du|^p + L_2(M)\big)\,|v|\,\eta^2\,dx \\
&\leq\; L_1(M)(1+\delta)(2M + |\Upsilon|\rho) \\
&\qquad\times \fint_{B_\rho(x_0)} \big(1 + (1+\delta^{-1})|\Upsilon|^2 + (1+\delta)|Du-\Upsilon|^2\big)^{\frac{p-2}{2}}\,|Du-\Upsilon|^2\,\eta^2\,dx \\
&\qquad+ \sqrt{2}\,\big[L_1(M)\,\big(1 + (1+\delta^{-1})\,|\Upsilon|^2\big) + L_2(M)\big]\fint_{B_\rho(x_0)} \big(\big|V\big(\tfrac{v}{\rho}\big)\big|^2 + \rho^{2\alpha}\big)\,dx \\
&=: \fint_{B_\rho(x_0)} (IV_a + IV_b)\,dx\,.
\end{aligned}$$

To absorb the first term on the right-hand side of the last inequality we choose

$$\delta = \frac{1-2L_1(M)M}{4L_1(M)M} \qquad \text{and} \qquad \rho_0 := \min\Big\{1\,,\ \frac{M-2L_1(M)M^2}{(1+2L_1(M)M)M_0}\Big\}.$$

This allows us to proceed again as in the boundary situation and we obtain in the first step that for all radii $\rho \in (0,\rho_0)$ there holds

$$\begin{aligned}\fint_{B_\rho(x_0)} \big[a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,\Upsilon)\big]\cdot (Du-\Upsilon)\,\eta^2\,dx - \fint_{B_\rho(x_0)} IV_a\,dx&\\ \geq\ c_2^{-1}\big(p,M,L_1(M)\big)\fint_{B_\rho(x_0)} |V(Du-\Upsilon)|^2\eta^2\,dx\,,&\end{aligned}$$

which yields for $\varepsilon = \frac{1}{6}\,c_2^{-1}(p,M,L_1(M))$ the inequality

$$\begin{aligned}\fint_{B_{\rho/2}(x_0)} |V(Du-\Upsilon)|^2\,dx &\leq\ 2^n \fint_{B_\rho(x_0)} |V(Du-\Upsilon)|^2\,\eta^2\,dx\\ &\leq\ c\,K(2M_0)^{\frac{p}{p-1}}\bigg[\fint_{B_\rho(x_0)}\Big|V\Big(\frac{v}{\rho}\Big)\Big|^2\,dx\ +\ \rho^{2\alpha}\bigg]\end{aligned}$$

where the constant c depends only on $n,p,L,M_0,M,L_1(M)$ and $L_2(M)$. Analogously to the remark after Lemma 3.6, the assertion follows also for radii $\rho \geq \rho_0$ in view of Lemma A.1 and the result for the case $\Upsilon = 0$. □

Remark: As already noted in Remark 3.3 we obtain the Caccioppoli inequalities for both the interior and the boundary situation under the almost natural growth condition (B3) on the inhomogeneity also for bounded weak solutions $u \in W^{1,p}(\Omega,\mathbb{R}^N)\cap L^\infty(\Omega,\mathbb{R}^N)$ and $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)\cap L^\infty(B^+,\mathbb{R}^N)$, respectively, without assuming any condition of the form $2L_1(M)M<1$. In fact, the integrand IV in Lemma 3.6 is then estimated by

$$\begin{aligned}IV\ &\leq\ \big(L_1(M)\,|Du|^{\tilde p} + L_2(M)\big)\,|u-\xi x_n|\,\eta^2\\ &\leq\ 2\,L_1(M)\,|Du-X|^{\tilde p}\,|u-\xi x_n|\,\eta^2 + c\big(p,M_0,L_1(M),L_2(M)\big)\big[\big|V\big(\tfrac{u-\xi x_n}{\rho}\big)\big|^2+\rho^{2\alpha}\big]\,.\end{aligned}$$

The first term on the right-hand side of the last inequality is bounded from above as follows: on the set $B_{(\leq)(\cdot)}$ we have

$$\begin{aligned}2\,L_1(M)\,|Du-X|^{\tilde p}\,|u-\xi x_n|\,\eta^2\ &\leq\ 2\,L_1(M)\,|u-\xi x_n|\\ &\leq\ 4\,L_1(M)\,\big[\big|V\big(\tfrac{u-\xi x_n}{\rho}\big)\big|^2+\rho^{2\alpha}\big]\,.\end{aligned}$$

On the remainder $B_{(>)(\cdot)}$ we find via Young's inequality and the fact that we consider bounded solutions $u \in L^\infty(B^+,\mathbb{R}^N)$:

$$\begin{aligned}2\,L_1(M)\,|Du-X|^{\tilde p}\,|u-\xi x_n|\,\eta^2\ &\leq\ \tfrac{\varepsilon}{\sqrt{2}}\,|Du-X|^p\,\eta^2 + c\big(p,\tilde p,L_1(M),\varepsilon\big)\,|u-\xi x_n|^{\frac{p}{p-\tilde p}}\\ &\leq\ \varepsilon\,|V(Du-\Upsilon)|^2\,\eta^2 + c(p,\tilde p,M_0,M,L_1(M),\|Dg\|_\infty,\varepsilon)\,\big|V\big(\tfrac{u-\xi x_n}{\rho}\big)\big|^2\end{aligned}$$

and the constant c blows up for $\tilde p \nearrow p$. Combining these estimates we infer

$$IV\ \leq\ \varepsilon\,|V(Du-\Upsilon)|^2\,\eta^2 + c(p,\tilde p,M_0,M,L_1(M),L_2(M),\|Dg\|_\infty,\varepsilon)\,\big[\big|V\big(\tfrac{u-\xi x_n}{\rho}\big)\big|^2+\rho^{2\alpha}\big]\,,$$

and the Caccioppoli inequality at the boundary follows in a standard way. In the interior situation we proceed analogously. We mention here that we do not need any further restriction on $\rho_{cacc} = 1-|x_0|$, and the constants c_{cacc} and $\widehat{c}_{cacc}$ now additionally depend on the exponent $\tilde p$.

3.5 Estimate for the excess quantity

3.5.1 Approximate $\mathcal{A}$-harmonicity

For every half-ball $B_\rho^+(y)$ with $y \in \Gamma$ and $B_\rho(y) \Subset B$, a fixed function $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$ we define the excess function by

$$\Phi(y, \rho, \xi) := \Big(\fint_{B_\rho^+(y)} |V(Du - \xi \otimes e_n)|^2 \, dx \Big)^{\frac{1}{2}} . \tag{3.26}$$

In this section we consider a solution $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ of the system (3.16). We will show that the function $u - \xi\, x_n$ is approximately $\mathcal{A}$-harmonic for some constant coefficients $\mathcal{A}$ which are derived from the original coefficients $a(\cdot, \cdot, \cdot)$. The application of Lemma 3.5 will then yield the existence of an $\mathcal{A}$-harmonic function, which is on the one hand comparable via the function W to the function $(u - \xi\, x_n)$ in the L^2-sense, and for which, on the other hand, good a priori estimates are available.

Lemma 3.8: *Let $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ be a weak solution of (3.16), where the conditions (H2) and (H4) are satisfied, and let $M_0 > 0$. Furthermore, assume that one of the following conditions holds:*

1. *the inhomogeneity fulfills a controllable growth condition (B1),*
2. *the inhomogeneity fulfills a natural growth condition (B2)*, and (3.17) is satisfied.*

Then for every $B_\rho^+(y)$ with $y \in \Gamma$ and $B_\rho(y) \Subset B$ and for every $\xi \in \mathbb{R}^N$ with $|\xi| \le M_0$ there holds

$$\Big| \fint_{B_\rho^+(y)} D_z a(y, 0, \xi \otimes e_n) \big(Du - \xi \otimes e_n, D\varphi\big) \, dx \Big| \le c_a \Big[\Phi^2 + \rho^\alpha + \chi_{M_0}\big(\Phi^2\big)\, \Phi \Big] \sup_{B_\rho^+(y)} |D\varphi|$$

for all $\varphi \in C_0^\infty(B_\rho^+(y), \mathbb{R}^N)$, where we have abbreviated $\Phi(y, \rho, \xi \otimes e_n)$ on the right-hand side by Φ. The constant c_a depends in Case 1 only on p, L, M_0 and $K(M_0)$, and in Case 2 additionally on $M, L_1(M)$ and $L_2(M)$. Here χ_{M_0} denotes the modulus of continuity from (3.2).

Proof: In the sequel, we will use the notation $X = \xi \otimes e_n$. Moreover, we will estimate in the various calculations below $|Du - X|^2$ and $|Du - X|^p$, respectively, by $\sqrt{2}|V(Du - X)|^2$ via Lemma A.1 (i).

Using $-\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du)$ and the fact that $a(y, 0, X)$ is constant, we infer the following identity for every test function $\varphi \in C_0^\infty(B_\rho^+(y), \mathbb{R}^N)$ satisfying $\sup_{B_\rho^+(y)} |D\varphi| \le 1$:

$$\begin{aligned} &\fint_{B_\rho^+(y)} \int_0^1 D_z a\big(y, 0, X + t(Du - X)\big) \, dt \, \big(Du - X, D\varphi\big) \, dx \\ &\quad = \fint_{B_\rho^+(y)} \big[a(y, 0, Du) - a(y, 0, X)\big] \cdot D\varphi \, dx \\ &\quad = \fint_{B_\rho^+(y)} \big[a(y, 0, Du) - a(\,\cdot\,, u, Du)\big] \cdot D\varphi \, dx + \fint_{B_\rho^+(y)} b(\,\cdot\,, u, Du) \cdot \varphi \, dx \, . \end{aligned}$$

This allows us to infer the estimate

$$
\begin{aligned}
&\Big| \fint_{B_\rho^+(y)} D_z a(y,0,X)\,\big(Du - X, D\varphi\big)\,dx \Big| \\
&\quad = \Big| \fint_{B_\rho^+(y)} \int_0^1 \big[D_z a(y,0,X) - D_z a(y,0,X + t(Du - X))\big]\,dt\,\big(Du - X, D\varphi\big)\,dx \\
&\qquad + \fint_{B_\rho^+(y)} \big[a(y,0,Du) - a(\,\cdot\,,u,Du)\big]\cdot D\varphi\,dx + \fint_{B_\rho^+(y)} b(\,\cdot\,,u,Du)\cdot\varphi\,dx \Big| \\
&\quad \le \fint_{B_\rho^+(y)} (I + II + III + IV)\,dx
\end{aligned} \tag{3.27}
$$

with

$$
\begin{aligned}
I &= \Big| \int_0^1 \big[D_z a\big(y,0,X\big) - D_z a\big(y,0,X + t(Du - X)\big)\big]\,dt\Big|\,\big|\big(Du - X\big)\big|\,, \\
II &= \big|a\big(y,0,Du\big) - a\big(x, X\,(x-y), Du\big)\big|\,, \\
III &= \big|a\big(x, X\,(x-y), Du\big) - a\big(x,u,Du\big)\big|\,, \\
IV &= \big|b(x,u,Du)\cdot\varphi(x)\big|\,.
\end{aligned}
$$

Estimate for I: On the set $B_\rho^+(y) \cap \big\{|Du - X| > 1\big\}$ we get from the boundedness of $D_z a(\cdot,\cdot,\cdot)$ in (H2):

$$
I \le 2\,L|Du - X| \;\le\; 2\,L|Du - X|^p \;\le\; 2\sqrt{2}\,L|V(Du - X)|^2\,.
$$

On the complement, we use the existence of the modulus of continuity χ_{M_0} for $D_z a(\cdot,\cdot,\cdot)$ to conclude

$$
\begin{aligned}
I &\le \int_0^1 \big|D_z a(y,0,X) - D_z a\big(y,0,X + t(Du - X)\big)\big|\,dt\,|Du - X| \\
&\le L\,\chi_{M_0}\big(\sqrt{2}|V(Du - X)|^2\big)\,2^{\frac14}\,|V(Du - X)|\,.
\end{aligned}
$$

Since $\chi_{M_0}^2$ is concave and monotone nondecreasing, we apply Hölder's and Jensen's inequality (note that we have $\chi_{M_0}^2(ct) \le c\chi_{M_0}^2(t)$ for $c \ge 1$) to arrive at

$$
\begin{aligned}
\frac{1}{|B_\rho^+(y)|} \int_{B_\rho^+(y) \cap \{|Du - X| \le 1\}} I\,dx &\le 2^{\frac14}\,L \fint_{B_\rho^+(y)} \chi_{M_0}\big(\sqrt{2}|V(Du - X)|^2\big)\,|V(Du - X)|\,dx \\
&\le \sqrt{2}\,L\,\chi_{M_0}\Big(\fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx\Big)\Big(\fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx\Big)^{\frac12}\,.
\end{aligned}
$$

Therefore, we achieve for the first integral:

$$
\begin{aligned}
\fint_{B_\rho^+(y)} I\,dx &\le 2\sqrt{2}\,L \fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx \\
&\qquad + \sqrt{2}\,L\,\chi_{M_0}\Big(\fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx\Big)\Big(\fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx\Big)^{\frac12} \\
&= 2\sqrt{2}\,L\,\Phi^2(y,\rho,X) + \sqrt{2}\,L\,\chi_{M_0}\big(\Phi^2(y,\rho,X)\big)\,\Phi(y,\rho,X)\,.
\end{aligned} \tag{3.28}
$$

Estimate for II: By assumption (H4) we have

$$\begin{aligned} II \,&\le\, L\,K(|X|)\,\omega\big(|x-y| + |X|\,|x-y|\big)\,\big(1+|Du|^2\big)^{\frac{p-1}{2}} \\ &\le\, L\,K(M_0)\,\omega\big(\rho\,(1+|X|)\big)\,\big(1+|Du-X|^{p-1}+|X|^{p-1}\big) \\ &\le\, L\,K(M_0)\,(1+M_0)^{\alpha}\,\rho^{\alpha}\,\big(1+|Du-X|^{p-1}+M_0^{p-1}\big) \\ &\le\, L\,K(M_0)\,c(M_0)\,\rho^{\alpha}\,\big(1+|Du-X|^{p-1}\big)\,. \end{aligned}$$

On $B_\rho^+(y)\cap\big\{|Du-X|>1\big\}$ we have (keeping in mind $\rho\le 1$)

$$\begin{aligned} II \,&\le\, L\,K(M_0)\,c(M_0)\,\rho^{\alpha}\,|Du-X|^{p-1} \\ &\le\, L\,K(M_0)\,c(M_0)\,|Du-X|^{p} \\ &\le\, L\,K(M_0)\,c(M_0)\,|V(Du-X)|^{2}\,, \end{aligned}$$

and on $B_\rho^+(y)\cap\big\{|Du-X|\le 1\big\}$ we find

$$II \,\le\, L\,K(M_0)\,c(M_0)\,\rho^{\alpha}\,.$$

Hence, for the second integral we obtain the estimate

$$\begin{aligned} \fint_{B_\rho^+(y)} II\,dx \le\, &L\,K(M_0)\,c(M_0)\,\Big(\fint_{B_\rho^+(y)}|V(Du-X)|^2\,dx+\rho^{\alpha}\Big) \\ =\, &L\,K(M_0)\,c(M_0)\,\big(\Phi^2(y,\rho,X)+\rho^{\alpha}\big)\,. \end{aligned} \tag{3.29}$$

Estimate for III: Taking into account the special form of $X=\xi\otimes e_n$ we see for the function appearing in III (note $y_n=0$ by assumption):

$$X\,(x-y) \,=\, \xi\,(x_n-y_n) \,=\, \xi\,x_n\,.$$

Therefore, similarly to the estimate for II we derive via (H4)

$$\begin{aligned} III \,&\le\, L\,K(M_0)\,\omega\big(|u-X(x-y)|\big)\,\big(1+|Du|^2\big)^{\frac{p-1}{2}} \\ &\le\, L\,K(M_0)\,c(M_0)\,\omega\big(|u-\xi x_n|\big)\,\big(1+|Du-X|^{p-1}\big)\,. \end{aligned} \tag{3.30}$$

In view of $\omega(t)\le 1$, we infer on the set $B_\rho^+(y)\cap\big\{|Du-X|>1\big\}$:

$$\begin{aligned} III \,&\le\, L\,K(M_0)\,c(M_0)\,|Du-X|^{p-1} \\ &\le\, L\,K(M_0)\,c(M_0)\,|V(Du-X)|^{2}\,. \end{aligned}$$

For an estimate of the right-hand side of (3.30) on the complement $B_\rho^+(y)\cap\big\{|Du-X|\le 1\big\}$ we first note that for $\big\{\big|\frac{u-\xi x_n}{\rho}\big|\le 1\big\}$ we have

$$|u-\xi x_n|^{\alpha} \,\le\, \rho^{\alpha}\,,$$

whereas for $\big\{\big|\frac{u-\xi x_n}{\rho}\big|>1\big\}$ we see (using $0<\rho<1$)

$$|u-\xi x_n|^{\alpha} \,\le\, \big|\tfrac{u-\xi x_n}{\rho}\big|^{\alpha} \,\le\, \big|\tfrac{u-\xi x_n}{\rho}\big|^{p} \,\le\, \sqrt{2}\,\big|V\big(\tfrac{u-\xi x_n}{\rho}\big)\big|^{2}\,.$$

Hence, we deduce on $B_\rho^+(y) \cap \{|Du - X| \le 1\}$:

$$III \le L\,K(M_0)\,c(M_0)\left(\rho^\alpha + \left|V\left(\tfrac{u-\xi x_n}{\rho}\right)\right|^2\right).$$

Since $u - \xi x_n$ vanishes on Γ, in particular on $\Gamma_\rho(y)$, we apply the Poincaré inequality from Lemma A.8 to deduce

$$\fint_{B_\rho^+(y)} \left|V\left(\frac{u-\xi x_n}{\rho}\right)\right|^2 dx \le c(p) \fint_{B_\rho^+(y)} |V(D_n u - \xi)|^2\,dx \le c(p) \fint_{B_\rho^+(y)} |V(Du - X)|^2\,dx\,.$$

This provides the following estimate for the third integral:

$$\begin{aligned}
\fint_{B_\rho^+(y)} III\,dx &\le L\,K(M_0)\,c(M_0)\left(\fint_{B_\rho^+(y)} |V(Du-X)|^2\,dx + \fint_{B_\rho^+(y)} \left|V\left(\tfrac{u-\xi x_n}{\rho}\right)\right|^2 dx + \rho^\alpha\right)\\
&\le L\,K(M_0)\,c(p, M_0)\left(\fint_{B_\rho^+(y)} |V(Du-X)|^2\,dx + \rho^\alpha\right)\\
&= L\,K(M_0)\,c(p,M_0)\left(\Phi^2(y,\rho,X) + \rho^\alpha\right). \qquad (3.31)
\end{aligned}$$

Estimate for IV: Due to $\sup_{B_\rho^+(y)} |D\varphi| \le 1$ and $\varphi = 0$ on $\partial B_\rho^+(y)$ there holds $\sup_{B_\rho^+(y)} |\varphi| \le \rho$. Hence, we may estimate the remaining term using the different growth conditions on the inhomogeneity $b(\cdot,\cdot,\cdot)$:

Controllable growth condition (B1): we proceed analogously to II and obtain

$$\begin{aligned}
\left|b(x,u,Du)\cdot\varphi(x)\right| &\le L\left(1 + |Du|^2\right)^{\frac{p-1}{2}} \rho\\
&\le L\left(1 + |X|^{p-1} + |Du - X|^{p-1}\right)\rho^\alpha\\
&\le L\,\rho^\alpha\,|Du - X|^{p-1} + L\,c(M_0)\,\rho^\alpha\\
&\le L\,c(M_0)\left(|V(Du-X)|^2 + \rho^\alpha\right).
\end{aligned}$$

Natural growth condition (B2)*: with the assumption $|u + g| \le M$ from (3.17) we may utilize (B2)* in order to derive via $|Du - X|^p \le 1 + |V(Du-X)|^2$:

$$\begin{aligned}
\left|b(x,u,Du)\cdot\varphi(x)\right| &\le \rho\left(L_1(M)\,|Du|^p + L_2(M)\right)\\
&\le \rho\left(2\,L_1(M)\,|Du - X|^p + 2\,L_1(M)\,|X|^p + L_2(M)\right)\\
&\le 2\,L_1(M)\,|V(Du-X)|^2 + c\big(M_0, M, L_1(M), L_2(M)\big)\,\rho^\alpha\,.
\end{aligned}$$

Therefore, in both cases we find for the last term:

$$\fint_{B_\rho^+(y)} IV\,dx \le c\left(\Phi^2(y,\rho,X) + \rho^\alpha\right), \qquad (3.32)$$

where the constant c depends only on L and M_0 in the case of controllable growth and on $M_0, M, L_1(M)$ and $L_2(M)$ in the case of natural growth, respectively.

Combining the estimates (3.28), (3.29), (3.31) and (3.32) with (3.27) we see that

$$\left|\fint_{B_\rho^+(y)} D_z a(y,0,X)\left(Du - X, D\varphi\right) dx\right| \le c\left[\Phi^2(y,\rho,X) + \rho^\alpha + \chi_{M_0}\left(\Phi^2(y,\rho,X)\right)\Phi(y,\rho,X)\right]$$

whenever $\sup_{B_\rho^+(y)} |D\varphi| \le 1$ with the dependencies of the constants as stated in the lemma. Rescaling by $\sup_{B_\rho^+(y)} |D\varphi|$ for a general test function $\varphi \in C_0^\infty(B_\rho^+(y), \mathbb{R}^N)$ yields the desired result:

$$\Big| \fint_{B_\rho^+(y)} D_z a(y,0,X) \big(Du - X, D\varphi\big)\, dx \Big| \\ \le c \Big[\Phi^2(y,\rho,X) + \rho^\alpha + \chi_{M_0}\big(\Phi^2(y,\rho,X)\big)\, \Phi(y,\rho,X)\Big] \sup_{B_\rho^+(y)} |D\varphi| . \qquad \square$$

3.5.2 Excess-decay estimate at the boundary

The right-hand side of the inequality in Lemma 3.8 must be small in order to apply the $\mathcal{A}$-harmonic approximation, Lemma 3.5, to the function $w = u - \xi x_n$. Combined with the a priori estimates for $\mathcal{A}$-harmonic functions (i. e., solutions of a linear elliptic system with constant coefficients $\mathcal{A}$) this provides an estimate for the excess function on smaller half-balls. We proceed in a manner close to [Gro02b, Section 3.3-3.4] and the case of homogeneous systems [Bec05, Chapter 6]. Hence, we will only sketch the proceeding and mention the modifications necessary or the new dependencies occurring in the choices of the constants.

For a solution $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ of the system (3.16) we fix $y \in \Gamma$, $\rho \in (0, \rho_{cacc})$ (with ρ_{cacc} determined in Lemma 3.6), $M_1 \ge 1$, $\xi \in \mathbb{R}^N$ with $|\xi| \le M_1$, and we set

$$\Phi(r,\xi) := \Phi(y,r,\xi) = \Big(\fint_{B_r^+(y)} |V(Du - \xi \otimes e_n)|^2\, dx \Big)^{\frac{1}{2}} ,$$
$$w := u - \xi\, x_n \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N) .$$

The bilinear form $\mathcal{A} := \nu^{p-2} D_z a(y,0,\xi \otimes e_n)$ is elliptic and bounded from above with

$$(1 + M_1^2)^{\frac{p-2}{2}} |B|^2 \le \mathcal{A}(B,B) \le L\, \nu^{p-2} |B|^2 \qquad \forall\, B \in \mathbb{R}^{nN} ,$$

see conditions (H2) and (H3)*. Applying Lemma 3.8 we obtain for all $\varphi \in C_0^\infty(B_\rho^+(y), \mathbb{R}^N)$:

$$\Big| \fint_{B_\rho^+(y)} \mathcal{A}(Dw, D\varphi)\, dx \Big| = \Big| \fint_{B_\rho^+(y)} \nu^{p-2} D_z a(y,0,\xi\otimes e_n) \big(Du - \xi \otimes e_n, D\varphi\big)\, dx \Big| \\ \le c_a \big[\Phi^2(\rho,\xi) + \rho^\alpha + \chi_{M_1}\big(\Phi^2(\rho,\xi)\big)\, \Phi(\rho,\xi)\big] \sup_{B_\rho^+(y)} |D\varphi| \\ \le 2\, c_a \sqrt{\Phi^2(\rho,\xi) + \delta^{-2}\rho^{2\alpha}} \sqrt{\Phi^2(\rho,\xi) + \tfrac{1}{2}\delta^2 + \chi_{M_1}^2\big(\Phi^2(\rho,\xi)\big)} \sup_{B_\rho^+(y)} |D\varphi| ,$$

where $\delta \in (0,1]$ is a parameter at our disposal, which will be chosen later. Here, we have used the elementary inequality $a + b + c \le 2\,(a^2 + \frac{1}{2}b^2 + c^2)^{\frac{1}{2}}$, and c_a depends only on p, L, ν, M_1 and $K(M_1)$ under the assumption (B1) of controllable growth, and additionally on $M, L_1(M)$ and $L_2(M)$ under the assumption (B2)* of natural growth.

For $\varepsilon > 0$ to be specified later, let $\delta = \delta(n,N,p,\nu,L,M_1,\varepsilon) \in (0,1]$ denote the constant from Lemma 3.5. Keep in mind that δ has to be chosen according to the ellipticity constant and the upper bound of $\mathcal{A}$. Assume

$$\Phi^2(\rho,\xi) + \chi_{M_1}^2\big(\Phi^2(\rho,\xi)\big) \le \tfrac{1}{2}\,\delta^2 , \tag{3.33}$$
$$\gamma := 2\, c_a \sqrt{\Phi^2(\rho,\xi) + \delta^{-2}\rho^{2\alpha}} \le 1 . \tag{3.34}$$

Then we have

$$\Big| \fint_{B_\rho^+(y)} \mathcal{A}(Dw, D\varphi)\,dx \Big| \le \gamma\,\delta \sup_{B_\rho^+(y)} |D\varphi| \qquad \text{for all } \varphi \in C_0^\infty(B_\rho^+(y),\mathbb{R}^N)\,,$$

and the application of the $\mathcal{A}$-harmonic approximation Lemma 3.5 ensures the existence of an $\mathcal{A}$-harmonic function $h \in W_\Gamma^{1,p}(B_{\rho/2}(y),\mathbb{R}^N)$ satisfying

$$\fint_{B_{\rho/2}^+(y)} \Big| V\Big(\frac{w-\gamma h}{\rho}\Big)\Big|^2 dx \le \gamma^2 \varepsilon\,, \quad \text{and} \quad \fint_{B_{\rho/2}^+(y)} \big|V(Dh)\big|^2 dx \le 2^{n+3}\,.$$

We next deduce some relevant properties of the function h: splitting the integration domain in $\{|Dh| > 1\}$ and $\{|Dh| \le 1\}$, we infer the following inequality (similarly to (6.19) in [Bec05]) using Lemma A.1 (i), Hölder's inequality and the second of the latter estimates:

$$\fint_{B_{\rho/2}^+(y)} |Dh|^p\,dx \le 2^{n+5}\,.$$

Moreover, in view of Lemma 3.4, h is smooth on $\overline{B_\sigma^+(y)}$ for all $\sigma < \frac{\rho}{2}$ and fulfills the a priori estimate

$$\sup_{B_{\rho/4}^+(y)} \big(|Dh| + \rho\,|D^2 h|\big) \le c\Big(\fint_{B_{\rho/2}^+(y)} |D_n h|^p\,dx\Big)^{\frac{1}{p}} \le c_h(n,N,p,\nu,L,M_1)\,.$$

Since h vanishes on $\Gamma_{\rho/2}$, there exists a constant vector $\zeta \in \mathbb{R}^N$ such that we have the representation

$$Dh(y) = \zeta \otimes e_n \quad \text{where} \quad |\zeta| \le c_h\,;$$

Taylor expansion of h in points $x \in B_{2\theta\rho}^+(y)$ with $\theta \in (0,\frac{1}{8}]$, Lemma A.1 and the choice $\varepsilon = \theta^{n+4}$ then yields (cf. [Bec05], p.65):

$$\fint_{B_{2\theta\rho}^+(y)} \Big| V\Big(\frac{w-\gamma\,\zeta x_n}{2\theta\rho}\Big)\Big|^2 dx \le c(p)\,c_h^2\,\theta^2\,\gamma^2\,. \tag{3.35}$$

We highlight that the choice of ε fixes $\delta = \delta(n,N,p,\nu,L,M_1,\varepsilon) \in (0,1]$ in terms of θ. In the next step we want to estimate the left-hand side of (3.35) by means of the Caccioppoli inequality. Since $w - \gamma\zeta x_n = u - (\xi+\gamma\zeta)x_n$ its application is only possible if $|\xi+\gamma\zeta|$ is bounded. Thus we choose $M_2 \ge M_1 + 1$ such that $|\xi+\gamma\zeta| \le M_2$. Observing that the constants c_h, δ and c_a depend monotone nondecreasingly on M_1 we note that it is sufficient to state the dependency on M_2. From the Caccioppoli inequality in Lemma 3.6 with radii $(\theta\rho, 2\theta\rho)$ instead of $(\frac{\rho}{2},\rho)$, we infer with (3.35) an estimate for the excess function on smaller half-balls $B_{\theta\rho}^+(y)$:

$$\begin{aligned}
\Phi^2(\theta\rho,\xi+\gamma\zeta) &= \fint_{B_{\theta\rho}^+(y)} \big|V\big(Du - (\xi+\gamma\zeta)\otimes e_n\big)\big|^2 dx \\
&\le c_{cacc}\Big(\fint_{B_{2\theta\rho}^+(y)} \Big|V\Big(\frac{w-\gamma\,\zeta x_n}{2\theta\rho}\Big)\Big|^2 dx + (2\theta\rho)^{2\alpha}\Big) \\
&\le c_{cacc}\big(c(p)\,c_h^2\,\theta^2\,\gamma^2 + \rho^{2\alpha}\big) \le c_{dec}^2\big(\theta^2\,\Phi^2(\rho,\xi) + \delta^{-2}\rho^{2\alpha}\big)\,,
\end{aligned}$$

where we have used the definition $\gamma = 2\,c_a\sqrt{\Phi^2 + \delta^{-2}\rho^{2\alpha}}$ in the last line and where the constant $c_{dec} = 2\sqrt{c_{cacc}\,c(p)}\,c_h c_a$ depends on n, N, p, L, ν, M_2 and $K(M_2)$ for a controllable growth condition, and additionally on $M, L_1(M)$ and $L_2(M)$ if we assume a natural growth condition. To an arbitrary exponent $\sigma \in (\alpha, 1)$ we fix $\theta \in (0, \frac{1}{8}]$ in dependency of σ and c_{dec} (meaning that we have $\theta = \theta(n, N, p, \nu, L, M_2, K(M_2), \sigma)$ and $\theta = \theta(n, N, p, \nu, L, M_2, K(M_2), M, L_1(M), L_2(M), \sigma)$, respectively) sufficiently small such that

$$c_{dec}^2 \theta^2 \leq \theta^{2\sigma}$$

is satisfied. Note that this fixes δ in dependency of exactly the same quantities as given for the parameter θ. Then we have

$$\Phi^2(\theta\rho, \xi + \gamma\zeta) \leq \theta^{2\sigma}\Phi^2(\rho, \xi) + c_{dec}^2 \delta^{-2} \rho^{2\alpha} .$$

With the definition $\widetilde{\Phi}^2(\rho, \xi) := \Phi^2(\rho, \xi) + \rho^{2\alpha}$ of the *modified excess function* we come to the conclusion that

$$\begin{aligned} \widetilde{\Phi}^2(\theta\rho, \xi + \gamma\zeta) &\leq \theta^{2\sigma}\Phi^2(\rho, \xi) + c_{dec}^2 \delta^{-2} \rho^{2\alpha} + (\theta\rho)^{2\alpha} \\ &\leq \theta^{2\sigma}\widetilde{\Phi}^2(\rho, \xi) + \widetilde{c}_{dec}^2 \delta^{-2} \rho^{2\alpha} , \end{aligned} \tag{3.36}$$

where $\widetilde{c}_{dec}^2 = 1 + c_{dec}^2$. If we now assume the smallness condition

$$\widetilde{\Phi}^2(\rho, \xi) + \chi_{M_2}^2(\widetilde{\Phi}^2(\rho, \xi)) \leq \frac{\delta^2}{4\, c_a^2\, c_h^2}$$

we easily compute that the previous assumptions (3.33), (3.34) and $|\xi + \gamma\zeta| \leq M_2$ are satisfied because $\chi_M(t)$ is monotone in M and t and the definition of γ shows:

- $\Phi^2(\rho, \xi) + \chi_{M_1}^2(\Phi^2(\rho, \xi)) \leq \widetilde{\Phi}^2(\rho, \xi) + \chi_{M_2}^2(\widetilde{\Phi}^2(\rho, \xi)) \leq \dfrac{\delta^2}{4\, c_a^2\, c_h^2} \leq \frac{1}{2}\, \delta^2$
- $\gamma^2 = 4\, c_a^2 \left(\Phi^2(\rho, \xi) + \delta^{-2}\rho^{2\alpha}\right) \leq 4\, \delta^{-2}\, c_a^2\, \widetilde{\Phi}^2(\rho, \xi) \leq c_h^{-2} \leq 1$
- $|\xi + \gamma\zeta| \leq M_1 + \gamma\, c_h \leq M_1 + 1 \leq M_2$.

In particular, we may choose $M_2 = 2M_1$. Taking into consideration the new dependencies of the quantities appearing above we may proceed as in the homogeneous situation and iterate the estimate (3.36); for this purpose we choose $t_0 > 0$ for fixed $M_2 > 0$ such that

$$t_0^2 + \chi_{M_2}^2(t_0^2) \leq \frac{\delta^2}{4\, c_a^2\, c_h^2} \quad \text{and} \quad t_0 \leq \frac{M_2(1 - \theta^\alpha)}{8\, c_a\, c_h} . \tag{3.37}$$

Furthermore, we choose a radius $\rho_0 \in (0, \rho_{cacc})$ satisfying

$$\frac{2\, \widetilde{c}_{dec}^2\, \delta^{-2}}{\theta^{2\alpha} - \theta^{2\sigma}}\, \rho_0^{2\alpha} \leq t_0^2 . \tag{3.38}$$

Hence, t_0 and ρ_0 depend only on $n, N, p, L, \nu, M_2, K(M_2), \alpha, \sigma$ and $\chi_{M_2}(\cdot)$ if we assume a controllable growth condition (B1), and additionally on $M, L_1(M)$, $L_2(M)$ and $\|Dg\|_\infty$ if we assume a natural growth condition (B2)*. Finally we conclude as in [Bec05, Lemma 6.3] the following excess improvement:

Lemma 3.9: *Let $M_2 \geq 2$. Choose t_0 and ρ_0 such that the smallness assumptions (3.37) are valid. Assume that for some $\rho \in (0, \rho_0]$ we have*

$$|\xi_0| \leq \tfrac{1}{2} M_2 , \qquad \text{and} \qquad \widetilde{\Phi}^2(\rho, \xi_0) \leq \tfrac{1}{2} t_0^2 . \tag{3.39}$$

Then there exists $\xi_\infty \in \mathbb{R}^N$ such that for every $r \in (0, \rho]$ there holds:

$$\fint_{B_r^+(y)} \big|V(Du - \xi_\infty \otimes e_n)\big|^2 \, dx \leq c_{it} \Big(\Big(\frac{r}{\rho}\Big)^{2\sigma} \Phi^2(\rho, \xi_0) + r^{2\alpha} \Big) \tag{3.40}$$

for a constant c_{it}, which depends only on $n, N, p, L, \nu, M_2, K(M_2), \alpha, \sigma$ and $\chi_{M_2}(\cdot)$ for a controllable growth condition (B1), and additionally on $M, L_1(M), L_2(M)$ and $\|Dg\|_\infty$ for a natural growth condition (B2).*

3.5.3 Excess-decay estimate in the interior

In the interior of Ω we define the excess function for a ball $B_\rho(x_0) \Subset \Omega$, a fixed function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and $C \in \mathbb{R}^{nN}$ by

$$\Psi(x_0, \rho, C) := \Big(\fint_{B_\rho(x_0)} \big|V(Du - C)\big|^2 \, dx \Big)^{\frac{1}{2}} . \tag{3.41}$$

To establish an excess-decay estimate in the interior as above at the boundary we now show that for a weak solution u of (3.1) there holds: the function $u - \Upsilon(x - x_0)$ is approximately $\mathcal{A}$-harmonic where $\mathcal{A}$ are again constant coefficients derived from the coefficients of the original system.

Lemma 3.10: *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of (3.1), where the conditions (H2) and (H4) are satisfied, and let $M_0 > 0$. Furthermore, assume that one of the following conditions holds:*

1. *the inhomogeneity fulfills a controllable growth condition (B1),*
2. *the inhomogeneity fulfills a natural growth condition (B2); additionally we suppose: $u \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq M$.*

Then for every ball $B_\rho(x_0) \Subset \Omega$ with $\rho \leq 1$ and for every $\Upsilon \in \mathbb{R}^{nN}$ with $|\Upsilon| \leq M_0$ there holds, provided that $|(u)_{x_0,\rho}| \leq M_0$, the following estimate:

$$\Big| \fint_{B_\rho(x_0)} D_z a\big(y, (u)_{x_0,\rho}, \Upsilon\big) \big(Du - \Upsilon, D\varphi\big) \, dx \Big| \leq \widehat{c}_a \Big[\Psi^2 + \rho^\alpha + \chi_{2M_0}\big(\Psi^2\big) \, \Psi \Big] \sup_{B_\rho(x_0)} |D\varphi|$$

for all $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$, where we have abbreviated $\Psi(x_0, \rho, \Upsilon)$ on the right-hand side by Ψ. The constant $\widehat{c}_a$ depends in Case 1 only on n, N, p, L, M_0 and $K(2M_0)$, and in Case 2 additionally on $M, L_1(M)$ and $L_2(M)$.

Proof: We proceed analogously to the proof of Lemma 3.8: for every test function $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ satisfying $\sup_{B_\rho(x_0)} |D\varphi| \leq 1$ we verify

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \int_0^1 D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon + t(Du - \Upsilon)\big) \, dt \, \big(Du - \Upsilon, D\varphi\big) \, dx \\
&\quad = \fint_{B_\rho(x_0)} \big[a\big(x_0, (u)_{x_0,\rho}, Du\big) - a\big(x_0, (u)_{x_0,\rho}, \Upsilon\big) \big] \cdot D\varphi \, dx \\
&\quad = \fint_{B_\rho(x_0)} \big[a\big(x_0, (u)_{x_0,\rho}, Du\big) - a\big(\cdot, u, Du\big) \big] \cdot D\varphi \, dx + \fint_{B_\rho(x_0)} b\big(\cdot, u, Du\big) \cdot \varphi \, dx
\end{aligned}$$

We note that in the last line we have employed the fact that $a(x_0, (u)_{x_0,\rho}, \Upsilon)$ is constant and $-\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du)$. Therefore, we find

$$\begin{aligned}
&\Big| \fint_{B_\rho(x_0)} D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon\big) \big(Du - \Upsilon, D\varphi\big)\, dx \Big| \\
&= \Big| \fint_{B_\rho(x_0)} \int_0^1 \big[D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon\big) - D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon + t(Du - \Upsilon)\big)\big]\, dt\, \big(Du - \Upsilon, D\varphi\big)\, dx \\
&\quad + \fint_{B_\rho(x_0)} \big[a\big(x_0, (u)_{x_0,\rho}, Du\big) - a\big(\cdot, u, Du\big)\big] \cdot D\varphi\, dx + \fint_{B_\rho(x_0)} b\big(\cdot, u, Du\big) \cdot \varphi\, dx \Big| \\
&\le \fint_{B_\rho(x_0)} (I + II + III + IV)\, dx
\end{aligned} \tag{3.42}$$

with the following abbreviations:

$$\begin{aligned}
I &= \Big| \int_0^1 \big[D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon\big) - D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon + t(Du - \Upsilon)\big)\big]\, dt \Big| \, \big|Du - \Upsilon\big|\,, \\
II &= \big|a\big(x_0, (u)_{x_0,\rho}, Du\big) - a\big(x, (u)_{x_0,\rho} + \Upsilon\,(x - x_0), Du\big)\big|\,, \\
III &= \big|a\big(x, (u)_{x_0,\rho} + \Upsilon\,(x - x_0), Du\big) - a\big(x, u, Du\big)\big|\,, \\
IV &= \big|b(x, u, Du) \cdot \varphi(x)\big|\,.
\end{aligned}$$

The first two terms and the last term are estimated exactly as in the boundary situation, where X is replaced by Υ; for the third term, we have to take into consideration that $|(u)_{x_0,\rho} + \Upsilon\,(x - x_0)| \le 2M_0$ is the new argument of the function K (instead of $|X(x - y)| \le M_0$). This yields:

$$\begin{aligned}
\fint_{B_\rho(x_0)} I\, dx &\le 2\sqrt{2}\, L\, \Psi^2(x_0, \rho, \Upsilon) + \sqrt{2}\, L\, \chi_{2M_0}\big(\Psi^2(x_0, \rho, \Upsilon)\big)\, \Psi(x_0, \rho, \Upsilon) \\
\fint_{B_\rho(x_0)} II\, dx &\le L\, K(M_0)\, c(M_0) \Big(\Psi^2(x_0, \rho, \Upsilon) + \rho^\alpha\Big) \\
\fint_{B_\rho(x_0)} III\, dx &\le L\, K(2M_0)\, c(M_0) \Big(\Psi^2(x_0, \rho, \Upsilon) + \rho^\alpha \\
&\qquad + \fint_{B_\rho(x_0)} \Big|V\Big(\frac{u - (u)_{x_0,\rho} - \Upsilon(x - x_0)}{\rho}\Big)\Big|^2 dx\Big) \\
\fint_{B_\rho(x_0)} IV\, dx &\le c\,\big(\Psi^2(y, \rho, X) + \rho^\alpha\big)\,,
\end{aligned}$$

where the last constant c depends, according to the growth condition imposed on the inhomogeneity $b(\cdot,\cdot,\cdot)$, on L and M_0 if we assume (B1), and on $M_0, M, L_1(M)$ and $L_2(M)$ if we assume (B2). The mean of $u - (u)_{x_0,\rho} - \Upsilon(x - x_0)$ vanishes on the ball $B_\rho(x_0)$. Hence, the application of Poincaré's inequality from Lemma A.6 gives:

$$\fint_{B_\rho(x_0)} \Big|V\Big(\frac{u - (u)_{x_0,\rho} - \Upsilon(x - x_0)}{\rho}\Big)\Big|^2 dx \;\le\; c_s(n, N, p) \fint_{B_\rho(x_0)} \big|V(Du - \Upsilon)\big|^2\, dx\,.$$

Combining this inequality with the estimates for the various terms above and the decomposition in (3.42), we obtain

$$\begin{aligned}
&\Big| \fint_{B_\rho(x_0)} D_z a\big(x_0, (u)_{x_0,\rho}, \Upsilon\big) \big(Du - \Upsilon, D\varphi\big)\, dx \Big| \\
&\qquad \le c \Big[\Psi^2(x_0, \rho, \Upsilon) + \rho^\alpha + \chi_{2M_0}\big(\Psi^2(x_0, \rho, \Upsilon)\big)\, \Psi(x_0, \rho, \Upsilon)\Big]\,,
\end{aligned}$$

and the constant c has the dependencies stated in the lemma. Rescaling then yields the desired result for general test functions $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$. □

The right-hand side of the bound in Lemma 3.10 must again be small in order to ensure that the second assumption (concerning the approximate $\mathcal{A}$-harmonicity) of Lemma 3.5 is satisfied: we fix $x_0 \in \Omega$, $\rho \in (0,1]$, $\widehat{M}_1 \in \mathbb{R}^+$ and $\Upsilon \in \mathbb{R}^{nN}$ such that $|(u)_{x_0,\rho}|$, $|\Upsilon| \leq \widehat{M}_1$ is fulfilled. If we choose e. g. $\widehat{M}_1 \geq M$ while assuming the growth condition (B2) then $|(u)_{\tilde{x},\tilde{\rho}}| \leq \widehat{M}_1$ trivially holds true for every ball $B_{\tilde{\rho}}(\tilde{x}) \Subset \Omega$. We set

$$\begin{aligned}\Psi(r,\Upsilon) := \Psi(x_0,r,\Upsilon) &= \Big(\fint_{B_r(x_0)} \big|V(Du-\Upsilon)\big|^2 \,dx\Big)^{\frac{1}{2}}, \\ \widehat{w} &:= u - (u)_{x_0,\rho} - \Upsilon\,(x-x_0)\,,\end{aligned} \tag{3.43}$$

where u denotes the weak solution of (3.1) and where we will always assume that the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys one of the growth conditions (B1) and (B2). Hence, we have $\widehat{w} \in W^{1,p}(\Omega,\mathbb{R}^N)$ with vanishing mean value on the ball $B_\rho(x_0)$. As above, in view of Lemma 3.10 on approximate $\mathcal{A}$-harmonicity, we obtain for the bilinear form $\widehat{\mathcal{A}} := D_z a(x_0,(u)_{x_0,\rho},\Upsilon)$, some free parameter $\widehat{\delta} \in (0,1]$ and all test functions $\varphi \in C_0^\infty(B_\rho(x_0),\mathbb{R}^N)$:

$$\begin{aligned}&\Big| \fint_{B_\rho^+(y)} \widehat{\mathcal{A}}(D\widehat{w}, D\varphi)\,dx \Big| \\ &\leq 2\,\widehat{c}_a \sqrt{\Psi^2(\rho,\Upsilon) + \widehat{\delta}^{-2}\rho^{2\alpha}}\,\sqrt{\Psi^2(\rho,\Upsilon) + \tfrac{1}{2}\widehat{\delta}^2 + \chi^2_{2\widehat{M}_1}(\Psi^2(\rho,\Upsilon))}\,\sup_{B_\rho(x_0)} |D\varphi|\,,\end{aligned} \tag{3.44}$$

and $\widehat{c}_a$ depends only on $n, N, p, L, \widehat{M}_1$ and $K(2\widehat{M}_1)$ under the assumption (B1) of controllable growth, and additionally on $M, L_1(M)$ and $L_2(M)$ under the assumption (B2) of natural growth. Let $\widehat{\varepsilon} > 0$ and let $\widehat{\delta} = \widehat{\delta}(n,N,p,L,\widehat{M}_1,\widehat{\varepsilon}) \in (0,1]$ denote the constant from Lemma 3.5. Assume

$$\Psi^2(\rho,\Upsilon) + \chi^2_{2\widehat{M}_1}(\Psi^2(\rho,\Upsilon)) \leq \tfrac{1}{2}\,\widehat{\delta}^2\,, \tag{3.45}$$

$$\widehat{\gamma} := 2\,\widehat{c}_a \sqrt{\Psi^2(\rho,\Upsilon) + \widehat{\delta}^{-2}\rho^{2\alpha}} \leq 1\,. \tag{3.46}$$

Due to (3.44) and the definition of $\Psi(\rho,\Upsilon)$ (keep in mind $D\widehat{w} = Du - \Upsilon$), these smallness assumptions allow us to verify that the assumptions of Lemma 3.5 are satisfied:

$$\begin{aligned}\Big| \fint_{B_\rho(x_0)} \widehat{\mathcal{A}}(D\widehat{w}, D\varphi)\,dx \Big| &\leq \widehat{\gamma}\,\widehat{\delta}\,\sup_{B_\rho(x_0)} |D\varphi|\,, \\ \fint_{B_\rho(x_0)} \big|V(D\widehat{w})\big|^2\,dx = \Psi^2(\rho,\Upsilon) &\leq \widehat{\gamma}^2\,.\end{aligned} \tag{3.47}$$

Hence, there exists an $\widehat{\mathcal{A}}$-harmonic function $\widehat{h} \in W^{1,p}(B_{\rho/2}(x_0),\mathbb{R}^N)$ satisfying

$$\fint_{B_\rho(x_0)} \Big|V\Big(\frac{\widehat{w} - \widehat{\gamma}\,\widehat{h}}{\rho}\Big)\Big|^2 dx \leq \widehat{\gamma}^2\,\widehat{\varepsilon} \quad \text{and} \quad \fint_{B_\rho(x_0)} \big|V(D\widehat{h})\big|^2\,dx \leq 2\,.$$

In view of the a priori estimate for harmonic functions in Lemma 3.4, we now find an excess-decay estimate on smaller balls: first, we see for every $\widehat{\theta} \in (0,\frac{1}{8}]$ and the choice $\widehat{\varepsilon} = \widehat{\theta}^{\,n+4}$ an inequality analogous to (3.35):

$$\fint_{B_{2\widehat{\theta}\rho}(x_0)} \Big|V\Big(\frac{\widehat{w} - \widehat{\gamma}\,(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0))}{2\widehat{\theta}\rho}\Big)\Big|^2 dx \leq c(p)\,\widehat{c}_h^{\,2}\,\widehat{\theta}^2\,\widehat{\gamma}^2\,. \tag{3.48}$$

The aim is to bound the left-hand side via Caccioppoli's inequality from below. Due to

$$\widehat{w} - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right) \;=\; u - \left((u)_{x_0,\rho} + \widehat{\gamma}\,\widehat{h}(x_0)\right) - \left(\Upsilon + \widehat{\gamma}D\widehat{h}(x_0)\right)(x-x_0)\,,$$

we next choose $\widehat{M}_2 \geq \widehat{M}_1 + 1$ such that $|(u)_{x_0,\rho} + \widehat{\gamma}\,\widehat{h}(x_0)|$ and $|\Upsilon + \widehat{\gamma}D\widehat{h}(x_0)|$ are bounded from above by $\widehat{M}_2$. Then, the application of Lemma 3.7 and the definition

$$\widetilde{\Psi}^2(\rho,\Upsilon) \;:=\; \Psi^2(\rho,\Upsilon) + \rho^{2\alpha}$$

of the *modified excess function* reveals

$$\begin{aligned}
\widetilde{\Psi}^2(\widehat{\theta}\rho, \Upsilon + \widehat{\gamma}D\widehat{h}(x_0)) &= ⨍_{B_{\widehat{\theta}\rho}(x_0)} \left|V\left(Du - \Upsilon - \widehat{\gamma}D\widehat{h}(x_0)\right)\right|^2 dx + (\widehat{\theta}\rho)^{2\alpha} \\
&\leq \widehat{c}_{cacc}\left(⨍_{B_{2\widehat{\theta}\rho}(y)} \left|V\Big(\frac{\widehat{w} - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right)}{2\widehat{\theta}\rho}\Big)\right|^2 dx + 2\rho^{2\alpha}\right) \\
&\leq \widehat{c}_{cacc}\left(c(p)\,\widehat{c}_h^2\,\widehat{\theta}^2\,\widehat{\gamma}^2 + 2\rho^{2\alpha}\right) \;\leq\; \widehat{c}_{dec}^2\left(\widehat{\theta}^2\,\Psi^2(\rho,\Upsilon) + \widehat{\delta}^{-2}\rho^{2\alpha}\right)
\end{aligned}$$

for a constant $\widehat{c}_{dec} = 2\sqrt{\widehat{c}_{cacc}\,c(p)}\,\widehat{c}_h\widehat{c}_a$ depending only on $n, N, p, L, \widehat{M}_2$ and $K(2\widehat{M}_2)$ for controllable growth (B1), and additionally on $M, L_1(M)$ and $L_2(M)$ for a natural growth condition (B2). To an arbitrary exponent $\sigma \in (\alpha, 1)$ we fix $\widehat{\theta} \in (0, \frac{1}{8}]$ sufficiently small such that

$$\widehat{c}_{dec}^2\,\widehat{\theta}^2 \;\leq\; \widehat{\theta}^{2\sigma}$$

is satisfied, which also fixes $\widehat{\delta}$ in dependency of $n, N, p, L, \widehat{M}_2, K(2\widehat{M}_2)$ and σ, and additionally of $M, L_1(M)$ and $L_2(M)$, respectively, for (B2). This gives

$$\widetilde{\Psi}^2(\widehat{\theta}\rho, \Upsilon + \widehat{\gamma}D\widehat{h}(x_0)) \;\leq\; \widehat{\theta}^{2\sigma}\widetilde{\Psi}^2(\rho,\Upsilon) + \widehat{c}_{dec}^2\,\widehat{\delta}^{-2}\,\rho^{2\alpha}\,. \tag{3.49}$$

To find a smallness condition which makes all the calculations above possible, we need an appropriate bound for $|\widehat{\gamma}\,\widehat{h}(x_0)|$. Since the mean value of $D\widehat{h}(x_0)(x-x_0)$ on every ball centred at x_0 as well as the mean value of $\widehat{w}$ on $B_\rho(x_0)$ vanishes (see the definition of $\widehat{w}$ in (3.43)), we apply the Poincaré inequality, denoting the related constant by $c_P(n,N,p)$, and we see

$$\begin{aligned}
\left|\widehat{\gamma}\,\widehat{h}(x_0)\right| \;&=\; \left|\widehat{\gamma} ⨍_{B_{\rho/2}(x_0)} \left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right) dx\right| \\
&\leq ⨍_{B_{\rho/2}(x_0)} \left|\widehat{w}(x) - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right)\right| dx + ⨍_{B_{\rho/2}(x_0)} |\widehat{w}(x)|\,dx \\
&\leq \tfrac{\rho}{2} ⨍_{B_{\rho/2}(x_0)} \left|\frac{\widehat{w}(x) - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right)}{\rho/2}\right| dx + 2^n\,c_P\,\rho ⨍_{B_\rho(x_0)} |D\widehat{w}(x)|\,dx\,.
\end{aligned}$$

Using the inequality $|v| \leq \sqrt{2}\left(|V(v)| + |V(v)|^2\right)$, $\widehat{\theta} = \frac{1}{8}$ and (3.47), we conclude from (3.48):

$$\begin{aligned}
|\widehat{\gamma}\,\widehat{h}(x_0)| \leq \rho\Big[& ⨍_{B_{\rho/2}(x_0)} \left|V\Big(\frac{\widehat{w} - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right)}{\rho/2}\Big)\right|^2 dx \\
&+ \Big(⨍_{B_{\rho/2}(x_0)} \left|V\Big(\frac{\widehat{w} - \widehat{\gamma}\left(\widehat{h}(x_0) + D\widehat{h}(x_0)(x-x_0)\right)}{\rho/2}\Big)\right|^2 dx\Big)^{\frac{1}{2}}\Big] \\
&+ 2^{n+\frac{1}{2}}\,c_P(n,p)\,\rho\Big[⨍_{B_\rho(x_0)} |V(D\widehat{w})|^2\,dx + \Big(⨍_{B_\rho(x_0)} |V(D\widehat{w})|^2\,dx\Big)^{\frac{1}{2}}\Big]
\end{aligned}$$

$$\begin{aligned}&\le c(p)\,\rho\left(\widehat{c}_h^2\widehat{\gamma}^{\,2}+\widehat{c}_h\widehat{\gamma}\right)+2^{n+\frac{1}{2}}\,c_P(n,N,p)\,\rho\left(\widehat{\gamma}^{\,2}+\widehat{\gamma}\right)\\&\le \tfrac{1}{4}\,\widehat{c}_i(n,N,p)\,\rho\left(\widehat{c}_h^2\widehat{\gamma}^{\,2}+\widehat{c}_h\widehat{\gamma}\right).\end{aligned}$$

In particular, we haven chosen $\widehat{c}_i \geq 2^{n+\frac{5}{2}} c_P \geq 1$. The smallness condition

$$\widetilde{\Psi}^2(\rho,\Upsilon)+\chi^2_{2\widehat{M}_2}\big(\widetilde{\Psi}^2(\rho,\Upsilon)\big)\;\le\;\frac{\widehat{\delta}^{\,2}}{4\,\widehat{c}_i^{\,2}\,\widehat{c}_a^{\,2}\,\widehat{c}_h^{\,2}}$$

ensures (cf. (3.45) in the situation at the boundary) that (3.46) and $|(u)_{x_0,\rho}+\widehat{\gamma}\,\widehat{h}(x_0)|$, $|\Upsilon+\widehat{\gamma}\,D\widehat{h}(x_0)|\le \widehat{M}_2$ hold true (keep in mind $\rho\le 1$):

- $\Psi^2(\rho,\Upsilon)+\chi^2_{2\widehat{M}_1}(\Psi^2(\rho,\Upsilon))\;\le \widetilde{\Psi}^2(\rho,\Upsilon)+\chi^2_{2\widehat{M}_2}(\widetilde{\Psi}^2(\rho,\Upsilon))\;\le\;\frac{1}{2}\widehat{\delta}^{\,2}$
- $\widehat{\gamma}^2\;=\;4\,\widehat{c}_a^{\,2}\left(\Psi^2(\rho,\Upsilon)+\widehat{\delta}^{\,-2}\rho^{2\alpha}\right)\;\le\;\widehat{\delta}^{\,-2}\,4\,\widehat{c}_a^{\,2}\,\widetilde{\Psi}^2(\rho,\Upsilon)\;\le\;\widehat{c}_h^{\,-2}\,\widehat{c}_i^{\,-2}\;\le\;1$
- $|(u)_{x_0,\rho}+\widehat{\gamma}\,\widehat{h}(x_0)|\;\le\;|(u)_{x_0,\rho}|+\frac{1}{4}\,\widehat{c}_i\left(\widehat{c}_h^{\,2}\,\widehat{\gamma}^{\,2}+\widehat{c}_h\widehat{\gamma}\right)\;\le\;\widehat{M}_1+\frac{1}{2}\;\le\;\widehat{M}_2$
- $|\Upsilon+\widehat{\gamma}\,D\widehat{h}(x_0)|\;\le\;|\Upsilon|+|\widehat{\gamma}D\widehat{h}(x_0)|\;\le\;\widehat{M}_1+\widehat{\gamma}\,\widehat{c}_h\;\le\;\widehat{M}_1+1\;\le\;\widehat{M}_2\;.$

In particular, we may take $\widehat{M}_2 = 2\widehat{M}_1$. In order to iterate the estimate (3.49) we next choose $t_1 > 0$ such that for fixed $\widehat{M}_2 \geq 2$ we have

$$t_1^2+\chi^2_{2\widehat{M}_2}(t_1^2)\;\le\;\frac{\widehat{\delta}^{\,2}}{4\,\widehat{c}_i^{\,2}\,\widehat{c}_a^{\,2}\,\widehat{c}_h^{\,2}}\qquad\text{and}\qquad t_1\;\le\;\frac{\widehat{M}_2(1-\widehat{\theta}^{\,\alpha})(2\widehat{\theta})^n}{8\,\widehat{c}_i\,\widehat{c}_a\,\widehat{c}_h}\,. \tag{3.50}$$

Furthermore, we choose a radius $\rho_1 \in (0,1)$ satisfying

$$\frac{2\,\widehat{c}_{dec}^{\,2}\,\widehat{\delta}^{\,-2}}{\widehat{\theta}^{\,2\alpha}-\widehat{\theta}^{\,2\sigma}}\,\rho_1^{2\alpha}\;\le\;t_1^2\,. \tag{3.51}$$

Hence, t_1 and ρ_1 depend only on $n, N, p, L, \widehat{M}_2, K(2\widehat{M}_2), \alpha, \sigma$ and $\chi_{2\widehat{M}_2}(\cdot)$ if we assume a controllable growth condition (B1), and additionally on $M, L_1(M)$ and $L_2(M)$ if we assume a natural growth condition (B2). Finally we conclude as in [Bec05, Lemma 6.6] the following excess improvement:

Lemma 3.11: *Let $\widehat{M}_2 \geq 2$. Choose t_1 and ρ_1 such that the smallness assumptions (3.50) are valid. Assume that for some $\rho \in (0,\rho_1]$ we have*

$$|\Upsilon_0|\;\le\;\tfrac{1}{2}\,\widehat{M}_2\,,\quad |(u)_{x_0,\rho}|\;\le\;\tfrac{1}{2}\,\widehat{M}_2\quad \textit{and}\quad \widetilde{\Psi}^2(\rho,\Upsilon_0)\;\le\;\tfrac{1}{2}\,t_1^2\,. \tag{3.52}$$

Then there exists $\Upsilon_\infty \in \mathbb{R}^{nN}$ such that for every $r \in (0,\rho]$ there holds

$$\fint_{B_r(x_0)}\big|V(Du)-V(\Upsilon_\infty)\big|^2\,dx\;\le\;\widehat{c}_{it}\left(\left(\frac{r}{\rho}\right)^{2\sigma}\Psi^2(\rho,\Upsilon_0)+r^{2\alpha}\right) \tag{3.53}$$

for a constant c_{it}, which depends only on $n, N, p, L, \widehat{M}_2, K(2\widehat{M}_2), \alpha, \sigma$ and $\chi_{2\widehat{M}_2}(\cdot)$ for a controllable growth condition (B1), and additionally on $M, L_1(M)$ and $L_2(M)$ for a natural growth condition (B2).

3.6 Regularity

We now prove partial Hölder continuity of Du for weak solutions u to system (3.1): in the first step we establish a partial regularity result in the interior of Ω, i. e., we prove Theorem 3.1; in the second step we deal with the characterization of regular boundary points, i. e., with Theorem 3.2 (for homogeneous systems we refer to [Bec05, Chapter 7]). The growth of the excess functions with respect to the radius of the ball, which we obtained in Lemma 3.11 and Lemma 3.9, will be the crucial point; namely it allows us to apply the following theorems going back to a more general result due to Campanato [Cam63, Teorema I.2] and providing an integral characterization of Hölder continuous functions (see also Theorem 2.1).

Theorem 3.12 (cf. [Sim96], Chapt. 1.1, Lemma 1): *Consider $n, N \in \mathbb{N}$, $n \geq 2$ and $x_0 \in \mathbb{R}^n$. Suppose $u \in L^2(B_{2R}(x_0), \mathbb{R}^N)$, $\alpha \in (0,1]$, $\kappa > 0$ and*

$$\inf_{\bar\mu \in \mathbb{R}} \Big\{ \fint_{B_\rho(y)} |u - \bar\mu|^2 \, dx \Big\} \leq \kappa^2 \Big(\frac{\rho}{R} \Big)^{2\alpha}$$

for every $y \in B_R(x_0)$ and $\rho \in (0, R]$. Then there exists a Hölder continuous representative $\bar u$ for the L^2-class of u with

$$|\bar u(x) - \bar u(z)| \leq c\,\kappa \Big(\frac{|x-z|}{R} \Big)^{\alpha}$$

for all $x, z \in B_R(x_0)$ and a constant c depending only on n, N and α.

For the proof of the characterization of regular boundary points in Theorem 3.2 we first consider the set of regular points $\mathrm{Reg}_{Du}(\Gamma)$ defined correspondingly to the definition of $\mathrm{Reg}_{Du}(\partial\Omega)$ in the model situation. Here we make use of a slight modification of Campanato's integral characterization of Hölder continuity:

Theorem 3.13 ([Gro02b], Theorem 2.3): *Consider $n, N \in \mathbb{N}$, $n \geq 2$ and $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. Suppose $v \in L^2(B^+_{6R}(x_0), \mathbb{R}^N)$, $\alpha \in (0,1]$, $\kappa > 0$ and*

$$\inf_{\bar\mu \in \mathbb{R}} \Big\{ \fint_{B^+_\rho(y)} |v - \bar\mu|^2 \, dx \Big\} \leq \kappa^2 \Big(\frac{\rho}{R} \Big)^{2\alpha}$$

for all $y \in \Gamma_{2R}(x_0)$ and $\rho \in (0, 4R]$; and

$$\inf_{\bar\mu \in \mathbb{R}} \Big\{ \fint_{B_\rho(y)} |v - \bar\mu|^2 \, dx \Big\} \leq \kappa^2 \Big(\frac{\rho}{R} \Big)^{2\alpha}$$

for all $y \in B^+_{2R}(x_0)$ with $B_\rho(y) \subset B^+_{2R}(x_0)$. Then there exists a Hölder continuous representative $\bar v$ of v on $\overline{B^+_R}(x_0)$, and for $\bar v$ there holds: $|\bar v(x) - \bar v(z)| \leq c\,\kappa \big(\frac{|x-z|}{R}\big)^{\alpha}$ for all $x, z \in \overline{B^+_R}(x_0)$, for a constant c depending only on n, N and α.

3.6.1 Proof of Theorem 3.1

Proof (of Theorem 3.1): Consider $x_0 \in \Omega \setminus (\Pi_1 \cup \Pi_2)$. Then there exist $\widehat M_2 \geq 2$ and $\rho \in (0, \rho_1]$ such that $B_{2\rho}(x_0) \Subset \Omega$ and

$$\big|(u)_{x_0,\rho}\big| < \tfrac12 \widehat M_2\,, \quad \big|\big(V(Du)\big)_{x_0,\rho}\big| < 2^{\frac{p-6}{4}}\, \widehat M_2^{\frac p2} \quad \text{and} \tag{3.54}$$

$$\fint_{B_\rho(x_0)} \big|V(Du) - \big(V(Du)\big)_{x_0,\rho}\big|^2 \, dx + \rho^{2\alpha} < \tfrac12\, c^{-2}(p, \widehat M_2)\, t_1^2\,, \tag{3.55}$$

where $c(p, \widehat{M}_2)$ is the constant originating from Lemma A.1 (vi). Since the functions

$$z \mapsto (u)_{z,\rho}\,, \qquad z \mapsto \big(V(Du)\big)_{z,\rho} \qquad \text{and} \qquad z \mapsto \fint_{B_\rho(z)} \big|V(Du) - \big(V(Du)\big)_{z,\rho}\big|^2\,dx$$

are continuous there exists a ball $B_{\tilde\rho}(x_0)$ such that for all points $z \in B_{\tilde\rho}(x_0)$ we have: $B_\rho(z) \Subset \Omega$, and the estimates (3.54) and (3.55) holds true with x_0 replaced by z. We next choose $\Upsilon_0(z) \in \mathbb{R}^{nN}$ such that

$$V\big(\Upsilon_0(z)\big) \,=\, \big(V(Du)\big)_{z,\rho}$$

(note: this is always possible because the function V is bijective). Combining these estimates with Lemma A.1 (i) and (vi) we find in view of $\widehat{M}_2 \geq 2$ that $|\Upsilon_0(z)| < \frac{1}{2}\widehat{M}_2$ and $\widetilde{\Psi}^2\big(z, \rho, \Upsilon_0(z)\big) < \frac{1}{2}\,t_1^2$. Thus the above assumptions in (3.52) in Lemma 3.11 are satisfied for all $z \in B_{\tilde\rho}(x_0)$ and we obtain: there exists $\Upsilon_\infty(z) \in \mathbb{R}^{nN}$ such that:

$$\fint_{B_r(z)} \big|V(Du) - V\big(\Upsilon_\infty(z)\big)\big|^2\,dx \,\leq\, c\left(\left(\frac{r}{\rho}\right)^{2\sigma} \Psi^2\big(z, \rho, \Upsilon_0(z)\big) + r^{2\alpha}\right)$$

for all $r \in (0, \rho]$ and all points $z \in B_{\tilde\rho}(x_0)$. The constant c depends only on n, N, p, ν, L, $\widehat{M}_2, K(2\widehat{M}_2), \alpha, \sigma$ and $\chi_{2\widehat{M}_2}(\cdot)$. Applying the integral characterization of Hölder continuous functions due to Campanato, Lemma 3.12, we conclude that there exists a representative of $V \circ Du$ which is Hölder continuous with exponent α $(< \sigma)$. Using Lemma A.4 as well as Lebesgue's Differentiation Theorem we obtain that Du is locally Hölder continuous with the same exponent α in a neighbourhood of x_0, and that $\mathcal{L}^n(\Pi_1) = \mathcal{L}^n(\Pi_2) = 0$ (since both u and $V \circ Du$ belong to the class $L^1(\Omega, \mathbb{R}^N)$). This completes the proof of the theorem. □

3.6.2 Regular boundary points in the model situation

In the sequel we consider the model situation of a half-ball and we characterize the set of regular boundary points on Γ. In the next section this will enable us to transform the model situation back to the general situation, where we deal with general domains and boundary values of class $C^{1,\alpha}$.

Theorem 3.14: *Let $u \in W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$ be a weak solution of*

$$-\operatorname{div} a(\,\cdot\,, u, Du) \,=\, b(\,\cdot\,, u, Du) \qquad \text{in } B^+$$

where the coefficients $a : B^+ \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfy the assumptions (H1), (H2), (H3) and (H4). Furthermore, assume that one of the following conditions hold*

1. *the inhomogeneity fulfills a controllable growth condition (B1),*
2. *the inhomogeneity fulfills a natural growth condition (B2)*, and (3.17) is satisfied.*

Then there holds for every $y \in \mathrm{Reg}_{Du}(\Gamma)$: Du is Hölder continuous with exponent α in a relative neighbourhood in $B^+ \cup \Gamma$, and the set of singular boundary points is contained in $\Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \left\{ y \in \Gamma \,:\, \liminf_{\rho \to 0^+} \fint_{B_\rho(y) \cap B^+} \big|V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y) \cap B^+}\big|^2\,dx \,>\, 0 \right\},$$

$$\Sigma_2 = \left\{ y \in \Gamma \,:\, \limsup_{\rho \to 0^+} \big|\big(V(D_n u)\big)_{B_\rho(y) \cap B^+}\big| \,=\, \infty \right\}.$$

Proof: In the first step of the proof we will find a different formulation for the set $\Sigma_1 \cup \Sigma_2$ which allows us to apply Lemma 3.9, where an assumption on the *total* weak derivative, instead of only the normal derivative of u, is required. To this end, let $y \in \Gamma \setminus (\Sigma_1 \cup \Sigma_2)$ and let $\{\rho_k\}_{k\in\mathbb{N}}$ be a monotone decreasing sequence of radii with $\rho_k \to 0$ for $k \to \infty$, $\rho_k \le \min\{\rho_{cacc}, 1-|y|\}$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \fint_{B^+_{\rho_k}(y)} \big|V(D_n u) - \big(V(D_n u)\big)_{B^+_{\rho_k}(y)}\big|^2\,dx \,=\, 0\,.$$

Since $y \notin \Sigma_2$ there exists $M_0 \ge 1$ such that

$$\big|\big(V(D_n u)\big)_{B^+_{\rho_k}(y)}\big| \,\le\, M_0 \qquad \forall\, k \in \mathbb{N}\,.$$

Similarly to the proof of the interior estimate in Theorem 3.1 we define $\{\xi(y,\rho_k)\} \in \mathbb{R}^N$ via

$$V\big(\xi(y,\rho_k)\big) \,=\, \big(V(D_n u)\big)_{B_{\rho_k}(y)\cap B^+} \,=\, \big(V(D_n u)\big)_{B^+_{\rho_k}(y)} \tag{3.56}$$

(and analogously for general radii σ). Then Lemma A.1 (i) yields $|\xi(y,\rho_k)| \le 2M_0^2$. Applying the Caccioppoli inequality in Lemma 3.6, the Poincaré inequality in Lemma A.8 (note that $u - \xi(y,\rho_k)x_n \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$) and Lemma A.1 (vi) we compute

$$\begin{aligned}
\fint_{B^+_{\rho_k/2}(y)} \big|V\big(Du - \xi(y,\rho_k)\otimes e_n\big)\big|^2\,dx \,&\le\, c_{cacc}\Big(\fint_{B^+_{\rho_k}(y)} \big|V\big(\tfrac{u-\xi(y,\rho_k)x_n}{\rho_k}\big)\big|^2\,dx + \rho_k^{2\alpha}\Big)\\
&\le\, c_{cacc}\Big(c_P(p)\fint_{B^+_{\rho_k}(y)} \big|V\big(D_n u - \xi(y,\rho_k)\big)\big|^2\,dx + \rho_k^{2\alpha}\Big)\\
&\le\, c\Big(\fint_{B^+_{\rho_k}(y)} \big|V(D_n u) - \big(V(D_n u)\big)_{B^+_{\rho_k}(y)}\big|^2\,dx + \rho_k^{2\alpha}\Big)
\end{aligned} \tag{3.57}$$

and the constant c depends only on n, p, L, ν, M_0 and $K(2M_0^2)$ under a controllable growth assumption (B1), and additionally on $M, L_1(M)$ and $L_2(M)$ under a natural growth assumption (B2)*. Due to the choice of the sequence $\{\rho_k\}$, the right-hand side of the last inequality vanishes as $k \to \infty$. Setting

$$\Sigma_1' = \Big\{y \in \Gamma \,:\, \liminf_{\rho\to 0^+} \fint_{B_\rho(y)\cap B^+} \big|V\big(Du - \xi(y,2\rho)\otimes e_n\big)\big|^2\,dx \,>\, 0\Big\}\,, \tag{3.58}$$

$$\Sigma_2' = \Big\{y \in \Gamma \,:\, \limsup_{\rho\to 0^+} |\xi(y,\rho)| \,=\, \infty\Big\}\,, \tag{3.59}$$

the calculation above yields the inclusion $\Gamma \setminus (\Sigma_1 \cup \Sigma_2) \subset \Gamma \setminus (\Sigma_1' \cup \Sigma_2')$.

We thus consider $y \in \Gamma \setminus (\Sigma_1 \cup \Sigma_2)$; without loss of generality, we may assume $y = 0$. In the second step of the proof we will show that $Du \in C^{0,\alpha}(\overline{B^+_\rho},\mathbb{R}^{nN})$ for some $\rho > 0$.

Let M_2 denote the upper bound on $2|\xi(0,\rho)|$ (note that $M_2 < \infty$ is guaranteed since $0 \notin \Sigma_2'$). We take t_0 to be the constant appearing in Lemma 3.9 and ρ_0 to be the corresponding radius, cf. (3.37) and (3.38). In order to apply Theorem 3.13 to end up with the Hölder continuity up to the boundary, we have to combine the excess-decay estimates in the interior and at the boundary. Thus, we define $\widehat{M}_2 = 2M_2$ and choose t_1 according to the smallness assumption (3.50) in Lemma 3.11 in the interior. For every $y \in B^+$ let $\rho_1 \in (0, \min\{1-|y|, y_n\})$ be the corresponding radius from (3.51). We now choose $t_2 > 0$ such that

$$t_2^2 \,\le\, \min\{t_0^2\,, 2^{1-n}\,3^{-2\sigma}\,c_{it}^{-1}\,t_1^2\}\,, \tag{3.60}$$

and

$$2^{n+1}\,3^{2\sigma}\,c_{it}\,t_2 \,\le\, M_2 \tag{3.61}$$

are satisfied; c_{it} denotes the constant in Lemma 3.9. We fix $R_0 > 0$ sufficiently small such that

$$6R_0 \,\le\, \min\{\rho_0, \rho_1\} \quad \text{and} \quad 3^n\,2^{3+4\alpha} R_0^{2\alpha} \,\le\, t_2^2\,. \tag{3.62}$$

Since $0 \notin \Sigma_1'$ we find a radius $R \in (0, R_0]$ such that, abbreviating $\xi_0(0) := \xi(0, 12R)$, we have

$$\Phi^2\big(0, 6R, \xi_0(0)\big) \,=\, \fint_{B_{6R}^+(0)} \big|V\big(Du - \xi_0(0) \otimes e_n\big)\big|^2\,dx \,\le\, 2^{-3}\,3^{-n}\,t_2^2\,, \tag{3.63}$$

and by assumption $|\xi_0(0)| \le \frac{1}{2} M_2$. The conditions in (3.60), (3.61) and (3.62) guarantee in particular that also the smallness assumption $\widetilde{\Phi}^2(0, 6R, \xi_0(0)) \le \frac{1}{2} t_0^2$ of Lemma 3.9 is satisfied on $B_{6R}^+(0)$, because the choices of t_2 and R_0 allow us to calculate

$$\begin{aligned}\widetilde{\Phi}^2(0, 6R, \xi_0(0)) &= \Phi^2(0, 6R, \xi_0(0)) + (6R)^{2\alpha} \\ &\le 2^{-3}\,3^{-n}\,t_2^2 + 2^{-1}\,3^{-n}\,t_2^2 \,\le\, \frac{1}{2}\,t_0^2\,.\end{aligned}$$

Thus we find $\xi_\infty(0) \in \mathbb{R}^N$ with $|\xi_\infty(0)| \le M_2$ such that for every $r \in (0, 6R]$ there holds:

$$\fint_{B_r^+} \big|V\big(Du - \xi_\infty(0) \otimes e_n\big)\big|^2\,dx \,\le\, c_{it}\,\Big[\Big(\frac{r}{6R}\Big)^{2\sigma}\,\Phi^2(0, 6R, \xi_0(0)) + r^{2\alpha}\Big]\,. \tag{3.64}$$

Using the smallness assumption (3.63) we next show that the conditions of Theorem 3.13 are fulfilled on all required balls and half-balls with centre $y \in \Gamma_{2R}$ and $y \in B_{2R}^+$, respectively. We distinguish several cases (cf. [Gro02b], p. 378-379):

Case 1: $y \in \Gamma_{2R}$, $|y| \le \rho \le 4R$:
Using $B_\rho^+(y) \subseteq B_{\rho+|y|}^+(0)$, the last estimate for $r = \rho + |y| \le 6R$ and $2^{2\alpha} \le 2^{2\sigma}$, we immediately see that an estimate corresponding to (3.64) also holds on $B_\rho^+(y)$:

$$\begin{aligned}\fint_{B_\rho^+(y)} \big|V\big(Du - \xi_\infty(0) \otimes e_n\big)\big|^2\,dx &\le \Big(\frac{\rho + |y|}{\rho}\Big)^n \fint_{B_{\rho+|y|}^+} \big|V\big(Du - \xi_\infty(0) \otimes e_n\big)\big|^2\,dx \\ &\le 2^n\,c_{it}\,\Big[\Big(\frac{\rho + |y|}{6R}\Big)^{2\sigma}\,\Phi^2(0, 6R, \xi_0(0)) + (\rho + |y|)^{2\alpha}\Big] \\ &\le 2^{n+2\sigma}\,c_{it}\,\Big[\Big(\frac{\rho}{6R}\Big)^{2\sigma}\,\Phi^2(0, 6R, \xi_0(0)) + \rho^{2\alpha}\Big]\,. \end{aligned} \tag{3.65}$$

Case 2: $y \in \Gamma_{2R}$, $0 < \rho < |y| < 2R$:
Here we calculate that the assumptions of Lemma 3.9 are also satisfied for the point y and radius $2R$: recalling the definition of $\Phi(0, 6R, \xi_0(0))$ we infer from (3.63) that

$$\begin{aligned}\fint_{B_{2R}^+(y)} \big|V\big(Du - \xi_0(0) \otimes e_n\big)\big|^2\,dx &\le \Big(\frac{6R}{2R}\Big)^n \fint_{B_{6R}^+} \big|V\big(Du - \xi_0(0) \otimes e_n\big)\big|^2\,dx \\ &= 3^n\,\Phi^2(0, 6R, \xi_0(0))\,. \end{aligned} \tag{3.66}$$

We have $|\xi_0(0)| \le \frac{1}{2} M_2$ (see above). Furthermore, by the condition (3.63) on $\Phi^2(0, 6R, \xi_0(0))$ and (3.62) on the radius we conclude

$$\widetilde{\Phi}^2(y, 2R, \xi_0(0)) \,\le\, \tfrac{1}{2}\,t_0^2\,.$$

Lemma 3.9 yields the existence of $\xi_\infty(y) \in \mathbb{R}^N$ with $|\xi_\infty(y)| \le M_2$ such that for all $0 < \rho \le 2R$ from (3.66) there follows:

$$\fint_{B_\rho^+(y)} \big|V\big(Du - \xi_\infty(y) \otimes e_n\big)\big|^2 \, dx \le c_{it} \Big[\Big(\frac{\rho}{2R}\Big)^{2\sigma} \Phi^2(0, 2R, \xi_0(0)) + \rho^{2\alpha} \Big]$$
$$\le 3^n \, c_{it} \Big[\Big(\frac{\rho}{6R}\Big)^{2\sigma} \Phi^2(0, 6R, \xi_0(0)) + \rho^{2\alpha} \Big] . \qquad (3.67)$$

In view of Lemma A.1 (v), combining the first two cases, i. e., (3.65) and (3.67), reveals (for $\bar\xi = \xi_\infty(0)$ or $\bar\xi = \xi_\infty(y)$ with $|\bar\xi| \le M_2$):

$$\fint_{B_\rho^+(y)} \big|V(Du) - V(\bar\xi \otimes e_n)\big|^2 \, dx \le c(n, N, p) \fint_{B_\rho^+(y)} \big|V\big(Du - \bar\xi \otimes e_n\big)\big|^2 \, dx$$
$$\le c(n, N, p) \, c_{it} \Big[\Big(\frac{\rho}{6R}\Big)^{2\sigma} \Phi^2(0, 6R, \xi_0(0)) + \rho^{2\alpha} \Big] ; \qquad (3.68)$$

hence, the assumptions of Theorem 3.13 are fulfilled for all points $y \in \Gamma$ and radii $\rho \in (0, 4R]$.

Case 3: $y \in B_{2R}^+, B_\rho(y) \subset B_{2R}^+$:
Recalling that $y'' = (y_1, ..., y_{n-1}, 0)$ denotes the projection of y onto $\mathbb{R}^{n-1} \times \{0\}$, we have the inclusions

$$B_\rho(y) \subset B_{y_n}(y) \subset B_{2y_n}^+(y'') .$$

We shall now show that the assumptions for the iteration and thus for the excess-decay estimate in the interior are satisfied on the ball $B_{y_n}(y)$. If $|y''| \le 2y_n \, (\le 4R)$ we can apply Case 1 with centre y'' and radius $2y_n$ to obtain

$$\fint_{B_{2y_n}^+(y'')} \big|V\big(Du - \hat\xi \otimes e_n\big)\big|^2 \, dx \le 3^{n+2\sigma} \, c_{it} \Big[\Big(\frac{2y_n}{6R}\Big)^{2\sigma} \Phi^2(0, 6R, \xi_0(0)) + (2y_n)^{2\alpha} \Big] . \qquad (3.69)$$

Here we have set $\hat\xi = \xi_\infty(0)$ and have replaced $2^{n+2\sigma}$ by $3^{n+2\sigma}$. Otherwise, if $2y_n < |y''| < 2R$ we have in particular $B_{2y_n}^+(y'') \subset B_{2R}^+(y'')$, and the application of Case 2 ensures that the smallness condition $\widetilde\Phi^2(y'', 2R, \xi_0(0)) \le \frac{1}{2} \, t_0^2$ is satisfied. Hence, Lemma 3.9 yields the existence of $\xi_\infty(y'') \in \mathbb{R}^N$ with $|\xi_\infty(y'')| \le M_2$ such that the above inequality holds setting $\hat\xi = \xi_\infty(y'')$.

Thus, for every $y \in B_{2R}^+$ and $B_\rho(y) \subset B_{2R}^+$ we conclude, with the appropriate choice $\hat\xi = \xi_\infty(0)$ or $\hat\xi = \xi_\infty(y'')$ that (keeping in mind $B_{y_n}(y) \subset B_{2y_n}^+(y')$):

$$\fint_{B_{y_n}(y)} \big|V\big(Du - \hat\xi \otimes e_n\big)\big|^2 \, dx \le 2^{n-1} \, 3^{n+2\sigma} \, c_{it} \Big[\Big(\frac{2y_n}{6R}\Big)^{2\sigma} \Phi^2(0, 6R, \xi_0(0)) + (2y_n)^{2\alpha} \Big] . \qquad (3.70)$$

Apart from the explicit estimates for the excess-functions in (3.69) and (3.70) in dependency of the radius, we use the choice in (3.62) for the radius R_0 and the smallness condition (3.63) for the excess function, and according to the choice of t_2 in (3.60) we obtain with $2y_n \le 4R_0$ that the following estimates hold:

$$\fint_{B_{2y_n}^+(y'')} \big|V\big(Du - \hat\xi \otimes e_n\big)\big|^2 \, dx \le 3^{n+2\sigma} \, c_{it} \Big[\Big(\frac{2y_n}{6R}\Big)^{2\sigma} 2^{-3} \, 3^{-n} \, t_2^2 + 2^{-3} \, 3^{-n} \, t_2^2 \Big]$$
$$\le \tfrac{1}{4} \, 3^{2\sigma} \, c_{it} \, t_2^2 , \qquad (3.71)$$
$$\fint_{B_{y_n}(y)} \big|V\big(Du - \hat\xi \otimes e_n\big)\big|^2 \, dx \le 2^{n-3} \, 3^{2\sigma} \, c_{it} \, t_2^2 \le \tfrac{1}{4} \, t_1^2 .$$

Since $|\hat{\xi} \otimes e_n| \leq M_2 = \frac{1}{2}\widehat{M}_2$, it remains to ensure that the mean value of u on the ball $B_{y_n}(y)$ is bounded by $\frac{1}{2}\widehat{M}_2$ for all assumptions in Lemma 3.11 to hold true. We note here that this is trivially satisfied for the assumption of a natural growth condition (B2)* on the inhomogeneity $b(\cdot,\cdot,\cdot)$ if we choose $\widehat{M}_2$ sufficiently large. Otherwise, if we consider the controllable growth situation, the Poincaré inequality in Lemma A.7, Lemma A.1 and (3.71) allow us to estimate (note $t_2 \leq 1$, $y_n \leq \frac{1}{2}$):

$$\begin{aligned}
|(u)_{y,y_n}| &\leq \fint_{B_{y_n}(y)} |u - \hat{\xi} x_n|\,dx + \Big|\fint_{B_{y_n}(y)} \hat{\xi} x_n\,dx\Big| \\
&\leq 2^n\, y_n \fint_{B^+_{2y_n}(y'')} |Du - \hat{\xi}\otimes e_n|\,dx + |\hat{\xi}|\,y_n \\
&\leq 2^n \Big[\fint_{B^+_{2y_n}(y'')} \big|V(Du - \hat{\xi}\otimes e_n)\big|^2\,dx + \Big(\fint_{B^+_{2y_n}(y'')} \big|V(Du - \hat{\xi}\otimes e_n)\big|^2\,dx\Big)^{\frac{1}{2}}\Big] + \tfrac{1}{2}\,|\hat{\xi}| \\
&\leq 2^n \Big[\tfrac{1}{4}\,3^{2\sigma}\,c_{it}\,t_2^2 + \big(\tfrac{1}{4}\,3^{2\sigma}\,c_{it}\,t_2^2\big)^{\frac{1}{2}}\Big] + \tfrac{1}{2}\,M_2 \\
&\leq 2^n\,3^{2\sigma}\,c_{it}\,t_2 + \tfrac{1}{2}\,M_2\,.
\end{aligned}$$

Condition (3.61) for t_2 now guarantees $|(u)_{y,y_n}| \leq M_2 = \frac{1}{2}\widehat{M}_2$. Therefore, all assumptions of Lemma 3.11 are satisfied and we conclude: there exists $\Upsilon_\infty(y) \in \mathbb{R}^{nN}$ with $|\Upsilon_\infty(y)| \leq \widehat{M}_2$ and for all $0 < r \leq y_n$ we deduce with (3.70) and $\alpha \leq \sigma$:

$$\begin{aligned}
\fint_{B_r(y)} \big|V(Du) - V(\Upsilon_\infty(y))\big|^2\,dx &\leq \widehat{c}_{it}\Big[\Big(\frac{r}{y_n}\Big)^{2\sigma} \fint_{B_{y_n}(y)} \big|V(Du - \hat{\xi}\otimes e_n)\big|^2\,dx + r^{2\alpha}\Big] \\
&\leq \widehat{c}_{it}\Big(\Big(\frac{r}{y_n}\Big)^{2\sigma} 2^{n-1}\,3^{n+2\sigma}\,c_{it}\Big[\Big(\frac{2y_n}{6R}\Big)^{2\sigma}\Phi^2(0,6R,\xi_0(0)) + (2y_n)^{2\alpha}\Big] + r^{2\alpha}\Big) \\
&\leq \widehat{c}_{it}\,c_{it}\,2^{n+2\sigma}\,3^{n+2\sigma}\Big(\Big(\frac{r}{6R}\Big)^{2\sigma}\Phi^2(0,6R,\xi_0(0)) + r^{2\alpha}\Big) \\
&\leq \widehat{c}_{it}\,c_{it}\,6^{n+2}\Big(\Big(\frac{r}{6R}\Big)^{2\sigma}\Phi^2(0,6R,\xi_0(0)) + r^{2\alpha}\Big)\,.
\end{aligned}$$

Combining the last estimate with (3.68) we have shown that the assumptions of Theorem 3.13 are satisfied for $V(Du)$. Thus $V(Du) \in C^{0,\alpha}(\overline{B^+_R}, \mathbb{R}^{nN})$, and due to Lemma A.4 we obtain: $Du \in C^{0,\alpha}(\overline{B^+_R}, \mathbb{R}^{nN})$. This completes the proof of the theorem. □

Remark: For the sets Σ_1' and Σ_2' introduced in the first part of the proof, we have not only the inclusion $\Sigma_1' \cup \Sigma_2' \subset \Sigma_1 \cup \Sigma_2$ but indeed equality. To see this, we first obtain via Lemma A.1 (i) that $\Sigma_2' = \Sigma_2$. Moreover, using the fact that for every function $v \in L^2(\Omega, \mathbb{R}^N)$ the mean value minimizes the map $\mathbb{R}^N \ni \bar{\mu} \mapsto \int_\Omega |v - \bar{\mu}|^2\,dx$, combined with Lemma A.1 (v), we derive

$$\begin{aligned}
&\fint_{B_\rho(y)\cap B^+} \big|V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y)\cap B^+}\big|^2\,dx \\
&\qquad \leq \fint_{B_\rho(y)\cap B^+} \big|V(D_n u) - V(\xi(y,2\rho))\big|^2\,dx \\
&\qquad \leq c(n,N,p) \fint_{B_\rho(y)\cap B^+} \big|V\big(Du - \xi(y,2\rho)\otimes e_n\big)\big|^2\,dx\,.
\end{aligned}$$

Choosing an appropriate subsequence $\{\rho_k\}$, the right-hand side of the last inequality vanishes. Hence, the desired equality follows (though not necessarily $\Sigma_1' = \Sigma_1$ holds true).

3.6.3 Proof of Theorem 3.2

Proof (of Theorem 3.2): In Section 3.2 we have already verified that the system (3.1) may be transformed via the map $\mathcal{T}$ locally at a point (without loss of generality at the origin and with $\nu_{\partial\Omega}(0) = e_n$) to the model situation of a half-ball, meaning that the transformed function $\tilde{v} = u \circ \mathcal{T}^{-1} - g \circ \mathcal{T}^{-1} \in W^{1,p}_{\Gamma}(B_r^+, \mathbb{R}^N)$ is weak solution of

$$-\operatorname{div} \hat{a}(\,\cdot\,, \tilde{v}, D\tilde{v}) \;=\; \hat{b}(\,\cdot\,, \tilde{v}, D\tilde{v}) \qquad \text{in } B_r^+ \,.$$

Here the radius r is chosen sufficiently small such that the smallness conditions in Section 3.2 are satisfied (in particular, the inclusions (3.8) hold true). Furthermore, the coefficients $\hat{a}(\cdot,\cdot,\cdot)$ satisfy structure conditions analogous to (H1), (H2), (H3)* and (H4), and the inhomogeneity $\hat{b}(\cdot,\cdot,\cdot)$ obeys either (B1) or (B2)* (with condition (3.17) being true in the latter case). Thus we are in the situation of the last theorem which characterizes the set of regular boundary points in the model situation of a half-ball, and we have to conclude that Du is Hölder continuous in a relative neighbourhood of 0 in $\overline{\Omega}$ with exponent α under the assumption that $0 \notin \widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2$ instead of $0 \notin \Sigma_1 \cup \Sigma_2$.

Analogously to (3.56) in the proof of Theorem 3.14, we define $\tilde{\xi}(0,\rho) \in \mathbb{R}^N$ by

$$V\big(\tilde{\xi}(0,\rho)\big) \;=\; \big(V(D_n(u-g))\big)_{\Omega\cap B_\rho} \;=\; \fint_{\Omega\cap B_\rho} V\big(D_n(u-g)\big)\,dx\,.$$

In view of $0 \notin \widetilde{\Sigma}_2$ there exists $M \geq 2$ and $r_1 \in (0,1)$ such that for all $\rho \leq r_1$ we have $\big|\big(V(D_n u)\big)_{\Omega\cap B_\rho}\big| \leq M$ and thus (similarly to the proof of Theorem 3.14) $|\tilde{\xi}(0,\rho)|$ is bounded from above by a constant $c(M, \|Dg\|_\infty)$. The special form of $\mathcal{T}$, i. e.,

$$\mathcal{T}(x) \;=\; \big(x', x_n - h(x')\big)$$

(where h is the local representation of the boundary defined in Section 3.2), then implies $D_n\mathcal{T}^{-1}(y) = e_n$; therefore, $D_n\tilde{v}$ may be rewritten as follows:

$$\begin{aligned} D_n\tilde{v}(y) &= D_n\big(u\circ\mathcal{T}^{-1} - g\circ\mathcal{T}^{-1}\big)(y) \\ &= Du\big(\mathcal{T}^{-1}(y)\big)D_n\mathcal{T}^{-1}(y) - Dg\big(\mathcal{T}^{-1}(y)\big)D_n\mathcal{T}^{-1}(y) \\ &= D_n u\big(\mathcal{T}^{-1}(y)\big) - D_n g\big(\mathcal{T}^{-1}(y)\big)\,. \end{aligned}$$

The change of variables formula and Lemma A.1 (v), (iii), (iv), (i) yield

$$\begin{aligned} &\fint_{B_\rho^+} \big|V(D_n\tilde{v}) - V\big(\tilde{\xi}(0,\sqrt{2}\rho)\big)\big|^2\,dy \\ &\quad= \fint_{B_\rho^+} \big|V\big(D_n(u-g)(\mathcal{T}^{-1}(y))\big) - \big(V\big(D_n(u-g)\big)\big)_{\Omega\cap B_{\sqrt{2}\rho}}\big|^2\,dy \\ &\quad= \fint_{\mathcal{T}^{-1}(B_\rho^+)} \big|V\big(D_n(u-g)\big) - \big(V\big(D_n(u-g)\big)\big)_{\Omega\cap B_{\sqrt{2}\rho}}\big|^2\,dx \\ &\quad\leq 2^n \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|V\big(D_n(u-g)\big) - \big(V\big(D_n(u-g)\big)\big)_{\Omega\cap B_{\sqrt{2}\rho}}\big|^2\,dx \\ &\quad\leq c(n,N,p) \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|V\big(D_n(u-g) - V^{-1}\big((V(D_n u))_{\Omega\cap B_{\sqrt{2}\rho}}\big) + (D_n g)_{\Omega\cap B_{\sqrt{2}\rho}}\big)\big|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&\leq c(n,N,p) \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|V\big(D_n u - V^{-1}\big((V(D_n u))_{\Omega\cap B_{\sqrt{2}\rho}}\big)\big)\big|^2\,dx \\
&\qquad + c(n,N,p) \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|V\big(D_n g - (D_n g)_{\Omega\cap B_{\sqrt{2}\rho}}\big)\big|^2\,dx \\
&\leq c(n,N,p,M) \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|V(D_n u) - (V(D_n u))_{\Omega\cap B_{\sqrt{2}\rho}}\big|^2\,dx \\
&\qquad + c(n,N,p) \fint_{\Omega\cap B_{\sqrt{2}\rho}} \big|D_n g - (D_n g)_{\Omega\cap B_{\sqrt{2}\rho}}\big|^2\,dx
\end{aligned}
$$

for radii $\sqrt{2}\rho \leq \min\{r_0, r_1\}$. In view of the assumption $0 \notin \widetilde{\Sigma}_1$ and $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$, the right-hand side of the last inequality vanishes for a subsequence as $\rho \to 0$. The application of the Caccioppoli and Poincaré's inequalities now shows, similarly to the calculation in (3.57), that the following inequality holds true:

$$
\fint_{B^+_{\rho/2}} \big|V\big(D\tilde{v} - \tilde{\xi}(0,\sqrt{2}\rho)\otimes e_n\big)\big|^2\,dy \;\leq\; c\Big(\fint_{B^+_\rho} \big|V(D_n\tilde{v}) - V\big(\tilde{\xi}(0,\sqrt{2}\rho)\big)\big|^2\,dy + \rho^{2\alpha}\Big),
$$

provided that $\rho \leq \min\{r_0, r_1\}$. Here the constant c depends only on $n, p, \nu, L, M, \widetilde{K}(2M^2)$, $\|g\|_{C^{1,\alpha}}$, $\|\mathcal{T}\|_{C^{1,\alpha}}$, $\|\mathcal{T}^{-1}\|_{C^{1,\alpha}}$ and α if we assume a controllable growth condition (B1), and additionally on $M, L_1(M)$ and $L_2(M)$ if we assume a natural growth condition (B2)*. We note that the dependencies on g, $\widetilde{K}$ and $\mathcal{T}$ occur due to the structure conditions for $\hat{a}(\cdot,\cdot,\cdot)$ (cf. Section 3.2). Hence, we end up with

$$
\liminf_{\rho\to 0^+} \fint_{B^+_\rho} \big|V\big(D\tilde{v} - \tilde{\xi}(0,\sqrt{8}\rho)\otimes e_n\big)\big|^2\,dy \;=\; 0 \quad\text{and}\quad \limsup_{\rho\to 0^+} \big|\tilde{\xi}(0,\rho)\big| \;\leq\; c(M)\,.
$$

Taking into account the additional smallness assumption concerning the transformation for the choice of the radii r_0, r_1, we may now proceed exactly as in the proof of Theorem 3.14 (with $\tilde{v}$, $\tilde{\xi}(0,\cdot)$ instead of u, $\xi(0,\cdot)$) and conclude: $D\tilde{v}$ is Hölder continuous with exponent α on the half-ball $\overline{B^+_R}$ for some $0 < R < 1$. Since $\mathcal{T}$ is a transformation of class $C^{1,\alpha}$, this means for two arbitrary points $x_1, x_2 \in \overline{\Omega} \cap B_{R/\sqrt{2}}$

$$
\begin{aligned}
\big|Du(x_1) - Du(x_2)\big| &\leq \big|D(u-g)(x_1) - D(u-g)(x_2)\big| + [Dg]_{C^{0,\alpha}(\Omega\cap B_R,\mathbb{R}^N)}|x_1 - x_2|^\alpha \\
&= \big|D\tilde{v}(\mathcal{T}(x_1))D\mathcal{T}(x_1) - D\tilde{v}(\mathcal{T}(x_2))D\mathcal{T}(x_2)\big| + [Dg]_{C^{0,\alpha}(\Omega\cap B_R,\mathbb{R}^N)}|x_1 - x_2|^\alpha \\
&\leq \big|D\tilde{v}(\mathcal{T}(x_1)) - D\tilde{v}(\mathcal{T}(x_2))\big|\,\big|D\mathcal{T}(x_1)\big| + \big|D\tilde{v}(\mathcal{T}(x_2))\big|\,\big|D\mathcal{T}(x_1) - D\mathcal{T}(x_2)\big| \\
&\qquad + [Dg]_{C^{0,\alpha}(\Omega\cap B_R,\mathbb{R}^N)}|x_1 - x_2|^\alpha \\
&\leq c\big([D\tilde{v}]_{C^{0,\alpha}(B^+_R,\mathbb{R}^N)}, [D\mathcal{T}]_{C^{0,\alpha}(\Omega\cap B_R,\mathbb{R}^N)}\big)\,|x_1 - x_2|^\alpha\,.
\end{aligned}
$$

Hence, since $0 \in \partial\Omega \setminus \big(\widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2\big)$ was chosen arbitrarily, the desired result follows. □

Chapter 4

Comparison estimates

In this section we provide an up-to-the-boundary comparison estimates in the setting of subquadratic growth both for degenerate and non-degenerate elliptic systems of partial differential equations in divergence form; we will utilize these estimates later when deriving Calderón-Zygmund type estimates in Section 5 and for the regularity theory for low dimensions in Section 6. To this end, we first turn our attention to weak solutions $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, $R < 1$ and $p \in (1, 2)$, of the *inhomogeneous* system

$$-\operatorname{div} a_0(x, Dv) = L\, G(x) \qquad \text{in } B^+_R(x_0)\,. \tag{4.1}$$

In the weak formulation this becomes

$$\int_{B^+_R(x_0)} a_0(x, Dv) \cdot D\varphi \, dx = L \int_{B^+_R(x_0)} G \cdot \varphi \, dx \qquad \forall\, \varphi \in C^\infty_0(B^+_R(x_0), \mathbb{R}^N)\,, \tag{4.2}$$

where the coefficients $a_0 \colon B^+_R(x_0) \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are Lipschitz continuous in x, but *independent of* v, and satisfy the following ellipticity and growth conditions: $z \mapsto a(x, z)$ is a vector field of class $C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus \{0\}, \mathbb{R}^{nN})$, and for some fixed $0 < \nu \le L$ and for $\mu \in [0, 1]$, there holds

(C1) Polynomial growth of a_0 :

$$|a_0(x, z)| \le L \left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}},$$

(C2) a_0 is differentiable in z with continuous and bounded derivatives:

$$\left|D_z a_0(x, z)\right| \le L \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}},$$

(C3) a_0 is uniformly elliptic, i. e., we have

$$D_z a_0(x, z)\, \lambda \cdot \lambda \ge \nu \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \qquad \forall\, \lambda \in \mathbb{R}^{nN},$$

(C4) a_0 is Lipschitz-continuous with respect to the first variable. More precisely, there exists a non-negative function $\gamma \in L^\infty(B_R^+(x_0))$ such that

$$\big|D_x a_0(x,z)\big| \;\le\; L\,\gamma(x)\,\big(\mu^2+|z|^2\big)^{\frac{p-1}{2}}$$

for all $(x,z) \in B_R^+(x_0) \times \mathbb{R}^{nN}$. Additionally, we suppose that $(x,z) \mapsto D_z a_0(x,z)$ and $(x,z) \mapsto D_x a_0(x,z)$ are Carathéodory maps. We emphasize that we have to exclude $z=0$ in conditions (C2) and (C3) when dealing with degenerate systems ($\mu = 0$). We further impose the following integrability condition on the inhomogeneity $G(\cdot)$:

(C5) $G \in L^{p^*}(B_R^+(x_0),\mathbb{R}^N)$ with $p^* = \frac{p}{p-1}$.

For the solution v of (4.1), we will prove the existence of second order derivatives using a difference quotients method, and we will derive a Caccioppoli-type estimate for second order derivatives. We mention that in the sequel, all estimates are considered on balls or intersection of balls, but they remain also valid if we replace the ball $B_R(x_0)$ by a cube $Q_R(x_0)$.

In the second part of this chapter, we will deal with *homogeneous* systems without x-dependency, i. e., with $\gamma = 0$ and $G = 0$, meaning that we consider weak solutions $v \in W^{1,p}_\Gamma(B_R^+(x_0),\mathbb{R}^N)$, $x_0 \in \mathbb{R}^{n-1}\times\{0\}$, $R<1$ and $p \in (1,2)$, to

$$\operatorname{div} a_0(Dv) \;=\; 0 \qquad \text{in } B_R^+(x_0)\,, \tag{4.3}$$

where the coefficients $a_0 \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfy the assumptions (C1)-(C3). In this special case we are in a position to improve the Caccioppoli-type inequality derived so far and infer an estimate where a certain integral involving second derivatives is bounded by only the tangential part of $V(Dv)$. This allows us to prove a higher integrability result via Gehring's Lemma, and we finally conclude a decay estimate for the weak derivative Dv.

4.1 A preliminary Caccioppoli-type inequality

In the first step, we will consider balls (centred at points y) which have a sufficiently large intersection with $\Gamma_R(x_0)$. In this situation we prove the existence of the tangential second derivatives of v by deriving a Caccioppoli-type estimate. In the interior, we will as well obtain the existence of second order derivatives using the same arguments without any constraint to the direction:

Lemma 4.1: *Let $v \in W^{1,p}_\Gamma(B_R^+(x_0),\mathbb{R}^N)$ be a weak solution to system (4.1) under the assumptions (C1)-(C5) and let $\mu \in [0,1]$ be arbitrary. Then, the tangential derivative $D'v = (D_1v,\dots,D_{n-1}v)$ belongs to $W^{1,p}(B_\rho^+(x_0),\mathbb{R}^{(n-1)N})$ for all $\rho < R$, and there exists a constant c depending only on n,p and $\frac{L}{\nu}$ such that*

a) (close to the boundary) for all $y \in B_R^+(x_0)\cup\Gamma_R(x_0)$ and $0<r<R-|y-x_0|$ with $y_n \le \frac{3}{4}r$ there holds

$$\int_{B^+_{3r/4}(y)} \big|D'(V_\mu(Dv))\big|^2\,dx \\ \le\; c\left(r^{-2}\,(1+\|\gamma\|_\infty^2)\int_{B_r^+(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + \int_{B_r^+(y)}|G|^{\frac{p}{p-1}}\,dx\right), \tag{4.4}$$

b) (in the interior) for all $y \in B_R^+(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n > \frac{3}{4}r$ there holds

$$\int_{B_{5r/8}(y)} \big|D(V_\mu(Dv))\big|^2\,dx \\ \leq c\Big(r^{-2}\,(1+\|\gamma\|_\infty^2)\int_{B^+_{3r/4}(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + \int_{B^+_{3r/4}(y)} |G|^{\frac{p}{p-1}}\,dx\Big)\,.$$

Proof: For the proof of a) we consider a standard cut-off function $\eta \in C_0^\infty(B_{7r/8}(y),[0,1])$ satisfying

$$\eta \equiv 1 \text{ on } B_{3r/4}(y) \qquad \text{and} \qquad |D\eta|^2 + |D^2\eta| \leq c\,r^{-2}\,. \tag{4.5}$$

Let f be a function defined in an open set $U \subset \mathbb{R}^n$, $V \subset U$. The difference quotient $\triangle_{s,h} f(x)$ of f with respect to x_s is defined as

$$\triangle_{s,h} f(x) \;:=\; \frac{f(x+he_s)-f(x)}{h}$$

for $x \in V$, $h \in \mathbb{R}$, with $0 < |h| < \operatorname{dist}(V,\partial U)$, where e_s, $s = 1,...,n$, denotes the standard basis of $\mathbb{R}^n$. Let $|h| < \frac{r}{8}$. We observe that $\eta^2\triangle_{s,h}v \in W_0^{1,p}(B_{7r/8}(y),\mathbb{R}^N)$ for all tangential directions $s = 1,\dots,n-1$, and we now choose

$$\varphi \;=\; \triangle_{s,-h}\big(\eta^2\triangle_{s,h}v\big) \qquad \in W_0^{1,p}(B_r^+(y),\mathbb{R}^N) \tag{4.6}$$

as a test function in (4.2). This is an admissible choice since we only consider the tangential difference quotients for which the zero boundary values on $\Gamma_{7r/8}(y)$ are preserved. Integration by parts for finite differences yields

$$\int_{B_r^+(y)} \triangle_{s,h}\,a_0(x,Dv)\cdot D\triangle_{s,h}v\,\eta^2\,dx \\ = \;-2\int_{B_r^+(y)} \triangle_{s,h}\,a_0(x,Dv)\cdot(\triangle_{s,h}v\otimes D\eta)\,\eta\,dx - L\int_{B_r^+(y)} G\cdot\triangle_{s,-h}\big(\eta^2\triangle_{s,h}v\big)\,dx\,. \tag{4.7}$$

The difference quotient $\triangle_{s,h}\,a_0(x,Dv) = \frac{1}{h}\big[a_0\big(x+he_s,Dv(x+he_s)\big) - a_0\big(x,Dv(x)\big)\big]$ can be rewritten as follows:

$$\begin{aligned}\triangle_{s,h}\,a_0\big(x,Dv(x)\big) \;&=\; \frac{1}{h}\big[a_0\big(x+he_s,Dv(x+he_s)\big) - a_0\big(x+he_s,Dv(x)\big)\big] \\ &\quad + \frac{1}{h}\big[a_0\big(x+he_s,Dv(x)\big) - a_0\big(x,Dv(x)\big)\big] \\ &= \frac{1}{h}\int_0^1 \frac{d}{dt}a_0\big(x+the_s,Dv(x)+th\triangle_{s,h}Dv(x)\big)\,dt \\ &\quad + \frac{1}{h}\int_0^1 \frac{d}{dt}a_0\big(x+the_s,Dv(x)\big)\,dt \\ &= \int_0^1 D_za_0\big(x+he_s,Dv(x)+th\triangle_{s,h}Dv(x)\big)\,dt\,\triangle_{s,h}Dv(x) \\ &\quad + \int_0^1 D_{x_s}a_0\big(x+the_s,Dv(x)\big)\,dt\,.\end{aligned} \tag{4.8}$$

At this stage it still remains to justify the formula in (4.8) for degenerate systems ($\mu = 0$) because the term involving the derivative $D_za_0(\cdot,\cdot)$ might not be well defined for some

$\tilde{t} \in [0,1]$. It suffices to show that for all $x \in B_R^+(x_0)$, $\lambda, \bar{\lambda} \in \mathbb{R}^{nN}$ not simultaneously equal to 0 (otherwise all integrands appearing in the estimates vanish) we have

$$a_0(x, \lambda + \bar{\lambda}) - a_0(x, \lambda) = \int_0^1 D_z a_0(x, \lambda + t\bar{\lambda}) \, dt \, \bar{\lambda} \,. \tag{4.9}$$

Following the arguments in [DM04a, p. 749] we consider the map $[0,1] \ni t \mapsto h(t) = a_0(x, \lambda + t\bar{\lambda}) \in \mathbb{R}^{nN}$. We first observe that the identity (4.9) is trivially fulfilled if the segment $[\lambda, \bar{\lambda}]$ does not contain the origin of $\mathbb{R}^{nN}$, because then, $h(t)$ is differentiable with respect to t on $[0,1]$. Therefore, we assume that for some $\tilde{t} \in [0,1]$ we have $\lambda + \tilde{t}\bar{\lambda} = 0$. We first suppose $\tilde{t} \in (0,1)$. Then, using the differentiability of h on $[0, \tilde{t})$ and on $(\tilde{t}, 1]$, we find for every $\varepsilon \in (0, \min\{\tilde{t}, 1 - \tilde{t}\})$:

$$h(1) - h(\tilde{t} + \varepsilon) = \int_{\tilde{t}+\varepsilon}^1 D_z a_0(x, \lambda + t\bar{\lambda}) \, dt \, \bar{\lambda} \,,$$
$$h(\tilde{t} - \varepsilon) - h(0) = \int_0^{\tilde{t}-\varepsilon} D_z a_0(x, \lambda + t\bar{\lambda}) \, dt \, \bar{\lambda} \,.$$

Observing that the function h is continuous by definition of the coefficients $a_0(\cdot, \cdot)$, the limit $\varepsilon \searrow 0$ reveals the identity (4.9), because the integrals converge due to the growth condition (C2), i. e., $|D_z a_0(x, \lambda + t\bar{\lambda})| \le L|\lambda + t\bar{\lambda}|^{p-2}$, and the fact $p - 2 > -1$ which allows us to employ Lemma A.2. Otherwise, if $\tilde{t} \in \{0, 1\}$ we only have to take into account one of the previous integrals and argue similarly. Thus, we have finished the proof of (4.9) for all $\lambda, \bar{\lambda} \in \mathbb{R}^{nN}$.

Using the ellipticity condition (C3), Young's inequality and $p-2 < 0$, we deduce the following inequality for the first integral on the right-hand side of the previous identity (4.8):

$$\begin{aligned}
&\int_0^1 D_z a_0\big(x + he_s, Dv + th\triangle_{s,h}Dv\big) \, dt \, \xi \cdot \xi \\
&\quad \ge \nu \int_0^1 \big(\mu^2 + |Dv + th\triangle_{s,h}Dv|^2\big)^{\frac{p-2}{2}} \, dt \, |\xi|^2 \\
&\quad \ge 2^{\frac{p-2}{2}} \nu \big(\mu^2 + |Dv(x)|^2 + |Dv(x + he_s)|^2\big)^{\frac{p-2}{2}} |\xi|^2 =: 2^{\frac{p-2}{2}} \nu \, Z_\mu(x)^{p-2} \, |\xi|^2
\end{aligned} \tag{4.10}$$

with the obvious abbreviation of $Z_\mu(x)$; the latter inequality holds true for $\xi = \triangle_{s,h}Dv$ (see the justification above), and for all $\xi \in R^{nN}$ whenever the segment $[Dv(x), Dv(x + he_s)]$ does not contain the origin of $\mathbb{R}^{nN}$. We now combine (4.10) with the identities (4.8) and (4.7), and we find

$$\begin{aligned}
&2^{\frac{p-2}{2}} \nu \int_{B_r^+(y)} Z_\mu(x)^{p-2} |\triangle_{s,h}Dv|^2 \, \eta^2 \, dx \\
&\le \int_{B_r^+(y)} \int_0^1 D_z a_0(x + he_s, Dv + th\triangle_{s,h}Dv) \, dt \, \triangle_{s,h}Dv \cdot \triangle_{s,h}Dv \, \eta^2 \, dx \\
&= \int_{B_r^+(y)} \triangle_{s,h} \, a_0(x, Dv) \cdot \triangle_{s,h}Dv \, \eta^2 \, dx - \int_{B_r^+(y)} \int_0^1 D_{x_s} a_0\big(x + the_s, Dv\big) \, dt \, \triangle_{s,h}Dv \, \eta^2 \, dx \\
&= -2 \int_{B_r^+(y)} \triangle_{s,h} \, a_0(x, Dv) \cdot (\triangle_{s,h}v \otimes D\eta) \, \eta \, dx - L \int_{B_r^+(y)} G \cdot \triangle_{s,-h}\big(\eta^2 \triangle_{s,h}v\big) \, dx \\
&\qquad - \int_{B_r^+(y)} \int_0^1 D_{x_s} a_0\big(x + the_s, Dv\big) \, dt \, \triangle_{s,h}Dv \, \eta^2 \, dx \\
&= I + II + III
\end{aligned} \tag{4.11}$$

with the obvious abbreviations. In view of $\operatorname{spt}\eta \subset B_{7r/8}(y)$ and the restriction $|h| < \frac{r}{8}$ we first rewrite term I using partial integration for finite differences, and we then apply the growth condition (C1), Young's inequality and standard properties of difference quotients (see e. g. [GT77, Chapter 7.11]) to find

$$\begin{aligned}
I &= 2\int_{B_r^+(y)} a_0(x,Dv)\cdot \triangle_{s,-h}\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\,dx\\
&\le 2\,L\int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p-1}{2}}\,\big|\triangle_{s,-h}\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\big|\,dx\\
&\le 2\,L\,r^{-2}\int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + 2\,L\,r^{2p-2}\int_{B_r^+(y)} \big|\triangle_{s,-h}\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\big|^p\,dx\\
&\le 2\,L\,r^{-2}\int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + 2\,L\,r^{2p-2}\int_{B_r^+(y)} \big|D_s\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\big|^p\,dx\,.
\end{aligned}$$

Via Young's inequality and the properties of the cut-off function η we next estimate the last integral on the right-hand side of the previous inequality:

$$\begin{aligned}
&\int_{B_r^+(y)} \big|D_s\big((\triangle_{s,h}v\otimes D\eta)\,\eta\big)\big|^p\,dx\\
&\qquad= \int_{B_r^+(y)} \big|\triangle_{s,h}v\otimes\big(D\eta\,D_s\eta+\eta D_sD\eta\big)+D_s\triangle_{s,h}v\otimes D\eta\,\eta\big|^p\,dx\\
&\qquad\le c\,r^{-2p}\int_{B_{7r/8}^+(y)} |\triangle_{s,h}v|^p\,dx + c\,r^{-p}\int_{B_r^+(y)} |\triangle_{s,h}D_sv|^p\,\eta^p\,dx\,.
\end{aligned}$$

Therefore, term I is estimated by

$$\begin{aligned}
I \le{}& 2\,L\,r^{-2}\int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx + c\,L\,r^{-2}\int_{B_{7r/8}^+(y)} |\triangle_{s,h}v|^p\,dx\\
&+ c\,L\,r^{p-2}\int_{B_r^+(y)} |\triangle_{s,h}D_sv|^p\,\eta^p\,dx\,.
\end{aligned}$$

For the second integral we proceed close to [Giu78], proof of Theorem III.3.5 (for $p=2$); we apply condition (C5) and use again the properties of finite difference quotients to compute

$$\begin{aligned}
II &\le L\int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx + L\int_{B_r^+(y)} \big|\triangle_{s,-h}\big(\eta^2\triangle_{s,h}v\big)\big|^p\,dx\\
&\le L\int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx + L\int_{B_r^+(y)} \big|D_s\big(\eta^2\triangle_{s,h}v\big)\big|^p\,dx\\
&\le L\int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx + c\,L\,r^{-p}\int_{B_r^+(y)} |\triangle_{s,h}v|^p\,\eta^p\,dx + c\,L\int_{B_r^+(y)} |D\triangle_{s,h}v|^p\,\eta^2\,dx\,.
\end{aligned}$$

Using assumption (C4) and Young's inequality (recall the definition of $Z_\mu(x)$ given in (4.10)), we calculate for the third integral:

$$\begin{aligned}
III &\le L\int_{B_r^+(y)} \|\gamma\|_\infty\,\big(\mu^2+|Dv|^2\big)^{\frac{p-1}{2}}\,|\triangle_{s,h}Dv|\,\eta^2\,dx\\
&\le L^2\,\varepsilon^{-1}\int_{B_r^+(y)} \|\gamma\|_\infty^2\,Z_\mu(x)^p\,\eta^2\,dx + \varepsilon\int_{B_r^+(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx
\end{aligned}$$

for every $\varepsilon \in (0,1)$. We now observe from Young's inequality (applied with $\frac{2}{2-p}$ and $\frac{2}{p}$) that we have

$$|\triangle_{s,h}Dv|^p = Z_\mu(x)^{\frac{p}{2}(2-p)}\, Z_\mu(x)^{\frac{p}{2}(p-2)}\, |\triangle_{s,h}Dv|^p \le Z_\mu(x)^p + Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2 \tag{4.12}$$

(note: if $Z_\mu(x) = 0$ then both sides vanish and the inequality trivially holds true). Combining the estimates for I, II and III with (4.11) and using adequate modifications of inequality (4.12), we find

$$\begin{aligned} 2^{\frac{p-2}{2}}\,\nu \int_{B_r^+(y)} & Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx \\ & \le c(\tfrac{L}{\varepsilon})\,L\,r^{-2}\Big(\int_{B^+_{7r/8}(y)} \big((1+\|\gamma\|^2_\infty)\,Z_\mu(x)^p + |\triangle_{s,h}v|^p\big)\,dx + \int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\Big) \\ & \quad + L\int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx + 3\,\varepsilon \int_{B_r^+(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx\,. \end{aligned} \tag{4.13}$$

Keeping in mind that $B^+_{7r/8}(y) \supset \operatorname{spt}(\eta)$ and $|h| \le \frac{r}{8}$ we first mention that

$$\int_{B^+_{7r/8}(y)} |\triangle_{s,h}v|^p\,dx \le \int_{B_r^+(y)} |D_s v|^p\,dx\,.$$

Furthermore, the integral over $Z_\mu(x)^p$ (see (4.10) for the definition of $Z_\mu(x)$) is estimated by

$$\begin{aligned} \int_{B^+_{7r/8}(y)} Z_\mu(x)^p\,dx &= \int_{B^+_{7r/8}(y)} \big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{\frac{p}{2}}\,dx \\ &\le 2\int_{B_r^+(y)} \big(\mu^2+|Dv(x)|^2\big)^{\frac{p}{2}}\,dx\,. \end{aligned} \tag{4.14}$$

Therefore, choosing $3\varepsilon = 2^{\frac{p-4}{2}}\nu$ in (4.13), dividing through by $2^{\frac{p-4}{2}}\nu$, recalling that $\eta = 1$ on $B_{3r/4}(y)$, we finally arrive at

$$\begin{aligned} \int_{B^+_{3r/4}(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,dx &\le \int_{B_r^+(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,\eta^2\,dx \\ &\le c\,r^{-2}\,(1+\|\gamma\|^2_\infty)\int_{B_r^+(y)} \big(\mu^2+|Dv(x)|^2\big)^{\frac{p}{2}}\,dx + c\int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx\,, \end{aligned} \tag{4.15}$$

and the constant c depends only on $\frac{L}{\nu}$ (note: the dependency on the parameter p is dropped due to $2^{(p-2)/2} \in (\frac{1}{2},1)$). We mention here: in order to conclude that the tangential derivatives belong to the space L^p, we deduce analogously to the proof of [Giu03, Theorem 8.1] from (4.12): the family $\big(\triangle_{s,h}Dv\big)_h$, $h \in \mathbb{R}$ with $|h| < \frac{r}{8}$, is bounded in $L^p(B_{3r/4}(y),\mathbb{R}^{nN})$ (see (4.14), (4.15)) and therefore converges in $L^p(B^+_{3r'/4}(y),\mathbb{R}^{nN})$ to D_sDv for all $r' < r$ (see e.g. [Eva98], Chapter 5.8.2, proof of Theorem 3 and the remark immediately after the proof). Thus we also conclude $D_sDv \in L^p(B^+_{3r'/4}(y),\mathbb{R}^{nN})$ (for $s \in \{1,\dots,n-1\}$), which proves $D'v \in W^{1,p}(B^+_\rho(x_0),\mathbb{R}^{(n-1)N})$ for all $\rho < R$.

We apply Lemma A.3 (i) and obtain

$$\begin{aligned}
\int_{B^+_{3r/4}(y)} \big|\triangle_{s,h} V_\mu(Dv)\big|^2\,dx &= \int_{B^+_{3r/4}(y)} h^{-2}\big|V_\mu\big(Dv(x+he_s)\big)-V_\mu\big(Dv(x)\big)\big|^2\,dx\\
&\le c(p)\,h^{-2}\int_{B^+_{3r/4}(y)} \big(\mu^2+|Dv(x)|^2+|Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\,\big|Dv(x+he_s)-Dv(x)\big|^2\,dx\\
&= c(p)\int_{B^+_{3r/4}(y)} Z_\mu(x)^{p-2}|\triangle_{s,h}Dv|^2\,dx\\
&\le c\big(p,\tfrac{L}{\nu}\big)\Big(r^{-2}\,(1+\|\gamma\|^2_\infty)\int_{B^+_r(y)}\big(\mu^2+|Dv(x)|^2\big)^{\frac{p}{2}}\,dx+\int_{B^+_r(y)}|G|^{\frac{p}{p-1}}\,dx\Big)\,.
\end{aligned}$$

As above, the sequence $\big(\triangle_{s,h}V_\mu(Dv)\big)_h$ is uniformly bounded in $L^2(B_{3r/4}(y),\mathbb{R}^{nN})$ and therefore converges strongly to $D_s(V_\mu(Dv))$. Thus we obtain the *tangential estimate* ($s=1,\dots,n-1$), and summing up this yields

$$\begin{aligned}
\int_{B^+_{3r/4}(y)} &\big|D'(V_\mu(Dv))\big|^2\,dx\\
&\le c\big(n,p,\tfrac{L}{\nu}\big)\Big(r^{-2}\,(1+\|\gamma\|^2_\infty)\int_{B^+_r(y)}\big(\mu^2+|Dv(x)|^2\big)^{\frac{p}{2}}\,dx+\int_{B^+_r(y)}|G|^{\frac{p}{p-1}}\,dx\Big)\,,
\end{aligned}$$

which is the desired inequality in a) for the boundary situation.

The proof of b) in the interior case is achieved in the same way: we here choose analogously to above a cut-off function η with support in $B_{11r/16}(y)$ which satisfies $\eta\equiv 1$ on $B_{5r/8}(y)$ with the same assumptions on the derivatives as in (4.5). Then we may use the same test function as in the boundary case, where this time $|h|<\frac{r}{16}$. Finally we note that in the interior we do not need any constraint of the direction, i. e., we can take $s=1,\dots,n$. □

Before going on we mention that the Caccioppoli-type estimate given in the last lemma can be rewritten in a slightly different but equivalent form. We define the j-th component of $V_\mu(Dv)$ via

$$V_{\mu,j}(Dv) \;=\; \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}\,D_jv \qquad j=1,\dots,n\,,$$

and the tangential part $V'_\mu(Dv):=\big(V_{\mu,1}(Dv),\dots,V_{\mu,n-1}(Dv)\big)$. Furthermore, the derivative of $V_\mu(Dv)$ is given by

$$D_s(V_\mu(Dv)) \;=\; \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}\,D_sDv+\tfrac{p-2}{2}\,\big(\mu^2+|Dv|^2\big)^{\frac{p-6}{4}}\,Dv\,Dv\cdot D_sDv$$

($s=1,\dots,n$) such that the absolute value of $D_s(V_\mu(Dv))$ is bounded by

$$\big|D_s(V_\mu(Dv))\big|\le 2\,\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}|D_sDv| \tag{4.16}$$

$$\big|D_s(V_\mu(Dv))\big|\ge \tfrac{1}{2}\,\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{4}}|D_sDv| \tag{4.17}$$

from below and above. Thus, we can reformulate the estimate in (4.4) in the boundary situation (as well as the corresponding estimate in the interior) by

$$\begin{aligned}
\int_{B^+_{3r/4}(y)} &\big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}}|D'Dv|^2\,dx\\
&\le c\big(n,p,\tfrac{L}{\nu}\big)\Big(r^{-2}\,(1+\|\gamma\|^2_\infty)\int_{B^+_r(y)}\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx+\int_{B^+_r(y)}|G|^{\frac{p}{p-1}}\,dx\Big)\,.
\end{aligned} \tag{4.18}$$

4.2 Inhomogeneous systems with x-dependency

We are now interested in improving the previous lemma for weak solutions of the system (4.1): we are going to give a sharper bound in the inequality given in Lemma 4.1 by an argument based on weak convergence. Employing the system (4.1), we further obtain the existence of the full derivative of $V_\mu(Dv)$ up to the boundary, a result, which was announced in [DKM07, Theorem 2.4]:

Theorem 4.2: *Let $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$ be a weak solution to system (4.1) under the assumptions (C1)-(C5) and let $\mu \in [0,1]$ be arbitrary. Then v is twice differentiable in the weak sense. Moreover, $v \in W^{2,p}(B^+_\rho(x_0), \mathbb{R}^N)$ for all $\rho < R$, and there exists a constant c depending only on n, N, p and $\frac{L}{\nu}$ such that for all $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $0 < r < R - |y - x_0|$ there holds:*

$$\int_{B^+_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2 \, dx \le c \Big(r^{-2} \int_{B^+_r(y)} (1 + \gamma(x)^2) \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}} \, dx + \int_{B^+_r(y)} |G|^{\frac{p}{p-1}} \, dx \Big) .$$

Proof: We first note that inequality in Lemma 4.1 is the desired estimate – at least for the tangential derivative of $V_\mu(Dv)$ – apart from the fact that the supremum of γ appears on the right-hand side. To prove the inequality in the final form we proceed similarly to the proof of the last lemma. The important difference is that we already may take advantage of the fact $D_s v \in W^{1,p}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ for all $0 < \rho < R$ and for all tangential derivatives ($s = 1, \ldots, n-1$). We first deal with the boundary situation and consider $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n \le \frac{3}{4} r$. We define

$$\varphi \, = \, \triangle_{s,-h}\big(\eta^2 D_s v\big) \qquad \in W^{1,p}_0(B^+_r(y), \mathbb{R}^N) \, , \tag{4.19}$$

where $\eta \in C^\infty_0(B_{3r/4}(y), [0,1])$ is a standard cut-off function satisfying $\eta \equiv 1$ on $B_{r/2}(y)$ and $D\eta \le c\, r^{-1}$, and $s \in \{1, \ldots, n-1\}$, $|h| < \frac{r}{4}$, cf. the test function in (4.6). In view of Lemma 4.1, φ is an admissible test function in (4.2). With integration by parts for finite differences we infer the identity

$$\int_{B^+_r(y)} \triangle_{s,h}\, a_0(x, Dv) \cdot \big(DD_s v\, \eta + 2\, D_s v \otimes D\eta\big) \eta \, dx \, = \, -L \int_{B^+_r(y)} G \cdot \triangle_{s,-h}\big(\eta^2 D_s v\big) \, dx \, .$$

Therefore, instead of inequality (4.11), we now obtain

$$\begin{aligned}
\nu & \int_{B^+_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} \, |DD_s v|^2 \, \eta^2 \, dx \\
& \le \int_{B^+_r(y)} D_z a_0(x, Dv)\, DD_s v \cdot DD_s v \, \eta^2 \, dx \\
& = \int_{B^+_r(y)} D_s a_0(x, Dv) \cdot DD_s v \, \eta^2 \, dx - \int_{B^+_r(y)} D_{x_s} a_0(x, Dv) \cdot DD_s v \, \eta^2 \, dx \\
& = \int_{B^+_r(y)} \big(D_s a_0(x, Dv) - \triangle_{s,h} a_0(x, Dv)\big) \cdot \big(DD_s v\, \eta + 2\, D_s v \otimes D\eta\big) \eta \, dx \\
& \qquad - 2 \int_{B^+_r(y)} D_s a_0(x, Dv) \cdot (D_s v \otimes D\eta)\, \eta \, dx \\
& \qquad + L \int_{B^+_r(y)} G \cdot \big[D_s\big(\eta^2 D_s v\big) - \triangle_{s,-h}\big(\eta^2 D_s v\big)\big] \, dx - L \int_{B^+_r(y)} G \cdot D_s\big(\eta^2 D_s v\big) \, dx \\
& \qquad - \int_{B^+_r(y)} D_{x_s} a_0(x, Dv) \cdot DD_s v \, \eta^2 \, dx
\end{aligned} \tag{4.20}$$

(note: all integrands vanish on the set $\{x \in B_r^+(y) : Dv = 0\}$). For the first integral on the right-hand side, called I_h in what follows, we next show that it vanishes as h tends to zero, using a weak convergence argument. For this purpose we abbreviate

$$\begin{aligned} f_h &:= \big(D_s a_0(x,Dv) - \triangle_{s,h} a_0(x,Dv)\big)\big(\mu^2 + |Dv|^2\big)^{\frac{2-p}{4}}\,\eta\,, \\ g &:= \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{4}}\big(DD_s v\,\eta + 2\,D_s v \otimes D\eta\big)\,. \end{aligned}$$

This means we can rewrite the integral $I_h = \int_{B_r^+(y)} f_h \cdot g\,dx$. From the last lemma (to be more precise, from (4.18)) we infer $g \in L^2(B_{3r/4}^+(y), \mathbb{R}^{nN})$. Furthermore, the sequence $\{f_h\}$ is uniformly bounded in $L^2(B_{3r/4}^+(y), \mathbb{R}^{nN})$: To this aim we first use condition (C2), the technical Lemma A.2 and the reasoning for the identity (4.9) to deduce

$$\begin{aligned} &\Big|\int_0^1 D_z a_0(x, Dv + th\triangle_{s,h}Dv)\,dt\,\triangle_{s,h}Dv\Big| \\ &\quad\le L\int_0^1 \big(\mu^2 + |Dv + th\triangle_{s,h}Dv|^2\big)^{\frac{p-2}{2}}\,dt\,|\triangle_{s,h}Dv| \\ &\quad\le L\,c(p)\,\big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s) - Dv(x)|^2\big)^{\frac{p-2}{2}}\,|\triangle_{s,h}Dv| \\ &\quad\le L\,c(p)\,\big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\,|\triangle_{s,h}Dv| \end{aligned} \tag{4.21}$$

(again, if $\mu = 0$, this inequality is trivially satisfied for $Dv(x) = \triangle_{s,h}Dv = 0$). Combined with the decomposition in (4.8) and condition (C4) we then obtain

$$\begin{aligned} \big|\triangle_{s,h} a_0(x,Dv(x))\big| \le\; & L\,c(p)\,\big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}\,|\triangle_{s,h}Dv| \\ & + L\,\|\gamma\|_\infty\,\big(\mu^2 + |Dv(x)|^2\big)^{\frac{p-1}{2}}\,, \end{aligned}$$

and from (C2) and (C4) we further infer

$$\big|D_s a_0(x,Dv(x))\big| \le L\,\|\gamma\|_\infty\,\big(\mu^2 + |Dv(x)|^2\big)^{\frac{p-1}{2}} + L\,\big(\mu^2 + |Dv(x)|^2\big)^{\frac{p-2}{2}}\,|DD_s v(x)|$$

for all $x \in B_{3r/4}^+(y)$ (note that if $Dv(x) = 0$ then $DD_s v(x) = 0$ and hence, the latter inequality trivially holds true). Hence, we end up with

$$\begin{aligned} &\int_{B_{3r/4}^+(y)} |f_h|^2\,dx \\ &\le 2\int_{B_{3r/4}^+(y)} \Big(\big|D_s a_0(x,Dv(x))\big|^2 + \big|\triangle_{s,h} a_0(x,Dv(x))\big|^2\Big)\big(\mu^2 + |Dv(x)|^2\big)^{\frac{2-p}{2}}\,dx \\ &\le L\,c(p)\int_{B_{3r/4}^+(y)} \Big(\|\gamma\|_\infty\big(\mu^2 + |Dv(x)|^2\big)^{p-1} + \big(\mu^2 + |Dv(x)|^2\big)^{p-2}|DD_s v(x)|^2 + \\ &\qquad\qquad \big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{p-2}|\triangle_{s,h}Dv(x)|^2\Big)\big(\mu^2 + |Dv(x)|^2\big)^{\frac{2-p}{2}}\,dx \\ &\le L\,c(p)\int_{B_{3r/4}^+(y)} \Big(\|\gamma\|_\infty\big(\mu^2 + |Dv(x)|^2\big)^{\frac{p}{2}} + \big(\mu^2 + |Dv(x)|^2\big)^{\frac{p-2}{2}}|DD_s v(x)|^2 + \\ &\qquad\qquad \big(\mu^2 + |Dv(x)|^2 + |Dv(x+he_s)|^2\big)^{\frac{p-2}{2}}|\triangle_{s,h}Dv(x)|^2\Big)\,dx \\ &\le L\,c\big(p, \tfrac{L}{\nu}, \|\gamma\|_\infty\big)\Big(r^{-2}\int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}}\,dx + \int_{B_r^+(y)} |G|^{\frac{p}{p-1}}\,dx\Big)\,, \end{aligned}$$

where we have applied the estimates (4.18) and (4.15) in the last inequality. Thus, we can find a function $f \in L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$ such that a subsequence of $\{f_h\}$ converges weakly in $L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$ to f. Furthermore, we estimate for every $\phi \in L^{p/(p-1)}(B^+_{3r/4}(y),\mathbb{R}^{nN})$ using Hölder's inequality with exponents 2, $\frac{2p}{2-p}$ and $\frac{p}{p-1}$:

$$\int_{B^+_{3r/4}(y)} |f_h\cdot\phi|\,dx \le \Big(\int_{B^+_{3r/4}(y)} \big|D_s a_0(x,Dv) - \triangle_{s,h} a_0(x,Dv)\big|^2\,dx\Big)^{\frac12} \cdot \Big(\int_{B^+_{3r/4}(y)} \big(\mu^2+|Dv|^2\big)^{\frac p2}\,dx\Big)^{\frac{2-p}{2p}} \Big(\int_{B^+_{3r/4}(y)} |\phi|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} .$$

Keeping in mind that $D_s a_0(x,Dv)$, $s\in\{1,\dots,n-1\}$, belongs to $L^2(B^+_\rho(x_0),\mathbb{R}^{nN})$ for all $\rho<R$ due to the last lemma, we obtain $\triangle_{s,h} a_0(x,Dv) \to D_s a_0(x,Dv)$ strongly in $L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$ as $h\to 0$, i. e., we have $\{f_h\}_h \rightharpoonup 0$ weakly in $L^p(B^+_{3r/4}(y),\mathbb{R}^{nN})$. Since weak limits are unique, we conclude $f\equiv 0$. Therefore, in view of $f_h \rightharpoonup 0$ in $L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$ and $g\in L^2(B^+_{3r/4}(y),\mathbb{R}^{nN})$, we finally arrive at

$$\lim_{h\to 0} I_h = \lim_{h\to 0}\int_{B^+_r(y)} f_h\cdot g\,dx \;=\; 0$$

Taking into account the strong convergence $\triangle_{s,-h}\big(\eta^2 D_s v\big) \to D_s\big(\eta^2 D_s v\big)$ in $L^p(B^+_r(y),\mathbb{R}^{nN})$ and $G\in L^{p/(p-1)}(B^+_r(y),\mathbb{R}^N)$, we observe that the first and the third integral on the right-hand side of (4.20) vanish as $h\to 0$ due to weak respectively strong convergence. Thus, we obtain

$$\begin{aligned}
\nu &\int_{B^+_r(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\,\eta^2\,dx \\
&\le -2\int_{B^+_r(y)} D_s a_0(x,Dv)\cdot(D_s v\otimes D\eta)\,\eta\,dx - L\int_{B^+_r(y)} G\cdot D_s\big(\eta^2 D_s v\big)\,dx \\
&\quad - \int_{B^+_r(y)} D_{x_s} a_0(x,Dv)\cdot DD_s v\,\eta^2\,dx \\
&= -2\int_{B^+_r(y)} D_z a_0(x,Dv)\,DD_s v\cdot(D_s v\otimes D\eta)\,\eta\,dx - L\int_{B^+_r(y)} G\cdot D_s\big(\eta^2 D_s v\big)\,dx \\
&\quad - \int_{B^+_r(y)} D_{x_s} a_0(x,Dv)\cdot\big(2\,D_s v\otimes D\eta + DD_s v\,\eta\big)\,\eta\,dx\,.
\end{aligned} \tag{4.22}$$

Evaluating the remaining integrals in a standard manner and keeping in mind (4.16), finally reveals the stronger tangential estimate

$$\begin{aligned}
\int_{B^+_{r/2}(y)} \big|D'(V_\mu(Dv))\big|^2\,dx &\le c\int_{B^+_{r/2}(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}} |DD'v|^2\,dx \\
&\le c\big(n,p,\tfrac{L}{\nu}\big)\Big(r^{-2}\int_{B^+_r(y)} (1+\gamma(x)^2)\,\big(\mu^2+|Dv|^2\big)^{\frac p2}\,dx + \int_{B^+_r(y)} |G|^{\frac{p}{p-1}}\,dx\Big)\,.
\end{aligned} \tag{4.23}$$

In contrast to (4.4) in Lemma 4.1, the function γ now appears in the integrand on the right-hand side; this will be a crucial point for later applications.

For interior balls $B_{3r/4}(y)\subset B^+$ all the calculations remain true for every $s\in\{1,\dots,n\}$, and hence, the last estimate holds for the full derivative. This proves the statement of the theorem in the interior. At the boundary we still have to find an estimate for the normal derivative. To this aim we differentiate the system (4.1) and get

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial z_j^\beta}\,D_{ij}v^\beta+\sum_{i=1}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial x_i}\;=\;-L\,G^\alpha$$

which implies

$$\sum_{\beta=1}^{N}\frac{\partial(a_0)_n^\alpha(x,Dv)}{\partial z_n^\beta}\,D_{nn}v^\beta\;=\;-\sum_{\beta=1}^{N}\sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial z_j^\beta}\,D_{ij}v^\beta-\sum_{i=1}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial x_i}-L\,G^\alpha$$

for $\alpha=1,\ldots,N$ almost everywhere in $B^+_{r/2}(y)\cap\{x_n>\varepsilon\}$ (for some $\varepsilon>0$). Finally, the estimate involving the derivative $D_{nn}v$ is derived as follows: We recall that in the interior all second derivatives exist. Then we multiply the previous relation by $D_{nn}v^\alpha$ and sum up upon α; using the growth (C2), the ellipticity condition (C3) and the Lipschitz continuity of $a_0(\cdot,\cdot)$ with respect to x in (C4), we get

$$\begin{aligned}
\nu\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_{nn}v|^2\;&\le\;\sum_{\alpha,\beta=1}^{N}\frac{\partial(a_0)_n^\alpha(x,Dv)}{\partial z_n^\beta}\,D_{nn}v^\beta\,D_{nn}v^\alpha\\
&=-\sum_{\alpha,\beta=1}^{N}\sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial z_j^\beta}\,D_{ij}v^\beta\,D_{nn}v^\alpha-\sum_{i=1}^{n}\frac{\partial(a_0)_i^\alpha(x,Dv)}{\partial x_i}\,D_{nn}v^\alpha-L\,G^\alpha\,D_{nn}v^\alpha\\
&\le\;c(n,N)\,L\Big(\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|DD'v|+\gamma(x)\left(\mu^2+|Dv|^2\right)^{\frac{p-1}{2}}+|G|\Big)\,|D_{nn}v| \qquad (4.24)
\end{aligned}$$

almost everywhere in $B^+_{r/2}(y)\cap\{x_n>\varepsilon\}$ (note that in order to apply (C2) and (C3), respectively, we have employed the fact that all terms above vanish if $Dv(x)=0$). Then Young's inequality and absorbing the term involving $|D_{nn}v|$ implies

$$\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_{nn}v|^2\;\le\;c\Big(\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|DD'v|^2+(1+\gamma(x)^2)\left(\mu^2+|Dv|^2\right)^{\frac{p}{2}}+|G|^{\frac{p}{p-1}}\Big)$$

for a constant c depending only on n,N and $\frac{L}{\nu}$. From (4.17) and the estimate (4.23) we know that the right-hand side of the last inequality exists and that there holds

$$\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|DD'v|^2\;\in\;L^1(B^+_{r/2}(y))\,.$$

Keeping in mind $G\in L^{p/(p-1)}$, we hence integrate the previous inequality on $B^+_{r/2}(y)\cap\{x_n>\varepsilon\}$. Letting $\varepsilon\to0$ we gain

$$\begin{aligned}
&\int_{B^+_{r/2}(y)}\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|D_{nn}v|^2\,dx\\
&\quad\le\;c\int_{B^+_{r/2}(y)}\Big(\left(\mu^2+|Dv|^2\right)^{\frac{p-2}{2}}|DD'v|^2+(1+\gamma(x)^2)\left(\mu^2+|Dv|^2\right)^{\frac{p}{2}}+|G|^{\frac{p}{p-1}}\Big)\,dx\\
&\quad\le\;c\Big(r^{-2}\int_{B^+_r(y)}(1+\gamma(x)^2)\left(\mu^2+|Dv|^2\right)^{\frac{p}{2}}dx+\int_{B^+_r(y)}|G|^{\frac{p}{p-1}}\,dx\Big)\,,
\end{aligned}$$

and the constant c still depends only on n,N,p and $\frac{L}{\nu}$. Combined with (4.17) and (4.23), this is the desired Caccioppoli-type inequality at the boundary.

Finally, we note that the decomposition

$$|D^2v|^p \le \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}} + \big(\mu^2+|Dv|^2\big)^{\frac{p-2}{2}}\,|D^2v|^2$$

cf. (4.12), finally gives $v \in W^{2,p}(B_\rho^+(x_0),\mathbb{R}^N)$ for all $\rho < R$. Thus the proof of the theorem is complete. □

4.3 Homogeneous systems without x-dependency

4.3.1 An improved version of Theorem 4.2

In the next step, we consider weak solutions $v \in W_\Gamma^{1,p}(B_R^+(x_0),\mathbb{R}^N)$ to the homogeneous system (4.3), where the coefficients $a_0(\cdot)$ do not depend explicitly on the x-variable. In this situation the previous Theorem 4.2 states that $v \in W^{2,p}(B_\rho^+(x_0),\mathbb{R}^N)$ for all $\rho < R$ with the estimate

$$\int_{B^+_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2\,dx \le c\,r^{-2}\int_{B_r^+(y)} \big(\mu^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\,.$$

for all $y \in B_R^+(x_0)\cup\Gamma_R(x_0)$ and $0<r<R-|y-x_0|$. In order to infer a higher integrability estimate for $D(V_\mu(Dv))$ we now show an improved version of this Caccioppoli-type estimate such that on the right-hand side only the tangential part of $V_\mu(Dv)$ shows up:

Theorem 4.3: *Let $v \in W_\Gamma^{1,p}(B_R^+(x_0),\mathbb{R}^N)$ be a weak solution to the system (4.3), whose coefficients $a_0(\cdot)$ satisfy the conditions (C1)-(C3), and let $\mu\in[0,1]$ be arbitrary. Then v is twice differentiable in the weak sense, more precisely $v \in W^{2,p}(B_\rho^+(x_0),\mathbb{R}^N)$ for all $\rho<R$, and there exists a constant c depending only on n,N,p and $\frac{L}{\nu}$ such that*

a) *(close to the boundary) for all $y \in B_R^+(x_0)\cup\Gamma_R(x_0)$ and $0<r<R-|y-x_0|$ with $y_n \le \frac{3}{4}r$ there holds*

$$\int_{B^+_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2\,dx \le c\,r^{-2}\int_{B_r^+(y)} \big|V'_\mu(Dv)\big|^2\,dx\,, \tag{4.25}$$

b) *(in the interior) for all $y \in B_R^+(x_0)$ and $0<r<R-|y-x_0|$ with $y_n > \frac{3}{4}r$ there holds*

$$\int_{B_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2\,dx \le c\,r^{-2}\int_{B_{3r/4}(y)} \big|V_\mu(Dv)-\big(V_\mu(Dv)\big)_{B_{3r/4}(y)}\big|^2\,dx\,. \tag{4.26}$$

Remark: We emphasize that in statement a) the normal derivative of v is not involved in the quadratic term of $|V'_\mu(Dv)|^2=(\mu^2+|Dv|^2)^{(p-2)/2}|D'v|^2$ on the right-hand side of (4.25). If we pass to coefficients which additionally depend explicitly on x (as in the previous Section 4.2), this result can no longer be obtained because a dependency only on the x_n-variable of the solution might occur: consider for example the coefficients $a(x,z)$ defined by

$$a(x,z) = \frac{\big(1+|z|^2\big)^{\frac{p-2}{2}}\,z}{\big(1+(1+x_n^\alpha)^2\big)^{\frac{p-2}{2}}(1+x_n^\alpha)}$$

for a number $\alpha\in(0,1)$. Then, $v(x)=\frac{1}{1+\alpha}\,x_n^{1+\alpha}+x_n$ is a weak solution of $\operatorname{div} a(x,Dv)=0$ in $B^+\subset\mathbb{R}^n, n\ge 2$, but the statement of the theorem obviously does not hold on any (half-)ball $B^+_{r/2}(y)\subset B^+$, and even $v\in W^{2,p}(B_\rho^+,\mathbb{R}^N)$ does not hold for every $\rho\in(0,1)$ (in fact, v only belongs to a suitable fractional Sobolev space).

Proof: We proceed analogously to the proof of the last theorem, taking advantage of the simpler structure of the coefficients in (4.3) in contrast to (4.1). We first recall $v \in W^{2,p}(B_\rho^+(x_0),\mathbb{R}^N)$ for all $\rho < R$ in view of Theorem 4.2. To prove inequality (4.25) we consider $y \in B_R^+(x_0) \cup \Gamma_R(x_0)$ and $0 < r < R - |y - x_0|$ with $y_n \leq \frac{3}{4}r$ and choose a cut-off function $\eta \in C_0^\infty(B_{3r/4}(y),[0,1])$ satisfying $\eta \equiv 1$ on $B_{r/2}(y)$ and $|D\eta| \leq \frac{8}{r}$. Now let $h \in \mathbb{R}$ with $|h| < \frac{r}{4}$ and choose $\varphi = \triangle_{s,-h}(\eta^2 D_s v) \in W_0^{1,p}(B_r^+(y),\mathbb{R}^N)$ (see (4.19) above), $s = 1,\dots,n-1$, as a test function for the system (4.3). Arguing exactly as in the proof of the previous theorem, we find (see (4.22)):

$$\nu \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2 \, \eta^2 \, dx \;\leq\; -2 \int_{B_r^+(y)} D_s a_0(Dv) \cdot (D_s v \otimes D\eta)\, \eta \, dx \,,$$

and from Young's inequality and the growth condition (C2) we thus infer

$$\begin{aligned} \nu & \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2 \, \eta^2 \, dx \\ & \leq\; 2L \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v| \, |D_s v| \, |D\eta| \, \eta \, dx \\ & \leq\; \frac{\nu}{2} \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2 \, \eta^2 \, dx + c \, \frac{L^2}{\nu} \, r^{-2} \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_s v|^2 \, dx \,. \end{aligned}$$

This allows us to find the following estimate in tangential direction ($s = 1,\dots,n-1$):

$$\int_{B_{r/2}^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2 \, dx \;\leq\; c\big(\tfrac{L}{\nu}\big) \, r^{-2} \int_{B_r^+(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_s v|^2 \, dx \,. \tag{4.27}$$

To estimate also the normal derivative we again make use of the differentiated system (4.3). Since $G = 0$ we end up with

$$\begin{aligned} \nu \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_{nn} v|^2 & \leq \sum_{\alpha,\beta=1}^N \frac{\partial (a_0)_n^\alpha(Dv)}{\partial z_n^\beta} \, D_{nn} v^\beta \, D_{nn} v^\alpha \\ & = - \sum_{\alpha,\beta=1}^N \sum_{\substack{i,j=1 \\ (i,j)\neq(n,n)}}^n \frac{\partial (a_0)_i^\alpha(Dv)}{\partial z_j^\beta} \, D_{ij} v^\beta \, D_{nn} v^\alpha \\ & \leq\; c(n,N)\, L \, \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD'v| \, |D_{nn} v| \end{aligned}$$

almost everywhere in $B_{r/2}^+(y) \cap \{x_n > \varepsilon\}$, see (4.24). At this stage, we apply Young's inequality to see

$$\big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |D_{nn} v|^2 \;\leq\; c(n,N,\tfrac{L}{\nu}) \, \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD'v|^2 \,.$$

Since the right-hand side is in $L^1(B_{r/2}^+(y))$, we may integrate the latter inequality on $B_{r/2}^+(y) \cap \{x_n > \varepsilon\}$. Then, keeping in mind (4.16) and the tangential estimate (4.27), the desired inequality in a) follows immediately from letting $\varepsilon \to 0$.

In the interior of $B_R^+(x_0)$ we proceed similarly, but we need a modification of the arguments to obtain the mean value version: Lemma 4.1 holds in the interior without any assumption on the direction of the derivative. We choose $\triangle_{s,-h}\big(\eta^2 \big(D_s v - \xi_s\big)\big)$ as a test function (here all directions $s = 1, \ldots, n$ are allowed), where $\eta \in C_0^\infty(B_{5r/8}(y), [0,1])$ is a cut-off function satisfying

$$\eta \equiv 1 \text{ on } B_{r/2}(y) \qquad \text{and} \qquad |D\eta|^2 + |D^2\eta| \leq c\, r^{-2}\,,$$

$h \in \mathbb{R}$ with $|h| < \frac{r}{8}$ and where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{nN}$ will be defined later. Calculating as in Theorem 4.2 when deriving the estimate (4.22), we use partial integration and the fact that $a_0(\xi)$ is constant to see

$$\begin{aligned}
\nu \int_{B_r(y)} & \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\, \eta^2\, dx \\
&\leq -2 \int_{B_r(y)} D_s a_0(Dv) \cdot \big((D_s v - \xi_s) \otimes D\eta\big)\, \eta\, dx \\
&= 2 \int_{B_r(y)} a_0(Dv) \cdot D_s\big[\big((D_s v - \xi_s) \otimes D\eta\big)\, \eta\big]\, dx \\
&= 2 \int_{B_r(y)} \big(a_0(Dv) - a_0(\xi)\big) \cdot D_s\big[\big((D_s v - \xi_s) \otimes D\eta\big)\, \eta\big]\, dx\,.
\end{aligned}$$

Using the properties of the test functions, the estimate

$$\begin{aligned}
a_0(Dv) - a_0(\xi) &= \int_0^1 D_z a_0\big(\xi + t(Dv - \xi)\big)\, dt\, (Dv - \xi) \\
&\leq c(p)\, L\, \big(\mu^2 + |Dv|^2 + |\xi|^2\big)^{\frac{p-2}{2}} |Dv - \xi|
\end{aligned}$$

(see the justification for (4.9)) and Young's inequality, we obtain

$$\begin{aligned}
\nu & \int_{B_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\, \eta^2\, dx \\
&\leq c \int_{B_r(y)} \big(\mu^2 + |Dv|^2 + |\xi|^2\big)^{\frac{p-2}{2}} |Dv - \xi| \,\big[|DD_s v|\, |D\eta|\, \eta + |Dv - \xi|\big(|D\eta|^2 + |D^2\eta|\, \eta\big)\big]\, dx \\
&\leq \tfrac{1}{2}\, \nu \int_{B_r(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\, \eta^2\, dx \\
&\qquad + c\big(p, \tfrac{L}{\nu}\big)\, L\, r^{-2} \int_{B_{3r/4}(y)} \big(\mu^2 + |Dv|^2 + |\xi|^2\big)^{\frac{p-2}{2}} |Dv - \xi|^2\, dx\,,
\end{aligned}$$

Absorbing the first integral on the the right-hand side and applying Lemma A.3 (i) yields

$$\int_{B_{r/2}(y)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |DD_s v|^2\, dx \;\leq\; c\big(p, \tfrac{L}{\nu}\big)\, r^{-2} \int_{B_{3r/4}(y)} \big|V_\mu(Dv) - V_\mu(\xi)\big|^2\, dx\,.$$

Since the function V_μ is surjective, we may choose ξ such that $V_\mu(\xi) = (V_\mu(Dv))_{B_{3r/4}(y)}$. Combined with the estimate in (4.16) this gives the desired Caccioppoli-inequality in the mean value version. □

4.3.2 Higher integrability of $\mathbf{D}(\mathbf{V}_\mu(\mathbf{Dv}))$

Starting from the Caccioppoli inequalities close to the boundary and in the interior in Theorem 4.3 we next derive reverse Hölder inequalities on balls and intersections of balls in $B_R^+(x_0)$ for weak solutions v of (4.3). This enables us to apply an up-to-the-boundary version of Gehring's Lemma which yields an appropriate higher integrability result.

First, we deal with the boundary situation and consider points $y \in B_R^+(x_0) \cup \Gamma_R(x_0)$ and radii $0 < r < R - |y - x_0|$ satisfying $y_n \leq \frac{3}{4}r$. We see that $V_\mu'(Dv)$ vanishes identically on $\Gamma_R(x_0) \supset \Gamma_r(y'')$ (recalling that y'' denotes the projection of y onto $\mathbb{R}^{n-1} \times \{0\}$ and that $D'v \equiv 0$ on $\Gamma_R(x_0)$ by assumption). Therefore, we can apply the Sobolev-Poincaré inequality in Lemma A.5 (with $\frac{2n}{n+2} < n$ instead of p) to the right-hand side of (4.25) and we obtain

$$\begin{aligned} r^2 \int_{B_{r/2}^+(y)} \big|D(V_\mu(Dv))\big|^2 \,dx &\leq c \int_{B_r^+(y)} \big|V_\mu'(Dv)\big|^2 \,dx \\ &\leq c \Big(\int_{B_r^+(y)} |D(V_\mu'(Dv))|^{\frac{2n}{n+2}} \,dx \Big)^{\frac{n+2}{n}} \\ &\leq c \Big(\int_{B_r^+(y)} |D(V_\mu(Dv))|^{\frac{2n}{n+2}} \,dx \Big)^{\frac{n+2}{n}} . \end{aligned} \tag{4.28}$$

Hence, taking mean values we have

$$\fint_{B_{r/2}^+(y)} \big|D(V_\mu(Dv))\big|^2 \,dx \leq c(n,N,p,\tfrac{L}{\nu}) \Big(\fint_{B_r^+(y)} |D(V_\mu(Dv))|^{\frac{2n}{n+2}} \,dx \Big)^{\frac{n+2}{n}} . \tag{4.29}$$

In the interior, we consider points $y \in B_R^+(x_0)$ and radii $0 < r < R - |y - x_0|$ with $y_n > \frac{3}{4}r$. As above we apply the Sobolev-Poincaré inequality to the Caccioppoli-type estimate (4.26) to see

$$\begin{aligned} r^2 \int_{B_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2 \,dx &\leq c \int_{B_{3r/4}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B_{3r/4}(y)}\big|^2 \,dx \\ &\leq c \Big(\int_{B_r^+(y)} |D(V_\mu(Dv))|^{\frac{2n}{n+2}} \,dx \Big)^{\frac{n+2}{n}} \end{aligned}$$

$$\Rightarrow \fint_{B_{r/2}(y)} \big|D(V_\mu(Dv))\big|^2 \,dx \leq c(n,N,p,\tfrac{L}{\nu}) \Big(\fint_{B_r^+(y)} |D(V_\mu(Dv))|^{\frac{2n}{n+2}} \,dx \Big)^{\frac{n+2}{n}} . \tag{4.30}$$

Hence, by (4.29) and (4.30), for every ball $B_\rho(z)$ with centre $z \in B_R^+(x_0) \cup \Gamma_R(x_0)$ and radius $0 < \rho < R - |x_0 - z|$ we have verified assumption (A.3) in Theorem A.14 for any ball $B_r(y) \cap \partial B_\rho(z) \cap B_R^+(x_0) = \emptyset$. As in the proof of [DGK04], Lemma 3.1, we apply Theorem A.14 with

$$g(x) = \big|DV_\mu(Dv)\big|^{\frac{2n}{n+2}}, \quad p = \tfrac{n+2}{n}, \quad \Omega = B_\rho(z) \cap B_R^+(x_0) \text{ and } A = \partial B_\rho(z) \cap B_R^+(x_0),$$

and we infer that there exists a positive number $\delta = \delta(n,N,p,\frac{L}{\nu})$ such that $|D(V_\mu(Dv))| \in L^{2t}(B_{\rho/2}(z) \cap B_R^+(x_0))$ with

$$\begin{aligned} &\Big(\fint_{B_{\rho/2}(z)\cap B_R^+(x_0)} |D(V_\mu(Dv))|^{2t} \,dx \Big)^{\frac{n}{(n+2)t}} \\ &\qquad \leq c\big(n,N,p,\tfrac{L}{\nu},t\big) \Big(\fint_{B_\rho(z)\cap B_R^+(x_0)} \big|D(V_\mu(Dv))\big|^2 \,dx \Big)^{\frac{n}{n+2}} \end{aligned}$$

for all $t \in [1, 1+\delta)$. Note that the dependence of k_Ω does not occur, as it can be chosen independent of ρ and R (note that every such Ω satisfies a uniform interior and exterior cone-condition). Thus we can choose a number $t_0 > 1$ such that for all $z \in B_R^+(x_0) \cup \Gamma_R(x_0)$ and $0 < \rho < R - |x_0 - z|$ there holds

$$\Big(\fint_{B^+_{\rho/2}(z)} |D(V_\mu(Dv))|^{2t_0}\, dx \Big)^{\frac{1}{t_0}} \leq c\big(n,N,p,\tfrac{L}{\nu}\big) \fint_{B^+_\rho(z)} \big|D(V_\mu(Dv))\big|^2\, dx\,. \tag{4.31}$$

This estimate remains valid if we consider (half-)balls $B^+_{\tilde\rho}(z), B^+_\rho(z)$ with $\tilde\rho < \rho$ instead of $B^+_{\rho/2}(z), B^+_\rho(z)$ (where an additional dependency on the ratio $\frac{\tilde\rho}{\rho}$ occurs) or if we consider cubes instead of balls as mentioned at the beginning of this chapter.

4.3.3 A decay estimate

The previous higher integrability result enables us to estimate $D(V_\mu(Dv))$ on half-balls of different radii and afterwards to deduce a decay estimate for Dv.

Lemma 4.4: *Let $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$ be a weak solution of the system (4.3) under the assumptions (C1)-(C3) and $\mu \in [0,1]$. Then for all $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$, $0 < \rho < R - |x_0 - y|$ and $\tau \in (0,1)$ we have*

$$\int_{B^+_{\tau\rho}(y)} \big|D(V_\mu(Dv))\big|^2\, dx \leq c\,\tau^\varepsilon \int_{B^+_\rho(y)} \big|D(V_\mu(Dv))\big|^2\, dx \tag{4.32}$$

with constants $c = c\big(n,N,p,\frac{L}{\nu}\big)$ and $\varepsilon := n\,(1 - \frac{1}{t_0}) > 0$, where $t_0 = t_0(n,N,p,\frac{L}{\nu}) > 1$ comes from the Gehring-Lemma.

Proof: We argue as follows: if $\tau \in [\frac12, 1)$, the estimate (4.32) is obvious for the constant $c = \tau^{-\varepsilon} \leq 2^\varepsilon \leq 2^n$, whereas in the case $\tau \in (0, \frac12)$ we estimate via Jensen's inequality and the higher integrability estimate (4.31) for $D(V_\mu(Dv))$:

$$\begin{aligned}
\int_{B^+_{\tau\rho}(y)} \big|D(V_\mu(Dv))\big|^2\, dx &\leq \alpha_n\,(\tau\rho)^n \fint_{B^+_{\tau\rho}(y)} \big|D(V_\mu(Dv))\big|^2\, dx \\
&\leq \alpha_n\,(\tau\rho)^n \Big(\fint_{B^+_{\tau\rho}(y)} \big|D(V_\mu(Dv))\big|^{2t_0}\, dx \Big)^{\frac{1}{t_0}} \\
&\leq \alpha_n\,(\tau\rho)^n\,(2\tau)^{-\frac{n}{t_0}} \Big(\fint_{B^+_{\rho/2}(y)} \big|D(V_\mu(Dv))\big|^{2t_0}\, dx \Big)^{\frac{1}{t_0}} \\
&\leq c\big(n,N,p,\tfrac{L}{\nu}\big)\,\alpha_n\,\rho^n\,\tau^{n - \frac{n}{t_0}} \fint_{B^+_\rho(y)} \big|D(V_\mu(Dv))\big|^2\, dx \\
&= c\big(n,N,p,\tfrac{L}{\nu}\big)\,\tau^\varepsilon \int_{B^+_\rho(y)} \big|D(V_\mu(Dv))\big|^2\, dx\,,
\end{aligned}$$

where α_n denotes the $\mathcal{L}^n$-measure of the unit ball in $\mathbb{R}^n$. □

In the next step, the last result for $D(V_\mu(Dv))$ is carried over to an estimate for $V_\mu(Dv)$:

Lemma 4.5: *Let $v \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$ be a weak solution of the system (4.3) under the assumptions (C1)-(C3) and $\mu \in [0,1]$. Then for every $B^+_\rho(y) \subset B^+_R(x_0)$ with $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $0 < \rho < R - |x_0 - y|$ and for all $\tau \in (0,1)$ we have*

$$\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv)\big|^2\, dx \leq c\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big|V_\mu(Dv)\big|^2\, dx \tag{4.33}$$

with $\gamma_0 = \min\{2+\varepsilon, n\}$ (with the definition of ε given in the previous lemma). Furthermore, we have the estimate

$$\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{\tau\rho}(y)}\big|^2\,dx \,\le\, c\,\tau^{2+\varepsilon} \int_{B^+_{\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx\,, \tag{4.34}$$

and both constants c depend only on n, N, p and $\frac{L}{\nu}$.

Proof: This result in (4.33) is achieved in a similar way as in the proof of [Cam87b, Theorem 3.I], where the corresponding estimate was shown for the interior situation in the superquadratic case. Note that our function V is called W in Campanato's paper. In the following, we will consider points $y \in B^+_R(x_0)\cup\Gamma_R(x_0)$ and radii $0<\rho<R-|x_0-y|$. We first use the usual Poincaré inequality, the last Lemma 4.4 and the Caccioppoli-type inequalities in Theorem 4.3, and we obtain for every $\tau\in(0,\frac12)$

$$\begin{aligned}
\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{\tau\rho}(y)}\big|^2\,dx &\le c\,(\tau\rho)^2 \int_{B^+_{\tau\rho}(y)} \big|D(V_\mu(Dv))\big|^2\,dx\\
&\le c\,\tau^{2+\varepsilon}\,\rho^2 \int_{B^+_{\rho/2}(y)} \big|D(V_\mu(Dv))\big|^2\,dx\\
&\le c\big(n,N,p,\tfrac{L}{\nu}\big)\,\tau^{2+\varepsilon} \int_{B^+_{\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx\,.
\end{aligned}$$

This is exactly the inequality given in (4.34) (otherwise if $\frac12\le\tau\le1$, the inequality is trivial, see below). Choosing ε possibly smaller, we may assume $\varepsilon\neq n-2$ and only distinguish the cases $0<\varepsilon<n-2$ and $n-2<\varepsilon<n$. In the first case, in view of Jensen's inequality there holds for all τ, t with $0<\tau<t<\frac12$:

$$\begin{aligned}
&\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx\\
&\quad\le 2\alpha_n\,(\tau\rho)^n\big|\big(V_\mu(Dv)\big)_{B^+_{t\rho}(y)}\big|^2 + 2\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{t\rho}(y)}\big|^2\,dx\\
&\quad\le 4\Big(\frac{\tau}{t}\Big)^n \int_{B^+_{t\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx + 2\int_{B^+_{t\rho}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{t\rho}(y)}\big|^2\,dx\\
&\quad\le 4\Big(\frac{\tau}{t}\Big)^n \int_{B^+_{t\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx + c\,t^{2+\varepsilon} \int_{B^+_{\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx\,,
\end{aligned}$$

where we have used (4.34) in the last line (with τ replaced by t), and the constant c depends only on n, N, p and $\frac{L}{\nu}$. Since we have $n>2+\varepsilon$, the technical Lemma A.11 then yields

$$\int_{B^+_{\tau\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx \,\le\, c\,\Big[\Big(\frac{\tau}{t}\Big)^{2+\varepsilon} + \tau^{2+\varepsilon}\Big] \int_{B^+_{\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx\,.$$

Taking the limit $t\to\frac12$, we obtain the desired inequality in the case $0<\tau<\frac12$, and the constant c still depends only on n, N, p and $\frac{L}{\nu}$. Otherwise, if $\frac12\le\tau<1$, the inequality in (4.33) holds trivially true for the constant $c = 2^{2+\varepsilon} = 2^{\gamma_0}$. This proves (4.33) in the case $0<\varepsilon<n-2$.

If, on the contrary, we consider the case $n-2<\varepsilon<n$, we see as above that

$$\int_{B^+_{t\rho}(y)} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{t\rho}(y)}\big|^2\,dx \,\le\, c\big(n,N,p,\tfrac{L}{\nu}\big)\,t^{2+\varepsilon} \int_{B^+_{\rho}(y)} \big|V_\mu(Dv)\big|^2\,dx \tag{4.35}$$

for all $t \in (0,1)$. Hence, by definition of the Campanato spaces (see Section 2.2) the last estimate implies that the map $V_\mu(Dv)$ belongs to the Campanato space $\mathcal{L}^{2,2+\varepsilon}(B^+_{R-\delta}(x_0),\mathbb{R}^N)$ for every $\delta > 0$ (note that the supremum defining the Campanato norm might blow up for points $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ with $|y-x_0| \nearrow R$). Thus, via the isomorphy of Campanato spaces and Hölder spaces given in Theorem 2.1, we conclude $V_\mu(Dv) \in C^{0,\alpha}(B^+_R(x_0)\cup\Gamma_R(x_0),\mathbb{R}^N)$ with Hölder exponent $\alpha = \frac{2+\varepsilon-n}{2} = 1 - \frac{n-\varepsilon}{2}$. Furthermore, for all balls $B^+_\rho(y) \subset B^+_R(x_0)$ considered above the Hölder norm of $V_\mu(Dv)$ and in particular its supremum norm on $\overline{B^+_{\rho/2}(y)}$ is bounded by the norm in the Campanato space on $B^+_{\rho/2}(y)$ (for the dependency on the radius, we use a rescaling argument); more precisely, we have the following estimate:

$$
\begin{aligned}
&\|V_\mu(Dv)\|^2_{\infty,\overline{B^+_{\rho/2}(y)}} \\
&\quad\leq c(\varepsilon,n)\,\rho^{-n}\Big(\|V_\mu(Dv)\|^2_{L^2(B^+_{\rho/2}(y),\mathbb{R}^N)} + \rho^{2+\varepsilon}\big[V_\mu(Dv)\big]^2_{\mathcal{L}^{2,2+\varepsilon}(B^+_{\rho/2}(y),\mathbb{R}^N)}\Big) \\
&\quad\leq c\big(n,N,\tfrac{L}{\nu}\big)\,\rho^{-n}\Big(\int_{B^+_{\rho/2}(y)}\big|V_\mu(Dv)\big|^2\,dx \\
&\qquad\quad + \sup_{\widetilde{y}\in B^+_{\rho/2}(y),0<\widetilde{\rho}\leq\rho}\Big(\frac{\widetilde{\rho}}{\rho}\Big)^{-2-\varepsilon}\int_{B_{\widetilde{\rho}}(\widetilde{y})\cap B^+_{\rho/2}(y)}\big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B_{\widetilde{\rho}}(\widetilde{y})\cap B^+_{\rho/2}(y)}\big|^2\,dx\Big) \\
&\quad\leq c\,\rho^{-n}\Big(\int_{B^+_\rho(y)}\big|V_\mu(Dv)\big|^2\,dx \\
&\qquad\quad + \sup_{\widetilde{y}\in B^+_{\rho/2}(y),0<\widetilde{\rho}\leq\rho-|\widetilde{y}-y|}\Big(\frac{\widetilde{\rho}}{\rho}\Big)^{-2-\varepsilon}\int_{B^+_{\widetilde{\rho}}(\widetilde{y})}\big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B_{\widetilde{\rho}}(\widetilde{y})}\big|^2\,dx\Big),
\end{aligned}
$$

where the radius $\widetilde{\rho}$ in the latter supremum is restricted to $\rho - |\widetilde{y}-y|$ because for every radius $\widetilde{\rho} \geq \rho - |\widetilde{y}-y| \geq \frac{\rho}{2}$ we have the following "monotonicity" estimate:

$$
\begin{aligned}
&\widetilde{\rho}^{-2-\varepsilon}\int_{B_{\widetilde{\rho}}(\widetilde{y})\cap B^+_{\rho/2}(y)}\big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B_{\widetilde{\rho}}(\widetilde{y})\cap B^+_{\rho/2}(y)}\big|^2\,dx \\
&\qquad\leq \widetilde{\rho}^{-2-\varepsilon}\int_{B^+_{\rho/2}(y)}\big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{\rho/2}(y)}\big|^2\,dx \\
&\qquad\leq \Big(\frac{\rho}{2}\Big)^{-2-\varepsilon}\int_{B^+_{\rho/2}(y)}\big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{B^+_{\rho/2}(y)}\big|^2\,dx\,.
\end{aligned}
$$

Thus, taking into account $\rho - |\widetilde{y}-y| \geq \frac{\rho}{2}$, we continue estimating the supremum of $V_\mu(Dv)$ using (4.35), and we finally arrive at

$$
\|V_\mu(Dv)\|^2_{\infty,\overline{B^+_{\rho/2}(y)}} \leq c\,\rho^{-n}\int_{B^+_\rho(y)}\big|V_\mu(Dv)\big|^2\,dx\,,
$$

where the constant c depends only on n, N, p and $\frac{L}{\nu}$. Then we have for all $0 < \tau < \frac{1}{2}$:

$$
\begin{aligned}
\int_{B^+_{\tau\rho}(y)}\big|V_\mu(Dv)\big|^2\,dx &\leq \alpha_n\,(\tau\rho)^n\,\|V_\mu(Dv)\|^2_{\infty,\overline{B^+_{\rho/2}(y)}} \\
&\leq c\big(n,N,p,\tfrac{L}{\nu}\big)\,\tau^n\int_{B^+_\rho(y)}\big|V_\mu(Dv)\big|^2\,dx\,.
\end{aligned}
$$

For $\frac{1}{2} \leq \tau < 1$ the last estimate holds true using the constant $c = 2^n = 2^{\gamma_0}$. Thus we have demonstrated the inequality (4.33) also in the case $n-2 < \varepsilon < n$ and we have completed the proof. □

We next state two important consequences of Lemma 4.5: first we obtain the following Morrey type decay-estimate:

Corollary 4.6: *Let the assumptions of Lemma 4.5 be satisfied. Then there exists a constant $c = c\big(n, N, p, \frac{L}{\nu}\big)$ independent of v such that for every $B^+_\rho(y) \subset B^+_R(x_0)$ with centre $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and radius $0 < \rho < R - |x_0 - y|$ there holds*

$$\int_{B^+_{\tau\rho}(y)} \big(\mu^p + |Dv|^p\big)\, dx \;\le\; c\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx \qquad \forall\, \tau \in (0,1]\,.$$

Furthermore, we have

$$\int_{B^+_\rho(x_0)} \big(\mu^p + |Dv|^p\big)\, dx \;\le\; c\left(\frac{\rho}{R}\right)^{\gamma_0} \int_{B^+_R(x_0)} \big(\mu^p + |Dv|^p\big)\, dx \qquad \forall\, \rho \in (0,R]\,. \tag{4.36}$$

Proof: Using (4.33) and keeping in mind $\gamma_0 \le n$, we infer these decay estimates for Dv as follows:

$$\begin{aligned}
\int_{B^+_{\tau\rho}(y)} \big(\mu^p + |Dv|^p\big)\, dx &\le 2 \int_{B^+_{\tau\rho}(y)} \big(\mu^p + \big|V_\mu(Dv)\big|^2\big)\, dx \\
&\le 4 \int_{B^+_\rho(y)} \Big[\tau^n \mu^p + c\,\tau^{\gamma_0} \big|V_\mu(Dv)\big|^2\Big]\, dx \\
&\le c\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + \big|V_\mu(Dv)\big|^2\big)\, dx \\
&\le c\big(n, N, p, \tfrac{L}{\nu}\big)\, \tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx\,.
\end{aligned}$$

Taking into account that Lemma 4.5 obviously holds for the choice $y = x_0$ and $\rho = R$, we have immediately the estimate in (4.36). □

As a second consequence we may state the following fundamental estimate which is analogous to [Cam87b, Theorem 1.II] for the superquadratic setting:

Corollary 4.7: *Under the assumptions of Lemma 4.5 there holds: if $n \in [2, p + \gamma_0)$, then for every $B^+_\rho(y) \subset B^+_R(x_0)$ with centre $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and radius $0 < \rho < R - |x_0 - y|$ and for all $\tau \in (0,1)$ there holds*

$$\int_{B^+_{\tau\rho}(y)} |v|^p\, dx \;\le\; c\,\tau^n \Big[\int_{B^+_\rho(y)} |v|^p\, dx + \rho^p \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx\Big] \tag{4.37}$$

with a constant c depending only on n, N, p and $\frac{L}{\nu}$.

Proof: We proceed similarly to [Cam87b, Chapter 4] (for the interior in the superquadratic case $p \ge 2$). We fix $B^+_\rho(y)$ with $y \in B^+_R(x_0) \cup \Gamma_R(x_0)$ and $\rho \in (0, R - |x_0 - y|\,)$. The Sobolev-Poincaré-inequality in Lemma A.5 and Corollary 4.6 yield for $\tau \in (0,1)$:

$$\begin{aligned}
\int_{B^+_{\tau\rho}(y)} |v - (v)_{B^+_{\tau\rho}(y)}|^p\, dx &\le c(n,N,p)\,(\tau\rho)^p \int_{B^+_{\tau\rho}(y)} |Dv|^p\, dx \\
&\le c\big(n, N, p, \tfrac{L}{\nu}\big)\,(\tau\rho)^p\,\tau^{\gamma_0} \int_{B^+_\rho(y)} \big(\mu^p + |Dv|^p\big)\, dx\,.
\end{aligned}$$

Thus, we obtain

$$(\tau\rho)^{-(\gamma_0+p)}\int_{B^+_{\tau\rho}(y)}|v-(v)_{B^+_{\tau\rho}(y)}|^p\,dx\;\le\;c\big(n,N,p,\tfrac{L}{\nu}\big)\,\rho^{-\gamma_0}\int_{B^+_\rho(y)}\big(\mu^p+|Dv|^p\big)\,dx\,.$$

Since the centre $y\in B^+_R(x_0)\cup\Gamma_R(x_0)$ and the radius ρ were chosen arbitrarily, the map v is in the Campanato-space $\mathcal{L}^{p,\gamma_0+p}(B^+_{R-\delta}(x_0),\mathbb{R}^N)$ for every $\delta>0$ (we recall $\gamma_0\le n$). Hence, using Theorem 2.1 we conclude (note $n<\gamma_0+p\le n+p$)

$$v\in C^{0,\alpha}(\overline{B^+_r(x_0)},\mathbb{R}^N)\qquad\text{with }\alpha=1-\frac{n-\gamma_0}{p}\,.$$

Similarly to the proof of Lemma 4.5 we find the estimate

$$\begin{aligned}[v]^p_{C^{0,\alpha}(\overline{B^+_{\rho/2}(x_0)},\mathbb{R}^N)}&\le\;c(n,p,\gamma_0)\,[v]^p_{\mathcal{L}^{p,\gamma_0+p}(B^+_{\rho/2}(y),\mathbb{R}^N)}\\&\le\;c\big(n,N,p,\tfrac{L}{\nu}\big)\,\rho^{-\gamma_0}\int_{B^+_\rho(y)}\big(\mu^p+|Dv|^p\big)\,dx\,.\end{aligned}$$

In particular, there holds for all $x,\widetilde{x}\in B^+_{\rho/2}(y)$

$$|v(\widetilde{x})|^p\;\le\;c(p)\,|v(x)|^p+c(p)\,\rho^{\alpha p}\,[v]^p_{C^{0,\alpha}(B^+_{\rho/2}(y),\mathbb{R}^N)}\,.$$

The point $\widetilde{x}\in B^+_{\rho/2}(y)$ is arbitrary, hence integration with respect to x gives

$$\rho^n\,\|v\|^p_{L^\infty(B^+_{\rho/2}(y),\mathbb{R}^N)}\;\le\;c\Big(\int_{B^+_\rho(y)}|v|^p\,dx+\rho^{n+\alpha p}\rho^{-\gamma_0}\int_{B^+_\rho(y)}\big(\mu^p+|Dv|^p\big)\,dx\Big)\,.$$

Therefore, in dimensions $n<p+\gamma_0=n+\alpha p$ we conclude with $\tau\in(0,\frac12)$

$$\begin{aligned}\int_{B^+_{\tau\rho}(y)}|v|^p\,dx&\le c(n)\,(\tau\rho)^n\,\|v\|^p_{L^\infty(B^+_{\rho/2}(y),\mathbb{R}^N)}\\&\le c\big(n,N,p,\tfrac{L}{\nu}\big)\,\tau^n\Big(\int_{B^+_\rho(y)}|v|^p\,dx+\rho^p\int_{B^+_\rho(y)}\big(\mu^p+|Dv|^p\big)\,dx\Big)\,,\end{aligned}$$

i. e., the desired estimate. Finally if $\tau\in[\frac12,1]$, the estimate trivially holds true with a constant $c=2^n$. □

Remark 4.8: For an appropriate reference estimate in the interior, we consider weak solution in $v\in W^{1,p}(B_R(x_0),\mathbb{R}^N)$, for a centre $x_0\in\mathbb{R}^n$, a radius $R<1$ and $p\in(1,2)$, to the homogeneous system

$$\operatorname{div}a_1(Dv)\;=\;0\qquad\text{in }B_R(x_0)\,.$$

It is easy to see that all estimates achieved above remain true in the interior of $B_R(x_0)$. In particular, the higher integrability estimate (4.31) is valid in this case, i. e., we have for all $y\in B_R(x_0)$ and $0<\rho<R-|x_0-y|$

$$\Big(\fint_{B_{\rho/2}(y)}|D(V_\mu(Dv))|^{2t_0}\,dx\Big)^{\frac{1}{t_0}}\;\le\;c\big(n,N,p,\tfrac{L}{\nu}\big)\fint_{B_\rho(y)}\big|D(V_\mu(Dv))\big|^2\,dx\,.\tag{4.38}$$

Moreover, the interior estimates analogous to the statements in Lemma 4.5 and Corollary 4.7 still hold true. Therefore, the decay estimates in (4.36) hold for balls $B_R(x_0)$ instead of half-balls $B^+_R(x_0)$, i. e., we have

$$\int_{B_\rho(x_0)}\big(\mu^p+|Dv|^p\big)\,dx\;\le\;c\big(n,N,p,\tfrac{L}{\nu}\big)\Big(\frac{\rho}{R}\Big)^{\gamma_0}\int_{B_R(x_0)}\big(\mu^p+|Dv|^p\big)\,dx\qquad\forall\,\rho\in(0,R]\,.$$

Chapter 5

Calderón-Zygmund estimates

Our aim in this chapter is to prove estimates of Calderón-Zygmund-type. For this purpose we consider weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1,2)$, of the inhomogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x, Du) = L\,G(x) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^1 and $L > 0$ is a constant. We further suppose $G \in L^{p/(p-1)}(\Omega, \mathbb{R}^N)$ and $g \in W^{1,p}(\Omega, \mathbb{R}^N)$. The coefficients $a : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are assumed to be continuous with respect to the first variable and of class C^1 with respect to the second variable, satisfying a standard $(p-1)$ growth condition (for the exact structure assumptions see Section 5.1 below). We shall now prove a global higher integrability result of the following form for the gradient Du: there exists a number $\delta > 0$ depending on n, N, p and the structure constants of the system such that

$$G \in L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^N) \text{ and } g \in W^{1,q}(\Omega, \mathbb{R}^N) \quad \Rightarrow \quad Du \in L^q(\Omega, \mathbb{R}^{nN}) \tag{5.1}$$

for all $q < \frac{np}{n-2} + \delta$ (and $n > 2$). This means that, in contrast to the application of Gehring's Lemma where this implication can only be deduced for exponents q "close" to p, we provide a *quantified* gain in the higher integrability exponent (which is bounded from below by $\frac{np}{n-2}$ independently of the structure constants).

To this aim we use a method which is based on Calderón-Zygmund type covering arguments and which was introduced by Caffarelli and Peral in [CP98]. In this paper the authors deduce a similar (interior) higher integrability result for elliptic equations (i. e., scalar-valued solution

u where $N = 1$): In the case of equations one can show by Moser iteration techniques L^∞-estimates for the gradient Dv of the weak solution to the frozen comparison system. This can be used to prove that the statement (5.1), i. e., the Calderón-Zygmund assertion, holds for every exponent $q > 1$. An analogous L^∞-estimate is available for systems with special structure such as the p-Laplacean. In this situation, consequently, the implication (5.1) is obtained without restriction on q, see [Iwa83]. We note that both results may be extended to non-standard $p(x)$-growth (where the function $p(x)$ obeys a quantitative continuity assumption), that is, we consider function u which belong to the generalized Sobolev space $W^{1,p(x)}(\Omega)$ and which are weak solutions of the equation

$$\operatorname{div} a(x, Du) = \operatorname{div}\big(|F|^{p(x)-2} F\big) \qquad \text{in } \Omega$$

(see [AM05], for the linear situation we also refer to [DR03]), or we consider weak solutions in $W^{1,p(x)}(\Omega, \mathbb{R}^N)$ of the non-homogeneous $p(x)$-Laplacean system

$$\operatorname{div}\big(|Du|^{p(x)-2} Du\big) = \operatorname{div}\big(|F|^{p(x)-2} F\big) \qquad \text{in } \Omega$$

(see [AM05]), respectively, for a function $F \in L^{p(x)}(\Omega, \mathbb{R}^{nN})$. Then there holds the implication

$$|F|^{p(x)} \in L^q_{loc}(\Omega) \Rightarrow |Du|^{p(x)} \in L^q_{loc}(\Omega) \quad \text{ for all } q \geq 1\,.$$

For general nonlinear elliptic systems the necessary L^∞ comparison estimates can no longer be expected and hence, some restriction on the exponent q will be required. In fact, taking into account the results of Chapter 4, it is still possible to deduce $Dv \in L^q$ for all $q < \frac{np}{n-2} + \delta$ and some (small) number $\delta > 0$. Then, via a comparison principle and the application of Calderón-Zygmund type estimates on level sets of the Hardy-Littlewood maximal function of $|Du|^p$ and of $|Dg|^p + |G|^{p/(p-1)}$, respectively, this estimate allows us to deduce the desired higher integrability of Du. Here we will follow the strategy of Kristensen and Mingione in [KM06] and extend their results to the subquadratic case. We mention that the literature does not provide appropriate counterexamples to judge the optimality of our restriction on the exponent q (or whether the bound given above is only required due to our method). We note that Habermann [Hab06, Hab08] has presented a local version of the higher integrability result for nonlinear elliptic systems of higher order with non-standard $p(x)$-growth.

Although these results may be considered of interest in their own right we mention their applications. Our results may be employed both in the non-degenerate and the degenerate elliptic case. In Section 7 we will consider weak solutions u to general non-degenerate elliptic systems $-\operatorname{div} a(\cdot, u, Du) = b(\cdot, u, Du)$ in Ω where the right-hand side obeys a controllable growth condition. While u is the fixed solution, we will apply the Calderón-Zygmund estimates to the weak solution u_h of a comparison problem of the form $-\operatorname{div} a_h(\cdot, Du_h) = b(\cdot, u, Du)$; the higher integrability of Du_h (originating from the higher integrability of Du via Gehring's Lemma) will then enable us to find appropriate fractional Sobolev estimates for the weak solution u to the original problem via an iteration procedure. In a second step this yields a suitable upper bound for the Hausdorff dimension of the singular set (see also [DKM07]; for minimizers of general variational integrals we refer to [KM06]), which in turn guarantees the existence of regular boundary points under additional assumptions concerning the Hölder continuity with respect to the (x, u) variables.

5.1 Structure conditions and result

In the sequel we assume that the following structure conditions are satisfied for the coefficients $a : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$: the mapping $z \mapsto a(x,z)$ is a vector field of class $C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus \{0\}, \mathbb{R}^{nN})$, and for fixed numbers $0 < \nu \leq L$, $1 < p < 2$, $\mu \in [0,1]$ and all tuples $(x,z), (\bar{x}, z) \in \Omega \times \mathbb{R}^{nN}$ there hold the following assumptions concerning growth, ellipticity and continuity:

(Z1) a has polynomial growth:

$$|a(x,z)| \,\leq\, L\left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}} ,$$

(Z2) a is differentiable with respect to z with bounded and continuous derivatives:

$$|D_z a(x,z)| \,\leq\, L\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} ,$$

(Z3) a is uniformly elliptic, i. e.,

$$D_z a(x,z)\,\lambda \cdot \lambda \,\geq\, \nu\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \qquad \forall\, \lambda \in \mathbb{R}^{nN} ,$$

(Z4) a is continuous with respect to its first argument, i. e., there exists $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing and continuous with $\omega(0) = 0$ such that

$$|a(x,z) - a(\bar{x},z)| \,\leq\, L\left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}} \omega\big(|x - \bar{x}|\big) .$$

We recall that the choice of the parameter μ specifies whether the system is non-degenerate ($\mu \neq 0$) or degenerate ($\mu = 0$). We note that we have to exclude the case $z = 0$ in conditions (Z2) and (Z3) when dealing with degenerate systems.

The main statement of this chapter is the following

Theorem 5.1: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 and let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a solution of the Dirichlet problem*

$$\begin{cases} -\operatorname{div} a(x, Du) \,=\, L\, G(x) & \text{in } \Omega\,, \\ \qquad\qquad u \,=\, g & \text{on } \partial\Omega\,, \end{cases} \tag{5.2}$$

where the vector field $a : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfies the assumptions (Z1)-(Z4) on Ω and where $g \in W^{1,q}(\Omega, \mathbb{R}^N), G \in L^{q/(p-1)}$ with $q \in [p, s_1]$ and

$$s_1 \,\in\, (p, \infty) \qquad \text{if } n = 2, \qquad \text{and} \qquad s_1 \,=\, \frac{np}{n-2} + \delta_1 \qquad \text{if } n > 2 \tag{5.3}$$

for some $\delta_1 = \delta_1(n, N, p, \frac{L}{\nu}) > 0$. Then there holds

$$\int_\Omega \left(\mu^2 + |Du|^2\right)^{\frac{q}{2}} dx \,\leq\, c \int_\Omega \left(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\right)^{\frac{q}{2}} dx$$

for a constant c depending only on $n, N, p, q, \frac{L}{\nu}, \omega(\cdot)$ and Ω.

In order to obtain these global estimates we proceed in a standard way and start with considering systems of the form

$$-\operatorname{div} a(x, Dg + Du) \,=\, L\, G(x)$$

in the model cases for the interior and the boundary situation, i. e., in the cube Q_{2R} centred at the origin with side length $l(Q_{2R}) = 4R$ or the corresponding upper half-cube Q^+_{2R} under the corresponding assumptions. In the latter case we additionally assume zero-boundary values on Γ_{2R}. In the sequel we will only consider cubes or rectangles with sides parallel to the coordinate axes and we will use the short-hand notation $\gamma Q := Q(x_0, \gamma R)$ for $\gamma > 0$ for cubes Q with side length $2R$ with the analogous definition for the upper cube γQ^+.

5.2 Preliminary results

5.2.1 Higher integrability of the comparison map

We first consider weak solutions $v \in W^{1,p}(3Q, \mathbb{R}^N)$ of systems without x-dependency of the form

$$\operatorname{div} a(Dv) \,=\, 0 \qquad \text{in } 3Q,$$

or weak solutions $v \in W^{1,p}_\Gamma((3Q)^+, \mathbb{R}^N)$ of

$$\operatorname{div} a(Dv) \,=\, 0 \qquad \text{in } (3Q)^+,$$

respectively, and we state an *a priori* estimate. We first remind the higher integrability result from Section 4.3.2, namely that $D(V_\mu(Dv)) \in W^{1,2t_0}$ for some $t_0 > 1$ depending only on n, N, p and $\frac{L}{\nu}$ on all smaller upper half-cubes $(\gamma Q)^+$ with $\gamma < 3$ as well as on all cubes in the interior of $(3Q)^{(+)}$ (denoting either the cube $3Q$ or the upper cube $(3Q)^+$); we note that the exponent t_0 comes from the application of the Gehring Lemma. Then we conclude (see below) the following reverse Hölder inequality for the comparison function v:

$$\Big(\fint_{(2Q)^{(+)}} \big(\mu^2 + |Dv|^2\big)^{\frac{s}{2}}\, dx \Big)^{\frac{1}{s}} \,\le\, c\big(n, N, p, \tfrac{L}{\nu}\big) \Big(\fint_{(3Q)^{(+)}} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}}\, dx \Big)^{\frac{1}{p}}, \tag{5.4}$$

where the exponent s is defined as

$$s \,\in\, (p, \infty) \qquad \text{if } n = 2, \qquad \text{and} \qquad s \,=\, \frac{np}{n-2} + \delta \qquad \text{if } n > 2 \tag{5.5}$$

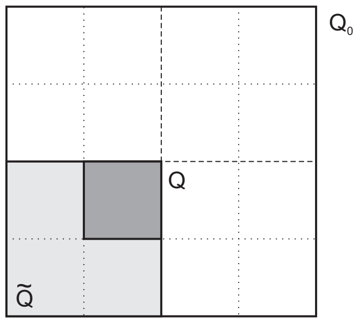
for some $\delta = \delta(n, N, p, \frac{L}{\nu}) > 0$ which can be chosen sufficiently small that $\frac{2s}{p} = \frac{2n}{n-2} + \frac{2\delta}{p} \le \frac{2nt_0}{n-2t_0} = (2t_0)^*$. We mention that δ is independent of the number $\mu \in [0, 1]$, of the particular solution considered and of the vector field $a(\cdot)$. We now prove inequality (5.4) for the cube and the upper half-cube simultaneously: using the Sobolev-Poincaré inequality from Lemma A.5, the higher integrability estimates (4.31) and (4.38) for $D(V_\mu(Dv))$ at the boundary and in the interior, respectively, and the Caccioppoli-inequality in Theorem 4.3 we obtain

$$\begin{aligned}
\Big(\fint_{(2Q)^{(+)}} \big|V_\mu(Dv) - \big(V_\mu(Dv)\big)_{(2Q)^{(+)}}\big|^{\frac{2s}{p}}\, dx \Big)^{\frac{1}{s}} &\le c\, l(Q)^{\frac{2}{p}} \Big(\fint_{(2Q)^{(+)}} \big|D(V_\mu(Dv))\big|^{2t_0}\, dx \Big)^{\frac{1}{pt_0}} \\
&\le c\, l(Q)^{\frac{2}{p}} \Big(\fint_{(2.5Q)^{(+)}} \big|D(V_\mu(Dv))\big|^{2}\, dx \Big)^{\frac{1}{p}} \\
&\le c\big(n, N, p, \tfrac{L}{\nu}\big) \Big(\fint_{(3Q)^{(+)}} \big|V_\mu(Dv)\big|^{2}\, dx \Big)^{\frac{1}{p}}.
\end{aligned}$$

Here, all inequalities are applied on (half-)cubes instead of on (half-)balls. Thus, using $p < 2$ and the definition of the function V, we conclude the desired reverse Hölder inequality (5.4):

$$\begin{aligned}
\Big(\fint_{(2Q)^{(+)}} (\mu^2 + |Dv|^2)^{\frac{s}{2}}\, dx \Big)^{\frac{1}{s}} &\leq \Big(\fint_{(2Q)^{(+)}} \big[(\mu^2 + |Dv|^2)^{\frac{p-2}{2}} |Dv|^2 + \mu^p \big]^{\frac{s}{p}}\, dx \Big)^{\frac{1}{s}} \\
&\leq 2 \Big(\fint_{(2Q)^{(+)}} \big| V_\mu(Dv) - \big(V_\mu(Dv)\big)_{(2Q)^{(+)}} \big|^{\frac{2s}{p}}\, dx \Big)^{\frac{1}{s}} + 2 \big| \big(V_\mu(Dv)\big)_{(2Q)^{(+)}} \big|^{\frac{2}{p}} + \mu \\
&\leq c\big(n, N, p, \tfrac{L}{\nu}\big) \Big(\fint_{(3Q)^{(+)}} \big| V_\mu(Dv) \big|^2\, dx \Big)^{\frac{1}{p}} + \mu \\
&\leq c\big(n, N, p, \tfrac{L}{\nu}\big) \Big(\fint_{(3Q)^{(+)}} (\mu^2 + |Dv|^2)^{\frac{p}{2}}\, dx \Big)^{\frac{1}{p}} .
\end{aligned}$$

5.2.2 Calderón-Zygmund coverings



Let $Q_0 \subset \mathbb{R}^n$ be a cube (centred in some arbitrary point x). By $\mathcal{D}(Q_0)$ we denote the class of all dyadic subcubes of Q_0, i.e., of all cubes with sides parallel to those of Q_0 that have been obtained by a positive, finite number of dyadic subdivisions of Q_0; in particular, Q_0 is not contained in $\mathcal{D}(Q_0)$. We now recall some basic properties of the class $\mathcal{D}(Q_0)$: If $Q_1, Q_2 \in \mathcal{D}(Q_0)$ then either the two cubes are disjoint, $Q_1 \cap Q_2 = \emptyset$, or one of the cubes contains the other one, $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. We call Q_p *a predecessor* of some cube Q if Q has been obtained by a finite number of dyadic subdivisions of Q_p; furthermore, we call $\widetilde{Q}$ *the predecessor* of Q if Q has been obtained by *exactly one* dyadic subdivision of the original cube $\widetilde{Q}$.

To deal with the boundary situation, we will also have to consider Calderón-Zygmund coverings involving rectangles of the form Q_R^+ and using the corresponding family of subrectangles. But this requires only minor modifications, which will be mentioned in the sequel.

We will use the following version of the Calderón-Zygmund decomposition:

Lemma 5.2 (Calderón-Zygmund; [CP98], Lemma 1.1): *Let $Q \subset \mathbb{R}^n$ be a bounded cube and $A \subset Q$ a measurable set satisfying*

$$0 < |A| < \delta\, |Q| \qquad \text{for some } \delta \in (0,1)\,.$$

Then there exists a sequence $\{Q_k\}_{k \in \mathbb{N}}$ of disjoint dyadic subcubes of Q such that there holds:

1. $\big| A \setminus \big(\bigcup Q_k \big) \big| = 0,$
2. $\big| A \cap Q_k \big| > \delta\, \big| Q_k \big|$, *and*
3. $\big| A \cap \bar{Q}_k \big| \leq \delta\, \big| \bar{Q}_k \big|$, *if Q_k is a dyadic subcube of $\bar{Q}_k$.*

The same result holds, if we replace the dyadic cubes by dyadic rectangles.

Proof: We divide Q (or the corresponding rectangle) into 2^n dyadic cubes $\{Q_1^j\}$ and choose those for which

$$\big|Q_1^j \cap A\big| \,>\, \delta\,\big|Q_1^j\big|$$

is satisfied. Now we divide each cube that has not been chosen before again into 2^n dyadic subcubes $\{Q_2^j\}$ and repeat the process above iteratively. Thus we obtain a disjoint sequence of dyadic subcubes called $\{Q_k\}$ for which the assumption 2 and 3 are fulfilled by construction. Now if $x \notin \bigcup_{k\in\mathbb{N}} Q_k$, then there exists a sequence of cubes $\{C_i(x)\}$ with $x \in C_{i+1}(x) \subset C_i(x)$ for all $i \in \mathbb{N}$ and with diameter $\operatorname{diam}(C_i(x)) \to 0$ as $i \to \infty$ such that

$$\big|C_i(x) \cap A\big| \,\leq\, \delta\,\big|C_i(x)\big| \,<\, \big|C_i(x)\big|$$

or, in an equivalent notion, meaning that $⨍_{C_i(x)} \mathbb{1}_A \,dx < 1$, where $\mathbb{1}_A$ denotes the characteristic function of the set A. By Lebesgue's differentiation theorem we have $⨍_{C_i(x)} \mathbb{1}_A \,dx \to \mathbb{1}_A(x)$ as $i \to \infty$ for almost every x. Thus, we conclude that for almost every $x \in Q \setminus \big(\bigcup_{k\in\mathbb{N}} Q_k\big)$ we have $x \in Q \setminus A$, and therefore $|A \setminus \big(\bigcup_{k\in\mathbb{N}} Q_k\big)| = 0$. □

Definition: *A sequence of cubes (or rectangles) with the properties of the Calderón-Zygmund lemma is called a* Calderón-Zygmund covering *for the set A.*

The next lemma, which is a consequence of the previous one, is the key to the proof of the Calderón-Zygmund-type estimates:

Lemma 5.3 ([CP98], Lemma 1.2): *Let $Q_0 \subset \mathbb{R}^n$ be a bounded cube and $\delta \in (0,1)$. Assume that $X \subset Y \subset Q_0$ are measurable sets satisfying the following two conditions:*

(i) $|X| < \delta\,|Q_0|$,

(ii) if $Q \in \mathcal{D}(Q_0)$, then there holds:

$$|X \cap Q| \,>\, \delta\,|Q| \quad \Rightarrow \quad \widetilde{Q} \subset Y\,,$$

where $\widetilde{Q}$ denotes the predecessor of Q. Then we have

$$|X| \,\leq\, \delta\,|Y|\,,$$

The result remains true if we replace the dyadic cubes by dyadic rectangles.

Proof: We consider the Calderón-Zygmund covering for the set X and then we choose a disjoint subcovering by predecessors $\widetilde{Q}_k$ which is denoted by $\{\widetilde{Q}_{k_j}\}_{j\in\mathbb{N}}$. By the last definition, i. e., the construction of the Calderón-Zygmund covering, we have $|X \cap \widetilde{Q}| \leq \delta|\widetilde{Q}|$ for all cubes (or rectangles, respectively) $\widetilde{Q} \in \{\widetilde{Q}_{k_j}\}_{j\in\mathbb{N}}$ but $\big|X \cap Q\big| > \delta\big|Q\big|$ for its successors. Furthermore, by assumption (ii) of the lemma, there holds $\widetilde{Q} \subset Y$ for all $\widetilde{Q} \in \{\widetilde{Q}_{k_j}\}_{j\in\mathbb{N}}$. Hence, we conclude

$$\big|X\big| \,=\, \big|X \cap \big(\bigcup_{k\in\mathbb{N}} Q_k\big)\big| \,\leq\, \sum_{j\in\mathbb{N}} \big|X \cap \widetilde{Q}_{k_j}\big| \,\leq\, \delta \sum_{j\in\mathbb{N}} \big|\widetilde{Q}_{k_j}\big| \,\leq\, \delta\,\big|Y\big|\,. \qquad \square$$

5.2.3 The restricted maximal operator

In order to show Calderón-Zygmund-type estimates, a main tool will be the Hardy Littlewood maximal function restricted on cubes and on rectangles. Let $Q_0 \subset \mathbb{R}^n$ be a cube and $f \in L^1(Q_0)$. Then the restricted maximal operator $M^*_{Q_0}$ relative to Q_0 is defined as

$$M^*_{Q_0}(f)(x) := \sup_{Q \subseteq Q_0, x \in Q} \fint_Q |f(y)|\, dy\,,$$

$x \in Q_0$, where the supremum is taken over all cubes Q contained in Q_0 with sides parallel to those of Q_0 and containing the point x (note: x is not necessarily the centre of Q). At the boundary, we will consider the maximal function restricted to a rectangle $\mathcal{R}_0 \subset \mathbb{R}^n$ with side length $2l_0 > 0$ of the form

$$\mathcal{R}_0 = x_0 + Q^+_{l_0}(0) = x_0 + (-l_0, l_0)^{n-1} \times (0, l_0)\,. \tag{5.6}$$

Now let $f \in L^1(\mathcal{R}_0)$. According to the restricted maximal operator $M^*_{Q_0}$ we define the restricted maximal operator $M^*_{\mathcal{R}_0}$ relative to the rectangle $\mathcal{R}_0$ as

$$M^*_{\mathcal{R}_0}(f)(x) := \sup_{\mathcal{R} \subseteq \mathcal{R}_0, x \in \mathcal{R}} \fint_{\mathcal{R}} |f(y)|\, dy\,,$$

$x \in \mathcal{R}_0$, where the supremum is this time taken analogously over all rectangles $\mathcal{R}$ contained in $\mathcal{R}_0$ (with $\mathcal{R}$ of the same type as $\mathcal{R}_0$) with sides parallel to those of $\mathcal{R}_0$ and containing the point x.

The next lemma provides weak type $(1,1)$ and L^q inequalities for the maximal operator:

Lemma 5.4: *Let Q_0, R_0 as well as the function $M^*_{Q_0}, M^*_{\mathcal{R}_0}$ be defined as above. Then there exists a constant c_w depending only on n and q such that for every function $f \in L^q(Q_0)$, $q \geq 1$, and for all $\lambda > 0$ there holds:*

$$\left|\left\{x \in Q_0 : M^*_{Q_0}(f)(x) > \lambda\right\}\right| \leq \frac{c_w}{\lambda^q} \int_{Q_0} |f(y)|^q\, dy\,.$$

Furthermore, if $q > 1$, we have $M^*_{Q_0}(f) \in L^q(Q_0)$ with

$$\int_{Q_0} \left|M^*_{Q_0}(f)(x)\right|^q dx \leq c(n,q) \int_{Q_0} \left|f(x)\right|^q dx\,.$$

The same estimates hold true, if we replace Q_0 by the rectangle $\mathcal{R}_0$.

Proof: A proof may be recovered from [Ste93, Chapter I.3, Theorem 1]. □

Remark: It is a well known fact that the standard maximal operator is not bounded as a map from L^1 to itself. It is easy to see that this statement also holds for the restricted maximal operator. Moreover, we emphasize that the latter constant $c(n,q)$ might diverge as $q \searrow 1$.

5.3 Local integrability estimates in the interior

We now study the interior situation and consider weak solutions $u \in W^{1,p}(Q_{2R}, \mathbb{R}^N)$ of the inhomogeneous system

$$-\operatorname{div} a(x, Dg + Du) \;=\; L\, G(x) \qquad\qquad \text{in } Q_{2R} \tag{5.7}$$

for functions $g \in W^{1,p}(Q_{2R}, \mathbb{R}^N)$ and $G \in L^{p/(p-1)}(Q_{2R}, \mathbb{R}^N)$. As noted above, the application of the Calderón-Zygmund Lemma 5.3 will be the crucial point in deriving the higher integrability estimates. The following Lemma provides a statement concerning superlevel sets of the maximal function of $|Du|^p$ which will be the central estimate in order to establish condition (ii) in Lemma 5.3 for suitable sets X and Y.

Lemma 5.5: *Let $u \in W^{1,p}(Q_{2R}, \mathbb{R}^N)$ be a weak solution of the Dirichlet problem (5.7) under the assumption (Z1)-(Z4). Let $B > 1$. Then there exists $\varepsilon = \varepsilon\big(n, N, p, \frac{L}{\nu}, B\big) > 0$ and a radius $R_0 = R_0\big(n, N, p, \frac{L}{\nu}, B, \omega(\cdot)\big) > 0$, such that there holds: if $2\sqrt{n}R \le R_0$, $\lambda > 0$ and $Q \subset Q_R$ is a dyadic subcube of Q_R such that*

$$\begin{aligned}\Big|Q \cap \Big\{x \in Q_R : M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) \;>\; AB\lambda\,,\\ M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) \;\le\; \varepsilon\lambda\Big\}\Big| \;>\; B^{-\frac{s}{p}}|Q|\,,\end{aligned} \tag{5.8}$$

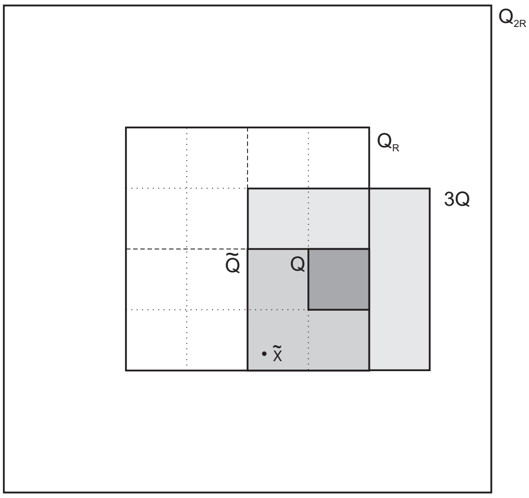
then its predecessor $\widetilde{Q}$ of Q satisfies

$$\widetilde{Q} \;\subseteq\; \big\{x \in Q_R : M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) \;>\; \lambda\big\}\,. \tag{5.9}$$

Here $M^* = M^*_{Q_{2R}}$ denotes the restricted maximal operator relative to Q_{2R}, s is the exponent defined in (5.5) and $A = A\big(n, N, p, \frac{L}{\nu}\big) \ge 2$ is an absolute constant. Furthermore, all constants and quantities involved are uniform with respect to $\mu \in [0, 1]$.

Remark: The superquadratic analogue of this Lemma is [KM06, Lemma 7.3], which was stated in this form only for the homogeneous situation. The inhomogeneity arising on the right-hand side of the system (5.7) now demands a straightforward modification of the statement: in order to be in a position to show the inclusion (5.9), the sublevel sets of the maximal function of both the function $|Dg|^p$ and the inhomogeneity $|G|^{p/(p-1)}$ have to be in a certain sense small. Furthermore, we note that we have included the degenerate case $\mu = 0$. This fact requires to pay attention whenever the system becomes degenerate. Furthermore, since the estimates are based on a comparison principle, the case $\mu = 0$ necessitates degenerate comparison estimates which are provided in Section 4. Lastly, we remark that the degenerate case $\mu = 0$ (as well as the presence of an inhomogeneity) was taken into account in the main higher integrability result [KM06, Lemma 7.8]: due to the uniformity of the preliminary estimates with respect to $\mu > 0$ the application of an approximation argument yields the desired higher integrability also for degenerate superquadratic systems.

Proof: We proceed similarly to [KM06, Lemma 7.3] and prove the lemma via contradiction; the constants A, ε and R_0 will be chosen later.



We suppose that (5.9) is not true although (5.8) is satisfied. Then there exists a point $\tilde{x} \in \widetilde{Q}$ such that

$$M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(\tilde{x}) \le \lambda\,. \tag{5.10}$$

Since $\widetilde{Q}$ is the predecessor of Q, we have in particular $\tilde{x} \in \widetilde{Q} \subset 3Q \subset Q_{2R}$ (note Q is a dyadic subcube of Q_R), and therefore by definition of the restricted maximal operator

$$\fint_{3Q} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\, dx \le \lambda\,. \tag{5.11}$$

Furthermore, we find by (5.8) a point $\bar{x} \in Q$ such that

$$M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(\bar{x}) \le \varepsilon\lambda\,. \tag{5.12}$$

and therefore

$$\fint_{3Q} \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\, dx \le \varepsilon\lambda\,. \tag{5.13}$$

We next define the comparison function $v \in W^{1,p}(3Q, \mathbb{R}^N)$ to be the unique solution of the Dirichlet problem with frozen coefficients $a(x_0, \cdot)$ and boundary values u, i. e., v solves

$$\begin{cases} \operatorname{div} a(x_0, Dv) &= 0 \qquad \text{in } 3Q\,, \\ \qquad u - v &\in W_0^{1,p}(3Q, \mathbb{R}^N)\,, \end{cases} \tag{5.14}$$

where x_0 denotes the centre of Q. The existence of v follows by means of monotone operators (see e.g. [Lio69, Théorème 2.1, page 171]). We first derive the following energy estimate

$$\int_{3Q} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}}\, dx \le c\big(n, N, p, \tfrac{L}{\nu}\big) \int_{3Q} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\, dx\,, \tag{5.15}$$

which states that the p-energy of Dv can be bounded from above by the p-energy of Du. In fact, due to the choice $v = u$ on the boundary, we may test the system $\operatorname{div} a(x_0, Dv) = 0$ with the function $u - v \in W_0^{1,p}(3Q, \mathbb{R}^N)$ to obtain

$$\begin{aligned} 0 &= \int_{3Q} a(x_0, Dv) \cdot (Du - Dv)\, dx \\ &= \int_{3Q} \big[a(x_0, Dv) - a(x_0, 0)\big] \cdot (Du - Dv)\, dx \\ &= \int_{3Q} \int_0^1 D_z a(x_0, tDv)\, dt\, Dv \cdot (Du - Dv)\, dx\,, \end{aligned}$$

where we have used the facts that $a(x_0, 0)$ is constant and that $u = v$ on the boundary of $3Q$. By assumptions (Z2), (Z3) and taking into account $p < 2$ we thus infer

$$\begin{aligned} \nu \int_{3Q} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|^2\, dx &\le \int_{3Q} \int_0^1 D_z a(x_0, tDv)\, dt\, Dv \cdot Dv\, dx \\ &= \int_{3Q} \int_0^1 D_z a(x_0, tDv)\, dt\, Dv \cdot Du\, dx \\ &\le c(p, L) \int_{3Q} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|\, |Du|\, dx\,. \end{aligned}$$

We recall that, when considering degenerate systems ($\mu = 0$), the structure conditions (Z2) and (Z3) must not be applied if $Dv = 0$. However, taking into account the growth of $z \mapsto a(\cdot, z)$ in (Z1), it is easy to see that the term $\int_0^1 D_z a(x_0, tDv)\,dt\,Dv$ vanishes on the set $\{x \in 3Q\colon Dv(x) = 0\}$. As a consequence, the previous inequality holds both for non-degenerate and degenerate systems. Hence, the Young-type inequality in Lemma A.3 (iii) yields

$$\int_{3Q} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|^2\,dx \le c\big(p, \tfrac{L}{\nu}\big) \int_{3Q} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx\,.$$

Distinguishing the cases $|Dv| \le \mu$ and $|Dv| > \mu$, we conclude the energy estimate (5.15) stated above which is independent of $\mu \in [0, 1]$. Since v is a solution of the frozen problem (5.14) where the vector field $a(x_0, \cdot)$ does not depend on x itself, it satisfies the reverse Hölder-type inequality (5.4), which in combination with the energy estimate (5.15) and the assumption (5.11) leads us to

$$\begin{aligned}
\fint_{2Q} \big(\mu^2 + |Dv|^2\big)^{\frac{s}{2}}\,dx &\le c\Big(\fint_{3Q} \big(\mu^2 + |Dv|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{s}{p}} \\
&\le c\Big(\fint_{3Q} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{s}{p}} \\
&\le c\big(n, N, p, \tfrac{L}{\nu}\big)\,\lambda^{\frac{s}{p}}\,. \qquad (5.16)
\end{aligned}$$

In the next step, we compare the weak solution u of the orginal problem to the weak solution v of the comparison problem by testing the original system $\operatorname{div} a(\cdot, Dg + Du) = LG(\cdot)$ as well as the frozen system $\operatorname{div} a(x_0, Dv) = 0$ introduced above with the difference $v - u$. Via the ellipticity condition and Lemma A.2 we obtain

$$\begin{aligned}
&c^{-1}(p)\,\nu \fint_{3Q} \big(\mu^2 + |Du|^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv - Du|^2\,dx \\
&\le \fint_{3Q} \big[a(x_0, Dv) - a(x_0, Du)\big] \cdot (Dv - Du)\,dx \\
&= \fint_{3Q} \big[a(x, Du) - a(x_0, Du)\big] \cdot (Dv - Du)\,dx \\
&\quad + \fint_{3Q} \big[a(x, Dg + Du) - a(x, Du)\big] \cdot (Dv - Du)\,dx - L \fint_{3Q} G \cdot (v - u)\,dx \\
&=: I + II + III \qquad (5.17)
\end{aligned}$$

with the obvious labelling. We next estimate the three terms arising on the right-hand side of (5.17):

Estimate for I: Here, we use that, according to hypothesis (Z4), the coefficients $a(\cdot, \cdot)$ are continuous with respect to the first variable. For all points $x \in 3Q$ we can estimate the distance $|x - x_0|$ in dependency of R and bound it from above by $2\sqrt{n}R \le R_0$. By Young's inequality and the energy inequality (5.15), we then find:

$$\begin{aligned}
|I| &\le L \fint_{3Q} \omega\big(|x - x_0|\big) \big(\mu^2 + |Du|^2\big)^{\frac{p-1}{2}} |Dv - Du|\,dx \\
&\le L \fint_{3Q} \omega\big(|x - x_0|\big) \big(\mu^2 + |Du|^2 + |Dv|^2\big)^{\frac{p}{4}} \big(\mu^2 + |Du|^2 + |Dv|^2\big)^{\frac{p-2}{4}} |Dv - Du|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le c(p,\tfrac{L}{\nu})\,L\,\omega^2(R_0)\fint_{3Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p}{2}}\,dx\\
&\qquad+\tfrac12\,c^{-1}(p)\,\nu\fint_{3Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv-Du|^2\,dx\\
&\le c(n,N,p,\tfrac{L}{\nu})\,L\,\omega^2(R_0)\fint_{3Q}\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx\\
&\qquad+\tfrac12\,c^{-1}(p)\,\nu\fint_{3Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv-Du|^2\,dx\,. \qquad (5.18)
\end{aligned}
$$

Estimate for II: For the second term, we first use assumption (Z2) on the growth of $D_z a(\cdot,z)$ combined with Lemma A.2. To incorporate degenerate systems we follow the arguments above (this time employing the fact that all integrals involving (Z2) vanish on the set $\{x\in 3Q : Dg(x)=0\}$) and we then conclude via Young's inequality, applied with $\tilde\varepsilon\in(0,1)$, and the energy estimate (5.15):

$$
\begin{aligned}
|II| &= \Big|\fint_{3Q}\big[a(x,Dg+Du)-a(x,Du)\big]\cdot(Dv-Du)\,\mathbb{1}_{\{Dg\neq 0\}}\,dx\Big|\\
&\le c(p)\,L\fint_{3Q}\big(\mu^2+|Du|^2+|Dg|^2\big)^{\frac{p-2}{2}}|Dg|\,|Dv-Du|\,dx\\
&\le \tilde\varepsilon\,L\fint_{3Q}|Dv-Du|^p\,dx+c(p)\,L\,\tilde\varepsilon^{1-p}\fint_{3Q}\Big[\big(\mu^2+|Du|^2+|Dg|^2\big)^{\frac{p-2}{2}}|Dg|\Big]^{\frac{p}{p-1}}\,dx\\
&\le c\,L\,\tilde\varepsilon\fint_{3Q}\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx+c\,L\,\tilde\varepsilon^{1-p}\fint_{3Q}\big(\mu^2+|Dg|^2\big)^{\frac{p}{2}}\,dx\,, \qquad (5.19)
\end{aligned}
$$

and the constant c depends only on n,N,p and $\frac{L}{\nu}$.

Estimate for III: For the last integral, we apply Young's inequality, the Poincaré inequality (note $R_0\le 1$) and (5.15) to find

$$
\begin{aligned}
|III| &\le L\fint_{3Q}|G|\,|v-u|\,dx\\
&\le \tilde\varepsilon\,c(n,N,p)\,L\fint_{3Q}|Dv-Du|^p\,dx+\tilde\varepsilon^{1-p}\,L\fint_{3Q}|G|^{\frac{p}{p-1}}\,dx\\
&\le c\big(n,N,p,\tfrac{L}{\nu}\big)\,L\,\tilde\varepsilon\fint_{3Q}\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx+\tilde\varepsilon^{1-p}\,L\fint_{3Q}\big(\mu^2+|G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx\,. \qquad (5.20)
\end{aligned}
$$

Altogether, we combine the decomposition in (5.17) with the estimates (5.18)-(5.20) to deduce

$$
\begin{aligned}
&\fint_{3Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv-Du|^2\,dx\\
&\qquad\le c\,\big(\omega^2(R_0)+\tilde\varepsilon\big)\fint_{3Q}\big(\mu^2+|Du|^2\big)^{\frac{p}{2}}\,dx+c\,\tilde\varepsilon^{1-p}\fint_{3Q}\big(\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx\,.
\end{aligned}
$$

Via (5.11) and (5.13) we finally arrive at

$$
\fint_{3Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Dv-Du|^2\,dx\;\le\;c\,\big[\omega^2(R_0)+\tilde\varepsilon+c(\tilde\varepsilon)\,\varepsilon\big]\,\lambda\,, \qquad (5.21)
$$

where the constant c depends only on n,N,p and $\frac{L}{\nu}$. In the next step, we will apply this comparison estimate as follows: we introduce the restricted maximal operator M^{**} relative to the reduced cube $2Q$. The next aim is to gain control over $M^{**}\big((\mu^2+|Du|^2)^{p/2}\big)$ on Q. Together with assumption (5.10) this will provide the desired contradiction to (5.8).

By Lemma A.3 we have

$$\big(\mu^2+|Du|^2\big)^{\frac{p}{2}} \le c\,\big(\mu^2+|Dv|^2\big)^{\frac{p}{2}} + c\,\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Du-Dv|^2$$

with a constant c depending only on p. Thus we conclude via the weak-type estimate in Lemma 5.4 that (note: due to (5.4) Dv is integrable with exponent s):

$$\begin{aligned}
&\Big|\Big\{x\in Q : M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > AB\lambda\Big\}\Big| \\
&\le \Big|\Big\{x\in Q : M^{**}\big((\mu^2+|Dv|^2)^{\frac{p}{2}}\big)(x) > \frac{AB\lambda}{2c_1}\Big\}\Big| \\
&\qquad + \Big|\Big\{x\in Q : M^{**}\big((\mu^2+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\big)(x) > \frac{AB\lambda}{2c_2}\Big\}\Big| \\
&\le \frac{c(n,s,p)}{(AB\lambda)^{s/p}}\int_{2Q}\big(\mu^2+|Dv|^2\big)^{\frac{s}{2}}\,dx + \frac{c(n,p)}{AB\lambda}\int_{2Q}\big(\mu^2+|Du|^2+|Dv|^2\big)^{\frac{p-2}{2}}|Du-Dv|^2\,dx \\
&=: I_M + II_M\,. \qquad (5.22)
\end{aligned}$$

The first integral on the right-hand side is estimated by (5.16)

$$\begin{aligned}
I_M &\le \frac{c}{(AB\lambda)^{s/p}}\,|2Q|\,\lambda^{\frac{s}{p}} \\
&= c_I\big(n,N,p,s,\tfrac{L}{\nu}\big)\,\frac{1}{(AB)^{s/p}}\,|Q| \le \frac{1}{8^{n+1}B^{s/p}}\,|Q|\,, \qquad (5.23)
\end{aligned}$$

where the last inequality is true provided that we have chosen A large enough, for instance $A := \max\{(8^{n+1}c_I)^{p/s},2\}$. This fixes the constant $A\ge 2$ in dependency of n,N,p and $\frac{L}{\nu}$ since the higher integrability exponent s is expressed in terms of the same quantities and all constants c are assumed to be greater than or equal to 1. For the second integral II_M we apply (5.21) and thus conclude that

$$\begin{aligned}
&\Big|\Big\{x\in Q : M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > AB\lambda\Big\}\Big| \\
&\qquad \le \frac{|Q|}{8^{n+1}B^{s/p}} + c_{II}\big(n,N,p,\tfrac{L}{\nu}\big)\,\frac{|Q|}{AB}\big[\omega^2(R_0)+\tilde\varepsilon + c(\tilde\varepsilon)\,\varepsilon\big]\,. \qquad (5.24)
\end{aligned}$$

Now we choose $R_0 = R_0(n,N,p,\frac{L}{\nu},B,\omega(\cdot))$ and $\tilde\varepsilon = \tilde\varepsilon(n,N,p,\frac{L}{\nu},B)$ sufficiently small such that

$$c_{II}\,\frac{\omega^2(R_0)}{AB} \le \frac{1}{8B^{s/p}} \qquad \text{and} \qquad c_{II}\,\frac{\tilde\varepsilon}{AB} \le \frac{1}{8B^{s/p}} \qquad (5.25)$$

is satisfied. Once $\tilde\varepsilon$ is determined, we can take $\varepsilon > 0$ depending on $n,N,p,\frac{L}{\nu}$ and B such that

$$c_{II}\,c(\tilde\varepsilon)\,\frac{\varepsilon}{AB} \le \frac{1}{8B^{s/p}}\,. \qquad (5.26)$$

Combining (5.25) and (5.26) with (5.24) we thus observe

$$\Big|\Big\{x\in Q : M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > AB\lambda\Big\}\Big| \le \frac{|Q|}{B^{s/p}}\Big(\frac{1}{8^{n+1}}+\frac{3}{8}\Big) \le \frac{|Q|}{2B^{s/p}}\,. \qquad (5.27)$$

It remains to show that this estimate for the restricted maximal function relative to the reduced cube combined with (5.10) suffices to control $M^*((\mu^2+|Du|^2)^{p/2})$. More precisely, we are going to calculate:

$$M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) \;\le\; \max\big\{M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x), 5^n\lambda\big\} \tag{5.28}$$

for every $x \in Q$. At this stage we recall that M^* and M^{**} denote the restricted maximal operators relative to Q_{2R} and relative to $2Q$, respectively. To prove the last inequality we consider an arbitrary point $y \in Q$ and a cube $C \subset Q_{2R}$ containing y. Then we have to distinguish:

Case $C \subset 2Q$: By the definition of M^{**} there holds

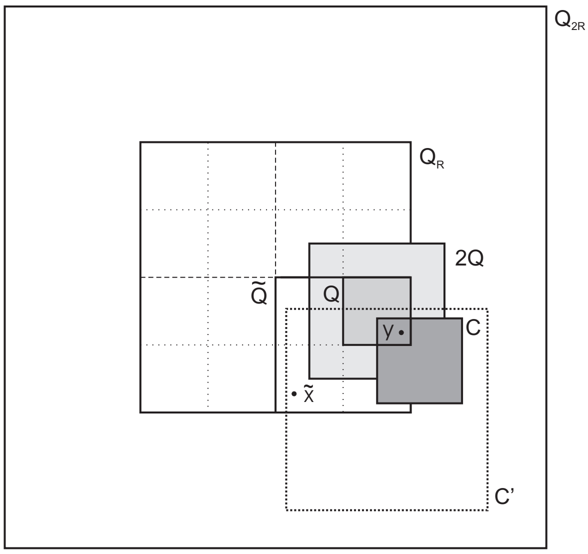
$$⨍_C (\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \;\le\; M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(y)\,.$$

Case $C \not\subset 2Q$: We have $C \setminus (2Q) \neq \emptyset$. In view of the fact that $y \in Q$ this implies the following inequality for the side lengths of the cubes:

$$l(C) \ge \tfrac{1}{2} l(Q)$$

(for illustration of this situation see the figure on the right). Then we may find a cube $C' \subset Q_{2R}$ containing the original cube C and the point $\tilde{x}$, where $\tilde{x} \in \widetilde{Q}$ is the point, for which the assumption (5.10) holds. Additionally, we require that the side length of C' is bounded by

$$l(C') \;\le\; 2l(Q) + l(C)\,.$$



A possible configuration for $C \not\subset 2Q$

Then we obtain with (5.10)

$$\begin{aligned} ⨍_C (\mu^2+|Du|^2)^{\frac{p}{2}}\,dx &\le \frac{(2l(Q)+l(C))^n}{(l(C))^n} ⨍_{C',\,\tilde{x}\in C'} (\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\ &\le 5^n\, M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(\tilde{x}) \;\le\; 5^n\lambda\,. \end{aligned}$$

Combined with the first case this implies (5.28).

Since $AB \ge A > 8^n$, we observe from (5.27) and (5.28):

$$\begin{aligned} &\Big|\Big\{x \in Q : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) \;>\; AB\lambda\Big\}\Big| \\ &\quad\le \Big|\Big\{x \in Q : \max\big\{M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x), 5^n\lambda\big\} \;>\; AB\lambda\Big\}\Big| \\ &\quad= \Big|\Big\{x \in Q : M^{**}\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) \;>\; AB\lambda\Big\}\Big| \;\le\; \frac{|Q|}{2B^{s/p}}\,, \end{aligned}$$

which is a contradiction to (5.8) and hence completes the proof of the lemma. □

Remark 5.6: In order to apply the previous lemma, we still need to fix the constant B, depending on the integrability exponent $q \in (p, s)$. For fixed q we choose B in a canonical way such that

$$B^{\frac{q-s}{p}} = \tfrac{1}{2} A^{-\frac{q}{p}} \tag{5.29}$$

is satisfied. Since the constant A depends only on n, N, p and $\frac{L}{\nu}$, this fixes the constant B in dependency of $n, N, p, \frac{L}{\nu}$ and $s - q$. We note that B diverges if $q \nearrow s$. Keeping in mind that R_0 and ε were chosen sufficiently small such that the inequalities in (5.25) and (5.26) are satisfied we further observe that R_0 and ε tend to zero if $q \nearrow s$. The choice of B in turn provides the following dependencies for the quantities involved in Lemma 5.5:

$$\begin{aligned} R_0 &= R_0\big(n, N, p, \tfrac{L}{\nu}, \omega(\cdot), s-q\big) \quad \text{and} \\ \varepsilon_0 := \varepsilon &= \varepsilon\big(n, N, p, \tfrac{L}{\nu}, s-q\big)\,. \end{aligned}$$

In the next lemma, we apply Lemma 5.5 on iterated level sets of the (restricted) maximal function to obtain an interior reverse Hölder inequality for weak solutions u of system (5.7):

Lemma 5.7: *Let $u \in W^{1,p}(Q_{2R}, \mathbb{R}^N)$ be a weak solution of (5.7) under the assumptions (Z1)-(Z4) with $2\sqrt{n}R \le R_0$, where R_0 is the radius according to the remark above, and let $\mu \in [0,1]$. For every exponent $q \in (p, s)$ there exists a constant c depending only on $n, N, p, \frac{L}{\nu}$ and $s - q$, such that there holds:*

$$\begin{aligned} &\Big(\fint_{Q_R} (\mu^2 + |Du|^2)^{\frac{q}{2}}\, dx \Big)^{\frac{1}{q}} \\ &\quad \le c \Big(\fint_{Q_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx \Big)^{\frac{1}{p}} + c \Big(\fint_{Q_{2R}} \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx \Big)^{\frac{1}{q}}. \end{aligned} \tag{5.30}$$

Proof: Without loss of generality we may assume $|Du| \not\equiv 0$ on Q_R, $g \in W^{1,q}(Q_{2R}, \mathbb{R}^N)$ and $G \in L^{q/(p-1)}(Q_{2R}, \mathbb{R}^N)$, otherwise estimate (5.30) is trivially satisfied. We use again the notation M^* for the restricted maximal operator relative to the cube Q_{2R}, and we define

$$\begin{aligned} \mu_1(t) &:= \Big|\big\{x \in Q_R : M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) > t\big\}\Big|\,, \\ \mu_2(t) &:= \Big|\big\{x \in Q_R : M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) > t\big\}\Big|\,. \end{aligned}$$

Then, with the parameter $B \ge 1$ defined in (5.29), we set:

$$\lambda_0 := 5^{n+2} c_w(n)\, B^{\frac{s}{p}} \fint_{Q_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx\,,$$

where c_w is the constant appearing in the weak L^1-estimate from Lemma 5.4; in particular, we see that λ_0 is positive. By Lemma 5.4 and the definition of λ_0 we find

$$\begin{aligned} \mu_1(\lambda_0) &\le \frac{c_w}{\lambda_0} \fint_{Q_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, dx\, |Q_{2R}| \\ &= \frac{2^n |Q_R|}{5^{n+2} B^{s/p}} < \frac{|Q_R|}{2B^{s/p}}\,. \end{aligned} \tag{5.31}$$

Since $A, B > 1$, we have in particular $AB > 1$ and thus we also obtain from the last inequality that for all $k \in \mathbb{N}_0$ the inequality

$$\mu_1\big((AB)^k \lambda_0\big) < \frac{|Q_R|}{2B^{s/p}} \tag{5.32}$$

is fulfilled, where A is the constant appearing in Lemma 5.5. We next show by induction that for every $k \in \mathbb{N}_0$ there holds

$$\mu_1\big((AB)^{k+1}\lambda_0\big) \leq B^{-\frac{s}{p}(k+1)}\mu_1\big(\lambda_0\big) + \sum_{i=0}^{k} B^{-\frac{s}{p}(k-i)}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)\,, \tag{5.33}$$

where ε_0 is chosen according to the previous remark. To prove (5.33) we define

$$\begin{aligned} X &:= \big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > (AB)^{k+1}\lambda_0\,, \\ &\qquad\qquad M^*\big((\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) \leq \varepsilon_0(AB)^k\lambda_0\big\} \\ Y &:= \big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > (AB)^k\lambda_0\big\}\,, \\ \delta &:= B^{-\frac{s}{p}}\,, \end{aligned}$$

and we show that both assumptions of Lemma 5.3 are satisfied on the cube Q_R:

- from (5.32) we see: $|X| \leq \mu_1((AB)^{k+1}\lambda_0) \leq \frac{|Q_R|}{2B^{s/p}} < \delta|Q_R|$.
- We consider the levels $\lambda := (AB)^k\lambda_0$ and assume that for a dyadic subcube $Q \in \mathcal{D}(Q_R)$ we have

$$\begin{aligned} \big|X \cap Q\big| &= \big|Q \cap \big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > AB\lambda\,, \\ &\qquad M^*\big((\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) \leq \varepsilon_0\lambda\big\}\big| > \delta\,|Q| = B^{-\frac{s}{p}}|Q|\,. \end{aligned}$$

  Then, according to Lemma 5.5, the predecessor $\widetilde{Q}$ of Q satisfies

$$\widetilde{Q} \subseteq \big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > \lambda\big\} = Y\,.$$

Thus, we may apply the Calderón-Zygmund Lemma 5.3 to conclude $|X| \leq \delta|Y|$, which transforms into

$$\begin{aligned} &\big|\big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > (AB)^{k+1}\lambda_0\big\}\big| \\ &\quad - \big|\big\{x \in Q_R : M^*\big((\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) > \varepsilon_0(AB)^k\lambda_0\big\}\big| \\ &\qquad \leq \delta\,\big|\big\{x \in Q_R : M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)(x) > (AB)^k\lambda_0\big\}\big|\,. \end{aligned}$$

Due to the definition of μ_1, μ_2 and δ, this is equivalent to the inequality

$$\mu_1\big((AB)^{k+1}\lambda_0\big) \leq B^{-\frac{s}{p}}\mu_1\big((AB)^k\lambda_0\big) + \mu_2\big((AB)^k\varepsilon_0\lambda_0\big)$$

for all $k \in \mathbb{N}_0$. Applying this inequality iteratively, we obtain the desired estimate (5.33) as follows:

$$\begin{aligned} \mu_1\big((AB)^{k+1}\lambda_0\big) &\leq B^{-2\frac{s}{p}}\mu_1\big((AB)^{k-1}\lambda_0\big) + B^{-\frac{s}{p}}\mu_2\big((AB)^{k-1}\varepsilon_0\lambda_0\big) + \mu_2\big((AB)^k\varepsilon_0\lambda_0\big) \\ &\leq \ldots \leq B^{-\frac{s}{p}(k+1)}\mu_1\big(\lambda_0\big) + \sum_{i=0}^{k} B^{-\frac{s}{p}(k-i)}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)\,. \end{aligned}$$

Summing up over k we infer for any $M \in \mathbb{N}$:

$$\sum_{k=0}^{M}(AB)^{\frac{q}{p}(k+1)}\mu_1\big((AB)^{k+1}\lambda_0\big) \le \sum_{k=0}^{M}(AB)^{\frac{q}{p}(k+1)}B^{-\frac{s}{p}(k+1)}\mu_1(\lambda_0) + \sum_{k=0}^{M}\sum_{i=0}^{k}(AB)^{\frac{q}{p}(k+1)}B^{-\frac{s}{p}(k-i)}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big) . \qquad (5.34)$$

To evaluate the right-hand side of the last inequality we notice that the choice of B in (5.29) provides:

$$\sum_{k=0}^{\infty}\big[(AB)^{\frac{q}{p}}B^{-\frac{s}{p}}\big]^{k+1} = \sum_{k=0}^{\infty}\big[A^{\frac{q}{p}}B^{\frac{q-s}{p}}\big]^{k+1} = \sum_{k=0}^{\infty}2^{-(k+1)} = 1 . \qquad (5.35)$$

Thus, interchanging the order of summation in the second term on the right-hand side of (5.34) we get:

$$\begin{aligned}
&\sum_{k=0}^{M}\sum_{i=0}^{k}(AB)^{\frac{q}{p}(k+1)}B^{-\frac{s}{p}(k-i)}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)\\
&\quad = (AB)^{\frac{q}{p}}\sum_{i=0}^{M}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)\sum_{k=i}^{M}(AB)^{\frac{q}{p}k}B^{-\frac{s}{p}(k-i)}\\
&\quad = (AB)^{\frac{q}{p}}\sum_{i=0}^{M}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)(AB)^{\frac{q}{p}i}\sum_{m=0}^{M-i}[(AB)^{\frac{q}{p}}B^{-\frac{s}{p}}]^m\\
&\quad \le 2(AB)^{\frac{q}{p}}\sum_{i=0}^{M}\mu_2\big((AB)^i\varepsilon_0\lambda_0\big)(AB)^{\frac{q}{p}i} .
\end{aligned}$$

Inserting the last two estimates in (5.34) and passing to the limit $M \to \infty$ we finally arrive at (with $k \mapsto k-1$ on the left-hand side):

$$\sum_{k=1}^{\infty}(AB)^{\frac{q}{p}k}\mu_1\big((AB)^k\lambda_0\big) \le \mu_1\big(\lambda_0\big) + 2(AB)^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{\frac{q}{p}k}\mu_2\big((AB)^k\varepsilon_0\lambda_0\big) . \qquad (5.36)$$

In order to conclude the higher integrability result (5.30) we will proceed as follows: the previous estimate (5.36) will next be used to control the $L^{q/p}$-norm of the restricted maximal operator $M^*\big((\mu^2+|Du|^2)^{p/2}\big)$. In the second step we then show that the two terms in (5.36) in turn are controlled by $\|(\mu^2+|Du|^2)^{p/2}\|_{L^1}$ and $\|M^*\big((\mu^2+|Dg|^2+|G|^{2/(p-1)})^{p/2}\big)\|_{L^{q/p}}$. For these computations, we make use of the following identity:

$$\int_Q |f|^{\tilde p}\,dx = \int_0^\infty \tilde p\,\lambda^{\tilde p-1}\big|\{x\in Q : |f(x)|>\lambda\}\big|\,d\lambda \qquad \forall f\in L^{\tilde p}(Q),\ \tilde p\ge 1 ,$$

i. e., the decomposition of the integral of $|f|^{\tilde p}$ into levelsets. This yields

$$\begin{aligned}
\int_{Q_R}\big[M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx &= \int_0^\infty \tfrac{q}{p}\,\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda\\
&= \int_0^{\lambda_0}\tfrac{q}{p}\,\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda + \int_{\lambda_0}^\infty \tfrac{q}{p}\,\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda\\
&=: I_{\lambda_0} + II_{\lambda_0} .
\end{aligned}$$

Using $\mu_1(\lambda) \leq |Q_R|$ we conclude for the first term

$$I_{\lambda_0} \leq |Q_R| \int_0^{\lambda_0} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \, d\lambda \, = \, |Q_R| \, \lambda_0^{\frac{q}{p}} \, .$$

The second integral II_{λ_0} is decomposed into integrals on intervals $\big[(AB)^k\lambda_0, (AB)^{k+1}\lambda_0\big)$. Furthermore, we use the fact that $\mu_1(\cdot)$ is monotone non-increasing to find

$$\begin{aligned} II_{\lambda_0} &= \sum_{k=0}^{\infty} \int_{(AB)^k\lambda_0}^{(AB)^{k+1}\lambda_0} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \mu_1(\lambda) \, d\lambda \\ &\leq \sum_{k=0}^{\infty} \mu_1\big((AB)^k\lambda_0\big) \, \big[\big((AB)^{k+1}\lambda_0\big)^{\frac{q}{p}} - \big((AB)^k\lambda_0\big)^{\frac{q}{p}}\big] \\ &\leq (AB\lambda_0)^{\frac{q}{p}} \, \sum_{k=0}^{\infty} (AB)^{\frac{q}{p}k} \mu_1\big((AB)^k\lambda_0\big) \\ &= (AB\lambda_0)^{\frac{q}{p}} \, \mu_1\big(\lambda_0\big) + (AB\lambda_0)^{\frac{q}{p}} \, \sum_{k=1}^{\infty} (AB)^{\frac{q}{p}k} \mu_1\big((AB)^k\lambda_0\big) \, . \end{aligned}$$

Using (5.36) we obtain altogether

$$\begin{aligned} &\int_{Q_R} \big[M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)\big]^{\frac{q}{p}} \, dx \\ &\qquad \leq |Q_R| \, \lambda_0^{\frac{q}{p}} + 2 \, (AB\lambda_0)^{\frac{q}{p}} \, \big[\mu_1\big(\lambda_0\big) + (AB)^{\frac{q}{p}} \sum_{k=0}^{\infty} (AB)^{\frac{q}{p}k} \mu_2\big((AB)^k\varepsilon_0\lambda_0\big)\big] \, . \end{aligned} \tag{5.37}$$

The last estimate shall now be estimated from above by the maximal operator of the functions g and G. Similarly to above we decompose the corresponding integral into

$$\begin{aligned} \int_{Q_R} \big[M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)\big]^{\frac{q}{p}} \, dx &= \int_0^{\varepsilon_0\lambda_0} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \mu_2(\lambda) \, d\lambda + \int_{\varepsilon_0\lambda_0}^{\infty} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \mu_2(\lambda) \, d\lambda \\ &=: \, III_{\lambda_0} + IV_{\lambda_0} \, . \end{aligned}$$

We then use the monotonicity of $\mu_2(\cdot)$ to find

$$\begin{aligned} III_{\lambda_0} &\geq \mu_2\big(\varepsilon_0\lambda_0\big) \int_0^{\varepsilon_0\lambda_0} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \, d\lambda \, = \, \mu_2\big(\varepsilon_0\lambda_0\big) \, (\varepsilon_0\lambda_0)^{\frac{q}{p}} \\ IV_{\lambda_0} &= \sum_{k=1}^{\infty} \int_{(AB)^{k-1}\varepsilon_0\lambda_0}^{(AB)^k\varepsilon_0\lambda_0} \tfrac{q}{p} \, \lambda^{\frac{q}{p}-1} \mu_2(\lambda) \, d\lambda \\ &\geq \sum_{k=1}^{\infty} \mu_2\big((AB)^k\varepsilon_0\lambda_0\big) \, \big[\big((AB)^k\varepsilon_0\lambda_0\big)^{\frac{q}{p}} - \big((AB)^{k-1}\varepsilon_0\lambda_0\big)^{\frac{q}{p}}\big] \\ &= (\varepsilon_0\lambda_0)^{\frac{q}{p}} \, \sum_{k=1}^{\infty} (AB)^{\frac{q}{p}k} \mu_2\big((AB)^k\varepsilon_0\lambda_0\big) \, \big(1 - (AB)^{-\frac{q}{p}}\big) \, . \end{aligned}$$

Due to $A \geq 2$ and $B \geq 1$ we have $\big(1 - (AB)^{-q/p}\big) \in (\frac{1}{2}, 1)$, and therefore

$$\int_{Q_R} \big[M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)\big]^{\frac{q}{p}} \, dx \, \geq \, \tfrac{1}{2} \, (\varepsilon_0\lambda_0)^{\frac{q}{p}} \sum_{k=0}^{\infty} (AB)^{\frac{q}{p}k} \mu_2\big((AB)^k\varepsilon_0\lambda_0\big)$$

follows. Since the last sum already appeared in inequality (5.37), this enables us to find a new estimate for the $L^{q/p}$-norm of $M^*\big((\mu^2+|Du|^2)^{p/2}\big)$:

$$\begin{aligned}
&\int_{Q_R} \big[M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx \\
&\le |Q_R|\,\lambda_0^{\frac{q}{p}} + 2\,(AB\lambda_0)^{\frac{q}{p}}\Big[\mu_1(\lambda_0) + (AB)^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{\frac{q}{p}k}\mu_2\big((AB)^k\varepsilon_0\lambda_0\big)\Big] \\
&\le |Q_R|\,\lambda_0^{\frac{q}{p}} + 2\,(AB\lambda_0)^{\frac{q}{p}}\mu_1(\lambda_0) + 4\,(AB)^{2\frac{q}{p}}\varepsilon_0^{-\frac{q}{p}}\int_{Q_R}\big[M^*\big((\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx\,.
\end{aligned}$$

Taking into account the dependencies of A, B and ε_0 and recalling the definition of λ_0, we calculate with (5.31) and the estimate of the norm of the maximal operator in Lemma 5.4 (for the exponent $\frac{q}{p} > 1$):

$$\begin{aligned}
&\int_{Q_R} \big[M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx \\
&\qquad\le c\,\Big(|Q_R|\,\lambda_0^{\frac{q}{p}} + \int_{Q_R}\big[M^*\big((\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx\Big) \\
&\qquad\le c\,|Q_R|\,\bigg[\Big(-\!\!\!\!\!\!\int_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{p}} + -\!\!\!\!\!\!\int_{Q_R}\big[(\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big]^{\frac{q}{p}}\,dx\bigg]\,,
\end{aligned}$$

and the constant c depends only on $n, N, p, \frac{L}{\nu}$ and $s-p$. Since every function f is bounded pointwise almost everywhere on Q_R by the maximal function $M^*(f)$, we thus infer

$$\begin{aligned}
-\!\!\!\!\!\!\int_{Q_R}(\mu^2+|Du|^2)^{\frac{q}{2}}\,dx &\le |Q_R|^{-1}\int_{Q_R}\big[M^*\big((\mu^2+|Du|^2)^{\frac{p}{2}}\big)\big]^{\frac{q}{p}}\,dx \\
&\le c\,\Big(-\!\!\!\!\!\!\int_{Q_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{p}} + c\,-\!\!\!\!\!\!\int_{Q_R}(\mu^2+|Dg|^2+|G|^{\frac{2}{p-1}})^{\frac{q}{2}}\,dx\,,
\end{aligned}$$

where the constant c still depends on the same quantities as above. Hence, we have completed the proof of Lemma 5.7. □

Remark 5.8: We used Lemma 5.4 for the estimate of the $L^{q/p}$-norm of the Hardy Littlewood maximal operator, which blows up if $\frac{q}{p} \to 1$. Therefore, the constant c in Lemma 5.7 might blow up for $q \to p$. Nevertheless, the estimate (5.30) is trivially satisfied in the case $q = p$.

5.4 Local integrability estimates up to the boundary

In order to achieve a boundary version of the higher integrability estimate in (5.30), we start by considering weak solutions $u \in W^{1,p}(Q_{2R}^+, \mathbb{R}^N)$ of the inhomogeneous system

$$\begin{cases} -\operatorname{div} a(x, Dg + Du) \;=\; L\,G(x) & \text{in } Q_{2R}^+\,, \\ \qquad\qquad\qquad\; u \;=\; 0 & \text{on } \Gamma\,, \end{cases} \tag{5.38}$$

for functions $g \in W^{1,p}(Q_{2R}^+, \mathbb{R}^N)$ and $G \in L^{p/(p-1)}(Q_{2R}^+, \mathbb{R}^N)$. We first obtain analogously to Lemma 5.5:

Lemma 5.9: *Let $u \in W^{1,p}(Q^+_{2R}, \mathbb{R}^N)$ be a weak solution of the Dirichlet problem (5.38) under the assumption (Z1)-(Z4). Let $B > 1$. Then there exists $\varepsilon = \varepsilon\big(n, N, p, \frac{L}{\nu}, B\big) > 0$ and a radius $R_0 = R_0\big(n, N, p, \frac{L}{\nu}, B, \omega(\cdot)\big) > 0$, such that there holds: if $2\sqrt{n}R \leq R_0$, $\lambda > 0$ and $Q \subset Q^+_R$ is a dyadic subrectangle of Q^+_R such that*

$$\Big|Q \cap \Big\{x \in Q^+_R : M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) > AB\lambda\,, \qquad M^*\big((\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}})^{\frac{p}{2}}\big)(x) \leq \varepsilon\lambda\Big\}\Big| > B^{-\frac{s}{p}}|Q|\,, \tag{5.39}$$

then its predecessor $\widetilde{Q}$ of Q satisfies

$$\widetilde{Q} \subseteq \big\{x \in Q^+_R : M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) > \lambda\big\}\,. \tag{5.40}$$

Here $M^* = M^*_{Q^+_{2R}}$ denotes the restricted maximal operator relative to Q^+_{2R}, s is the exponent defined in (5.5) and $A = A\big(n, N, p, \frac{L}{\nu}\big)$ is an absolute constant. Furthermore, all constants and quantities involved are uniform with respect to $\mu \in [0, 1]$.

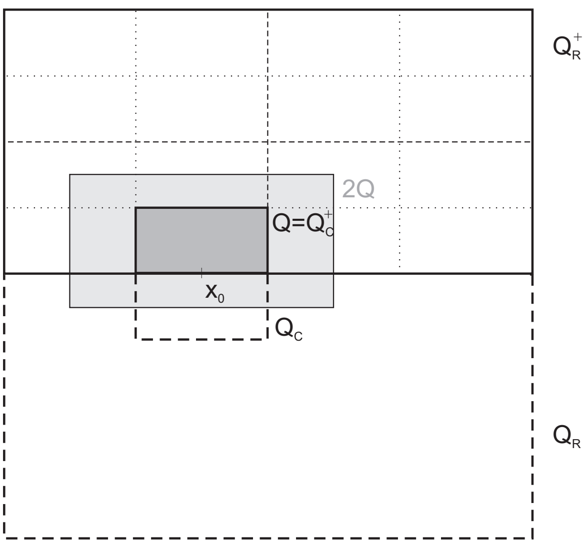
Proof: We prove the Lemma by contradiction. We proceed analogously to the proof of Lemma 5.5 and we only state the modifications due to the boundary situation. Instead of cubes we now consider dyadic rectangles (also called Q for easier comparability) of the type (5.6). We distinguish the two cases whether the closure of Q intersects Γ or not:

The case $\overline{Q} \cap \{x_n = 0\} = \emptyset$: Since Q is a dyadic subrectangle we also have $3Q \subset Q^+_{2R}$ (therefore, we are in fact in the interior situation). We next use the higher integrability estimate (5.4) of the comparison map in the rectangle-version. Keeping in mind that all the computations here have to be performed on (dyadic sub-)rectangles instead of on (dyadic sub-)cubes, we may repeat the arguments leading to (5.28). This allows us to infer for all points $x \in Q$

$$M^*\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x) \leq \max\big\{M^{**}\big((\mu^2 + |Du|^2)^{\frac{p}{2}}\big)(x), 5^n\lambda\big\}$$

where M^{**} denotes the restricted maximal operator relative to $2Q$. This provides again the desired contradiction.

The case $\overline{Q} \cap \{x_n = 0\} \neq \emptyset$: We first recall that Q is a dyadic subrectangle of Q^+_R which by definition means that Q has sides parallel to the coordinate axes. Hence, in this case one side of Q is lying on Γ. Thus we find a cube $Q_c \subset Q_{2R}$ with centre x_0 on Γ such that $Q = Q^+_c$ (see the illustration for the involved cubes and rectangles). The reason for introducing Q_c is that $2Q \not\subset Q^+_{2R}$ whereas $(2Q_c)^+$ (which is indeed only a shifted version of $2Q$ with respect to the normal direction e_n) satisfies $(2Q_c)^+ \subset Q^+_{2R}$. We then may go on as in the proof of Lemma 5.5 with Q^+_c instead of Q. In particular, we have



to replace (5.11) and (5.13) (coming from the assumptions of the lemma) by

$$\begin{aligned} &\fint_{(3Q_c)^+} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx \,\le\, \lambda \qquad \text{and} \\ &\fint_{(3Q_c)^+} \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx \,\le\, \varepsilon\,\lambda\,, \end{aligned}$$

for some $\varepsilon > 0$ to be determined later and where we have used for the first inequality the fact that $\widetilde{Q} \subset (3Q_c)^+$. The comparison map $v \in W^{1,p}((3Q_c)^+, \mathbb{R}^N)$ is then defined as the unique solution of the Dirichlet problem with frozen coefficients

$$\begin{cases} \operatorname{div} a(x_0, Dv) \,=\, 0 & \text{in } (3Q_c)^+\,, \\ \qquad\qquad v \,=\, u & \text{on } \partial(3Q_c)^+\,. \end{cases}$$

Testing the system with $v - u$ and using the higher integrability estimate (5.4) of Dv in the up to the boundary version, we obtain analogously to (5.16)

$$\fint_{2Q} \big(\mu^2 + |Dv|^2\big)^{\frac{s}{2}}\,dx \,\le\, c\big(n, N, p, \tfrac{L}{\nu}\big)\,\lambda^{\frac{s}{p}}\,.$$

Then, the conclusion follows as in Lemma 5.5. □

Remark 5.10: In order to apply the previous lemma, we again have to fix the constant B, depending on the integrability exponent $q \in (p, s)$, which in turn determines the quantities R_0 and ε. Choosing B as in (5.29), we then pick the smaller radius R_0 and the smaller number ε such that R_0 and ε are appropriate for both the Lemma 5.5 in the interior and Lemma 5.9 at the boundary. Then we have the following dependencies for the quantities involved:

$$\begin{aligned} R_0 \,&=\, R_0\big(n, N, p, \tfrac{L}{\nu}, \omega(\cdot), s - q\big) \quad \text{and} \\ \varepsilon_0 \,&:=\, \varepsilon = \varepsilon\big(n, N, p, \tfrac{L}{\nu}, s - q\big)\,. \end{aligned}$$

In the next lemma we apply Lemma 5.9 exactly as in deriving the higher integrability estimate in Lemma 5.7 in the interior situation; this gives a reverse Hölder inequality up to the boundary for solutions u of the system (5.38):

Lemma 5.11: *Let $u \in W^{1,p}(Q_{2R}^+, \mathbb{R}^N)$ be a weak solution of (5.38) under the assumptions (Z1)-(Z4) with $2\sqrt{n}R \le R_0$, where R_0 is the radius as above in Lemma 5.7, and let $\mu \in [0,1]$. Then for every exponent $q \in (p, s)$ there holds:*

$$\begin{aligned} &\Big(\fint_{Q_R^+} \big(\mu^2 + |Du|^2\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}} \\ &\quad \le\, c\Big(\fint_{Q_{2R}^+} \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} + c\Big(\fint_{Q_{2R}^+} \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}\,. \end{aligned} \tag{5.41}$$

for a constant c depending only on $n, N, p, \frac{L}{\nu}$ and $s - q$. □

5.5 The global higher integrability result

With the local estimates of Lemma 5.7 for cubes Q_R in the interior and of Lemma 5.11 for upper half-cubes Q_R^+ at the boundary we are in a position to prove the higher integrability result on general bounded domains $\Omega \subset \mathbb{R}^n$ of class C^1 stated in Theorem 5.1 above:

Proof (of Theorem 5.1): We first consider the case $q = p$. Here we obtain the desired result by arguing similarly to the estimates (5.15) and (5.20): testing the system (5.2) with the function $u - g \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ we have

$$\begin{aligned}
&\int_\Omega \big[a(x,Du) - a(x,0)\big] \cdot Du\,dx \\
&\quad = \int_\Omega a(x,Du) \cdot (Du - Dg)\,dx - \int_\Omega a(x,0) \cdot Du\,dx + \int_\Omega a(x,Du) \cdot Dg\,dx \\
&\quad = L \int_\Omega G \cdot (u-g)\,dx - \int_\Omega a(x,0) \cdot Du\,dx + \int_\Omega a(x,Du) \cdot Dg\,dx\,.
\end{aligned}$$

Using the ellipticity condition (Z3) on the left-hand side of the last equation (in order to cover also the degenerate case $\mu = 0$ we argue only on the set $\{x \in \Omega : Du(x) \neq 0\}$ and note that on the remaining set the integrand does not contribute to the integral), we obtain via Lemma A.2 and the growth condition (Z1):

$$\begin{aligned}
&c(p)\,\nu \int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{p-2}{2}} |Du|^2\,dx \\
&\quad \le L \int_\Omega \Big(|G|\,|u-g| + \mu^{p-1}\,|Du| + \big(\mu^2 + |Du|^2\big)^{\frac{p-1}{2}} |Dg|\Big)\,dx \\
&\quad \le 3\varepsilon \int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx + c(n,p,\tfrac{L}{\varepsilon},\Omega)\,L \int_\Omega \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx\,,
\end{aligned}$$

where we have applied Young's and the Poincaré-inequality in the last line. With the inequality $(\mu^2 + |Du|^2)^{\frac{p}{2}} \le (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |Du|^2 + \mu^p$ and the choice $\varepsilon = \frac{c(p)\nu}{4}$ we thus get

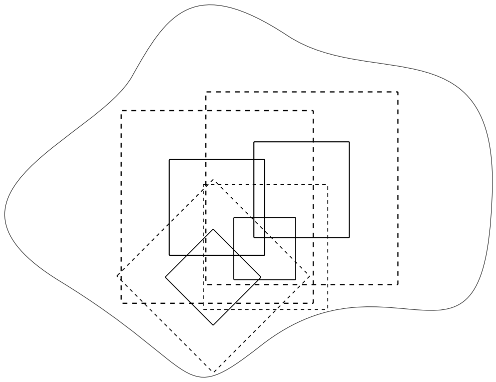
$$\int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx \;\le\; c\big(n,p,\tfrac{L}{\nu},\Omega\big) \int_\Omega \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx\,, \tag{5.42}$$

i. e., we have proved the assertion of the theorem in the case $q = p$.

Thus, we may now assume $q > p$; first we define $w = u - g$ and see that $w \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ is a solution of the system

$$-\operatorname{div} a(x, Dg + Dw) \;=\; L\,G(x) \qquad \text{in } \Omega \tag{5.43}$$

since $u = g + w$ solves the Dirichlet problem (5.2). Systems of this form were already considered in Lemma 5.7 on cubes in the interior case and in Lemma 5.11 on half-cubes at the boundary. We next flatten the boundary of the domain Ω in a standard way which we will explain in detail:



Some sets $\rho_i(Q_{2r_i})$ and $\rho_i(Q_{r_i})$ in the interior

Since by assumption of the theorem $\overline{\Omega}$ is a compact set, we find a covering of $\overline{\Omega}$ by a finite number of C^1-regular charts $(\rho_i, A_i)_{1\leq i\leq k}$ and $(\sigma_i, B_i)_{1\leq i\leq k}$ with

$$\rho_i : Q_{2r_i} \to A_i$$
$$\sigma_i : Q^+_{2s_i} \to B_i$$

for numbers r_i, $s_i > 0$, the side lengths of the cubes and half-cubes, respectively, for $i = 1, \ldots, k$. Furthermore, without loss of generality, we may assume that the inclusion

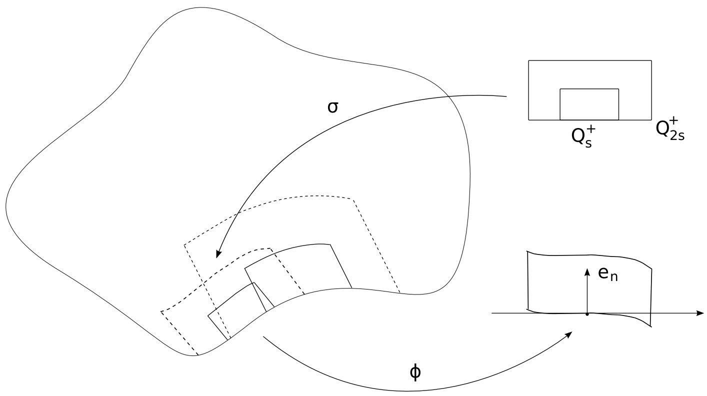
$$\Omega \subset \bigcup_{i=1}^{k} \big(\rho_i(Q_{r_i}) \cup \sigma_i(Q^+_{s_i})\big) .$$

is satisfied.

By construction, the charts ρ_i map into the interior of Ω (they may be assumed to be isometries, cf. the figure above), and for the boundary charts σ_i we assume

$$\sigma_i(Q^+_{2s_i}) = B_i \cap \Omega \quad \text{and} \quad \sigma_i(\Gamma_{2s_i}) = B_i \cap \partial\Omega ,$$

i. e., the ρ_i do not intersect the boundary $\partial\Omega$ whereas the σ_i do (for all indices $i = 1, \ldots, k$).



Some sets $\sigma_i(Q^+_{2s_i})$ and $\sigma_i(Q^+_{s_i})$ at the boundary

For the boundary situation we employ an additional assumption (which is in fact a standard assumption for the boundary situation and, in particular, for the transformation of the original coefficients): For this purpose we consider for any arbitrary chart σ_i the boundary point $x_i = \sigma_i(0)$, the "centre" of the distorted half-cube; we then define ϕ_i to be the isometry which maps x_i to 0 in such a way that there holds $\nu_{\partial(\phi_i(\Omega\cap B_i))}(0) = e_n$ for the inner unit normal vector. We note that this implies $\nabla h(0) = \nabla h((\phi_i(x_i))') = 0$ where $h : \mathbb{R}^{n-1} \to \mathbb{R}$ denotes the function which represents the boundary $\partial\Omega \cap B_i$ after application of the isometry ϕ_i (for illustration see the figure above). Having introduced these quantities, we may now assume that for all indices $i = 1, \ldots, k$ there holds

$$\big|\nabla h\big(\phi_i'(x)\big)\big| < \tfrac{1}{2} \qquad \forall\, x \in \sigma_i(\Gamma_{2s_i}) \tag{5.44}$$

(cf. Section 3.2). Moreover, we can choose the charts σ_i such that we have for the volumes of the corresponding sets:

$$\big|\sigma_i(Q^+_{2s_i})\big| = \big|B_i \cap \Omega\big| \qquad \forall\, i \in \{1, \ldots, k\} .$$

Transforming the system (5.43) above via the maps ρ_i and σ_i for $i \in \{1, \dots, k\}$, we obtain a finite number of systems which are of the types given in (5.7) and (5.38), respectively, and which are solved by the transformed functions defined on (half-) cubes. In detail, we introduce for $i \in \{1, \dots, k\}$ the functions

$$\begin{aligned} \widehat{w}_i &:= w \circ \rho_i & \widetilde{w}_i &:= w \circ \sigma_i \\ \widehat{g}_i &:= g \circ \rho_i & \widetilde{g}_i &:= g \circ \sigma_i \\ \widehat{G}_i &:= G \circ \rho_i & \widetilde{G}_i &:= G \circ \sigma_i \end{aligned}$$

as well as the transformed coefficients

$$\begin{aligned} \widehat{a}_i(x, z) &:= a\big(\rho_i(x), z\, D\rho_i^{-1}(\rho(x))\big) D(\rho_i^{-1})^t(\rho_i(x)) \\ \widetilde{a}_i(x, z) &:= a\big(\sigma_i(x), z\, D\sigma_i^{-1}(\sigma(x))\big) D(\sigma_i^{-1})^t(\sigma_i(x)) \,, \end{aligned}$$

which are defined on Q_{2r_i} and on $Q^+_{2s_i}$, respectively. Due to the fact that the transformation mappings are of class C^1, the transformed functions $\widehat{w}_i$, $\widetilde{w}_i$ belong to the space $W^{1,p}$, $\widehat{g}_i$, $\widetilde{g}_i$ to $W^{1,q}$, and $\widehat{G}_i$, $\widetilde{G}_i$ to $L^{q/(p-1)}$. Furthermore, in view of (5.44), the new coefficients $\widehat{a}_i(\cdot,\cdot)$ and $\widetilde{a}_i(\cdot,\cdot)$, $i = 1, \dots, k$, have the same structure as the original coefficients, i. e., they satisfy conditions of the form (Z1)-(Z4) with structure constants $c(\nu), c(L)$ instead of ν, L. Moreover, according to the fact that the number of charts is finite, we may assume that all the systems have the same modulus of continuity $\widetilde{\omega}(\cdot)$. We easily infer via a transformation argument that, for every $1 \le i \le k$, the function $\widehat{w}_i$ is a weak solution of the system

$$- \operatorname{div} \widehat{a}_i(x, D\widehat{g}_i + D\widehat{w}_i) \;=\; \widehat{L}\, \widehat{G}_i(x) \qquad \text{in } Q_{2r_i} \,,$$

and the function $\widetilde{w}_i$ is a weak solution of

$$- \operatorname{div} \widetilde{a}_i(x, D\widetilde{g}_i + D\widetilde{w}_i) \;=\; \widetilde{L}\, \widetilde{G}_i(x) \qquad \text{in } Q^+_{2s_i}$$

with $\widetilde{w}_i = 0$ on Γ_{2s_i}. This transformation allows us to apply Lemma 5.7 and Lemma 5.11, respectively: we first fix δ_1 with the given dependencies by choosing $\delta_1 = \frac{\delta}{2}$, where δ is the number representing the higher integrability exponent of the comparison map Dv which was determined in (5.5). We next choose the radius R_0 in dependency of $n, N, p, \frac{L}{\nu}$ and $\widetilde{\omega}(\cdot)$ according to the remarks above. We note that we can skip here the dependency of $s - q \ge \delta_1 > 0$. Then we divide all the cubes Q_{r_i} and half-cubes $Q^+_{s_i}$ for $i \in \{1, \dots, k\}$ into (disjoint) subcubes $Q_{R_i} \subset Q_{r_i}$ and $Q^+_{S_i} \subset Q^+_{s_i}$ (centred at points x_{ij} and y_{ij} for $1 \le j \le m_i$) with $2\sqrt{n}R_i \le R_0$ and $2\sqrt{n}S_i \le R_0$. On each of the inner cubes Q_{R_i} we may apply the estimate (5.30) (with u replaced by $\widehat{w}_i$) such that we arrive at

$$\begin{aligned} \int_{Q_{r_i}} \big(\mu^2 + |D\widehat{w}_i|^2\big)^{\frac{q}{2}}\, dx \;&=\; \sum_{j=1}^{m_i} \int_{Q_{R_i}(x_{ij})} \big(\mu^2 + |D\widehat{w}_i|^2\big)^{\frac{q}{2}}\, dx \\ &\le\; c \sum_{j=1}^{m_i} \Big[\Big(\int_{Q_{2R_i}(x_{ij})} \big(\mu^2 + |D\widehat{w}_i|^2\big)^{\frac{p}{2}}\, dx \Big)^{\frac{q}{p}} + \int_{Q_{2R_i}(x_{ij})} \big(\mu^2 + |D\widehat{g}_i|^2 + |\widehat{G}_i|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx \Big] \\ &\le\; c \Big[\Big(\int_{Q_{2r_i}} \big(\mu^2 + |D\widehat{w}_i|^2\big)^{\frac{p}{2}}\, dx \Big)^{\frac{q}{p}} + \int_{Q_{2r_i}} \big(\mu^2 + |D\widehat{g}_i|^2 + |\widehat{G}_i|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx \Big] \,, \end{aligned}$$

where the constant c depends only on $n, N, p, \frac{L}{\nu}$, $\widetilde{\omega}(\cdot)$ and Ω. Here, we have omitted the meanvalues for the integrals as we have $R_i = R_i(n, N, p, \frac{L}{\nu}, \widetilde{\omega}(\cdot), \Omega) > 0$ for all $i \in \{1, \dots, k\}$, and in the last line we have used the fact that each point in Q_{2r_i} is covered by at most $2n$ small cubes $Q_{2R_i}(x_{ij})$. Analogously we apply on each rectangle $Q^+_{S_i}$ at the boundary the estimate (5.41) and on each inner cube Q_{S_i} the estimate (5.30) (with u replaced by $\widetilde{w}_i$), respectively. Thus, we conclude

$$\begin{aligned}&\int_{Q^+_{s_i}} \big(\mu^2 + |D\widetilde{w}_i|^2\big)^{\frac{q}{2}}\,dx \\ &\qquad\qquad \leq c\,\Big[\Big(\int_{Q^+_{2s_i}} \big(\mu^2 + |D\widetilde{w}_i|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{p}} + \int_{Q^+_{2s_i}} \big(\mu^2 + |D\widetilde{g}_i|^2 + |\widetilde{G}_i|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\,dx\Big]\,.\end{aligned}$$

We recall that Ω is covered by $\rho_i(Q_{r_i})$ in the interior and by $\sigma_i(Q^+_{s_i})$ at the boundary. This allows us to go back to the original system on Ω via the transformations ρ_i and σ_i. Taking into account that the number of charts is finite (and depending on the domain Ω) we thus obtain

$$\begin{aligned}&\int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{q}{2}}\,dx \,\leq\, c(q)\,\Big[\int_\Omega \big(\mu^2 + |Dw|^2\big)^{\frac{q}{2}}\,dx + \int_\Omega \big(\mu^2 + |Dg|^2\big)^{\frac{q}{2}}\,dx\Big] \\ &\leq c(q)\sum_{i=1}^k \Big[\int_{Q_{r_i}} \big(\mu^2 + |D\widehat{w}_i|^2\big)^{\frac{q}{2}}\,dx + \int_{Q^+_{s_i}} \big(\mu^2 + |D\widetilde{w}_i|^2\big)^{\frac{q}{2}}\,dx\Big] + c(q)\int_\Omega \big(\mu^2 + |Dg|^2\big)^{\frac{q}{2}}\,dx \\ &\leq c\big(n,N,p,q,\tfrac{L}{\nu},\omega(\cdot),\Omega\big)\,\Big[\Big(\int_\Omega \big(\mu^2 + |Dw|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{p}} + \int_\Omega \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\,dx\Big]\,,\end{aligned} \tag{5.45}$$

where we have used the definition $w = u - g$ to rewrite the inequalities given above in terms of $\widehat{w}$ on Q_{r_i} and of $\widetilde{w}$ on $Q^+_{s_i}$; furthermore, we recall the fact that the modulus of continuity $\widetilde{\omega}(\cdot)$ in the transformed setting depends only on $\partial\Omega$ and $\omega(\cdot)$. In the last step we estimate the first integral of the right-hand side of the last inequality from above via applying the estimate (5.42) achieved before in the case $q = p$ and Jensen's inequality, and we see:

$$\begin{aligned}\Big(\int_\Omega \big(\mu^2 + |Dw|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} &\leq 2\Big(\int_\Omega \big(\mu^2 + |Du|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} + 2\Big(\int_\Omega \big(\mu^2 + |Dg|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} \\ &\leq c\big(n,p,\tfrac{L}{\nu},\Omega\big)\Big(\int_\Omega \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}} \\ &\leq c\big(n,p,q,\tfrac{L}{\nu},\Omega\big)\Big(\int_\Omega \big(\mu^2 + |Dg|^2 + |G|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}\,.\end{aligned}$$

Combined with (5.45) this yields the result of the theorem. □

Remark 5.12: Let us consider the previous system in the special case of a ball $B_R(x_0)$ with coefficients not explicitly depending on x. The next aim is to study the constant c of Theorem 5.1 and its dependency with respect to the domain $\Omega = B_R(x_0)$. To this end we suppose that $u \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ is a weak solution of the system

$$\begin{cases} -\operatorname{div} a(Du) = L\,G(x) & \text{in } B_R(x_0)\,, \\ u = g & \text{on } \partial B_R(x_0)\,, \end{cases}$$

for functions $g \in W^{1,q}(B_R(x_0), \mathbb{R}^N)$ and $G \in L^{q/(p-1)}(B_R(x_0), \mathbb{R}^N)$ for a fixed exponent $q \in [p, s_1]$. Rescaling via

$$u_r(x) := \frac{u(Rx + x_0)}{R}, \quad g_r(x) := \frac{g(Rx + x_0)}{R} \quad \text{and} \quad G_r(x) := R\,G(Rx + x_0)$$

for x in the unit ball B, we find: u_r is a weak solution of the system

$$\begin{cases} -\operatorname{div} a(Du_r) = L\, G_r(x) & \text{in } B\,, \\ u_r = g_r & \text{on } \partial B\,. \end{cases}$$

Taking into account that in this case the number of charts of the covering $(\rho_i, A_i)_{1\le i\le k}$ and $(\sigma_i, B_i)_{1\le i\le k}$ of B is a constant depending only on the dimension n, we may apply Theorem 5.1 with $\Omega = B$ to obtain

$$\int_\Omega \big(\mu^2 + |Du_r|^2\big)^{\frac{q}{2}}\, dx \;\le\; c \int_\Omega \big(\mu^2 + |Dg_r|^2 + |G_r|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx\,,$$

where the constant c depends only on n, N, p, q and $\frac{L}{\nu}$. Scaling back to the original solution u, we end up with the following higher integrability estimate:

$$\begin{aligned} \fint_{B_R(x_0)} \big(\mu^2 + |Du|^2\big)^{\frac{q}{2}}\, dx &= \fint_{B_1} \big(\mu^2 + |Du_r|^2\big)^{\frac{q}{2}}\, dx \\ &\le c\big(n, N, p, q, \tfrac{L}{\nu}\big) \fint_{B_1} \big(\mu^2 + |Dg_r|^2 + |G_r|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx \\ &\le c\big(n, N, p, q, \tfrac{L}{\nu}\big) \fint_{B_R(x_0)} \big(\mu^2 + |Dg|^2 + R^{\frac{2}{p-1}}\, |G|^{\frac{2}{p-1}}\big)^{\frac{q}{2}}\, dx\,. \end{aligned}$$

Hence, in the case $\Omega = B_R(x_0)$ the constant c in Theorem 5.1 also depends on the radius R, but the dependency on R is only due to the term involving the function G. Nevertheless, since $\frac{2}{p-1} > 0$, we can neglect this R-terms if we are on small balls or cubes, respectively.

Chapter 6

Low dimensions: partial regularity of the solution

We now consider weak solutions $u \in W^{1,p}(\Omega,\mathbb{R}^N)$, $1 < p < 2$, of the elliptic system

$$\begin{cases} -\operatorname{div} a(\,\cdot\,,u,Du) = b(\,\cdot\,,u,Du) & \text{in } \Omega\,, \\ \qquad\qquad\quad u = g & \text{on } \partial\Omega\,. \end{cases} \tag{6.1}$$

Here $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ denotes a bounded domain of class C^1 and we suppose boundary values $g \in C^1(\overline{\Omega},\mathbb{R}^N)$. As usual this boundary condition is to be understood in the sense of traces. The coefficients $a\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are assumed to be uniformly continuous with respect to the first and the second variable, of class C^1 with respect to the last variable, and satisfy a standard $(p-1)$-growth condition. We further require the vector field $b\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to \mathbb{R}^N$ to obey either a controllable or a natural growth condition (for the precise structure assumptions see Section 6.1 below).

The present chapter is devoted to Morrey-type estimates up to the boundary and the question of (partial) regularity of the weak solution u in low dimensions, meaning that $n \in (p, p+2]$. For this purpose, we define the set of regular and singular points of u via

$$\begin{aligned} \operatorname{Reg}_u(\overline{\Omega}) &:= \big\{x \in \overline{\Omega} : u \in C^0(\overline{\Omega}\cap A,\mathbb{R}^N) \text{ for some neighbourhood } A \text{ of } x\big\}\,, \\ \operatorname{Sing}_u(\overline{\Omega}) &:= \overline{\Omega} \setminus \operatorname{Reg}_u(\overline{\Omega})\,. \end{aligned}$$

We are going to prove that the weak solution u to the nonlinear system (6.1) is Hölder continuous on $\operatorname{Reg}_u(\overline{\Omega})$ with some Hölder exponent $\lambda > 0$. Moreover, we show that the set of singular points is of Hausdorff dimension strictly less than $n-p$, which immediately implies the existence of regular boundary points and, in fact, that $\mathcal{H}^{n-1}$-almost every boundary point is regular. For general dimension n, under such a mild continuity assumption on the coefficients, this property has only been proved for quasilinear systems, see for example [Col71, Pep71, GG78, Gro02a].

Taking into account the counterexamples to full regularity given in [DG68, GM68b, Gia78, NJS80] (for $n \geq 3$) and the general form of the coefficients (i. e., their u-dependency), it is well-known that we cannot expect a global Hölder continuity result to hold. In contrast, due to the global higher integrability of the weak solution and the Sobolev embedding theorem, we see that full Hölder regularity up to the boundary holds true provided that $p \in (n-\varepsilon, n)$ with a (small) number $\varepsilon > 0$ depending only on the structure constants, cf. [GI05] for $n = 2$. However, since the literature lacks an appropriate counterexample in the two-dimensional case (keep in mind that all the counterexamples mentioned above are for codimension ≥ 3), it is still an open question whether there might exist a singular point in dimension $n = 2$ and arbitrary $p \in (1, 2)$.

We note that we also cover partial Hölder continuity of weak solutions to degenerate systems. A model case of the degenerate situation is given by the p-Laplacean, i. e., by the degenerate system

$$\operatorname{div}\left(|Du|^{p-2}\, Du\right) = 0 \qquad \text{in } \Omega\,.$$

The strategy for the proof of the partial regularity result stated in Theorem 6.1 below relies on the so-called direct method and is essentially based on the techniques due to Campanato, applied e. g. in [Cam82b, Cam83, Cam87a, Cam87b]. In [Cam82b] Campanato derived interior estimates under a controllable growth assumption, and in [Cam83] he obtained similar results for systems of higher order. Moreover, Campanato presented in [Cam87a, Cam87b] global estimates for coefficients not depending explicitly on u, i. e., $a(x, u, z) \equiv a(x, z)$, in the superquadratic case. These results were extended recently by Idone to systems with inhomogeneities which may also depend on u and Du, see [Ido04a, Ido04b].

For the examination of both the boundary situation and the interior, we define adequate comparison maps which are solutions of the frozen (homogeneous) system and for which good a priori estimates are available (see Chapter 4). This allows us to deduce Morrey-type estimates for the gradient Du, namely that Du belongs to a suitable Morrey space $L^{p,\gamma}(\Omega, \mathbb{R}^{nN})$, which in view of the Campanato-Meyer embedding Theorem immediately yields the desired Hölder continuity of u. In case of natural growth of the inhomogeneity, these techniques require some modifications for which we adapt Arkhipova's cut-off procedure from the proof of [Ark97, Ark03, Theorem 1], where the corresponding result (for non-degenerate systems) was proved in the superquadratic case.

6.1 Structure conditions and result

We impose standard structure conditions on $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$: the mapping $z \mapsto a(x, u, z)$ is a vector field of class $C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus \{0\}, \mathbb{R}^{nN})$, and for fixed numbers $0 < \nu \leq L$, $1 < p < 2$, $\mu \in [0, 1]$ and all triples $(x, u, z), (\bar{x}, \bar{u}, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$, there hold the following growth, ellipticity and continuity assumptions:

(H1) Polynomial growth of a :

$$|a(x, u, z)| \leq L \left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}},$$

(H2) a is differentiable in z with continuous and bounded derivatives:

$$|D_z a(x, u, z)| \leq L \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}},$$

(H3) a is uniformly strongly elliptic, i. e.,

$$D_z a(x,u,z)\,\lambda\cdot\lambda \,\geq\, \nu\left(\mu^2+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\,\lambda\in\mathbb{R}^{nN}\,,$$

(H4) There exists a nondecreasing, concave modulus of continuity $\omega:\mathbb{R}^+\to[0,1]$ such that

$$|a(x,u,z)-a(\bar{x},\bar{u},z)| \,\leq\, L\left(\mu^2+|z|^2\right)^{\frac{p-1}{2}}\omega\big(|x-\bar{x}|+|u-\bar{u}|\big)\,,$$

We recall that the parameter μ specifies whether our system is non-degenerate ($\mu\neq 0$) or degenerate ($\mu=0$), and we note that we have to exclude $z=0$ in conditions (H2) and (H3) when dealing with degenerate systems. Condition (H4) means that the coefficients $a(x,u,z)$ are continuous with respect to (x,u), uniformly for fixed z. Moreover, we assume the inhomogeneity $b(\cdot,\cdot,\cdot)$ to be a Carathéodory map, that is, it is continuous with respect to (u,z) and measurable with respect to x, and to satisfy one of the following growth conditions:

(B1) Controllable growth condition:

$$|b(x,u,z)| \,\leq\, L\left(\mu^2+|z|^2\right)^{\frac{p-1}{2}}$$

for all $(x,u,z)\in\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$,

(B2) Natural growth condition: there exists a constant L_2 (possibly depending on $M>0$) such that

$$|b(x,u,z)| \leq L_2\,|z|^p + L$$

for all $(x,u,z)\in\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}$ with $|u|\leq M$.

If we pass to vector fields $\frac{a}{\nu}$ and $\frac{b}{\nu}$, we see that the dependency on the constants ν, L and L_2 will only show up in terms of the ratio $\frac{L}{\nu}$ and $\frac{L_2}{\nu}$. Therefore, the dependency on these constants in the various estimates below will be of this type.

For the right-hand side $b(\cdot,\cdot,\cdot)$ we are going to treat both the controllable and the natural growth condition listed above. In the second case, we will have to restrict ourselves to *bounded* weak solutions $u\in W^{1,p}(\Omega,\mathbb{R}^N)\cap L^\infty(\Omega,\mathbb{R}^N)$. More precisely, we assume $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)}\leq M<\infty$ for some constant $M>0$ such that

$$2\,L_2\,M \,<\, \nu\,. \tag{6.2}$$

The regularity proofs in both situations are largely similar. Therefore we will start with a Morrey-type excess-decay estimate under a controllable growth condition (B1) to illustrate the general approach via a comparison principle. In a second step, we will concentrate on the modifications necessary for the natural growth condition. The conclusion of the partial regularity of the weak solution then rests on a classical iteration lemma (thus, it is only carried out for the natural growth situation). For ease of notation, some of the constants are labelled by the superscript (i) and refer to the growth condition (Bi) for $i=1,2$.

The aim of this chapter is the proof of the following

Theorem 6.1: *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^1 and $g\in C^1(\overline{\Omega},\mathbb{R}^N)$. Let $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ be a weak solution of (6.1) with coefficients $a:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ satisfying the assumptions (H1)-(H4), and inhomogeneity $b:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^N$. If one of the following assumptions is fulfilled:*

1. *$b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition (B1),*

2. *$b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2); additionally, we assume $u \in L^\infty(\Omega,\mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq M$ and $2L_2M < \nu$,*

then there exists a constant $\delta_2 > 0$ depending only on n, N, p and $\frac{L}{\nu}$ such that for $n > p > n-2-\delta_2$ there holds

$$\dim_{\mathcal{H}}\big(\overline{\Omega} \setminus \mathrm{Reg}_u(\overline{\Omega})\big) \; < \; n-p\,.$$

Moreover,

$$u \in C^{0,\lambda}_{\mathrm{loc}}\big(\mathrm{Reg}_u(\overline{\Omega}),\mathbb{R}^N\big) \qquad \forall\, \lambda \in \big(0, \min\{1-\tfrac{n-2-\delta_2}{p}, 1\}\big)$$

and the singular set $\mathrm{Sing}_u(\overline{\Omega})$ of u is contained in

$$\Sigma \;:=\; \Big\{x \in \overline{\Omega} : \liminf_{R\searrow 0}\; R^{p-n} \int_{B_R(x)\cap\Omega} \big(1+|Du|^p\big)\,dx \;>\; 0\Big\}\,.$$

As in the proof of partial regularity for weak solutions of inhomogeneous systems in Chapter 3 where we characterized the regular set for the gradient Du, we have to consider on the one hand the interior situation and on the other hand the boundary situation. The latter case is treated by reducing the original system (6.1) in a (by now) standard way to the model situation of a unit half-ball, i. e., we consider weak solutions $u \in W^{1,p}(B^+,\mathbb{R}^N)$ (or $u \in W^{1,p}(B^+,\mathbb{R}^N) \cap L^\infty(B^+,\mathbb{R}^N)$) of the system:

$$\begin{cases} -\operatorname{div} a(\,\cdot\,,u,Du) \;=\; b(\,\cdot\,,u,Du) & \text{in } B^+\,,\\ \qquad\qquad\qquad u \;=\; g & \text{on } \Gamma\,. \end{cases} \tag{6.3}$$

We mention that – in contrast to Section 3, when considering inhomogeneous systems for arbitrary dimension n – in order to cover also degenerate systems, we do not reduce to boundary values 0. Hence, the function g will appear in most of the estimates below. For ease of notation we will then use the abbreviation $\|Dg\|_{L^\infty}$ instead of $\|Dg\|_{L^\infty(B^+,\mathbb{R}^N)}$.

6.2 Higher integrability

In this section, we will prove a higher integrability result up to the boundary for the gradient of the weak solution u of system (6.3). We mention that this estimate is valid in all dimensions. The procedure is standard and only needs to be adjusted to the boundary situation. For this purpose, we will first of all deduce a weak version of a Caccioppoli-type inequality where an additional additive constant may occur on the right-hand side, both in the interior and close to the boundary part Γ of the domain B^+. Via the Poincaré inequalities stated in Section 2, we will infer a reverse Hölder inequality. Then we are in a position to apply the Gehring Lemma A.14 in the up-to-the-boundary version to finally deduce the desired higher integrability of Du.

Lemma 6.2 (Higher integrability): *Let $u \in g + W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a weak solution of (6.3), where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the growth and ellipticity conditions (H1) and (H3) and where $g \in C^1(B^+ \cup \Gamma,\mathbb{R}^N)$. If one of the following assumptions is fulfilled:*

1. *the inhomogeneity* $b(\cdot,\cdot,\cdot)$ *obeys the controllable growth condition (B1),*
2. *the inhomogeneity* $b(\cdot,\cdot,\cdot)$ *obeys the natural growth condition (B2); additionally, there holds* $u \in L^\infty(B^+,\mathbb{R}^N)$ *with* $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M$ *and* $2L_2M < \nu$,

then there exists an exponent $s > p$ *depending only on* $n, N, p, \frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, *and in case 2 additionally on* $\frac{L_2}{\nu}$ *and* M *such that* $u \in W^{1,s}(B^+_\rho,\mathbb{R}^N)$ *for all* $\rho < 1$. *Furthermore, for* $y \in B^+ \cup \Gamma$ *and* $0 < \rho < 1 - |y|$ *there holds:*

$$\Big(\fint_{B^+_{\rho/2}(y)} \big(1 + |Du|\big)^s \, dx \Big)^{\frac{p}{s}} \le c^{(i)} \fint_{B^+_\rho(y)} \big(1 + |Du|^p\big) \, dx$$

(for $i = 1, 2$*) with constants* $c^{(1)} = c^{(1)}(n, N, p, \frac{L}{\nu}, \|Dg\|_{L^\infty})$ *and* $c^{(2)} = c^{(2)}(n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M)$.

Proof: We start by proving the following Caccioppoli-type inequalities:

$$\fint_{B^+_{r/2}(z)} \big(1 + |Du|^p\big) \, dx \le c_{cacc} \fint_{B^+_r(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big) \, dx \tag{6.4}$$

for all $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ with $z_n \le \frac{3}{4} r$, and

$$\fint_{B_{r/2}(z)} \big(1 + |Du|^p\big) \, dx \le c_{cacc} \fint_{B_{3r/4}(z)} \Big(1 + \Big|\frac{u - (u)_{B_{3r/4}(z)}}{r}\Big|^p\Big) \, dx \tag{6.5}$$

for all $z \in B^+$ and $0 < r < 1 - |z|$ with $z_n > \frac{3}{4} r$. Here the constant c_{cacc} depends only on $p, \frac{L}{\nu}, \|Dg\|_{L^\infty}$ when considering (B1), and on $n, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M$ when considering (B2), respectively. To prove inequality (6.4) close to the boundary Γ, we choose a standard cut-off function $\eta \in C^\infty_0(B_r(z),[0,1])$ satisfying $\eta \equiv 1$ on $B_{r/2}(z)$ and $|\nabla\eta| \le \frac{4}{r}$. First we note that u coincides with the function $g \in C^1(\overline{B^+},\mathbb{R}^N)$ on Γ and therefore the function $\varphi = (u-g)\eta^2$ belongs to $W^{1,p}_0(B^+,\mathbb{R}^N)$. Under the natural growth condition, there also holds $\varphi \in L^\infty(B^+,\mathbb{R}^N)$, and thus in both situations – when dealing with (B1) or (B2) – φ can be taken as a test function in the weak formulation (6.3). Hence, we obtain

$$\begin{aligned} \fint_{B^+_r(z)} b(\,\cdot\,,u,Du) \cdot \varphi \, dx &= \fint_{B^+_r(z)} a(\,\cdot\,,u,Du) \cdot D\varphi \, dx \\ &= \fint_{B^+_r(z)} a(\,\cdot\,,u,Du) \cdot \big((Du - Dg)\,\eta^2 + 2(u-g) \otimes \nabla\eta\,\eta\big) \, dx\,, \end{aligned}$$

and therefore, we have the identity

$$\begin{aligned} &\fint_{B^+_r(z)} \big(a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,0)\big) \cdot Du\,\eta^2 \, dx \\ &\quad = -\fint_{B^+_r(z)} a(\,\cdot\,,u,0) \cdot Du\,\eta^2 \, dx - \fint_{B^+_r(z)} 2\,a(\,\cdot\,,u,Du) \cdot (u-g) \otimes \nabla\eta\,\eta \, dx \\ &\qquad + \fint_{B^+_r(z)} a(\,\cdot\,,u,Du) \cdot Dg\,\eta^2 \, dx + \fint_{B^+_r(z)} b(\,\cdot\,,u,Du) \cdot (u-g)\,\eta^2 \, dx \\ &\quad = I + II + III + IV \end{aligned} \tag{6.6}$$

with the obvious labelling. The left-hand side of (6.6) is bounded from below via the ellipticity assumption (H3)

$$\begin{aligned}
\fint_{B_r^+(z)} \big(a(\cdot\,,u,Du)-a(\cdot\,,u,0)\big)\cdot Du\,\eta^2\,dx &= \fint_{B_r^+(z)}\int_0^1 D_z a(\cdot\,,u,t\,Du)\,Du\cdot Du\,\eta^2\,dt\,dx\\
&\geq \fint_{B_r^+(z)}\int_0^1 \nu\,\big(\mu^2+t^2|Du|^2\big)^{\frac{p-2}{2}}|Du|^2\eta^2\,dt\,dx\\
&\geq \nu\fint_{B_r^+(z)}\big|V_\mu(Du)\big|^2\eta^2\,dx\,, \qquad (6.7)
\end{aligned}$$

where we have used the basic inequality $(\mu^2+t^2|Du|^2)^{(p-2)/2}\geq(\mu^2+|Du|^2)^{(p-2)/2}$ for all $t\in[0,1]$. Note that, in order to apply (H3) also for degenerate systems, we have employed the fact that all integrals above vanish on the set $\{x\in B_r^+(z): Du(x)=0\}$. To estimate term I in (6.6) we use (H1) and Young's inequality (for a positive ε to be determined later) and obtain

$$I\,\leq\, L\,\mu^{p-1}\fint_{B_r^+(z)}|Du|\,\eta^2\,dx\,\leq\,\varepsilon\fint_{B_r^+(z)}|Du|^p\,\eta^2\,dx+\varepsilon^{\frac{1}{1-p}}L^{\frac{p}{p-1}}\,\mu^p\,.$$

Using (H1) (taking into account $\frac{p}{p-1}\geq 2$) and Young's inequality, the second and the third term can be handled similarly, and we get

$$\begin{aligned}
II\,&\leq\,2\fint_{B_r^+(z)} L\big(\mu^2+|Du|^2\big)^{\frac{p-1}{2}}\,|u-g|\,|\nabla\eta|\,\eta\,dx\\
&\leq\,8\,L\fint_{B_r^+(z)}\big(\mu^2+|Du|^2\big)^{\frac{p-1}{2}}\Big|\frac{u-g}{r}\Big|\,\eta\,dx\\
&\leq\,\varepsilon\fint_{B_r^+(z)}\big(\mu^p+|Du|^p\big)\,\eta^2\,dx+8^p\,\varepsilon^{1-p}L^p\fint_{B_r^+(z)}\Big|\frac{u-g}{r}\Big|^p\,dx
\end{aligned}$$

and

$$III\,\leq\,\varepsilon\fint_{B_r^+(z)}\big(\mu^p+|Du|^p\big)\,\eta^2\,dx+\varepsilon^{1-p}L^p\fint_{B_r^+(z)}|Dg|^p\,dx\,.$$

For further calculations, i. e., the estimate for term IV, we have to distinguish the different growth conditions concerning the inhomogeneity $b(\cdot,\cdot,\cdot)$:

Controllable growth condition (B1): using (B1) we proceed as in integral II (note here $r\leq 1$) and we obtain

$$\begin{aligned}
IV\,&\leq\,L\fint_{B_r^+(z)}\big(\mu^2+|Du|^2\big)^{\frac{p-1}{2}}\,|u-g|\,\eta^2\,dx\\
&\leq\,\varepsilon\fint_{B_r^+(z)}\big(\mu^p+|Du|^p\big)\,\eta^2\,dx+\varepsilon^{1-p}L^p\fint_{B_r^+(z)}\Big|\frac{u-g}{r}\Big|^p\,dx\,.
\end{aligned}$$

In view of the inequality $\mu^p+|Du|^p\leq 2\,\big(\mu^p+|V_\mu(Du)|^2\big)$, setting $\varepsilon=\frac{\nu}{8}$ and dividing through by ν, we combine the last estimates for the various terms arising in (6.6) and conclude

$$\begin{aligned}\fint_{B_r^+(z)} (\mu^p + |Du|^p)\,\eta^2\,dx &\le 2\fint_{B_r^+(z)} \Big(\mu^p + |V_\mu(Du)|^2\Big)\,\eta^2\,dx\\ &\le 2\fint_{B_r^+(z)} \Big(\mu^p + \nu^{-1}\big(a(\,\cdot\,,u,Du) - a(\,\cdot\,,u,0)\big)\cdot Du\Big)\,\eta^2\,dx\\ &\le c\big(p,\tfrac{L}{\nu}\big)\fint_{B_r^+(z)} \Big(\mu^p + |Dg|^p + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx + \frac{1}{2}\fint_{B_r^+(z)} (\mu^p + |Du|^p)\,\eta^2\,dx\,. \qquad (6.8)\end{aligned}$$

Absorbing the integral over $\big(\mu^p + |Du|^p\big)\eta^2$ on the left-hand side in the last inequality, we thus find

$$\fint_{B_r^+(z)} (\mu^p + |Du|^p)\,\eta^2\,dx \le c\big(p,\tfrac{L}{\nu}\big)\fint_{B_r^+(z)} \Big(\mu^p + |Dg|^p + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\,. \qquad (6.9)$$

Natural growth condition (B2): Here the estimate for the remaining integral arising from the inhomogeneity is similar to the one in Lemma 3.6 (we note that in the present situation it is not necessary to introduce a number δ as on p. 36 since we do not consider a linear perturbation of u). We assume that the radius $r \le r_0$ is sufficiently small with

$$r_0 := \min\Big\{1 - |z|\,,\, \frac{\nu - 2L_2M}{4L_2(\|Dg\|_{L^\infty}+1)}\Big\}.$$

Applying the growth condition (B2) for the integral IV and paying attention to the smallness assumption $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \le M < \infty$ with $2L_2M < \nu$, we infer from the inequality $|Du|^p \le \mu^p + |V_\mu(Du)|^2$ that

$$\begin{aligned} IV &\le \fint_{B_r^+(z)} \big(L_2|Du|^p + L\big)\,|u-g|\,\eta^2\,dx\\ &\le L_2\fint_{B_r^+(z)} (\mu^p + |V_\mu(Du)|^2)\,\big(|u-g(z'')| + |g(z'')-g(x)|\big)\,\eta^2\,dx + L\fint_{B_r^+(z)} |u-g|\,\eta^2\,dx\\ &\le L_2\,(2\,M + 2\,r\,\|Dg\|_{L^\infty})\fint_{B_r^+(z)} (\mu^p + |V_\mu(Du)|^2)\,\eta^2\,dx + L\fint_{B_r^+(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\\ &\le \Big(L_2\,M + \frac{\nu}{2}\Big)\fint_{B_r^+(z)} \big|V_\mu(Du)\big|^2\eta^2\,dx + \nu\,\mu^p + L\fint_{B_r^+(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\\ &\le \Big(L_2\,M + \frac{\nu}{2}\Big)\fint_{B_r^+(z)} \big|V_\mu(Du)\big|^2\eta^2\,dx + L\fint_{B_r^+(z)} \Big(2 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\,,\end{aligned}$$

where z'' denotes the projection of $z \in \mathbb{R}^n$ onto $\mathbb{R}^{n-1}\times\{0\}$. We further note that, in view of $u = g$ on Γ, we have bounded $g(z'')$ by M from above. We recall $L_2M + \frac{\nu}{2} < \nu$; then we subtract $(L_2M + \frac{\nu}{2})\fint_{B_r^+(z)} \big|V_\mu(Du)\big|^2\eta^2\,dx$ on the right-hand side in (6.7) and combine it with the estimates for I, II and III to get

$$\begin{aligned}\Big(\frac{\nu}{2} - L_2M\Big)&\fint_{B_r^+(z)} \big|V_\mu(Du)\big|^2\,\eta^2\,dx\\ &\le \Big((8^p+1)\,\varepsilon^{1-p}L^p + L + \varepsilon^{\frac{1}{1-p}}L^{\frac{p}{p-1}}\Big)\fint_{B_r^+(z)} \Big(1 + |Dg|^p + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\\ &\quad + 3\,\varepsilon\fint_{B_r^+(z)} (\mu^p + |Du|^p)\,\eta^2\,dx\,;\end{aligned}$$

with the choice $\varepsilon = \frac{\nu - 2L_2 M}{24} = \frac{\nu}{24}(1 - \frac{L_2}{\nu}M)$ this yields analogously to inequality (6.9) in the controllable situation the estimate

$$\fint_{B_r^+(z)} \big(\mu^p + |Du|^p\big)\,\eta^2\,dx \;\le\; c\big(p, \tfrac{L}{\nu}, \tfrac{L_2}{\nu}, M\big) \fint_{B_r^+(z)} \Big(1 + |Dg|^p + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx\,. \tag{6.10}$$

Starting from the inequalities (6.9) in the case of controllable growth and (6.10) in the case of natural growth, we first note that both inequalities still hold true if we replace $\mu \in [0,1]$ by 1. Keeping in mind that g is of class C^1 and using the properties of the cut-off function η, we then end up with the desired Caccioppoli-type estimate in (6.4) with a constant c depending only on $p, \frac{L}{\nu}, \|Dg\|_{L^\infty}$ for controllable growth, and on $p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M$ for natural growth (provided that $r \le r_0$).

The estimate (6.5) in the interior is achieved in the same way using a standard cut-off function η with support in the ball $B_{3r/4}(z) \subset B^+$ and choosing $\varphi = \eta^2(u - (u)_{B_{3r/4}(z)})$ as a test function instead of $\varphi = (u-g)\eta^2$. For inhomogeneities obeying a natural growth condition, we further observe that $|u - (u)_{B_{3r/4}(z)}| \le 2M$. Therefore, we obtain in the estimate of IV the term $2ML_2 \fint_{B_r^+(z)} |V_\mu(Du)|^2\eta^2\,dx$ such that the constant depends also in the interior on the factor $1 - 2M\frac{L_2}{\nu}$ (which is strictly positive by assumption).

In order to finish the proof of the Caccioppoli-type inequalities it still remains to remove the condition $r \le r_0$ required in the calculations above leading to the boundary version (6.4) (under the natural growth condition). Thus, we consider an arbitrary centre $z \in B^+ \cup \Gamma$ and a radius $r \in (r_0, 1-|z|)$ satisfying $z_n \le \frac{3}{4}r$. We now choose a finite number of points z_i satisfying $(z_i)_n \le \frac{3}{4}r_0$ for $i = 1, \ldots, k_1$, and $(z_i)_n > \frac{3}{4}r_0$ for $i = k_1+1, \ldots, k_2$ such that the inclusion

$$B^+_{r/2}(z) \,\subset\, \bigcup_{i=1}^{k_2} B^+_{r_0/2}(z_i)$$

holds. Keeping in mind that $r_0 = r_0(L_2, M, \|Dg\|_{L^\infty})$, the numbers $k_1 \le k_2$ depend only on n, L_2, M and $\|Dg\|_{L^\infty}$. Then, applying (6.5) and (6.4), respectively, we find in a standard way

$$\begin{aligned}
&\fint_{B^+_{r/2}(z)} \big(1 + |Du|^p\big)\,dx \\
&\le r_0^{-p}\, r^{-n}\, c\,\Big(\sum_{i=1}^{k_1} \int_{B^+_{r_0}(z_i)} \big(1 + |u-g|^p\big)\,dx + \sum_{i=k_1+1}^{k_2} \int_{B_{3r_0/4}(z_i)} \big(1 + |u - (u)_{B_{3r_0/4}(z_i)}|^p\big)\,dx\Big) \\
&\le c\,\Big(\sum_{i=1}^{k_1} \int_{B^+_{r_0}(z_i)} \big(1 + |u-g|^p\big)\,dx + \sum_{i=k_1+1}^{k_2} \int_{B_{3r_0/4}(z_i)} \big(1 + |g - (g)_{B_{3r_0/4}(z_i)}|^p + |u-g|^p\big)\,dx\Big) \\
&\le c \sum_{i=1}^{k_2} \int_{B^+_{r_0}(z_i)} \big(1 + |u-g|^p\big)\,dx \;\le\; c_{cacc} \fint_{B_r^+(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx
\end{aligned}$$

with a constant c_{cacc} admitting exactly the dependencies stated above. Lastly, we want to remark how the constant c_{cacc} in (6.4) and (6.5) depends upon the parameters $\frac{L}{\nu}$, $\frac{L_2}{\nu}$ and M: from the choices of ε above we see that c_{cacc} becomes larger and larger as $\frac{L}{\nu}$ increases or as $1 - 2\,\frac{L_2}{\nu}\,M$ approaches 0.

In the next step we apply the Sobolev-Poincaré Lemma A.5 in the zero-boundary-data version, to the inequalities (6.4) in order to get for $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ with $z_n \le \frac{3}{4} r$

$$\begin{aligned} \fint_{B^+_{r/2}(z)} \big(1 + |Du|^p\big)\, dx &\le c_{cacc} \fint_{B^+_r(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\, dx \\ &\le c\Big[1 + \Big(\fint_{B^+_r(z)} |Du - Dg|^{\frac{np}{n+p}}\, dx\Big)^{\frac{n+p}{n}}\Big] \\ &\le c\Big(\fint_{B^+_r(z)} \big(1 + |Du|^p\big)^{\frac{n}{n+p}}\, dx\Big)^{\frac{n+p}{n}}, \end{aligned} \tag{6.11}$$

and the constant c depends on $n, N, p, \frac{L}{\nu}, \|Dg\|_{L^\infty}$ for controllable growth, and on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M$ for natural growth, respectively. We remark that we here have absorbed the term $|Dg|$ in the constant c. In the interior (for $z_n > \frac{3}{4} r$) we apply the Sobolev-Poincaré Lemma A.5 in the mean value version to (6.5) and increase the domain of integration to $B^+_r(z)$; thus, we end up with (6.11) for all $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ (using the generalized notation $B^+_r(z) \equiv B_r(z)$ for balls in the interior). Therefore, we have established a so-called reverse Hölder inequality for the function $x \mapsto (1 + |Du|^p)^{\frac{n}{n+p}}$. In the next step the application of the Gehring Lemma A.14 is performed as in [DGK04, Lemma 3.1] or on p. 75 above: we consider an arbitrary ball $B_\rho(y)$ with $y \in B^+ \cup \Gamma$, $0 < \rho < 1 - |y|$, and define $\Omega := B^+_\rho(y)$ and $A := \partial B_\rho(y) \cap B^+$. Then, in view of inequality (6.11) which is in particular valid for all balls $B_r(z) \cap A = \emptyset$ with $z \in \Omega$, the prerequisite (A.3) of Gehring's Lemma is fulfilled for the function $x \mapsto (1 + |Du|^p)^{\frac{n}{n+p}}$ and exponent $\frac{n+p}{n}$ instead of g and p. Thus, there exist a constant c and an exponent $s > p$ depending on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$ when considering condition (B1), and depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M when considering condition (B2) (we note that we can choose the constant k_Ω independent of ρ because every half-ball satisfies a uniform interior and exterior cone-condition) such that $(1 + |Du|^p)^{\frac{n}{n+p}} \in L^{\frac{n+p}{n}\frac{s}{p}}(B^+_{\rho/2}(y))$ with the estimate

$$\begin{aligned} \Big(\fint_{B^+_{\rho/2}(y)} \big(1 + |Du|\big)^s\, dx\Big)^{\frac{p}{s}} &\le 2^p \Big(\fint_{B^+_{\rho/2}(y)} \big(1 + |Du|^p\big)^{\frac{s}{p}}\, dx\Big)^{\frac{p}{s}} \\ &\le 2^{n(1+\frac{p}{s})+p} \Big(\fint_{B^+_\rho(y)} \frac{\mathcal{L}^n\big(B_{d(x,A)}(x) \cap B^+_\rho(y)\big)}{\mathcal{L}^n\big(B^+_\rho(y)\big)} \big(1 + |Du|^p\big)^{\frac{s}{p}}\, dx\Big)^{\frac{p}{s}} \\ &\le c \fint_{B^+_\rho(y)} \big(1 + |Du|^p\big)\, dx\,. \end{aligned}$$

In the second last line we have used the fact that $\mathcal{L}^n(B^+_\rho(y)) \le 2^n \mathcal{L}^n(B^+_{\rho/2}(y))$ and the inequality

$$\mathcal{L}^n\big(B_{d(x,A)}(x) \cap B^+_\rho(y)\big) \ge \mathcal{L}^n\big(B_{\rho/2}(x) \cap B^+_\rho(y)\big) = \mathcal{L}^n\big(B^+_{\rho/2}(x)\big)$$

for all $x \in B^+_\rho(y) \setminus B^+_{\rho/2}(y)$. Hence, we have finished the proof of the desired higher integrability estimate. □

For bounded weak solutions of systems with inhomogeneities under a natural growth condition the previous calculations allow us to state the following Morrey-type estimate (cf. [Ark03, Lemma 2] for the superquadratic case):

Corollary 6.3: *Assume $u \in g + W^{1,p}_{\Gamma}(B^+,\mathbb{R}^N) \cap L^\infty(B^+,\mathbb{R}^N)$ to be a weak solution to (6.3) with $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$, $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \leq M$ and $2L_2 M < \nu$, where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the conditions (H1) and (H3) and where the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys the natural growth condition (B2). Then for fixed $\sigma \in (0,1)$ we have $Du \in L^{p,n-p}(B^+_{1-\sigma},\mathbb{R}^N)$ with*

$$\|Du\|^p_{L^{p,n-p}(B^+_{1-\sigma},\mathbb{R}^N)} \leq c_\sigma\,(1 + M^p)$$

and the constant c_σ depends on σ and the same parameters as the constant $c^{(2)}$ in the previous Lemma 6.2.

Proof: This is a direct consequence of the Caccioppoli-estimates (6.4) and (6.5) combined with the bounds $\|u\|_{L^\infty(B^+,\mathbb{R}^N)} \leq M$ and $\|u - (u)_{B_r(z)}\|_{L^\infty(B^+,\mathbb{R}^N)} \leq 2M$, respectively. □

6.3 Decay estimate for the solution

In this section we deduce an appropriate decay estimate for the solution u of the original system (6.3) by comparing u with the solution $v \in W^{1,p}(B^+_R(x_0),\mathbb{R}^N)$ of the frozen system

$$\begin{cases} \operatorname{div} a_0(Dv) = 0 & \text{in } B^+_R(x_0)\,, \\ v = u - g & \text{on } \partial B^+_R(x_0)\,, \end{cases} \tag{6.12}$$

where $a_0(z) := a(x_0, (u)_{B^+_R(x_0)}, z)$ are the frozen coefficients, $x_0 \in \Gamma$, and $2R < 1 - |x_0|$. We note that freezing in the average of u as opposed to in 0 turns out to be of advantage also at the boundary (this is due to the fact that our transformation to the model situation does not force u to vanish on Γ). Testing the latter system with $u - g - v$, which is admissible, since the functions $u - g$ and v have the same boundary values, we obtain

$$\begin{aligned} 0 &= \int_{B^+_R(x_0)} a_0(Dv)\cdot(Du - Dg - Dv)\,dx \\ &= \int_{B^+_R(x_0)} \big(a_0(Dv) - a_0(0)\big)\cdot(Du - Dg - Dv)\,dx \\ &= \int_{B^+_R(x_0)} \int_0^1 D_z a_0(tDv)\,Dv\cdot(Du - Dg - Dv)\,dt\,dx\,. \end{aligned}$$

The ellipticity condition (H3) and the growth condition (H2) (applied on the set $\{x \in B^+_R(x_0) : Dv \neq 0\}$), Young's inequality, the technical Lemmas A.2 and A.3 (iii) now yield:

$$\begin{aligned} \nu \int_{B^+_R(x_0)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|^2\,dx &\leq \nu \int_{B^+_R(x_0)} \int_0^1 \big(\mu^2 + |tDv|^2\big)^{\frac{p-2}{2}} |Dv|^2\,dt\,dx \\ &\leq \int_{B^+_R(x_0)} \int_0^1 D_z a_0(tDv)\,Dv\cdot Dv\,dt\,dx \\ &= \int_{B^+_R(x_0)} \int_0^1 D_z a_0(tDv)\,Dv\cdot(Du - Dg)\,dt\,dx \\ &\leq c(p)\,L \int_{B^+_R(x_0)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|\,|Du - Dg|\,dx \\ &\leq \varepsilon \int_{B^+_R(x_0)} \big(\mu^2 + |Dv|^2\big)^{\frac{p-2}{2}} |Dv|^2\,dx + c(p)\,\varepsilon^{1-p} L^p \int_{B^+_R(x_0)} \big(\mu^p + |Du - Dg|^p\big)\,dx\,. \end{aligned}$$

Choosing $\varepsilon = \frac{\nu}{2}$, absorbing the first integral on the right-hand side and reasoning as in (6.8), we end up with an estimate for the p-Dirichlet functional of Dv:

$$\begin{aligned}\int_{B_R^+(x_0)} |Dv|^p\,dx &\le c\big(p,\tfrac{L}{\nu}\big)\int_{B_R^+(x_0)} \big(\mu^p + |Du-Dg|^p\big)\,dx\\ &\le c\big(p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\int_{B_R^+(x_0)} \big(1+|Du|^p\big)\,dx\,. \end{aligned} \tag{6.13}$$

Now, we have in the weak sense

$$\operatorname{div}\big(-a_0(Dv)+a(\cdot\,,u,Du)\big)+b(\cdot\,,u,Du)\;=\;0 \qquad \text{in } B_R^+(x_0)$$

and therefore it also weakly holds that

$$\begin{aligned}&\operatorname{div}\big(a_0(Dv+Dg)-a_0(Du)\big)\\ &\quad=\operatorname{div}\big(a_0(Dv+Dg)-a_0(Dv)\big)+\operatorname{div}\big(a(\cdot\,,u,Du)-a_0(Du)\big)+b(\cdot\,,u,Du)\end{aligned} \tag{6.14}$$

in $B_R^+(x_0)$. To go on we next distinguish the different growth conditions concerning the inhomogeneity $b(\cdot,\cdot,\cdot)$.

6.3.1 Controllable growth of $\mathbf{b}(\cdot,\cdot,\cdot)$

The procedure here is quite similar to the one established in [Cam82b, Section 4], where (partial) Hölder continuity of the solution was discussed in the interior in low dimensions under similar assumptions concerning the coefficients. By Young's inequality combined with the ellipticity condition (H3) (applied on the set where $Dv+Dg-Du\neq 0$, otherwise all the relevant integrals vanish) we first infer

$$\begin{aligned}&2^{\frac{p-2}{2}}\nu\int_{B_R^+(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dx\\ &\le \nu\int_{B_R^+(x_0)}\int_0^1 \big(\mu^2+2(1-t)^2|Du|^2+2t^2|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dt\,dx\\ &\le \nu\int_{B_R^+(x_0)}\int_0^1 \big(\mu^2+\big|Du+t(Dv+Dg-Du)\big|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dt\,dx\\ &\le \int_{B_R^+(x_0)}\int_0^1 D_z a_0\big(Du+t(Dv+Dg-Du)\big)\big(Dv+Dg-Du\big)\cdot\big(Dv+Dg-Du\big)\,dt\,dx\\ &= \int_{B_R^+(x_0)} \big(a_0(Dv+Dg)-a_0(Du)\big)\cdot\big(Dv+Dg-Du\big)\,dx\,.\end{aligned}$$

Using $u-g-v\in W_0^{1,p}(B_R^+(x_0),\mathbb{R}^N)$ in relation (6.14) as a test function, we may rewrite the last line of the previous inequality and we get

$$\begin{aligned}&2^{\frac{p-2}{2}}\nu\int_{B_R^+(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}|Du-Dv-Dg|^2\,dx\\ &\le \int_{B_R^+(x_0)} \big(a_0(Dv+Dg)-a_0(Dv)\big)\cdot\big(Dv+Dg-Du\big)\,dx\\ &\qquad+\int_{B_R^+(x_0)} \big(a(\cdot\,,u,Du)-a_0(Du)\big)\cdot\big(Dv+Dg-Du\big)\,dx\\ &\qquad-\int_{B_R^+(x_0)} b(\cdot\,,u,Du)\cdot(v+g-u)\,dx\;=:\;I+II+III\,.\end{aligned} \tag{6.15}$$

The terms on the right-hand side are bounded from above separately: via the growth condition (H2) on the set $\{x \in B_R^+(x_0) : Dg \neq 0\}$, Lemma A.2, Young's inequality and the energy estimate (6.13), we estimate term I and, in view of $p < 2$, we obtain

$$\begin{aligned} I &\le c(p)\, L \int_{B_R^+(x_0)} \big(\mu^2 + |Dv|^2 + |Dg|^2\big)^{\frac{p-2}{2}} |Dg|\, |Dv + Dg - Du|\, dx \\ &\le c(p, \|Dg\|_{L^\infty})\, L \int_{B_R^+(x_0)} \big(1 + |Dv - Du|\big)\, dx \\ &\le c(p, \tfrac{L}{\nu}, \|Dg\|_{L^\infty})\, L \Big(\delta \int_{B_R^+(x_0)} \big(1 + |Du|^p\big)\, dx + R^n\, \delta^{1-p}\Big) \end{aligned} \tag{6.16}$$

for every $\delta \in (0,1)$. For the second term we first use assumption (H4) (recalling the definition $a_0(\cdot) := a(x_0, (u)_{B_R^+(x_0)}, \cdot)$ of the frozen coefficients) and Hölder's inequality (note $\omega(\cdot) \le 1$) with

$$\tfrac{p-1}{p}\,\tfrac{s-p}{s} + \tfrac{p-1}{p}\,\tfrac{p}{s} + \tfrac{1}{p} = 1\,,$$

where $s > p$ denotes the (up-to-the-boundary) higher integrability exponent of the gradient Du from Lemma 6.2 depending only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. In view of Young's inequality we then obtain

$$\begin{aligned} II &\le L \int_{B_R^+(x_0)} \omega\big(|x - x_0| + |u - (u)_{B_R^+(x_0)}|\big)\big(\mu^2 + |Du|^2\big)^{\frac{p-1}{2}} |Du - Dg - Dv|\, dx \\ &\le \big|B_R^+(x_0)\big|\, L \Big(\fint_{B_R^+(x_0)} \omega\big(|x - x_0| + |u - (u)_{B_R^+(x_0)}|\big)\, dx\Big)^{\frac{p-1}{p}\,\frac{s-p}{s}} \\ &\quad \times \Big(\fint_{B_R^+(x_0)} \big(\mu^2 + |Du|^2\big)^{\frac{p-1}{2}\,\frac{p}{p-1}\,\frac{s}{p}}\, dx\Big)^{\frac{p-1}{p}\,\frac{p}{s}} \Big(\fint_{B_R^+(x_0)} |Du - Dg - Dv|^p\, dx\Big)^{\frac{1}{p}} \\ &\le \big|B_R^+(x_0)\big|\, L \Big(\fint_{B_R^+(x_0)} \omega\big(R + |u - (u)_{B_R^+(x_0)}|\big)\, dx\Big)^{\frac{p-1}{p}\,\frac{s-p}{s}} \\ &\quad \times \Big(\fint_{B_R^+(x_0)} \big(\mu^p + |Du|^p\big)^{\frac{s}{p}}\, dx\Big)^{\frac{p-1}{p}\,\frac{p}{s}} \Big(3^{p-1} \fint_{B_R^+(x_0)} \big(|Du|^p + \|Dg\|_{L^\infty}^p + |Dv|^p\big)\, dx\Big)^{\frac{1}{p}}. \end{aligned}$$

To continue estimating term II we define

$$\beta := \frac{p-1}{p}\,\frac{s-p}{s} \tag{6.17}$$

(where $s = s(n, N, p, \frac{L}{\nu}, \|Dg\|_{L^\infty})$ is given in Lemma 6.2), and we recall that $\omega(\cdot)$ is concave and monotone non-decreasing. Making use of the higher integrability estimate for $1 + |Du|^p$, which was proved in Lemma 6.2, the energy estimate (6.13) and Jensen's inequality we then find:

$$\begin{aligned} II &\le \big|B_R^+(x_0)\big|\, L\, c\, \omega^\beta \Big(\fint_{B_R^+(x_0)} \big(R + |u - (u)_{B_R^+(x_0)}|\big)\, dx\Big) \\ &\quad \times \Big(\fint_{B_{2R}^+(x_0)} \big(1 + |Du|^p\big)\, dx\Big)^{\frac{p-1}{p}} \Big(\fint_{B_R^+(x_0)} \big(1 + |Du|^p\big)\, dx\Big)^{\frac{1}{p}} \\ &\le L\, c\, \omega^\beta \Big(\Big(\fint_{B_R^+(x_0)} \big(R^p + |u - (u)_{B_R^+(x_0)}|^p\big)\, dx\Big)^{\frac{1}{p}}\Big) \int_{B_{2R}^+(x_0)} \big(1 + |Du|^p\big)\, dx \end{aligned}$$

$$\leq L\, c\big(n, N, p, \tfrac{L}{\nu}, \|Dg\|_{L^\infty}\big)\, \omega^\beta\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac{1}{p}}\Big)$$
$$\times \int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx\,, \tag{6.18}$$

where we have used the Poincaré inequality in the last line.

Finally we estimate the remaining term III appearing on the right-hand side in inequality (6.15): by the growth condition imposed on $b(x,u,Du)$ in (B1) and Hölder's inequality we have

$$\begin{aligned} III &= -\int_{B_R^+(x_0)} b(\,\cdot\,,u,Du)\cdot(v+g-u)\,dx \\ &\leq L\int_{B_R^+(x_0)}\big(\mu^2+|Du|^2\big)^{\frac{p-1}{2}}\,|v+g-u|\,dx \\ &\leq L\Big(\int_{B_R^+(x_0)}\big(\mu^p+|Du|^p\big)\,dx\Big)^{\frac{p-1}{p}}\Big(\int_{B_R^+(x_0)}|v+g-u|^p\,dx\Big)^{\frac{1}{p}}\,. \end{aligned}$$

Keeping in mind that the functions $u-g$ and v have the same values on the boundary $\partial B_R^+(x_0)$, the second term is estimated via the Poincaré inequality and then (6.13), and we obtain

$$\begin{aligned} \int_{B_R^+(x_0)}|v+g-u|^p\,dx &\leq c(n,N,p)\,R^p\int_{B_R^+(x_0)}|Dv+Dg-Du|^p\,dx \\ &\leq c(n,N,p)R^p\int_{B_R^+(x_0)}\big(|Dv|^p+\|Dg\|_{L^\infty}^p+|Du|^p\big)\,dx \\ &\leq c\big(n,N,p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,R^p\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\,. \end{aligned}$$

Therefore, we conclude

$$III \leq L\,c\big(n,N,p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,R\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\,. \tag{6.19}$$

Merging the estimates for I, II and III, i.e., (6.16), (6.18) and (6.19), with (6.15), we find the comparison estimate

$$\begin{aligned} &\int_{B_R^+(x_0)}\big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}\,|Du-Dv-Dg|^2\,dx \\ &\quad\leq c\Big[\omega^\beta\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac{1}{p}}\Big)+R+\delta\Big]\int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx+c\,R^n\,\delta^{1-p} \end{aligned} \tag{6.20}$$

for every $\delta\in(0,1)$, and the constant c depends only on $n,N,p,\frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. In the next step, in a standard way we transfer the decay properties of v to the weak solution u of the original Dirichlet problem (6.3). We first recall the exponent γ_0 defined by

$$\gamma_0 = \min\{2+\varepsilon, n\} \tag{6.21}$$

for some $\varepsilon>0$ depending only on n,N,p and $\frac{L}{\nu}$ (for the precise derivation of γ_0 we refer to Lemma 4.5). Then, Corollary 4.6 provides the decay estimate

$$\int_{B_\rho^+(x_0)}|Dv|^p\,dx \leq c\big(n,N,p,\tfrac{L}{\nu}\big)\Big(\frac{\rho}{R}\Big)^{\gamma_0}\int_{B_R^+(x_0)}\big(1+|Dv|^p\big)\,dx$$

for all radii $\rho \in (0, R]$ for the solution v of the comparison problem (6.12) with constant (frozen) coefficients (keeping in mind $v = 0$ on $\Gamma_\rho(x_0)$ by definition). In view of $\gamma_0 \leq n$ we further note that

$$\int_{B_\rho^+(x_0)} \big(1 + |Dg|^p\big)\, dx \;\leq\; c\big(\|Dg\|_{L^\infty}\big) \Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_R^+(x_0)} 1\, dx$$

for all $\rho \in (0, R]$. We now observe from Lemma A.3 (ii) that the inequality

$$1 + |Du|^p \;\leq\; c\big(n, N, p\big) \Big[\big(1 + |Dv + Dg|^p\big) + \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\Big]$$

holds true. Thus, combining the last three inequalities and taking advantage of the energy inequality (6.13) gives

$$\begin{aligned}
\int_{B_\rho^+(x_0)} \big(1 + |Du|^p\big)\, dx &\leq c \int_{B_\rho^+(x_0)} \big(1 + |Dv + Dg|^p\big)\, dx \\
&\quad + c \int_{B_\rho^+(x_0)} \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\, dx \\
&\leq c \Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_R^+(x_0)} \big(1 + |Du|^p\big)\, dx \\
&\quad + c \int_{B_R^+(x_0)} \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\, dx
\end{aligned}$$

for every radius $\rho \in (0, R]$, and the constant c depends on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. Replacing the second integral appearing on the right-hand side of the previous inequality by the estimate in (6.20), we finally arrive at a decay estimate for the gradient Du:

$$\begin{aligned}
&\int_{B_\rho^+(x_0)} \big(1 + |Du|^p\big)\, dx \\
&\quad \leq c \Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\Big((2R)^{p-n} \int_{B_{2R}^+(x_0)} \big(1 + |Du|^p\big)\, dx\Big)^{\frac{1}{p}}\Big) + R + \delta\Big] \\
&\qquad\qquad \times \int_{B_{2R}^+(x_0)} \big(1 + |Du|^p\big)\, dx + c\, R^n\, \delta^{1-p}\,,
\end{aligned}$$

with the constant c depending on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$. The same inequality trivially holds if $\rho \in (R, 2R]$. If we define the *Excess function*

$$\Phi(x_0, r) \;:=\; \int_{B_r^+(x_0)} \big(1 + |Du|^p\big)\, dx\,,$$

the last estimate can be rewritten in the following form:

$$\Phi(x_0, \rho) \;\leq\; c \Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\big((2R)^{p-n}\Phi(x_0, 2R)\big)^{\frac{1}{p}}\Big) + R + \delta\Big]\, \Phi(x_0, 2R) + c\, R^n\, \delta^{1-p}$$

for all $x_0 \in \Gamma$, $2R < 1 - |x_0|$ and every $\rho \in (0, 2R]$. This estimate is similar to inequality (4.23) achieved in [Cam82b], where regularity up to the boundary of weak solutions was considered in the low-dimensional (non-degenerate) case with $p > 2$.

We note that the latter estimate also follows in the interior, i. e., for balls $B_R(x_0)$ contained in B^+ (or in the interior of Ω). Here we do not need to take into account the function g

(which specifies the boundary values of u on Γ), and hence term I does not appear in the calculations corresponding to (6.15) in the interior. All other estimates above as well as the conclusion of (6.22) below remain valid, and we can choose the same constant c. Replacing $2R$ by R we thus conclude altogether (note that the Excess function $\Phi(x_0, r)$ is defined for arbitrary centre $x_0 \in B^+ \cup \Gamma$ and radius $0 < r < 1 - |x_0|$ on the set $B_r(x_0) \cap B^+$):

Lemma 6.4: *Let β, γ_0 be chosen as above in (6.17), (6.21), and let $\delta \in (0,1)$. Furthermore, let $u \in g + W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$, $1 < p < 2$, be a weak solution of the system (6.3) under the assumptions (H1)-(H4), (B1), and $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$. Then, if $x_0 \in \Gamma$, $R < 1 - |x_0|$ or if $x_0 \in B^+$, $R < \min\{1 - |x_0|, (x_0)_n\}$, there holds*

$$\Phi(x_0, \rho) \le c^{(1)}_{ex} \Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\big(R^{p-n}\Phi(x_0,R)\big)^{\frac{1}{p}}\Big) + R + \delta \Big] \Phi(x_0, R) + c^{(1)}_{ex} R^n \delta^{1-p} \quad (6.22)$$

for every $\rho \in (0, R]$, and the constant $c^{(1)}_{ex}$ depends only on $n, N, p, \frac{L}{\nu}$ and $\|Dg\|_{L^\infty}$.

6.3.2 Natural growth of $\mathbf{b}(\cdot,\cdot,\cdot)$

In what follows, we proceed analogously to the situation of the controllable growth condition (B1) for the inhomogeneity $b(\cdot,\cdot,\cdot)$. Therefore, we sometimes refer to the corresponding estimates in the last section. For the modifications necessary for natural growth we adapt the techniques used in [Ark03, proof of Theorem 1].

Fix $\sigma \in (0,1)$. We consider the unique solution $v \in W^{1,p}(B^+_R(x_0), \mathbb{R}^N)$ to the Dirichlet problem (6.12), where $x_0 \in \Gamma_{1-\sigma}$, $2R < 1 - \sigma - |x_0|$, and again aim for a comparison of the functions u and v. Furthermore, let $n < p + \gamma_0$. It still holds (6.14), i. e., we have

$$\begin{aligned} &\operatorname{div}\big(a_0(Dv + Dg) - a_0(Du)\big) \\ &\quad = \operatorname{div}\big(a_0(Dv + Dg) - a_0(Dv)\big) + \operatorname{div}\big(a(\,\cdot\,, u, Du) - a_0(Du)\big) + b(\,\cdot\,, u, Du) \end{aligned} \quad (6.23)$$

in $B^+_R(x_0)$ in the weak sense, but, in contrast to above, we may test the system only with bounded functions in $W^{1,p}_0(B^+_R(x_0), \mathbb{R}^N) \cap L^\infty(B^+_R(x_0), \mathbb{R}^N)$ (according to the growth condition (B2)). Hence, in order to be allowed to test our system with the function $u - v - g$ as before under the controllable growth assumption, we start by proving a qualitative L^∞-estimate for v on $B^+_{R/2}(x_0)$:

Consider a ball $B_\rho(y)$ with centre $y \in \overline{B^+_{R/2}}(x_0)$ and radius $\rho < \frac{R}{2}$. According to Corollary 4.7 we have

$$⨍_{B^+_\rho(y)} |v|^p\,dx \le c\big(n, N, p, \tfrac{L}{\nu}\big) \Big[R^{-n} \int_{B^+_{R/2}(y)} |v|^p\,dx + R^{p-n} \int_{B^+_{R/2}(y)} \big(\mu^p + |Dv|^p\big)\,dx \Big]$$

(it is obvious that we may allow $|y - x_0| = R/2$). Hence, taking advantage of $B^+_{R/2}(y) \subset B^+_R(x_0)$, the Poincaré inequality (keeping in mind $v = 0$ on $\Gamma_R(x_0)$ by definition), and the estimate (6.13) for the p-Dirichlet functional of Dv, we estimate the mean values of $|v|^p$ as follows:

$$\begin{aligned}\sup_{\substack{y\in B^+_{R/2}(x_0)\\ \rho\in(0,R/2)}} \fint_{B^+_\rho(y)} |v|^p\,dx &\le c\Big[R^{-n}\int_{B^+_R(x_0)} |v|^p\,dx + R^{p-n}\int_{B^+_R(x_0)} \big(\mu^p+|Dv|^p\big)\,dx\Big]\\ &\le c\,R^{p-n}\int_{B^+_R(x_0)} \big(\mu^p+|Dv|^p\big)\,dx\\ &\le c\,R^{p-n}\int_{B^+_R(x_0)} \big(1+|Du|^p\big)\,dx\\ &\le c\big(n,N,p,\tfrac{L}{\nu},\tfrac{L_2}{\nu},\|Dg\|_{L^\infty},M,\sigma\big) \;=:\; m_0^p\,,\end{aligned}$$

where we have used Corollary 6.3 in the last line. According to Lebesgue's Differentiation Theorem this yields $v\in L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)$, see also Remark 2.2, with

$$\|v\|_{L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)} \;\le\; m_0^p\,. \tag{6.24}$$

Therefore we have $u-v-g\in W^{1,p}_0(B^+_R(x_0),\mathbb{R}^N)\cap L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)$ with

$$\begin{aligned}\|u-v-g\|_{L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)} &\le \|u-g(x_0)\|_{L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)} + \|g-g(x_0)\|_{L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)}\\ &\quad+\|v\|_{L^\infty(B^+_{R/2}(x_0),\mathbb{R}^N)}\\ &\le\; 2M+\|Dg\|_{L^\infty}+m_0 \;=:\; m\;>\;0\,.\end{aligned}$$

To obtain an admissible test-function for the system (6.23), we next modify the function $u-v-g$ on $B^+_R(x_0)$ (for which we cannot expect an L^∞-estimate) as follows: we set

$$h \;:=\; (v+g-u)\,\big(T^\delta-(|v+g-u|+m)^\delta\big)_+$$

for some exponent $\delta>0$ to be determined later and a number $T=T(\delta,m)>0$ determined by the condition

$$T^\delta-(2m)^\delta=\tfrac12\,T^\delta \quad\Leftrightarrow\quad T\;=\;2^{1+\frac1\delta}\,m\,. \tag{6.25}$$

In particular, $\delta\to0$ implies $T\to\infty$, and via the estimate $|u-v-g|\le m$ on $B^+_{R/2}(x_0)$ found above we note that we have

$$\big(T^\delta-(|v+g-u|+m)^\delta\big)_+ \;\ge\; \tfrac12\,T^\delta \qquad \text{on } B^+_{R/2}(x_0)\,.$$

Keeping in mind that the function h vanishes outside of the set $\theta_+ := \{x\in B^+_R(x_0) : |v+g-u|<T-m\}$, we observe that the weak differentiability of $v+g-u$ is transferred to h, and hence, by construction we have $h\in W^{1,p}_0(B^+_R(x_0),\mathbb{R}^N)\cap L^\infty(B^+_R(x_0),\mathbb{R}^N)$. In particular, this implies that testing the system (6.23) with the function h is allowed. We next proceed similarly to (6.15), but we have to take into account a new term which arises by this modification:

$$\begin{aligned}&2^{\frac{p-4}{2}}T^\delta\,\nu\int_{B^+_{R/2}(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}\,|Du-Dv-Dg|^2\,dx\\ &\quad\le\; 2^{\frac{p-2}{2}}\nu\int_{B^+_R(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}\,|Du-Dv-Dg|^2\\ &\qquad\qquad\times\big(T^\delta-(|v+g-u|+m)^\delta\big)_+\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \nu \int_{B_R^+(x_0)} \int_0^1 \big(\mu^2 + \big|Du + t(Dv + Dg - Du)\big|^2\big)^{\frac{p-2}{2}} |Du - Dv - Dg|^2 \, dt \\
&\qquad\qquad\qquad\qquad \times \big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \, dx \\
&\leq \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot \big(Dv + Dg - Du\big) \big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \, dx \\
&= \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot Dh \, dx \\
&\quad + \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot (v + g - u) \otimes \frac{(Dv + Dg - Du) \cdot (v + g - u)}{|v + g - u|} \\
&\qquad\qquad\qquad\qquad \times \delta \, (|v + g - u| + m)^{\delta - 1} \, \mathbb{1}_{\theta_+} \, dx \, .
\end{aligned}
$$

Using the system (6.23) given above with the test function h, we further estimate the first integral on the right-hand side of the last inequality. Hence, we find exactly as in the calculations leading to (6.15):

$$
\begin{aligned}
&2^{\frac{p-4}{2}} T^\delta \, \nu \int_{B_{R/2}^+(x_0)} \big(\mu^2 + |Du|^2 + |Dv + Dg|^2\big)^{\frac{p-2}{2}} \, |Du - Dv - Dg|^2 \, dx \\
&\quad \leq \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Dv)\big) \cdot Dh \, dx \\
&\qquad + \int_{B_R^+(x_0)} \big(a(\,\cdot\,, u, Du) - a_0(Du)\big) \cdot Dh \, dx - \int_{B_R^+(x_0)} b(\,\cdot\,, u, Du) \cdot h \, dx \\
&\qquad + \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot (v + g - u) \otimes \frac{(Dv + Dg - Du) \cdot (v + g - u)}{|v + g - u|} \\
&\qquad\qquad\qquad\qquad \times \delta \, (|v + g - u| + m)^{\delta - 1} \, \mathbb{1}_{\theta_+} \, dx \\
&\quad = \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Dv)\big) \cdot \big(Dv + Dg - Du\big) \big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \, dx \\
&\qquad + \int_{B_R^+(x_0)} \big(a(\,\cdot\,, u, Du) - a_0(Du)\big) \cdot \big(Dv + Dg - Du\big) \big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \, dx \\
&\qquad - \int_{B_R^+(x_0)} b(\,\cdot\,, u, Du) \cdot (v + g - u) \big(T^\delta - (|v + g - u| + m)^\delta\big)_+ \, dx \\
&\qquad + \delta \int_{B_R^+(x_0)} \big(a_0(Dv) - a(\,\cdot\,, u, Du)\big) \cdot (v + g - u) \otimes \frac{(v + g - u) \cdot (Dv + Dg - Du)}{|v + g - u|} \\
&\qquad\qquad\qquad\qquad \times (|v + g - u| + m)^{\delta - 1} \, \mathbb{1}_{\theta_+} \, dx \\
&\quad =: I' + II' + III' + IV'
\end{aligned}
\tag{6.26}
$$

with the obvious abbreviations. We first note $(T^\delta - (|v + g - u| + m)^\delta)_+ \leq T^\delta$. Therefore, term I' and term II' are estimated as term I in (6.16) and term II in (6.18), respectively, in the controllable growth situation, and we get

$$
\begin{aligned}
|I'| &\leq T^\delta \, c(p, \tfrac{L}{\nu}, \|Dg\|_{L^\infty}) \, L\Big(\delta \int_{B_R^+(x_0)} \big(1 + |Du|^p\big) \, dx + R^n \, \delta^{1-p}\Big) \, , \\
|II'| &\leq T^\delta \, L \, c\big(n, N, p, \tfrac{L}{\nu}, \|Dg\|_{L^\infty}\big) \, \omega^\beta\Big(\Big(R^{p-n} \int_{B_R^+(x_0)} \big(1 + |Du|^p\big) \, dx\Big)^{\frac{1}{p}}\Big) \\
&\qquad\qquad\qquad\qquad \times \int_{B_{2R}^+(x_0)} \big(1 + |Du|^p\big) \, dx \, .
\end{aligned}
$$

The growth condition (B2) yields for the third term:

$$\begin{aligned}|III'| &\leq \int_{B_R^+(x_0)} \big(L_2|Du|^p + L\big)\, |v+g-u|\, \big(T^\delta - (|v+g-u|+m)^\delta\big)_+\, dx \\ &\leq T^\delta\, (L_2+L) \int_{B_R^+(x_0)} \big(1+|Du|^p\big)\, |v+g-u|\, \mathbb{1}_{\theta_+}\, dx\,.\end{aligned}$$

Taking into account Hölder's inequality, Lemma 6.2 on higher integrability (where s denotes the higher integrability exponent depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M), the basic inequality $|v+g-u|\,\mathbb{1}_{\theta_+} < T-m \leq T$ and the Poincaré inequality, term III is further estimated by

$$\begin{aligned}&|III'| \\ &\leq T^\delta\,(L_2+L)\,|B_R^+(x_0)| \Big(\fint_{B_R^+(x_0)} (1+|Du|^p)^{\frac{s}{p}}\, dx\Big)^{\frac{p}{s}} \Big(\fint_{B_R^+(x_0)} (|v+g-u|\,\mathbb{1}_{\theta_+})^{\frac{s}{s-p}}\, dx\Big)^{\frac{s-p}{s}} \\ &\leq T^\delta\, c^{(2)}\big(n,N,p,\tfrac{L}{\nu},\tfrac{L_2}{\nu},M,\|Dg\|_{L^\infty}\big)\,(L_2+L) \int_{B_{2R}^+(x_0)} \big(1+|Du|^p\big)\, dx \\ &\qquad\qquad \times \big(|v+g-u|\,\mathbb{1}_{\theta_+}\big)^{(\frac{s}{s-p}-p)\,\frac{s-p}{s}} \Big(\fint_{B_R^+(x_0)} |v+g-u|^p\, dx\Big)^{\frac{s-p}{s}} \\ &\leq T^\delta\, c\big(n,N,p,\tfrac{L}{\nu},\tfrac{L_2}{\nu},M,\|Dg\|_{L^\infty}\big)\,(L_2+L)\, T^{1-\frac{p(s-p)}{s}} \Big(R^{p-n}\int_{B_R^+(x_0)} \big(1+|Du|^p\big)\, dx\Big)^{\frac{s-p}{s}} \\ &\qquad\qquad \times \int_{B_{2R}^+(x_0)} \big(1+|Du|^p\big)\, dx\,.\end{aligned}$$

In the last line we have used once again the energy estimate (6.13). For the last integral IV', we obtain via (H1), Young's inequality and (6.13):

$$\begin{aligned}|IV'| &\leq 2\,\delta\, L \int_{B_R^+(x_0)} \big(\mu^{p-1}+|Du|^{p-1}+|Dv|^{p-1}\big)\,\big(|Du|+|Dv|+\|Dg\|_{L^\infty}\big) \\ &\qquad\qquad \times |v+g-u|\,(|v+g-u|+m)^{\delta-1}\,\mathbb{1}_{\theta_+}\, dx \\ &\leq T^\delta\, c\big(\|Dg\|_{L^\infty}\big)\,\delta\, L \int_{B_R^+(x_0)} \big(1+|Du|^p+|Dv|^p\big)\, dx \\ &\leq T^\delta\, c\big(p,\tfrac{L}{\nu},\|Dg\|_{L^\infty}\big)\,\delta\, L \int_{B_R^+(x_0)} \big(1+|Du|^p\big)\, dx\,.\end{aligned}$$

Hence, combining the estimates for the terms I', II', III' and IV' with (6.26) we finally arrive at

$$\begin{aligned}&\int_{B_{R/2}^+(x_0)} \big(\mu^2+|Du|^2+|Dv+Dg|^2\big)^{\frac{p-2}{2}}\,|Du-Dv-Dg|^2\, dx \\ &\leq c\,\Big[\omega^\beta\Big(\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac{1}{p}}\Big) \\ &\qquad + T^{1-\frac{p(s-p)}{s}}\Big(R^{p-n}\int_{B_R^+(x_0)}\big(1+|Du|^p\big)\,dx\Big)^{\frac{s-p}{s}} + \delta\Big] \int_{B_{2R}^+(x_0)}\big(1+|Du|^p\big)\,dx + c\,R^n\,\delta^{1-p}\end{aligned}$$

with a constant c depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M. This estimate corresponds to (6.20) above for systems under a controllable growth assumption. For a similar up-to-the-boundary estimate (concerning the superquadratic case for non-degenerate systems) we refer to inequality (36) in [Ark03]. Furthermore, we note that the same reasoning leading to the latter inequality also applies for balls $B_R(x_0) \subset B^+_{1-\sigma}$, and thus, a corresponding estimate (without the function g) also holds in the interior. Following the arguments of the comparison principle in the last section and recalling the definition $\Phi(x_0,r) = \int_{B_r(x_0)\cap B^+} \big(1+|Du|^p\big)\,dx$ of the Excess function, we then deduce the following decay estimate for the gradient Du:

Lemma 6.5: *Let β, γ_0 be chosen as above in (6.17), (6.21), and let $M<\infty$, $\delta\in(0,1)$, $\sigma\in(0,1)$ and $n<p+\gamma_0$. Furthermore, let $u\in g+W^{1,p}_\Gamma(B^+,\mathbb{R}^N)\cap L^\infty(B^+,\mathbb{R}^N)$, $1<p<2$, satisfying $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)}\le M$ be a weak solution of the system (6.3) under the assumptions (H1)-(H4), (B2), (6.2), and $g\in C^1(B^+\cup\Gamma,\mathbb{R}^N)$. Then, if $x_0\in\Gamma_{1-\sigma}$, $R<1-\sigma-|x_0|$ or if $x_0\in B^+$, $R<\min\{1-\sigma-|x_0|,(x_0)_n\}$, there holds*

$$\Phi(x_0,\rho) \le c_{ex}^{(2)}\Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0} + \omega^\beta\Big(\big(R^{p-n}\Phi(x_0,R)\big)^{\frac{1}{p}}\Big) + T^{1-\frac{p(s-p)}{s}}\big(R^{p-n}\Phi(x_0,R)\big)^{\frac{s-p}{s}} + \delta\Big]\,\Phi(x_0,R) + c_{ex}^{(2)}\,R^n\,\delta^{1-p} \tag{6.27}$$

for every $\rho\in(0,R]$. Here, the constant $c_{ex}^{(2)}$ depends only on $n,N,p,\frac{L}{\nu},\frac{L_2}{\nu},\|Dg\|_{L^\infty}$ and M, s is the higher integrability exponent from Lemma 6.2 admitting the same dependencies, and T is a positive number additionally depending on σ and δ.

6.4 Proof of Theorem 6.1

Now, we prove a (partial) regularity result in the model situation of the unit half-ball. This in turn yields the statement of Theorem 6.1 using a transformation which flattens the boundary locally and a covering argument in a standard way (see Chapter 3.2).

Theorem 6.6: *Let $u\in W^{1,p}(B^+,\mathbb{R}^N)$ be a weak solution of*

$$-\operatorname{div} a(\,\cdot\,,u,Du) = b(\,\cdot\,,u,Du) \qquad \text{in } B^+$$

with $u=g$ on Γ, $g\in C^1(B^+\cup\Gamma,\mathbb{R}^N)$, and coefficients $a: B^+\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ satisfying the assumptions (H1)-(H4), and inhomogeneity $b: B^+\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}^N$. If one of the following assumptions is fulfilled:

1. *$b(\cdot,\cdot,\cdot)$ obeys a controllable growth condition (B1),*
2. *$b(\cdot,\cdot,\cdot)$ obeys a natural growth condition (B2); additionally, we assume $u\in L^\infty(B^+,\mathbb{R}^N)$ with $\|u\|_{L^\infty(B^+,\mathbb{R}^N)}\le M$ and $2L_2M<\nu$,*

then there exists a constant $\delta_2>0$ depending only on n,N,p and $\frac{L}{\nu}$ such that if $n>p>n-2-\delta_2$, then there holds

$$\dim_{\mathcal{H}}\big((B^+\cup\Gamma)\setminus \operatorname{Reg}_u(B^+\cup\Gamma)\big) < n-p\,.$$

Moreover,

$$u\in C^{0,\lambda}_{\mathrm{loc}}\big(\operatorname{Reg}_u(B^+\cup\Gamma),\mathbb{R}^N\big) \qquad \forall\,\lambda\in\big(0,\min\{1-\tfrac{n-2-\delta_2}{p},1\}\big)$$

and the singular set $\operatorname{Sing}_u(B^+\cup\Gamma)$ of u is contained in

$$\widetilde{\Sigma} := \Big\{x\in B^+\cup\Gamma : \liminf_{R\searrow 0} R^{p-n}\int_{B_R(x)\cap B^+}\big(1+|Du|^p\big)\,dy > 0\Big\}.$$

Proof: In the sequel we will discuss only the case of natural growth. The result for the controllable growth condition follows completely analogously (the proof is actually simpler).

We first fix ε in dependency of n, N, p and $\frac{L}{\nu}$ to be the positive number stemming from the application of Gehring's Lemma (see also Lemma 4.5) if $n \geq 3$ and $\varepsilon = 2p(1-\lambda)$, $\lambda \in (0,1)$ arbitrary, if $n = 2$. We set $\gamma_0 = \min\{2+\varepsilon, n\}$ admitting the same dependencies and choose $\kappa_0 < 1$ according to Lemma A.11 in dependency of the exponents γ_0, $\gamma_0 - \frac{\varepsilon}{2}$ instead of α, β and the constant $c_{ex}^{(2)}$ in (6.22) instead of A. Furthermore, let s be the higher integrability exponent from Lemma 6.2 depending on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}$ and M, and set $\beta = \frac{p-1}{p}\frac{s-p}{s}$.

Furthermore, we fix $\sigma \in (0,1)$, and set $\delta = \frac{\kappa_0}{4}$, which in turn fixes a number $T > 0$ (according to Lemma 6.5) in dependency of $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, \|Dg\|_{L^\infty}, M, \sigma$ and δ. Since $\omega(\cdot)$ is a modulus of continuity, we then find a positive number ς such that

$$\omega^\beta\big(\varsigma^{\frac{1}{p}}\big) < \frac{\kappa_0}{4} \qquad \text{and} \qquad T^{1-\frac{p(s-p)}{s}}\,\varsigma^{\frac{s-p}{s}} < \frac{\kappa_0}{4}\,.$$

We now consider a *regular point* $x_0 \in B^+_{1-\sigma}$, this means a point $x_0 \in B^+_{1-\sigma} \setminus \widetilde{\Sigma}$ where the excess quantity $R^{p-n}\Phi(x_0,R)$ becomes arbitrarily small for $R \searrow 0$. Hence there exists a radius $R_0 > 0$ such that $B_{R_0}(x_0) \Subset B_{1-\sigma}$ and

$$R_0^{p-n} \int_{B^+_{R_0}(x_0)} \big(1+|Du|^p\big)\,dx \;=\; R_0^{p-n}\,\Phi(x_0,R_0) \;<\; \varsigma\,.$$

Since the function $z \mapsto R_0^{p-n}\,\Phi(z,R_0)$ is continuous, there exists a ball $B_r(x_0)$ such that for all $z \in B_r(x_0) \cap (B^+ \cup \Gamma)$ we have $B_{R_0}(z) \Subset B_{1-\sigma}$ and such that the previous inequality is also satisfied when we replace x_0 by z, i. e., there holds

$$R_0^{p-n}\,\Phi(z,R_0) \;<\; \varsigma \qquad \text{for all } z \in B_r(x_0) \cap (B^+ \cup \Gamma)\,.$$

Our next goal is to show that the gradient Du belongs to an appropriate Morrey space on $B_r(x_0) \cap (B^+ \cup \Gamma)$. To this aim we will show Morrey-type estimates of the form

$$\Phi(z,\rho) \;\leq\; c\left[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\right] \tag{6.28}$$

for all balls $B^+_\rho(z)$ with centre $z \in B_r(x_0) \cap (B^+ \cup \Gamma)$, radius $\rho \leq R_0$, and a constant c which depends only on $n, N, p, \frac{L}{\nu}, \frac{L_2}{\nu}, M$ and $\|Dg\|_{L^\infty}$. We next have to combine the estimates at the boundary and in the interior and thus, we need to distinguish several cases:

Case 1: $z \in \Gamma$, $0 < \rho \leq R_0$:
In view of the choices of $\sigma, \delta, \kappa_0, \varsigma$ and R_0 made above, the boundary version of Lemma 6.5 gives

$$\begin{aligned}
\Phi(z,\rho) &\leq c_{ex}^{(2)}\left[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\kappa_0}{4}\right]\Phi(x_0,R_0) + 4^{p-1}\,c_{ex}^{(2)}\,R_0^n\,\kappa_0^{1-p} \\
&\leq c\left[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\kappa_0}{4}\right]\Phi(x_0,R_0) + c\,R_0^{\gamma_0-\varepsilon/2}
\end{aligned}$$

for all $\rho \leq R_0$, and the constant c has the dependencies stated above. Thus we are in a position to apply Lemma A.11, an iteration scheme to be able to neglect κ_0 by choosing the exponent γ_0 slightly smaller, to deduce the claimed inequality (6.28) for every such centre z.

Case 2: $z \in B^+$, $0 < \rho \le R_0 \le z_n$:
There holds $B_{R_0}(z) \subset B^+$, hence we apply the interior version of Lemma 6.5 and inequality (6.28) follows identically to Case 1.

Case 3: $z \in B^+$, $0 < z_n < \rho \le R_0$:
Without loss of generality we may assume $\rho \le R_0/4$, otherwise (6.28) is trivially satisfied. Then we have the inclusions

$$B^+_\rho(z) \subset B^+_{2\rho}(z'') \subset B^+_{R_0/2}(z'') \subset B^+_{R_0}(z)$$

where z'' denotes the projection of z onto $\mathbb{R}^{n-1} \times \{0\}$, and the boundary estimate in Case 1 yields the desired inequality:

$$\begin{aligned} \Phi(z,\rho) \le \; \Phi(z'',2\rho) \; &\le \; c\left[\Big(\frac{4\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',\tfrac{1}{2}R_0) + (2\rho)^{\gamma_0-\varepsilon/2}\right] \\ &\le \; c\left[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\right] \end{aligned}$$

where we have used the monotonicity of Φ with respect to the domain of integration.

Case 4: $z \in B^+$, $0 < \rho \le z_n < R_0$:
Without loss of generality we may assume $z_n < R_0/4$, otherwise we apply Case 2 for the inner ball $B_{R_0/4}(z) \subset B^+$. We then take advantage of the inclusions

$$B_\rho(z) \subset B_{z_n}(z) \subset B^+_{2z_n}(z'') \subset B^+_{R_0/2}(z'') \subset B^+_{R_0}(z),$$

the interior estimates in Case 2 and the boundary estimates in Case 1, and we find

$$\begin{aligned} \Phi(z,\rho) &\le c\left[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,z_n) + \rho^{\gamma_0-\varepsilon/2}\right] \\ &\le c\left[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',2z_n) + \rho^{\gamma_0-\varepsilon/2}\right] \\ &\le c\left[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2} c\left[\Big(\frac{4z_n}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'',\tfrac{1}{2}R_0) + (2z_n)^{\gamma_0-\varepsilon/2}\right] + \rho^{\gamma_0-\varepsilon/2}\right] \\ &\le c\left[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z,R_0) + \rho^{\gamma_0-\varepsilon/2}\right]. \end{aligned}$$

Combining the estimates above we see that we have covered all the cases required to prove inequality (6.28). Recalling the definition of the Excess function Φ, this yields

$$Du \in L^{p,\gamma_0-\varepsilon/2}\big(B_r(x_0) \cap (B^+ \cup \Gamma), \mathbb{R}^{nN}\big).$$

We define $\delta_2 = \frac{\varepsilon}{2}$ (with exactly the dependencies asserted in the statement of the theorem) and observe that the low dimensional assumption prescribes that

$$n \; < \; p + 2 + \delta_2 \; = \; p + 2 + \varepsilon/2\,.$$

We recall $\gamma_0 = 2$ if $n = 2$ and $\gamma_0 = 2 + \varepsilon$ if $n > 2$. As a consequence (taking ε smaller if required) we have $\gamma_0 - \varepsilon/2 \in (n-p, n]$, and, according to the Campanato-Meyer embedding in Theorem 2.3, we arrive at the conclusion that u is Hölder continuous on $B_r(x_0) \cap (B^+ \cup \Gamma)$, more precisely, we have

$$u \in C^{0,\lambda}\big(B_r(x_0) \cap (B^+ \cup \Gamma), \mathbb{R}^N\big) \qquad \text{with } \lambda = 1 - \frac{n - \gamma_0 + \varepsilon/2}{p}\,.$$

Using a covering argument and the fact that $\sigma \in (0,1)$ was chosen arbitrarily, we immediately conclude the desired regularity result.

Since we have shown higher integrability of Du in Lemma 6.2 we can improve the condition of x being a regular point via

$$R^{p-n} \int_{B_R^+(x)} \big(1 + |Du|^p\big)\, dx \leq \, c \Big(R^{s-n} \int_{B_R^+(x)} \big(1 + |Du|^s\big)\, dx\Big)^{\frac{p}{s}}$$

for R sufficiently small. As a consequence we get

$$B^+ \setminus \widetilde{\Sigma} \supseteq \Big\{x \in B^+ \cup \Gamma : \liminf_{R \to 0} \, R^{s-n} \int_{B_R(x) \cap B^+} \big(1 + |Du|^s\big)\, dy \, = \, 0\Big\}$$

which, in view of Lemma A.12, in turn provides the upper bound for the Hausdorff dimension of the singular set given in the theorem. □

Chapter 7

Existence of regular boundary points I

In this chapter we are concerned with the existence of regular boundary points for the gradient of weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1,2)$, of nonlinear, inhomogeneous elliptic systems of the form

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) \;=\; b(\,\cdot\,, u, Du) & \text{in } \Omega\,, \\ u \;=\; g & \text{on } \partial\Omega\,. \end{cases} \tag{7.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,\alpha}$ and $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in (0,1)$. The coefficients $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are assumed to be Hölder continuous with exponent α with respect to the first two variables and of class C^1 in the last variable, satisfying a standard $(p-1)$-growth condition. Furthermore, the right-hand side $b\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ is assumed to obey a controllable growth condition.

Let us recall the usual notation concerning regularity theory. We denote by

$$\operatorname{Reg}_{Du}(\overline{\Omega}) \;:=\; \big\{x \in \overline{\Omega} : Du \in C^0(\overline{\Omega} \cap A, \mathbb{R}^{nN}) \text{ for some neighbourhood } A \text{ of } x\big\}$$

the set of regular points for Du (in the interior and at the boundary), and by $\operatorname{Sing}_{Du}(\overline{\Omega}) := \overline{\Omega} \setminus \operatorname{Reg}_{Du}(\overline{\Omega})$ the set of singular points of Du. For the subquadratic case we have already obtained a characterization of the singular set in Chapter 3, stating that x_0 is a regular point for Du, i. e., $x_0 \in \operatorname{Reg}_{Du}(\overline{\Omega})$, if and only if the excess quantity

$$\fint_{\Omega \cap B_\rho(x_0)} \big|V(Du) - \big(V(Du)\big)_{\Omega \cap B_\rho(x_0)}\big|^2 \, dx$$

is sufficiently small and $|(V(Du))_{\Omega \cap B_\rho(x_0)}| + |(u)_{\Omega \cap B_\rho(x_0)}|$ does not diverge for $\rho \searrow 0$. Moreover, we have proved that the gradient Du of the weak solution to the inhomogeneous system (7.1) is locally Hölder continuous with exponent α in a (small) neighbourhood of every point

$x_0 \in \mathrm{Reg}_{Du}(\overline{\Omega})$. By Lebesgue's differentiation Theorem, the regularity criterion applies to almost every point in $\overline{\Omega}$, meaning that $|\overline{\Omega} \setminus \mathrm{Reg}_{Du}(\overline{\Omega})| = 0$. However, this does not yield the existence of even one single regular boundary point for general nonlinear elliptic systems, since the boundary $\partial\Omega$ itself is a set of Lebesgue measure zero. We recall that, due to the counterexample in [Gia78], it is a well-known fact that singularities may occur at the boundary even if the boundary data is smooth.

Consequently, the main objective is to improve the almost-everywhere regularity result in the sense that the singular set $\mathrm{Sing}_{Du}(\overline{\Omega})$ is not only negligible with respect to the Lebesgue measure but that its Hausdorff dimension is also small enough. Under additional assumptions on the regularity of the coefficients our aim is to prove that the Hausdorff dimension is less than $n-1$ because this would immediately yield that almost every boundary point is regular. We want to start the discussion of the size of the singular set by briefly stating some significant results for special systems: considering quasilinear systems of the form

$$- \operatorname{div} \big(a(\,\cdot\,, u)\, Du\big) \;=\; b(\,\cdot\,, u, Du)\,,$$

various partial regularity results were established, stating that the weak solution u (instead of its first derivative) is locally Hölder continuous. To bound the Hausdorff dimension of the singular set $\mathrm{Sing}_u(\overline{\Omega})$, we recall that the regular (boundary) points $x_0 \in \overline{\Omega}$ of u are characterized as the ones where the lower order excess functional

$$\fint_{\Omega\cap B_\rho(x_0)} \big|V(u) - \big(V(u)\big)_{\Omega\cap B_\rho(x_0)}\big|^2\, dx$$

is small, see e. g. [GM68a, Col71, Pep71, Gro02a, Ark96]. Since the set of non-Lebesgue points of every $W^{1,p}$-map has Hausdorff dimension not larger than $n-p$, this yields that the Hausdorff dimension of $\mathrm{Sing}_u(\overline{\Omega})$ may not exceed $n-p$. If the coefficient matrix $a(\cdot,\cdot)$ of the quasilinear system is further assumed to be of diagonal form, it is even known that the weak solution u is in fact a classical solution, i. e., of class C^2 (see [Wie76] where boundary regularity is included). Useful estimates for the singular set are also available for nonlinear elliptic systems obeying special structure assumptions: for instance, Uhlenbeck established in her fundamental paper [Uhl77] a strong maximum principle for the gradient Du of weak solutions to nonlinear systems depending in the nonlinear portion of the coefficient function only on the modulus $|Du|$. This was the key to an everywhere-regularity result for Du. For an extension to the nonquadratic case we refer to [Tol83, AF89]. However, these techniques could not yet be carried over to the boundary, leaving the question of full boundary regularity open in this case.

Turning our attention to general nonlinear elliptic systems, we observe that a direct comparison technique allows us to infer local Hölder continuity of the weak solution u outside a set of Hausdorff dimension $n-p$, provided that the assumption $n \le p+2$ on low dimensions holds, see e. g. the results in [Cam82b, Cam87b, Ark97, Ark03, Ido04a, Ido04b] and in Chapter 6. In contrast, in arbitrary dimensions n the reduction of the Hausdorff dimension of the singular set $\mathrm{Sing}_{Du}(\overline{\Omega})$ for the gradient Du was a long-standing unsolved problem. It was finally tackled by Mingione in [Min03b] where he introduced a remarkable new technique: he studied the (interior) singular set $\mathrm{Sing}_{Du}(\Omega)$ in the superquadratic case $p \ge 2$ for systems without u-dependencies and with inhomogeneities obeying a controllable growth condition, and he succeeded in showing that the Hausdorff dimension of $\mathrm{Sing}_{Du}(\Omega)$ is not larger than $n-2\alpha$. In [Min03a] he extended these results to systems with inhomogeneities under a natural growth condition, and he also covered systems depending additionally on the weak solution u, provided that the low dimensional assumption $n \leq p+2$ is satisfied. Recently, Duzaar, Kristensen and Mingione [DKM07] considered weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1, \infty)$, of the homogeneous Dirichlet problem corresponding to (7.1) and developed a technique which allowed them to carry these estimates up to the boundary, implying in particular the existence of regular boundary points provided that $n - 2\alpha < n - 1$ (or equivalently $\alpha > \frac{1}{2}$) is satisfied. To be more precise, the authors obtained for every $\alpha \in (\frac{1}{2}, 1]$ that almost every boundary point is regular if the coefficients $a(x, z)$ have no u-dependency or if the low dimensional assumption $n \leq p + 2$ holds. In the quadratic case this result was improved in two different ways: on the one hand, inhomogeneities with controllable growth were included, and on the other hand the condition on α was sharpened to $\alpha > \frac{1}{2} - \varepsilon$ for some number $\varepsilon > 0$ stemming from an application of Gehring's lemma. We mention that various results establishing better estimates for the (interior) singular set of minimizers of variational integral can be found in [KM06].

In the following we shall extend the results in [DKM07] for $p \in (1, 2)$ to inhomogeneous systems under a controllable growth condition and the assumption $\alpha > \frac{1}{2}$. To this end, in a first step we will ensure the existence of regular boundary points for systems with $a(x, u, z) \equiv a(x, z)$. In a second step, we will use an iteration procedure to extend this conclusion to systems also depending on the weak solution u, provided that $n \leq p + 2$. For the exact statements see Theorem 7.1 and Theorem 7.2 in the next section.

We close this introductory part with some remarks about the ideas behind the arguments and the techniques used within this section. Roughly speaking, the strategy can be described as follows: To simplify matters we initially consider coefficients of the form $a(x, z)$ which are Lipschitz continuous (or even differentiable) with respect to the x-variable. We may differentiate the system similarly as in Chapter 4.2 and obtain the existence of second order derivatives of the weak solution in a suitable Sobolev space. Hence, we find

$$\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\overline{\Omega})) \leq n - 2\,,$$

see [GM79, Theorem 4.2] and [Ive79]. Weakening the regularity condition on the coefficients by imposing only a Hölder continuity condition with an arbitrarily small exponent, we trivially know that

$$\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\overline{\Omega})) \leq n\,,$$

i. e., the upper bound on the dimension of the singular set reflects the the regularity of the coefficients with respect to x. This gives the impression that the degree of Hölder continuity of the coefficients is related not only to the regularity of the solution (namely that Du is locally Hölder continuous with the same exponent), but also to the size of the singular set. Working from this observation, Mingione [Min03b] accomplished in some sense an interpolation between Lipschitz continuity on the one hand and Hölder continuity on the other: for arbitrary exponents $\alpha \in (0, 1)$ the existence of higher order derivatives of u cannot be ensured, but it is still possible to differentiate the system (7.1) in a fractional sense. This leads to the desired estimate, namely that the Hausdorff dimension of the set of (interior) singular points does not exceed $n - 2\alpha$, via suitable fractional Sobolev spaces and a measure density result. If we allow the coefficients $a(x, u, z)$ to depend additionally on u itself, the situation becomes more complex and the estimates are technically much more involved. To follow the line of arguments from above, we have to investigate the regularity of

$x \mapsto (x, u(x))$. If the weak solution u is a priori known to be everywhere Hölder continuous then $x \mapsto (x, u(x))$ is also Hölder continuous and the arguments apply with only marginal modifications. However, in general this map is no longer continuous, because u may exhibit irregularities. At this stage the fact that u solves the Dirichlet problem (7.1) comes into play: in low dimensions partial Hölder continuity of u is already ensured outside of closed subsets of Hausdorff dimension less than $n-p$ (see Chapter 6), and therefore the map $x \mapsto (x, u(x))$ is regular at least on a "large" subset of $\overline{\Omega}$. In other words, the set of points where u is not continuous has sufficiently small Hausdorff dimension, hence, we may restrict the analysis of Du to the regular set $\mathrm{Reg}_u(\Omega)$ of u, and we still arrive at a good result for $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Omega))$. The fact that the Hölder continuity of the coefficients with respect to x is decreased by the presence of u is compensated for in a last step by an iteration technique which leads to $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Omega)) \leq \min\{n-2\alpha, n-p\}$.

This method relying upon fractional differentiability estimates for the gradient Du was developed by Mingione [Min03b, Min03a] for elliptic systems and is based on an interpolation technique dating back to Campanato, cf. [CC81, Cam82a]. It was later extended to parabolic systems by Duzaar and Mingione [DM05], and by Bögelein [Bög07, Bög] to higher order parabolic systems. In this chapter we are interested in the situation at the boundary, so we will explain the main ingredients for the up to the boundary approach introduced by Duzaar, Kristensen and Mingione [DKM07], and in the sequel we adapt them to inhomogeneous systems. We highlight that, by testing the system (7.1), up-to-the-boundary estimates for classical differences of the form $|V(Du)(x+he_s)-V(Du)(x)|$ can only be found for tangential directions. When working with partial derivatives, an estimate for the normal direction follows immediately by differentiating the system. This is no longer possible for our system when considering derivatives of only *fractional* order. To overcome this difficulty, i. e., to prove an appropriate difference estimate also for the normal direction, an indirect technique was introduced in [DKM07]: a family of comparison maps $u_h \in u + W_0^{1,p}$, $h \in (-1, 1)$, is constructed. Here, u_h stands for the unique solution of some regularized system

$$- \operatorname{div} a_h(\,\cdot\,, Du_h) \;=\; b(\,\cdot\,, u, Du)\,,$$

with continuous coefficients $a_h(\cdot,\cdot)$ which satisfy growth conditions analogous to $a(\cdot,\cdot,\cdot)$. Such systems are obtained via a regularization procedure involving both the original coefficients $a(\cdot,\cdot,\cdot)$ and the specific solution u. Due to the comparison results in Chapter 4.2, we infer that $DV(Du_h)$ exists and hence, appropriate estimates for the differences $|V(Du_h)(x+he_s) - V(Du_h)(x)|$ are available for *all* directions and every *fixed* number h. By a standard estimate for $|V(Du_h) - V(Du)|$, this allows us to deduce the missing normal estimate for Du. These estimates have to be iterated: in each step, additional higher integrability is gained for Du and is then carried over to Du_h via Calderón-Zygmund type estimates (provided in Chapter 5) in order to enable the next iterative step. We arrive at $V(Du) \in W^{s,2}$ for every $s < \alpha$, and the statement concerning the Hausdorff dimension of the singular set then follows immediately.

Finally, we remark that it is not clear to what extent the estimates for the Hausdorff dimension of the singular set may be improved. Up to now, the bound depends on the parameter α. While one cannot rule out that the dependence on α is only due to technique, it is believed that this dependence is a structural feature of the problem concerning the Hausdorff dimension of the singular set. As a consequence, the question of the existence of regular boundary points for Hölder exponents $\alpha \in (0, \frac{1}{2}]$ remains open for general nonlinear systems.

7.1 Structure conditions and results

We impose on the coefficients $a\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ standard conditions of subquadratic growth: the mapping $z \mapsto a(x,u,z)$ is a continuous vector field, and for fixed numbers $0 < \nu \le L$, $1 < p < 2$ and all triples $(x,u,z), (\bar{x},\bar{u},z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$, the following growth, ellipticity and continuity assumptions hold:

(H1) a has polynomial growth:

$$|a(x,u,z)| \le L\left(1+|z|^2\right)^{\frac{p-1}{2}},$$

(H2) a is differentiable in z with continuous and bounded derivatives:

$$|D_z a(x,u,z)| \le L\left(1+|z|^2\right)^{\frac{p-2}{2}},$$

(H3) a is uniformly strongly elliptic, i. e.,

$$D_z a(x,u,z)\,\lambda \cdot \lambda \ge \nu\left(1+|z|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \qquad \forall\, \lambda \in \mathbb{R}^{nN},$$

(H4) There exists a nondecreasing, concave modulus of continuity $\omega : \mathbb{R}^+ \to [0,1]$ such that $\omega(s) \le \min\{1, s^\alpha\}$ for all $s \in \mathbb{R}^+$ and

$$|a(x,u,z) - a(\bar{x},\bar{u},z)| \le L\left(1+|z|^2\right)^{\frac{p-1}{2}} \omega\big(|x-\bar{x}| + |u-\bar{u}|\big),$$

i. e., the conditions (H1)-(H4) of the last chapter with $\mu = 1$. We remark that the latter condition (H4) will be of importance in the following. It prescribes uniform Hölder continuity (for fixed z) with respect to the (x,u)-variable with Hölder exponent α. Moreover, we assume the inhomogeneity $b\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ to be a Carathéodory map, that is, it is continuous with respect to (u,z) and measurable with respect to x, and to satisfy for all $(x,u,z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ a controllable growth condition of the form:

(B1) $$|b(x,u,z)| \le L\left(1+|z|^2\right)^{\frac{p-1}{2}}.$$

Our main theorems in the chapter provide appropriate upper bounds for the singular set $\mathrm{Sing}_{Du}(\partial\Omega)$ which in turn guarantees the existence of regular boundary points. The first result is concerned with systems of type (7.1) where the coefficients do not depend on u:

Theorem 7.1 (cf. [DKM07], Theorem 1.1): *Let Ω be a domain of class $C^{1,\alpha}$ and let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a weak solution of the Dirichlet problem (7.1) under the assumptions (H1)-(H4), (B1) and $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$. Furthermore, let the vector field $a(\cdot,\cdot,\cdot)$ be independent of u, i. e., $a(x,u,z) \equiv a(x,z)$. If*

$$\alpha > \frac{1}{2}, \tag{7.2}$$

then $\mathcal{H}^{n-1}$-almost every boundary point is a regular point for Du.

Furthermore, for general systems we obtain the following result in low dimensions:

Theorem 7.2 (cf. [DKM07], Theorem 1.2): *Let Ω be a domain of class $C^{1,\alpha}$ and let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a weak solution of the Dirichlet problem (7.1) under the assumptions (H1)-(H4), (B1) and $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$. Assume further*

$$\alpha > \frac{1}{2} \qquad \text{and} \qquad 1 < \gamma_1 \le p \le \gamma_2 < \infty. \tag{7.3}$$

Then there exists a positive number δ depending only on $n, N, \gamma_1, \gamma_2, \frac{L}{\nu}$, $\|g\|_{C^{1,\alpha}(\Omega,\mathbb{R}^N)}$ and $\partial\Omega$ such that if

$$p > n - 2 - \delta \qquad \text{for } n > 2\,, \tag{7.4}$$

then $\mathcal{H}^{n-1}$-almost every boundary point is a regular point for Du.

7.2 Smoothing

In what follows we concentrate on the (partially) boundary value problem

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du) & \text{in } Q_2^+\,, \\ u = 0 & \text{on } \Gamma_2\,, \end{cases} \tag{7.5}$$

where all assumptions mentioned above are fulfilled with Ω replaced by Q_2^+. We next construct a family of regularized vector fields $a_h\colon Q_1^+ \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$, $|h| \in (0,1]$, out of both the original coefficients and the weak solution $u \in W_\Gamma^{1,p}(Q_2^+,\mathbb{R}^N)$ such that the new coefficients depend only on (x,z) and are smooth with respect to x for every fixed h. Moreover, the dependency with respect to h reflects the regularity properties of $x \mapsto u(x)$ in a quantifiable way.

We first note that due to the uniform continuity of the coefficients with respect to the (x,u)-variable for fixed z, i.e., the condition (H4), we can extend $a(\cdot,\cdot,\cdot)$ for every fixed $(u,z) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ to a vector field $a\colon \overline{Q_2^+} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ still satisfying all the assumptions. Now we extend $a(\cdot,\cdot,\cdot)$ to a new vector field still denoted by $a(\cdot,\cdot,\cdot)$ defined on $Q_2 \times \mathbb{R}^N \times \mathbb{R}^{nN}$. This extension is as usually performed by even reflection:

$$a(x,u,z) := \begin{cases} a(x,u,z) & x \in \overline{Q_2^+}\,, \\ a(i(x),u,z) & x \in \overline{Q_2} \setminus \overline{Q_2^+}\,, \end{cases}$$

where $i\colon \mathbb{R}^n \ni (x',x_n) \mapsto (x',-x_n)$. The extended vector field obviously still satisfies the assumptions (H1)-(H3). For condition (H4), it only remains to verify the case where $x \in \overline{Q_2^+}$ and $\bar{x} \in \overline{Q_2} \setminus \overline{Q_2^+}$. We then find a point $\tilde{x} \in \Gamma_2$ such that $|x-\tilde{x}|, |\tilde{x}-\bar{x}| \le |x-\bar{x}|$ and

$$\begin{aligned} \big|a(x,u,z) - a(\bar{x},\bar{u},z)\big| &\le \big|a(x,u,z) - a(\tilde{x},u,z)\big| + \big|a(\tilde{x},u,z) - a(\bar{x},\bar{u},z)\big| \\ &\le L\left[\omega(|x-\tilde{x}|) + \omega(|\tilde{x}-\bar{x}| + |u-\bar{u}|)\right]\big(1+|z|^2\big)^{\frac{p-1}{2}} \\ &\le 2L\,\omega(|x-\bar{x}| + |u-\bar{u}|)\big(1+|z|^2\big)^{\frac{p-1}{2}}. \end{aligned}$$

Thus also (H4) is satisfied replacing L by $2L$ if required. In the same way we extend the map u (still denoted by u) preserving the regularity properties of the original one, i.e., $u \in W^{1,p}(Q_2,\mathbb{R}^N)$, setting

$$u(x) := \begin{cases} u(x) & x \in \overline{Q_2^+} \\ u(i(x)) & x \in \overline{Q_2} \setminus \overline{Q_2^+}\,. \end{cases}$$

For the construction of an appropriate smoothing of the coefficients $a(\cdot,\cdot,\cdot)$ we proceed as follows: we fix a smooth, positive, radially symmetric convolution kernel $\phi \in C_0^\infty(B_1)$ such that $\int_{B_1} \phi \, dx = 1$. For $0 < |h| \le 1$ we define

$$a_h(x,z) := \int_{B_1} a\big(x + |h|y, u(x+|h|y), z\big)\, \phi(y)\, dy$$
$$= |h|^{-n} \int_{B_{|h|}(x)} a\big(y, u(y), z\big)\, \phi\big(\tfrac{y-x}{|h|}\big)\, dy$$

for every $x \in \overline{Q_{2-|h|}}$ and $z \in \mathbb{R}^{nN}$. Finally, we define the difference-averaged operator pointwise by

$$\pi_h(u)(x) := \fint_{B_1} \big|\tau_{y,|h|}(u)(x)\big|\, dy = \fint_{B_1} \big|u(x+|h|y) - u(x)\big|\, dy\,.$$

In the next step we prove the following *properties of the smoothed coefficients* (see [DKM07, Section 3]):

Proposition 7.3: *Assume that the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the conditions (H1)-(H4) on Q_2. Then the following statements for the smoothed coefficients $a_h(\cdot,\cdot)$ defined above hold true:*

(h1) $|a_h(x,z)| \le L\,(1+|z|^2)^{\frac{p-1}{2}}$,

(h2) $|D_z a_h(x,z)| \le L\,(1+|z|^2)^{\frac{p-2}{2}}$,

(h3) $D_z a_h(x,z)\lambda\cdot\lambda \ge \nu\,(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2$,

(h4) $(a_h(x,z_2) - a_h(x,z_1))\cdot(z_2-z_1) \ge c^{-1}(p)\,\nu\,(1+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2$,

(h5) $|D_x a_h(x,z)| \le c(n,\|D\phi\|_{L^\infty(B_1)})\,L\,|h|^{-1}\big[\omega(|h|)+\omega(\pi_h(u)(x))\big]\,(1+|z|^2)^{\frac{p-1}{2}}$,

(h6) $|a_h(x,z) - a(x,u(x),z)| \le c(n,\|D\phi\|_{L^\infty(B_1)})\,L\,\big[\omega(|h|)+\omega(\pi_h(u)(x))\big]\,(1+|z|^2)^{\frac{p-1}{2}}$,

(h7) $a_h(x,z)\cdot z \ge c^{-1}(p)\,\nu\,|z|^p - c(p,\tfrac{L}{\nu})\,\nu$

for all $z, z_1, z_2, \lambda \in \mathbb{R}^{nN}$ and $x \in Q_{2-|h|}$, and all constants c are independent of h.

Proof: The first three properties follow immediately from (H1)-(H3) since the smoothing procedure affects only the (x,u)-variable. For the proof of (h4) we note that for all $z_1, z_2 \in \mathbb{R}^{nN}$ and $y \in Q_2$, due to (H3) and Lemma A.2, there holds the pointwise inequality

$$\begin{aligned}
&c^{-1}(p)\,\nu\,(1+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2\\
&\quad\le \nu\int_0^1 (1+|z_1+t(z_2-z_1)|)^{\frac{p-2}{2}}\,dt\,|z_2-z_1|^2\\
&\quad\le \int_0^1 D_z a(y,u(y),z_1+t(z_2-z_1))\,dt\,(z_2-z_1)\cdot(z_2-z_1)\\
&\quad= \big[a(y,u(y),z_2) - a(y,u(y),z_1)\big]\cdot(z_2-z_1)\,.
\end{aligned}$$

Convolution then yields the desired inequality:

$$\begin{aligned}
&c^{-1}(p)\,\nu\,(1+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2\\
&\quad\le |h|^{-n}\int_{B_{|h|}(x)} \big[a(y,u(y),z_2) - a(y,u(y),z_1)\big]\,\phi\big(\tfrac{y-x}{|h|}\big)\,dy\cdot(z_2-z_1)\\
&\quad= \big[a_h(x,z_2) - a_h(x,z_1)\big]\cdot(z_2-z_1)\,.
\end{aligned}$$

In order to infer (h6) we use (H4), subadditivity and Jensen's inequality (keeping in mind the concavity of ω) and see

$$
\begin{aligned}
&\big|a_h(x,z) - a(x,u(x),z)\big| \\
&\quad = \Big| \int_{B_1} \big[a(x+|h|y, u(x+|h|y), z) - a(x,u(x),z)\big]\, \phi(y)\, dy\Big| \\
&\quad \le L \int_{B_1} \omega\big(|h||y| + |u(x+|h|y) - u(x)|\big)\, \phi(y)\, dy\, (1+|z|^2)^{\frac{p-1}{2}} \\
&\quad \le L \Big(\omega(|h|) + c(n) ⨍_{B_1} \omega\big(|\tau_{y,|h|}(u)(x)|\big)\, \phi(y)\, dy\Big)\, (1+|z|^2)^{\frac{p-1}{2}} \\
&\quad \le L \Big(\omega(|h|) + c(n)\, \|D\phi\|_{L^\infty(B_1)}\, \omega\Big(⨍_{B_1} |\tau_{y,|h|}(u)(x)|\, dy\Big)\Big)\, (1+|z|^2)^{\frac{p-1}{2}} \\
&\quad = L\, \big(\omega(|h|) + c(n, \|D\phi\|_{L^\infty(B_1)})\, \omega(\pi_h(u)(x))\big)\, (1+|z|^2)^{\frac{p-1}{2}} .
\end{aligned}
$$

For (h5) we have to differentiate the kernel and use the fact that $\int_{B_1} a(x,u(x),z)\partial_y\phi(y)\, dy$ vanishes. This finally allows us to proceed analogously to above:

$$
\begin{aligned}
|D_x a_h(x,z)| &= \Big| -|h|^{-n-1} \int_{B_{|h|}(x)} a(y,u(y),z)\, (D_x\phi)\big(\tfrac{y-x}{|h|}\big)\, dy\Big| \\
&= \Big| |h|^{-1} \int_{B_1} a(x+|h|y, u(x+|h|y), z)\, \partial_y \phi(y)\, dy\Big| \\
&= \Big| |h|^{-1} \int_{B_1} \big[a(x+|h|y, u(x+|h|y), z) - a(x,u(x),z)\big]\, \partial_y\phi(y)\, dy\Big| \\
&\le L\, c(n)\, |h|^{-1}\, \|D\phi\|_{L^\infty(B_1)} ⨍_{B_1} \omega\big(|h||y| + |\tau_{y,|h|}(u)(x)|\big)\, dy\, (1+|z|^2)^{\frac{p-1}{2}} \\
&\le c(n, \|D\phi\|_{L^\infty(B_1)})\, L\, |h|^{-1}\big[\omega(|h|) + \omega(\pi_h(u)(x))\big]\, (1+|z|^2)^{\frac{p-1}{2}} .
\end{aligned}
$$

The last property (h7) follows immediately from (h1), (h4) and Young's inequality:

$$
\begin{aligned}
a_h(x,z)\cdot z &= (a_h(x,z) - a_h(x,0))\cdot z + a_h(x,0)\cdot z \\
&\ge c^{-1}(p)\, \nu\, (1+|z|^2)^{\frac{p-2}{2}}|z|^2 - L|z| \\
&\ge c^{-1}(p)\, \nu\, |z|^p - c(p)\, \nu - \nu\big(\varepsilon|z|^p + c(p, \tfrac{L}{\nu}, \varepsilon)\big) ,
\end{aligned}
$$

which is the desired estimate for a suitable choice of ε. □

Remark 7.4: In this chapter we also consider the particular situations where the vector field $a(x,u,z) \equiv a(x,z)$ does not explicitly depend on u or where we argue under the low dimensional assumption, for which the weak solution u is a priori known to be Hölder continuous (at least outside a set of $\mathcal{H}^{n-p}$ measure zero and therefore in particular outside a set of $\mathcal{H}^{n-1}$ measure zero, see Theorem 6.1). This allows us to simplify or improve the representation in the proposition above, and therefore, to obtain even Hölder continuity of $a_h(\cdot,\cdot)$ with respect to x:

1. The case $a(x,u,z) \equiv a(x,z)$: we observe that the comparison of $u(x+|h|y)$ and $u(x)$ does not appear in the proof of (h5), (h6) and thus leads to

   (h5)^1 $\quad |D_x a_h(x,z)| \le c(n)\, L\, |h|^{-1}\omega(|h|)\, (1+|z|^2)^{\frac{p-1}{2}},$

   (h6)^1 $\quad |a_h(x,z) - a(x,u(x),z)| \le c(n)\, L\, \omega(|h|)\, (1+|z|^2)^{\frac{p-1}{2}}.$

Furthermore, the Hölder continuity of $a(\cdot,\cdot)$ with respect to x is preserved, and we have

$$\text{(h8)}^1 \qquad |a_h(x,z) - a_h(y,z)| \,\leq\, c(n)\, L\, |x-y|^\alpha\, (1+|z|^2)^{\frac{p-1}{2}}.$$

2. The case $u \in C^{0,\lambda}(Q_{2d}^+ \cup \Gamma_{2d}, \mathbb{R}^N)$ for some $\lambda, d \in (0,1)$: we note that u is extended via even reflection to a map $u \in C^{0,\lambda}(Q_{2d}, \mathbb{R}^N)$ and obtain

$$\text{(h8)}^2 \qquad |a_h(x,z) - a_h(y,z)| \,\leq\, c(n, [u]_{C^{0,\lambda}})\, L\, |x-y|^{\alpha\lambda}\, (1+|z|^2)^{\frac{p-1}{2}}$$

for all $x, y \in Q_d^+$ and $0 < |h| < d$.

The dependency on $\|D\phi\|_{L^\infty(B_1)}$ is omitted here since the convolution kernel ϕ was chosen fixed. We further remark that in both situations the constants are independent of h.

For the proof of the Hölder continuity (h8) we first observe that case 2, where Hölder continuity of the weak solution u is already known, is easily traced back to case 1 by setting $\widetilde{a}(x,z) := a(x,u(x),z)$: here we have to keep in mind that the constant L and the modulus of continuity ω in condition (H4) then have to be replaced by some new constant $\widetilde{L} = L\, c([u]_{C^{0,\lambda}})$ and a new modulus of continuity $\widetilde{\omega}$ satisfying $\widetilde{\omega}(t) \leq \min\{1, t^{\alpha\lambda}\}$.

To prove (h8)^1 we have to distinguish two cases: using the simplified representation (h5)^1 we infer in the case when $|x-y| \leq h$:

$$\begin{aligned} |a_h(x,z) - a_h(y,z)| \;&=\; \Big| \int_0^1 D_x a_h(y + t(x-y), z) \cdot (x-y)\, dt \Big| \\ &\leq\; c\, |h|^{\alpha-1} |x-y|\, (1+|z|^2)^{\frac{p-1}{2}} \\ &\leq\; c\, |x-y|^\alpha\, (1+|z|^2)^{\frac{p-1}{2}}\,. \end{aligned}$$

Otherwise if $|x-y| > |h|$ we conclude from (h6)^1 and (H4)

$$\begin{aligned} |a_h(x,z) - a_h(y,z)| \;&\leq\; |a_h(x,z) - a(x,z)| + |a(x,z) - a(y,z)| + |a(y,z) - a_h(y,z)| \\ &\leq\; c\, \big[|h|^\alpha + |x-y|^\alpha\big]\, (1+|z|^2)^{\frac{p-1}{2}} \\ &\leq\; c\, |x-y|^\alpha\, (1+|z|^2)^{\frac{p-1}{2}}\,, \end{aligned}$$

which is the desired estimate (h8)^1. We note that for vector field of type $a(x,u,z) \equiv a(x,z)$ we did not need the assumption that the map u is Hölder continuous.

7.3 A comparison estimate

This section provides a comparison estimate which will be the crucial point for the derivation of an appropriate fractional Sobolev estimate and therefore for the proof of our main theorems. Let A be a bounded Lipschitz domain such that $Q_{4d}^+ \subset A \subseteq Q_1^+$, $d \in (0, \frac{1}{4}]$ and let $u \in W_\Gamma^{1,p}(Q_2^+, \mathbb{R}^N)$ be the fixed solution of the boundary value problem (7.5) used in the construction of the vector fields $\{a_h\}$ above. We further assume that the map u is defined on the whole cube using even reflection and that the inhomogeneity $b(\cdot,\cdot,\cdot)$ obeys the controllable growth condition (B1).

Let $u_h \in u + W_0^{1,p}(A, \mathbb{R}^N)$ be the unique solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} a_h(\,\cdot\,, Du_h) = b(\,\cdot\,, u, Du) & \text{in } A\,, \\ u_h = u & \text{on } \partial A\,. \end{cases} \tag{7.6}$$

Since the right-hand side satisfies $b(x, u, Du) \in L^{p/(p-1)}(A, \mathbb{R}^N) \subset W^{-1,p}(A, \mathbb{R}^N)$, the existence of u_h follows in a standard way via the theory of monotone operators (in view of (h7) the monotonicity property is guaranteed), see [Lio69, Théorème 2.1, page 171]. Moreover, uniqueness follows from (h4). In the first step, we shall find an energy-estimate for the p-Dirichlet-functional with more or less the same calculations which led to (6.13), more precisely, we derive:

$$\int_A |Du_h|^p \, dx \le c \int_A (1 + |Du|^2)^{\frac{p}{2}} \, dx\,, \tag{7.7}$$

where the constant $c = c(n, N, p, \frac{L}{\nu})$ is independent of h. Testing the system (7.6) with $u_h - u$ we infer from (h7), (h1) and Young's inequality:

$$\begin{aligned} \int_A |Du_h|^p \, dx &\le c \int_A \big(a_h(x, Du_h) \cdot Du_h + 1\big)\, dx \\ &= c \int_A \big(a_h(x, Du_h) \cdot Du + b(x, u, Du) \cdot (u_h - u) + 1\big)\, dx \\ &\le c\,\varepsilon \int_A |Du_h|^p \, dx + c(\varepsilon) \int_A (1 + |Du|^2)^{\frac{p}{2}} \, dx + c \int_A |b(x, u, Du|\, |u_h - u| \, dx\,. \end{aligned}$$

Applying (B1), Young's inequality and the Poincaré inequality (note here $A \subseteq Q_1^+$ such that the constant c remains independent of A) we estimate the last integral via

$$\begin{aligned} \int_A |b(x, u, Du)|\, |u_h - u| \, dx &\le \varepsilon \int_A |u_h - u|^p \, dx + c(L, \varepsilon) \int_A (1 + |Du|^2)^{\frac{p}{2}} \, dx \\ &\le c_P\, \varepsilon \int_A |Du_h|^p \, dx + c(L, \varepsilon) \int_A (1 + |Du|^2)^{\frac{p}{2}} \, dx\,. \end{aligned}$$

Choosing ε sufficiently small yields the desired energy-estimate (7.7).

Exploiting the facts that u_h solves the system (7.6) and that u solves the system (7.5) we compute via Lemma A.1 (iv) and (h4)

$$\begin{aligned} \int_A \big|V(Du_h) - V(Du)\big|^2 \, dx &\le c \int_A \big(1 + |Du|^2 + |Du_h|^2\big)^{\frac{p-2}{2}} |Du_h - Du|^2 \, dx \\ &\le c \int_A \big[a_h(x, Du_h) - a_h(x, Du)\big] \cdot (Du_h - Du)\, dx \\ &= c \int_A b(x, u, Du) \cdot (u_h - u)\, dx - c \int_A a_h(x, Du) \cdot (Du_h - Du)\, dx \\ &= c \int_A \big[a(x, u, Du) - a_h(x, Du)\big] \cdot (Du_h - Du)\, dx\,, \end{aligned} \tag{7.8}$$

where $c = c(n,N,p)\nu^{-1}$. This last integral is estimated applying (h6), Young's inequality, Lemma A.1 (iv) and (7.7), and we arrive at

$$\begin{aligned} c\int_A \big[a(x,u,Du) - a_h(x,Du)\big]\cdot(Du_h - Du)\,dx \\ &\leq c\,L\int_A \big[\omega(|h|) + \omega(\pi_h(u)(x))\big]\,(1+|Du|^2)^{\frac{p-1}{2}}|Du_h - Du|\,dx \\ &\leq \varepsilon\int_A (1+|Du|^2+|Du_h|^2)^{\frac{p-2}{2}}|Du_h - Du|^2\,dx \\ &\quad + c(\varepsilon)\int_A \big[\omega(|h|) + \omega(\pi_h(u)(x))\big]^2(1+|Du|^2+|Du_h|^2)^{\frac{p}{2}}\,dx \\ &\leq c\,\varepsilon\int_A \big|V(Du_h) - V(Du)\big|^2\,dx + c\,|h|^{2\alpha}\int_A (1+|Du|^2)^{\frac{p}{2}}\,dx \\ &\quad + c\int_A \omega(\pi_h(u)(x))^2(1+|Du|^2+|Du_h|^2)^{\frac{p}{2}}\,dx\,, \end{aligned}$$

and the constant c depends only on n, N, p and $\frac{L}{\nu}$. Choosing ε in dependency of these quantities sufficiently small, we can absorb the integral of $|V(Du_h) - V(Du)|^2$ in the last inequality on the left-hand side of (7.8) and we finally arrive at the conclusion

$$\begin{aligned} \int_A \big|V(Du_h) - V(Du)\big|^2\,dx \\ &\leq c\Big(|h|^{2\alpha}\int_A (1+|Du|^2)^{\frac{p}{2}}\,dx + \int_A \omega(\pi_h(u)(x))^2(1+|Du|^2+|Du_h|^2)^{\frac{p}{2}}\,dx\Big)\,, \end{aligned} \tag{7.9}$$

where $c = c(n,N,p,\frac{L}{\nu})$. As noted in the remark at the end of the last section the estimates become much less complicated in the case of vector fields of type $a(x,u,z) \equiv a(x,z)$. Therefore, applying $(h6)^1$ instead of (h6), the last integral on the right-hand side in (7.9) disappears, and we find the inequality

$$\int_A \big|V(Du_h) - V(Du)\big|^2\,dx \;\leq\; c\big(n,N,p,\tfrac{L}{\nu}\big)|h|^{2\alpha}\int_A (1+|Du|^2)^{\frac{p}{2}}\,dx\,. \tag{7.10}$$

7.4 A decay estimate and proof of Theorem 7.1

In the next step we derive a decay estimate for the integral of $\tau_{s,h}(V(Du))$. Here the map u is again the fixed weak solution to the Dirichlet problem (7.5) used for the construction of the family $\{a_h\}$. In what follows $u_h \in u + W_0^{1,p}(A,\mathbb{R}^N)$ denotes the unique solution of the Dirichlet problem (7.6) where A is a bounded Lipschitz domain with $Q_{4d}^+ \subset A \subseteq Q_1^+$, $d \in (0,\frac{1}{4}]$ to be specified later. For the finite difference operator $\tau_{s,h}$ we will always assume $h \in \mathbb{R}$, $0 < |h| < d$ with $h > 0$ when dealing with the normal direction $s = n$.

The crucial point for the decay estimate is the following: the system (7.5), which we have introduced above, is exactly of the form (4.1) considered in Chapter 4 where we have derived comparison estimates for inhomogeneous systems with x-dependency; due to the properties (h1)-(h3) and (h5) stated in Proposition 7.3 the smoothed coefficients $a_h(\cdot,\cdot)$ satisfy all the required conditions with

$$\gamma(x) \;:=\; |h|^{-1}\big[\omega(|h|) + \omega(\pi_h(u)(x))\big]$$

(and with appropriate constants which were also computed in Proposition 7.3). Furthermore, setting $G(x) = L^{-1}\, b(x, u(x), Du(x))$, the controllable growth condition (B1) guarantees also condition (C5) to be fulfilled. Now the application of Theorem 4.2 allows us to obtain the existence of second derivatives for the comparison map u_h, and we find $V(Du_h) \in W^{1,2}(Q^+_{2d}, \mathbb{R}^{nN})$ with

$$\int_{Q^+_{2d}} \big|D(V(Du_h))\big|^2\, dx \le \frac{c}{d^2}\int_{Q^+_{4d}} \big(1+|Du|^2+|Du_h|^2\big)^{\frac{p}{2}}\, dx + \frac{c}{d^2\,|h|^2}\int_{Q^+_{4d}} \big[\omega(|h|)+\omega(\pi_h(u)(x))\big]^2 \big(1+|Du_h|^2\big)^{\frac{p}{2}}\, dx$$

for a constant c depending only on n, N, p and $\frac{L}{\nu}$. The existence of $D(V(Du_h))$ then yields (note $h > 0$ for the normal difference operator $\tau_{n,h}$):

$$\int_{Q^+_d} \big|\tau_{s,h}(V(Du_h))\big|^2\, dx \le c(n)\,|h|^2 \int_{Q^+_{2d}} \big|D(V(Du_h))\big|^2\, dx\,.$$

Keeping in mind $\omega(t) \le t^\alpha$, we conclude from the last two estimates and the energy estimate (7.7) (with $Q^+_{4d} \subset A$) that

$$\int_{Q^+_d} \big|\tau_{s,h}(V(Du_h))\big|^2\, dx \le \frac{c\,|h|^{2\alpha}}{d^2}\int_A \big(1+|Du|^2\big)^{\frac{p}{2}}\, dx + \frac{c}{d^2}\int_{Q^+_{4d}} \big(\omega(\pi_h(u)(x))\big)^2 \big(1+|Du_h|^2\big)^{\frac{p}{2}}\, dx\,, \tag{7.11}$$

and the constant $c = c(n, N, p, \frac{L}{\nu})$ is independent of h. By the following comparison argument this yields the desired decay estimate for $\tau_{s,h}(V(Du_h))$ replaced by $\tau_{s,h}(V(Du))$:

$$\int_{Q^+_d} \big|\tau_{s,h}(V(Du))(x)\big|^2\, dx \le 3\int_{Q^+_d} \big|V(Du(x+he_s)) - V(Du_h(x+he_s))\big|^2\, dx + 3\int_{Q^+_d} \big|\tau_{s,h}(V(Du_h))(x)\big|^2\, dx + 3\int_{Q^+_d} \big|V(Du_h(x)) - V(Du(x))\big|^2\, dx\,.$$

Since $|h| < d$, we can estimate the first and the last integral on the right-hand side of the last inequality by $6\int_A |V(Du_h) - V(Du)|^2\, dx$, for which in turn we apply the comparison estimate (7.9) from in the previous section. Hence, in view of (7.11) we finally derive

$$\int_{Q^+_d} \big|\tau_{s,h}(V(Du))\big|^2\, dx \le \frac{c\,|h|^{2\alpha}}{d^2}\int_A \big(1+|Du|^2\big)^{\frac{p}{2}}\, dx + \frac{c}{d^2}\int_{Q^+_{4d}} \big(\omega(\pi_h(u)(x))\big)^2 \big(1+|Du|^2+|Du_h|^2\big)^{\frac{p}{2}}\, dx\,, \tag{7.12}$$

and c depends only on n, N, p and $\frac{L}{\nu}$.

For vector field of the form $a(x, u, z) \equiv a(x, z)$ we derive a simplified version of the last inequality: via (h5)[1] we may apply Theorem 4.2 for $\gamma = \omega(|h|)/|h| \le |h|^{\alpha-1}$, and the application of (7.10) instead of (7.9) then yields

$$\int_{Q^+_d} \big|\tau_{s,h}(V(Du))\big|^2\, dx \le \frac{c\,|h|^{2\alpha}}{d^2}\int_A \big(1+|Du|^2\big)^{\frac{p}{2}}\, dx \tag{7.13}$$

for c having the same dependencies as above. Choosing $A = Q^+_1$, the estimate (7.13) for vector fields of this special type leads to

Proposition 7.5: *Let $u \in W^{1,p}(Q_2^+, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem (7.5) for coefficients of the form $a(x,u,z) \equiv a(x,z)$ under the assumptions (H1)-(H4) and (B1). Then $V(Du) \in W^{s,2}(Q_d^+, \mathbb{R}^{nN})$ for every $s < \alpha$ and every $d \in (0, 1/4)$. Moreover, we have*

$$Du \in W^{s,p}(Q_d^+, \mathbb{R}^{nN}).$$

Proof: The fact that $V(Du) \in W^{s,2}(Q_d^+, \mathbb{R}^{nN})$ for any $s < \alpha$ and $d \in (0,1/4)$ is easily inferred from (7.13) and Lemma 2.4 applied with the choice $G = V(Du)$. The length d of the cube has now to be chosen in $(0,1/4)$ instead of $d \in (0,1/4]$ (for which the estimate (7.13) holds), because the conclusion of Lemma 2.4 only follows on smaller (half-) cubes. In order to obtain the assertion concerning Du we first pass from (7.13) to the corresponding decay estimate for $\tau_{s,h}(Du)$ via Lemma 2.6 and then apply Lemma 2.4 with the choice $G = Du$. This yields the desired estimate (the assumption $\alpha > \frac{1}{2}$ is not needed here). □

The previous proposition allows us to prove our result concerning the existence of regular boundary points in the situation without an explicit dependency on the u-variable:

Proof (of Theorem 7.1): Following the reasoning in Section 3.2, see also [DKM07, proof of Theorem 1.1], we first observe that, due to the regularity assumption on g, we are in a position to reduce the Dirichlet problem (7.1) to the study of systems with zero boundary values $g = 0$. Furthermore, the regularity of $\partial\Omega$ allows us to flatten the boundary locally around every boundary point $x_0 \in \partial\Omega$ by a transformation whose regularity is determined by that of $\partial\Omega$. We again refer to Section 3.2 and the arguments leading to the associated Dirichlet problem (3.15). Thus, it is sufficient to assume in the sequel the model situation of an upper cube $\Omega = Q_2^+$, and to prove that almost every point on Γ is in fact a regular boundary point, i.e., that it belongs to the set $\mathrm{Reg}_{Du}(\Gamma)$. Since the Hausdorff dimension is invariant under bi-Lipschitz transformations, a standard covering argument then yields that an estimate for the Hausdorff dimension of the set of singular boundary points on Γ (for a solution of a problem of type (7.5)) implies a corresponding estimate for the singular boundary points on $\partial\Omega$, i.e., for $\mathrm{Sing}_{Du}(\partial\Omega)$.

We recall from Chapter 3, Theorem 3.14 that we have the following inclusions for weak solutions $u \in W_\Gamma^{1,p}(Q_2^+, \mathbb{R}^N)$ of the model situation: $\mathrm{Sing}_{Du}(\Gamma_2) \subset \Sigma_1 \cup \Sigma_2 \subset \Sigma_1^* \cup \Sigma_2^*$ where

$$\Sigma_1 = \Big\{ y \in \Gamma_2 \, : \, \liminf_{\rho \to 0^+} \fint_{B_\rho(y) \cap Q_2^+} \big| V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y) \cap Q_2^+} \big|^2 \, dx > 0 \Big\},$$

$$\Sigma_2 = \Big\{ y \in \Gamma_2 \, : \, \limsup_{\rho \to 0^+} \big| \big(V(D_n u)\big)_{B_\rho(y) \cap Q_2^+} \big| = \infty \Big\},$$

and analogously with the full derivative Du instead of only the normal derivative $D_n u$

$$\Sigma_1^* = \Big\{ y \in \Gamma_2 \, : \, \liminf_{\rho \to 0^+} \fint_{B_\rho(y) \cap Q_2^+} \big| V(Du) - \big(V(Du)\big)_{B_\rho(y) \cap Q_2^+} \big|^2 \, dx > 0 \Big\},$$

$$\Sigma_2^* = \Big\{ y \in \Gamma_2 \, : \, \limsup_{\rho \to 0^+} \big| \big(V(Du)\big)_{B_\rho(y) \cap Q_2^+} \big| = \infty \Big\}$$

(for the second inclusion see the remark below). Our next aim is to show the upper bound $\dim_{\mathcal{H}} \big(\Sigma_1^* \cup \Sigma_2^*\big) < n-1$ on the Hausdorff dimension of the sets Σ_1^* and Σ_2^*. We note that it is sufficient to prove

$$\dim_{\mathcal{H}} \big((\Sigma_1^* \cup \Sigma_2^*) \cap \Gamma_d \big) < n-1$$

for some fixed number $d \in (0, \frac{1}{4})$ because we may cover $\partial\Omega$ by a larger number of charts, depending on the smallness of d. Keeping in mind the assumption (7.2), i.e., $\alpha > \frac{1}{2}$, we fix a number $s \in (\frac{1}{2}, \alpha)$, and conclude from Proposition 7.5 that $V(Du) \in W^{s,2}(Q_d^+, \mathbb{R}^{nN})$. Lastly, the application of Proposition A.13 (with θ, q replaced by $s, 2$) yields

$$\dim_{\mathcal{H}} \big((\Sigma_1^* \cup \Sigma_2^*) \cap \Gamma_d \big) \le n - 2s < n - 1 .$$

This finishes the proof of the theorem. □

Remark: For the sake of completeness, we sketch the proof for the inclusion $\Sigma_1 \cup \Sigma_2 \subset \Sigma_1^* \cup \Sigma_2^*$ (see [DKM07, Remark 5.1]): we consider $y \in \Gamma_2 \setminus (\Sigma_1^* \cup \Sigma_2^*)$ and show $y \in \Gamma_2 \setminus (\Sigma_1 \cup \Sigma_2)$. By assumption we find $M < \infty$ such that

$$\sup_{\rho > 0} \big| \big(V(Du)\big)_{B_\rho(y) \cap Q_2^+} \big| \le M .$$

Furthermore, since the function $V \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is a bijection we find $A = A(\rho) \in \mathbb{R}^{nN}$ such that

$$V(A) = \big(V(Du)\big)_{B_\rho(y) \cap Q_2^+}$$

holds true, and via Lemma A.1 (i) we have $|A| \le c(M)$. Then, in view of Lemma A.1 (v), we compute

$$\begin{aligned} \fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y)\cap Q_2^+}\big|^2 \, dx &\le \fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u) - V(A_n)\big|^2 \, dx \\ &\le c(n,N,p) \fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u - A_n)\big|^2 \, dx . \end{aligned}$$

Furthermore, the fact that $t \mapsto V(t)$ is monotone nondecreasing on $\mathbb{R}^+$ and $|D_n u - A_n| \le |Du - A|$ allows us to calculate further

$$\begin{aligned} \fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y)\cap Q_2^+}\big|^2 \, dx &\le c(n,N,p) \fint_{B_\rho(y)\cap Q_2^+} \big|V(Du - A)\big|^2 \, dx \\ &\le c(n,N,p,M) \fint_{B_\rho(y)\cap Q_2^+} \big|V(Du) - V(A)\big|^2 \, dx \end{aligned}$$

where we have taken into account Lemma A.1 (vi). Recalling the definition of A we finally obtain for all radii $\rho > 0$:

$$\begin{aligned} &\fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u) - \big(V(D_n u)\big)_{B_\rho(y)\cap Q_2^+}\big|^2 \, dx \\ &\qquad \le c(n,N,p,M) \fint_{B_\rho(y)\cap Q_2^+} \big|V(Du) - \big(V(Du)\big)_{B_\rho(y)\cap Q_2^+}\big|^2 \, dx . \end{aligned}$$

Hence, $y \notin \Sigma_1$. It still remains to bound the mean values $|(V(D_n u))_{B_\rho(y)\cap Q_2^+}|$: here we proceed similarly and arrive at the conclusion that for all radii $\rho > 0$ sufficiently small there holds (keeping in mind $y \in \Gamma_2 \setminus (\Sigma_1^* \cup \Sigma_2^*)$)

$$\begin{aligned} \big|\big(V(D_n u)\big)_{B_\rho(y)\cap Q_2^+}\big| &\le \fint_{B_\rho(y)\cap Q_2^+} \big|V(D_n u) - V(A_n)\big| \, dx + \big|V(A_n)\big| \\ &\le c(n,N,p,M) \Big(\fint_{B_\rho(y)\cap Q_2^+} \big|V(Du) - V(A)\big|^2 \, dx \Big)^{\frac{1}{2}} + \big|V(A)\big| \\ &\le c(n,N,p,M) . \end{aligned}$$

Combining the latter inequality with the previous estimate we have shown $y \in \Gamma_2 \setminus (\Sigma_1 \cup \Sigma_2)$ and thus the asserted inclusion.

All the calculations leading to Proposition 7.5 where based on a comparison principle which works as well for cubes in the interior. Hence, we also obtain $V(Du) \in W^{s,2}(Q_d, \mathbb{R}^{nN})$ for every $s < \alpha$ and every $d \in (0, 1/4)$ for cubes $Q_d \subset \Omega$. Arguing exactly as in the proof of Theorem 5.1 on global estimates of Calderón-Zygmund type we may combine the estimates in the interior with the estimates at the boundary and use a standard covering argument in order to infer the following global estimate. We mention that in this situation we have to keep in mind the fact that the fractional Sobolev norm is super-additive with respect to the domain of integration.

Theorem 7.6 (cf. [DKM07], Theorem 5.1): *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of the Dirichlet problem (7.1) with $g \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ under the assumptions (H1)-(H4) and (B1). Furthermore, assume that the coefficients are independent of u, i. e., $a(x, u, z) \equiv a(x, z)$. Then $V(Du) \in W^{s,2}(\Omega, \mathbb{R}^{nN})$ for every $s < \alpha$. Moreover, we have*

$$Du \in W^{s,p}(\Omega, \mathbb{R}^{nN}) .$$

As a consequence, in view of the Sobolev embedding theorem for fractional Sobolev spaces, we obtain the following higher integrability result which provides, in contrast to the application of Gehring's Lemma, a quantitative improvement of the higher integrability exponent:

Corollary 7.7: *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be as in the previous Theorem 7.6. Then there holds:*

$$Du \in L^t(\Omega, \mathbb{R}^{nN}) \qquad \textit{for all } t < \frac{np}{n - 2\alpha} .$$

Proof: Applying the embedding Theorem A.10 for the function $V(Du) \in W^{s,2}(\Omega, \mathbb{R}^{nN})$ for every $s < \alpha$, we obtain $V(Du) \in L^{\tilde{t}}(\Omega, \mathbb{R}^{nN})$ for all $\tilde{t} < \frac{2n}{n-2\alpha}$. Hence, in view of Lemma A.1 (i), the statement of the corollary follows. □

7.5 Proof of Theorem 7.2

We proceed here analogously to [DKM07, proof of Theorem 1.2]. First we define the number δ introduced in (7.4) in the statement of the theorem as follows:

$$\delta := \min\Big\{\frac{\delta_1(n-2)p}{2}, \delta_2\Big\} > 0 \qquad \text{if } n \geq 3 , \tag{7.14}$$

where the number δ_1 is given in Theorem 5.1, and δ_2 comes from Theorem 6.1. We emphasize that a condition of type (7.4) is not required if $n = 2$. Thus, (keeping in mind that we consider inhomogeneities obeying a controllable growth condition) δ depends only on n, N, p and $\frac{L}{\nu}$. Furthermore, we assume for all the estimates below the low dimensional assumption

$$p > n - 2 - \delta . \tag{7.15}$$

We next fix a sequence of domains $\{\Omega_k\}_{k\in\mathbb{N}}$ of class C^2 such that for all $k \in \mathbb{N}$ we have the inclusions:

$$Q^+_{4d_{k+1}} \subset \Omega_k \subset Q^+_{s_k} \subset Q^+_{\rho_k} \subset Q^+_{d_k} , \tag{7.16}$$

where
$$d_k := \frac{1}{32^k}, \qquad \rho_k := \frac{d_k}{2}, \qquad s_k := \frac{d_k}{4}.$$

In particular, this means $\Gamma_{d_{k+1}} \subset \overline{\Omega_k}$ and $\Omega_k \subset Q_1^+$ for all $k \in N$. We now start with a higher integrability estimate for the derivative Du of the weak solution to our model system (7.5) of an upper half-cube:

Lemma 7.8 (cf. [DKM07], Lemma 6.2): *Let $u \in W^{1,p}(Q_2^+, \mathbb{R}^N) \cap C^{0,\lambda}(Q_1^+, \mathbb{R}^N)$, $\lambda \in (0,1]$, be a weak solution of the Dirichlet problem (7.5) under the assumptions (H1)-(H4), (B1) and (7.15). Then, for every $t < p + 2\alpha$ there exists $\bar{k} = \bar{k}(t) \in \mathbb{N}$ such that $Du \in L^t(Q^+_{d_{\bar{k}}}, \mathbb{R}^{nN})$.*

Proof: For $k \in \mathbb{N}$ we define the comparison maps $u_h^k \in u + W_0^{1,p}(\Omega_k, \mathbb{R}^N)$ as the unique solution to the Dirichlet problem
$$\begin{cases} -\operatorname{div} a_h(\,\cdot\,, Du_h^k) = b(\,\cdot\,, u, Du) & \text{in } \Omega_k\,, \\ u_h^k = u & \text{on } \partial\Omega_k\,, \end{cases} \tag{7.17}$$
i. e., the Dirichlet problem (7.6) with the choice $A := \Omega_k$. In the sequel, we restrict ourselves to $0 < |h| \leq \frac{d_k}{4}$. We define the sequence
$$\eta_1 := 0\,, \qquad \eta_{k+1} := \eta_k + \frac{p\lambda}{2}\big(\alpha(2-p) - \eta_k\big)\,,$$
and, accordingly,
$$\theta_k := \frac{p\alpha\lambda}{p+\eta_k} + \frac{p\eta_k(1-\lambda)}{(2-p)(p+\eta_k)}$$
for $k \in \mathbb{N}$. We easily check that the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is increasing with $\eta_k \nearrow (2-p)\alpha$. The strategy of the proof will be the following:
$$Du \in L^{p+\frac{2\eta_k}{2-p}}(Q^+_{\rho_k}, \mathbb{R}^{nN}) \;\rightarrow\; Du \in W^{\gamma\theta_k, p+\eta_k}(Q^+_{\rho_{k+1}}, \mathbb{R}^{nN}) \;\rightarrow\; Du \in L^{p+\frac{2\eta_{k+1}}{2-p}}(Q^+_{\rho_{k+1}}, \mathbb{R}^{nN})$$
for all $\gamma \in (0,1)$ and every $k \in \mathbb{N}$. The first implication is performed via employing the decay estimate (7.12) in an appropriate form, taking advantage of the Hölder continuity of u and applying the Calderón-Zygmund Theorem 5.1; for the latter step, we need the low dimensional assumption (7.15). The second implication is then a direct consequence from the interpolation Theorem 2.7. More precisely, we prove by induction that for every $k \in \mathbb{N}$ there holds:
$$(\mathbf{B_k}) \qquad \int_{Q^+_{\rho_k}} |Du|^{p+\frac{2\eta_k}{2-p}}\,dx \leq c_k\,.$$
For $k \geq 2$ the constant c_k depends only on $n, N, p, \frac{L}{\nu}, \alpha, \lambda, \Omega_{k-1}, k-1, c_{k-1}$ and $[u]_{0,\lambda}$.

Proof of (B_1): Since $\eta_1 = 0$ the assertion of (B_1) is satisfied with $c_1 = \|Du\|^p_{L^p(Q_1^+, \mathbb{R}^{nN})}$.

Proof of $(B_k) \Rightarrow (B_{k+1})$: In order to derive (B_{k+1}) we first show that (B_k) implies the following fractional Sobolev estimate
$$(\mathbf{B'_k}) \qquad \int_{Q^+_{\rho_{k+1}}} \int_{Q^+_{\rho_{k+1}}} \frac{|Du(x) - Du(y)|^{p+\eta_k}}{|x-y|^{n+\gamma(p+\eta_k)\,\theta_k}}\,dx\,dy \leq \widetilde{c}_k\,,$$

for all $\gamma \in (0,1)$ and $\widetilde{c}_k$ depends only on $n, N, p, \frac{L}{\nu}, \alpha, \lambda, \Omega_k, k, c_k, [u]_{C^{0,\lambda}(\Omega,\mathbb{R}^N)}$ and γ (i. e., the first implication above). For the proof in the case $k = 1$ we infer from (7.12) for the choices $A = \Omega_1$ and $d = d_2$, from the Hölder continuity of u with exponent λ and the energy estimate (7.7) that there holds

$$\int_{Q^+_{d_2}} \big|\tau_{s,h}(V(Du))\big|^2\,dx \,\le\, c\,|h|^{2\alpha\lambda} \int_{\Omega_1} \big(1+|Du|^2\big)^{\frac{p}{2}}\,dx\,,$$

for every $s \in \{1,\dots,n\}$ with a constant $c = c(n, N, p, \frac{L}{\nu})$. Lemma 2.6 then allows us to conclude

$$\begin{aligned}\int_{Q^+_{d_2}} \big|\tau_{s,h}(Du)\big|^p\,dx \,&\le\, c\,|h|^{p\alpha\lambda}\Big(\int_{Q^+_1} \big(1+|Du|^2\big)^{\frac{p}{2}}\,dx\Big)^{\frac{p}{2}+\frac{2-p}{2}}\\ &\le\, c\,|h|^{p\alpha\lambda}\int_{Q^+_1} \big(1+|Du|^2\big)^{\frac{p}{2}}\,dx\,,\end{aligned}$$

for every $s \in \{1,\dots,n\}$, and the constant c has the same dependencies as above. The application of Lemma 2.4 then yields $Du \in W^{\gamma\alpha\lambda,p}(Q^+_{\rho_2},\mathbb{R}^{nN})$ for all $\gamma \in (0,1)$ with the desired fractional Sobolev estimate (B'_1). For the proof of (B'_k), $k \ge 2$, we take advantage of Hölder's inequality, Lemma A.3 (i), the decay estimate (7.12) for the choices $A = \Omega_k$ and $d = d_{k+1}$ and the inclusions (7.16). Thus, we infer for every $s \in \{1,\dots,n\}$

$$\begin{aligned}&\int_{Q^+_{d_{k+1}}} \big|\tau_{s,h}(Du)\big|^{p+\eta_k}\,dx\\ &\quad= \int_{Q^+_{d_{k+1}}} \big(1+|Du(x)|^2+|Du(x+he_s)|^2\big)^{\frac{p-2}{2}\frac{p}{2}+\frac{2-p}{2}\frac{p}{2}} |\tau_{s,h}(Du)(x)|^{p+\eta_k}\,dx\\ &\quad\le 2\Big(\int_{Q^+_{d_{k+1}}} \big(1+|Du(x)|^2+|Du(x+he_s)|^2\big)^{\frac{p-2}{2}} |\tau_{s,h}(Du)(x)|^{2}\,dx\Big)^{\frac{p}{2}}\\ &\qquad\qquad\qquad \times\Big(\int_{Q^+_{\rho_k}} \big(1+|Du(x)|\big)^{p+\frac{2\eta_k}{2-p}}\,dx\Big)^{1-\frac{p}{2}}\\ &\quad\le c(n,N,p)\Big(\int_{Q^+_{d_{k+1}}} \big|\tau_{s,h}(V(Du))\big|^2\,dx\Big)^{\frac{p}{2}}\Big(\int_{Q^+_{\rho_k}} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx\Big)^{1-\frac{p}{2}}\\ &\quad\le c\,|h|^{p\alpha}\int_{Q^+_{\rho_k}} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx + cI^{\frac{p}{2}}\Big(\int_{Q^+_{\rho_k}} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx\Big)^{1-\frac{p}{2}}.\end{aligned} \tag{7.18}$$

Here we have abbreviated

$$I \,=\, \int_{\Omega_k} \big(\omega(\pi_h(u))\big)^2\,\big(1+|Du|^2+|Du^k_h|^2\big)^{\frac{p}{2}}\,dx\,,$$

and the constant c depends only on $n, N, p, \frac{L}{\nu}$ and k but is independent of h. We note that the finiteness of the right-hand side of (7.18) is guaranteed by the induction hypothesis (B_k). In order to estimate the latter integral denoted by I we will apply the Calderón-Zygmund Theorem 5.1 in the next step. For this purpose, we first have to check that all the assumptions of this theorem are fulfilled: in view of Proposition 7.3 (h1),(h2),(h3) combined with $(\mathrm{h8})^2$ in Remark 7.4 (valid for Hölder continuous maps u) we observe: (Z1)-(Z4) are satisfied for the coefficients $a_h(\cdot,\cdot)$ of the system in (7.17). Therefore, keeping in mind the growth condition (B1) on the inhomogeneity, we see that if we have $u \in W^{1,q}(\Omega_k,\mathbb{R}^N)$ for a number $q \in [p, s_1]$, then $g := u \in L^q$ and $LG := b(x,u,Du) \in L^{\frac{q}{p-1}}$. As a consequence of Theorem 5.1, the higher integrability of Du may be carried over to Du_h^k, and we obtain the following estimate

$$\begin{aligned}\int_{\Omega_k} \big(1+|Du_h^k|\big)^q\,dx &\le c\int_{\Omega_k} \big(1+|Du|+|\tfrac{1}{L}\,b(x,u,Du)|^{\frac{1}{p-1}}\big)^q\,dx\\ &\le c\int_{\Omega_k} \big(1+|Du|\big)^q\,dx \end{aligned}\tag{7.19}$$

for a constant c depending only on $n, N, p, \frac{L}{\nu}, \alpha, \lambda, \Omega_k$ and $[u]_{0,\lambda}$. In the present situation, (B_k) ensures the higher integrability with exponent $q = p + \frac{2\eta_k}{2-p}$, and with $\eta_k < (2-p)\alpha$, (7.15) and the choice of δ in (7.14) we easily see for $n \ge 3$ the following inequality:

$$p + \frac{2\eta_k}{2-p} \;<\; p+2 \;<\; \frac{1}{n-2}(pn+2\delta) \;\le\; \frac{np}{n-2} + \delta_1\,.$$

Therefore, the assumption $q \in [p, s_1]$ in Theorem 5.1 holds true. Combining the estimate in (7.19) with Hölder's inequality and the fact that $\omega(s) \le \min\{1, s^\alpha\}$ for all $s \in \mathbb{R}^+$, we further find

$$I \le c\Big(\int_{\Omega_k} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx\Big)^{\frac{p(2-p)}{p(2-p)+2\eta_k}} \Big(\int_{\Omega_k} \big(\pi_h(u)\big)^{2\alpha(1+\frac{p(2-p)}{2\eta_k})}\,dx\Big)^{\frac{2\eta_k}{p(2-p)+2\eta_k}}$$

where the constant c has the same dependencies as above. The Hölder continuity of u allows us to write

$$\begin{aligned}\big(\pi_h(u)\big)^{2\alpha(1+\frac{p(2-p)}{2\eta_k})} &= \big(\pi_h(u)\big)^{2\alpha\left[1+\frac{p(2-p)}{2\eta_k}\right]-p-\frac{2\eta_k}{2-p}}\,\big(\pi_h(u)\big)^{p+\frac{2\eta_k}{2-p}}\\ &\le c(\lambda,[u]_{0,\lambda})\,|h|^{\lambda\left[2\alpha(1+\frac{p(2-p)}{2\eta_k})-p-\frac{2\eta_k}{2-p}\right]}\,\big(\pi_h(u)\big)^{p+\frac{2\eta_k}{2-p}}\,.\end{aligned}$$

In order to estimate $(\pi_h(u))^{p+\frac{2\eta_k}{2-p}}$ on Ω_k we apply Fubini's Theorem and Jensen's inequality to infer for every $\tilde p > 1$:

$$\begin{aligned}\int_{\Omega_k} (\pi_h(u))^{\tilde p}\,dx &= \int_{\Omega_k}\Big(\fint_{B_1} |u(x+|h|y)-u(x)|\,dy\Big)^{\tilde p}\,dx\\ &= \int_{\Omega_k}\Big(\fint_{B_1}\Big|\int_0^1 Du(x+s|h|y)\,ds\cdot|h|y\Big|\,dy\Big)^{\tilde p}\,dx\\ &\le \int_{\Omega_k}\fint_{B_1}\int_0^1 |Du(x+s|h|y)|^{\tilde p}\,ds\,dy\,dx\,|h|^{\tilde p}\\ &\le \int_0^1\fint_{B_1}\int_{s|h|y+\Omega_k} |Du(x)|^{\tilde p}\,dx\,dy\,ds\,|h|^{\tilde p}\\ &\le 2\int_{Q^+_{\rho_k}} |Du(x)|^{\tilde p}\,dx\,|h|^{\tilde p}\end{aligned}\tag{7.20}$$

where in the last line we have used $s|h|y + \Omega_k \subset Q_{\rho_k}$, see (7.16) and the restriction on h, and the fact that u is extended to Q_2 by even reflection. Combining the last two estimates (setting $\tilde p = p + \frac{2\eta_k}{2-p}$) we find

$$I \;\le\; c\,|h|^{2\alpha\lambda+(1-\lambda)\frac{2\eta_k}{2-p}} \int_{Q^+_{\rho_k}} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx$$

for $c = c(n, N, p, \frac{L}{\nu}, \alpha, \lambda, \Omega_k, [u]_{0,\lambda})$. The latter inequality enables us to calculate further in (7.18): with

$$\theta_k(p+\eta_k) \;=\; p\alpha\lambda + (1-\lambda)\frac{p\eta_k}{2-p} \;<\; p\alpha \tag{7.21}$$

(since we have $\eta_k < (2-p)\alpha$) we infer for every $s \in \{1, \dots, n\}$ and $0 < |h| \le \frac{d_k}{4}$:

$$\int_{Q^+_{d_{k+1}}} \big|\tau_{s,h}(Du)\big|^{p+\eta_k}\,dx \;\le\; c\,|h|^{\theta_k(p+\eta_k)} \int_{Q^+_{\rho_k}} \big(1+|Du|\big)^{p+\frac{2\eta_k}{2-p}}\,dx$$

by definition of θ_k, and the constant c depends only on $n, N, p, \frac{L}{\nu}\alpha, \lambda, \Omega_k, k$ and $[u]_{0,\lambda}$ but is independent of h. The application of Lemma 2.4 yields $Du \in W^{\gamma\theta_k, p+\eta_k}(Q^+_{\rho_{k+1}}, \mathbb{R}^{nN})$ for all $\gamma \in (0,1)$ with the desired fractional Sobolev estimate (B'_k). Moreover, the constant $\widetilde{c}_k$ has exactly the dependencies stated in (B'_k). This finishes the proof of (B'_k).

It remains to prove (B_{k+1}): to this end we choose $\gamma \in (0,1)$ sufficiently close to 1 such that

$$p + \frac{2\eta_{k+1}}{2-p} \;=\; (p+\eta_k)\,(1+\theta_k) \;<\; \frac{n(p+\eta_k)(1+\gamma\theta_k)}{n-(p+\eta_k)\gamma\theta_k\lambda}\,.$$

Here, we have used the definitions of η_k and θ_k to obtain the first equality. In view of the fact that $(p+\eta_k)\theta_k\gamma < p\alpha < p < n$ (see (7.21)), we may apply Theorem 2.7 and we obtain (B_{k+1}).

Finally, the statement of the lemma follows from the convergence $p + \frac{2\eta_k}{2-p} \nearrow p + 2\alpha$. □

This higher integrability result for Du allows us to deduce fractional differentiability for $V(Du)$:

Lemma 7.9: *Let $u \in W^{1,p}(Q^+_2, \mathbb{R}^N) \cap C^{0,\lambda}(Q^+_1, \mathbb{R}^N)$, $\lambda \in (0,1]$, be a weak solution of the Dirichlet problem (7.5) under the assumptions (H1)-(H4), (B1) and (7.15). Then, for every $t_2 < \alpha$ there exists $\bar{k} = \bar{k}(t_2)$ such that $V(Du) \in W^{t_2,2}(Q^+_{\rho_{\bar{k}}}, \mathbb{R}^{nN})$.*

Proof: For fixed $\bar{t}_2 \in (t_2, \alpha)$ we determine $\gamma \in (0,1)$ such that $\bar{t}_2 = \alpha\gamma$. The application of the previous Lemma 7.8 for $t := p + 2\alpha\gamma$ yields the existence of $\bar{k} = \bar{k}(t)$ for which $Du \in L^t(Q^+_{d_{\bar{k}-1}}, \mathbb{R}^{nN})$. Keeping in mind

$$\big(\omega(\pi_h(u))\big)^{\frac{p+2\alpha\gamma}{\alpha\gamma}} \;\le\; \big(\omega(\pi_h(u))\big)^{\frac{p+2\alpha\gamma}{\alpha}} \;\le\; (\pi_h(u))^{p+2\alpha\gamma}\,,$$

we infer from the decay estimate (7.12) (with $Q^+_{d_{\bar{k}}}, \Omega_{\bar{k}-1}$ instead of Q^+_d, A), Hölder's inequality, the computations in (7.20) and (7.19) with $\tilde{p} = q = p + 2\alpha\gamma$ the following line of inequalities:

$$\begin{aligned}
&\int_{Q^+_{d_{\bar{k}}}} \big|\tau_{s,h}(V(Du))\big|^2\,dx \\
&\quad\le c\,|h|^{2\alpha} \int_{\Omega_{\bar{k}-1}} \big(1+|Du|^2\big)^{\frac{p}{2}}\,dx + c\int_{\Omega_{\bar{k}-1}} \big(\omega(\pi_h(u)(x))\big)^2 \big(1+|Du|^2+|Du_h^k|^2\big)^{\frac{p}{2}}\,dx \\
&\quad\le c\,|h|^{2\alpha} \int_{\Omega_{\bar{k}-1}} \big(1+|Du|^2\big)^{\frac{p}{2}}\,dx \\
&\qquad + c\Big(\int_{\Omega_{\bar{k}-1}} (\pi_h(u))^{p+2\alpha\gamma}\,dx\Big)^{\frac{2\alpha\gamma}{p+2\alpha\gamma}} \Big(\int_{\Omega_{\bar{k}-1}} \big(1+|Du|^2+|Du_h^k|^2\big)^{\frac{p+2\alpha\gamma}{2}}\,dx\Big)^{\frac{p}{p+2\alpha\gamma}} \\
&\quad\le c\,\big(|h|^{2\alpha}+|h|^{2\alpha\gamma}\big) \int_{Q^+_{\rho_{\bar{k}-1}}} \big(1+|Du|\big)^{p+2\alpha\gamma}\,dx \;\le\; c\,|h|^{2\bar{t}_2}
\end{aligned}$$

for every $s \in \{1, \ldots, n\}$, $0 < |h| \leq \frac{d_{\bar{k}}}{4}$ and for a constant c depending only on $n, N, p, \frac{L}{\nu}, \alpha, \lambda$, $\Omega_{\bar{k}-1}, \bar{k}-1$, $\|Du\|_{L^p}$, $[u]_{0,\lambda}$ and $\bar{t}_2$. Lemma 2.4 then yields the assertion of the lemma. □

Proof (of Theorem 7.2): We will proceed close to the proof of Theorem 7.1: first, we reduce our problem (7.1) to the analysis of an associated Dirichlet problem with zero boundary values on $\partial\Omega$. Then, by a covering argument and a local flattening procedure, we reduce it to the study a finite number of problems of type (7.5) on cubes. As a consequence of these transformations, the new structure conditions L and ν (see e. g. Section 3.2) depend on the regularity of the boundary data, i. e. on $\|g\|_{C^{1,\alpha}(\Omega,\mathbb{R}^N)}$ and $\partial\Omega$, which in turn is reflected in the dependencies of the number δ given in the statement of Theorem 7.2.

We again denote by $\mathrm{Sing}_{Du}(\Gamma)$ the set of singular points of Du on Γ, and we will now show the estimate $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Gamma)) < n-1$ on the Hausdorff dimension of the singular set. The crucial point in the present situation is the following: the fact that we consider only the low dimensional case, see (7.4), ensures via Theorem 6.1 that u is known to be Hölder continuous on the regular set $\mathrm{Reg}_u(Q^+ \cup \Gamma)$ of u with every exponent $\lambda \in \big(0, 1 - \frac{n-2-\delta_2}{p}\big)$ (cf. Theorem 6.6 for the model case); moreover, we have $\dim_{\mathcal{H}}(\mathrm{Sing}_u(Q^+ \cup \Gamma) < n-p$. Hence, it suffices to confine our attention to the regular set of u and hence, to prove

$$\dim_{\mathcal{H}}\big(\mathrm{Sing}_{Du}(\Gamma) \cap \mathrm{Reg}_u(Q^+ \cup \Gamma)\big) \; < \; n-1 \, .$$

We next choose an increasing sequence of sets $B_k \nearrow \mathrm{Reg}_u(Q^+ \cup \Gamma)$ with $B_k \subset \mathrm{Reg}_u(Q^+ \cup \Gamma)$ such that B_k is relatively open in $Q^+ \cup \Gamma$ for every $k \in \mathbb{N}$, i. e., such that for every $k \in \mathbb{N}$ there exists an open set $A_k \subset \mathbb{R}^n$ with $B_k = (Q^+ \cup \Gamma) \cap A_k$. Therefore, in view of the continuity of $\mathcal{H}^{n-1}$ with respect to monotone sequences of measurable sets, we find: in order to prove Theorem 7.2 it is sufficient to show

$$\dim_{\mathcal{H}}\big(\mathrm{Sing}_{Du}(\Gamma) \cap B_k\big) \; < \; n-q \tag{7.22}$$

for all $k \in \mathbb{N}$ and some $q > 1$. We observe that Lemma 7.8 and Lemma 7.9 still hold if we replace the cube Q^+ by any other cube $Q^+_R(x_0)$ for some $x_0 \in \Gamma \cap B_k$; as a consequence of these lemmas, we then obtain $V(Du) \in W^{t_2,2}(Q^+_{\rho_{\bar{k}}R}(x_0), \mathbb{R}^{nN})$, and the number $\rho_{\bar{k}}$ depends only on $t_2 \in (0, \alpha)$. Hence, taking $t_2 \in (\frac{1}{2}, \alpha)$ (keeping in mind the assumption (7.3) on α), the application of Proposition A.13 yields

$$\dim_{\mathcal{H}}\big(\mathrm{Sing}_{Du}(\Gamma) \cap Q^+_{\rho_{\bar{k}}R}(x_0)\big) \; \leq \; n - 2t_2 \; < \; n-1 \, .$$

Since $x_0 \in \Gamma \cap B_k$ is arbitrary and B_k is relatively open in $Q^+ \cup \Gamma$, a standard covering argument yields (7.22) which in turn implies $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Gamma)) < n-1$, meaning that $\mathcal{H}^{n-1}$-almost every boundary point in Γ is a regular point for Du. This finishes the proof of Theorem 7.2. □

Chapter 8

Existence of regular boundary points II

In this section we continue to study the existence of regular boundary points. We consider bounded weak solutions $u \in W^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ of the quadratic nonlinear elliptic system

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du) & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,. \end{cases} \tag{8.1}$$

Here Ω is a domain of class $C^{1,\alpha}$, $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha \in (0,1)$. The coefficients $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ are assumed to be Hölder continuous with exponent α with respect to the first two variables and of class C^1 with respect to the last variable, satisfying a standard quadratic growth condition. We shall now work on the existence of regular boundary points under the prerequisite that the right-hand side $b\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ obeys a natural growth condition (see (B2) in Chapter 8.1 further below) and that the smallness assumption $|u| \le M$ for some $M > 0$ with $2L_2M < \nu$ is satisfied. The latter condition ensures, see e. g. [DG00] combined with [Gro02b], that every weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ is partially $C^{1,\alpha}$-regular. More precisely, we have:

$$u \in C^{1,\alpha}_{\text{loc}}(\operatorname{Reg}_{Du}(\overline{\Omega}), \mathbb{R}^N) \qquad \text{and} \qquad |\operatorname{Sing}_{Du}(\overline{\Omega})| = 0\,.$$

Therefore, the situation with inhomogeneities under a natural growth condition seems to be closely connected to that under a controllable growth condition, and the same line of arguments as in the last chapter might be expected to lead to the desired results, but several critical difficulties arise: The definition of the comparison maps u_h in (7.17) would require some modifications; however, even if this problem were solved, it is yet not clear how the higher integrability of Du could be carried over to the weak solution Du_h of the regularized Dirichlet problem (cf. p. 146), because the necessary Calderón-Zygmund theory developed in Chapter 5 only applies when the right-hand side belongs to the Lebesgue space $L^{q/(p-1)}$ for some $q > p$. This prerequisite is not fulfilled in this case since the natural growth of $b(\cdot,\cdot,\cdot)$ merely gives $L^{1+\delta}$ for some (small) value $\delta > 0$ (coming from the higher integrability of Du). This motivates why we present a different technique introduced by Kronz in [Kro] where it is a promising approach to up to the boundary regularity results including upper bounds for the Hausdorff dimension of the singular set, with the flexibility to attack even higher order systems.

To overcome the difficulties arising from the fact that differences $|Du(x+he_s) - Du(x)|$ can be estimated up to the boundary only for the tangential directions, Kronz [Kro] suggested to replace the indirect comparison principle from the previous chapter by a direct method. Introducing slicewise mean values on slices in tangential direction he observed that estimates for the tangential differences suffice to control the averaged mean deviation with respect to these slicewise mean values. Using an alternative definition of fractional Sobolev spaces based on pointwise inequalities, this allows us to derive a fractional Sobolev estimate for the map $a_n(\cdot, u, Du)$, which in turn is transferred to the normal derivative of the weak solution u. Combined with a corresponding estimate for the tangential derivatives of u this leads to a higher integrability statement for the full gradient Du. Via a standard iteration argument combined with a measure density result and a partial Hölder continuity result for u (outside a set of Hausdorff dimension less than $n-2$) in low dimensions, we then reach the desired result that almost every boundary point is a regular one for Du.

Finally, we mention that, to this date, the existence of regular boundary points for elliptic systems with inhomogeneities under a natural growth condition is established only for the quadratic case $p = 2$. A positive answer to the same question also for elliptic systems fulfilling standard assumptions of p-growth with $p \in (1,\infty)$ arbitrary should be obtainable from an adaptation of the techniques used within this chapter.

8.1 Structure conditions and result

We impose on the coefficients $a\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ standard conditions of quadratic growth: the functions $(x,u,z) \mapsto a(x,u,z)$ and $(x,u,z) \mapsto D_z a(x,u,z)$ are continuous, and for fixed $0 < \nu \le L$ and all triples $(x,u,z), (\bar{x},\bar{u},z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ there holds:

(H1) a has linear growth:

$$|a(x,u,z)| \le L\big(1+|z|\big),$$

(H2) a is differentiable with respect to z with bounded and continuous derivatives:

$$|D_z a(x,u,z)| \le L,$$

(H3) a is uniformly strongly elliptic:

$$D_z a(x,u,z)\,\widetilde{\lambda}\cdot\widetilde{\lambda} \ge \nu\,|\widetilde{\lambda}|^2 \qquad \forall\,\widetilde{\lambda} \in \mathbb{R}^{nN},$$

(H4) There exists a modulus of continuity $\omega\colon \mathbb{R} \to [0,1]$ with $\omega(t) \le \min(1, t^{\alpha})$ such that

$$|a(x,u,z) - a(\bar{x},\bar{u},z)| \;\le\; L\,\big(1+|z|\big)\,\omega\big(|x-\bar{x}| + |u-\bar{u}|\big)\,.$$

For the inhomogeneity $b\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ we assume for all $(x,u,z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$

(B2) a natural growth condition: there exists a constant L_2 (possibly depending on $M > 0$) such that

$$|b(x,u,z)| \le L + L_2\,|z|^p$$

for all $(x,u,z) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ with $|u| \le M$.

The aim of this chapter is to improve $|\operatorname{Sing}_{Du}(\overline{\Omega})| = 0$ in the following sense:

Theorem 8.1: *Consider $n \in \{2,3,4\}$ and $\alpha > 1/2$. Let $\Omega \subset \mathbb{R}^n$ be a domain of class $C^{1,\alpha}$ and $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$. Assume further that $u \in W^{1,2}(\Omega,\mathbb{R}^N) \cap L^{\infty}(\Omega,\mathbb{R}^N)$ is a weak solution of the Dirichlet problem (8.1) under the assumptions (H1)-(H4) and (B2), and suppose $\|u\|_{L^{\infty}(\Omega,\mathbb{R}^N)} \le M$ for some $M > 0$ such that $2L_2M < \nu$. Then $\mathcal{H}^{n-1}$-almost every boundary point is a regular point for Du.*

In the sequel we restrict ourselves again to the model case $\Omega = Q_2^+$, cf. Chapter 7, and study weak solutions $u \in W^{1,2}_{\Gamma}(Q_2^+,\mathbb{R}^N) \cap L^{\infty}(Q_2^+,\mathbb{R}^N)$ of the system

$$-\operatorname{div} a(\,\cdot\,,u,Du) \;=\; b(\,\cdot\,,u,Du) \qquad \text{in } Q_2^+. \tag{8.2}$$

By a transformation argument this covers the situation of general inhomogeneous systems of type (8.1) on arbitrary domains Ω of class $C^{1,\alpha}$, see Chapter 3.2.

8.2 Slicewise mean values and a Caccioppoli inequality

8.2.1 A statement concerning slicewise mean values

Before introducing slicewise mean values we need some more notation: it will be convenient to work on cylinders; hence, for $\rho > 0$, we define $(n-1)$-dimensional balls

$$D_\rho(z') := \big\{y \in \mathbb{R}^{n-1} : |z' - y'| < \rho\big\}$$

for $z' \in \mathbb{R}^{n-1}$, and cylinders on the upper half-plane

$$Z_\rho(z) := D_\rho(z') \times \big(\max\{0, z_n - \rho\}, z_n + \rho\big) \;=:\; D_\rho(z') \times I_\rho(z_n)$$

for centres $z = (z', z_n) \in \mathbb{R}^n$ with $z_n \ge 0$. Given a function $v \in L^1(Z_R(z),\mathbb{R}^N)$, $Z_\rho(x_0) \subset Z_R(z)$, we denote the mean value $(v)_{Z_\rho(x_0)}$ by $(v)_{x_0,\rho}$. Furthermore, we define the slicewise mean value at almost every height $x_n \in I_\rho((x_0)_n)$ via

$$(v)_{x_0',\rho}(x_n) \;:=\; \fint_{D_\rho((x_0)')} v(x', x_n)\,dx'\,.$$

The next lemma enables us to conclude from difference estimates for a map u an appropriate estimate for the averaged mean deviation with respect to slicewise mean values (see [Kro]):

Lemma 8.2: *Let $\sigma < \frac{1}{3}$, $n \geq 2$, $\tau > 0$, $Z_\rho(x_0) \subset Q^+$ for some $x_0 \in Q^+ \cup \Gamma$. Furthermore, assume that $v \in L^p(Z_\rho(x_0), \mathbb{R}^N)$, $p > 1$, satisfies*

$$\int_{Z_{\sigma\rho}(x_0)} |\tau_{h,e} v|^p \, dx \leq K^p \, |h|^{\tau p}$$

for some $K > 0$, all $e \in S^{n-1}$ with $e \perp e_n$ and $h \in \mathbb{R}$ with $|h| < 2\sigma\rho$. Then, for every $\beta \in (0, \tau)$ there exists a function $F \in L^p(Z_{\sigma\rho}(x_0))$ such that

$$\int_{Z_{\sigma\rho}(x_0)} |F|^p \, dx \leq c(n, p, \tau, \beta) \, K^p \, \rho^{(\tau-\beta)p}$$

and

$$\begin{aligned} &\Big(-\!\!\!\!\!\!\int_{Z_r(z)} \big|v(x) - (v)_{z',r}(x_n)\big|^{\widetilde{p}} \, dx \Big)^{\frac{1}{\widetilde{p}}} \\ &\qquad \leq \Big(-\!\!\!\!\!\!\int_{Z_r(z)} -\!\!\!\!\!\!\int_{D_r(z')} |v(x', x_n) - v(y', x_n)|^{\widetilde{p}} \, dy' \, dx \Big)^{\frac{1}{\widetilde{p}}} \leq c(n, \beta) \, r^\beta \, F(z) \end{aligned}$$

for every exponent $\widetilde{p} \in [1, p)$, almost all $z \in Q^+ \cup \Gamma$ and all $r > 0$ such that $Z_r(z) \subset Z_{\sigma\rho}(x_0)$.

Proof: The proof of this lemma is taken from [Kro]. We choose exponents $\beta < \tau$, $q \in \big(\max\{\widetilde{p}, \frac{(n-1)p}{(\tau-\beta)p+(n-1)}\}, p\big)$ and an arbitrary cylinder $Z_r(z) \subset Z_{\sigma\rho}(x_0)$ with $z \in Q^+ \cup \Gamma$. Using the definition of slicewise mean values, Jensen's inequality and the inclusion $D_r(z') \subset D_{\sigma\rho}((x_0)')$ we obtain

$$\begin{aligned} \Big(-\!\!\!\!\!\!\int_{Z_r(z)} \big|v(x) - (v)_{z',r}(x_n)\big|^{\widetilde{p}} \, dx \Big)^{\frac{1}{\widetilde{p}}} &\leq \Big(-\!\!\!\!\!\!\int_{Z_r(z)} -\!\!\!\!\!\!\int_{D_r(z')} |v(x', x_n) - v(y', x_n)|^q \, dy' \, dx \Big)^{\frac{1}{q}} \\ &\leq c \, r^\beta \Big(-\!\!\!\!\!\!\int_{Z_r(z)} \int_{D_{\sigma\rho}((x_0)')} \frac{|v(x', x_n) - v(y', x_n)|^q}{|x' - y'|^{n-1+\beta q}} \, dy' \, dx \Big)^{\frac{1}{q}} \end{aligned}$$

for a constant c depending only on n and β. Defining

$$f(x) = \int_{D_{\sigma\rho}((x_0)')} \frac{|v(x', x_n) - v(y', x_n)|^q}{|x' - y'|^{n-1+\beta q}} \, dy'$$

for $x = (x', x_n) \in Z_{\sigma\rho}(x_0)$ we further find

$$\begin{aligned} \Big(-\!\!\!\!\!\!\int_{Z_r(z)} \big|v(x) - (v)_{z',r}(x_n)\big|^{\widetilde{p}} \, dx \Big)^{\frac{1}{\widetilde{p}}} &\leq c \, r^\beta \Big(-\!\!\!\!\!\!\int_{Z_r(z)} f(x) \, dx \Big)^{\frac{1}{q}} \\ &\leq c \, r^\beta \sup_{Z_{\tilde{r}}(\tilde{z}) \subseteq Z_{\sigma\rho}(x_0), \, z \in Z_{\tilde{r}}(\tilde{z})} \Big(-\!\!\!\!\!\!\int_{Z_{\tilde{r}}(\tilde{z})} f(x) \, dx \Big)^{\frac{1}{q}} \\ &= c \, r^\beta \big(M^*(f)(z) \big)^{\frac{1}{q}} =: c(n, \beta) \, r^\beta \, F(z) \, , \end{aligned}$$

where $M^*(f)(z)$ is the maximal function restricted to the cylinder $Z_{\sigma\rho}(x_0)$, cf. Chapter 5.2.3; note that the supremum is taken over all cylinders containing the point z. Lemma 5.4 then yields the desired result $(M^*(f))^{1/q} \in L^p(Z_{\sigma\rho}(x_0))$, provided that we can show $f \in L^{p/q}(Z_{\sigma\rho}(x_0))$. We next claim: $f \in L^{p/q}(Z_{\sigma\rho}(x_0))$ with

$$\int_{Z_{\sigma\rho}(x_0)} f^{\frac{p}{q}} \, dx \leq c \, K^p \, \rho^{(\tau-\beta)p} \tag{8.3}$$

for a constant c depending only on n, p, τ and β. To this end we apply Hölder's inequality, the co area formula and Fubini's Theorem to find

$$\begin{aligned}
\int_{Z_{\sigma\rho}(x_0)} f^{\frac{p}{q}}\,dx &= \int_{Z_{\sigma\rho}(x_0)} \Big(\int_{D_{\sigma\rho}((x_0)')} \frac{|v(x',x_n)-v(y',x_n)|^q}{|x'-y'|^{n-1+\beta q}}\,dy'\Big)^{\frac{p}{q}}\,dx \\
&\leq \big(\mathcal{L}^{n-1}(D_{\sigma\rho}((x_0)'))\big)^{\frac{p-q}{q}} \int_{Z_{\sigma\rho}(x_0)} \int_{D_{\sigma\rho}((x_0)')} \frac{|v(x',x_n)-v(y',x_n)|^p}{|x'-y'|^{(n-1)\frac{p}{q}+\beta p}}\,dy'\,dx \\
&\leq c\,(\sigma\rho)^{(n-1)(\frac{p}{q}-1)} \int_{Z_{\sigma\rho}(x_0)} \int_0^{2\sigma\rho} \int_{S_h^{n-2}(x')} \frac{|v(x',x_n)-v(y',x_n)|^p}{h^{(n-1)\frac{p}{q}+\beta p}}\,d\mathcal{H}^{n-2}(y')\,dh\,dx \\
&= c\,(\sigma\rho)^{(n-1)(\frac{p}{q}-1)} \int_{Z_{\sigma\rho}(x_0)} \int_0^{2\sigma\rho} \int_{S^{n-2}(x')} \frac{|v(x',x_n)-v(x'+he,x_n)|^p}{h^{(n-1)\frac{p}{q}+\beta p+2-n}}\,de\,dh\,dx \\
&= c\,(\sigma\rho)^{(n-1)(\frac{p}{q}-1)} \int_0^{2\sigma\rho} \int_{S^{n-2}(x')} \int_{Z_{\sigma\rho}(x_0)} \frac{|\tau_{h,e}v|^p}{h^{(n-1)\frac{p}{q}+\beta p+2-n}}\,dx\,de\,dh\,,
\end{aligned}$$

and the constant c depends only on n, p and q. Using the assumption of the lemma, $\sigma < 1$ and taking into account the fact that $q = q(n,p,\tau,\beta)$, we hence conclude

$$\begin{aligned}
\int_{Z_{\sigma\rho}(x_0)} f^{\frac{p}{q}}\,dx &\leq c(n,p,q)\,K^p\,(\sigma\rho)^{(n-1)(\frac{p}{q}-1)}\mathcal{H}^{n-2}(S^{n-2}) \int_0^{2\sigma\rho} h^{(\tau-\beta)p+(n-1)(1-\frac{p}{q})-1}\,dh \\
&= c(n,p,\tau,\beta)\,K^p\,\rho^{(\tau-\beta)p}\,,
\end{aligned}$$

provided that $(\tau-\beta)p+(n-1)(1-\frac{p}{q}) > 0$ which is equivalent to our choice $q > \frac{(n-1)p}{(\tau-\beta)p+(n-1)}$ above. This proves (8.3) and therefore the assertion of the lemma. □

8.2.2 Caccioppoli inequality revised

In the sequel we will argue under the permanent assumption that the weak solution u of system (8.2) is Hölder continuous on Q^+ with Hölder exponent λ for some $\lambda \in (0,1)$. This assumption will later be justified by the fact that in low dimensions the weak solution u is a priori known to be Hölder continuous outside a set of Hausdorff dimension $n-2$ (and we are interested in the behaviour of Du on the boundary which is of Hausdorff dimension $n-1$). The fact that the oscillations of u are hence arbitrarily small in a cylinder – provided that the side length of the cylinder is chosen sufficiently small – allows us to deduce an up-to-the-boundary version of the Caccioppoli inequality in a more or less standard way: the proof follows the line of arguments in the proof of the Caccioppoli-type inequality in Lemma 3.6, but with simplified estimates because on the one hand we consider the quadratic case $p = 2$ and on the other hand, due to the Hölder continuity of u, we do not need to involve the smallness assumption $|u| \leq M$ with $2L_2M < \nu$.

Lemma 8.3 (Caccioppoli inequality revised): *Let $u \in W^{1,2}_\Gamma(Q_2^+,\mathbb{R}^N) \cap L^\infty(Q_2^+,\mathbb{R}^N)$ be a bounded weak solution of (8.2) with coefficients $a(\cdot,\cdot,\cdot)$ and inhomogeneity $b(\cdot,\cdot,\cdot)$ satisfying the assumptions (H1)-(H4) and (B2), respectively. Assume further $u \in C^{0,\lambda}(Q^+,\mathbb{R}^N)$. Then there exist positive constants $\widetilde{c}_{cacc} = \widetilde{c}_{cacc}(\frac{L}{\nu},\frac{L_2}{\nu})$ and $\widetilde{\rho}_{cacc} = \widetilde{\rho}_{cacc}(\frac{L_2}{\nu},\lambda,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)})$ such that for every $\xi \in \mathbb{R}^N$ and every cylinder $Z_\rho(y)$ with $y \in Q^+ \cup \Gamma$ and $y_n < \rho \leq \widetilde{\rho}_{cacc}$ there holds:*

$$\fint_{Z_{\rho/2}(y)} |Du - \xi \otimes e_n|^2\,dx \leq \widetilde{c}_{cacc}\Big(\fint_{Z_\rho(y)} \Big|\frac{u-\xi x_n}{\rho}\Big|^2\,dx + \rho^{2\beta}\,\big(1+|\xi|\big)^{2+2\beta}\Big)$$

for all $\beta \in (0,\alpha]$.

8.3 A preliminary estimate

Our starting point for all further calculations is the following inequality concerning finite tangential differences of Du and which is the up to the boundary analogue of [Min03a], estimate (4.7). More precisely, we consider $\delta \in (0,1)$ and assume $u \in W^{1,2}_{\Gamma}(Q_2^+,\mathbb{R}^N) \cap L^\infty(Q_2^+,\mathbb{R}^N)$ to be a weak solution of system (8.2); then for every cut-off function $\eta \in C_0^\infty(Q_{1-\delta},[0,1])$ and every tangential direction $e \in S^{n-1}$ with $e \perp e_n$ there holds

$$\begin{aligned}\int_{Q^+} \eta^2 |\tau_{e,h} Du|^2 \,dx \leq c \bigg(|h|^{2\alpha} & \int_{Q^+\cap \operatorname{spt}(\eta)} \big(1+|Du(x+he)|^2 + |h|^{-2}|\tau_{e,h}u(x)|^2\big)\,dx \\ &+ \int_{Q^+\cap \operatorname{spt}(\eta)} \big(1+|Du(x+he)|^2\big)\,|\tau_{e,h}u(x)|^{2\alpha}\,dx \\ &+ \int_{Q^+} \big(1+|Du(x)|^2\big)\,|\tau_{e,-h}(\eta^2\tau_{e,h}u(x))|\,dx \bigg) \end{aligned} \tag{8.4}$$

for all $h \in \mathbb{R}$ with $|h| < \delta$, and the constant c depends only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}, \|u\|_\infty$ and $\|D\eta\|_\infty$. For the sake of completeness we here give the proof of inequality (8.4): we test the system (8.2) with the function $\varphi = \tau_{e,-h}(\eta^2\tau_{e,h}u)$. Using partial integration for finite differences, we then obtain

$$\begin{aligned}\int_{Q^+} \tau_{e,h}\big(a(x,u(x),Du(x))\big) \cdot \big(\tau_{e,h}Du(x)\,\eta^2 + 2\,\tau_{e,h}u \otimes D\eta\,\eta\big)\,dx \\ = \int_{Q^+} b(x,u(x),Du(x)) \cdot \tau_{e,-h}(\eta^2\tau_{e,h}u)\,dx\,. \end{aligned} \tag{8.5}$$

We next decompose the finite differences $\tau_{s,h}\big(a(x,u(x),Du(x))\big)$ as follows:

$$\begin{aligned}&\tau_{e,h}\big(a(x,u(x),Du(x))\big) \\ &\quad = a(x+he,u(x+he),Du(x+he)) - a(x,u(x+he),Du(x+he)) \\ &\qquad + a(x,u(x+he),Du(x+he)) - a(x,u(x),Du(x+he)) \\ &\qquad + a(x,u(x),Du(x+he)) - a(x,u(x),Du(x)) \\ &\quad =: \mathcal{A}(h) + \mathcal{B}(h) + \mathcal{C}(h) \end{aligned} \tag{8.6}$$

with the obvious notation. Hence, (8.5) may be rewritten as

$$\begin{aligned}&\int_{Q^+} \mathcal{C}(h) \cdot \tau_{e,h}Du(x)\,\eta^2\,dx \\ &\quad = -\int_{Q^+} \mathcal{A}(h) \cdot \big(\tau_{e,h}Du(x)\,\eta^2 + 2\,\tau_{e,h}u \otimes D\eta\,\eta\big)\,dx \\ &\qquad - \int_{Q^+} \mathcal{B}(h) \cdot \big(\tau_{e,h}Du(x)\,\eta^2 + 2\,\tau_{e,h}u \otimes D\eta\,\eta\big)\,dx \\ &\qquad - \int_{Q^+} \mathcal{C}(h) \cdot 2\,\tau_{e,h}u \otimes D\eta\,\eta\,dx + \int_{Q^+} b(x,u(x),Du(x)) \cdot \tau_{e,-h}(\eta^2\tau_{e,h}u)\,dx \\ &\quad =: I + II + III + IV\,. \end{aligned} \tag{8.7}$$

In the next step we estimate the various terms arising in the last inequality:

Estimate for I: Using Young's inequality and (H4) we obtain for every $\varepsilon \in (0,1)$:

$$\int_{Q^+} \mathcal{A}(h) \cdot \tau_{e,h} Du(x)\, \eta^2\, dx \;\le\; L\, \omega(|h|) \int_{Q^+} \big(1 + |Du(x+he)|\big)\, |\tau_{e,h} Du(x)|\, \eta^2\, dx$$
$$\le\; \varepsilon \int_{Q^+} |\tau_{e,h} Du(x)|^2\, \eta^2\, dx + 2\, L^2\, \varepsilon^{-1}\, |h|^{2\alpha} \int_{Q^+ \cap \operatorname{spt}(\eta)} \big(1 + |Du(x+he)|^2\big)\, dx\,.$$

Similarly, in view of $|h| < 1$, we conclude for the second term

$$\int_{Q^+} \mathcal{A}(h) \cdot 2\, \tau_{e,h} u \otimes D\eta\, \eta\, dx \;\le\; c(\|D\eta\|_\infty)\, L\, |h|^{\alpha+1} \int_{Q^+} \big(1 + |Du(x+he)|\big)\, |h|^{-1} |\tau_{e,h} u|\, dx$$
$$\le\; c(\|D\eta\|_\infty)\, L\, |h|^{2\alpha} \int_{B_R(x_0)} \big(1 + |Du(x+he)|^2 + |h|^{-2} |\tau_{e,h} u|^2\big)\, dx\,.$$

Estimate for II: Applying (H4) and Young's inequality we find

$$\int_{Q^+} \mathcal{B}(h) \cdot \tau_{e,h} Du(x)\, \eta^2\, dx \;\le\; L \int_{Q^+} \big(1 + |Du(x+he)|\big)\, |\tau_{e,h} u(x)|^{\alpha}\, |\tau_{e,h} Du(x)|\, \eta^2\, dx$$
$$\le\; \varepsilon \int_{Q^+} |\tau_{e,h} Du(x)|^2\, \eta^2\, dx + 2\, L^2\, \varepsilon^{-1} \int_{Q^+ \cap \operatorname{spt}(\eta)} \big(1 + |Du(x+he)|^2\big)\, |\tau_{e,h} u(x)|^{2\alpha}\, dx\,,$$

and due to the boundedness of u we see for the second term in II:

$$\int_{Q^+} \mathcal{B}(h) \cdot 2\, \tau_{e,h} u \otimes D\eta\, \eta\, dx \;\le\; 2\, L \int_{Q^+} |D\eta|\, \eta\, \big(1 + |Du(x+he)|\big)\, |\tau_{e,h} u(x)|^{1+\alpha}\, dx$$
$$\le\; c\big(\, \|u\|_\infty\, , \|D\eta\|_\infty \,\big)\, L \int_{Q^+ \cap \operatorname{spt}(\eta)} \big(1 + |Du(x+he)|^2\big)\, |\tau_{e,h} u(x)|^{2\alpha}\, dx\,.$$

Before estimating term III and the left-hand side of (8.7) we observe that $\mathcal{C}(h)$ may be rewritten as follows

$$\mathcal{C}(h) = \; a(x, u(x), Du(x+he)) - a(x, u(x), Du(x))$$
$$= \int_0^1 D_z a\big(x, u(x), Du(x) + t\tau_{e,h} Du(x)\big)\big)\, dt\, \tau_{e,h} Du(x) \;=:\; \widetilde{\mathcal{C}}(h)\, \tau_{e,h} Du(x)\,. \qquad (8.8)$$

Keeping in mind the conditions (H2) and (H3) on $D_z a(\cdot,\cdot,\cdot)$ we easily check that the following upper and lower bounds are available for $\widetilde{\mathcal{C}}(h)$:

$$|\widetilde{\mathcal{C}}(h)| \;\le\; L \qquad \text{and} \qquad \widetilde{\mathcal{C}}(h)\, \tau_{e,h} Du(x) \cdot \tau_{e,h} Du(x) \;\ge\; \nu\, |\tau_{e,h} Du(x)|^2\,.$$

Estimate for III: Using the upper bound, we compute for term III in the same way as in the estimates of the previous integrals I and II that we have

$$\int_{Q^+} \mathcal{C}(h) \cdot 2\, \tau_{e,h} u \otimes D\eta\, \eta\, dx \;\le\; 2 \int_{Q^+} |\widetilde{\mathcal{C}}(h)|\, |\tau_{e,h} Du(x)|\, |\tau_{e,h} u|\, |D\eta|\, \eta\, dx$$
$$\le\; \varepsilon \int_{Q^+} |\tau_{e,h} Du(x)|^2\, \eta^2\, dx + c(\|u\|_\infty\, , \|D\eta\|_\infty)\, L^2\, \varepsilon^{-1} \int_{Q^+ \cap \operatorname{spt}(\eta)} |\tau_{e,h} u(x)|^{2\alpha}\, dx\,.$$

Estimate for the left-hand side of (8.7): The lower bound of $\widetilde{\mathcal{C}}(h)$ is used to estimate

$$\int_{Q^+} \mathcal{C}(h) \cdot \tau_{e,h} Du(x)\, \eta^2\, dx \;\ge\; \nu \int_{Q^+} |\tau_{e,h} Du(x)|^2\, \eta^2\, dx\,.$$

Estimate for IV: For the term with the inhomogeneity $b(\cdot,\cdot,\cdot)$ we only need to apply the growth condition (B2) to infer

$$\int_{Q^+} b(x,u(x),Du(x))\cdot\tau_{e,-h}(\eta^2\tau_{e,h}u)\,dx \;\le\; (L+L_2)\int_{Q^+}\big(1+|Du|^2\big)|\tau_{e,-h}(\eta^2\tau_{e,h}u)|\,dx\,.$$

We now combine all the estimates found for the integrals appearing in (8.7) and choose $\varepsilon=\frac{\nu}{6}$ to end up with the desired inequality (8.4). We note that the constant c has the dependencies stated above.

8.4 Higher integrability of finite differences of Du

We next assume $\theta\in(0,1)$, $u\in C^{0,\lambda}(Q^+,\mathbb{R}^N)$, $Z_r(x_0)\subset Q^+$ with $x_0\in Q^+\cup\Gamma$. Then we choose a standard cut-off function $\eta\in C_0^\infty(Z_{(1+\theta)r/2}(x_0),[0,1])$ satisfying $\eta\equiv1$ on $Z_{\theta r}(x_0)$ and $|D\eta|\le\frac{c}{(1-\theta)r}$. We easily infer from (8.4):

$$\int_{Z_{\theta r}(x_0)}|\tau_{e,h}Du|^2\,dx \;\le\; c\,|h|^{\alpha\lambda}\int_{Z_r(x_0)}\big(1+|Du|^2\big)\,dx \tag{8.9}$$

for all $e\in S^{n-1}$ with $e\perp e_n$, $h\in\mathbb{R}$ satisfying $|h|<\frac{r(1-\theta)}{2}$, and the constant c depends only on $n,N,\frac{L}{\nu},\frac{L_2}{\nu},[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)},\theta$ and r. Note that we have $\|u\|_{L^\infty(Q^+,\mathbb{R}^N)}\le[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ because u is assumed to vanish on Γ. Moreover, the coefficients $a(\cdot,\cdot,\cdot)$ and the inhomogeneity $b(\cdot,\cdot,\cdot)$ satisfy the hypotheses of Lemma 6.2 which ensures the existence of a higher integrability exponent $\widetilde{s}>2$ depending only on $n,N,\frac{L}{\nu},\frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ such that we have $u\in W^{1,\widetilde{s}}(Q^+_\rho,\mathbb{R}^N)$ for all $\rho<1$. Furthermore, for every centre $x_0\in Q^+\cup\Gamma$ and every radius $\rho\in(0,1-|x_0|)$ there holds:

$$\Big(\fint_{Z_{\rho/2}(x_0)}|Du|^{\widetilde{s}}\,dx\Big)^{\frac{1}{\widetilde{s}}} \;\le\; c\big(n,N,\tfrac{L}{\nu},\tfrac{L_2}{\nu},[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\big)\Big(\fint_{Z_\rho(x_0)}\big(1+|Du|^2\big)\,dx\Big)^{\frac12}. \tag{8.10}$$

Employing the previous two estimates we obtain similarly to [Min03a, Section 5, step 2] a higher integrability result for $\tau_{e,h}Du$:

Proposition 8.4: *Let $u\in W^{1,2}_\Gamma(Q_2^+,\mathbb{R}^N)\cap L^\infty(Q_2^+,\mathbb{R}^N)\cap C^{0,\lambda}(Q^+,\mathbb{R}^N)$ be a weak solution of (8.2) under the assumptions (H1)-(H4) and (B2). Furthermore, let $Z_\rho(x_0)\subset Q^+$ for some $x_0\in Q^+\cup\Gamma$, $\sigma\in(0,\frac{1}{10})$, $e\in S^{n-1}$ with $e\perp e_n$ and $h\in\mathbb{R}$ with $|h|\in(0,2\sigma\rho)$. Then there exists a higher integrability exponent $s\in(2,\widetilde{s})$ depending only on $n,N,\frac{L}{\nu},\frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ such that*

$$\fint_{Z_{\sigma\rho}(x_0)}|\tau_{e,h}Du|^s\,dx \;\le\; c\,|h|^{\frac{\alpha\lambda s}{2}}\Big(\fint_{Z_\rho(x_0)}\big(1+|Du|^2\big)\,dx\Big)^{\frac{s}{2}}$$

for a constant $c=c\big(n,N,\frac{L}{\nu},\frac{L_2}{\nu},[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)},\rho,\sigma\big)$. Here, $\widetilde{s}=\widetilde{s}\big(n,N,\frac{L}{\nu},\frac{L_2}{\nu},[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\big)$ is the higher integrability exponent of Du.

Proof: We consider in the sequel the tangential directions $e\in S^{n-1}$, i.e., $e\perp e_n$, and we initially look at numbers $h\in\mathbb{R}$ satisfying $|h|<1$. Taking $\tau_{e,-h}\varphi$ with $\varphi\in C_0^\infty(Q^+_{1-|h|},\mathbb{R}^N)$ as a test function and making use of the partial integration formula for finite differences we rewrite the system (8.2) in its weak form as follows:

$$\int_{Q^+}\big[\mathcal{A}(h)+\mathcal{B}(h)+\mathcal{C}(h)\big]\cdot D\varphi\,dx \;=\; \int_{Q^+} b(x,u,Du)\cdot\tau_{e,-h}\varphi\,dx\,,$$

where the abbreviations for $\mathcal{A}(h)$, $\mathcal{B}(h)$ and $\mathcal{C}(h)$, representing the differences of the coefficients $a(\cdot,\cdot,\cdot)$ with respect to each variable, were introduced in (8.6). We set

$$v_h := \frac{\tau_{e,h}u}{|h|^{\frac{\alpha\lambda}{2}}}, \qquad \widetilde{\mathcal{A}}(h) := \frac{-\mathcal{A}(h)}{|h|^{\frac{\alpha\lambda}{2}}}, \qquad \widetilde{\mathcal{B}}(h) := \frac{-\mathcal{B}(h)}{|h|^{\frac{\alpha\lambda}{2}}},$$

and we recall the definition of $\widetilde{\mathcal{C}}(h) = \int_0^1 D_z a\big(x, u(x), Du(x) + t\tau_{e,h}Du(x)\big)\big)\,dt$ in (8.8). Dividing the previous identity by $|h|^{\alpha\lambda/2}$ we get

$$\int_{Q^+} \widetilde{\mathcal{C}}(h)\, Dv_h \cdot D\varphi\, dx = \int_{Q^+} \big[\widetilde{\mathcal{A}}(h) + \widetilde{\mathcal{B}}(h)\big]\cdot D\varphi\, dx + \int_{Q^+} |h|^{-\frac{\alpha\lambda}{2}}\, b(x,u,Du)\cdot \tau_{e,-h}\varphi\, dx \tag{8.11}$$

for all functions $\varphi \in C_0^\infty(Q^+_{1-|h|}, \mathbb{R}^N)$, i.e., the map $v_h \in W^{1,2}(Q^+_{1-|h|}, \mathbb{R}^N)$ is a weak solution to the linear system (8.11) for every $h \in \mathbb{R}$ with $|h| < 1$. In the next step we are going to infer Caccioppoli-type inequalities for the functions v_h, where the constants may be chosen independently of the parameter h. For this purpose we first observe some simple properties of the new system: taking into account the assumptions (H1)-(H4) and the Hölder continuity of u with exponent λ, we immediately find the following upper and lower bounds:

$$\begin{aligned}
|\widetilde{\mathcal{A}}(h)| &\le L\,\big(1 + |Du(x+he)|\big)\,,\\
|\widetilde{\mathcal{B}}(h)| &\le L\,[u]^{\alpha}_{C^{0,\lambda}(B^+,\mathbb{R}^N)}\,\big(1 + |Du(x+he)|\big)\,,\\
\nu\,|\widetilde{\lambda}|^2 &\le \widetilde{\mathcal{C}}(h)\widetilde{\lambda}\otimes\widetilde{\lambda} \le L\,|\widetilde{\lambda}|^2 \qquad \forall\,\widetilde{\lambda}\in\mathbb{R}^{nN}\,.
\end{aligned}$$

For σ, ρ and x_0 fixed according to the assumptions of the proposition, we next choose $h \in \mathbb{R}$ such that $|h| \in (0, 2\sigma\rho)$ and consider intersections of balls $B^+_R(y)$ with the upper half-plane $\mathbb{R}^{n-1}\times\mathbb{R}^+$ for centres $y \in \overline{Z_{(1-\sigma)\rho/2}(x_0)}$ satisfying $B^+_R(y) \subset Q^+_{1-|h|}$ (implying that $0 < R < 1 - |h| - \max_{k\in\{1,\dots,n\}} |y_k|$) and $y_n \le \frac{3R}{4}$, i.e., we first study the situation for centres close to the boundary. Furthermore, we take a cut-off function $\eta \in C_0^\infty(B_{3R/4}(y), [0,1])$ satisfying $\eta \equiv 1$ on $B_{R/2}(y)$ and $|D\eta| \le \frac{8}{R}$, and we choose $\varphi := \eta^2 v_h$ as a test function in (8.11) which is admissible by a standard approximation argument. Taking into account

$$D\varphi = \eta^2\, Dv_h + 2\,\eta\, v_h \otimes D\eta$$

we estimate the various terms arising in (8.11): using Young's inequality with $\varepsilon \in (0,1)$ and the estimates for $\widetilde{\mathcal{A}}(h)$, $\widetilde{\mathcal{B}}(h)$ and $\widetilde{\mathcal{C}}(h)$ given above we see

- $\displaystyle \nu \int_{B^+_R(y)} \eta^2\,|Dv_h|^2\,dx \le \int_{B^+_R(y)} \eta^2\,\widetilde{\mathcal{C}}(h)\,Dv_h\cdot Dv_h\,dx\,,$

- $\displaystyle \int_{B^+_R(y)} 2\,\eta\,|\widetilde{\mathcal{C}}(h)Dv_h\cdot v_h\otimes D\eta|\,dx \le \varepsilon \int_{B^+_R(y)} \eta^2\,|Dv_h|^2\,dx + \frac{c\,L^2}{\varepsilon\,R^2}\int_{B^+_R(y)} |v_h|^2\,dx\,,$

- $$\begin{aligned}\int_{B^+_R(y)} |\widetilde{\mathcal{A}}(h)\cdot D\varphi|\,dx &\le \varepsilon \int_{B^+_R(y)} \eta^2\,|Dv_h|^2\,dx + \frac{L}{R^2}\int_{B^+_R(y)} |v_h|^2\,dx\\ &\quad + c\,\big(\varepsilon^{-1}L^2 + L\big)\int_{B^+_R(y)} \big(1 + |Du(x+he)|^2\big)\,dx\,,\end{aligned}$$

- $$\begin{aligned}\int_{B^+_R(y)} |\widetilde{\mathcal{B}}(h)\cdot D\varphi|\,dx &\le \varepsilon \int_{B^+_R(y)} \eta^2\,|Dv_h|^2\,dx + \frac{c\,\varepsilon}{R^2}\int_{B^+_R(y)} |v_h|^2\,dx\\ &\quad + c\big([u]_{C^{0,\lambda}(B^+,\mathbb{R}^N)}\big)\,\varepsilon^{-1}L^2\int_{B^+_R(y)} \big(1 + |Du(x+he)|^2\big)\,dx\,.\end{aligned}$$

In order to estimate the last integral on the right-hand side of (8.11) we first recall the definition $v_h = |h|^{-\frac{\alpha\lambda}{2}} \tau_{e,h} u$ and calculate

$$\begin{aligned}\big|\tau_{e,-h}\varphi\big| &= \big|\tau_{e,-h}(\eta^2 v_h)\big| \;=\; |h|^{-\frac{\alpha\lambda}{2}} \big|\tau_{e,-h}(\eta^2 \tau_{e,h} u)\big| \\ &\le |h|^{-\frac{\alpha\lambda}{2}} \big(|\tau_{e,h}u(x-he)| + |\tau_{e,h}u(x)|\big) \;\le\; 2\,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\,|h|^{\lambda-\frac{\alpha\lambda}{2}}\,. \end{aligned} \tag{8.12}$$

This yields

- $$\int_{B_R^+(y)} |h|^{-\frac{\alpha\lambda}{2}}\, |b(x,u,Du)\cdot \tau_{e,-h}\varphi|\,dx \;\le\; c\big([u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\big) \int_{B_R^+(y)} \big(L + L_2\,|Du(x)|^2\big)\,dx\,.$$

Collecting the estimates for all terms arising in equation (8.11) and choosing $\varepsilon = \frac{\nu}{6}$, we finally conclude the Caccioppoli-type estimate

$$\int_{B_{R/2}^+(y)} |Dv_h|^2\,dx \;\le\; c\,R^{-2} \int_{B_R^+(y)} |v_h|^2\,dx + c \int_{B_R^+(y)} \big(1 + |Du(x)|^2 + |Du(x+he)|^2\big)\,dx\,,$$

and the constant c depends only on $\frac{L}{\nu}$, $\frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$. With the boundary version of the Sobolev-Poincaré inequality, Lemma A.5, we deduce

$$\begin{aligned}\fint_{B_{R/2}^+(y)} |Dv_h|^2\,dx \le c\Big(\fint_{B_R^+(y)} |Dv_h|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{n}} \\ + c\fint_{B_R^+(y)} \big(1 + |Du(x)|^2 + |Du(x+he)|^2\big)\,dx\,,\end{aligned}$$

and the constant c depends only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$. We here note that the integrand of the second integral on the right-hand side of the last inequality belongs to $L^{\widetilde{s}/2}$ due to the higher integrability result for Du from (8.10).

In the interior we proceed analogously and consider $B_R^+(y)$ with centres $y \in Z_{(1-\sigma)\rho/2}(x_0)$ satisfying $B_R^+(y) \subset Q_{1-|h|}^+$ and $y_n > \frac{3R}{4}$. If we choose $\varphi := \eta^2\big(v_h - (v_h)_{y,3R/4}\big)$ as a test function all the computations above remain valid (with 2 replaced by 4 in inequality (8.12)). Then, after applying the Sobolev-Poincaré inequality in the interior in the mean value version on the ball $B_{3R/4}(y)$, we obtain the corresponding inequality

$$\begin{aligned}\fint_{B_{R/2}(y)} |Dv_h|^2\,dx \le c\Big(\fint_{B_R^+(y)} |Dv_h|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{n}} \\ + c\fint_{B_R^+(y)} \big(1 + |Du(x)|^2 + |Du(x+he)|^2\big)\,dx\,,\end{aligned}$$

and c has exactly the same dependencies as in the previous reverse Hölder-type inequality; in particular, the constant c is independent of the parameter h. Applying the global Gehring Lemma, Theorem A.14, on the cylinder $Z_{(1-\sigma)\rho/2}(x_0)$ for the choices of σ, ρ and x_0 made in the assumptions of the proposition, we obtain that there exist a constant c depending only on $n, N, q, \frac{L}{\nu}, \frac{L_2}{\nu}, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ and σ and a positive number δ depending only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}$

and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ such that there holds

$$\begin{aligned}
&\Big(\fint_{Z_{\sigma\rho}(x_0)} |Dv_h|^q\,dx\Big)^{\frac{1}{q}} \\
&\le c\Big[\Big(\fint_{Z_{(1-8\sigma)\rho/2}(x_0)} |Dv_h|^2\,dx\Big)^{\frac{1}{2}} + \Big(\fint_{Z_{(1-8\sigma)\rho/2}(x_0)} \big(1+|Du(x)|^2+|Du(x+he)|^2\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}\Big] \\
&\le c\Big[|h|^{-\frac{\alpha\lambda}{2}}\Big(\fint_{Z_{(1-8\sigma)\rho/2}(x_0)} |\tau_{e,h}Du|^2\,dx\Big)^{\frac{1}{2}} + \Big(\fint_{Z_{\rho/2}(x_0)} \big(1+|Du(x)|^2\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}\Big] \\
&\le c\Big[\Big(\fint_{Z_{\rho/2}(x_0)} \big(1+|Du|^2\big)\,dx\Big)^{\frac{1}{2}} + \Big(\fint_{Z_{\rho/2}(x_0)} \big(1+|Du|^2\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}\Big] \\
&\le c\Big(\fint_{Z_{\rho/2}(x_0)} \big(1+|Du|^2\big)^{\frac{q}{2}}\,dx\Big)^{\frac{1}{q}}
\end{aligned}$$

for all $q \in [2, 2+\delta)$. Here, we have also used the bound $|h| < 2\sigma\rho$ (with $|\sigma| < \frac{1}{10}$), the estimate (8.9) on finite differences and Jensen's inequality. We notice that, in view of the dependencies appearing in (8.9), the constant c depends additionally on the radius ρ. Hence, for all $s \in (2, \min\{\widetilde{s}, 2+\delta\})$, where $\widetilde{s} > 2$ is the higher integrability exponent of Du from (8.10), the previous inequality holds true; moreover, keeping in mind the definition of v_h and the higher integrability result (8.10), we finally arrive at

$$\begin{aligned}
\Big(\fint_{Z_{\sigma\rho}(x_0)} |\tau_{e,h}Du|^s\,dx\Big)^{\frac{1}{s}} &= |h|^{\frac{\alpha\lambda}{2}}\Big(\fint_{Z_{\sigma\rho}(x_0)} |Dv_h|^s\,dx\Big)^{\frac{1}{s}} \\
&\le c\,|h|^{\frac{\alpha\lambda}{2}}\Big(\fint_{Z_{\rho/2}(x_0)} \big(1+|Du|^2\big)^{\frac{\widetilde{s}}{2}}\,dx\Big)^{\frac{1}{\widetilde{s}}} \\
&\le c\,|h|^{\frac{\alpha\lambda}{2}}\Big(\fint_{Z_{\rho}(x_0)} \big(1+|Du|^2\big)\,dx\Big)^{\frac{1}{2}},
\end{aligned}$$

which finishes the proof of the proposition. □

Moreover, we want to mention two direct consequences of Proposition 8.4. The first one follows from Lemma 8.2 and concerns the slicewise mean-square deviation of Du:

Corollary 8.5: *Let $u \in W^{1,2}_\Gamma(Q_2^+,\mathbb{R}^N) \cap L^\infty(Q_2^+,\mathbb{R}^N) \cap C^{0,\lambda}(Q^+,\mathbb{R}^N)$ be a weak solution of (8.2) under the assumptions (H1)-(H4) and (B2). Furthermore, let $Z_\rho(x_0) \subset Q^+$ for some $x_0 \in Q^+ \cup \Gamma$ and $\sigma \in (0, \frac{1}{10})$. Then for every $\gamma \in (0,1)$ there exists a function $F_1 \in L^s(Z_{\sigma\rho}(x_0))$ ($s > 2$ still denotes the higher integrability exponent from Proposition 8.4) such that the following estimate holds true:*

$$\begin{aligned}
&\Big(\fint_{Z_r(z)} \big|Du(x) - (Du)_{z',r}(x_n)\big|^2\,dx\Big)^{\frac{1}{2}} \\
&\qquad \le \Big(\fint_{Z_r(z)} \fint_{D_r(z')} |Du(x',x_n) - Du(y',x_n)|^2\,dy'\,dx\Big)^{\frac{1}{2}} \le c\,r^{\frac{\gamma\alpha\lambda}{2}}\,F_1(z)
\end{aligned}$$

for all cylinders $Z_r(z) \subset Z_{\sigma\rho}(x_0)$ with $z \in Q^+ \cup \Gamma$, and the constant c depends only on n, α, λ and γ.

Remarks: We note that the L^s-norm of F_1 might blow up if $\gamma \nearrow 1$ because in that case the application of the L^q-inequality for the maximal operator in the proof of Lemma 8.2 becomes critical (meaning that $q \searrow 1$), cf. Lemma 5.4 and the Remark thereafter.

Moreover, when verifying the assumptions of Lemma 8.2, we observe that the number K (resulting from the inequality in Proposition 8.4) depends on the radius ρ and on σ. This dependency is reflected only in the L^s-norm of F_1. However, in the sequel this is not of importance because ρ and σ may be chosen fixed in every step of the subsequent iteration. More precisely, in the next section the cylinders $Z_{\sigma\rho}(x_0)$ will be used to infer appropriate fractional Sobolev estimates on them and then, via a covering argument, also on Q^+ (respectively on smaller half-cubes in the course of the iteration).

As a second consequence of Proposition 8.4 we obtain that the tangential derivative is already known to be in a suitable fractional Sobolev space. This follows immediately from Lemma 2.5 and the inclusion $W^{\theta,s} \subseteq M^{\theta,s}$ (for $\theta \in (0,1)$, $s \in (1,\infty)$) given in Remark 2.9.

Corollary 8.6: *Let $u \in W^{1,2}_\Gamma(Q_2^+,\mathbb{R}^N) \cap L^\infty(Q_2^+,\mathbb{R}^N) \cap C^{0,\lambda}(Q^+,\mathbb{R}^N)$ be a weak solution of (8.2) under the assumptions (H1)-(H4) and (B2). Then for every $\gamma \in (0,1)$ there holds*

$$D'u = (D_1 u, \dots, D_{n-1}u) \in M^{\gamma\alpha\lambda/2,s}(Q_\rho^+, \mathbb{R}^{(n-1)N})$$

for every $\rho < 1$. In particular, there exists a function $H_1 \in L^s(Q_{1/2}^+)$ such that

$$|D'u(x) - D'u(y)| \leq |x-y|^{\frac{\gamma\alpha\lambda}{2}} \big(H_1(x) + H_2(y)\big)$$

for almost all $x, y \in Q_{1/2}^+$.

8.5 An estimate for the full derivative

So far, we can estimate finite differences close to the boundary only with respect to tangential directions. Therefore, we have obtained that the tangential derivative $D'u$ belongs to a fractional Sobolev space. In order to find a fractional Sobolev estimate of type (2.2) also with respect to normal direction we next choose a cylinder $Z_\rho(x_0) \subset Q^+$, $x_0 \in Q^+ \cup \Gamma$, $\rho \leq \widetilde{\rho}_{cacc}$ where $\widetilde{\rho}_{cacc}$ is from Lemma 8.3, and $\sigma \in (0, \frac{1}{10})$. Furthermore, we fix a number $\gamma \in (0,1)$ to be specified later. In the sequel we study the model system (8.2) on cylinders $Z_r(z)$ with $z \in Q^+ \cup \Gamma$ such that $Z_{2r}(z) \subset Z_{\sigma\rho}(x_0)$, and by M^* we will always denote the maximal operator restricted to the cylinder $Z_{\sigma\rho}(x_0)$, i. e.,

$$M^*(f)(z) := \sup_{Z_{\tilde r}(\tilde z) \subseteq Z_{\sigma\rho}(x_0),\, z \in Z_{\tilde r}(\tilde z)} \fint_{Z_{\tilde r}(\tilde z)} |f(x)|\, dx\,.$$

for every $f \in L^1(Z_{\sigma\rho}(x_0),\mathbb{R}^k)$, $k \geq 1$, and $z \in Z_{\sigma\rho}(x_0)$. In coordinates we have the following representation of the weak formulation for the system (8.2):

$$\sum_{j=1}^N \sum_{\kappa=1}^n \fint_{Z_r(z)} a_\kappa^j(x,u(x),Du(x))\, D_\kappa \varphi^j\, dx = \sum_{j=1}^N \fint_{Z_r(z)} b^j(x,u(x),Du(x))\, \varphi^j\, dx \tag{8.13}$$

for all $\varphi \in C_0^\infty(Z_r(z),\mathbb{R}^N)$.

8.5.1 A fractional Sobolev estimate for $\mathbf{a_n}(\,\cdot\,,\mathbf{u},\mathbf{Du})$

In the first step we are going to derive a weak differentiability result for the function

$$A_r^j(x_n) := \fint_{D_r(z')} a_n^j(x',x_n,u(x',x_n),Du(x',x_n))dx' \tag{8.14}$$

for every $j \in \{1,\dots,N\}$ and $x_n \in I_r(z_n)$. For this purpose we choose a "splitting" test function of the form $\varphi(x) = \phi_1(x')\,\phi_2(x_n)\,E_j$ where $\phi_1 \in C_0^\infty(D_r(z'))$ with $\phi_1 \equiv 1$ on the $(n-1)$-dimensional ball $D_{\tau r}(z')$ for some $\tau \in (0,1)$, $\phi_2 \in C_0^\infty(I_r(z_n))$, and where E_j denotes the standard unit coordinate vector in $\mathbb{R}^N$. Testing (8.13) with φ then yields for $j \in \{1,\dots,N\}$:

$$\begin{aligned}
&\fint_{I_r(z_n)}\fint_{D_r(z')} a_n^j(x,u(x),Du(x))\,\phi_1(x')\,D_n\phi_2(x_n)\,dx'\,dx_n \\
&\quad= -\fint_{I_r(z_n)}\fint_{D_r(z')}\sum_{\kappa=1}^{n-1} a_\kappa^j(x,u(x),Du(x))\,D_\kappa\phi_1(x')\,\phi_2(x_n)\,dx'\,dx_n \\
&\qquad+\fint_{I_r(z_n)}\fint_{D_r(z')} b^j(x,u(x),Du(x))\,\phi_1(x')\,\phi_2(x_n)\,dx'\,dx_n \\
&\quad= -\fint_{I_r(z_n)}\frac{1}{|D_r(z')|}\int_{D_r(z')\setminus D_{\tau r}(z')}\sum_{\kappa=1}^{n-1} a_\kappa^j(x,u(x),Du(x))\,D_\kappa\phi_1(x')\,\phi_2(x_n)\,dx'\,dx_n \\
&\qquad+\fint_{I_r(z_n)}\fint_{D_r(z')} b^j(x,u(x),Du(x))\,\phi_1(x')\,\phi_2(x_n)\,dx'\,dx_n \\
&\quad= -\fint_{I_r(z_n)}\frac{1}{|D_r(z')|}\int_{\tau r}^{r}\int_{\partial D_{\widetilde r}(z')}\sum_{\kappa=1}^{n-1}\big[a_\kappa^j(x,u(x),Du(x)) - a_\kappa^j(z,(u)_{z,r},(Du)_{z',r}(x_n))\big] \\
&\qquad\qquad\qquad\times D_\kappa\phi_1(x')\,d\mathcal{H}^{n-2}(x')\,d\widetilde r\,\phi_2(x_n)\,dx_n \\
&\qquad+\fint_{I_r(z_n)}\fint_{D_r(z')} b^j(x,u(x),Du(x))\,\phi_1(x')\,dx'\,\phi_2(x_n)\,dx_n\,,
\end{aligned}$$

where we have used the co area formula in the last line. In particular, we may choose by approximation

$$\phi_1(x') = \begin{cases} 1 & \text{if } |x'-z'| \le \tau r\,, \\ \frac{r-|x'-z'|}{(1-\tau)r} & \text{if } \tau r < |x'-z'| < r\,, \\ 0 & \text{if } |x'-z'| \ge r\,. \end{cases}$$

We note that this implies $D_\kappa\phi_1(x') = -\frac{1}{(1-\tau)r}\frac{x_\kappa - z_\kappa}{|x'-z'|}$ for every $\kappa \in \{1,\dots,n-1\}$ provided that $|x'-z'| \in (\tau r, r)$. Setting

$$B_\kappa^j(x) = a_\kappa^j(x,u(x),Du(x)) - a_\kappa^j(z,(u)_{z,r},(Du)_{z',r}(x_n)) \tag{8.15}$$

for $j \in \{1,\dots,N\}$ and $\kappa \in \{1,\dots,n-1\}$, we calculate with this particular choice for the cut-off function ϕ_1:

$$\begin{aligned}
&\fint_{I_r(z_n)}\fint_{D_r(z')} a_n^j(x,u(x),Du(x))\,\phi_1(x')\,dx'\,D_n\phi_2(x_n)\,dx_n \\
&\quad= \fint_{I_r(z_n)}\frac{1}{|D_r(z')|}\fint_{\tau r}^{r}\int_{\partial D_{\widetilde r}(z')} B^j(x)\cdot\frac{x'-z'}{|x'-z'|}\,d\mathcal{H}^{n-2}(x')\,d\widetilde r\,\phi_2(x_n)\,dx_n \\
&\qquad+\fint_{I_r(z_n)}\fint_{D_r(z')} b^j(x,u(x),Du(x))\,\phi_1(x')\,dx'\,\phi_2(x_n)\,dx_n\,.
\end{aligned}$$

Recalling the definition of $A_r^j(x_n)$ given in (8.14) we consider the limit $\tau \nearrow 1$ and conclude from Lebesgue's differentiation Theorem for almost every radius r (and fixed centre $z \in Z_{\sigma\rho}(x_0)$) such that $Z_r(z) \subset Z_{\sigma\rho}(x_0)$:

$$\begin{aligned}\int_{I_r(z_n)} A_r^j(x_n)\, D_n\phi_2(x_n)\, dx_n = &\int_{I_r(z_n)} \frac{1}{|D_r(z')|}\int_{\partial D_r(z')} B^j(x)\cdot \frac{x'-z'}{|x'-z'|}\, d\mathcal{H}^{n-2}(x')\,\phi_2(x_n)\, dx_n\\ &+\int_{I_r(z_n)} \fint_{D_r(z')} b^j(x,u(x),Du(x))\, dx'\,\phi_2(x_n)\, dx_n\,.\end{aligned}$$

Hence, for almost every radius r with $Z_r(z) \subset Z_{\sigma\rho}(x_0)$ we find that the function $A_r(x_n) = (A_r^1(x_n),\dots,A_r^N(x_n))$ is weakly differentiable on $I_r(z_n)$ (note that the index $j \in \{1,\dots,N\}$ and ϕ_2 are arbitrary in the latter identity), and its weak derivative is given by

$$\begin{aligned}A_r'(x_n) = &-\frac{1}{|D_r(z')|}\int_{\partial D_r(z')} B(x)\cdot \frac{x'-z'}{|x'-z'|}\, d\mathcal{H}^{n-2}(x')\\ &-\fint_{D_r(z')} b(x,u(x),Du(x))\, dx'\,.\end{aligned}\tag{8.16}$$

We next consider for any fixed r all radii $\widetilde{\rho} \in (0,r]$ and we define the set J via

$$J = \Big\{\widetilde{\rho} : \widetilde{\rho} \in (0,r] \quad\text{and}\quad \int_{I_{\widetilde{\rho}}(z_n)}\int_{\partial D_{\widetilde{\rho}}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, dx_n > \frac{2}{r}\int_{Z_r(z)} |B(x)|\, dx\Big\}\,.$$

The following computations reveals that there holds $\mathcal{L}^1(J) < \frac{r}{2}$: employing the co area formula and Fubini's Theorem yields

$$\begin{aligned}\int_{Z_r(z)} |B(x)|\, dx &= \int_{I_r(z_n)}\int_0^r\int_{\partial D_{\widetilde{\rho}}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, d\widetilde{\rho}\, dx_n\\ &\ge \int_0^r\int_{I_{\widetilde{\rho}}(z_n)}\int_{\partial D_{\widetilde{\rho}}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, dx_n\, d\widetilde{\rho}\\ &\ge \int_J\int_{I_{\widetilde{\rho}}(z_n)}\int_{\partial D_{\widetilde{\rho}}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, dx_n\, d\widetilde{\rho}\\ &> \int_J \frac{2}{r}\int_{Z_r(z)} |B(x)|\, dx\, d\widetilde{\rho} = \mathcal{L}^1(J)\,\frac{2}{r}\int_{Z_r(z)} |B(x)|\, dx\,.\end{aligned}$$

Therefore, we find some radius $\bar\rho \in [\frac{r}{2}, r]$ such that on the one hand $A_{\bar\rho}(x_n)$ is weakly differentiable and on the other hand $\bar\rho \notin J$, i. e., we have

$$\int_{I_{\bar\rho}(z_n)}\int_{\partial D_{\bar\rho}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, dx_n \le \frac{2}{r}\int_{Z_r(z)} |B(x)|\, dx\,.$$

Hence, in view of Poincaré's inequality and identity (8.16), we obtain for this choice of $\bar\rho$:

$$\begin{aligned}\fint_{I_{\bar\rho}(z_n)} &\big|A_{\bar\rho}(x_n) - (A_{\bar\rho})_{z_n,\bar\rho}\big|\, dx_n \le c(N)\int_{I_{\bar\rho}(z_n)} \big|A_{\bar\rho}'(x_n)\big|\, dx_n\\ &\le \frac{c(N)}{|D_{\bar\rho}(z')|}\int_{I_{\bar\rho}(z_n)}\int_{\partial D_{\bar\rho}(z')} |B(x)|\, d\mathcal{H}^{n-2}(x')\, dx_n\\ &\qquad + c(N)\int_{I_{\bar\rho}(z_n)}\fint_{D_{\bar\rho}(z')} |b(x,u(x),Du(x))|\, dx'\, dx_n\\ &\le c(N)\Big[\frac{1}{|D_{\bar\rho}(z')|\, r}\int_{Z_r(z)} |B(x)|\, dx + \bar\rho\fint_{Z_{\bar\rho}(z)} |b(x,u(x),Du(x))|\, dx\Big]\\ &\le c(n,N)\Big[\fint_{Z_r(z)} |B(x)|\, dx + r\fint_{Z_r(z)} |b(x,u(x),Du(x))|\, dx\Big]\,.\end{aligned}\tag{8.17}$$

In the next step we want to control the integrals arising on the right-hand side of the last inequality by using the growth conditions on $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ and by exploiting the assumption that u is Hölder continuous with exponent λ.

For the first integral in (8.17) we use the definition of $B(x)$ in (8.15), the assumptions (H2), (H4), the Hölder continuity of u, and Corollary 8.5 to see

$$\begin{aligned}
⨍_{Z_r(z)} \big|B(x)\big|\,dx &\le ⨍_{Z_r(z)} \Big[\,\big|a(x,u(x),Du(x)) - a(x,u(x),(Du)_{z',r}(x_n))\big| \\
&\qquad + \big|a(x,u(x),(Du)_{z',r}(x_n)) - a(z,(u)_{z,r},(Du)_{z',r}(x_n))\big|\,\Big]\,dx \\
&\le L ⨍_{Z_r(z)} \big|Du(x) - (Du)_{z',r}(x_n)\big|\,dx \\
&\quad + 4\,L\big(r^{\alpha} + [u]^{\alpha}_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\,r^{\alpha\lambda}\big) ⨍_{Z_r(z)} \big(1 + |(Du)_{z',r}(x_n)|\big)\,dx \\
&\le c\,r^{\frac{\gamma\alpha\lambda}{2}}\,\big(F_1(z) + M^*\big(1+|Du|\big)(z)\big)\,,
\end{aligned}$$

and the constant c depends only on $n, L, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \alpha, \lambda$ and γ. Moreover, the functions F_1 and $M^*\big(1+|Du|\big)$ belong to the space $L^s(Z_{\sigma\rho}(x_0))$, due to Corollary 8.5 and the higher integrability of Du combined with Lemma 5.4 on the maximal function, respectively.

For the second integral in (8.17), we initially assume that we are close to the boundary, meaning that $z_n < 2r$. Then, we infer the following estimate from the natural growth condition (B2) on the inhomogeneity, the Caccioppoli inequality from Lemma 8.3 (note that $2r \le \widetilde{\rho}_{cacc}$), the Hölder continuity of u and the Poincaré inequality in the boundary version:

$$\begin{aligned}
r ⨍_{Z_r(z)} |b(x,u(x),Du(x))|\,dx &\le r ⨍_{Z_r(z)} (L + L_2\,|Du|^2)\,dx \\
&\le r\,L_2\,\widetilde{c}_{cacc}\Big(⨍_{Z_{2r}(z)} \Big|\frac{u}{r}\Big|^2\,dx + r^{2\alpha}\Big) + r\,L \\
&\le c\,\Big(r^{1-1+\lambda} ⨍_{Z_{2r}(z)} |Du|\,dx + r^{2\alpha+1}\Big) + r\,L \\
&\le c\,r^{\lambda}\,M^*\big(1+|Du|\big)(z)\,,
\end{aligned}$$

and the constant c depends only on n, N, L, L_2, ν and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$. For cylinders in the interior, where $z_n \ge 2r$, we end up with exactly the same estimate using an interior Caccioppoli-type inequality corresponding to the statement in Lemma 8.3 and the Poincaré inequality where in both cases $|u|$ is replaced by $|u-(u)_{z,2r}|$.

Hence, combining the last two estimates, we conclude from (8.17)

$$\begin{aligned}
&⨍_{Z_{\bar\rho}(z_n)} \Big| ⨍_{D_{\bar\rho}(z')} a_n(y',x_n,u(y',x_n),Du(y',x_n))\,dy' - ⨍_{Z_{\bar\rho}(z)} a_n(\tilde y,u(\tilde y),Du(\tilde y))\,d\tilde y\Big|\,dx \\
&\quad = ⨍_{I_{\bar\rho}(z_n)} \big|A_{\bar\rho}(x_n) - (A_{\bar\rho})_{z_n,\bar\rho}\big|\,dx_n \;\le\; c\,r^{\frac{\gamma\alpha\lambda}{2}}\,\big[F_1(z) + M^*\big(1+|Du|\big)(z)\big]\,, \qquad (8.18)
\end{aligned}$$

and the constant c depends only on $n, N, L, L_2, \nu, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \alpha, \lambda$ and γ. Besides, we have F_1, $M^*\big(1+|Du|\big) \in L^s(Z_{\sigma\rho}(x_0))$ for some $s > 2$. We mention here that the L^s-norm of F_1 might diverge for $\gamma \nearrow 1$, see the comments after Corollary 8.5. Furthermore, applying

Jensen's inequality, the Hölder continuity of $a(\cdot,\cdot,\cdot)$ with respect to the first two variables in (H4), condition (H2), the Hölder continuity of u and Corollary 8.5 we find

$$\begin{aligned}
&\fint_{Z_{\bar\rho}(z)} \Big|a_n(x,u(x),Du(x)) - \fint_{D_{\bar\rho}(z')} a_n(y',x_n,u(y',x_n),Du(y',x_n))\,dy'\Big|\,dx \\
&\le \fint_{Z_{\bar\rho}(z)}\fint_{D_{\bar\rho}(z')} \big|a_n(x',x_n,u(x',x_n),Du(x',x_n)) - a_n(y',x_n,u(y',x_n),Du(x',x_n))\big|\,dy'\,dx \\
&\qquad + \fint_{Z_{\bar\rho}(z)}\fint_{D_{\bar\rho}(z')} \big|a_n(y',x_n,u(y',x_n),Du(x',x_n)) - a_n(y',x_n,u(y',x_n),Du(y',x_n))\big|\,dy'\,dx \\
&\le c\big(L,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\big)\,\bar\rho^{\alpha\lambda}\fint_{Z_{\bar\rho}(z)}\big(1+|Du|\big)\,dx \\
&\qquad + L\fint_{Z_{\bar\rho}(z)}\fint_{D_{\bar\rho}(z')}\big|Du(x',x_n)-Du(y',x_n)\big|\,dy'\,dx \\
&\le c\big(n,L,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)},\alpha,\lambda,\gamma\big)\,\bar\rho^{\frac{\gamma\alpha\lambda}{2}}\,\big[M^*\big(1+|Du|\big)(z)+F_1(z)\big]\,. \qquad (8.19)
\end{aligned}$$

Hence, combining (8.18) and (8.19), we conclude

$$\fint_{Z_{\bar\rho}(z)} \big|a_n(x,u(x),Du(x)) - \big(a_n(\,\cdot\,,u,Du)\big)_{z,\bar\rho}\big|\,dx \;\le\; c\,r^{\frac{\gamma\alpha\lambda}{2}}\,\big[M^*\big(1+|Du|\big)(z)+F_1(z)\big]$$

for every r with $Z_r(z)\subset Z_{\sigma\rho}(x_0)$ and an appropriate radius $\bar\rho\in[\frac{r}{2},r]$ for which $A_{\bar\rho}(x_n)$ is weakly differentiable on $I_r(z_n)$ and $\bar\rho\notin J$. The constant c here depends only on n,N,L,L_2,ν, $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)},\alpha,\lambda$ and γ. In particular, this yields

$$\fint_{Z_{r/2}(z)} \big|a_n(x,u(x),Du(x)) - \big(a_n(\,\cdot\,,u,Du)\big)_{z,r/2}\big|\,dx \;\le\; c\,r^{\frac{\gamma\alpha\lambda}{2}}\,\big[M^*\big(1+|Du|\big)(z)+F_1(z)\big]\,,$$

and the constant c admits the same dependencies as in the previous inequality. This allows us to apply the characterization of fractional Sobolev spaces given in Lemma 2.8 and Remark 2.9 (note that these results also hold true if we replace the balls by cubes or cylinders). Since the cylinders $Z_\rho(x_0)\subset Q^+$ were chosen arbitrarily we infer via a covering argument

$$a_n(\,\cdot\,,u,Du)\in M^{\frac{\gamma\alpha\lambda}{2},s}(Q^+_{1/2},\mathbb{R}^N)\,.$$

Furthermore, there exists a function $G_1\in L^s(Q^+_{1/2},\mathbb{R}^N)$ which satisfies

$$|a_n(x,u(x),Du(x)) - a_n(y,u(y),Du(y))| \;\le\; |x-y|^{\frac{\gamma\alpha\lambda}{2}}\,\big(G_1(x)+G_1(y)\big)$$

for almost every $x,y\in Q^+_{1/2}$. We finally note that G_1 can be calculated from c, $M^*\big(1+|Du|\big)$, $F_1(z)$ and the restriction on the radius ρ.

We close this section with some remarks concerning the components $a_k(\cdot,u,Du)$ of the coefficients, $k\in\{1,\dots,n-1\}$, and the interior situation:

Remarks 8.7: We first note that testing the system (8.2) with finite differences in normal direction of the weak solution u is not allowed. Hence, the statement in Proposition 8.4 cannot be expected to cover (via a modified proof) also differences of Du in *any* arbitrary direction $e\in S^{n-1}$ up to the boundary. This reveals the crucial point for the up-to-the-boundary estimates derived in this section: our method makes only an up to the boundary estimate for $a_n(\cdot, u, Du)$ available – which is still sufficient to enable us later to find an appropriate fractional Sobolev estimate for Du – but a corresponding estimate for $a_k(\cdot, u, Du)$, $k \in \{1, \dots, n-1\}$, does not follow.

For cylinders in the interior, however, Proposition 8.4 holds true for every direction $e \in S^{n-1}$. As a consequence, we may repeat the arguments above line-by-line and end up with an interior fractional estimate for the full coefficients $a(\cdot, u, Du)$. We here mention that fractional Sobolev estimates for the coefficients $a(\cdot, u, Du)$ are not necessary in this situation. In fact, interior fractional Sobolev estimates for weak solutions of quadratic systems with inhomogeneities obeying a natural growth condition can be obtained directly by exploiting the fundamental estimate (8.4); for this we refer to [Min03a].

8.5.2 A fractional Sobolev estimate for Du

Following the approach of [Kro] we next derive a fractional Sobolev estimate for $D_n u$ from the last section: The ellipticity condition (H3) and the upper bound in (H2) allow us to estimate

$$
\begin{aligned}
&\big[a_n(x, u(x), Du(x)) - a_n(x, u(x), Du(y))\big] \cdot \big(D_n u(x) - D_n u(y)\big) \\
&\quad = \int_0^1 D_z a_n\big(x, u(x), Du(y) + t(Du(x) - Du(y))\big)\, dt \\
&\qquad\qquad \big(Du(x) - Du(y)\big) \cdot \big(D_n u(x) - D_n u(y)\big) \\
&\quad \geq \nu\, |D_n u(x) - D_n u(y)|^2 - L\, |D' u(x) - D' u(y)|\, |D_n u(x) - D_n u(y)|
\end{aligned}
$$

for almost all $x, y \in Q^+_{1/2}$. Dividing by $|D_n u(x) - D_n u(y)|$ (provided that $D_n u(x) \neq D_n u(y)$ which is the trivial case) and taking into account the fractional Sobolev estimates for both $a_n(\cdot, u, Du)$ and the tangential derivative $D'u$ given in Corollary 8.6, condition (H4) and the Hölder continuity of u, the latter inequality implies

$$
\begin{aligned}
\nu\, |D_n u(x) - D_n u(y)| &\leq \big|a_n(x, u(x), Du(x)) - a_n(x, u(x), Du(y))\big| + L\, |D' u(x) - D' u(y)| \\
&\leq \big|a_n(y, u(y), Du(y)) - a_n(x, u(x), Du(y))\big| + \big|a_n(x, u(x), Du(x)) - a_n(y, u(y), Du(y))\big| \\
&\qquad + L\, |D' u(x) - D' u(y)| \\
&\leq L\, \big(|x-y|^\alpha + [u]^\alpha_{C^{0,\lambda}(Q^+, \mathbb{R}^N)} |x-y|^{\alpha\lambda}\big)\, \big(1 + |Du(y)|\big) \\
&\qquad + |x-y|^{\frac{\gamma\alpha\lambda}{2}}\, \big(G_1(x) + G_1(y)\big) + L\, |x-y|^{\frac{\gamma\alpha\lambda}{2}}\, \big(H_1(x) + H_1(y)\big) \\
&\leq c(L, [u]_{C^{0,\lambda}(Q^+, \mathbb{R}^N)})\, |x-y|^{\frac{\gamma\alpha\lambda}{2}}\, \big(1 + |Du(y)| + G_1(x) + G_1(y) + H_1(x) + H_1(y)\big)
\end{aligned}
$$

for almost every $x, y \in Q^+_{1/2}$, meaning that we have

$$
D_n u \in M^{\frac{\gamma\alpha\lambda}{2}, s}(Q^+_{1/2}, \mathbb{R}^N)\,.
$$

Combined with Corollary 8.6 stating that $D'u$ belongs to the same fractional Sobolev space, we end up with

$$
Du \in M^{\frac{\gamma\alpha\lambda}{2}, s}(Q^+_{1/2}, \mathbb{R}^{nN})
$$

which is the desired estimate for the full derivative Du. We remind the embedding for the fractional Sobolev spaces, namely that

$$
M^{\gamma\alpha\lambda/2, s}(Q^+_{1/2}, \mathbb{R}^{nN}) \subset W^{\gamma'\gamma\alpha\lambda/2, s}(Q^+_{1/2}, \mathbb{R}^{nN})
$$

for all $\gamma' \in (0,1)$. Then, in view of the fact that γ and γ' may be chosen arbitrarily close to 1 and the interpolation Theorem 2.7 due to Campanato, we finally arrive at the higher integrability result

$$Du \in L^{s(1+\alpha\lambda/2)}(Q^+_{1/2}, \mathbb{R}^{nN}).$$

Calculating the limiting exponent in Theorem 2.7 reveals that setting $\gamma = \gamma' = (\frac{n}{n+2\lambda})^{1/2}$, for example, is an appropriate choice.

8.6 Iteration

In the next step we are going to iterate the fractional Sobolev estimate for Du. To this aim we define a sequence $\{b_k\}_{k\in\mathbb{N}}$ as follows:

$$b_0 := 0, \qquad b_{k+1} := \frac{\alpha\lambda}{2} + b_k\left(1 - \frac{\lambda}{2}\right) = b_k + \frac{\alpha\lambda}{2}(\alpha - b_k)$$

for all $k \in \mathbb{N}_0$. We observe that the sequence $\{b_k\}$ is increasing with $b_k \nearrow \alpha$. The strategy of the proof will be the following: For every $k \in \mathbb{N}_0$ we will show the following inclusions:

$$Du \in L^{s_k(1+b_k)} \quad\rightarrow\quad Du \in M^{\gamma b_{k+1}, s_{k+1}} \quad\rightarrow\quad Du \in L^{s_{k+1}(1+b_{k+1})}$$

(on appropriate half-cubes with decreasing radius), where $\gamma \in (0,1)$ is an arbitrary number and where $(s_k)_{k\in\mathbb{N}}$ is a decreasing sequence of higher integrability exponents with $s_k > 2$ for every $k \in \mathbb{N}_0$. We will next establish suitable estimates by induction. The first step of the induction, $k = 0$, was already performed above (with $s_0 = \widetilde{s}$, $s_1 = s$). We now proceed to the inductive step and suppose that for some fixed number $k \in \mathbb{N}$ we have proved $Du \in L^{s_k(1+b_k)}(Q^+_{1/2^k}, \mathbb{R}^{nN})$. The objective is to conclude in a first step $Du \in M^{\gamma b_{k+1}, s_{k+1}}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN})$ by improving the estimates reached in Section 8.5.1. In the second step we will then deduce the higher integrability result $Du \in L^{s_{k+1}(1+b_{k+1})}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN})$ from the fractional Sobolev estimate by applying the interpolation Theorem 2.7.

8.6.1 Higher integrability

In the first step (cf. Proposition 8.4) we again deduce a higher integrability result for the tangential differences $\tau_{e,h}Du$ which now incorporates the fact that Du is assumed to be higher integrable with exponent $s_k(1+b_k)$. In what follows we will frequently use the inequality

$$\alpha\lambda + b_k(1-\lambda) \geq \frac{\alpha\lambda}{2} + b_k\left(1 - \frac{\lambda}{2}\right) = b_{k+1}$$

which we infer from the fact fact that $b_k \leq \alpha$.

Proposition 8.8: *Let $u \in W^{1,2}_\Gamma(Q^+_2, \mathbb{R}^N) \cap L^\infty(Q^+_2, \mathbb{R}^N) \cap C^{0,\lambda}(Q^+, \mathbb{R}^N)$ be a weak solution to the inhomogeneous system (8.2) under the assumptions (H1)-(H4) and (B2). Assume further $u \in W^{1,s_k(1+b_k)}_\Gamma(Q^+_{1/2^k}, \mathbb{R}^N)$ for some $k \in \mathbb{N}$, $s_k > 2$, and let $Z_\rho(x_0) \subset Q^+_{1/2^k}$ for some $x_0 \in \Gamma_{1/2^k} \cup Q^+_{1/2^k}$, $\sigma \in (0, \frac{1}{5})$, $e \in S^{n-1}$ with $e \perp e_n$ and $h \in \mathbb{R}$ satisfying $|h| \in (0, 2\sigma\rho)$.*

Then there exists a higher integrability exponent $s_{k+1} \in (2, s_k)$ *depending only on* $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}$ *and* $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ *such that*

$$\fint_{Z_{\sigma\rho}(x_0)} |\tau_{e,h} Du|^{s_{k+1}}\, dx \;\le\; c\, |h|^{s_{k+1} b_{k+1}} \Big(\fint_{Z_\rho(x_0)} \big(1+|Du(x)|\big)^{s_k(1+b_k)}\, dx \Big)^{\frac{s_{k+1}}{s_k}}$$

for a constant $c = c\big(n, N, \frac{L}{\nu}, \frac{L_2}{\nu}, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \rho, \sigma\big)$.

Proof: We start by deriving an estimate for tangential differences similar to (8.9), namely we show that for every $\theta \in (0,1)$ and every cylinder $Z_r(x_0) \subset Q^+_{1/2^k}$ there holds

$$\int_{Z_{\theta r}(x_0)} |\tau_{e,h} Du|^2\, dx \;\le\; c\, |h|^{2b_{k+1}} \int_{Z_r(x_0)} \big(1+|Du|\big)^{2+2b_k}\, dx \tag{8.20}$$

for all $e \in S^{n-1}$ with $e \perp e_n$ and $h \in \mathbb{R}$ satisfying $|h| < \frac{r(1-\theta)}{2}$. Furthermore, the constant c depends only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \theta$ and r. For this purpose we choose a standard cut-off function

$$\eta \in C_0^\infty\big(D_{(1+\theta)r/2}(x_0') \times ((x_0)_n - (1+\theta)r/2, (x_0)_n + (1+\theta)r/2), [0,1]\big)$$

satisfying $\eta \equiv 1$ on $Z_{\theta r}(x_0)$ and $|D\eta| \le \frac{c(n)}{(1-\theta)r}$, and we then study the different terms arising on the right-hand side of the preliminary estimate (8.4): for the first integral we find by standard properties of differences for every $h \in \mathbb{R}$ with $|h| \le \frac{(1-\theta)r}{2}$:

$$|h|^{2\alpha} \int_{Q^+\cap \operatorname{spt}(\eta)} \big(1+|Du(x+he)|^2 + |h|^{-2}|\tau_{e,h} u(x)|^2\big)\, dx \;\le\; |h|^{2\alpha} \int_{Z_r(x_0)} \big(1+|Du(x)|\big)^2\, dx\,;$$

for the second integral we further argue with Hölder's inequality and the Hölder continuity of u, and we calculate

$$\begin{aligned}
&\int_{Q^+\cap \operatorname{spt}(\eta)} \big(1+|Du(x+he)|^2\big)\, |\tau_{e,h} u(x)|^{2\alpha}\, dx \\
&\le \Big(\int_{Z_{(1+\theta)r/2}(x_0)} \big(1+|Du(x+he)|\big)^{2+2b_k}\, dx \Big)^{\frac{1}{1+b_k}} \Big(\int_{Z_{(1+\theta)r/2}(x_0)} |\tau_{e,h} u(x)|^{2\alpha\,\frac{1+b_k}{b_k}}\, dx \Big)^{\frac{b_k}{1+b_k}} \\
&\le c([u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)})\, |h|^{2\alpha\lambda + 2b_k(1-\lambda)} \int_{Z_r(x_0)} \big(1+|Du(x)|\big)^{2+2b_k}\, dx \\
&\le c([u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)})\, |h|^{2b_{k+1}} \int_{Z_r(x_0)} \big(1+|Du(x)|\big)^{2+2b_k}\, dx\,.
\end{aligned}$$

For the last integral on the right-hand side of (8.4) we apply Young's inequality with exponents $\frac{1+b_k}{b_k}, 1+b_k$ and the standard estimate for the difference operator as above. Hence, we see for every $\varepsilon \in (0,1)$:

$$\begin{aligned}
&\int_{Q^+} \big(1+|Du|^2\big)\, |\tau_{e,-h}(\eta^2\, \tau_{e,h} u)|\, dx \qquad\qquad (8.21)\\
&= \int_{Q^+} |h|^{-\frac{2b_k}{1+b_k}}\, |\tau_{e,-h}(\eta^2\, \tau_{e,h} u)|^{\frac{2b_k}{1+b_k}} \cdot |h|^{\frac{2b_k}{1+b_k}} \big(1+|Du|^2\big)\, |\tau_{e,-h}(\eta^2\, \tau_{e,h} u)|^{\frac{1-b_k}{1+b_k}}\, dx \\
&\le \varepsilon\, |h|^{-2} \int_{Q^+} |\tau_{e,-h}(\eta^2\, \tau_{e,h} u)|^2\, dx + c(\varepsilon)\, |h|^{2b_k} \int_{Q^+} \big(1+|Du|\big)^{2+2b_k}\, |\tau_{e,-h}(\eta^2\, \tau_{e,h} u)|^{1-b_k}\, dx
\end{aligned}$$

$$\begin{aligned}
&\leq 2\,\varepsilon \int_{Q^+} \eta^2\,|\tau_{e,h}Du|^2\,dx + c(\|D\eta\|_\infty)\,|h|^2 \int_{Z_r(x_0)} \big(1+|Du|\big)^2\,dx \\
&\qquad + c(\varepsilon,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)})\,|h|^{2b_k+\lambda(1-b_k)} \int_{Z_r(x_0)} \big(1+|Du|\big)^{2+2b_k}\,dx \\
&\leq 2\,\varepsilon \int_{Q^+} \eta^2\,|\tau_{e,h}Du|^2\,dx + c(\varepsilon,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)},\|D\eta\|_\infty)\,|h|^{2b_{k+1}} \int_{Z_r(x_0)} \big(1+|Du|\big)^{2+2b_k}\,dx\,.
\end{aligned}$$

Choosing $\varepsilon = \frac{c}{4}$ where c is the constant coming from (8.4) and merging the previous estimates together with the inequality in (8.4), we obtain the assertion in (8.20).

In the next step we proceed similarly to the case $k=0$ and estimate the $L^{s_{k+1}}$-norm of $|\tau_{e,h}Du|$ for some exponent $s_{k+1}>2$ in terms of an appropriate power of $|h|$. To this end we consider in the sequel directions $e \in S^{n-1}$ with $e \perp e_n$ and $h \in \mathbb{R}$ satisfying $|h| < 1/2^k$; furthermore, we set analogously to the proof of Proposition 8.4

$$v_h^{(k)} := \frac{\tau_{e,h}u}{|h|^{b_{k+1}}}\,, \qquad \widetilde{\mathcal{A}}^{(k)}(h) := \frac{-\mathcal{A}(h)}{|h|^{b_{k+1}}}\,, \qquad \widetilde{\mathcal{B}}^{(k)}(h) := \frac{-\mathcal{B}(h)}{|h|^{b_{k+1}}}\,,$$

and $\widetilde{\mathcal{C}}^{(k)}(h) = \widetilde{\mathcal{C}}(h) = \int_0^1 D_z a\big(x,u(x),Du(x)+t\tau_{e,h}Du(x))\big)\,dt$ as above. Analogously to the derivation of (8.11) we obtain

$$\begin{aligned}
\int_{Q^+_{1/2^k}} \widetilde{\mathcal{C}}^{(k)}(h)\,Dv_h^{(k)} \cdot D\varphi\,dx &= \int_{Q^+_{1/2^k}} \big[\widetilde{\mathcal{A}}^{(k)}(h) + \widetilde{\mathcal{B}}^{(k)}(h)\big] \cdot D\varphi\,dx \\
&\quad + \int_{Q^+_{1/2^k}} |h|^{-b_{k+1}}\,b(x,u,Du)\cdot \tau_{e,-h}\varphi\,dx \qquad (8.22)
\end{aligned}$$

for all functions $\varphi \in C_0^\infty(Q^+_{1/2^k-|h|},\mathbb{R}^N)$, i.e., the map $v_h^{(k)} \in W^{1,2+2b_k}(Q^+_{1/2^k-|h|},\mathbb{R}^N)$ is a weak solution to the linear system (8.22). For σ,ρ and x_0 fixed according to the assumptions of the proposition, we next choose $h\in\mathbb{R}$ sufficiently small such that $|h| \in (0,2\sigma\rho)$ and look at intersections of balls $B_R^+(y)$ with the upper half-plane $\mathbb{R}^{n-1}\times\mathbb{R}^+$ for centres $y \in \overline{Z_{(1-\sigma)\rho/2}(x_0)}$ at the boundary Γ which satisfy $B_R^+(y) \subset Q^+_{1/2^k-|h|}$ and $y_n \leq \frac{3R}{4}$. Furthermore, we take a cut-off function $\eta_k \in C_0^\infty(B_{3R/4}(y),[0,1])$ satisfying $\eta_k \equiv 1$ on $B_{R/2}(y)$ and $|D\eta_k| \leq \frac{8}{R}$, and we choose $\varphi := \eta_k^2 v_h^{(k)}$ as a test function. Taking into account

$$D\varphi = \eta_k^2 Dv_h^{(k)} + 2\eta_k v_h^{(k)} \otimes D\eta_k$$

and the assumptions (H1)-(H4) we again estimate the various terms arising in (8.22); firstly we remind that for every $\varepsilon \in (0,1)$ there holds (cf. the proof of Proposition 8.4):

- $\nu \displaystyle\int_{B_R^+(y)} \eta_k^2\,|Dv_h^{(k)}|^2\,dx \leq \int_{B_R^+(y)} \eta_k^2\,\widetilde{\mathcal{C}}^{(k)}(h)\,Dv_h^{(k)} \cdot Dv_h^{(k)}\,dx\,,$
- $\displaystyle\int_{B_R^+(y)} 2\eta_k\,\big|\widetilde{\mathcal{C}}^{(k)}(h)Dv_h^{(k)} \cdot v_h^{(k)} \otimes D\eta_k\big|\,dx \leq \varepsilon \int_{B_R^+(y)} \eta_k^2\,|Dv_h^{(k)}|^2\,dx + \frac{c\,L^2}{\varepsilon\,R^2} \int_{B_R^+(y)} |v_h^{(k)}|^2\,dx\,,$
- $\displaystyle\int_{B_R^+(y)} \big|\widetilde{\mathcal{A}}^{(k)}(h)\cdot D\varphi\big|\,dx \leq \varepsilon \int_{B_R^+(y)} \eta_k^2\,|Dv_h^{(k)}|^2\,dx + \frac{c\,\varepsilon}{R^2} \int_{B_R^+(y)} |v_h^{(k)}|^2\,dx$
  $\displaystyle\qquad + c\,\varepsilon^{-1}L^2 \int_{B_R^+(y)} \big(1+|Du(x+he)|^2\big)\,dx\,.$

To find an adequate estimate for the integral involving $\widetilde{\mathcal{B}}^{(k)}(h)$ we first take advantage of the Hölder continuity of u and Young's inequality and we see

$$\begin{aligned}
&\int_{B_R^+(y)} \big(1 + |Du(x+he)|\big)^2 \, |\tau_{e,h} u|^{2\alpha} \, dx \\
&\qquad \le c\big([u]_{C^{0,\lambda}(B^+,\mathbb{R}^N)}\big) \, |h|^{2\alpha\lambda - 2b_k\lambda} \int_{B_R^+(y)} \big(1 + |Du(x+he)|\big)^2 \, |\tau_{e,h} u|^{2b_k} \, dx \\
&\qquad \le c\big([u]_{C^{0,\lambda}(B^+,\mathbb{R}^N)}\big) \, |h|^{2b_{k+1}} \int_{B_R^+(y)} \big(1 + |Du(x+he)| + |G_h(x)|\big)^{2+2b_k} \, dx \,,
\end{aligned}$$

where we have used the fact that

$$|\tau_{e,h} u| \;\le\; |h| \int_0^1 |Du(x+the)| \, dt \;=:\; |h| \, G_h(x) \,.$$

In view of Fubini's Theorem, the fact that $u \in W_\Gamma^{1,s_k(1+b_k)}(Q^+_{1/2^k}, \mathbb{R}^N)$ and the inclusion $B_R^+(y) \subset Q^+_{1/2^k - |h|}$ (see the choices for y and R above), we note that the function G_h is $L^{s_k(1+b_k)}$-integrable on $B_R^+(y)$ and satisfies

$$\int_{B_R^+(y)} |G_h|^{s_k(1+b_k)} \, dx \;\le\; \int_{Q^+_{1/2^k}} |Du|^{s_k(1+b_k)} \, dx \;<\; \infty \,.$$

Hence, we find with Young's inequality for every $\varepsilon \in (0,1)$

- $$\begin{aligned}
&\int_{B_R^+(y)} \big|\widetilde{\mathcal{B}}^{(k)}(h) \cdot D\varphi\big| \, dx \;\le\; L \int_{B_R^+(y)} |h|^{-b_{k+1}} \big(1 + |Du(x+he)|\big) \, |\tau_{e,h} u|^{\alpha} \, |D\varphi| \, dx \\
&\qquad \le \varepsilon \int_{B_R^+(y)} \eta_k^2 \, |Dv_h^{(k)}|^2 \, dx + \frac{c\,\varepsilon}{R^2} \int_{B_R^+(y)} |v_h^{(k)}|^2 \, dx \\
&\qquad\quad + c\big([u]_{C^{0,\lambda}(B^+,\mathbb{R}^N)}\big) \, \varepsilon^{-1} L^2 \int_{B_R^+(y)} \big(1 + |Du(x+he)| + |G_h(x)|\big)^{2+2b_k} \, dx.
\end{aligned}$$

Exactly as in (8.12) there holds $|\tau_{e,-h}(\eta_k^2 v_h^{(k)})| \le 2\,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)} \, |h|^{\lambda - b_{k+1}}$. Therefore, the remaining term in (8.22) can be bounded from above by calculations similar to those performed in (8.21), which means by Young's inequality, standard properties concerning difference quotients and the Hölder continuity of u, and we obtain:

- $$\begin{aligned}
&\int_{B_R^+(y)} |h|^{-b_{k+1}} \, |b(x,u,Du)| \, |\tau_{e,-h}\varphi| \, dx \\
&\qquad \le |h|^{-b_{k+1}} \int_{B_R^+(y)} \big(L + L_2 \, |Du(x)|^2\big) |\tau_{e,-h}(\eta_k^2 v_h^{(k)})| \, dx \\
&\qquad \le \varepsilon \, |h|^{-2} \int_{B_R^+(y)} |\tau_{e,-h}(\eta_k^2 v_h^{(k)})|^2 \, dx \\
&\qquad\quad + c\big(\tfrac{L}{\varepsilon}, \tfrac{L_2}{\varepsilon}\big) \, |h|^{2b_k - b_{k+1}(1+b_k)} \int_{B_R^+(y)} \big(L + L_2 \, |Du|\big)^{2+2b_k} \, |\tau_{e,-h}(\eta_k^2 v_h^{(k)})|^{1-b_k} \, dx \\
&\qquad \le \varepsilon \int_{B_R^+(y)} \eta_k^2 \, |Dv_h^{(k)}|^2 \, dx + \frac{c}{R^2} \int_{B_R^+(y)} |v_h^{(k)}|^2 \, dx \\
&\qquad\quad + c\big(\tfrac{L}{\varepsilon}, \tfrac{L_2}{\varepsilon}, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}\big) \int_{B_R^+(y)} \big(L + L_2 \, |Du|\big)^{2+2b_k} \, dx \,.
\end{aligned}$$

In the last line we have made use of the fact that $2b_k - b_{k+1}(1+b_k) + (1-b_k)(\lambda - b_{k+1}) = \lambda(1-\alpha) > 0$. We now argue exactly as in the proof of Proposition 8.4: Collecting all the terms we infer with the choice $\varepsilon = \frac{\nu}{8}$ a Caccioppoli-type estimate from which, in turn, we deduce via the Sobolev-Poincaré inequality the following reverse Hölder-type inequality:

$$\fint_{B^+_{R/2}(y)} |Dv_h^{(k)}|^2\,dx \le c\Big(\fint_{B^+_R(y)} |Dv_h^{(k)}|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{n}} + c\fint_{B^+_R(y)} \big(1+|Du(x)|+|Du(x+he)|+|G_h(x)|\big)^{2+2b_k}\,dx\,,$$

and the constant c depends only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ but is independent of h. We note that the latter inequality is also valid in the interior situation if we consider balls $B^+_R(y)$ with centres $y \in Z_{(1-\sigma)\rho/2}(x_0)$ satisfying $B^+_R(y) \subset Q^+_{1-|h|}$ and $y_n > \frac{3R}{4}$ (see the proof of Proposition 8.4 for the necessary modifications).

We finally apply the global Gehring Lemma, Theorem A.14, on the cylinder $Z_{(1-\sigma)\rho/2}(x_0)$ for the choices of σ, ρ and x_0 made in the proposition; hence, we find a constant c depending only on $n, N, q, \frac{L}{\nu}, \frac{L_2}{\nu}, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$ and σ and a positive number $\delta_{k+1} < s_k - 2$ depending only on $n, N, \frac{L}{\nu}, \frac{L_2}{\nu}$ and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$, both independent of the parameter h, such that for all $q \in [2, 2+\delta_{k+1})$

$$\begin{aligned}\Big(\fint_{Z_{\sigma\rho}(x_0)} |Dv_h^{(k)}|^q\,dx\Big)^{\frac{1}{q}} &\le c\Big[\Big(\fint_{Z_{(1-4\sigma)\rho}(x_0)} |Dv_h^{(k)}|^2\,dx\Big)^{\frac12} \\ &\quad + \Big(\fint_{Z_{(1-4\sigma)\rho}(x_0)} \big(1+|Du(x)|+|Du(x+he)|+|G_h(x)|\big)^{(1+b_k)q}\,dx\Big)^{\frac1q}\Big] \\ &\le c\Big[|h|^{-b_{k+1}}\Big(\fint_{Z_{(1-4\sigma)\rho}(x_0)} |\tau_{e,h}Du|^2\,dx\Big)^{\frac12} + \Big(\fint_{Z_\rho(x_0)} \big(1+|Du(x)|\big)^{(1+b_k)q}\,dx\Big)^{\frac1q}\Big] \\ &\le c\Big(\fint_{Z_\rho(x_0)} \big(1+|Du(x)|\big)^{s_k(1+b_k)}\,dx\Big)^{\frac{1}{s_k}}.\end{aligned}$$

Here, we have also used the definition of the function G_h, the bound $|h| < 2\sigma\rho$ (with $\sigma < \frac15$), the estimate (8.20) on finite differences and Jensen's inequality. Hence, we find an exponent $s_{k+1} \in (2, s_k)$ with the dependencies stated in the proposition such that the inequality above holds true; keeping in mind the definition of $v_h^{(k)}$, i.e., its normalization by the factor $|h|^{b_{k+1}}$, this immediately yields the desired assertion. □

Again, Proposition 8.8 combined with Lemma 8.2 and with Lemma 2.5, respectively, allows us to state two direct consequences concerning the slicewise mean-square deviation of Du and a suitable fractional differentiability of the tangential derivative $D'u$:

Corollary 8.9: *Let $u \in W^{1,2}_\Gamma(Q^+_2,\mathbb{R}^N) \cap L^\infty(Q^+_2,\mathbb{R}^N) \cap C^{0,\lambda}(Q^+,\mathbb{R}^N)$ be a weak solution to the inhomogeneous system (8.2) under the assumptions (H1)-(H4) and (B2). Assume further $u \in W^{1,s_k(1+b_k)}_\Gamma(Q^+_{1/2^k},\mathbb{R}^N)$ for some $k \in \mathbb{N}$, $s_k > 2$, and let $Z_\rho(x_0) \subset Q^+_{1/2^k}$ for some $x_0 \in \Gamma_{1/2^k} \cup Q^+_{1/2^k}$ and $\sigma \in (0,\frac15)$. Then for every $\gamma \in (0,1)$ there exists a function $F_{k+1} \in L^{s_{k+1}}(Z_{\sigma\rho}(x_0))$ where $s_{k+1} \in (2, s_k)$ is the higher integrability exponent determined* *in Proposition 8.8 such that the following estimate holds true:*

$$\Big(\fint_{Z_r(z)} \big|Du(x) - (Du)_{z',r}(x_n)\big|^2 \, dx \Big)^{\frac{1}{2}}$$
$$\leq \Big(\fint_{Z_r(z)} \fint_{D_r(z')} |Du(x',x_n) - Du(y',x_n)|^2 \, dy' \, dx \Big)^{\frac{1}{2}} \leq c \, r^{\gamma b_{k+1}} \, F_{k+1}(z)$$

for all cylinders $Z_r(z) \subset Z_{\sigma\rho}(x_0)$ with $z \in Q^+ \cup \Gamma$, and the constant c depends only on n, α, λ and γ.

Corollary 8.10: *Let $u \in W^{1,2}_\Gamma(Q_2^+, \mathbb{R}^N) \cap L^\infty(Q_2^+, \mathbb{R}^N) \cap C^{0,\lambda}(Q^+, \mathbb{R}^N)$ be a weak solution to the inhomogeneous system (8.2) under the assumptions (H1)-(H4) and (B2). Assume further $u \in W^{1,s_k(1+b_k)}_\Gamma(Q^+_{1/2^k}, \mathbb{R}^N)$ for some $k \in \mathbb{N}$, $s_k > 2$. Then for every $\gamma \in (0,1)$ there holds*

$$D'u \in M^{\gamma b_{k+1}, s_{k+1}}(Q_\rho^+, \mathbb{R}^{(n-1)N})$$

for every $\rho < \frac{1}{2^{k+1}}$. In particular, there exists a function $H_{k+1} \in L^{s_{k+1}}(Q^+_{1/2^{k+1}})$ such that

$$|D'u(x) - D'u(y)| \leq |x-y|^{\gamma b_{k+1}} \big(H_{k+1}(x) + H_{k+1}(y)\big)$$

for almost all $x, y \in Q^+_{1/2^{k+1}}$.

8.6.2 An improved fractional Sobolev estimate for $\mathbf{a_n}(\,\cdot\,, \mathbf{u}, \mathbf{Du})$

Taking into account that Du is assumed to be higher integrable with exponent $s_k(1+b_k)$, we next proceed similarly to Section 8.5: We choose a cylinder $Z_\rho(x_0) \subset Q^+_{1/2^k}$ with centre $x_0 \in Q^+_{1/2^k} \cup \Gamma_{1/2^k}$ and radius ρ sufficiently small , i. e., $\rho \leq \widetilde{\rho}_{cacc}$ where $\widetilde{\rho}_{cacc}$ is from the Caccioppoli-type inequality in Lemma 8.3, and $\sigma \in (0, \frac{1}{5})$. Furthermore, we fix a number $\gamma \in (0,1)$. In the sequel we again study the model system (8.2) on cylinders $Z_r(z)$ with $z \in Q^+_{1/2^k} \cup \Gamma_{1/2^k}$ such that $Z_{2r}(z) \subset Z_{\sigma\rho}(x_0)$, and by M^* we still denote the maximal operator restricted to the cylinder $Z_{\sigma\rho}(x_0)$. We use the notation from Section 8.5, in particular, the definitions of $A_{\bar\rho}$ and B from (8.14) and (8.15). We first improve the estimate (8.18). To this aim we once again start with inequality (8.17), i. e., with

$$\fint_{I_{\bar\rho}(z_n)} \big|A_{\bar\rho}(x_n) - (A_{\bar\rho})_{z_n,\bar\rho}\big| \, dx_n \leq c \Big[\fint_{Z_r(z)} |B(x)| \, dx + r \fint_{Z_r(z)} |b(x, u(x), Du(x))| \, dx \Big] \tag{8.23}$$

for a constant $c = c(n, N)$ and where $\bar\rho \in [\frac{r}{2}, r]$ is chosen in such a way that on the one hand $A_{\bar\rho}(x_n)$ is weakly differentiable in $I_{\bar\rho}(z_n)$ and on the other hand $\bar\rho \notin J$ (see p. 162).

For the first integral on the right-hand side of (8.23) we recall the definition of $B(x)$ in (8.15) and take advantage of conditions (H2) and (H4) to infer

$$\begin{aligned}
\fint_{Z_r(z)} \big|B(x)\big| \, dx \leq \fint_{Z_r(z)} \Big[& \big|a(x, u(x), Du(x)) - a(x, u(x), (Du)_{z',r}(x_n))\big| \\
& + \big|a(x, u(x), (Du)_{z',r}(x_n)) - a(z, (u)_{z,r}, (Du)_{z',r}(x_n))\big| \Big] \, dx \\
\leq L \fint_{Z_r(z)} & \big|Du(x) - (Du)_{z',r}(x_n)\big| \, dx \\
& + L \fint_{Z_r(z)} \big(|x-z|^\alpha + |u(x) - (u)_{z,r}|^\alpha\big) \big(1 + |(Du)_{z',r}(x_n)|\big) \, dx \, .
\end{aligned}$$

In view of Hölder's and Jensen's inequality, the Hölder continuity of u and Poincaré's Lemma, we derive

$$\begin{aligned}
&\fint_{Z_r(z)} |u(x)-(u)_{z,r}|^{\alpha}\,\big(1+|(Du)_{z',r}(x_n)|\big)\,dx \\
&\quad\le \Big(\fint_{Z_r(z)} |u(x)-(u)_{z,r}|^{\alpha\frac{1+b_k}{b_k}}\,dx\Big)^{\frac{b_k}{1+b_k}}\Big(\fint_{Z_r(z)} (1+|Du|)^{1+b_k}\,dx\Big)^{\frac{1}{b_k+1}} \\
&\quad\le c\,r^{\alpha\lambda-b_k\lambda}\Big(\fint_{Z_r(z)} |u(x)-(u)_{z,r}|^{1+b_k}\,dx\Big)^{\frac{b_k}{1+b_k}}\Big(\fint_{Z_r(z)} (1+|Du|)^{1+b_k}\,dx\Big)^{\frac{1}{1+b_k}} \\
&\quad\le c\,r^{\alpha\lambda+b_k(1-\lambda)}\fint_{Z_r(z)} (1+|Du|)^{1+b_k}\,dx \\
&\quad\le c\,r^{\gamma b_{k+1}}\,M^*\big((1+|Du|)^{1+b_k}\big)(z) \qquad (8.24)
\end{aligned}$$

for $c=c(n,[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)})$. Furthermore, we trivially have

$$\begin{aligned}
\fint_{Z_r(z)} |x-z|^{\alpha}\,\big(1+|(Du)_{z',r}(x_n)|\big)\,dx &\le c(n)\,r^{\alpha}\fint_{Z_r(z)} \big(1+|Du|\big)^{1+b_k}\,dx \\
&\le c(n)\,r^{\gamma b_{k+1}}\,M^*\big((1+|Du|)^{1+b_k}\big)(z)\,.
\end{aligned}$$

Keeping in mind Corollary 8.9 we finally arrive at the following estimate for the integral of $|B(x)|$:

$$\fint_{Z_r(z)} \big|B(x)\big|\,dx \le c\,r^{\gamma b_{k+1}}\big(F_{k+1}(z)+M^*\big((1+|Du|)^{1+b_k}\big)(z)\big)\,, \qquad (8.25)$$

where the constant c depends only on $n, L, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \alpha, \lambda$ and γ. We note that the functions F_{k+1} and $M^*\big((1+|Du|)^{1+b_k}\big)$ belong to the space $L^{s_{k+1}}(Z_{\sigma\rho}(x_0))$, due to Corollary 8.9 and the higher integrability of Du combined with Lemma 5.4 on the maximal function, respectively (we here recall $s_{k+1}\in(2,s_k)$).

For the second integral on the right-hand side of (8.23) we argue similarly to above on p. 163: we initially assume that we are close to the boundary, i. e., $z_n<2r$. Then, we infer the following estimate from the growth condition (B2) on the inhomogeneity, the Caccioppoli inequality (note that $2r\le\rho\le\widetilde{\rho}_{cacc}$), the Hölder continuity of u and Poincaré's inequality in the boundary version:

$$\begin{aligned}
r\fint_{Z_r(z)} |b(x,u(x),Du(x))|\,dx &\le r\fint_{Z_r(z)} (L+L_2\,|Du|^2)\,dx \\
&\le r\,L_2\,\widetilde{c}_{cacc}\Big(\fint_{Z_{2r}(z)} \Big|\frac{u}{r}\Big|^2\,dx+r^{2\alpha}\Big)+r\,L \\
&\le c\,r\fint_{Z_{2r}(z)} \Big(1+\Big|\frac{u}{r}\Big|^{1+b_k} r^{(1-b_k)(\lambda-1)}\Big)\,dx \\
&\le c\,r^{1+(1-b_k)(\lambda-1)}\fint_{Z_{2r}(z)} \big(1+|Du|\big)^{1+b_k}\,dx \\
&\le c\,r^{b_{k+1}}\,M^*\big((1+|Du|)^{1+b_k}\big)(z)\,, \qquad (8.26)
\end{aligned}$$

where in the last line we have employed the fact that

$$1+(1-b_k)\,(\lambda-1) = \lambda+b_k\,(1-\lambda) \ge b_{k+1}$$

and where the constant c depends only on n, N, L, L_2, ν and $[u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}$. For cylinders in the interior, meaning that $z_n \geq 2r$, we end up with exactly the same estimate using both the Caccioppoli inequality and the Poincaré inequality with $|u|$ replaced by $|u-(u)_{z,2r}|$.

Merging the estimates found in (8.25) and (8.26) together with (8.17) hence yields

$$\begin{aligned}
&\fint_{Z_{\bar\rho}(z)} \Big| \fint_{D_{\bar\rho}(z')} a_n(y', x_n, u(y', x_n), Du(y', x_n))\, dy' - \big(a_n(\,\cdot\,, u, Du)\big)_{z,\bar\rho} \Big|\, dx \\
&\quad = \fint_{I_{\bar\rho}(z_n)} \big|A_{\bar\rho}(x_n) - (A_{\bar\rho})_{z_n,\bar\rho}\big|\, dx_n \;\leq\; c\, r^{\gamma b_{k+1}} \big[F_{k+1}(z) + M^*\big((1+|Du|)^{1+b_k}\big)(z)\big]
\end{aligned}$$

for a constant c depending only on $n, N, L, L_2, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \alpha, \lambda$ and γ. This is the desired improvement of inequality (8.18). Moreover, $F_{k+1}, M^*\big((1+|Du|)^{1+b_k}\big) \in L^{s_{k+1}}(Z_{\sigma\rho}(x_0))$ holds true. In order to find a fractional Sobolev estimate for the map $x \mapsto a_n(x, u(x), Du(x))$ it still remains to deduce an estimate corresponding to (8.19). To this aim we follow the line of arguments leading to (8.19) and (8.24): we use Corollary 8.9, Hölder's inequality and the Hölder continuity of u, and we see

$$\begin{aligned}
&\fint_{Z_{\bar\rho}(z)} \Big| a_n(x, u(x), Du(x)) - \fint_{D_{\bar\rho}(z')} a_n(y', x_n, u(y', x_n), Du(y', x_n))\, dy' \Big|\, dx \\
&\quad \leq L \fint_{Z_{\bar\rho}(z)} \fint_{D_{\bar\rho}(z')} \big|Du(x', x_n) - Du(y', x_n)\big|\, dy'\, dx \\
&\qquad + 4L \fint_{Z_{\bar\rho}(z)} \fint_{D_{\bar\rho}(z')} \big(\bar\rho^{\,\alpha} + |u(x', x_n) - u(y', x_n)|^\alpha\big)\big(1 + |Du(x)|\big)\, dy'\, dx \\
&\quad \leq c\, r^{\gamma b_{k+1}} F_{k+1}(z) + 4L\,\bar\rho^{\,\alpha} \fint_{Z_{\bar\rho}(z)} \big(1 + |Du(x)|\big)\, dx \\
&\qquad + 8L \Big(\fint_{Z_r(z)} |u(x) - (u)_{z,r}|^{\alpha\frac{1+b_k}{b_k}}\, dx\Big)^{\frac{b_k}{1+b_k}} \Big(\fint_{Z_r(z)} (1+|Du|)^{1+b_k}\, dx\Big)^{\frac{1}{b_k+1}} \\
&\quad \leq c\, r^{\gamma b_{k+1}} \big(F_{k+1}(z) + M^*\big((1+|Du|)^{1+b_k}\big)(z)\big),
\end{aligned}$$

and the constant c depends only on $n, L, [u]_{C^{0,\lambda}(Q^+,\mathbb{R}^N)}, \alpha, \lambda$ and γ. In particular, taking into account $\bar\rho \in [\frac{r}{2}, r]$, we infer from the latter two estimates that we have

$$\begin{aligned}
&\fint_{Z_{r/2}(z)} \big|a_n(x, u(x), Du(x)) - \big(a_n(\,\cdot\,, u, Du)\big)_{z,r/2}\big|\, dx \\
&\qquad\qquad \leq c\, r^{\gamma b_{k+1}} \Big(F_{k+1}(z) + M^*\big((1+|Du|)^{1+b_k}\big)(z)\Big),
\end{aligned}$$

where the constant c admits the same dependencies as in the preceding inequalities. In view of $F_{k+1}, M^*\big((1+|Du|)^{1+b_k}\big) \in L^{s_{k+1}}(Z_{\sigma\rho}(x_0))$, we may apply the characterization of fractional Sobolev spaces in Lemma 2.8 and Remark 2.9, and we obtain

$$a_n(\,\cdot\,, u, Du) \in M^{\gamma b_{k+1}, s_{k+1}}(Q^+_{1/(2\cdot 2^k)}, \mathbb{R}^N).$$

Furthermore, there exists a function $G_{k+1} \in L^{s_{k+1}}(Q^+_{1/(2^{k+1})}, \mathbb{R}^N)$ which satisfies

$$|a_n(x, u(x), Du(x)) - a_n(y, u(y), Du(y))| \;\leq\; |x-y|^{\gamma b_{k+1}} \big(G_{k+1}(x) + G_{k+1}(y)\big)$$

for almost every $x, y \in Q^+_{1/(2^{k+1})}$. We note that G_{k+1} can be calculated from the constant c, the functions $M^*\big((1+|Du|)^{1+b_k}\big)$, $F_{k+1}(z)$ and the restriction on the radius ρ which in turn result in a dependency on the iteration step k. For the interior situation we observe that the statements of the Remarks 8.7 remain valid, which means in particular that the coefficients $a(\cdot, u, Du)$ satisfy a corresponding interior fractional Sobolev estimate.

8.6.3 Final conclusion for Du

Exactly as in Section 8.5.2 we make $D_n u$ inherit the fractional Sobolev estimate of both the coefficients $a_n(\cdot, u, Du)$ and the tangential derivative $D'u$ (see Corollary 8.10), and we find

$$D_n u \in M^{\gamma b_{k+1}, s_{k+1}}(Q^+_{1/2^{k+1}}, \mathbb{R}^N)\,.$$

Due to the fact that $D'u$ belongs to the same fractional Sobolev space, we arrive at the conclusion

$$Du \in M^{\gamma b_{k+1}, s_{k+1}}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN})\,.$$

At this point we are in the position to use the embeddding

$$M^{\gamma b_{k+1}, s_{k+1}}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN}) \subset W^{\gamma'\gamma b_{k+1}, s_{k+1}}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN})$$

for all $\gamma' \in (0,1)$. Since γ and γ' may be chosen arbitrarily close to 1, the application of Theorem 2.7 yields $Du \in L^{s_{k+1}(1+b_{k+1})}(Q^+_{1/2^{k+1}}, \mathbb{R}^{nN})$. We note, that the choice $\gamma = \gamma' = (\frac{n}{n+2\lambda})^{1/2}$ is appropriate for every $k \in \mathbb{N}$. This finishes the iteration. Keeping in mind $b_k \nearrow \alpha$, the iteration scheme immediately implies the following fractional differentiability result for Du:

Lemma 8.11: *Let $\alpha \in (0,1)$ and let $u \in W^{1,2}_\Gamma(Q^+_2, \mathbb{R}^N) \cap L^\infty(Q^+_2, \mathbb{R}^N) \cap C^{0,\lambda}(Q^+, \mathbb{R}^N)$, $\lambda \in (0,1]$, be a weak solution of the Dirichlet problem (8.2) under the assumptions (H1)-(H4) and (B2). Then, for every $t < \alpha$ there exists $\bar{k} = \bar{k}(t)$ such that $Du \in W^{t,2}(Q^+_{1/2^{\bar{k}}}, \mathbb{R}^{nN})$.*

Remark: We mention that in Lemma 7.9 in the previous chapter we have derived the same statement for weak solutions to subquadratic nonlinear elliptic systems with inhomogeneities satisfying a controllable growth condition, see also [DKM07, Lemma 6.1] for the quadratic case. We easily observe that the method presented in this chapter does not only apply to inhomogeneities obeying a natural growth condition, but also to those obeying a controllable growth condition. As an advantage of the technique presented in this chapter, we note that in the formulation of the previous Lemma 8.11 the low dimensional assumption $p > n-2-\delta$ for some positive number δ is not necessary, whereas it was required in the proof of [DKM07, Lemma 6.1].

PROOF (OF THEOREM 8.1): All the arguments required here can be recovered from the proof of Theorem 7.2 on p. 148; for the sake of completeness we sketch briefly the procedure: First, we reduce the general Dirichlet problem (8.1) to the corresponding boundary value problem with zero boundary values, i.e., $g = 0$ on $\partial\Omega$. Then we employ a covering argument and a local flattening procedure to end up with a finite number of problems of type (8.2) on cubes.

In the model situation, [Ark03, Theorem 1] then guarantees that u is Hölder continuous on the regular set $\mathrm{Reg}_u(Q^+_2 \cup \Gamma)$ of u with any exponent $\lambda \in \big(0, 1 - \frac{n-2}{2}\big)$ and that

$\dim_{\mathcal{H}}(\mathrm{Sing}_u(Q_2^+ \cup \Gamma) < n-2$. In particular, if $n = 2$, we note that the set of singular points is empty. We next observe that the statement in Lemma 8.11 still holds true if we replace the cube Q_1^+ by any smaller cube $Q_R^+(x_0)$, meaning that in this case we obtain $Du \in W^{t,2}(Q_{\delta R}^+(x_0), \mathbb{R}^{nN})$ for some $\delta(t) > 0$ for all $t < \alpha$. Therefore, choosing an increasing sequence of sets $B_k \nearrow \mathrm{Reg}_u(Q^+ \cup \Gamma)$ with $B_k \subset \mathrm{Reg}_u(Q^+ \cup \Gamma)$ such that B_k is relatively open in $Q^+ \cup \Gamma$ for every $k \in \mathbb{N}$, Lemma 8.11 allows us to infer that for every $t < \alpha$ and every point $x_0 \in \Gamma \cap B_k$ there holds $Du \in W^{t,2}(Q_{\delta R}^+(x_0), \mathbb{R}^{nN})$ for some $\delta(t) > 0$. Taking $t \in (\frac{1}{2}, \alpha)$ and applying Proposition A.13 thus yields

$$\dim_{\mathcal{H}}\big(\mathrm{Sing}_{Du}(\Gamma) \cap Q_\delta^+(x_0)\big) \le n - 2t < n - 1$$

which in turn implies $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Gamma) \cap B_k) < n - 2t$ for every $k \in \mathbb{N}$ via a covering argument. Hence, keeping in mind $\dim_{\mathcal{H}}(\mathrm{Sing}_u(Q^+ \cup \Gamma) < n-2$, we finally conclude the desired estimate $\dim_{\mathcal{H}}(\mathrm{Sing}_{Du}(\Gamma)) < n-1$ on the Hausdorff dimension of the singular set of the gradient Du on the boundary. This completes the proof of our main result. □

Remark: It is not clear whether the result of Theorem 8.1 can be improved for arbitrary vector fields $a(x, u, z) \equiv a(x, z)$ which do not explicitly depend on u, in the sense that the existence of regular boundary points is in this case valid for all dimensions $n \ge 2$. To me, there seems to be no hope to produce any positive power of h for the last integral in (8.4) with the techniques presented so far such that in turn no quantitative gain in the higher integrability exponent via fractional Sobolev estimates is achieved. However, if for some reason the weak solution u is a priori known to be Hölder continuous in an open set of Ω outside a set of Hausdorff dimension less than $n-1$, then the statement obviously holds true without any restriction on the dimension n.

Appendix A

Additional Lemmas

A.1 The function $\mathbf{V}_\mu(\xi)$

To handle the subquadratic case the V-function is very useful. For $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$, $\mu \in [0,1]$ and $p > 1$ it is defined by

$$V_\mu(\xi) = \big(\mu^2 + |\xi|^2\big)^{\frac{p-2}{4}}\, \xi\,, \tag{A.1}$$

which is a locally bi-Lipschitz bijection on $\mathbb{R}^k$. Actually, only the cases $\mu = 0$ (for the degenerate case) and $\mu = 1$ (for the non-degenerate case) are interesting because for every $\mu \in (0,1)$ the functions $V_\mu(\xi)$ and $V_1(\xi)$ are equivalent. Therefore, we introduce the abbreviation $V(\xi) = V_1(\xi)$. The crucial point of the V-function is its property concerning growth: it behaves linearly for $|\xi|$ very small, but grows like $|\xi|^{p/2}$ for $|\xi| \to \infty$. Some useful algebraic properties of V we shall frequently use can be found in [CFM98]:

Lemma A.1 ([CFM98], Lemma 2.1): *Let $p \in (1,2)$ and $V : \mathbb{R}^k \to \mathbb{R}^k$ be the function defined in (A.1). Then for all $\xi, \eta \in \mathbb{R}^k$ and $t > 0$ there holds:*

(i) $2^{\frac{p-2}{4}} \min\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$,

(ii) $|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)|$,

(iii) $|V(\xi + \eta)| \le c(p)\,\big(|V(\xi)| + |V(\eta)|\big)$,

(iv) $\frac{p}{2}\,|\xi - \eta| \le \dfrac{|V(\xi) - V(\eta)|}{(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}} \le c(k,p)\,|\xi - \eta|$,

(v) $|V(\xi) - V(\eta)| \le c(k,p)\,|V(\xi - \eta)|$,

(vi) $|V(\xi - \eta)| \le c(p,M)\,|V(\xi) - V(\eta)|$, *provided* $|\eta| \le M$.

We will also need some technical lemmas when dealing with the V_μ-function:

Lemma A.2: *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0,1]$ and $q > -1$. Then there exist constants $c_1, c_2 \ge 1$, which depend only on q but are independent of μ, such that*

$$c_1^{-1}\,\big(\mu + |\xi| + |\eta|\big)^q \le \int_0^1 \big(\mu + |\xi + t\eta|\big)^q\, dt \le c_2\,\big(\mu + |\xi| + |\eta|\big)^q\,.$$

Proof: A proof can be found in [AF89, Lemma 2.1], and for the case $\mu = 1$ also in [Cam82a, Lemma 2.VI]. Without loss of generality we may assume $|\eta| \neq 0$, otherwise both inequalities are trivially satisfied. We first study negative exponents $q \in (-1, 0)$: the lower bound holds true for the constant $c_1 = 1$; for the upper bound, we distinguish three different cases:

The case $\mu, |\xi| < |\eta|$: we use the fact that $q \in (-1, 0)$ and decompose the integral as follows

$$\begin{aligned}\int_0^1 \big(\mu + |\xi + t\eta|\big)^q \, dt &\le \int_0^{|\xi|/|\eta|} \big(\mu + |\xi| - t|\eta|\big)^q \, dt + \int_{|\xi|/|\eta|}^1 \big(\mu - |\xi| + t|\eta|\big)^q \, dt \\ &= -\frac{2\,\mu^{q+1}}{(q+1)\,|\eta|} + \frac{(\mu + |\xi|)^{q+1}}{(q+1)\,|\eta|} + \frac{(\mu - |\xi| + |\eta|)^{q+1}}{(q+1)\,|\eta|} \\ &\le 2\,\frac{(\mu + |\xi| + |\eta|)^{q+1}}{(q+1)\,|\eta|} \le \frac{6}{q+1} \big(\mu + |\xi| + |\eta|\big)^q .\end{aligned}$$

The case $\mu, |\eta| \le |\xi|$: here we proceed similarly and obtain

$$\begin{aligned}\int_0^1 \big(\mu + |\xi + t\eta|\big)^q \, dt &\le \int_0^1 \big(\mu + |\xi| - t|\eta|\big)^q \, dt \le \int_0^1 \big(\mu + |\xi| - t|\xi|\big)^q \, dt \\ &= -\frac{\mu^{q+1}}{(q+1)\,|\xi|} + \frac{(\mu + |\xi|)^{q+1}}{(q+1)\,|\xi|} \le \frac{3}{q+1} \big(\mu + |\xi| + |\eta|\big)^q .\end{aligned}$$

The case $|\xi|, |\eta| \le \mu$: neglecting the term $|\xi + t\eta|$ we get

$$\int_0^1 \big(\mu + |\xi + t\eta|\big)^q \, dt \le \mu^q \le 3^{-q} \big(\mu + |\xi| + |\eta|\big)^q .$$

Therefore, we have shown the desired estimate for the constant $c_2(q) = \max\{\frac{6}{q+1}, 3^{-q}\} = \frac{6}{q+1}$ provided that $q \in (-1, 0)$. For nonnegative exponents q we have to differ the same cases, using opposite signs instead and the fact that

$$a^{q+1} + b^{q+1} \le (a+b)^{q+1} \le 2^q \big(a^{q+1} + b^{q+1}\big)$$

for $a, b \ge 0$. This yields the result with $c_1^{-1}(q) = \min\{\frac{6^{-q}}{q+1}, 3^{-q}\} = \frac{6^{-q}}{q+1}$ and $c_2 = 1$ for nonnegative exponents q. □

Lemma A.3: *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0, 1]$ and $p \in (1, 2)$. Then there exist constants c_1 and c_2 depending only on k, p and on p, respectively, such that the following inequalities hold:*

(i) $c_1^{-1}\,|\xi - \eta|\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}} \le |V_\mu(\xi) - V_\mu(\eta)| \le c_1\,|\xi - \eta|\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}$

(ii) $(\mu^2 + |\xi|^2)^{\frac{p}{2}} \le c_2\,(\mu^2 + |\eta|^2)^{\frac{p}{2}} + c_2\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2$,

(iii) $(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi|\,|\eta| \le \varepsilon\,(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi|^2 + \varepsilon^{1-p}(\mu^2 + |\eta|^2)^{\frac{p}{2}}$ *for* $\varepsilon \in (0, 1)$.

(iv) $(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi|\,|\eta| \le \varepsilon\,(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi|^2 + \varepsilon^{-1}(\mu^2 + |\eta|^2)^{\frac{p-2}{2}}|\eta|^2$ *for* $\varepsilon \in (0, 1)$.

Proof: The inequality in (i) is proved in [AF89, Lemma 2.2], while the other inequalities are easily obtained by distinguishing cases: for (ii) we consider $\max\{\mu, |\eta|\} > \frac{1}{2}|\xi|$ and $\max\{\mu, |\eta|\} \le \frac{1}{2}|\xi|$, and for (iii), (iv) we study the cases $|\eta| > \varepsilon|\xi|$ and $|\eta| \le \varepsilon|\xi|$. □

Employing Lemma A.1 we lastly state another important property of V_μ (cf. [DM04b, Lemma 3] for the proof carried out for the V_0-function):

Lemma A.4: *Let $\mu \in [0,1]$ and let $f : \overline{\Omega} \to \mathbb{R}^{nN}$ be a function for which $V_\mu \circ f$ is Hölder continuous with exponent $\alpha \in (0,1)$. Then also f is Hölder continuous on $\overline{\Omega}$ with the same exponent α.*

A.2 Sobolev-Poincaré inequalities

We first state the Sobolev-Poincaré inequality on balls and appropriate sections of balls in a convenient form. A proof can be found by modifying the arguments in [Giu03, Chapter 3.6].

Lemma A.5 (Sobolev-Poincaré): *Let $p < n$, $p^* = \frac{np}{n-p}$ and $B_r(z) \subset \mathbb{R}^n$. Then there exists a constant $c = c(n,N,p)$ such that for every $u \in W^{1,p}(B_r(z),\mathbb{R}^N)$*

$$\Big(\int_{B_r(z)} |u-(u)_{B_r(z)}|^{p^*}\,dx\Big)^{1/p^*} \le c\Big(\int_{B_r(z)} |Du|^p\,dx\Big)^{1/p},$$

and such that for every $u \in W^{1,p}_\Gamma(B_r^+(z),\mathbb{R}^N)$ with $0 \le z_n \le \frac{3}{4}r$

$$\Big(\int_{B_r^+(z)} |u|^{p^*}\,dx\Big)^{1/p^*} \le c\Big(\int_{B_r^+(z)} |Du|^p\,dx\Big)^{1/p}.$$

Furthermore, we want to consider a $W^{1,p}$-function u in the subquadratic case and state some inequalities of Sobolev-Poincaré-type, both for the interior and the boundary, which are appropriate for our situation. For the interior estimates we also refer to [DGK05, Theorem 2].

Lemma A.6 ([Bec07], Lemma 3.3): *Let $p \in (1,2)$, $B_\rho(x_0) \subset \mathbb{R}^n$ with $n \ge 2$ and set $p^\sharp = \frac{2n}{n-p}$. Moreover, let V be the function defined in (A.1). Then there exists a constant c_s depending only on n, N and p such that for every $u \in W^{1,p}(B_\rho(x_0),\mathbb{R}^N)$*

$$\Big(\fint_{B_\rho(x_0)} \Big|V\Big(\frac{u-(u)_{x_0,\rho}}{\rho}\Big)\Big|^{p^\sharp}\,dx\Big)^{\frac{1}{p^\sharp}} \le c_s\Big(\fint_{B_\rho(x_0)} \big|V(Du)\big|^2\,dx\Big)^{\frac{1}{2}}$$

and such that for every $u \in W^{1,p}_\Gamma(B_r^+(x_0),\mathbb{R}^N)$ with $x_0 \in \mathbb{R}^{n-1}\times\{0\}$

$$\Big(\fint_{B_\rho^+(x_0)} \Big|V\Big(\frac{u}{\rho}\Big)\Big|^{p^\sharp}\,dx\Big)^{\frac{1}{p^\sharp}} \le c_s\Big(\fint_{B_\rho^+(x_0)} \big|V(Du)\big|^2\,dx\Big)^{\frac{1}{2}}.$$

In the next step we will have a closer look at the Poincaré inequality for $u \in W^{1,p}_\Gamma(B_R^+,\mathbb{R}^N)$. Since u vanishes on Γ, the L^p-norm of u is estimated by the L^p-norm of only the normal derivative $D_n u$ rather than the full derivative:

Lemma A.7 ([Bec07], Lemma 3.4): *For functions $u \in W^{1,p}_\Gamma(B_R^+(x_0),\mathbb{R}^N)$ with $x_0 \in \mathbb{R}^{n-1}\times\{0\}$, $p \ge 1$, there holds:*

$$\int_{B_R^+(x_0)} |u|^p\,dx \le \frac{R^p}{p}\int_{B_R^+(x_0)} |D_n u|^p\,dx.$$

Furthermore, we have an analogous result involving the function V:

Lemma A.8 ([Bec07], Lemma 3.6): *Let $p \in (1,2)$ and $B^+_\rho(x_0) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, $n \geq 2$. Then for all $u \in W^{1,p}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ there holds*

$$- \!\!\!\!\!\! \int_{B^+_\rho(x_0)} \Big| V\Big(\frac{u}{\rho}\Big)\Big|^2 \, dx \,\leq\, c(p) \, - \!\!\!\!\!\! \int_{B^+_\rho(x_0)} |V(D_n u)|^2 \, dx \,.$$

Also in the setting of fractional Sobolev spaces we can state a Poincaré-type inequality extending the results for the Sobolev spaces $W^{m,p}$ for integer values of m:

Lemma A.9 (see e. g. [Min03b], (4.2)): *Let $u \in W^{\theta,q}(B_r(z), \mathbb{R}^N)$ where $q \geq 1$, $\theta \in (0,1)$ and $B_r(z) \subset \mathbb{R}^n$. Then we have for a constant $c = c(n,q)$*

$$\int_{B_r(z)} |u - (u)_{B_r(z)}|^q \, dx \,\leq\, c\, r^{\theta q} \int_{B_r(z)} \int_{B_r(z)} \frac{|u(x) - u(y)|^q}{|x-y|^{n+\theta q}} \, dx\, dy \,.$$

Moreover, there holds a corresponding Sobolev embedding theorem:

Theorem A.10 ([Ada75], Theorem 7.57): *Let Ω be a domain in $\mathbb{R}^n$ having the cone property. Furthermore, let $s > 0$ and $p \in (1,n)$. Assume $u \in W^{s,p}(\Omega, \mathbb{R}^N)$. Then we have the following embeddings:*

(i) If $n > sp$, then $u \in L^t(\Omega, \mathbb{R}^N)$ for all $t \in [p, \frac{np}{n-sp}]$.

(ii) If $n = sp$, then $u \in L^t(\Omega, \mathbb{R}^N)$ for all $t \in [p, \infty)$.

(iii) If $n < (s-j)p$ for some noninteger j, then $u \in C^j(\overline{\Omega}, \mathbb{R}^N)$.

A.3 Further technical lemmas

The next lemma due to Campanato is of technical nature: instead of iterating the decay, it may be applied to yield directly the desired decay estimate, and it will be applied when proving (partial) regularity in low dimensions in Chapter 6 (here, Φ will be the *Excess function*).

Lemma A.11 ([Gia83], Chapter III, Lemma 2.1; [DGK04], Lemma 2.2): *Let A, B, R_1, α and β be non-negative numbers with $\alpha > \beta$. Then there exist a positive constant κ_0 and a constant c depending only on α, β and A such that the following is true: whenever Φ is nonnegative and nondecreasing on $(0, R_1)$ and satisfies*

$$\Phi(\rho) \,\leq\, \Big[A\Big(\frac{\rho}{R}\Big)^\alpha + \kappa\Big] \Phi(R) + B\, R^\beta \qquad \textit{for all } \rho \in (0,R) \tag{A.2}$$

for some $R < R_1$ and some $\kappa \in (0, \kappa_0)$, then there holds for all $\rho \in (0,R)$

$$\Phi(\rho) \,\leq\, c\Big[\Big(\frac{\rho}{R}\Big)^\beta \Phi(R) + B\,\rho^\beta\Big] \,.$$

Lastly, we give a measure density result tracing back to Giusti which allows us to control the Hausdorff-dimension $\dim_{\mathcal{H}}$ of the singular set, when we consider partial regularity for weak solutions to some nonlinear system:

Lemma A.12 (cf. [Giu03], Proposition 2.7, [Min03b], Section 4): *Let A be an open set in $\mathbb{R}^n$, and let λ be a finite, non-negative and increasing function defined on the family of open subsets of A which is also countably superadditive in the following sense that*

$$\sum_{i\in\mathbb{N}} \lambda(O_i) \le \lambda\big(\bigcup_{i\in\mathbb{N}} O_i\big)$$

whenever $\{O_i\}_{i\in\mathbb{N}}$ is a family of pairwise disjoint open subsets of A. Then, for $0<\alpha<n$, we have $\dim_{\mathcal{H}}(E^\alpha)\le\alpha$ where

$$E^\alpha = \Big\{x\in A : \limsup_{\rho\to 0^+} \rho^{-\alpha}\,\lambda\big(B_\rho(x)\big) > 0\Big\}.$$

In the original formulation due to Giusti instead of λ a Radon measure μ on A such that $\mu(A)<\infty$ was considered. The new formulation allows us to deduce the following estimate for the set of non-Lebesgue-points of fractional Sobolev functions which is essentially based on the arguments in [Min03b, Section 4]:

Proposition A.13 ([DKM07], Proposition 2.1): *Suppose that $v\in W^{\theta,q}(Q_d^+,\mathbb{R}^N)$ for $d>0$ is a fixed number, $\theta\in(0,1]$, $q\ge 1$, $N\in\mathbb{N}$. Moreover, let*

$$\begin{aligned} A &:= \Big\{x\in Q_d^+\cup\Gamma_d : \limsup_{\rho\to 0^+} \fint_{B_\rho(x)\cap Q_d^+} \big|v(y)-(v)_{B_\rho(x)\cap Q_d^+}\big|^q\,dy > 0\Big\},\\ B &:= \Big\{x\in Q_d^+\cup\Gamma_d : \limsup_{\rho\to 0^+} \big|(v)_{B_\rho(x)\cap Q_d^+}\big| = \infty\Big\}. \end{aligned}$$

Then

$$\dim_{\mathcal{H}}(A)\le n-\theta q \qquad \textit{and} \qquad \dim_{\mathcal{H}}(B)\le n-\theta q\,.$$

Proof: We first note that we can restrict ourselves to prove the proposition for the interior case where we replace the half-cube Q_d^+ by the full cube Q_d. Otherwise we extend a given function $v\in W^{\theta,q}(Q_d^+,\mathbb{R}^N)$ by even reflection; then, an easy calculation reveals that the extended function $\bar v$ belongs to $W^{\theta,q}(Q_d,\mathbb{R}^N)$ and satisfies

$$\|\bar v\|_{W^{\theta,q}(Q_d,\mathbb{R}^N)} \le 4\,\|v\|_{W^{\theta,q}(Q_d^+,\mathbb{R}^N)}\,.$$

Therefore, we consider a function $v\in W^{\theta,q}(Q_d,\mathbb{R}^N)$ and we define a set-function λ defined by

$$\lambda(O) := \int_O\int_O \frac{|v(x)-v(y)|^q}{|x-y|^{n+\theta q}}\,dx\,dy$$

on every open subset $O\subset Q_d$. We observe that all the assumptions on λ in Lemma A.12 are fulfilled. To estimate the dimensions of the sets A and B we define

$$S_A := \Big\{x\in Q_d : \limsup_{\rho\to 0^+} \rho^{\theta q-n}\lambda\big(B_\rho(x)\big) > 0\Big\}.$$

Now let $\varepsilon > 0$. Then, the previous lemma implies $\mathcal{H}^{n-\theta q+\varepsilon}(S_A) = 0$. By the Poincaré-type inequality in Lemma A.9 we conclude that if $x_0 \in A$, then $x_0 \in S_A$, and therefore $A \subseteq S_A$ and $\mathcal{H}^{n-\theta q+\varepsilon}(A) = 0$. To infer the analogous estimate for the set B we fix $\varepsilon_0 \in (0, \varepsilon)$ and define

$$S_B := \big\{x \in Q_d : \limsup_{\rho \to 0^+} \rho^{\theta q - n - \varepsilon_0} \lambda\big(B_\rho(x)\big) \, > \, 0\big\} \, .$$

Again, from Lemma A.12 follows that $\mathcal{H}^{n-\theta q+\varepsilon}(S_B) = 0$. To prove $B \subseteq S_B$ we next consider centres $x_0 \in Q_d \setminus S_B$ and radii $R < 1$ such that $B_R(x_0) \subset Q_d$. Then, we use Jensen's inequality and the fractional Poincaré inequality in Lemma A.9 to estimate

$$\begin{aligned}
\big|(v)_{x_0,2^{-k-1}R} - (v)_{x_0,2^{-k}R}\big|^q &\le 2^{-n} \fint_{B_{2^{-k}R}(x_0)} \big|v - (v)_{x_0,2^{-k}R}\big|^q \, dx \\
&\le c(n,q) \left(\frac{R}{2^k}\right)^{\theta q - n} \int_{B_{2^{-k}R}(x_0)} \int_{B_{2^{-k}R}(x_0)} \frac{|v(x) - v(y)|^q}{|x-y|^{n+\theta q}} \, dx \, dy \\
&= c(n,q) \left(\frac{R}{2^k}\right)^{\varepsilon_0} \left(\frac{R}{2^k}\right)^{\theta q - n - \varepsilon_0} \lambda\big(B_{2^{-k}R}(x_0)\big) \\
&\le \tilde{c}(n,q) \, 2^{-k\varepsilon_0}
\end{aligned}$$

for every $k \in \mathbb{N}_0$ sufficiently large. Summing up these terms finally yields

$$\lim_{k \to \infty} |(v)_{x_0,2^{-k}R}| \, \le \, c(n,q,\varepsilon_0) \, < \, \infty \, .$$

Hence, since $\varepsilon_0 \in (0, \varepsilon)$ was chosen arbitrarily, we obtain $\mathcal{H}^{n-\theta q+\varepsilon}(B) = 0$; this completes the proof of the proposition. □

A.4 A global version of Gehring's Lemma

We will use the following version of the Gehring lemma which was proved in [DGK04]. It gives conditions easy to verify to prove higher integrability up to the boundary of some bounded Lipschitz-domain $\Omega \subset \mathbb{R}^n$ which satisfies an Ahlfors regularity condition (K_Ω) with positive constant k_Ω (see p. 12).

Theorem A.14 ([DGK04], Theorem 2.4): *Let A be a closed subset of $\overline{\Omega}$. Consider two nonnegative function $g, f \in L^1(\Omega)$ and p with $1 < p < \infty$ such that there holds*

$$\fint_{B_{r/2}(z) \cap \Omega} g^p \, dx \, \le \, b^p \Big[\Big(\fint_{B_r(z) \cap \Omega} g \, dx \Big)^p + \fint_{B_r(z) \cap \Omega} f^p \, dx \Big] \tag{A.3}$$

for almost all $z \in \Omega \setminus A$ with $B_r(z) \cap A = \emptyset$, for some constant b. Then there exist constants $c = c(n,p,q,b,k_\Omega)$ and $\delta = \delta(n,p,b,k_\Omega)$ such that

$$\Big(\fint_\Omega \widetilde{g}^q \, dx \Big)^{\frac{1}{q}} \, \le \, c \Big[\Big(\fint_\Omega g^p \, dx \Big)^{\frac{1}{p}} + \Big(\fint_\Omega f^q \, dx \Big)^{\frac{1}{q}} \Big]$$

for all $q \in [p, p+\delta)$, where $\widetilde{g}(x) = \frac{\mathcal{L}^n(B_{d(x,A)}(x) \cap \Omega)}{\mathcal{L}^n(\Omega)} \, g(x)$.

As in [DGK04] we use the convention $d(x, \emptyset) = \infty$. In particular, if $A = \emptyset$, we have $\widetilde{g} \equiv g$, and the Theorem then provides a global version of the usual Gehring Lemma.

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