Minimal rearrangements, strict convexity and critical points

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Dedicated to the memory of Carlo Pucci

1 Introduction and statement of the result

Given a convex function $A : [0, +\infty) \to [0, +\infty)$ vanishing at 0, briefly a *Young function*, consider the Dirichlet-type functional associated with A and defined as

$$\int_{\mathbb{R}^n} A(|\nabla u|)\, dx \tag{1.1}$$

at any real-valued function u from the Sobolev space $W^{1,1}(\mathbb{R}^n)$.
We are concerned with the problem of minimizing the functional (1.1) in the class of all weakly differentiable rearrangements of some prescribed nonnegative function $u \in W^{1,1}(\mathbb{R}^n)$ at which (1.1) is finite. Recall that a measurable nonnegative function in $\mathbb{R}^n$ is called a rearrangement of u if it has the same distribution function as u. Such a distribution function is denoted by $\mu : [0, +\infty) \to [0, +\infty)$ and is given by

$$\mu(t) = \mathcal{L}^n(\{u > t\}) \qquad \text{for } t \geq 0\,,$$

where $\mathcal{L}^n$ stands for the Lebesgue measure. Obviously, μ is well-defined provided that

$$\mathcal{L}^n(\{u > 0\}) < +\infty\,, \tag{1.2}$$

an assumption which will be always kept in force throughout.

A classical result, known as the *Pólya–Szegö principle*, tells us that if u is nonnegative and belongs to $W^{1,1}(\mathbb{R}^n)$, then its *spherically symmetric rearrangement* $u^\star$, defined as

$$u^\star(x) = \sup\{t \geq 0 : \mu(t) > \omega_n |x|^n\} \qquad \text{for } x \in \mathbb{R}^n\,,$$

is also in $W^{1,1}(\mathbb{R}^n)$, and is a minimizer of the constrained variational problem under consideration (here, ω_n is the measure of the unit ball in $\mathbb{R}^n$). In other words, $u^\star \in W^{1,1}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} A(|\nabla u^\star|)\, dx \leq \int_{\mathbb{R}^n} A(|\nabla u|)\, dx \tag{1.3}$$

for every nonnegative $u \in W^{1,1}(\mathbb{R}^n)$ (see [Bae, BZ, H, K1, S, T]). Thus, every solution to the relevant minimum problem has to fulfill

$$\int_{\mathbb{R}^n} A(|\nabla u^\star|)\, dx = \int_{\mathbb{R}^n} A(|\nabla u|)\, dx < +\infty. \tag{1.4}$$

Accordingly, any nonnegative function $u \in W^{1,1}(\mathbb{R}^n)$ satisfying (1.2) and (1.4) will be called a *minimal rearrangement* relative to A.

In recent years, the study of minimal rearrangements (and of related topics) has attracted the attention of various authors working on symmetrizations (see e.g. [BZ, Bu, CF1, CF2, ET, FV1, FV2, FM, K1, K2, U]). A major contribution to this issue was given by Brothers and Ziemer in [BZ]. The result states that if

$$A \quad \text{is strictly convex} \tag{1.5}$$

and u is a minimal rearrangement relative to A satisfying

$$\mathcal{L}^n(\{\nabla u = 0\} \cap \{0 < u < \operatorname{ess\,sup} u\}) = 0\,, \tag{1.6}$$

then

$$u = u^\star \qquad \mathcal{L}^n\text{–a.e.} \quad \text{(up to translations)}. \tag{1.7}$$

Note that this theorem is proved in [BZ] under a somewhat stronger assumption than (1.5); the conclusion is however true for any strictly convex Young function A, as observed e.g. in [Bu, CF1]. Notice also that, in the original statement of [BZ], hypothesis (1.6) is replaced by

$$\mathcal{L}^n(\{\nabla u^\star = 0\} \cap \{0 < u^\star < \operatorname{ess\,sup} u\}) = 0\,,$$

a condition equivalent to the absolute continuity of μ in $[0, \operatorname{ess\,sup} u)$, which is in general weaker than (1.6), but which is in fact equivalent to (1.6) for functions u satisfying (1.4) ([CF1, Lemma 3.3]).

The aim of the present paper is to investigate minimal rearrangements in the general situation where even (1.5) and (1.6) are dropped. Counterexamples show that (1.7) need not hold in this case. Actually, if either (1.5) or (1.6) is removed, then functions u, satisfying (1.4) and whose level sets $\{u > t\}$ are non concentric balls for $t > 0$, can be exhibited.

To see that (1.5) is indispensable, suppose first that A is linear in the whole of $[0, +\infty)$; namely, $A(s) = as$ for some $a > 0$ and for $s \geq 0$. Then, as a consequence of the coarea formula, any nonnegative function $u \in W^{1,1}(\mathbb{R}^n)$ having (non necessarily concentric) balls as level sets fulfills (1.4). If A is affine just in some interval $[s_1, s_2] \subset [0, +\infty)$, then, for the same reason, any $u \in W^{1,1}(\mathbb{R}^n)$ whose spherically symmetric rearrangement satisfies $|\nabla u^\star| \equiv (s_1 + s_2)/2$ in $\{0 < u^\star < \operatorname{ess\,sup} u\}$, and whose level sets are balls with centers so close to 0 that $|\nabla u| \in [s_1, s_2]$ in $\{0 < u < \operatorname{ess\,sup} u\}$, is still a minimal rearrangement.

As for the necessity of (1.6), observe that, if at least one plateau $\{u = t_0\}$ with $\mathcal{L}^n(\{u = t_0\}) > 0$ is allowed for some $t_0 > 0$, then any function which is not necessarily globally symmetric, but which is separately symmetric in $\{0 < u < t_0\}$ and in $\{t_0 < u\}$ satisfies (1.4) for every A. More subtle examples of (smooth) non symmetric minimal rearrangements u, not fulfilling (1.6), but yet with $\mathcal{L}^n(\{u = t\}) = 0$ for every $t > 0$, can also be worked out (see [BZ]). Thus, strict convexity of A and absence of critical points, as required in (1.6), are essentially sharp assumptions for every minimal rearrangement to be necessarily symmetric.

Our main result tells us that, although a minimal rearrangement can be asymmetric if these assumptions are not in force, nevertheless the asymmetry of u can be measured by the size of the set of those points x where either (1.6) or (1.5) is violated, i.e. which are either critical for u, or for which A fails to be strictly convex at $|\nabla u(x)|$. More precisely, denote by $\{J_i\}$ the family of all open maximal intervals in $(0, +\infty)$ on which A is affine, and set

$$L = \cup_i \overline{J_i}\,. \tag{1.8}$$

Then an estimate for the distance in $L^1(\mathbb{R}^n)$ of $u^\star$ from a suitable translate of u is provided in terms of $\mathcal{L}^n(\{|\nabla u| \in \{0\} \cup L\} \cap \{0 < u < \operatorname{ess\,sup} u\})$, $\mathcal{L}^n(\{u > 0\})$ and $\int_{\mathbb{R}^n} A(|\nabla u|)\, dx$.

Theorem 1.1 *A constant $C(n)$, depending only on n, exists such that, if A is any Young function vanishing only at 0, and u is any minimal rearrangement relative to A, then*

$$\inf_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |u(x+y) - u^\star(x)|\, dx \tag{1.9}$$

$$\leq C(n)\Big(\mathcal{L}^n(\{u>0\})\Big)^{1+\frac{2n-1}{2n^2}} \Big(\mathcal{L}^n(\{|\nabla u| \in \{0\} \cup L\} \cap \{0<u<\operatorname{ess\,sup} u\})\Big)^{\frac{1}{2n^2}} A^{-1}\left(\frac{\int_{\mathbb{R}^n} A(|\nabla u|)dx}{\mathcal{L}^n(\{u>0\})}\right).$$

Remarks (i) Inequality (1.9) can be regarded as a quantitative version of the theorem of Brothers and Ziemer. In particular, the stability of conclusion (1.7) under perturbations of assumption (1.6) follows from (1.9).

(ii) Estimates like (1.9) with the L^1-norm replaced on the left-hand side by stronger Lebesgue (or Orlicz) norms could also be derived. This requires an additional step to the proof of (1.9), which makes use of classical Sobolev inequalities when $A(t) = t^p$ for some $p > 1$, and of suitable extensions to Orlicz-Sobolev spaces ([C1, C2]) in the general case.

(iii) If A is not even strictly positive in $(0, +\infty)$, then any information about minimal rearrangements is lost. Indeed, assume that a positive number t_0 exists such that $A(t) = 0$ for $t \in [0, t_0]$. Then any Lipschitz continuous function u satisfying (1.2) and such that $|\nabla u(x)| \leq t_0$ for $\mathcal{L}^n$- a.e. $x \in \mathbb{R}^n$ is a minimal rearrangement. Indeed, $\int_{\mathbb{R}^n} A(|\nabla u|)dx = 0$, and hence, by (1.3), $\int_{\mathbb{R}^n} A(|\nabla u^\star|)dx = \int_{\mathbb{R}^n} A(|\nabla u|)dx = 0$.

The remaining part of the paper is devoted to the proof of Theorem 1.1. An analysis of the level sets $\{u > t\}$ of minimal rearrangements u, and of the behaviour of $|\nabla u|$ restricted to the level surfaces $\{u = t\}$, is carried out in the next section. The results there established refine the first part of the argument of [BZ]. The core of the proof of Theorem 1.1, where this preliminary study is used to derive information about the mutual displacement of the level sets $\{u > t\}$ as t ranges in $[0, \operatorname{ess\,sup} u)$, is contained in the last section, and differs substantially from that of [BZ]. Our approach is rather inspired by the methods of [FV2] (and [FV1]), where an alternate proof of Brothers and Ziemer's result is given in an even more general framework.

2 Preliminary results

In this section we establish some basic properties of minimal rearrangements that constitute our starting point in the proof of Theorem 1.1. These are a variant, concerning the case of non necessarily strictly convex A, of certain properties which are by now well-known under assumption (1.5). The key tool here is a sharp version of Jensen's inequality (Lemma 2.1), which enables us to characterize all the equality cases (Corollary 2.2).

Let A be a Young function. We denote by $A' : [0,+\infty) \to [0,+\infty)$ the unique left-continuous function such that

$$A(s) = \int_0^s A'(t)\,dt \qquad \text{for all } s \geq 0\,.$$

Recall that, since A is convex, then A' is non-decreasing, and hence its distributional derivative D^2A is a positive Radon measure. Let us denote by $\mathrm{supp}(D^2A)$ the complement in $(0,+\infty)$ of the largest open set where D^2A vanishes. Then we have

$$(0,+\infty) \setminus \mathrm{supp}(D^2A) = \bigcup_i J_i\,,$$

where the J_i's are defined as in Section 1.

Lemma 2.1 *Let (E,ν) be a measure space, with $\nu(E) < +\infty$ and let A be a Young function. Given any ν-integrable function $f : E \to [0,+\infty)$, set $f_E = \fint_E f(x)d\nu$. Then*

$$\int_E A(f(x))d\nu(x) = A(f_E)\nu(E) + \int_{[f_E,+\infty)} dD^2A(t)\int_{\{f>t\}}(f(x)-t)d\nu(x) + \int_{(0,f_E)} dD^2A(t)\int_{\{f<t\}}(t-f(x))d\nu(x).$$

Proof. We may assume, without loss of generality, that $\nu(E) = 1$. Let us fix $s_0 \geq 0$. Since

$$A(s) - A(s_0) = \int_{s_0}^s A'(t)\,dt, \qquad \text{if } s \geq 0\,, \text{ and} \quad A'(t) = A'(s_0) + \int_{[s_0,t)} dD^2A(\tau) \qquad \text{if } t \geq s_0\,,$$

an application of Fubini's theorem yields

$$A(s) - A(s_0) = A'(s_0)(s - s_0) + \int_{[s_0,s)} (s-\tau)\,dD^2A(\tau) \qquad \text{if } s \geq s_0\,. \tag{2.1}$$

Similarly, we have that

$$A(s) - A(s_0) = A'(s_0)(s - s_0) + \int_{[s,s_0)} (\tau - s)\,dD^2A(\tau) \qquad \text{if } s \leq s_0\,. \tag{2.2}$$

Now, choose $s = f(x)$ and $s_0 = f_E$ and integrate (2.1) on $\{f \geq f_E\}$ and (2.2) on $\{f < f_E\}$. On adding up the two resulting equations, one obtains

$$\begin{aligned}
&\int_E A(u(x))\,d\nu(x) - A(f_E) = \int_E A'(f_E)(f(x) - f_E)\,d\nu(x)\\
&\qquad + \int_{\{f>f_E\}} d\nu(x)\int_{[f_E,f(x))}(f(x)-t)\,dD^2A(t) + \int_{\{f<f_E\}} d\nu(x)\int_{[f(x),f_E)}(t - f(x))\,dD^2A(t)\\
&= \int_{[f_E,+\infty)} dD^2A\int_{\{f>f_E\}} \chi_{[f_E,f(x))}(t)(f(x)-t)d\nu(x) + \int_{[0,f_E)} dD^2A\int_{\{f<f_E\}} \chi_{[f(x),f_E)}(t)(f(x)-t)d\nu(x)\\
&= \int_{[f_E,+\infty)} dD^2A(t)\int_{\{f>t\}}(f(x)-t)\,d\nu(x) + \int_{(0,f_E)} dD^2A(t)\int_{\{f<t\}}(f(x)-t)\,d\nu(x)\,,
\end{aligned}$$

where χ_S denotes the characteristic function of a set S. The conclusion follows. □

Corollary 2.2 *Let (E,ν), A and f be as in Lemma 2.1. If*

$$\fint_E A(f(x))\, d\nu(x) = A\left(\fint_E f(x)\, d\nu(x)\right), \tag{2.3}$$

then either
(i) $f \equiv const.$ *ν-a.e. in E*
or
(ii) *an index i exists such that* $(\operatorname{ess\,inf} f, \operatorname{ess\,sup} f) \subset J_i$.

Proof. Assume that (i) is not true, i.e. f is not constant on E, and set

$$\Phi(t) = \begin{cases} \displaystyle\int_{\{f>t\}} (f(x)-t)\, d\nu(x) & \text{if } \ t \geq f_E \\ \displaystyle\int_{\{f<t\}} (t-f(x))\, d\nu(x) & \text{if } \ 0 < t < f_E \,. \end{cases}$$

Then $\Phi(t) > 0$ for all $t \in (\operatorname{ess\,inf} f, \operatorname{ess\,sup} f)$. Assumption (2.3) and Lemma 2.1 yield

$$\int_0^{+\infty} \Phi(t)\, dD^2A(t) = 0\,.$$

Therefore, $(\operatorname{ess\,inf} f, \operatorname{ess\,sup} f)$ is contained in $(0,+\infty) \setminus \operatorname{supp}(D^2A)$, since the latter set is the largest open subset of $(0,+\infty)$ where the positive measure D^2A vanishes. Hence (ii) follows. □

Let u be a function from $W^{1,1}(\mathbb{R}^n)$ and let $\phi : \mathbb{R}^n \to [0,+\infty)$ be a Borel function. The *coarea formula* states that

$$\int_{\mathbb{R}^n} \phi(x)|\nabla u(x)|\, dx = \int_{-\infty}^{+\infty} dt \int_{\{u=t\}} \phi(x)\, d\mathcal{H}^{n-1}(x)\,, \tag{2.4}$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. Observe that the standard version of this formula involves the essential boundary of $\{u > t\}$ instead of $\{u = t\}$. However, every Sobolev function u possesses representatives $\hat{u}$ with the property that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the level surface $\{\hat{u} = t\}$ and the essential boundary of the level set $\{u > t\}$ differ by a set of zero-$\mathcal{H}^{n-1}$ measure. In (2.4), and in what follows, it is understood that we are dealing with such representatives.

The main result of this Section, stated in Theorem 2.3 below, provides us with a distinguished representative of any minimal rearrangement. Henceforth, any minimal rearrangement will be always identified with such a representative.

Theorem 2.3 *Let A be any Young function vanishing only at 0, and let u be a minimal rearrangement relative to A. Then there exist a function equivalent to u, and still denoted by u, and a family of open balls $\{U_t\}_{t\geq 0}$ such that:*

(i) $\{u > t\} = U_t$ *for* $t \in [0, \operatorname{ess\,sup} u)$;

(ii) $\{u = \operatorname{ess\,sup} u\} = \bigcap_{0 \leq t < \operatorname{ess\,sup} u} U_t,$ *and is a closed ball;*

(iii) *u is lower semicontinuous;*

(iv) *if $u(x) \in (0, \operatorname{ess\,sup} u)$ and $\mathcal{L}^n(\{u = u(x)\}) = 0$, then $x \in \partial U_{u(x)}$;*

(v) *for every $t \in (0, \operatorname{ess\,sup} u)$ there exists at most one point $x \in \partial U_t$ such that $u(x) \neq t$;*

(vi) *the coarea formula (2.4) holds for u;*

(vii) *for $\mathcal{L}^1$-a.e. $t \in (0, \operatorname{ess\,sup} u)$, either*

$$|\nabla u(x)| = |\nabla u^\star|_{|\{u^\star = t\}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial U_t\,, \tag{2.5}$$

or

$$\text{there exists an index } i \text{ such that } |\nabla u(x)| \in \overline{J}_i \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial U_t\,. \tag{2.6}$$

Proof. Fix any representative of u, still denoted by u, for which (2.4) holds. An application of (2.4) with $\phi = \chi_{\{\nabla u = 0\}}$ yields

$$\mathcal{H}^{n-1}(\{\nabla u = 0\} \cap \{u = t\}) = 0 \qquad\qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, +\infty)\ . \tag{2.7}$$

Hence, by Jensen's inequality,

$$\int_{\{u=t\}} \frac{A(|\nabla u|)}{|\nabla u|}\, d\mathcal{H}^{n-1} \geq A\left(\frac{\displaystyle\int_{\{u=t\}} \frac{|\nabla u|}{|\nabla u|} d\mathcal{H}^{n-1}}{\displaystyle\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}} \right) \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} = A\left(\frac{\mathcal{H}^{n-1}(\{u=t\})}{\displaystyle\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}} \right) \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \tag{2.8}$$

for $\mathcal{L}^1$-a.e. $t \in [0, \operatorname{ess\,sup} u)$. The derivative of the distribution function μ satisfies

$$\frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star = t\}}} = -\mu'(t) \geq \int_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} \qquad\qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, \operatorname{ess\,sup} u)\ . \tag{2.9}$$

(see e.g. [BZ] and [CF1, Lemma 3.2 and (3.19)]). Since $A(s)/s$ non decreasing, (2.9) yields

$$A\left(\frac{\mathcal{H}^{n-1}(\{u=t\})}{\displaystyle\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}} \right) \int_{\{u=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} \geq A\left(\frac{\mathcal{H}^{n-1}(\{u=t\})}{\mathcal{H}^{n-1}(\{u^\star = t\})} |\nabla u^\star|_{|\{u^\star = t\}} \right) \frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star = t\}}} \tag{2.10}$$

for $\mathcal{L}^1$-a.e. $t \in [0, \operatorname{ess\,sup} u)$. The standard isoperimetric inequality ensures that

$$\mathcal{H}^{n-1}(\{u = t\}) \geq \mathcal{H}^{n-1}(\{u^\star = t\}) \tag{2.11}$$

for $\mathcal{L}^1$-a.e. $t \in [0, \operatorname{ess\,sup} u)$, and that equality holds in (2.11) if and only if $\{u > t\}$ is equivalent to a ball. From (2.11) and the monotonicity of A we infer that

$$\begin{aligned} A\left(\frac{\mathcal{H}^{n-1}(\{u=t\})}{\mathcal{H}^{n-1}(\{u^\star = t\})} |\nabla u^\star|_{|\{u^\star = t\}} \right) \frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star = t\}}} &\geq \frac{A\Big(|\nabla u^\star|_{|\{u^\star = t\}}\Big)}{|\nabla u^\star|_{|\{u^\star = t\}}} \mathcal{H}^{n-1}(\{u^\star = t\}) \\ &= \int_{\{u^\star = t\}} \frac{A(|\nabla u^\star|)}{|\nabla u^\star|}\, d\mathcal{H}^{n-1}\ . \end{aligned} \tag{2.12}$$

Combining (2.8), (2.10), (2.12) and making use of the coarea formula (2.4) entails that

$$\int_{\mathbb{R}^n} A(|\nabla u|) dx = \int_0^{+\infty} dt \int_{\{u=t\}} \frac{A(|\nabla u|)}{|\nabla u|} d\mathcal{H}^{n-1} \geq \int_0^{+\infty} dt \int_{\{u^\star = t\}} \frac{A(|\nabla u^\star|)}{|\nabla u^\star|} d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} A(|\nabla u^\star|) dx. \tag{2.13}$$

Since u is a minimal rearrangement, equality holds in the last inequality. Hence, equality has to hold also in (2.8), (2.10) and (2.12) for a.e. $t \in [0, \operatorname{esssup} u)$. In particular, since A is strictly increasing, from (2.12) we get that

$$\{u > t\} \text{ is equivalent to a ball for } \mathcal{L}^1\text{-a.e. } t \in [0, \operatorname{esssup} u)\,.$$

With this property of u in place, the existence of a representative of u satisfying (i), (ii), (iii) and (iv) follows quite easily (see [CF1, Lemma 4.1] for details).

As far as (v) is concerned, assume that $x \in \partial U_t$ and $u(x) \neq t$. Since $x \notin U_t$, then $u(x) < t$. Let τ be any number in $(u(x), t)$. We have $x \in \overline{U}_t \subset \overline{U}_\tau$ and $x \notin U_\tau$, whence $x \in \partial U_\tau$. Thus, the balls U_t and U_τ are tangent at x. If there were another point $y \in \partial U_t$, with $u(y) \neq t$, then all the balls U_τ with $\tau \in (\max\{u(x), u(y)\}, t)$ would be tangent at x and y, and hence all these balls would coincide. Thus μ would be constant in $(\max\{u(x), u(y)\}, t)$ and, consequently, u^*, the decreasing rearrangement of u defined as

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\} \qquad \text{for } s \geq 0\,, \tag{2.14}$$

would be discontinuous at $\mu(\max\{u(x), u(y)\})$. This is impossible, since

$$u^\star(x) = u^*(\omega_n |x|^n)$$

and $u^\star$ is a Sobolev function.

Property (vi) is a straightforward consequence of (v), which ensures that for $\mathcal{L}^1$-a.e. $t \in [0, \operatorname{esssup} u)$ $\partial\{u > t\}$ and $\{u = t\}$ differ at most for one point.

Finally, to prove assertion (vii), apply inequalities (2.8), (2.10), (2.12) and (2.13) to the representative constructed above. Since equality holds in (2.8), Corollary 2.2 ensures that for $\mathcal{L}^1$-a.e. $t \in [0, \operatorname{esssup} u)$, there either exists a positive constant c_t such that

$$|\nabla u(x)| = c_t \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{u = t\}\,, \tag{2.15}$$

or an index i such that

$$|\nabla u(x)| \in \overline{J}_i \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{u = t\}\,. \tag{2.16}$$

Thus, to conclude the proof we need only to show that if (2.15) is in force and (2.16) does not hold, then

$$c_t = |\nabla u^\star|_{|\{u^\star = t\}}\,. \tag{2.17}$$

To this aim, let us denote by J_1 the (possibly empty) open interval from the family $\{J_i\}$, having the form $(0, s_0)$ and such that $A(s) = as$ for some $a \in \mathbb{R}$ and for every $s \in (0, s_0)$. Since

$$\frac{A(s)}{s} \quad \text{is strictly increasing for } s \geq s_0\,,$$

and $c_t \notin \overline{J}_1 = [0, s_0]$, then

$$\frac{A(s)}{s} \neq \frac{A(c_t)}{c_t} \qquad \text{for all } s \neq c_t\,. \tag{2.18}$$

Thus, inasmuch as (2.10) holds as an equality, (2.17) follows from (2.18). □

The next corollary is a consequence of Theorem 2.3. In the statement, we denote by R_t and C_t the radius and the center, respectively, of the ball U_t, and set $R_{t^-} = \lim_{\tau\to t^-} R_\tau$.

Corollary 2.4 *Let $x \in \mathbb{R}^n$ and let $t = u(x)$. If $t \in (0, \operatorname{ess\,sup} u)$, then*

$$\operatorname{dist}(x, \partial U_t) \le \frac{2\mathcal{L}^n(\{u=t\})}{\omega_n R_{t^-}^{n-1}}\,. \tag{2.19}$$

Proof. If $\mathcal{L}^n(\{u = t\}) = 0$, then (2.19) follows from assertion (iv) in Theorem 2.3. If, on the contrary, $\mathcal{L}^n(\{u = t\}) > 0$, then $\{u = t\} = \Big(\bigcap_{0<\tau<t} U_\tau\Big) \setminus U_t$, and $\bigcap_{0<\tau<t} U_\tau$ is equivalent to a ball of radius R_{t^-}. Thus

$$\operatorname{dist}(x, \partial U_t) \le 2R_{t^-} - 2R_t\,.$$

Since $2R_{t^-} - 2R_t \le \dfrac{2(R_{t^-}^n - R_t^n)}{R_{t^-}^{n-1}}$, inequality (2.19) follows. □

3 Proof of Theorem 1.1

Throughout this section u denotes (the representative provided by Theorem 2.3 of) a minimal rearrangement relative to a Young function A vanishing only at 0.

Define the functions $\sigma : \mathbb{R}^n \to [0, +\infty)$ as

$$\sigma(x) = \mu(u(x)) \qquad \text{for } x \in \mathbb{R}^n\,. \tag{3.1}$$

and $\Phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\Phi(x) = \left(\frac{\sigma(x)}{\omega_n}\right)^{1/n} \qquad \text{for } x \in \mathbb{R}^n\,. \tag{3.2}$$

Since the decreasing rearrangement u^* is continuous, then $u^*(\mu(t)) = t$ for $t \in [0, \operatorname{ess\,sup} u]$. Hence, u can be factored as in [FV1, FV2] by

$$u(x) = u^*(\mu(u(x))) = u^*(\omega_n \Phi(x)^n) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n\,. \tag{3.3}$$

Starting from this expression for u, the proof consists in showing that, if u is suitably translated, then the function $\Phi(x)$ does not differ too much from $|x|$ in a large subset of the support of u, and consequently u is close to $u^\star$ in $L^1(\mathbb{R}^n)$. A difficulty here is that, in contrast to the case of Brothers and Ziemer's theorem, the function μ need not be absolutely continuous. This is apparent from the representation formula

$$\mu(t) = \mathcal{L}^n(\{u = \operatorname{ess\,sup} u\}) + \mathcal{L}^n(\{\nabla u = 0\} \cap \{t < u < \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} d\tau \int_{\{u=\tau\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u(x)|} \tag{3.4}$$

for every $t \in [0, \operatorname{ess\,sup} u)$, which follows from the coarea formula (2.4). Further complications arise due to the presence of a set of non strict convexity of A.

The idea is then to work with appropriate modifications $\widetilde{\mu}$, $\widetilde{\sigma}$ and $\widetilde{\Phi}$ of μ, σ and Φ, respectively, which enjoy suitable regularity properties and take into account only the set of strict convexity of A, and whose deviation from μ, σ and Φ can be quantified. To this purpose, we set

$$F = \{x \in \mathbb{R}^n : |\nabla u(x)| \in \{0\} \cup L\} \cap \{0 < u < \operatorname{ess\,sup} u\}\,,$$

where L is given by (1.8), and we define

$$\widetilde{\mu}(t) = \mathcal{L}^n(\{u = \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} d\tau \int_{\{u=\tau\}} \frac{\chi_{\mathbb{R}^n \setminus F}(x)}{|\nabla u(x)|}\, d\mathcal{H}^{n-1} \quad \textit{for } t \geq 0. \tag{3.5}$$

Observe that, on setting

$$G = \{t \geq 0 : \mathcal{H}^{n-1}(\{u = t\} \cap F) = 0\},$$

then, owing to (vii) of Theorem 2.3, the function $\widetilde{\mu}$ admits the alternate representation

$$\widetilde{\mu}(t) = \mathcal{L}^n(\{u = \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} \chi_G(\tau) \frac{\mathcal{H}^{n-1}(\{u = \tau\})}{|\nabla u|_{|\{u=\tau\}}}\, d\tau \quad \textit{for } t \geq 0. \tag{3.6}$$

Parallely to (3.1) and (3.2), we define

$$\widetilde{\sigma}(x) = \widetilde{\mu}(u(x)) \qquad \text{for } x \in \mathbb{R}^n, \tag{3.7}$$

and

$$\widetilde{\Phi}(x) = \left(\frac{\widetilde{\sigma}(x)}{\omega_n}\right)^{1/n} \qquad \text{for } x \in \mathbb{R}^n.$$

We begin with an estimate for μ in terms of $\widetilde{\mu}$ and of the ratio

$$\varepsilon = \frac{\mathcal{L}^n(F)}{\mathcal{L}^n(\{u > 0\})}.$$

Lemma 3.1 *Assume that $\varepsilon < 1$. Then there exists $t_\varepsilon \in (0, +\infty]$ such that*

$$\mathcal{L}^n(\{u > t_\varepsilon\}) \leq \sqrt{\varepsilon}\mathcal{L}^n(\{u > 0\}), \tag{3.8}$$

$$(1 - \sqrt{\varepsilon})\mu(t) < \widetilde{\mu}(t) \qquad \textit{if } 0 \leq t < t_\varepsilon, \tag{3.9}$$

and

$$\mathcal{L}^n(\{\nabla u = 0\} \cap \{t < u < \operatorname{ess\,sup} u\}) < \sqrt{\varepsilon}\mathcal{L}^n(\{u > t\}) \qquad \textit{if } 0 \leq t < t_\varepsilon. \tag{3.10}$$

Proof. Set

$$t_\varepsilon = \inf\{t \geq 0 : \widetilde{\mu}(t) \leq (1 - \sqrt{\varepsilon})\mu(t)\},$$

where we agree that $\inf \emptyset = +\infty$. Since

$$\begin{aligned} \mu(t) - \widetilde{\mu}(t) &= \mathcal{L}^n(\{\nabla u = 0\} \cap \{t < u < \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} d\tau \int_{\{u=\tau\}} \frac{\chi_{\{\nabla u \neq 0\} \cap F}(x)}{|\nabla u(x)|}\, d\mathcal{H}^{n-1} \\ &= \mathcal{L}^n(\{u > t\} \cap F) \leq \varepsilon \mathcal{L}^n(\{u > 0\}) \end{aligned} \tag{3.11}$$

for every $t \in [0, \operatorname{ess\,sup} u)$, and since $\varepsilon < 1$, then $\mu(t) - \widetilde{\mu}(t) < \sqrt{\varepsilon}\mu(t)$ if t is sufficiently small, then $t_\epsilon > 0$. Now, (3.9) holds by the very definition of t_ϵ. As for (3.8), if $t_\varepsilon = +\infty$, there is nothing to prove. Otherwise, note that, since μ is right continuous and $\widetilde{\mu}$ is continuous, t_ε is in fact a minimum. Therefore, by (3.11),

$$\sqrt{\varepsilon}\mu(t_\varepsilon) \leq \mu(t_\varepsilon) - \widetilde{\mu}(t_\varepsilon) \leq \varepsilon \mathcal{L}^n(\{u > 0\}).$$

Hence, (3.8) follows. Finally, (3.10) is a consequence of (3.9) and of the first equality in (3.11). □

The next lemma tells us that $\widetilde{\sigma}$ is Lipschitz continuous, and gives a formula for its gradient.

Lemma 3.2 *The function $\widetilde{\sigma} \in W^{1,\infty}(U_0)$. Moreover,*

$$\nabla \widetilde{\sigma}(x) = -\mathcal{H}^{n-1}(\{u = u(x)\}) \frac{\nabla u(x)}{|\nabla u(x)|} \chi_{\{\nabla u \neq 0\} \setminus F}(x) \qquad \textit{for } \mathcal{L}^n\textit{-a.e. } x \in U_0 . \tag{3.12}$$

Proof. Set

$$\widetilde{\mu}_\eta(t) = \mathcal{L}^n(\{u = \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} d\tau \int_{\{u=\tau\}} \frac{\chi_{\mathbb{R}^n \setminus F}(x)}{\eta + |\nabla u(x)|} \, d\mathcal{H}^{n-1} \quad \textit{for } t \geq 0 , \tag{3.13}$$

and note that, analogously to (3.6), $\widetilde{\mu}_\eta(t)$ can be alternatively written as

$$\widetilde{\mu}_\eta(t) = \mathcal{L}^n(\{u = \operatorname{ess\,sup} u\}) + \int_t^{\operatorname{ess\,sup} u} \chi_G(\tau) \frac{\mathcal{H}^{n-1}(\{u=\tau\})}{\eta + |\nabla u|_{|\{u=\tau\}}} \, d\tau \qquad \text{for } t \geq 0 . \tag{3.14}$$

Clearly, $\widetilde{\mu}_\eta(t) \uparrow \widetilde{\mu}(t)$ for every $t \geq 0$ as $\eta \downarrow 0$. Moreover, $\widetilde{\mu}_\eta$ is Lipschitz continuous in $[0, +\infty)$, and, by (3.14),

$$\widetilde{\mu}_\eta'(t) = -\frac{\mathcal{H}^{n-1}(\{u=t\})}{\eta + |\nabla u|_{|\{u=t\}}} \chi_G(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 ,$$

whence

$$|\widetilde{\mu}_\eta'(t) - \widetilde{\mu}'(t)| \leq \frac{\eta \mathcal{H}^{n-1}(\{u=t\}) \chi_G(t)}{|\nabla u|_{|\{u=t\}} \left(\eta + |\nabla u|_{|\{u=t\}}\right)} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 .$$

Thus, $\widetilde{\mu}_\eta'(t) \to \widetilde{\mu}'(t)$ $\mathcal{L}^1$-a.e. in $[0, +\infty)$ as $\eta \to 0$, since $|\nabla u|_{|\{u=t\}} \neq 0$ for $\mathcal{L}^1$-a.e. $t \in G$, inasmuch as $\dfrac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}}} \chi_G(t) \in L^1(0, \infty)$. This membership and the fact that

$$|\widetilde{\mu}_\eta'(t) - \widetilde{\mu}'(t)| \leq \frac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}}} \chi_G(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0$$

entail that $\widetilde{\mu}_\eta' \to \widetilde{\mu}'$ in $L^1(0, \infty)$. Hence, $\widetilde{\mu}_\eta \to \widetilde{\mu}$ uniformly in $(0, \infty)$. Consequently, the real-valued functions $\widetilde{\sigma}_\eta$ in $\mathbb{R}^n$ defined as $\widetilde{\sigma}_\eta(x) = \widetilde{\mu}_\eta(u(x))$ for $x \in \mathbb{R}^n$, converge uniformly to $\widetilde{\sigma}$. Furthermore, by (3.13) and by the chain rule for Sobolev functions (see e.g. [AFP, Theorem 3.96]),

$$\nabla \widetilde{\sigma}_\eta(x) = \widetilde{\mu}_\eta'(u(x)) \nabla u(x) = -\mathcal{H}^{n-1}(\{u = u(x)\}) \frac{\nabla u(x)}{\eta + |\nabla u|_{|\{u=u(x)\}}} \chi_{\mathbb{R}^n \setminus F}(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U_0 .$$

Clearly, the last expression converges to the right-hand side of (3.12) $\mathcal{L}^n$-a.e. as $\eta \to 0$. Now, recall that, for $\mathcal{L}^n$-a.e. $x \in U_0$ satisfying $\mathcal{L}^n(\{u = u(x)\}) > 0$, we have $\nabla u(x) = 0$; on the other hand, by Theorem 2.3, $\mathcal{L}^n$-a.e. $x \in U_0$ such that $\mathcal{L}^n(\{u = u(x)\}) = 0$ fulfills $x \in \partial U_{u(x)}$, and hence $\mathcal{H}^{n-1}(\{u = u(x)\}) = n\omega_n (R_{u(x)})^{n-1} \leq n\omega_n R_0^{n-1}$. Thus,

$$|\nabla \widetilde{\sigma}_\eta(x)| \leq n\omega_n R_0^{n-1} \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in U_0 .$$

By dominated convergence, $\nabla \widetilde{\sigma}_\eta$ converges to the right-hand side of (3.12) in $L^1(U_0)$. The conclusion follows. □

We conclude our preparatory results with a lemma dealing with perturbations of radial Sobolev functions. In the statement B_R denotes the ball, centered at 0, having radius R.

Lemma 3.3 *Let R be a positive number, and let g be a monotone function from $W^{1,1}_{loc}(0,R)$. Let r_1, r_2 and δ be real numbers satisfying $0<\delta<r_1<r_2<r_2+\delta<R$, and let $\psi: B_R \to [0,R]$ be a measurable function such that*

$$|\psi(x)-|x||<\delta \qquad \text{for } \mathcal{L}^n\text{-a.e. } x\in B_R\,. \tag{3.15}$$

Then

$$\int_{B_{r_2}\setminus B_{r_1}} |g(\psi(x))-g(|x|)|\,dx \le 2\delta\Big(\frac{r_1}{r_1-\delta}\Big)^{n-1}\int_{B_{r_2+\delta}\setminus B_{r_1-\delta}} |\nabla(g(|x|))|\,dx\,. \tag{3.16}$$

Proof. Assume, for instance, that g is non-decreasing (the case where g is non-increasing being completely analogous). Owing to (3.15),

$$g(|x|-\delta)\le g(|x|)\le g(|x|+\delta) \quad \text{and} \quad g(|x|-\delta)\le g(\psi(x))\le g(|x|+\delta) \quad \text{for } \mathcal{L}^n\text{-a.e. } x\in B_{r_2}\setminus B_{r_1}\,,$$

whence

$$|g(\psi(x))-g(|x|)|\le g(|x|+\delta)-g(|x|-\delta) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x\in B_{r_2}\setminus B_{r_1}\,.$$

Thus, (3.16) holds thanks to the following chain of inequalities:

$$\begin{aligned}\int_{B_{r_2}\setminus B_{r_1}} |g(\psi(x))-g(|x|)|dx &\le \int_{r_1}^{r_2} d\rho \int_{\partial B_\rho} |g(\rho+\delta)-g(\rho-\delta)|d\mathcal{H}^{n-1} = n\omega_n\int_{r_1}^{r_2}\varrho^{n-1}|g(\rho+\delta)-g(\rho-\delta)|d\rho\\
&= n\omega_n\delta\int_{r_1}^{r_2}\varrho^{n-1}d\rho\int_{-1}^{1}g'(\rho+\delta t)dt \le n\omega_n\delta\Big(\frac{r_1}{r_1-\delta}\Big)^{n-1}\int_{-1}^{1}dt\int_{r_1}^{r_2}g'(\rho+\delta t)(\rho+\delta t)^{n-1}d\rho\\
&\le n\omega_n\delta\Big(\frac{r_1}{r_1-\delta}\Big)^{n-1}\int_{-1}^{1}dt\int_{r_1-\delta}^{r_2+\delta}g'(r)r^{n-1}dr = 2\delta\Big(\frac{r_1}{r_1-\delta}\Big)^{n-1}\int_{B_{r_2+\delta}\setminus B_{r_1-\delta}}|\nabla(g(|x|))|dx\,.\end{aligned}$$

□

We are now in position to accomplish the proof of Theorem 1.

Proof of Theorem 1. Throughout the proof, we assume that U_0 is translated in such a way to coincide with B_{R_0}. Hence, both u and $u^\star$ are supported in U_0. Suppose, for the time being, that $\varepsilon\le 1/2$. By (3.9),

$$0\le\sigma(x)-\widetilde{\sigma}(x)<\sqrt{\varepsilon}\sigma(x) \qquad \text{if } u(x)<t_\varepsilon\,,$$

whence

$$|\Phi(x)-\widetilde{\Phi}(x)|\le\frac{1}{\omega_n^{1/n}}(\sigma(x)-\widetilde{\sigma}(x))^{1/n}\le\Phi(x)\varepsilon^{\frac{1}{2n}} \qquad \text{if } u(x)<t_\varepsilon\,. \tag{3.17}$$

By Lemma 3.2, we have that

$$\nabla\widetilde{\Phi}(x)=\frac{1}{n\omega_n^{1/n}}(\widetilde{\sigma}(x))^{\frac{1}{n}-1}\nabla\widetilde{\sigma}(x)=-\frac{1}{n\omega_n^{1/n}}(\widetilde{\sigma}(x))^{\frac{1}{n}-1}\mathcal{H}^{n-1}(\{u=u(x)\})\frac{\nabla u(x)}{|\nabla u(x)|}\chi_{\{\nabla u\neq 0\}\setminus F}(x)\,,$$

for $x \in \mathcal{L}^n$-a.e. $x \in U_0$. Thus, for $\mathcal{L}^n$-a.e. $x \in U_0$ such that $u(x) < t_\varepsilon$,

$$(3.18)\quad |\nabla\widetilde{\Phi}(x)| \leq \frac{1}{n\omega_n^{1/n}}(\widetilde{\mu}(u(x)))^{\frac{1}{n}-1} n\omega_n^{1/n}(\mu(u(x)))^{1-1/n} = \left(\frac{\mu(u(x))}{\widetilde{\mu}(u(x))}\right)^{1-1/n} \leq \left(\frac{1}{1-\sqrt{\varepsilon}}\right)^{1-1/n}.$$

Observe that the first inequality holds owing to (iv) of Theorem 2.3 and to the fact that $\nabla u = 0$ $\mathcal{L}^n$-a.e. in $\{u = u(x)\}$ if $\mathcal{L}^n(\{u = u(x)\}) > 0$, whereas the second inequality is a consequence of (3.9). Thus, if $u(x), u(y) < t_\varepsilon$, then by (3.17)-(3.18)

$$(3.19)\quad |\Phi(x) - \Phi(y)| \leq |\Phi(x) - \widetilde{\Phi}(x)| + |\Phi(y) - \widetilde{\Phi}(y)| + |\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| \leq \frac{|x-y|}{(1-\sqrt{\varepsilon})^{\frac{n-1}{n}}} + (\Phi(x)+\Phi(y))\varepsilon^{\frac{1}{2n}}.$$

Fix t and τ such that $0 \leq \tau < t < t_\varepsilon$. By (v) of Theorem 2.3, sequences $\{x_m\} \subset \partial U_t$ and $\{y_m\} \subset \partial U_\tau$ exist such that $u(x_m) = t$, $u(y_m) = \tau$ and $\lim_{m\to\infty} |x_m - y_m| =$dist$(\partial U_t, \partial U_\tau)$. Since $\Phi(x_m) = R_t$ and $\Phi(y_m) = R_\tau$, applying (3.19) with x and y replaced by x_m and y_m, respectively, and passing to the limit as $m \to +\infty$, yield

$$R_\tau - R_t \leq \frac{\operatorname{dist}(\partial U_t, \partial U_\tau)}{(1-\sqrt{\varepsilon})^{\frac{n-1}{n}}} + (R_t + R_\tau)\varepsilon^{\frac{1}{2n}}.$$

Thus,

$$|C_t - C_\tau| = R_\tau - R_t - \operatorname{dist}(\partial U_t, \partial U_\tau) \leq \operatorname{dist}(\partial U_t, \partial U_\tau)\left[\frac{1}{(1-\sqrt{\varepsilon})^{\frac{n-1}{n}}} - 1\right] + (R_t + R_\tau)\varepsilon^{\frac{1}{2n}}.$$

Hence, a constant $c_1(n)$, depending only on n, exists such that

$$(3.20)\qquad |C_t - C_\tau| \leq c_1(n) R_0 \varepsilon^{\frac{1}{2n}}$$

We claim that there exists a constant $c_2(n)$ such that

$$(3.21)\qquad |\Phi(x) - |x|| \leq c_2(n) R_0 \varepsilon^{\frac{1}{2n}} \qquad \text{if } 0 < u(x) < t_\varepsilon,$$

where t_ε is provided by Lemma 3.1. Indeed, fix any x satisfying $0 < u(x) < t_\varepsilon$, and set $t = u(x)$. Then, $\Phi(x) = R_t$, and

$$(3.22)\quad |\Phi(x) - |x|| \leq |R_t - |x - C_t|| + ||x - C_t| - |x|| \leq \operatorname{dist}(x, \partial U_t) + |C_t| \leq \frac{2\mathcal{L}^n(\{u = t\})}{\omega_n R_{t-}^{n-1}} + c_1(n) R_0 \varepsilon^{\frac{1}{2n}},$$

where, in the last inequality, we have made use of Corollary 2.4 and of (3.20) with $\tau = 0$. Since

$$\frac{\mathcal{L}^n(\{u = t\})}{\omega_n R_{t-}^{n-1}} \leq R_{t-} \lim_{\tau\to t-} \frac{\mathcal{L}^n(\{\nabla u = 0\} \cap \{\tau < u < \operatorname{ess\,sup} u)\})}{\mathcal{L}^n(\{u > \tau\})},$$

then (3.21) follows from (3.22) and (3.10).
We next notice that

$$(3.23)\qquad \mathcal{L}^n(\{u \geq t_\varepsilon\}) \leq 2\omega_n \sqrt{\varepsilon} R_0^n \qquad \text{if } t_\varepsilon < \operatorname{ess\,sup} u.$$

Actually, by (3.10),

$$(3.24)\quad \mathcal{L}^n(\{u = t_\varepsilon\}) \leq \lim_{t\to t_\varepsilon^-} \mathcal{L}^n(\{\nabla u = 0\} \cap \{t < u < \operatorname{ess\,sup} u\}) \leq \lim_{t\to t_\varepsilon^-} \sqrt{\varepsilon}\mathcal{L}^n(\{u > t\}) \leq \sqrt{\varepsilon}\mathcal{L}^n(\{u > 0\}).$$

Combining (3.24) with (3.8) yields (3.23).
We are now ready to estimate $\int_{U_0}|u-u^\star|dx$. We split U_0 in the subsets $\{u^\star \geq t_\varepsilon\}$ and $\{u^\star < t_\varepsilon\}$. Consider the former first. If $t_\varepsilon < \operatorname{ess\,sup} u$, then an application of Hölder inequality, of the Sobolev inequality, and of (3.23) tells us that

$$\text{(3.25)} \qquad \int_{\{u^\star \geq t_\varepsilon\}} |u-u^\star|\,dx \leq \Big(\int_{\{u^\star\geq t_\varepsilon\}} |u-u^\star|^{n'}\,dx\Big)^{\frac{1}{n'}} \Big(\mathcal{L}^n(\{u^\star \geq t_\varepsilon\})\Big)^{\frac{1}{n}}$$
$$\leq \frac{1}{n\omega_n^{1/n}}(2\omega_n)^{1/n}\varepsilon^{\frac{1}{2n}}R_0\int_{U_0}|\nabla u-\nabla u^\star|\,dx \leq \frac{2^{1+1/n}}{n}\varepsilon^{\frac{1}{2n}}R_0\int_{U_0}|\nabla u|\,dx\,.$$

Note that here we have made use of the fact that u and $u^\star$ have the same distribution function and of (1.3). If $t_\varepsilon = \operatorname{ess\,sup} u$, then by (ii) of Theorem 2.3, the set $\{u \geq t_\varepsilon\}$ is a closed ball whose center is the limit of C_t as $t \uparrow \operatorname{ess\,sup} u$. Hence, the ball $\{u\geq t_\varepsilon\}$ has radius smaller than or equal to R_0 and, by (3.20) with $\tau = 0$, its center has norm not exceeding $c_1(n)R_0\varepsilon^{\frac{1}{2n}}$. Easy geometric considerations entail that, if two balls have the same radius r and centers with distance equal to d, then the Lebesgue measure of their symmetric difference does not exceed $c_3(n)r^{n-1}d$ for some constant $c_3(n)$. Consequently, there exists a constant $c_4(n)$ such that $\mathcal{L}^n(\{u^\star = \operatorname{ess\,sup} u\} \setminus \{u = \operatorname{ess\,sup} u\}) \leq c_4(n)R_0^n\varepsilon^{\frac{1}{2n}}$. Since

$$\int_{\{u^\star=\operatorname{ess\,sup} u\}}|u-u^\star|\,dx = \int_{\{u^\star=\operatorname{ess\,sup} u\}\setminus\{u=\operatorname{ess\,sup} u\}}|u-u^\star|\,dx\,,$$

then an analogous chain of inequalities as in (3.25) yields

$$\text{(3.26)} \qquad \int_{\{u^\star=\operatorname{ess\,sup} u\}}|u-u^\star|\,dx \leq c_5(n)\varepsilon^{\frac{1}{2n^2}}R_0\int_{U_0}|\nabla u|\,dx$$

for some constant $c_5(n)$.
Let us finally consider the subset $\{u^\star < t_\varepsilon\}$. By (3.8), $R_{t_\varepsilon} \leq \varepsilon^{\frac{1}{2n}}R_0$. Set $r_\varepsilon = 2c_2(n)R_0\varepsilon^{\frac{1}{2n}}$, where $c_2(n)$ is the constant appearing in (3.21). Assume first that $R_{t_\varepsilon} \geq r_\varepsilon$. By (3.3),

$$\text{(3.27)} \qquad \int_{\{u^\star<t_\varepsilon\}}|u-u^\star|\,dx \leq \int_{\{u^\star\leq t_\varepsilon\}}|u-u^\star|\,dx = \int_{B_{R_0}\setminus B_{R_{t_\varepsilon}}}|u^*(\omega_n\Phi^n(x))-u^*(\omega_n|x|^n)|\,dx\,.$$

By Lemma 3.3 applied with $g(s) = u^*(\omega_n s^n)$, $\psi(x) = \Phi(x)$, $r_2 = R_0$, $r_1 = R_{t_\varepsilon}$ and $\delta = c_2(n)R_0\varepsilon^{\frac{1}{2n}}$, and by (3.21), the last integral in (3.27) is smaller than or equal to

$$2c_2(n)R_0\varepsilon^{\frac{1}{2n}}\Big(\frac{R_{t_\varepsilon}}{R_{t_\varepsilon}-c_2(n)R_0\varepsilon^{\frac{1}{2n}}}\Big)^{n-1}\int_{B_{R_0}}|\nabla u^\star|\,dx\,.$$

In conclusion,

$$\text{(3.28)} \qquad \int_{\{u^\star<t_\varepsilon\}}|u-u^\star|\,dx \leq 2^n c_2(n)R_0\varepsilon^{\frac{1}{2n}}\int_{U_0}|\nabla u^\star|\,dx \leq 2^n c_2(n)R_0\varepsilon^{\frac{1}{2n}}\int_{U_0}|\nabla u|\,dx\,.$$

In the case where $r_\varepsilon > R_{t_\varepsilon}$, we write

$$\text{(3.29)} \qquad \int_{\{u^\star<t_\varepsilon\}}|u-u^\star|\,dx = \int_{B_{R_0}\setminus B_{r_\varepsilon}}|u-u^\star|\,dx + \int_{B_{r_\varepsilon}\setminus B_{R_{t_\varepsilon}}}|u-u^\star|\,dx\,.$$

The former integral on the right-hand side of (3.29) can be estimated as in (3.28). As for the latter, an analogous chain of inequalities as in (3.25) yields

$$\int_{B_{r_\varepsilon}} |u-u^\star|\,dx \le \left(\int_{B_{r_\varepsilon}} |u-u^\star|^{n'}\,dx\right)^{1/n'} (\omega_n r_\varepsilon^n)^{1/n} \le c_6(n)\varepsilon^{\frac{1}{2n}} R_0 \int_{U_0} |\nabla u|\,dx$$

for some constant $c_6(n)$. Thus,

$$\int_{\{u^\star<t_\varepsilon\}} |u-u^\star|\,dx < (2^n c_2(n)+c_6(n))\varepsilon^{\frac{1}{2n}} R_0 \int_{U_0} |\nabla u|\,dx\,. \tag{3.30}$$

Combining either (3.25) or (3.26) with either (3.28) or (3.30) ensures that there exists a constant $c_7(n)$ such that

$$\int_{\mathbb{R}^n} |u-u^\star|\,dx \le c_7(n) R_0 \varepsilon^{\frac{1}{2n^2}} \int_{U_0} |\nabla u|\,dx\,.$$

Note that here we have made use of the fact that ε does not exceed 1. Since, by Jensen's inequality,

$$\int_{U_0} |\nabla u|\,dx \le \mathcal{L}^n(U_0) A^{-1}\left(\frac{\int_{U_0} A(|\nabla u|)dx}{\mathcal{L}^n(U_0)}\right), \tag{3.31}$$

inequality (1.9) follows in the case where $0 \le \varepsilon \le 1/2$.

If $\varepsilon > 1/2$, then an application of Hölder's inequality, of the Sobolev inequality and of the Pólya–Szegö inequality with $A(s) = s$ as in (3.25) tells us that

$$\int_{U_0} |u-u^\star|\,dx \le \frac{2}{n\omega_n^{1/n}} \mathcal{L}^n(U_0)^{1/n} \int_{U_0} |\nabla u|\,dx\,,$$

whence, by (3.31),

$$\int_{U_0} |u-u^\star|\,dx \le \frac{2}{n\omega_n^{1/n}} \mathcal{L}^n(U_0)^{1+1/n} A^{-1}\left(\frac{\int_{U_0} A(|\nabla u|)dx}{\mathcal{L}^n(U_0)}\right).$$

Since we are assuming that $\varepsilon > 1/2$, this inequality obviously implies that

$$\int_{U_0} |u-u^\star|\,dx \le \frac{2^{1+\frac{1}{n^2}}}{n\omega_n^{1/n}} \mathcal{L}^n(U_0)^{1+1/n} \varepsilon^{\frac{1}{n^2}} A^{-1}\left(\frac{\int_{U_0} A(|\nabla u|)dx}{\mathcal{L}^n(U_0)}\right),$$

and (1.9) holds also in this case. □

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