Boundary regularity for elliptic problems with continuous coefficients

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Abstract: We consider weak solutions of second order nonlinear elliptic systems in divergence form or of quasi-convex variational integrals with continuous coefficients under superquadratic growth conditions. Via the method of $\mathcal{A}$-harmonic approximation we give a characterization of regular boundary points using and extending some new techniques recently developed by M. Foss & G. Mingione in [15].

1 Introduction and results

In this paper we present a characterization of regular boundary points in the regularity theory of vectorial elliptic and variational problems by extending the techniques and the results of Foss & Mingione in [15] to the boundary. We first consider weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of a general homogeneous system of second order elliptic equations in divergence form

$$\operatorname{div}\, a(\,\cdot\,, u, Du) = 0 \qquad \text{in } \Omega\,, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $a\colon \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is a continuous vector field on which we impose standard boundedness, differentiability, growth and ellipticity conditions: $z \mapsto a(\cdot,\cdot,z)$ is of class C^1, and for fixed $0 < \nu \leq L$ and all $x, \bar{x} \in \overline{\Omega}$, $u, \bar{u} \in \mathbb{R}^N$, and $z, \bar{z}, \lambda \in \mathbb{R}^{nN}$ there holds:

$$\begin{cases} \big|a(x,u,z)\big| + \big|D_z a(x,u,z)\big|\,\big(1+|z|\big) \,\leq\, L\,\big(1+|z|\big)^{p-1}\,, \\ D_z a(x,u,z)\,\lambda\cdot\lambda \,\geq\, \nu\,\big(1+|z|\big)^{p-2}|\lambda|^2\,, \\ \big|a(x,u,z) - a(\bar{x},\bar{u},z)\big| \,\leq\, L\,\big(1+|z|\big)^{p-1}\,\omega\big(|x-\bar{x}|^2 + |u-\bar{u}|^2\big)\,. \\ \big|D_z a(x,u,z) - D_z a(x,u,\bar{z})\big| \,\leq\, L\,\mu\big(\tfrac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\big)\,\big(1+|z|+|\bar{z}|\big)^{p-2}\,. \end{cases} \tag{1.2}$$

Here $n, N \geq 2$, $p \geq 2$, and $\mu, \omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ are two moduli of continuity, i. e. bounded by 1 (without loss of generality), concave and non-decreasing such that $\lim_{\rho\to 0}\omega(\rho) = 0 = \lim_{\rho\to 0}\mu(\rho)$.

The role of the modulus of continuity $\omega(\cdot)$ will be the crucial point in our paper; we remark that, in the sequel, we confine ourselves to the vectorial case. For the scalar case we refer to [15] and the references therein. If we assume a Hölder condition of the form $\omega(t) \leq t^{\frac{\alpha}{2}}$ for some $\alpha \in (0,1)$, $t \in \mathbb{R}^+$, i. e., $(1+|z|)^{1-p}a(x,u,z)$ is Hölder continuous in the variables (x,u) uniformly with respect to z, then it is known (see [17]; [14] for the variational case) that standard growth and ellipticity assumptions on the coefficients imply partially Hölder continuous first derivatives of the weak solution u, which means Hölder continuity outside the *singular* set of Lebesgue measure 0, with optimal Hölder exponent α. Moreover, assuming that the boundary data are sufficiently smooth, general criteria for Du to be regular in a neighbourhood of a given boundary point were obtained by Grotowski and Hamburger (see [20, 21]) using boundary versions of the method of $\mathcal{A}$-harmonic approximation and of the blow-up technique, respectively. The assumption on $\omega(\cdot)$ was weakened to Dini-continuous coefficients, where $\int_0^r \frac{\omega(\rho)}{\rho}d\rho < \infty$ is fulfilled for some $r > 0$, which still allows to conclude a partial regularity result for Du (see [8, 27]; [9] for the variational case). Moreover, a condition of the form $\limsup_{\rho\to 0}\omega(\rho)\log(\frac{1}{\rho}) = 0$ ensures in the case of variational functionals under non-standard growth without u-dependency (see [3], Theorem 2.1) to infer $u \in C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ for every $\alpha \in (0,1)$.

Assuming merely the continuity of the coefficients with respect to the variable (x,u) without any further structural assumptions, Campanato proved low order partial regularity in [5], namely that the weak

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solution u is Hölder continuous with every exponent $\alpha \in (0,1)$ outside a negligible closed subset of Ω, for the low dimensional case, where $n \leq p+2$ (cf. [6] for similar estimates up to the boundary; see also [23] for the variational setting). Moreover, the Hausdorff dimension of the singular set is bounded by $n-p$ from above implying that actually almost every boundary point is verified to be a regular one. In contrast, in the case of quasi-convex variational integrals, these methods do not apply, and a similar low dimensional result was obtained only under the assumption $\omega(t) \leq t^{\frac{\alpha}{2}}$ (cf. [23], Theorem 1.5). However, for general dimensions, the question of low order partial regularity under a continuity assumption remained unsolved for a long time, until Foss & Mingione gave a positive answer in [15] both for weak solutions of elliptic systems and for local minimizers of quasi-convex variational integrals. The aim of our paper is now to extend the characterization of regular points up to the boundary. For this purpose we denote by $\mathrm{Reg}_{\partial\Omega}\, u$ the regular boundary points of u in the sense that

$$\begin{aligned}\mathrm{Reg}_{\partial\Omega}\, u \,:=\, \big\{ x_0 \in \partial\Omega \colon\, & u \in C^{0,\alpha}(U(x_0) \cap \bar\Omega, \mathbb{R}^N) \text{ for every } \alpha \in (0,1) \\ & \text{and some neighbourhood } U(x_0) \text{ of } x_0 \big\}\,,\end{aligned}$$

and the set of singular boundary points by $\mathrm{Sing}_{\partial\Omega}\, u := \partial\Omega \setminus \mathrm{Reg}_{\partial\Omega}\, u$. Analogously for fixed $\alpha \in (0,1)$ we define

$$\mathrm{Reg}_{\partial\Omega,\alpha}\, u \,:=\, \big\{ x_0 \in \Gamma \colon\, u \in C^{0,\alpha}(U(x_0) \cap \bar\Omega, \mathbb{R}^N) \text{ for some neighbourhood } U(x_0) \text{ of } x_0 \big\}$$

and $\mathrm{Sing}_{\partial\Omega,\alpha}\, u := \partial\Omega \setminus \mathrm{Reg}_{\partial\Omega,\alpha}\, u$. Our first theorem then provides a characterization of the regular boundary points analogous to the characterization of regular points in the interior of Ω (see [15], Theorem 1.1):

Theorem 1.1: *Consider $p \geq 2$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a bounded domain of class C^1 and a map $g \in C^1(\bar\Omega, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution of system (1.1) under the assumptions (1.2) with boundary values $u = g$ on $\partial\Omega$. Then there holds:*

$$\begin{aligned}\mathrm{Sing}_{\partial\Omega}\, u \subseteq \Big\{ x_0 \in \partial\Omega \colon\, & \liminf_{\rho \to 0^+} \fint_{\Omega \cap B_\rho(x_0)} \frac{|D_{\nu_{\partial\Omega}(x_0)} u - (D_{\nu_{\partial\Omega}(x_0)} u)_{\Omega \cap B_\rho(x_0)}|^p}{(1 + |(D_{\nu_{\partial\Omega}(x_0)} u)_{\Omega \cap B_\rho(x_0)}|)^p}\, dx > 0 \\ & \text{or } \liminf_{\rho \to 0^+} \rho^\beta \fint_{\Omega \cap B_\rho(x_0)} |D_{\nu_{\partial\Omega}(x_0)} u|^2\, dx > 0 \Big\}\end{aligned}$$

for every $\beta \in (0,2)$; here $\nu_{\partial\Omega}(x_0)$ denotes the inward-pointing unit normal vector to $\partial\Omega$ in x_0. Moreover, for every $\alpha \in (0,1)$ there exists $s > 0$ depending only on $n, N, p, \nu, L, \alpha, \beta, \partial\Omega, g$, $\omega(\cdot)$ and $\mu(\cdot)$ such that the following inclusion holds for every $\beta \in (0,2)$:

$$\begin{aligned}\mathrm{Sing}_{\partial\Omega,\alpha}\, u \subseteq \Big\{ x_0 \in \partial\Omega \colon\, & \liminf_{\rho \to 0^+} \fint_{\Omega \cap B_\rho(x_0)} \frac{|D_{\nu_{\partial\Omega}(x_0)} u - (D_{\nu_{\partial\Omega}(x_0)} u)_{x_0,\rho}|^p}{(1 + |(D_{\nu_{\partial\Omega}(x_0)} u)_{\Omega \cap B_\rho(x_0)}|)^p}\, dx \geq s \\ & \text{or } \liminf_{\rho \to 0^+} \rho^\beta \fint_{\Omega \cap B_\rho(x_0)} |D_{\nu_{\partial\Omega}(x_0)} u|^2\, dx \geq s \Big\}\,.\end{aligned}$$

We note that for general dimensions the problem of knowing whether there might exist regular boundary points, even in the case of Hölder continuous coefficients with exponent $\alpha < \frac{1}{2}$, remains open (cf. [12]), unless we have some additional structural condition (as e. g. a splitting condition, see [22] for minima).

In the second part of the paper we consider variational integrals of the form

$$\mathcal{F}[u] \,:=\, \int_\Omega F(x, u, Du)\, dx\,, \tag{1.3}$$

where the integrand $F \colon \bar\Omega \times \mathbb{R}^N \times R^{nN} \to \mathbb{R}$ is strictly quasi-convex, continuous and grows polynomially. More precisely, we assume that $z \mapsto F(\cdot,\cdot,z)$ is of class C^2 and that F satisfies for fixed $0 < \nu \leq L$ and all $x, \bar x \in \overline{\Omega}$, $u, \bar u \in \mathbb{R}^N$, and $z, \bar z \in \mathbb{R}^{nN}$ the following assumptions:

$$\left\{\begin{array}{l} \nu\,(1+|z|)^p \,\leq\, F(x,u,z) \,\leq\, L\,(1+|z|)^p \\ \nu \int_{(0,1)^n} (1 + |z| + |D\varphi(y)|)^{p-2} |D\varphi(y)|^2\, dy \,\leq\, \int_{(0,1)^n} \big[F(x,u,z+D\varphi(y)) - F(x,u,z)\big]\, dy \\ \big|F(x,u,z) - F(\bar x, \bar u, z)\big| \,\leq\, L\,\big(1+|z|\big)^p\, \omega\big(|x-\bar x|^p + |u - \bar u|^p\big)\,, \\ \big|D_{zz}F(x,u,z) - D_{zz}F(x,u,\bar z)\big| \,\leq\, L\,\mu\big(\frac{|z-\bar z|}{1+|z|+|\bar z|}\big)\,\big(1+|z|+|\bar z|\big)^{p-2}\,. \end{array}\right. \tag{1.4}$$

The functions $\mu(\cdot)$ and $\omega(\cdot)$ are those already considered in the elliptic case, and for $(1.4)_2$, which is called strict quasi-convexity condition, we assume $\varphi \in C_0^\infty((0,1)^n, \mathbb{R}^N)$. We note here that quasi-convexity is an extension of convexity to a global property and is essentially equivalent to lower semicontinuity (cf. [1]). Applying step 2 of page 6 in [25], we may also assume a growth condition on the first derivatives of the form $D_z f(x,u,z) \le L(1+|z|)^{p-1}$. Moreover, it can be verified that the conditions (1.4) above (see [26], Theorem 4.3) imply the strict ellipticity of the matrix $D^2 f$ in the sense of Legendre-Hadamard, and therefore we may also assume

$$\nu\,(1+|z|)^{p-2}|\xi|^2\,|\eta|^2 \;\le\; D_{zz}F(x,u,z)\,\xi\otimes\eta\cdot\xi\otimes\eta \;\le\; L\;(1+|z|)^{p-2}|\xi|^2\,|\eta|^2$$

for all $\xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n$.

We note that in the case of Hölder continuity of coefficients a partial regularity theory for the gradient of minimizers has been established in the by now classical papers [14, 2, 7], while partial Hölder continuity in the interior in the case of general continuous coefficients has been again proved in [15].

Our second theorem now yields a characterization of the regular boundary points of minimizers of quasi-convex integrals corresponding to the elliptic case:

Theorem 1.2: *Consider $p \ge 2$, $\Omega \subset \mathbb{R}^n$, $n \ge 2$, a bounded domain of class C^1 and a map $g \in C^1(\bar\Omega, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$ in (1.3) under the assumptions (1.4) with boundary values $u = g$ on $\partial\Omega$. Then there holds:*

$$\begin{aligned}\operatorname{Sing}_{\partial\Omega} u \subseteq \Big\{x_0 \in \partial\Omega\colon\ &\liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \frac{|D_{\nu_{\partial\Omega}(x_0)}u - (D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|^p}{(1+|(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|)^p}\,dx > 0\\ &\text{or } \liminf_{\rho\to 0^+} \rho^\beta \fint_{\Omega\cap B_\rho(x_0)} |D_{\nu_{\partial\Omega}(x_0)}u|^p\,dx > 0\Big\}\end{aligned}$$

for every $\beta \in (0,p)$. Moreover, for every $\alpha \in (0,1)$ there exists $s > 0$ depending only on $n, N, p, \nu, L, \alpha, \beta$, $\partial\Omega, g, \omega(\cdot)$ and $\mu(\cdot)$ such that for every $\beta \in (0,p)$ the following inclusion holds:

$$\begin{aligned}\operatorname{Sing}_{\partial\Omega,\alpha} u \subseteq \Big\{x_0 \in \partial\Omega\colon\ &\liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \frac{|D_{\nu_{\partial\Omega}(x_0)}u - (D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|^p}{(1+|(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|)^p}\,dx \ge s\\ &\text{or } \liminf_{\rho\to 0^+} \rho^\beta \fint_{\Omega\cap B_\rho(x_0)} |D_{\nu_{\partial\Omega}(x_0)}u|^p\,dx \ge s\Big\}\,.\end{aligned}$$

Finally we briefly comment on the techniques used in the proofs and on the modifications necessary to handle the boundary situation: regularity proofs for both nonlinear systems and functionals are usually based on a comparison principle in order to establish an excess decay estimate. In the present situation, the excess quantity introduced by Foss & Mingione consists of three terms: the first involving the averaged mean deviation of the derivative of the weak solution (re-normalized by the factor $(1 + |(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|)^p$ which might diverge due to the fact that we cannot expect to obtain Lipschitz estimates even if the boundary data are smooth), the second involving the radius of the ball $B_\rho(x_0)$ and finally the Morrey-type excess $M(x_0,\rho)$ quantifying the oscillations of the weak solution u. The appropriate decay of this excess quantity is now obtained by a linearization argument (combined with the Ekeland variational principle in the case of variational integrals), namely by freezing the coefficients and the functional, respectively, in order to obtain an elliptic system $\mathcal{A}$ with constant coefficients. In the second step, the comparison with an $\mathcal{A}$-harmonic map (for which good a priori estimates are available up to the boundary) is made possible by the technique of $\mathcal{A}$-harmonic approximation (for details we refer to [10] and, for a more general form in the setting of geometric measure theory [13]). Our main goal, the Hölder continuity of u on a relative neighbourhood of a given boundary point x_0 where the excess is small, can then be established in the model case of the upper half unit ball (which is sufficient for the general situation) by proceeding similarly to existing papers concerned with boundary regularity (see e. g. [20]); here, we mention that our boundary excess describes only the behaviour of the normal derivative of u but appropriate boundary versions of the Caccioppoli and the Poincaré inequality allow us to control the full derivative of u. Hence, the excess at the boundary is now used to get control over the corresponding excess quantity on balls in the interior within a neighbourhood of x_0, and a combination with the interior case then yields the desired regularity result.

In the sequel, we set our main focus on the treatment of the boundary situation, but, as indicated in [15], the methods also apply to cover inhomogeneous elliptic system and almost minimizers of integral functionals.

2 Preliminaries

We start with some remarks on the notation used below: we write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ and $B^+_\rho(x_0) = \{x \in \mathbb{R}^n : x_n > 0, |x - x_0| < \rho\}$ for a ball or an upper half-ball, respectively, centred on a point x_0 ($\in \mathbb{R}^{n-1} \times \{0\}$ in the latter case) with radius $\rho > 0$. Sometimes it will be convenient to treat the n-th component of $x \in \mathbb{R}^n$ separately; therefore, we set $x = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1})$. Furthermore, we write

$$\Gamma_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho,\, x_n = 0\},$$

for $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. In the case $x_0 = 0$ we set $B_\rho := B_\rho(0)$, $B := B_1$ as well as $B^+_\rho := B^+_\rho(0)$, $B^+ := B^+_1$ with $\Gamma_\rho := \Gamma_\rho(0)$, $\Gamma := \Gamma_1$. We also introduce the following notation for $W^{1,p}$-functions defined on some half-ball $B^+_\rho(x_0)$ and which vanish (in the sense of traces) on the flat part of the boundary:

$$W^{1,p}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N) := \{u \in W^{1,p}(B^+_\rho(x_0), \mathbb{R}^N) : u = 0 \text{ on } \Gamma_\rho(x_0)\}.$$

Let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure. For any bounded, measurable set $X \subset \mathbb{R}^n$ with $\mathcal{L}^n(X) =: |X| > 0$, we denote the mean value of a function $h \in L^1(X, \mathbb{R}^N)$ by $(h)_X = -\!\!\!\!\!\!\int_X h\,dx$, and, in particular, we use the abbreviation $(h)_{x_0,\rho}$ for the mean value on $B_\rho(x_0)$ or on $B^+_\rho(x_0)$, respectively. The constants c appearing in the different estimates will all be chosen greater than or equal to 1, and they may vary from line to line.

We consider a bounded domain Ω in $\mathbb{R}^n$, for some $n \geq 2$. The boundary of Ω is assumed to be of class C^1 with modulus of continuity $\tau(\cdot)$; this means that for every point $x_0 \in \partial\Omega$ there exist a radius $r > 0$ and a function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^1 such that (up to an isometry) Ω is locally represented by $\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > h(x')\}$. Thus we can locally straighten the boundary $\partial\Omega$ via a C^1-transformation $\mathcal{T}$ defined by $\mathcal{T}(x', x_n) = (x', x_n - h(x'))$.

We recall that u is a weak solution of (1.1) with boundary values g under the assumptions (1.2) if u is a $W^{1,p}(\Omega, \mathbb{R}^N)$-map such that

$$\int_\Omega a(x, u, Du)\, D\varphi\, dx = 0 \qquad \text{for every } \varphi \in W^{1,p}_0(\Omega, \mathbb{R}^N)$$

and if $u = g$ on $\partial\Omega$ in the sense of traces. Further, u is a local minimizers of the functional $\mathcal{F}[\cdot]$ with boundary values g under the assumptions (1.4) if u is a $W^{1,p}(\Omega, \mathbb{R}^N)$-map such that

$$\mathcal{F}[u] \leq \mathcal{F}[v] \qquad \text{for every } v \in u + W^{1,p}_0(\Omega, \mathbb{R}^N)$$

and if $u = g$ on $\partial\Omega$ in the sense of traces.

Firstly we recall a boundary version of Poincaré's inequality $W^{1,p}_\Gamma(B^+_R, \mathbb{R}^N)$-maps. The fact that u vanishes on Γ allows to estimate the integral over u by the integral of the normal derivative $D_n u$ only rather than the full derivative.

Lemma 2.1 ([4], Lemma 3.4): *For functions $u \in W^{1,p}_\Gamma(B^+_R(x_0), \mathbb{R}^N)$ with $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ there holds:*

$$\int_{B^+_R(x_0)} |u|^p\, dx \leq \frac{R^p}{p} \int_{B^+_R(x_0)} |D_n u|^p\, dx.$$

We next want to state results from the linear theory. Firstly, we recall the following up-to-the-boundary version of the $\mathcal{A}$-harmonic approximation lemma:

Lemma 2.2 ([19], Lemma 2.4; [20], Lemma 2.3.): *Consider fixed positive ν and L, and n, $N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ with the following property: if $\mathcal{A}$ is a bilinear form on $\mathbb{R}^{nN}$ satisfying*

$$\nu\, |\xi|^2\, |\eta|^2 \leq \mathcal{A}(\xi \otimes \eta, \xi \otimes \eta) \leq L\, |\xi|^2\, |\eta|^2 \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$, and if $w \in W^{1,2}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ (for some $\rho > 0$, $x_0 \in \mathbb{R}^{n-1} \times \{0\}$) with $-\!\!\!\!\!\!\int_{B^+_\rho(x_0)} |Dw|^2\, dx \leq 1$ is approximately $\mathcal{A}$-harmonic in the sense that

$$\Big| -\!\!\!\!\!\!\int_{B^+_\rho(x_0)} \mathcal{A}(Dw, D\varphi)\, dx \Big| \leq \delta \sup_{B^+_\rho(x_0)} |D\varphi|$$

for all $\varphi \in C_0^1(B_\rho^+(x_0), \mathbb{R}^N)$, *then there exists an* $\mathcal{A}$*-harmonic function* $h \in W_\Gamma^{1,2}(B_\rho^+(x_0), \mathbb{R}^N)$ *such that*

$$\fint_{B_\rho^+(x_0)} |Dh|^2\,dx \,\le\, 1 \quad \textit{and} \quad \rho^{-2} \fint_{B_\rho^+(x_0)} |w-h|^2\,dx \,\le\, \varepsilon\,.$$

That $\mathcal{A}$-harmonic maps are indeed smooth, is the statement of the next lemma:

Lemma 2.3 ([19], Theorem 2.3): *Consider fixed positive* ν *and* L, *and* n, $N \in \mathbb{N}$ *with* $n \ge 2$. *Then there exists a constant* c_h *depending only on* n, N, L *and* ν *such that for every bilinear form* $\mathcal{A}$ *on* $\mathbb{R}^{nN}$ *with upper bound* L *and ellipticity constant* ν *and any* $\mathcal{A}$ *harmonic map* $h \in W_\Gamma^{1,2}(B_\rho^+(x_0), \mathbb{R}^N)$ *(for some* $\rho > 0$, $x_0 \in \mathbb{R}^{n-1} \times \{0\}$*) there holds:*

$$\rho^{-2} \sup_{B_{\rho/2}^+(x_0)} |Dh|^2 + \sup_{B_{\rho/2}^+(x_0)} |D^2 h|^2 \,\le\, c_h\, \rho^{-2} \fint_{B_\rho^+(x_0)} |Dh|^2\,dx\,.$$

Given a functions $u \in L^2(B_\rho^+(x_0), \mathbb{R}^N)$ we denote by $P_{x_0,\rho}^+$ the unique function minimizing the functional $P \mapsto \int_{B_\rho^+(x_0)} |u-P|^2\,dx$ amongst all functions P of the form $P(x) = Q\,x_n$ for some $Q \in \mathbb{R}^N$. $P_{x_0,\rho}^+ = Q_{x_0,\rho}^+\,x_n$ is then given via

$$Q_{x_0,\rho}^+ \,:=\, c_Q(n)\,\rho^{-2} \fint_{B_\rho^+(x_0)} u(x)\,x_n\,dx$$

for $c_Q = (\fint_{B_1^+} x_n^2\,dx)^{-1} = n+2$. The following lemma provides explicit estimates for $Q_{x_0,\rho}^+$ similar to [24], Lemma 2:

Lemma 2.4: *Let* $u \in W_\Gamma^{1,2}(B_\rho^+(x_0), \mathbb{R}^N)$ *for some* $x_0 \in \Gamma$, $0 < \theta \le 1$, $P_{x_0,\rho}^+$ *the polynomial defined above. Then the following estimates hold:*

$$(i) \quad \big|Q_{x_0,\theta\rho}^+ - Q_{x_0,\rho}^+\big|^2 \,\le\, c(n)\,(\theta\rho)^{-2} \fint_{B_{\theta\rho}^+(x_0)} |u - P_{x_0,\rho}^+|^2\,dx$$

$$(ii) \quad \big|Q_{x_0,\rho}^+ - \xi\big|^2 \,\le\, c(n) \fint_{B_\rho^+(x_0)} |D_n u - \xi|^2\,dx$$

where $\xi \in \mathbb{R}^N$.

Proof: Using both the definitions of $Q_{x_0,\theta\rho}^+$ and of the constant c_Q and Hölder's inequality we compute

$$\begin{aligned}
\big|Q_{x_0,\theta\rho}^+ - Q_{x_0,\rho}^+\big|^2 &= \Big|c_Q\,(\theta\rho)^{-2} \fint_{B_{\theta\rho}^+(x_0)} u(x)\,x_n\,dx - Q_{x_0,\rho}^+\Big|^2 \\
&= \Big|c_Q\,(\theta\rho)^{-2} \fint_{B_{\theta\rho}^+(x_0)} \big[u(x)\,x_n - Q_{x_0,\rho}^+\,x_n^2\big]\,dx\Big|^2 \\
&\le c_Q^2\,(\theta\rho)^{-4} \fint_{B_{\theta\rho}^+(x_0)} \big|u(x) - Q_{x_0,\rho}^+\,x_n\big|^2\,dx \fint_{B_{\theta\rho}^+(x_0)} x_n^2\,dx \\
&\le c_Q\,(\theta\rho)^{-2} \fint_{B_{\theta\rho}^+(x_0)} \big|u(x) - Q_{x_0,\rho}^+\,x_n\big|^2\,dx\,.
\end{aligned}$$

For the second inequality we proceed analogously and apply at the end the Poincaré-inequality in the zero-boundary-data-version in order to derive:

$$\begin{aligned}
\big|Q_{x_0,\rho}^+ - \xi\big|^2 &= \Big|c_Q\,\rho^{-2} \fint_{B_\rho^+(x_0)} \big[u(x)\,x_n - \xi\,x_n^2\big]\,dx\Big|^2 \\
&\le c_Q\,\rho^{-2} \fint_{B_\rho^+(x_0)} \big|u(x) - \xi\,x_n\big|^2\,dx \\
&\le c_P\,c_Q \fint_{B_\rho^+(x_0)} \big|D_n u(x) - \xi\big|^2\,dx\,.
\end{aligned}$$

□

Moreover, we will need an iteration result (cf. [18], Lemma 7.3):

Lemma 2.5: *Let $\varphi\colon [0,\rho] \to \mathbb{R}$ be a positive non-decreasing function satisfying*

$$\varphi(\theta^{k+1}\rho) \;\le\; \theta^{\gamma}\,\varphi(\theta^{k}\rho) + B\,(\theta^{k}\rho)^{n}$$

for every $k \in \mathbb{N}$, where $\theta \in (0,1)$ and $\gamma \in (0,n)$. Then there exists a constant c depending only on n, θ and γ such that for every $t \in (0,\rho]$ the following holds:

$$\varphi(t) \;\le\; c\left[\Big(\frac{t}{\rho}\Big)^{\gamma}\varphi(\rho) + B\,t^{\gamma}\right].$$

For the proof of the characterization of regular boundary points we will concentrate on the model situation of a half ball and we will make use of a slight modification of Campanato's integral characterization of Hölder-continuity up to the boundary:

Theorem 2.6 ([20], Theorem 2.3): *Consider $n \in \mathbb{N}$, $n \ge 2$ and $x_0 \in \mathbb{R}^{n-1}\times\{0\}$. Suppose that there are positive constants $\alpha \in (0,1]$, $\kappa > 0$ such that, for some $v \in L^2(B^+_{6R}(x_0))$, there holds the following:*

$$\inf_{\mu\in\mathbb{R}}\Big\{ -\!\!\!\!\!\!\int_{B^+_\rho(y)} |v-\mu|^2\,dx\Big\} \;\le\; \kappa^2\Big(\frac{\rho}{R}\Big)^{2\alpha}$$

for all $y \in \Gamma_{2R}(x_0)$ and $\rho \le 4R$; and

$$\inf_{\mu\in\mathbb{R}}\Big\{ -\!\!\!\!\!\!\int_{B_\rho(y)} |v-\mu|^2\,dx\Big\} \;\le\; \kappa^2\Big(\frac{\rho}{R}\Big)^{2\alpha}$$

for all $y \in B^+_{2R}(x_0)$ with $B_\rho(y) \subset B^+_{2R}(x_0)$. Then there exists a Hölder-continuous representative $\bar v$ of v on $\overline{B^+_R}(x_0)$, and for $\bar v$ there holds: $|\bar v(x) - \bar v(z)| \le c\,\kappa\,\big(\frac{|x-z|}{R}\big)^{\alpha}$ for all $x, z \in \overline{B^+_R}(x_0)$, for a constant c depending only on n and α.

3 Elliptic Systems

3.1 Decay estimate

The first step in proving a regularity theorem for solutions u of elliptic systems is to establish a suitable reverse-Poincaré or Caccioppoli inequality. In the case of continuous coefficients $a(\cdot,\cdot,\cdot)$ with respect to the first two variables (instead of Hölder or Dini continuous coefficients) we have to state here the exact dependency for some linear disturbance of the weak solution u for the system

$$\operatorname{div} a(\,\cdot\,, u, Du) \;=\; 0 \qquad \text{in } B^+. \tag{3.1}$$

Lemma 3.1 (Caccioppoli inequality): *Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a weak solution to (3.1) under the assumptions (1.2), $\xi \in \mathbb{R}^N$ and $B^+_\rho(x_0)$, $x_0 \in \Gamma$, $\rho < 1 - |x_0|$ be an upper half ball. Then there exists a constant $c = c(n,N,p,L,\nu)$ such that*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} &\big[(1+|\xi|)^{p-2}|Du - \xi\otimes e_n|^2 + |Du-\xi\otimes e_n|^p\big]\,dx \\
&\le c\,-\!\!\!\!\!\!\int_{B^+_\rho(x_0)} \Big[(1+|\xi|)^{p-2}\Big|\frac{u-\xi x_n}{\rho}\Big|^2 + \Big|\frac{u-\xi x_n}{\rho}\Big|^p\Big]\,dx \\
&\qquad + c\,(1+|\xi|)^p -\!\!\!\!\!\!\int_{B^+_\rho(x_0)} \big[\omega(\rho^2) + \omega(|u|^2) + \omega(\rho^2\,|\xi|^2)\big]\,dx\,.
\end{aligned}$$

Proof: Since there holds $u - \xi\,x_n = 0$ on Γ, the map $\eta^p(u-\xi\,x_n)$ with $\eta \in C^\infty_0(B_\rho(x_0),[0,1])$ a standard cut-off function may be taken as a test function in the weak formulation of (3.1). Now we refer to the proof of the Caccioppoli inequality for the interior case, see [15], Proposition 3.1. □

In the next step we define the excess functionals analogously to [15], Section 3.2, in a boundary version (i. e., replacing full balls by half balls and restricting ourselves to the mean value of the normal derivative instead of the full derivative of u): For any half-ball $B_\rho^+(x_0) \subset B^+$ with $x_0 \in \Gamma$, a fixed function $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$ we define the Campanato-type excess

$$C(x_0, \rho) := \fint_{B_\rho^+(x_0)} \left[\frac{|Du - (D_n u)_{x_0,\rho} \otimes e_n|^2}{(1 + |(D_n u)_{x_0,\rho}|)^2} + \frac{|Du - (D_n u)_{x_0,\rho} \otimes e_n|^p}{(1 + |(D_n u)_{x_0,\rho}|)^p} \right] dx \,,$$

the Morrey-type excess

$$M(x_0, \rho) := \rho^\beta \fint_{B_\rho^+(x_0)} |D_n u|^2 \, dx \qquad \text{for } \beta \in (0, 2)$$

and finally the excess functional

$$E(x_0, \rho) := C(x_0, \rho) + \sqrt{\omega(M(x_0, \rho))} + \sqrt{\omega(\rho)} \,.$$

The next proposition provides a suitable decay estimate, under the assumption that the excess $E(x_0, \rho)$ and the radius ρ are sufficiently small, and will be an essential tool for the iteration later on.

Proposition 3.2 (cf. [15], Propostion 3.2): *For each $\beta \in (0, 2)$ and $\theta \in (0, \frac{1}{4})$ there exist two positive numbers*

$$\varepsilon_0 = \varepsilon_0\big(n, N, p, \nu, L, \beta, \theta, \mu(\cdot)\big) > 0 \qquad \text{and} \qquad \varepsilon_1(n, p, \beta, \theta) > 0 \tag{3.2}$$

such that the following is true: If $u \in W^{1,p}(B^+, \mathbb{R}^N)$ is a weak solution to (3.1) under the assumptions (1.2), and if $B_\rho^+(x_0)$, $x_0 \in \Gamma$, $\rho < 1 - |x_0|$, is a half ball satisfying the smallness conditions

$$E(x_0, \rho) < \varepsilon_0 \qquad \text{and} \qquad \rho < \varepsilon_1 \,, \tag{3.3}$$

then we have

$$C(x_0, \theta\rho) \le c_* \, \theta^2 \, E(x_0, \rho) \tag{3.4}$$

for a constant c_* depending only on n, N, p, ν and L.

Proof: In the first step, we deduce an approximate $\mathcal{A}$-harmonicity result following the estimates in the proof of [15], Proposition 3.2, Step 1 and obtain: for every $B_\rho^+(x_0)$, $x_0 \in \Gamma$, $\rho < 1 - |x_0|$, and all functions $\varphi \in C_0^1(B_\rho^+(x_0), \mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B_\rho^+(x_0))} \le 1$ there holds

$$\Big| \fint_{B_\rho^+(x_0)} D_z a\big(x_0, 0, (D_n u)_{x_0,\rho} \otimes e_n\big)\big(Du - (D_n u)_{x_0,\rho} \otimes e_n, D\varphi\big) \, dx \Big|$$
$$\le c \, \big(1 + |(D_n u)_{x_0,\rho}|\big)^{p-1} \big[\mu\big(\sqrt{E(x_0,\rho)}\big)^{\frac{1}{p}} + \sqrt{E(x_0,\rho)}\big] \, \big[E(x_0,\rho)^{\frac{1}{2}} + E(x_0,\rho)^{1-\frac{1}{p}}\big] \,,$$

and the constant c depends only on n, N, p and L (note $\mu(\cdot) \le 1$ and the fact that $u = 0$ on Γ in order to apply the Poincaré inequality in the boundary version). Now we define

$$\mathcal{A} := \frac{D_z a(x_0, 0, (D_n u)_{x_0,\rho} \otimes e_n)}{(1 + |(D_n u)_{x_0,\rho}|)^{p-2}} \,,$$
$$w := \frac{u - (D_n u)_{x_0,\rho} \, x_n}{\sqrt{E(x_0,\rho)} \, (1 + |(D_n u)_{x_0,\rho}|)} \,,$$
$$H(t) := [\mu(\sqrt{t})^{\frac{1}{p}} + \sqrt{t}] \, [1 + t^{\frac{1}{2} - \frac{1}{p}}] \,; \tag{3.5}$$

we note here that $\mathcal{A}$ fulfills condition (2.1), i. e., it is bounded from below and above, and further, by the definition of the excess functional $E(x_0, \rho)$, there holds $\fint_{B_\rho^+(x_0)} |Dw|^2 \, dx \le 1$. These definitions enable us to rewrite the previous estimate after a rescaling argument:

$$\Big| \fint_{B_\rho^+(x_0)} \mathcal{A}\left(Dw, D\varphi\right) dx \Big| \le c_1(n, N, p, L) \, H(E(x_0, \rho)) \, \|D\varphi\|_{L^\infty(B_\rho^+(x_0))}$$

for all $\varphi \in C_0^1(B_\rho^+(x_0), \mathbb{R}^N)$. For $\varepsilon > 0$ to be determined later, we now take $\delta = \delta(n, N, \nu, L, \varepsilon)$ to be the corresponding constant from the $\mathcal{A}$-harmonic approximation Lemma 2.2. Provided that the smallness condition

$$H(E(x_0, \rho)) \le \delta / c_1 \tag{SC.1}$$

holds, we find, according to Lemma 2.2, an $\mathcal{A}$-harmonic map $h \in W_\Gamma^{1,2}(B_\rho^+(x_0), \mathbb{R}^N)$ such that

$$\fint_{B_\rho^+(x_0)} |Dh|^2 \, dx \le 1 \quad \text{and} \quad \rho^{-2} \fint_{B_\rho^+(x_0)} |w - h|^2 \, dx \le \varepsilon \,, \tag{3.6}$$

and by Lemma 2.3 on $\mathcal{A}$-harmonic maps h is indeed smooth and satisfies, due to the last line, the estimate $\sup_{B_{\rho/2}^+(x_0)} |D^2 h|^2 \le c_h(n, N, \nu, L) \rho^{-2}$. We now consider $\theta \in (0, \frac{1}{4})$ fixed, to be specified later, and deduce from Taylor's theorem (keep in mind $h = 0$ on $\Gamma_\rho(x_0)$):

$$\begin{aligned} \sup_{x \in B_{2\theta\rho}^+(x_0)} \big| h(x) - h(x_0) - Dh(x_0)(x - x_0) \big|^2 &= \sup_{x \in B_{2\theta\rho}^+(x_0)} \big| h(x) - D_n h(x_0) x_n \big|^2 \\ &\le c_h \, \rho^{-2} \, (2\theta\rho)^4 = c \, \theta^4 \rho^2 \,, \end{aligned} \tag{3.7}$$

and the constant c depends only on n, N, ν and L. (3.6) and (3.7) now ensure that

$$\begin{aligned} &(2\theta\rho)^{-2} \fint_{B_{2\theta\rho}^+(x_0)} |w(x) - D_n h(x_0) x_n|^2 \, dx \\ &\le 2 \, (2\theta\rho)^{-2} \Big(\fint_{B_{2\theta\rho}^+(x_0)} |w(x) - h(x)|^2 \, dx + \fint_{B_{2\theta\rho}^+(x_0)} |h(x) - D_n h(x_0) x_n|^2 \, dx \Big) \\ &\le 2 \, (2\theta\rho)^{-2} \big((2\theta)^{-n} \rho^2 \varepsilon + c \, \theta^4 \rho^2 \big) \le c(n, N, \nu, L) \, \theta^2 \end{aligned}$$

where we have chosen $\varepsilon = \theta^{n+4}$ in the last inequality. By the definition of w we easily conclude

$$\begin{aligned} (2\theta\rho)^{-2} \fint_{B_{2\theta\rho}^+(x_0)} \big| u(x) - (D_n u)_{x_0,\rho} \, x_n - \sqrt{E(x_0, \rho)} \, (1 + |(D_n u)_{x_0,\rho}|) \, D_n h(x_0) x_n \big|^2 \, dx \\ \le c(n, N, \nu, L) \, \theta^2 \, (1 + |(D_n u)_{x_0,\rho}|)^2 E(x_0, \rho) \,. \end{aligned} \tag{3.8}$$

Denoting by $Q_{x_0,2\theta\rho}^+$ the value minimizing the functional $Q \mapsto \int_{B_\rho^+(x_0)} |u - Q \, x_n|^2 \, dx$ amongst all $Q \in \mathbb{R}^N$, and $P_{x_0,\rho}^+ = Q_{x_0,\rho}^+ \, x_n$ we obtain from the last inequality that

$$(2\theta\rho)^{-2} \fint_{B_{2\theta\rho}^+(x_0)} \big| u - P_{x_0,2\theta\rho}^+ \big|^2 \, dx \le c \, \theta^2 \, (1 + |(D_n u)_{x_0,\rho}|)^2 E(x_0, \rho) \tag{3.9}$$

with c still depending only on n, N, ν and L. To derive a corresponding estimate with exponent p instead of 2 in (3.9) (in the case $p > 2$) we use an interpolation argument. To this end we define p^* the usual Sobolev-Exponent (i. e. $p^* = \frac{np}{n-p}$ if $p < n$ and $p^* > p$ arbitrary if $p \ge n$) and choose $t \in (0, 1)$ such that $\frac{1}{p} = \frac{1-t}{2} + t \frac{1}{p^*}$ is satisfied. Using the inequalities of Hölder and of Sobolev-Poincaré and (3.9) we infer

$$\begin{aligned} &\fint_{B_{2\theta\rho}^+(x_0)} \big| u - P_{x_0,2\theta\rho}^+ \big|^p \, dx \\ &\le \Big(\fint_{B_{2\theta\rho}^+(x_0)} \big| u - P_{x_0,2\theta\rho}^+ \big|^2 \, dx \Big)^{(1-t)\frac{p}{2}} \Big(\fint_{B_{2\theta\rho}^+(x_0)} \big| u - P_{x_0,2\theta\rho}^+ \big|^{p^*} \, dx \Big)^{t \, \frac{p}{p^*}} \\ &\le c \, \rho^p \, \theta^{(2-t)p} (1 + |(D_n u)_{x_0,\rho}|)^{p(1-t)} \, E(x_0, \rho)^{(1-t)\frac{p}{2}} \Big(\fint_{B_{2\theta\rho}^+(x_0)} \big| Du - DP_{x_0,2\theta\rho}^+ \big|^p \, dx \Big)^t \end{aligned}$$

where c depends now on n, N, p, ν and L. Applying Minkowski, Lemma 2.4 and the Poincaré-inequality at the boundary (keep in mind that $P^+_{x_0,\rho} = Q^+_{x_0,\rho} x_n$ vanishes on Γ) we obtain for the latter integral

$$\begin{aligned}
&\Big(\fint_{B^+_{2\theta\rho}(x_0)} \big|Du - DP^+_{x_0,2\theta\rho}\big|^p \, dx \Big)^{\frac{1}{p}} \\
&\le \Big(\fint_{B^+_{2\theta\rho}(x_0)} \big|Du - (D_n u)_{x_0,\rho} \otimes e_n\big|^p \, dx \Big)^{\frac{1}{p}} + \big|(D_n u)_{x_0,\rho} - Q^+_{x_0,2\theta\rho}\big| \\
&\le \Big(\fint_{B^+_{2\theta\rho}(x_0)} \big|Du - (D_n u)_{x_0,\rho} \otimes e_n\big|^p \, dx \Big)^{\frac{1}{p}} + c(n) \Big(\fint_{B^+_{2\theta\rho}(x_0)} |D_n u - (D_n u)_{x_0,\rho}|^p \, dx \Big)^{\frac{1}{p}} \\
&\le c(n,p)\, \theta^{-\frac{n}{p}} \Big(\fint_{B^+_{\rho}(x_0)} \big|Du - (D_n u)_{x_0,\rho} \otimes e_n\big|^p \, dx \Big)^{\frac{1}{p}} \\
&= c(n,p)\, \theta^{-\frac{n}{p}} \, (1 + |(D_n u)_{x_0,\rho}|) \, E(x_0,\rho)^{\frac{1}{p}} \, .
\end{aligned}$$

Hence, inserting this in the inequality above, we get

$$(2\theta\rho)^{-p} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^p \, dx \;\le\; c\, \theta^{(1-t)p-nt} \, E(x_0,\rho)^{(1-t)\frac{p-2}{2}} \, (1 + |(D_n u)_{x_0,\rho}|)^p \, E(x_0,\rho) \, .$$

We now assume the additional smallness assumption

$$E(x_0,\rho) \;\le\; \theta^{\frac{2[(t-1)p+tn+2]}{(1-t)(p-2)}} \qquad \text{if } p > 2 \, . \tag{SC.2}$$

One easily checks that this condition becomes void when p approaches 2 from above, because then the exponent becomes (in dependency of the dimension n) negative. We finally arrive at

$$(2\theta\rho)^{-p} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^p \, dx \;\le\; c\, \theta^2 \, (1 + |(D_n u)_{x_0,\rho}|)^p E(x_0,\rho) \tag{3.10}$$

with a constant $c = c(n, N, p, \nu, L)$. In the end, we want to produce mainly via Caccioppoli's inequality and dividing by $(1 + |(D_n u)_{x_0,\rho}|)^p$ the Campanato Excess quantity $C(x_0, \theta\rho)$. Hence we have to estimate $(D_n u)_{x_0,\widetilde{\rho}}$ in terms of $(D_n u)_{x_0,\theta\rho}$ in the following form:

$$1 + |(D_n u)_{x_0,\widetilde{\rho}}| \;\le\; 2 \big(1 + |(D_n u)_{x_0,\theta\rho}|\big) \qquad \text{for all } \widetilde{\rho} \in [\theta\rho, \rho] \, . \tag{3.11}$$

To this end, we compute for all $\widetilde{\rho} \in [\theta\rho, \rho]$ via the minimizing property of the meanvalue:

$$\begin{aligned}
1 + |(D_n u)_{x_0,\widetilde{\rho}}| &\le 1 + |(D_n u)_{x_0,\theta\rho} - (D_n u)_{x_0,\widetilde{\rho}}| + |(D_n u)_{x_0,\theta\rho}| \\
&\le \Big(\frac{\widetilde{\rho}}{\theta\rho}\Big)^{\frac{n}{2}} \Big(\fint_{B^+_{\widetilde{\rho}}(x_0)} |D_n u - (D_n u)_{x_0,\widetilde{\rho}}|^2 \, dx \Big)^{\frac{1}{2}} + 1 + |(D_n u)_{x_0,\theta\rho}| \\
&\le \theta^{-\frac{n}{2}} \Big(\fint_{B^+_{\rho}(x_0)} |D_n u - (D_n u)_{x_0,\rho}|^2 \, dx \Big)^{\frac{1}{2}} + 1 + |(D_n u)_{x_0,\theta\rho}| \\
&\le \theta^{-\frac{n}{2}} \sqrt{E(x_0,\rho)} \, (1 + |(D_n u)_{x_0,\rho}|) + 1 + |(D_n u)_{x_0,\theta\rho}| \, .
\end{aligned}$$

Therefore if we assume the smallness condition

$$2 \sqrt{E(x_0,\rho)} \;\le\; \theta^{\frac{n}{2}} \tag{SC.3}$$

we obtain firstly by absorption in a standard way the result (3.11) in the special case $\widetilde{\rho} = \rho$. Secondly we consider $\widetilde{\rho} \in [\theta\rho, \rho]$ arbitrary and now take into account (3.11) for $\widetilde{\rho} = \rho$ to infer (3.11) for all possible choices of $\widetilde{\rho}$. Hence we may rewrite (3.9) and (3.10) in the following form:

$$\left\{ \begin{aligned}
&(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^2 \, dx \;\le\; c\, \theta^2 \, (1 + |(D_n u)_{x_0,\theta\rho}|)^2 \, E(x_0,\rho) \\
&(2\theta\rho)^{-p} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^p \, dx \;\le\; c\, \theta^2 \, (1 + |(D_n u)_{x_0,\theta\rho}|)^p \, E(x_0,\rho)
\end{aligned} \right. \tag{3.12}$$

for a constant c depending only on n, N, p, ν and L.

In the last step we have to derive the full decay estimate for the Campanato-type excess $C(x_0,\rho)$. We apply the Caccioppoli inequality (Lemma 3.1) with the choice $\xi = Q^+_{2\theta\rho}$ to derive

$$\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^2 + \big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^p\big]\,dx \\
&\qquad \le c \fint_{B^+_{2\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\Big|\frac{u-Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^2 + \Big|\frac{u-Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^p\Big]\,dx \\
&\qquad\qquad + c\,(1+|Q^+_{x_0,2\theta\rho}|)^p \fint_{B^+_{2\theta\rho}(x_0)} \big[\omega(\rho^2)+\omega(|u|^2)+\omega(\rho^2\,|Q^+_{x_0,2\theta\rho}|^2)\big]\,dx
\end{aligned} \tag{3.13}$$

with $c = c(n,N,p,\nu,L)$. We will now get the expressions comprising ω and $Q^+_{x_0,2\theta\rho}$, which appear in the last formula, under control. Using Lemma 2.4 we have (keep in mind $\rho \le 1$):

$$\begin{aligned}
\rho^2\,|Q^+_{x_0,2\theta\rho}|^2 &\le 2\,\rho^2\,\big(|Q^+_{x_0,2\theta\rho} - (D_nu)_{x_0,2\theta\rho}|^2 + |(D_nu)_{x_0,2\theta\rho}|^2\big) \\
&\le c\,\rho^2\Big(\fint_{B^+_{2\theta\rho}(x_0)} |D_nu - (D_nu)_{x_0,\rho}|^2\,dx + \fint_{B^+_{2\theta\rho}(x_0)} |D_nu|^2\,dx\Big) \\
&\le c\,\theta^{-n}\rho^2\Big((1+|(D_nu)_{x_0,\rho}|)^2 \fint_{B^+_{\rho}(x_0)} \frac{|D_nu-(D_nu)_{x_0,\rho}|^2}{(1+|(D_nu)_{x_0,\rho}|)^2}\,dx + \fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx\Big) \\
&\le c_2(n)\,\theta^{-n}\Big(\rho^2\,E(x_0,\rho) + \rho^2\,E(x_0,\rho)\fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx + \rho^2\fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx\Big) \\
&\le \rho^2 + \frac{\rho^2}{2}\fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx + \frac{\rho^\beta}{2}\fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx \\
&\le \rho + M(x_0,\rho)
\end{aligned} \tag{3.14}$$

provided that the smallness conditions

$$E(x_0,\rho) \le \frac{\theta^n}{2c_2} \qquad\text{and}\qquad \rho \le \Big(\frac{\theta^n}{2c_2}\Big)^{\frac{1}{2-\beta}} \tag{SC.4}$$

hold true. We further have, by possibly increasing the value of c_2, but keeping the dependency on only n, via Poincaré's inequality (Lemma 2.1) and (SC.4)

$$\fint_{B^+_{2\theta\rho}(x_0)} |u|^2\,dx \le (2\theta)^{-n}\fint_{B^+_{\rho}(x_0)}|u|^2\,dx \le c_2\,\theta^{-n}\rho^2\fint_{B^+_{\rho}(x_0)}|D_nu|^2\,dx \le M(x_0,\rho)\,;$$

combined with the concavity of $\omega(\cdot)$ this implies immediately

$$\fint_{B^+_{2\theta\rho}(x_0)} \omega(|u|^2)\,dx \le \omega\Big(\fint_{B^+_{2\theta\rho}(x_0)}|u|^2\,dx\Big) \le \omega(M(x_0,\rho))$$

meaning that we have (note that ω is sublinear)

$$\fint_{B^+_{2\theta\rho}(x_0)} \big[\omega(\rho^2)+\omega(|u|^2)+\omega(\rho^2\,|Q^+_{x_0,2\theta\rho}|^2)\big]\,dx \le 2\,\omega(\rho) + 2\,\omega(M(x_0,\rho))\,. \tag{3.15}$$

Now it still remains to bound $Q^+_{x_0,2\theta\rho}$ in terms of $(D_nu)_{x_0,\theta\rho}$. Using again Lemma 2.4, the smallness condition (SC.4) (possibly increasing c_2), and (3.11) with $\widetilde\rho = 2\theta\rho$ and $\widetilde\rho = \rho$, respectively, we see that

$$\begin{aligned}
|Q^+_{x_0,2\theta\rho}| &\le |Q^+_{x_0,2\theta\rho} - (D_nu)_{x_0,2\theta\rho}| + |(D_nu)_{x_0,2\theta\rho}| \\
&\le \frac{c_2(n)\,(1+|(D_nu)_{x_0,\theta\rho}|)}{\theta^n}\fint_{B^+_{\rho}(x_0)} \frac{|D_nu-(D_nu)_{x_0,\rho}|^2}{(1+|(D_nu)_{x_0,\rho}|)^2}\,dx + 2\,\big(1+|(D_nu)_{x_0,\theta\rho}|\big) \\
&\le 3\,\big(1+|(D_nu)_{x_0,\theta\rho}|\big)\,.
\end{aligned} \tag{3.16}$$

Hence, if we employ the following smallness estimate $\sqrt{\omega(M(x_0,\rho))} + \sqrt{\omega(\rho)} \le E(x_0,\rho) \le \theta^n \le \theta^2$ being derived from the latter smallness condition (SC.4) and combine this with (3.16) and (3.15), we may estimate the second integral on the right-hand side of (3.13) by

$$(1+|Q^+_{x_0,2\theta\rho}|)^p \fint_{B^+_{2\theta\rho}(x_0)} \big[\omega(\rho^2)+\omega(|u|^2)+\omega(\rho^2\,|Q^+_{x_0,2\theta\rho}|^2)\big]\,dx \\ \le\, c\,\theta^2\,\big(1+|(D_nu)_{x_0,\theta\rho}|\big)^p\,E(x_0,\rho)\,. \tag{3.17}$$

Next we turn to the left-hand side of (3.13) and find for $p>2$ using Young's inequality and Lemma 2.4

$$\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \big(1+|(D_nu)_{x_0,\theta\rho}|\big)^{p-2}\,|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^2\,dx\\
&\le\, c(p)\fint_{B^+_{\theta\rho}(x_0)} \big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\,|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^2+|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^p\big]\,dx\\
&\qquad + c(p)\,|(D_nu)_{x_0,\theta\rho}-Q^+_{x_0,\theta\rho}|^p + c(p)\,|Q^+_{x_0,\theta\rho}-Q^+_{x_0,2\theta\rho}|^p\\
&\le\, c(n,p)\fint_{B^+_{\theta\rho}(x_0)} \big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\,|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^2+|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^p\big]\,dx\\
&\qquad + c(n,p)\,(2\theta\rho)^{-p}\fint_{B^+_{2\theta\rho}(x_0)} |u-P^+_{2\theta\rho}|^p\,dx\,.
\end{aligned} \tag{3.18}$$

If we take into account the following two inequalities

$$\fint_{B^+_{\theta\rho}(x_0)} |Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^p\,dx \,\le\, 2^p\fint_{B^+_{\theta\rho}(x_0)} |Du-Q^+_{x_0,2\theta\rho}\otimes e_n|^p\,dx$$

and

$$\fint_{B^+_{\theta\rho}(x_0)} |Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^2\,dx \,\le\, \fint_{B^+_{\theta\rho}(x_0)} |Du-Q^+_{x_0,2\theta\rho}\otimes e_n|^2\,dx\,,$$

we further calculate combining (3.18) with (3.13) and (3.17)

$$\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \big[\big(1+|(D_nu)_{x_0,\theta\rho}|\big)^{p-2}\,|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^2+|Du-(D_nu)_{x_0,\theta\rho}\otimes e_n|^p\big]\,dx\\
&\le\, c\fint_{B^+_{2\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\Big|\frac{u-Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^2+\Big|\frac{u-Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^p\Big]\,dx\\
&\qquad + c\,\theta^2\,\big(1+|(D_nu)_{x_0,\theta\rho}|\big)^p\,E(x_0,\rho)\\
&\le\, c_*\,\theta^2\,\big(1+|(D_nu)_{x_0,\theta\rho}|\big)^p\,E(x_0,\rho)
\end{aligned} \tag{3.19}$$

where we took advantage of (3.12) and (3.16) in the last inequality and where the constant c_* depends only on n,N,p,ν and L. Dividing both sides by $(1+|(D_nu)_{x_0,\theta\rho}|)^p$ and taking into account the definition of $C(x_0,\theta\rho)$ this is exactly the desired excess decay estimate stated in the proposition provided that all smallness conditions (SC.1), (SC.2) and (SC.4) hold true (observe that the smallness assumption (SC.3) is weaker than (SC.4)). The dependency of the constants ε_0 and ε_1 claimed in (3.2) is now obtained by taking into consideration the dependencies in all the smallness conditions needed within the proof. □

3.2 Proof of Theorem 1.1

In what follows we are going to combine the results concerning the decay estimates of the interior situation in [15], Proposition 3.2, and the boundary situation, Proposition 3.2, in which the constants c_* and smallness parameters ε_0 and ε_1 have the same dependencies (at least for the model situation). Therefore, we may assume without loss of generality that both propositions are valid for the same set of parameters.

Step 1: Choice of the constants. We fix $\beta\in(0,2)$ and $\alpha\in(0,1)$. We choose $\gamma=\gamma(\alpha)\in(n-2,n)$ such that

$$\alpha \,=\, 1-\frac{n-\gamma}{2} \tag{3.20}$$

and $\theta \in (0, \frac{1}{4})$ such that

$$\theta \, := \, \min \Big\{ \Big(\frac{1}{4}\Big)^{\frac{1}{\beta}}, \Big(\frac{1}{4\, c_*}\Big)^{\frac{1}{2}}, \Big(\frac{1}{4}\Big)^{\frac{1}{n-\gamma}} \Big\}, \tag{3.21}$$

for c_* being the constant according to Proposition 3.2. This choice fixes θ in dependency of n, N, p, ν, L, α and β. We further fix a constant ε_2 and an iteration quantity $\varepsilon_{it} \leq 1$:

$$\varepsilon_2 \, := \, \min \Big\{ \frac{\theta^n}{4}, \frac{\varepsilon_0}{4} \Big\}, \qquad \varepsilon_{it} \, := \, \frac{\theta^n}{4^{3p}\, 6^n} \tag{3.22}$$

where ε_0 appears in Proposition 3.2, and therefore they depend on n, N, p, ν, L, α and β, and ε_2 additionally on $\mu(\cdot)$. Next, we fix $\delta_1 > 0$ such that

$$\sqrt{\omega(\delta_1)} \, < \, \varepsilon_{it}\, \varepsilon_2 \tag{3.23}$$

(note that this implies due to the monotonicity of ω that $\sqrt{\omega(t)} < \varepsilon_{it}\varepsilon_2$ whenever $t \in [0, \delta_1]$). This fixes δ_1 in dependency of $n, N, p, \nu, L, \alpha, \beta$, $\mu(\cdot)$ and $\omega(\cdot)$. Lastly we define the maximally admissible radius

$$\rho_m \, := \, \min \big\{ \delta_1^{\frac{1}{\beta}}, \delta_1, \varepsilon_1 \big\} \, > \, 0\,. \tag{3.24}$$

Here, ε_1 is the radius from Proposition 3.2 and ρ_m depends on $n, N, p, \nu, L, \alpha, \beta$, $\mu(\cdot)$ and $\omega(\cdot)$. For what follows we will always assume $\rho \leq \rho_m < 1$.

Step 2: An almost BMO-estimate. We now consider a boundary point $x_0 \in \Gamma$ and a radius $\rho \leq \min\{\rho_m, 1 - |x_0|\}$ for which

$$C(x_0, \rho) \, < \, \widetilde{\varepsilon}_{it}\, \varepsilon_2 \qquad \text{and} \qquad M(x_0, \rho) \, < \, \widetilde{\varepsilon}_{it}\, \delta_1 \tag{3.25}$$

is satisfied for some iteration parameter $\widetilde{\varepsilon}_{it} \in [\varepsilon_{it}, 1]$. Without loss of generality we may assume $x_0 = 0$. We shall now show that due to the choice of constants above and due to the decay estimate in Proposition 3.2 this smallness estimate is also valid on smaller radii, namely for every $k \in \mathbb{N}_0$ there holds

$$C(0, \theta^k \rho) \, < \, \widetilde{\varepsilon}_{it}\, \varepsilon_2 \qquad \text{and} \qquad M(0, \theta^k \rho) \, < \, \widetilde{\varepsilon}_{it}\, \delta_1\,. \tag{I}$$

We shall establish $(I)_k$ by induction: $k = 0$ is given by (3.25), and therefore we assume $(I)_k$ and prove $(I)_{k+1}$. We begin by noting that by definition of $C(0, \theta^k \rho)$ and the assumption $(I)_k$ of the induction we calculate

$$\begin{aligned} -\!\!\!\!\!\!\int_{B^+_{\theta^k \rho}} |Du - (D_n u)_{0,\theta^k \rho} \otimes e_n|^2 \, dx &\leq \big(1 + |(D_n u)_{0,\theta^k \rho}|\big)^2\, C(0, \theta^k \rho) \\ &< \, 2\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 + 2\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 -\!\!\!\!\!\!\int_{B^+_{\theta^k \rho}} |D_n u|^2 \, dx\,. \end{aligned} \tag{3.26}$$

The latter estimate enables us to derive the second inequality in $(I)_{k+1}$ exploiting the choices of θ, ε_2 and ρ_m in (3.21), (3.22) and (3.24), respectively:

$$\begin{aligned} M(0, \theta^{k+1}\rho) &\leq \, 2\, (\theta^{k+1}\rho)^\beta -\!\!\!\!\!\!\int_{B^+_{\theta^{k+1}\rho}} |Du - (D_n u)_{0,\theta^k \rho} \otimes e_n|^2 \, dx + 2\, (\theta^{k+1}\rho)^\beta\, |(D_n u)_{0,\theta^k \rho}|^2 \\ &\leq \, 2\, \theta^{\beta - n}\, (\theta^k \rho)^\beta -\!\!\!\!\!\!\int_{B^+_{\theta^k \rho}} |Du - (D_n u)_{0,\theta^k \rho} \otimes e_n|^2 \, dx + 2\, \theta^\beta\, (\theta^k \rho)^\beta -\!\!\!\!\!\!\int_{B^+_{\theta^k \rho}} |D_n u|^2 \, dx \\ &< \, 4\, \theta^{\beta - n}\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 \, \big((\theta^k \rho)^\beta + M(0, \theta^k \rho)\big) + 2\, \theta^\beta\, M(0, \theta^k \rho) \\ &\leq \, 3\, \theta^\beta\, M(0, \theta^k \rho) + \widetilde{\varepsilon}_{it}\, \theta^\beta\, \rho^\beta \, \leq \, \widetilde{\varepsilon}_{it}\, \delta_1\,. \end{aligned}$$

Moreover, the first inequality in $(I)_{k+1}$ is a direct consequence of Proposition 3.2: the first assumptions in (3.3) is satisfied since $(I)_k$ and the choices in (3.22), (3.23) and (3.24) imply

$$\begin{aligned} E(0, \theta^k \rho) &= \, C(0, \theta^k \rho) + \sqrt{\omega(M(0, \theta^k \rho))} + \sqrt{\omega(\theta^k \rho)} \\ &\leq \, \widetilde{\varepsilon}_{it}\, \varepsilon_2 + \sqrt{\omega(\delta_1)} + \sqrt{\omega(\rho)} \, < \, 3\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 \, < \, \varepsilon_0\,, \end{aligned}$$

and the second assumptions in (3.3) is fulfilled by the choice of ρ_m in (3.24). Hence, the statement of Proposition 3.2 implies, taking into account (3.21), the following inequality:

$$C(0,\theta^{k+1}\rho) \le c_* \theta^2 E(0,\theta^k\rho) \le 3 c_* \theta^2 \widetilde{\varepsilon}_{it} \varepsilon_2 < \widetilde{\varepsilon}_{it} \varepsilon_2$$

which finishes the proof of $(I)_{k+1}$ such that (I) holds for every $k \in \mathbb{N}_0$.

Step 3: Iteration. We still consider $0 \in \Gamma$ and a radius $\rho \le \rho_m$ such that (3.25) (and hence (I) for all $k \in \mathbb{N}_0$) is satisfied. Then the calculation (3.26) above and the choices of ε_2 in (3.22) and of θ in (3.21) yield with $\omega_n = |B_1|$ that

$$\begin{aligned}
\int_{B^+_{\theta^{k+1}\rho}} \big|Du\big|^2\,dx &\le \omega_n\,(\theta^{k+1}\rho)^n\,\big|(D_n u)_{0,\theta^k\rho}\big|^2 + 2\int_{B^+_{\theta^{k+1}\rho}} \big|Du - (D_n u)_{0,\theta^k\rho}\otimes e_n\big|^2\,dx \\
&\le 2\,(\theta^n + 2\,\varepsilon_2)\int_{B^+_{\theta^k\rho}} |D_n u|^2\,dx + 2\,\omega_n\,\varepsilon_2\,(\theta^k\rho)^n \\
&\le 3\,\theta^{n-\gamma}\,\theta^\gamma \int_{B^+_{\theta^k\rho}} |D_n u|^2\,dx + 2\,\omega_n\,(\theta^k\rho)^n \\
&\le \theta^\gamma \int_{B^+_{\theta^k\rho}} |Du|^2\,dx + 2\,\omega_n\,(\theta^k\rho)^n
\end{aligned}$$

for γ defined via equation (3.20) and where we have neglected the factor $\varepsilon_{it} \le 1$. Setting $\varphi(t) := \int_{B^+_t} |Du|^2\,dx$ the last inequality may be rewritten by

$$\varphi(\theta^{k+1}\rho) \le \theta^\gamma\,\varphi(\theta^{k+1}\rho) + 2\,\omega_n\,(\theta^k\rho)^n$$

and the application of the iteration Lemma 2.5 yields

$$\varphi(t) \le c_3\Big[\Big(\frac{t}{\rho}\Big)^\gamma \varphi(\rho) + t^\gamma\Big]$$

for all $t \le \rho$ for a constant $c_3 = c_3(n,N,p,\nu,L,\alpha,\beta)$, i.e. there holds

$$\int_{B^+_t} |Du|^2\,dx \le \frac{c_3}{\rho^\gamma}\Big[\int_{B^+_\rho} |Du|^2\,dx + 1\Big]\,t^\gamma \qquad \text{for all } t \le \rho\,. \tag{3.27}$$

Step 4: Hölder continuity at boundary points. Now we are going to combine the estimates in the interior and at the boundary. For fixed $\alpha \in (0,1)$ we define

$$\operatorname{Reg}_{\Gamma,\alpha} u := \big\{x_0 \in \Gamma\colon u \in C^{0,\alpha}(U(x_0)\cap \overline{B^+},\mathbb{R}^N) \text{ for some neighbourhood } U(x_0) \text{ of } x_0\big\}\,.$$

and $\operatorname{Sing}_{\Gamma,\alpha} u := \Gamma \setminus \operatorname{Reg}_{\Gamma,\alpha} u$. Now we set

$$s := \min\Big\{\varepsilon_{it}\,\delta_1, \Big(\frac{\varepsilon_{it}\,\varepsilon_2}{2}\Big)^{\frac{p}{2}}\Big\} \tag{3.28}$$

and choose a point $x_0 \in \Gamma$, w.l.o.g. $x_0 = 0$, for which the following two estimates hold true:

$$\liminf_{\rho\to 0^+} \fint_{B^+_\rho(0)} \frac{|Du - (D_n u)_{0,\rho}\otimes e_n|^p}{(1+|(D_n u)_{0,\rho}|)^p}\,dx < s \quad \text{and} \quad \liminf_{\rho\to 0^+} \rho^\beta \fint_{B^+_\rho(0)} |D_n u|^2\,dx < s\,.$$

The aim is now to show that $0 \in \operatorname{Reg}_{\Gamma,\alpha} u$, i.e., that u is Hölder continuous with exponent α in a neighbourhood of 0 in B^+. For this purpose we first determine a radius $\rho_0 \le \frac{\rho_m}{6}$ such that

$$\fint_{B^+_{6\rho_0}(0)} \frac{|Du - (D_n u)_{0,6\rho_0}\otimes e_n|^p}{(1+|(D_n u)_{0,6\rho_0}|)^p}\,dx < s \quad \text{and} \quad (6\rho_0)^\beta \fint_{B^+_{6\rho_0}(0)} |D_n u|^2\,dx < s\,.$$

(which is possible due to the two estimates above). Then, taking into account the definitions of $C(0,6\rho_0)$, $M(0,6\rho_0)$ and the parameter $s<1$ in (3.28), a straightforward calculation yields

$$C(0,6\rho_0) < \varepsilon_{it}\,\varepsilon_2 \qquad \text{and} \qquad M(0,6\rho_0) < \varepsilon_{it}\,\delta_1\,,$$

and therefore, the assumptions in (3.25) of step 2 are satisfied for $x_0 = 0$ such that also

$$C(0, 6\,\theta^k \rho_0) \,<\, \varepsilon_{it}\, \varepsilon_2 \qquad \text{and} \qquad M(0, 6\,\theta^k \rho_0) \,<\, \varepsilon_{it}\, \delta_1 \tag{3.29}$$

is fulfilled for all $k \in \mathbb{N}_0$. In the remainder of the proof we will take advantage of the following fact which is derived analogously to (3.11) in the proof of Proposition 3.2: whenever we have two (half-) balls $B^{(+)}_{\rho_2}(x_2) \subset B^+_{\rho_1}(x_1)$ (with $x_1 \in \Gamma$), for which $C(x_1, \rho_1) \le \frac{\theta^n}{4}$ (cf. (SC.3)) and $\frac{\rho_2}{\rho_1} \ge \theta$ is satisfied, then we have:

$$1 + |(D_n u)_{x_1,\rho_1}| \,\le\, 2\left(1 + |(D_n u)_{x_2,\rho_2}|\right). \tag{3.30}$$

This allows us to estimate the Campanato-type and the Morrey-type excess in 0 also for intermediate radii: for any $\widetilde{\rho} \in (0, 6\rho_0]$ there exists a unique $k \in \mathbb{N}_0$ such that there holds $6\,\theta^{k+1}\rho_0 < \widetilde{\rho} \le 6\,\theta^k \rho_0$. Applying (3.30) with the centre $x_1 = x_2 = 0$ and radii $\rho_1 = 6\,\theta^k \rho_0$. $\rho_2 = \widetilde{\rho}$ we find:

$$\begin{aligned}
C(0, \widetilde{\rho}) &= \fint_{B^+_{\widetilde{\rho}}} \left[\frac{|Du - (D_n u)_{0,\widetilde{\rho}} \otimes e_n|^2}{(1 + |(D_n u)_{0,\widetilde{\rho}}|)^2} + \frac{|Du - (D_n u)_{0,\widetilde{\rho}} \otimes e_n|^p}{(1 + |(D_n u)_{0,\widetilde{\rho}}|)^p} \right] dx \\
&\le \fint_{B^+_{\widetilde{\rho}}} \left[2^2\, \frac{|Du - (D_n u)_{0,6\theta^k\rho_0} \otimes e_n|^2}{(1 + |(D_n u)_{0,6\theta^k\rho_0}|)^2} + 2^{2p}\, \frac{|Du - (D_n u)_{0,6\theta^k\rho_0} \otimes e_n|^p}{(1 + |(D_n u)_{0,6\theta^k\rho_0}|)^p} \right] dx \\
&\le 2^{2p}\, \theta^{-n}\, C(0, 6\theta^k \rho_0) \,<\, 2^{2p}\, \theta^{-n}\, \varepsilon_{it}\, \varepsilon_2
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
M(0, \widetilde{\rho}) &= \widetilde{\rho}^{\beta} \fint_{B^+_{\widetilde{\rho}}} |D_n u|^2\, dx \,\le\, \Big(\frac{6\theta^k \rho_0}{\widetilde{\rho}}\Big)^{n-\beta} (6\theta^k \rho_0)^\beta \fint_{B^+_{6\theta^k\rho_0}} |D_n u|^2\, dx \\
&\le \theta^{\beta-n}\, M(0, 6\theta^k \rho_0) \,<\, \theta^{-n}\, \varepsilon_{it}\, \delta_1 \,.
\end{aligned} \tag{3.32}$$

Similar to [20], p. 378-379, we now have to show decay estimates on a variety of balls $B_\rho(y)$ and half balls $B^+_\rho(y)$:

Case 1: $y \in \Gamma_{2\rho_0}$, $|y| \le \rho \le 4\rho_0$:
Here, we may compare the excess functionals in $B^+_\rho(y)$ via (3.30) and (3.31) with the excess functionals in $B^+_{\rho+|y|}$ and we obtain similar to the last computation

$$\begin{aligned}
C(y, \rho) &\le 2^{2p+n}\, C(0, \rho + |y|) \,<\, 2^{4p+n} \theta^{-n}\, \varepsilon_{it}\, \varepsilon_2 \,, \\
M(y, \rho) &\le 2^{n-\beta}\, M(0, \rho + |y|) \,<\, 2^n\, \theta^{-n}\, \varepsilon_{it}\, \delta_1
\end{aligned}$$

and via (3.27) we find

$$\int_{B^+_\rho(y)} |Du|^2\, dx \,\le\, \int_{B^+_{\rho+|y|}} |Du|^2\, dx \,\le\, \frac{2^\gamma\, c_3}{(6\rho_0)^\gamma} \Big[\int_{B^+} |Du|^2\, dx + 1 \Big] \rho^\gamma \,. \tag{3.33}$$

Case 2: $y \in \Gamma_{2\rho_0}$, $0 < \rho < |y| \le 2\rho_0$:
Here we have to verify that the assumptions (3.25) for the iteration are satisfied for the half-ball $B^+_{2\rho_0}(y)$ (keep in mind $\theta \in (0, \frac{1}{4})$ in order to apply (3.30)):

$$\begin{aligned}
C(y, 2\rho_0) &\le 2^{2p}\, 3^n\, C(0, 6\rho_0) \,<\, 2^{2p}\, 3^n\, \varepsilon_{it}\, \varepsilon_2 \,, \\
M(y, 2\rho_0) &\le 3^{n-\beta}\, M(0, 6\rho_0) \,<\, 3^n\, \varepsilon_{it}\, \delta_1 \,.
\end{aligned}$$

i.e., (3.25) is valid for the iteration parameter $\widetilde{\varepsilon}_{it} := 2^{2p}\, 3^n\, \varepsilon_{it} \le 1$ by definition of ε_{it}. Therefore we may conclude for all $k \in \mathbb{N}_0$ there holds $C(y, 2\theta^k \rho_0)$, $M(y, 2\theta^k \rho_0) < 2^{2p}\, 3^n\, \varepsilon_{it}\, \varepsilon_2$. From the calculations in (3.31) and (3.32) we now easily infer that the excess functionals for intermediate radii can only be increased by the factor $2^{2p}\theta^{-n}$ such that for all radii $\widetilde{\rho} \in (0, 2\rho_0]$ we have

$$C(y, \widetilde{\rho}) \,<\, 4^{2p}\, 3^n\, \theta^{-n}\, \varepsilon_{it}\, \varepsilon_2 \quad \text{and} \quad M(y, \widetilde{\rho}) \,<\, 3^n\, 2^{2p}\, \theta^{-n}\, \varepsilon_{it}\, \delta_1 \,.$$

Moreover, as a consequence of (3.25) on the half-ball $B^+_{2\rho_0}(y)$, we note that due to the calculations in step 3 (see (3.27)) there holds

$$\int_{B^+_\rho(y)} |Du|^2\, dx \,\le\, \frac{c_3}{(2\rho_0)^\gamma} \Big[\int_{B^+} |Du|^2\, dx + 1 \Big] \rho^\gamma \,. \tag{3.34}$$

Case 3: $y \in B^+_{2\rho_0}$, $B_\rho(y) \subset B^+_{2\rho_0}$:
Let $y'' = (y_1, ..., y_{n-1}, 0)$ be the projection of y onto $\mathbb{R}^{n-1} \times \{0\}$. Here we have the inclusions

$$B_\rho(y) \,\subset\, B_{y_n}(y) \,\subset\, B^+_{2y_n}(y'') \,.$$

We shall now show that the assumptions for the iteration and thus for the excess-decay estimate in the interior (see [15], (3.50) and (3.57)), which are analogous to (3.25) and (3.27), respectively, are satisfied on the ball $B_{y_n}(y)$. If $|y''| \leq 2y_n$ ($\leq 4\rho_0$) we can apply case 1 with centre y'' and radius $2y_n$, otherwise if $2y_n < |y''| < 2\rho_0$ we can apply case 2 (note that we have in particular $B^+_{2y_n}(y'') \subset B^+_{2\rho_0}(y'')$) and we obtain for both cases

$$C(y'', 2y_n) \,<\, 4^{2p}\, 3^n\, \theta^{-n}\, \varepsilon_{it}\, \varepsilon_2 \qquad \text{and} \qquad M(y'', 2y_n) \,<\, 3^n\, 2^{2p}\, \theta^{-n}\, \varepsilon_{it}\, \delta_1 \,.$$

Then, recalling (3.30) and the definition of ε_{it}, we arrive at the conclusion that

$$\begin{aligned} C(y, y_n) &:= \fint_{B_{y_n}(y)} \Big[\frac{|Du - (Du)_{y,y_n}|^2}{(1 + |(Du)_{y,y_n}|)^2} + \frac{|Du - (Du)_{y,y_n}|^p}{(1 + |(Du)_{y,y_n}|)^p} \Big]\, dx \\ &\leq \fint_{B_{y_n}(y)} \Big[2^2\, \frac{|Du - (D_n u)_{y'',2y_n} \otimes e_n|^2}{(1 + |(D_n u)_{y'',2y_n}|)^2} + 2^{2p}\, \frac{|Du - (D_n u)_{y'',2y_n} \otimes e_n|^p}{(1 + |(D_n u)_{y'',2y_n}|)^p} \Big]\, dx \\ &\leq\, 2^{n+2p}\, C(y'', 2y_n) \,<\, 4^{3p}\, 6^n\, \theta^{-n}\, \varepsilon_{it}\, \varepsilon_2 \,\leq\, \varepsilon_2 \,. \end{aligned}$$

Moreover, we have

$$M(y, y_n) := y_n^\beta \fint_{B_{y_n}(y)} |D_n u|^2\, dx \,\leq\, 2^n\, M(y'', 2y_n) \,<\, 6^n\, 2^{2p}\, \theta^{-n}\, \varepsilon_{it}\, \delta_1 \,\leq\, \delta_1 \,.$$

Hence, the smallness conditions for the iteration in the interior are satisfied, and step 2 in the Proof of [15], Theorem 1.1, combined with (3.33) and (3.34), respectively, yields

$$\begin{aligned} \int_{B_\rho(y)} |Du|^2\, dx &\leq \widetilde{c}_3 \Big[y_n^{-\gamma} \int_{B_{y_n}(y)} |Du|^2\, dx + 1 \Big]\, \rho^\gamma \\ &\leq \widetilde{c}_3 \Big[y_n^{-\gamma} \int_{B^+_{2y_n}(y'')} |Du|^2\, dx + 1 \Big]\, \rho^\gamma \,\leq\, \frac{c_3\, \widetilde{c}_3}{\rho_0^\gamma} \Big[\int_{B^+} |Du|^2\, dx + 2 \Big]\, \rho^\gamma \end{aligned} \tag{3.35}$$

where the constant $\widetilde{c}_3 = \widetilde{c}_3(n, N, p, \nu, L, \alpha, \beta)$ denotes the corresponding constant appearing in the interior.

Combining (3.33), (3.34) and (3.35) and applying Poincaré's inequality on the left-hand side of each inequality, we may apply Theorem 2.6 to conclude: $u \in C^{0,\alpha}(B^+_{\rho_0}, \mathbb{R}^N)$ for $\alpha = 1 - \frac{n-\gamma}{2}$, and therefore, $0 \in \mathrm{Reg}_{\Gamma,\alpha}\, u$. This means, we have proved so far a model analogon of Theorem 1.1 (in the sequel, we will now denote by $\tilde{v}$ the solution of the corresponding problem on a half-ball), i. e., we have

Theorem 3.3: *Let $p \geq 2$ and $\tilde{v} \in W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$ be a weak solution of system (3.1) under the assumptions (1.2). Then if $y \in \mathrm{Reg}_{\Gamma,\alpha}\, \tilde{v}$ there holds: for every $\alpha \in (0,1)$ there exists $\tilde{s} > 0$ depending only on $n, N, p, \nu, L, \alpha, \beta$, $\omega(\cdot)$ and $\mu(\cdot)$ such that for every $\beta \in (0,2)$ the following inclusion holds:*

$$\begin{aligned} \mathrm{Sing}_{\Gamma,\alpha}\, \tilde{v} \subseteq \Big\{ y_0 \in \Gamma \colon \liminf_{\rho \to 0^+} \fint_{B^+ \cap B_\rho(y_0)} \frac{|D\tilde{v} - (D_n \tilde{v})_{B^+ \cap B_\rho(y_0)} \otimes e_n|^p}{(1 + |(D_n \tilde{v})_{B^+ \cap B_\rho(y_0)}|)^p}\, dx \geq \tilde{s} \\ \text{or } \liminf_{\rho \to 0^+} \rho^\beta \fint_{B^+ \cap B_\rho(y_0)} |D_n \tilde{v}|^2\, dx \geq \tilde{s} \Big\} \,. \end{aligned}$$

Step 5: Transformation of the system to the model situation. In the next step we sketch for convenience of the reader why the handling of the model case of a half ball is sufficient in order to deduce a criterion for a weak solution of a general elliptic system of type (1.1) with boundary data g to be regular in the neighbourhood of a given boundary point $z \in \partial\Omega$. Without loss of generality we may assume $z = 0$ and $\nu_{\partial\Omega}(z) = e_n$ where $\nu_{\partial\Omega}(z)$ denotes the inward-pointing unit normal vector to $\partial\Omega$ in z. The regularity assumption on $\partial\Omega$ ensures the existence of a function $h \colon \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^1 with modulus of continuity $\tau(\cdot)$, satisfying $h(0) = 0$ and $\nabla h(0) = 0$, and the existence of a radius $r > 0$ such that $\Omega \cap B_r = \{x \in B_r : x_n > h(x')\}$. For ease of notation also the modulus of continuity of Dg is denoted by $\tau(\cdot)$. We further choose r sufficiently small such that $|\nabla h(x')| < \frac{1}{2}$ for all $x' \in B_{\sqrt{2}r}$. For the functions $\mathcal{T}(x) = \big(x', x_n - h(x')\big)$ introduced in Section 2 and its inverse $\mathcal{T}^{-1}(y) = \big(y', y_n + h(y')\big)$ (both of class C^1 with modulus of continuity $\tau(\cdot)$) we thus obtain Lipschitz constants between $\frac{1}{\sqrt{2}}$ and $\sqrt{2}$ as well as $\|D\mathcal{T}\|_\infty, \|D\mathcal{T}^{-1}\|_\infty \leq \sqrt{2}$. Furthermore, we have $\det D\mathcal{T} = 1 = \det D\mathcal{T}^{-1}$ and the inclusions

$$B^+_{\rho/\sqrt{2}} \subset \mathcal{T}(\Omega \cap B_\rho) \subset B^+_{\sqrt{2}\rho}.$$

for all $\rho \leq \sqrt{2}r$ (cf. e.g. [19], Chapt. 3.7). Note that this also implies $|B^+_{\rho/\sqrt{2}}| \leq |\Omega \cap B_\rho| \leq |B^+_{\sqrt{2}\rho}|$. Setting $\tilde{v}(y) := u \circ \mathcal{T}^{-1}(y) - g \circ \mathcal{T}^{-1}(y)$ allows us to calculate that $\tilde{v} \in W^{1,p}_\Gamma(B^+_r, \mathbb{R}^N)$ is weak solution of

$$\operatorname{div} \tilde{a}(\cdot, \tilde{v}, D\tilde{v}) = 0 \qquad \text{in } B^+_r$$

for coefficients $\tilde{a}(\cdot,\cdot,\cdot)$ defined by

$$\tilde{a}(y, v, z) := a\big(\mathcal{T}^{-1}(y), v + \tilde{g}(y), z\, D\mathcal{T}(\mathcal{T}^{-1}(y)) + Dg(\mathcal{T}^{-1}(y))\big)\, D\mathcal{T}^t(\mathcal{T}^{-1}(y))$$

for all $(y, v, z) \in B^+_r \times \mathbb{R}^N \times \mathbb{R}^{nN}$. Keeping in mind the the assumptions on $a(\cdot,\cdot,\cdot)$ given in (1.2) we easily calculate that the new coefficients satisfy structure conditions analogous to (1.2), namely that there holds for all $y, \bar{y} \in B^+_r$, $v, \bar{v} \in \mathbb{R}^N$ and $z, \bar{z}, C \in \mathbb{R}^{nN}$:

- $\big|\tilde{a}(y, v, z)\big| \leq 2^{\frac{p}{2}}\, L\, (1 + \|Dg\|_\infty)^{p-1} \big(1 + |z|\big)^{p-1}$,
- $\big|D_z\tilde{a}(y, v, z)\big| \leq 2^{\frac{p}{2}}\, L\, (1 + \|Dg\|_\infty)^{p-2} \big(1 + |z|\big)^{p-2}$,
- $D_z\tilde{a}(y, v, z)\,(C, C) \geq \nu\, (1 + \|Dg\|_\infty)^{2-p}\, 2^{-2p+2} \big(1 + |z|\big)^{p-2} |C|^2$,
- $\big|\big(\tilde{a}(y, v, z) - \tilde{a}(\bar{y}, \bar{v}, z)\big)\big| \leq 2^{p+2}\, L \big(1 + \|Dg\|_\infty\big)^p \big(1 + |z|\big)^{p-1} \big(\omega\big(|y - \bar{y}| + |v - \bar{v}|\big) + \tau\big(|y - \bar{y}|\big)\big)$,
- $\big|D_z\tilde{a}(y, v, z) - D_z\tilde{a}(y, v, \bar{z})\big| \leq 2^{\frac{p}{2}+3}\, L\, (1 + \|Dg\|_\infty)^{p-1} \big(1 + |z| + |\bar{z}|\big)^{p-2} \mu\Big(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\Big)$.

Therefore, the corresponding assumptions in (1.2) are also satisfied for the new coefficients $\tilde{a}(\cdot,\cdot,\cdot)$ for $\tilde{L} = L\, c(p, \partial\Omega, g)$, $\tilde{\nu} = \nu\, c(p, g)$ and $\tilde{\omega}(\cdot) = \omega(\cdot) + \tau(\cdot)$.

Step 6: Transformation of the smallness conditions and final conclusion. In the last step there still remains to show that the smallness conditions in Theorem 3.3 for the transformed system are satisfied provided that the smallness conditions required in Theorem 1.1 are fulfilled. We still assume $z = 0$, $\nu_{\partial\Omega}(z) = e_n$ and use the notation introduced in step 5. Keeping in mind $D_n\mathcal{T}^{-1} = 1$, hence $D_n(u \circ \mathcal{T}^{-1})(y) = D_n u(\mathcal{T}^{-1}(y))$, and the inclusion $\mathcal{T}^{-1}(B^+_\rho) \subset \Omega \cap B_{\sqrt{2}\rho}$ we find for the second of the smallness conditions

$$\begin{aligned}
\rho^\beta \fint_{B^+_\rho} |D_n\tilde{v}|^2\, dx &\leq \frac{|\Omega \cap B_{\sqrt{2}\rho}|}{|B^+_\rho|}\, \rho^\beta \fint_{\Omega \cap B_{\sqrt{2}\rho}} |D_n(u - g)|^2\, dx \\
&\leq c(n) \Big[(\sqrt{2}\rho)^\beta \fint_{\Omega \cap B_{\sqrt{2}\rho}} |D_n u|^2\, dx + \rho^\beta\, \|Dg\|_\infty\Big]. \qquad (3.36)
\end{aligned}$$

Choosing s sufficiently small and using the change of variables formula we may proceed similar to (3.11) to obtain

$$1 + |(D_n u)_{\Omega \cap B_{\sqrt{2}\rho}}| + |(D_n\tilde{v})_{B^+_\rho}| \leq c(\|Dg\|_\infty) \big(1 + |(D_n\tilde{v})_{B^+_{\rho/2}}|\big) \qquad (3.37)$$

for all $\rho \leq \frac{r}{\sqrt{2}}$. The latter inequality combined with the application of Caccioppoli's and Poincaré's inequality (in the boundary version) yields for the first of the smallness conditions in Theorem 3.3 that

$$\begin{aligned}
\fint_{B^+_{\rho/2}} \frac{|D\tilde{v} - (D_n\tilde{v})_{B^+_{\rho/2}} \otimes e_n|^p}{(1 + |(D_n\tilde{v})_{B^+_{\rho/2}}|)^p}\, dx &\leq c \fint_{B^+_\rho} \Big[\frac{|D_n\tilde{v} - (D_n\tilde{v})_{B^+_\rho}|^2}{(1 + |(D_n\tilde{v})_{B^+_\rho}|)^2} + \frac{|D_n\tilde{v} - (D_n\tilde{v})_{B^+_\rho}|^p}{(1 + |(D_n\tilde{v})_{B^+_\rho}|)^p}\Big]\, dx \\
&\quad + c \fint_{B^+_\rho} \big[\tilde{\omega}(\rho^2) + \tilde{\omega}(|\tilde{v}|^2) + \tilde{\omega}(\rho^2\, |(D_n\tilde{v})_{B^+_{\rho/2}}|^2)\big]\, dx \qquad (3.38)
\end{aligned}$$

for a constant c depending only on $n, N, p, \tilde{\nu}, \tilde{L}$ and $\|Dg\|_\infty$. Moreover, applying the diffeomorphism $\mathcal{T}$ and (3.37) we infer

$$\begin{aligned}\fint_{B_\rho^+} \frac{|D_n\tilde{v} - (D_n\tilde{v})_{B_\rho^+}|^p}{(1+|(D_n\tilde{v})_{B_\rho^+}|)^p}\,dx &\le c \fint_{B_\rho^+} \frac{|D_n((u-g)\circ\mathcal{T}^{-1}) - (D_n((u-g)\circ\mathcal{T}^{-1}))_{B_\rho^+}|^p}{(1+|(D_nu)_{\Omega\cap B_{\sqrt{2}\rho}}|)^p}\,dx \\ &\le c \fint_{\Omega\cap B_{\sqrt{2}\rho}} \frac{|D_nu - (D_nu)_{\Omega\cap B_{\sqrt{2}\rho}}|^p}{(1+|(D_nu)_{\Omega\cap B_{\sqrt{2}\rho}}|)^p}\,dx + c\,\tau(\rho) \qquad (3.39)\end{aligned}$$

where the constant depends only on n, p and $\|Dg\|_\infty$. Keeping in mind

$$\fint_{B_\rho^+} \big[\tilde{\omega}(\rho^2) + \tilde{\omega}(|\tilde{v}|^2) + \tilde{\omega}(\rho^2\,|(D_n\tilde{v})_{B_{\rho/2}^+}|^2)\big]\,dx \;\le\; \tilde{\omega}(\rho) + 2^{n+1}\,\tilde{\omega}\Big(\rho^\beta \fint_{B_\rho^+} |D_n\tilde{v}|^2\,dx\Big)$$

we may combine (3.36), (3.38) and (3.39) and infer

$$\liminf_{\rho\to 0^+} \fint_{B_\rho^+} \frac{|D\tilde{v} - (D_n\tilde{v})_{B_\rho^+}\otimes e_n|^p}{(1+|(D_n\tilde{v})_{B_\rho^+}|)^p}\,dx < \tilde{s} \quad\text{and}\quad \liminf_{\rho\to 0^+} \rho^\beta \fint_{B_\rho^+} |D_n\tilde{v}|^2\,dx < \tilde{s}$$

provided that

$$\liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho} \frac{|Du - (D_nu)_{\Omega\cap B_\rho}\otimes e_n|^p}{(1+|(D_nu)_{\Omega\cap B_\rho}|)^p}\,dx < s \quad\text{and}\quad \liminf_{\rho\to 0^+} \rho^\beta \fint_{\Omega\cap B_\rho} |D_nu|^2\,dx < s$$

for s sufficiently small. As a consequence of step 4 we then obtain $\tilde{v} \in C^{0,\alpha}$ locally in a neighbourhood of 0 in $\overline{B^+}$, and therefore, via transformation, $u \in C^{0,\alpha}$ in a neighbourhood of 0 in $\overline{\Omega}$, i. e., $0 \in \mathrm{Reg}_{\partial\Omega,\alpha}\,u$. This finishes the proof of Theorem 1.1. □

Remark 3.4: Similar to the situation in the interior there are better inclusions available for the singular set in the case $p \ge n$. Via the Sobolev embedding we obtain that u is Hölder continuous everywhere with exponent $1-\frac{n}{p}$ if $p > n$. Otherwise if $p = n$ we first deduce higher integrability of Du, i. e., $Du \in L^{q_1}(\bar{\Omega}, \mathbb{R}^{nN})$ for some $q_1 > p$, and then conclude that u is Hölder continuous everywhere with exponent $1-\frac{n}{q_1}$ (cf. [18], Remark 6.13). Hence the existence of regular boundary points is ensured in this case, actually we have $\mathrm{Reg}_{\partial\Omega,\alpha}\,u = \partial\Omega$ for all $\alpha \le 1-\frac{n}{q_1}$. In contrast, Theorem 1.1 gives a characterization of regular boundary points where u is locally Hölder continuous with *any* exponent $\alpha < 1$, even though now the question of existence of regular boundary points is open. Moreover, we have the better inclusion for the singular set $\mathrm{Sing}_{\partial\Omega}\,u$ given by

$$\mathrm{Sing}_{\partial\Omega}\,u \subseteq \Big\{x_0 \in \partial\Omega\colon\ \liminf_{\rho\to 0^+} \fint_{\Omega\cap B_\rho(x_0)} \frac{|D_{\nu_{\partial\Omega}(x_0)}u - (D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|^p}{(1+|(D_{\nu_{\partial\Omega}(x_0)}u)_{\Omega\cap B_\rho(x_0)}|)^p}\,dx > 0\Big\}\,.$$

To this aim we go back to the model situation: the application of Caccioppoli's (note $\omega(\cdot) \le 1$) and Poincaré's inequality and the Hölder continuity $\tilde{v} \in C^{0,1-\frac{n}{q_1}}(\overline{B^+}, \mathbb{R}^N)$ reveals

$$\begin{aligned}\rho^\beta \fint_{B_\rho^+(x_0)} |D_n\tilde{v}|^2\,dx &\le \rho^\beta \Big(\fint_{B_\rho^+(x_0)} |D_n\tilde{v}|^p\,dx\Big)^{\frac{2}{p}} \\ &\le c\,\rho^\beta \Big(\fint_{B_{2\rho}^+(x_0)} \Big|\frac{\tilde{v}}{\rho}\Big|^p\,dx + 1\Big)^{\frac{2}{p}} \;\le\; c\,\rho^{\beta-2\frac{n}{q_1}}\end{aligned}$$

where the constant c depends on $n, N, p, \tilde{L}$ and $\tilde{\nu}$. Choosing $\beta \in (2\frac{n}{q_1}, 2)$ the left-hand side of the last inequality converges to 0 for $\rho \to 0$, and the smallness condition on $M(x_0,\rho)$ in Theorem 3.3 and hence in Theorem 1.1 is trivially satisfied.

4 Quasi-convex functionals

4.1 Decay estimate

Also in the case of minimizers we consider the model case of the unit half-ball, i. e., we deal with local minimizers $u \in W_\Gamma^{1,p}(B^+, \mathbb{R}^N)$ of the functional

$$\mathcal{F}_B[u] := \int_{B^+} F(x, u, Du)\,dx\,. \qquad (4.1)$$

We start again by stating a Caccioppoli inequality involving the exact dependency for some linear disturbance of u:

Lemma 4.1 (Caccioppoli inequality): *Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a local minimizer of (4.1) under the assumptions (1.4), $\xi \in \mathbb{R}^N$ and $B^+_\rho(x_0)$, $x_0 \in \Gamma$, $\rho < 1 - |x_0|$ be an upper half ball. Then there exists a constant $c = c(n,N,p,L,\nu)$ such that*

$$\begin{aligned}
&\fint_{B^+_{\rho/2}(x_0)} \big[(1+|\xi|)^{p-2}|Du - \xi\otimes e_n|^2 + |Du - \xi\otimes e_n|^p\big]\,dx \\
&\qquad \le c \fint_{B^+_\rho(x_0)} \Big[(1+|\xi|)^{p-2}\Big|\frac{u-\xi x_n}{\rho}\Big|^2 + \Big|\frac{u-\xi x_n}{\rho}\Big|^p\Big]\,dx \\
&\qquad\qquad + c \fint_{B^+_\rho(x_0)} \big[\omega(\rho^p) + \omega(|u|^p) + \omega(\rho^p\,|\xi|^p)\big](1+|\xi|+|Du|)^p\,dx\,.
\end{aligned}$$

Proof: This lemma is proved as in the interior situation. We only have to pay attention to the application of the quasi-convexity condition $(1.4)_2$, where zero-boundary data of the testfunction φ is requested: hence we have to choose $\varphi = \eta(u - \xi\,x_n)$ for a standard cut-off function (for the later application of the hole-filling argument) $\eta \in C^\infty_0(B_t(x_0),[0,1])$, $0 \le \rho/2 \le s < t \le \rho$. □

The second ingredient are two higher integrability results (in order to enable an appropriate estimate for the last integral arising on the right-hand side of the Caccioppoli inequality): firstly, we may prove analogously to [16], Theorem 4.1, via the application of a Gehring-Lemma in the up-to-the-boundary version, see [11], Theorem 2.4:

Lemma 4.2: *Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a local minimizer of (4.1) under the assumptions $(1.4)_1$. Then there exists a higher integrability exponent $q_1 > p$ and a constant c both depending only on n,N,p,ν and L such that for any half-ball $B^+_\rho(x_0) \subset B^+$, $x_0 \in \Gamma$ there holds*

$$\Big(\fint_{B^+_{\rho/2}(x_0)} |Du|^{q_1}\,dx\Big)^{\frac{1}{q_1}} \le c\Big(\fint_{B^+_\rho(x_0)} (1+|Du|)^p\,dx\Big)^{\frac1p}.$$

The second higher integrability result provides an estimate up to the boundary (provided that the boundary values are higher integrable), concerns solutions of functionals without (x,u)-dependency and will later be applied for the frozen functional $\mathcal{F}$. For a proof we refer to the similar result [11], Lemma 3.2.

Lemma 4.3: *Let $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ be a local minimizer of (4.1) under the assumptions $(1.4)_1$, and let $v_0 \in u + W^{1,p}_0(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ be a solution of the following Dirichlet problem:*

$$v_0 \mapsto \min_w \fint_{B^+_{\rho/2}(x_0)} G(Dw)\,dx \qquad \text{with } w \in u + W^{1,p}_0(B^+_{\rho/2}(x_0),\mathbb{R}^N)\,, \tag{4.2}$$

where $G\colon \mathbb{R}^{nN} \to \mathbb{R}$ is continuous and satisfies $\nu|z|^p \le G(z) \le L(1+|z|)^p$, and $B^+_{\rho/2}(x_0) \subset B^+$, $x_0 \in \Gamma$ a half-ball. Then there exists another higher integrability exponent $q \in (p, q_1]$ depending only on n,N,p,ν and L such that

$$\Big(\fint_{B^+_{\rho/2}(x_0)} |Dv_0|^q\,dx\Big)^{\frac1q} \le c\Big(\fint_{B^+_{\rho/2}(x_0)} |Dv_0|^p\,dx\Big)^{\frac1p} + c\Big(\fint_{B^+_{\rho/2}(x_0)} \big(1+|Du|\big)^{q_1}\,dx\Big)^{\frac{1}{q_1}}.$$

In the next step we define the excess functionals analogously to the elliptic case (cf. [15], Section 4.2): For any half-ball $B^+_\rho(x_0) \subset B^+$ with $x_0 \in \Gamma$, a fixed function $u \in W^{1,p}_\Gamma(B^+,\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$ we define the Campanato-type excess $C(x_0,\rho)$ and as in the elliptic setting, but we redefine the Morrey-type excess $M(x_0,\rho)$ and the excess functional $E(x_0,\rho)$ by setting

$$\begin{aligned}
M(x_0,\rho) &:= \rho^\beta \fint_{B^+_\rho(x_0)} |Du|^p\,dx \qquad \text{for } \beta \in (0,p)\\
E(x_0,\rho) &:= C(x_0,\rho) + \omega(M(x_0,\rho))^{\frac{q-p}{qp}} + \omega(\rho)^{\frac{q-p}{qp}}\,,
\end{aligned}$$

where $q \in (p, q_1]$ is the higher integrability exponent introduced in Lemma 4.3.

The technique for deriving a partial regularity result and the characterization of regular boundary points now consists in comparing the minimal map u on some half-ball $B^+_{\rho/2}(x_0)$ with the minimizer of the functional frozen in the first two variables (amongst all functions $w \in u + W^{1,p}_0(B^+_{\rho/2}(x_0), \mathbb{R})$). Via an approximation Theorem based on a variational principle due to Ekeland (see e. g. [18], Chapter 5), we may now proceed as in [15], Proposition 4.4 and obtain the existence of a function v which is close with respect to the L^2-distance to the original minimizer u and which is an almost minimizer of the frozen functional; this provides the following comparison result:

Lemma 4.4: *Let $u \in W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$ be a local minimizer of (4.1) under the assumptions (1.4), and let $B^+_\rho(x_0) \subset B^+$, $x_0 \in \Gamma$, be such a half ball that*

$$E(x_0, \rho) + \rho \le 1 . \tag{4.3}$$

Then there exists a map $v \in u + W^{1,p}_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$ such that

$$-\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} |Dv - Du|^p \, dx \le K(x_0, \rho) \tag{4.4}$$

and

$$-\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} G(Dv)\, dx \le -\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} G(Dv + D\varphi)\, dx + c_e\, K(x_0, \rho)^{1-\frac{1}{p}} \Big(-\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} |D\varphi|^p \, dx \Big)^{\frac{1}{p}} , \tag{4.5}$$

for every $\varphi \in W^{1,p}_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$, and c_e is a constant depending only on n, N, p, ν and L. The exponent $q > p$ is the higher integrability exponent in Lemma 4.3, and the integrand $G(z) := F(x_0, 0, z)$ is defined by freezing $F(\cdot, \cdot, z)$ and

$$K(x_0, \rho) := \big(1 + |(D_n u)_{x_0,\rho}|\big)^p \Big(\omega(M(x_0, \rho))^{\frac{q-p}{q}} + \omega(\rho)^{\frac{q-p}{q}} \Big)$$

Analogously to the elliptic case we next prove the excess decay estimate for the corresponding excess quantities defined above:

Proposition 4.5 (cf. [15], Proposition 4.5): *For each $\beta \in (0, p)$ and $\theta \in (0, \frac{1}{8})$ there exist two positive numbers*

$$\varepsilon_0 = \varepsilon_0\big(n, N, p, \nu, L, \beta, \theta, \mu(\cdot)\big) > 0 \qquad \text{and} \qquad \varepsilon_1(n, p, \beta, \theta) > 0 \tag{4.6}$$

such that the following is true: If $u \in W^{1,p}(B^+, \mathbb{R}^N)$ is a local minimizer to (4.1) under the assumptions (1.4), and if $B^+_\rho(x_0)$, $x_0 \in \Gamma$, $\rho < 1 - |x_0|$, is a half ball satisfying the smallness conditions

$$E(x_0, \rho) < \varepsilon_0 \qquad \text{and} \qquad \rho < \varepsilon_1 , \tag{4.7}$$

then we have

$$C(x_0, \theta\rho) \le c_*\, \theta^2\, E(x_0, \rho) \tag{4.8}$$

for a constant c_* depending only on n, N, p, ν and L.

Proof: In the first step of our proof, we infer approximate $\mathcal{A}$-harmonicity (with $\mathcal{A}$ an adequate freezing of $D_{zz}F$ introduced later on). For this linearization we will use the last Proposition 4.4: Hence we assume the smallness condition (4.3), i. e.,

$$E(x_0, \rho) + \rho \le 1 . \tag{SCF-1}$$

Then the map $v \in u + W^{1,p}_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$ found in Proposition 4.4 minimizes the functional

$$\xi \mapsto -\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} G(D\xi)\, dx + c_e\, K(x_0, \rho)^{1-\frac{1}{p}} \Big(-\!\!\!\!\!\!\int_{B^+_{\rho/2}(x_0)} |D\xi - Dv|^p \, dx \Big)^{\frac{1}{p}} ,$$

for every $\xi \in u + W_0^{1,p}(B^+_{\rho/2}(x_0),\mathbb{R}^N)$, $G(z) := F(x_0,0,z)$, $K(x_0,\rho)$ and c_e chosen according to Proposition 4.4. Deriving the Euler-Lagrange equation for this variational integral we see that there holds

$$0 = \fint_{B^+_{\rho/2}(x_0)} D_zG(Dv)\, D\varphi\, dx + c_e\, K(x_0,\rho)^{1-\frac{1}{p}} \Big(\fint_{B^+_{\rho/2}(x_0)} |D\varphi|^p\, dx\Big)^{\frac{1}{p}}$$

for all $\varphi \in W_0^{1,p}(B^+_{\rho/2}(x_0),\mathbb{R}^N)$. Assuming $\varphi \in C_0^1(B^+_{\rho/2}(x_0),\mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B^+_{\rho/2}(x_0),\mathbb{R}^N)} \leq 1$ we infer

$$\Big| \fint_{B^+_{\rho/2}(x_0)} D_zG(Dv)\, D\varphi\, dx\Big| \,\leq\, c_e\, K(x_0,\rho)^{1-\frac{1}{p}}\,,$$

and therefore, introducing the abbreviation $\Lambda_\rho := (D_n u)_{x_0,\rho}$ and taking into account $(1.4)_4$ and the fact that $\fint_{B^+_{\rho/2}(x_0)} D_zG(\Lambda_\rho \otimes e_n)\, D\varphi\, dx = 0$, we have

$$\begin{aligned}
&\Big| \fint_{B^+_{\rho/2}(x_0)} D_{zz}G\big(\Lambda_\rho \otimes e_n\big)\, \big(Dv - \Lambda_\rho\otimes e_n, D\varphi\big)\, dx\Big| \\
&\leq \Big| \fint_{B^+_{\rho/2}(x_0)} \int_0^1 \big[D_{zz}G\big(\Lambda_\rho\otimes e_n\big) - D_{zz}G\big(\Lambda_\rho\otimes e_n + t(Dv-\Lambda_\rho\otimes e_n)\big) \\
&\qquad \big(Dv - \Lambda_\rho\otimes e_n, D\varphi\big)\, dt\, dx\Big| + \Big| \fint_{B^+_{\rho/2}(x_0)} \big[D_zG\big(Dv\big) - D_zG\big(\Lambda_\rho\otimes e_n\big)\, D\varphi\, dx\Big| \\
&\leq L \fint_{B^+_{\rho/2}(x_0)} |Dv - \Lambda_\rho\otimes e_n|\, \big(1 + |\Lambda_\rho| + |Dv - \Lambda_\rho\otimes e_n|\big)^{p-2}\, \mu\Big(\frac{|Dv-\Lambda_\rho\otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx \\
&\qquad + \Big| \fint_{B^+_{\rho/2}(x_0)} D_zG\big(Dv\big)\, D\varphi\, dx\Big| \\
&\leq I + c_e\, K(x_0,\rho)^{1-\frac{1}{p}}
\end{aligned} \tag{4.9}$$

with the obvious labelling. To estimate I we first have to derive some preliminary estimates: via (4.4) we find

$$\begin{aligned}
\fint_{B^+_{\rho/2}(x_0)} |Dv - \Lambda_\rho\otimes e_n|^p\, dx &\leq\, c \fint_{B^+_{\rho/2}(x_0)} |Dv - Du|^p\, dx + c \fint_{B^+_{\rho/2}(x_0)} |Du - \Lambda_\rho\otimes e_n|^p\, dx \\
&\leq\, c\, K(x_0,\rho) + c\,\big(1+|\Lambda_\rho|\big)^p\, C(x_0,\rho) \\
&\leq\, c\,\big(1+|\Lambda_\rho|\big)^p\, E(x_0,\rho)
\end{aligned}$$

for a constant $c = c(n,p)$, and where we used the estimate $K(x_0,\rho) \leq (1+|\Lambda_\rho|)^p E(x_0,\rho)^p \leq (1+|\Lambda_\rho|)^p E(x_0,\rho)$ in the last line. Applying Hölder's inequality and the same calculations, we obtain the analogous result for the exponent p replaced by 2:

$$\begin{aligned}
\fint_{B^+_{\rho/2}(x_0)} |Dv - \Lambda_\rho\otimes e_n|^2\, dx &\leq\, c\Big(\fint_{B^+_{\rho/2}(x_0)} |Dv - Du|^p\, dx\Big)^{\frac{2}{p}} + c \fint_{B^+_{\rho/2}(x_0)} |Du - \Lambda_\rho\otimes e_n|^2\, dx \\
&\leq\, c(n,p)\,\big(1+|\Lambda_\rho|\big)^2\, E(x_0,\rho)\,.
\end{aligned}$$

Combining the last two estimates we find an estimate for the excess functional concerning Dv instead of Du:

$$\fint_{B^+_{\rho/2}(x_0)} \Big[\frac{|Dv-\Lambda_\rho\otimes e_n|^2}{(1+|\Lambda_\rho|)^2} + \frac{|Dv-\Lambda_\rho\otimes e_n|^p}{(1+|\Lambda_\rho|)^p} \Big]\, dx \,\leq\, c_v\, E(x_0,\rho)\,, \tag{4.10}$$

and the constant c_v depends only on the parameters n and p. (4.10) now allows us to estimate I using

Hölder's and Jensen's inequality (note $\mu(\cdot) \le 1$):

$$
\begin{aligned}
I =\; & L \fint_{B^+_{\rho/2}(x_0)} |Dv - \Lambda_\rho \otimes e_n| \big(1 + |\Lambda_\rho| + |Dv - \Lambda_\rho \otimes e_n|\big)^{p-2} \mu\Big(\frac{|Dv - \Lambda_\rho \otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx \\
\le\; & c\,(1+|\Lambda_\rho|)^{p-1} \fint_{B^+_{\rho/2}(x_0)} \frac{|Dv - \Lambda_\rho \otimes e_n|}{(1+|\Lambda_\rho|)}\, \mu\Big(\frac{|Dv - \Lambda_\rho \otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx \\
& + c\,(1+|\Lambda_\rho|)^{p-1} \fint_{B^+_{\rho/2}(x_0)} \frac{|Dv - \Lambda_\rho \otimes e_n|^{p-1}}{(1+|\Lambda_\rho|)^{p-1}}\, \mu\Big(\frac{|Dv - \Lambda_\rho \otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx \\
\le\; & c\,(1+|\Lambda_\rho|)^{p-1} \Big(\fint_{B^+_{\rho/2}(x_0)} \frac{|Dv - \Lambda_\rho \otimes e_n|^2}{(1+|\Lambda_\rho|)^2}\, dx\Big)^{\frac12} \Big(\fint_{B^+_{\rho/2}(x_0)} \mu\Big(\frac{|Dv - \Lambda_\rho \otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx\Big)^{\frac12} \\
& + c\,(1+|\Lambda_\rho|)^{p-1} \Big(\fint_{B^+_{\rho/2}(x_0)} \frac{|Dv - \Lambda_\rho \otimes e_n|^p}{(1+|\Lambda_\rho|)^p}\, dx\Big)^{\frac{p-1}{p}} \Big(\fint_{B^+_{\rho/2}(x_0)} \mu\Big(\frac{|Dv - \Lambda_\rho \otimes e_n|}{1+|\Lambda_\rho|}\Big)\, dx\Big)^{\frac1p} \\
\le\; & c\,(1+|\Lambda_\rho|)^{p-1} \Big(E(x_0,\rho)^{\frac12}\, \mu\big(\sqrt{E(x_0,\rho)}\big)^{\frac12} + E(x_0,\rho)^{\frac{p-1}{p}}\, \mu\big(\sqrt{E(x_0,\rho)}\big)^{\frac1p}\Big) \\
\le\; & c\,(1+|\Lambda_\rho|)^{p-1} \big[E(x_0,\rho)^{\frac12} + E(x_0,\rho)^{1-\frac1p}\big]\, \mu\big(\sqrt{E(x_0,\rho)}\big)^{\frac1p},
\end{aligned}
$$

where the constant c depends only on n, p and L. Combining the latter estimate for I with (4.9) and

$$
K(x_0,\rho)^{1-\frac1p} \;\le\; (1+|\Lambda_\rho|)^{p-1} E(x_0,\rho)^{p-1} \;\le\; (1+|\Lambda_\rho|)^{p-1} E(x_0,\rho)
$$

we finally arrive at

$$
\begin{aligned}
&\Big| \fint_{B^+_{\rho/2}(x_0)} D_{zz}G\big(\Lambda_\rho \otimes e_n\big)\big(Dv - \Lambda_\rho \otimes e_n, D\varphi\big)\, dx \Big| \\
&\qquad\qquad \le\; c\,\big(1+|\Lambda_\rho|\big)^{p-1} \Big[\mu\big(\sqrt{E(x_0,\rho)}\big)^{\frac1p} + \sqrt{E(x_0,\rho)}\Big]\, \Big[E(x_0,\rho)^{\frac12} + E(x_0,\rho)^{1-\frac1p}\Big]
\end{aligned}
$$

for all $\varphi \in C^1_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B^+_{\rho/2}(x_0))} \le 1$ and $c = c(n,N,p,\nu,L)$. Now we define the functions $\mathcal{A}$ and w analogously to the elliptic case

$$
\begin{aligned}
\mathcal{A} &:= \frac{D_{zz}F(x_0, 0, \Lambda_\rho \otimes e_n)}{(1+|\Lambda_\rho|)^{p-2}}\,, \\
w &:= \frac{v - \Lambda_\rho\, x_n}{\sqrt{c_v\, E(x_0,\rho)}\,(1+|\Lambda_\rho|)}
\end{aligned}
$$

and $H(t)$ as in (3.5); we note that $\mathcal{A}$ fulfills condition (2.1), i. e., it is bounded from below (in the sense of Legendre-Hadamard) and above, and further, by the definition of the excess functional $E(x_0,\rho)$ and the constant c_v, see (4.10), there holds $\fint_{B^+_{\rho/2}(x_0)} |Dw|^2\, dx \le 1$. These definitions enable us to rewrite the previous estimate after a rescaling argument:

$$
\Big| \fint_{B^+_{\rho/2}(x_0)} \mathcal{A}\,\big(Dw, D\varphi\big)\, dx \Big| \;\le\; c_4(n,N,p,\nu,L)\, H(E(x_0,\rho))\, \|D\varphi\|_{L^\infty(B^+_{\rho/2}(x_0))}
$$

for all $\varphi \in C^1_0(B^+_{\rho/2}(x_0), \mathbb{R}^N)$, which is completely analogous to the elliptic situation, apart from the fact that we have the take the radius $\frac{\varrho}{2}$ instead of ρ due to the comparison technique. For $\varepsilon > 0$ to be determined later, we now take $\delta = \delta(n,N,\nu,L,\varepsilon)$ to be the corresponding constant from the $\mathcal{A}$-harmonic approximation Lemma 2.2. Provided that the smallness condition

$$
H(E(x_0,\rho)) \;\le\; \delta / c_4 \tag{SCF.2}
$$

holds, we find, according to Lemma 2.2, an $\mathcal{A}$-harmonic map $h \in W^{1,2}_\Gamma(B^+_{\rho/2}(x_0), \mathbb{R}^N)$ such that

$$
\fint_{B^+_{\rho/2}(x_0)} |Dh|^2\, dx \le 1 \quad \text{and} \quad \rho^{-2} \fint_{B^+_{\rho/2}(x_0)} |w - h|^2\, dx \le \varepsilon\,, \tag{4.11}
$$

and via Lemma 2.3 we see $\sup_{B^+_{\rho/4}(x_0)} |D^2 h|^2 \le c_h(n,N,\nu,L)\rho^{-2}$. We now consider $\theta \in (0,1/8)$ fixed, to be specified later, choose $\varepsilon = \theta^{n+4}$ and deduce for the polynomial

$$\widetilde{P} := \Lambda_\rho\, x_n + \sqrt{c_v\, E(x_0,\rho)}\,(1+|\Lambda_\rho|)\, D_n h(x_0)\, x_n$$

exactly as in (3.8)

$$(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|v - \widetilde{P}\big|^2\, dx \le c(n,N,\nu,L)\,\theta^2\,(1+|\Lambda_\rho|)^2 E(x_0,\rho)\,.$$

Provided that the smallness condition

$$E(x_0,\rho) \le \theta^{n+4} \tag{SCF.3}$$

is fulfilled, this estimate is easily carried over to u using Poincaré's and Jensen's inequality and the comparison estimate (4.4):

$$\begin{aligned}
(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - \widetilde{P}\big|^2\, dx &\le 2\,(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - v\big|^2\, dx + 2\,(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|v - \widetilde{P}\big|^2\, dx \\
&\le c\,\theta^{-n-2} \fint_{B^+_{\rho/2}(x_0)} \big|Du - Dv\big|^2\, dx + c\,\theta^2\,(1+|\Lambda_\rho|)^2 E(x_0,\rho) \\
&\le c(n,N,\nu,L)\,\theta^2\,(1+|\Lambda_\rho|)^2 E(x_0,\rho)\,.
\end{aligned}$$

Denoting by $Q^+_{x_0,2\theta\rho}$ the value minimizing the functional $Q \mapsto \int_{B^+_\rho(x_0)} |u - Q\,x_n|^2\, dx$ amongst all $Q \in \mathbb{R}^N$, and $P^+_{x_0,\rho} = Q^+_{x_0,\rho}\, x_n$, we can proceed as in (3.9)-(3.12) to derive under the additional smallness assumption (see (SC.2) if $p > 2$)

$$E(x_0,\rho) \le \theta^{\frac{2[(t-1)p+tn+2]}{(1-t)(p-2)}} \tag{SCF.4}$$

and (SCF.3) that there holds

$$1 + |\Lambda_\rho| + |\Lambda_{2\theta\rho}| \le 4\,\big(1 + |\Lambda_{\theta\rho}|\big) \tag{4.12}$$

and further

$$\begin{cases}
(2\theta\rho)^{-2} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^2\, dx \le c\,\theta^2\,(1+|\Lambda_{\theta\rho}|)^2\, E(x_0,\rho) \\
(2\theta\rho)^{-p} \fint_{B^+_{2\theta\rho}(x_0)} \big|u - P^+_{x_0,2\theta\rho}\big|^p\, dx \le c\,\theta^2\,(1+|\Lambda_{\theta\rho}|)^p\, E(x_0,\rho)
\end{cases} \tag{4.13}$$

for a constant c depending only on n, N, p, ν and L.

In the last step we have to derive the full decay estimate for the Campanato-type excess $C(x_0,\rho)$. We apply the Caccioppoli Lemma 4.1 with the choice $\xi = Q^+_{2\theta\rho}$ to derive

$$\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^2 + \big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^p\Big]\, dx \\
&\le c \fint_{B^+_{2\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\Big|\frac{u - Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^2 + \Big|\frac{u - Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^p\Big]\, dx \\
&\quad + c\,\omega(\rho^p) \fint_{B^+_{2\theta\rho}(x_0)} (1+|Q^+_{2\theta\rho}|+|Du|)^p\, dx + c \fint_{B^+_{2\theta\rho}(x_0)} \omega(|u|^p)\,(1+|Q^+_{2\theta\rho}|+|Du|)^p\, dx \\
&\quad + \fint_{B^+_{2\theta\rho}(x_0)} \omega(\rho^p\,|Q^+_{2\theta\rho}|^p)\,(1+|Q^+_{2\theta\rho}|+|Du|)^p\, dx \\
&=: II + III + IV + V
\end{aligned} \tag{4.14}$$

with $c = c(n,N,p,\nu,L)$ and the obvious labelling. In what follows we shall assume smallness conditions of the type (SC.4):

$$E(x_0,\rho) \le \frac{\theta^n}{2c_5} \qquad \text{and} \qquad \rho \le \Big(\frac{\theta^n}{2c_5}\Big)^{\frac{1}{p-\beta}} \tag{SCF.5}$$

for a constant c_5 being determined in the course of the proof in dependency of n, N, p, ν and L analogously to the proof of Proposition 3.2. First we calculate using the definition of $E(x_0, \rho)$ and (4.12)

$$\begin{aligned}
\fint_{B^+_{2\theta\rho}(x_0)} (1+|Du|)^p\,dx &\le c(p) \fint_{B^+_{2\theta\rho}(x_0)} |Du - \Lambda_\rho \otimes e_n|^p\,dx + c(p)\,(1+|\Lambda_\rho|)^p \\
&\le c(n,p)\Big[\theta^{-n}\,E(x_0,\rho)+1\Big]\,(1+|\Lambda_\rho|)^p \\
&\le c(p)\,(1+|\Lambda_{\theta\rho}|)^p
\end{aligned} \tag{4.15}$$

and, with $4\theta \le 1$, we also have

$$\fint_{B^+_{4\theta\rho}(x_0)} (1+|Du|)^p\,dx \le c(p)\,(1+|\Lambda_{\theta\rho}|)^p\,. \tag{4.16}$$

The smallness assumptions in (SCF.5) allow us in particular to proceed as in (3.14) and (3.16) to derive

$$\rho^p\,|Q^+_{2\theta\rho}|^p \le \rho + M(x_0,\rho) \qquad \text{and} \qquad |Q^+_{2\theta\rho}|^p \le c\,(1+|\Lambda_{\theta\rho}|)^p\,.$$

We shall start to estimate the terms III, IV and V: the latter estimate combined with (4.16) allows us to compute

$$III \le c\,\omega(\rho^p) \fint_{B^+_{2\theta\rho}(x_0)} (1+|Q^+_{2\theta\rho}|+|Du|)^p\,dx \le c\,\omega(\rho)\,(1+|\Lambda_{\theta\rho}|)^p\,.$$

Moreover, by concavity of the modulus of continuity ω, Jensen's inequality, Poincaré and (SCF.4) we find $\fint_{B^+_{2\theta\rho}(x_0)} \omega(|u|^p)\,dx \le \omega(M(x_0,\rho))$ and hence, applying Hölder and Lemma 4.2 on the higher integrability of Du (keep in mind $q \le q_1$), we see that

$$\begin{aligned}
IV &\le c\Big(\fint_{B^+_{2\theta\rho}(x_0)} \omega(|u|^p)^{\frac{q}{q-p}}\,dx\Big)^{\frac{q-p}{q}} \Big(\fint_{B^+_{2\theta\rho}(x_0)} (1+|Q^+_{2\theta\rho}|+|Du|)^q\,dx\Big)^{\frac{p}{q}} \\
&\le c\Big(\fint_{B^+_{2\theta\rho}(x_0)} \omega(|u|^p)\,dx\Big)^{\frac{q-p}{q}} \fint_{B^+_{4\theta\rho}(x_0)} (1+|Q^+_{2\theta\rho}|+|Du|)^p\,dx \\
&\le c\,\omega\big(M(x_0,\rho)\big)^{\frac{q-p}{q}}\,(1+|\Lambda_{\theta\rho}|)^p\,.
\end{aligned}$$

Finally, the last term is estimated via the bound for $\rho^p|Q^+_{2\theta\rho}|^p$ above and the sublinearity of ω by

$$\begin{aligned}
V &\le c\,\omega(\rho+M(x_0,\rho)) \fint_{B^+_{2\theta\rho}(x_0)} (1+|Q^+_{2\theta\rho}|+|Du|)^p\,dx \\
&\le c\left[\omega(\rho)^{\frac{q-p}{q}} + \omega\big(M(x_0,\rho)\big)^{\frac{q-p}{q}}\right](1+|\Lambda_{\theta\rho}|)^p\,,
\end{aligned}$$

where all the constants c depend only on n, N, p, ν and L. Collecting now the estimates for the various terms (note $\omega(\cdot) \le 1$) we come to the conclusion that

$$\begin{aligned}
III + IV + V &\le c\left[\omega(\rho)^{\frac{q-p}{q}} + \omega\big(M(x_0,\rho)\big)^{\frac{q-p}{q}}\right](1+|\Lambda_{\theta\rho}|)^p \\
&\le c\,E(x_0,\rho)^{p-1}\,(1+|\Lambda_{\theta\rho}|)^p\,E(x_0,\rho) \\
&\le c\,\theta^2\,(1+|\Lambda_{\theta\rho}|)^p\,E(x_0,\rho)\,.
\end{aligned}$$

In the last line we used the fact that $E(x_0,\rho)^{p-1} \le \theta^{(n+4)\,(p-1)} \le \theta^2$ due to (SCF.3). Combining the latter inequality with (4.14) we arrive at

$$\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^2 + \big|Du - Q^+_{x_0,2\theta\rho}\otimes e_n\big|^p\Big]\,dx \\
&\le c \fint_{B^+_{2\theta\rho}(x_0)} \Big[(1+|Q^+_{x_0,2\theta\rho}|)^{p-2}\Big|\frac{u - Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^2 + \Big|\frac{u - Q^+_{x_0,2\theta\rho}x_n}{2\theta\rho}\Big|^p\Big]\,dx \\
&\quad + c\,\theta^2\,(1+|\Lambda_{\theta\rho}|)^p\,E(x_0,\rho)
\end{aligned}$$

and the constant c depends only on n, N, p, ν and L. At this stage we argue exactly as we did to achieve (3.19) in the elliptic situation, i. e., we replace in the inequality above $Q^+_{x_0,2\theta\rho}$ by $\Lambda_{\theta\rho} = (D_n u)_{x_0,\theta\rho}$, and via (4.13) we obtain

$$\fint_{B^+_{\theta\rho}(x_0)} \big[\big(1 + |(D_n u)_{x_0,\theta\rho}|\big)^{p-2} |Du - (D_n u)_{x_0,\theta\rho} \otimes e_n|^2 + |Du - (D_n u)_{x_0,\theta\rho} \otimes e_n|^p\big]\, dx$$
$$\leq c_*\, \theta^2 \left(1 + |(D_n u)_{x_0,\theta\rho}|\right)^p E(x_0,\rho)\,,$$

where the constant c_* depends only on n, N, p, ν and L. Dividing both sides by $(1 + |(D_n u)_{x_0,\theta\rho}|)^p$ and taking into account the definition of $C(x_0,\theta\rho)$ this is the desired excess decay estimate provided that all smallness conditions (SCF-1), (SCF.2), (SCF.3), (SCF.4) and (SCF.5) hold true. The dependency of the constants ε_0 and ε_1 claimed in (4.6) is obtained by taking into consideration the dependencies in all the smallness conditions needed within the proof. □

4.2 Proof of Theorem 1.2

Here, we will only sketch the proof and remark the necessary modification with respect to the proof of Theorem 1.1 in Section 3.2.

Step 1: Choice of the constants. We first fix $\beta \in (0,p)$, and for the choice of α we now distinguish two situations: if $n > p$ we choose $\alpha \in (0,1)$ arbitrary. Otherwise if $n \leq p$, due to the Sobolev embedding, only the case $\alpha \in (1 - \frac{n}{p}, 1)$ has to be considered. Hence, we choose $\gamma = \gamma(\alpha) \in (\max\{0, n-p\}, n)$ such that

$$\alpha \;=\; 1 - \frac{n-\gamma}{p} \tag{4.17}$$

and $\theta \in (0, \frac{1}{8})$ such that

$$\theta \;:=\; \min\Big\{ \Big(\frac{1}{2^p}\Big)^{\frac{1}{\beta}}, \Big(\frac{1}{4\,c_*}\Big)^{\frac{1}{2}}, \Big(\frac{1}{2^p}\Big)^{\frac{1}{n-\gamma}} \Big\}, \tag{4.18}$$

for c_* from Proposition 4.5. Further we fix a constant ε_2 and an iteration quantity $\varepsilon_{it} \leq 1$ via

$$\varepsilon_2 \;:=\; \min\Big\{ \frac{\theta^n}{2^p}, \frac{\varepsilon_0}{4} \Big\}, \qquad \varepsilon_{it} \;:=\; \frac{\theta^n}{4^{3p}\, 6^n} \tag{4.19}$$

with ε_0 from Proposition 4.5. Next, we fix $\delta_1 > 0$ such that

$$\omega(\delta_1)^{\frac{q-p}{qp}} \;<\; \varepsilon_{it}\, \varepsilon_2$$

with q the higher integrability exponent determined in Lemma 4.3. Lastly we define the maximally admissible radius as in (3.24) via

$$\rho_m \;:=\; \min\big\{ \delta_1^{\frac{1}{\beta}}, \delta_1, \varepsilon_1 \big\} \;>\; 0\,.$$

Here, ε_1 is the radius from Proposition 4.5 and in conclusion, all the quantities $\theta, \varepsilon_2, \delta_1$ and ρ_m have the same dependencies as in Section 3.2 for the elliptic case. Note that in the definitions of α, θ and ε_2 there appears the the parameter p (instead of 2 in (3.20),(3.21) and (3.22), respectively) due to the fact that the Campanato-type excess $C(x_0,\rho)$ was redefined with exponent p instead of 2 for the case of variational integrals. For what follows we will always assume $\rho \leq \rho_m < 1$.

Step 2: An almost BMO-estimate. We consider a boundary point $x_0 \in \Gamma$, without loss of generality we may assume $x_0 = 0$, and a radius ρ for which

$$C(x_0,\rho) \;<\; \widetilde{\varepsilon}_{it}\, \varepsilon_2 \qquad \text{and} \qquad M(x_0,\rho) \;<\; \widetilde{\varepsilon}_{it}\, \delta_1 \tag{4.20}$$

is satisfied for some iteration parameter $\widetilde{\varepsilon}_{it} \in [\varepsilon_{it}, 1]$. Taking into account that this time we have

$$\fint_{B^+_{\theta^k\rho}} |Du - (D_n u)_{\theta^k\rho} \otimes e_n|^2\, dx \;<\; 2^{p-1}\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 + 2^{p-1}\, \widetilde{\varepsilon}_{it}\, \varepsilon_2 \fint_{B^+_{\theta^k\rho}} |D_n u|^2\, dx \tag{4.21}$$

we prove exactly as in Section 3.2 that, due to the choice of constants above and due to the decay estimate in Proposition 4.5, there holds for every $k \in \mathbb{N}_0$

$$C(\theta^k\rho) < \widetilde{\varepsilon}_{it}\,\varepsilon_2 \qquad \text{and} \qquad M(\theta^k\rho) < \widetilde{\varepsilon}_{it}\,\delta_1 \,.$$

Step 3: Iteration. We still consider $0 \in \Gamma$ and a radius $\rho \leq \rho_m$ such that (4.20) is satisfied. Then the inequality (4.21) above and the choices of ε_2 in (4.19) and of θ in (4.18) yield that

$$\int_{B^+_{\theta^{k+1}\rho}} \big|Du\big|^p\,dx \leq \theta^\gamma \int_{B^+_{\theta^k\rho}} |Du|^p\,dx + 4^p\,\omega_n\,(\theta^k\rho)^n$$

for γ defined via equation (4.17). Setting $\varphi(t) := \int_{B^+_t} |Du|^p\,dx$ and applying the iteration Lemma 2.5 in a standard way we obtain

$$\int_{B^+_t} |Du|^p\,dx \leq \frac{c_6}{\rho^\gamma}\Big[\int_{B^+_\rho} |Du|^p\,dx + 1\Big]\,t^\gamma \qquad \text{for all } t \leq \rho. \tag{4.22}$$

Step 4: Hölder continuity at boundary points. Exactly as we proceeded in step 4 of the proof of Theorem 1.1 we combine the estimates in the interior and at the boundary. For fixed $\alpha \in (0,1)$ we consider the quantities defined in step 1, we set

$$s := \min\Big\{\varepsilon_{it}\,\delta_1, \Big(\frac{\varepsilon_{it}\,\varepsilon_2}{2}\Big)^{\frac{p}{2}}\Big\}$$

and we look at a point $x_0 \in \Gamma$, w.l.o.g. $x_0 = 0$, for which

$$\liminf_{\rho\to 0^+} \fint_{B^+_\rho(0)} |Du - (D_n u)_{0,\rho} \otimes e_n|^p\,dx < s \quad \text{and} \quad \liminf_{\rho\to 0^+} \rho^\beta \fint_{B^+_\rho(0)} |Du|^2\,dx < s$$

holds true. Then we conclude, again by deriving decay estimates of the form (4.22) on various balls and half-balls and applying Theorem 2.6, that $0 \in \mathrm{Reg}_{\Gamma,\alpha}\,u$, i.e., that u is Hölder continuous with exponent α in a neighbourhood of 0 in B^+. Hence, we have proved a result corresponding to Theorem 3.3 for variational integrals in the model situation of the half ball.

Step 5: Transformation to the model situation. We assume $z = 0$ and $\nu_{\partial\Omega}(z) = e_n$. Keeping the notation of step 5 in Section 3.2, we choose $r > 0$ sufficiently small such that $|\nabla h(x')| < \frac{1}{2}$ for all $x' \in B_{\sqrt{2}r}$. Setting $\tilde{v}(y) := u \circ \mathcal{T}^{-1}(y) - g \circ \mathcal{T}^{-1}(y)$ here allows us to calculate that $\tilde{v} \in W^{1,p}_\Gamma(B^+_r, \mathbb{R}^N)$ is a local minimizer of

$$\mathcal{F}[v] := \int_\Omega \widetilde{F}(y, v, Dv)\,dy\,,$$

where the integrand is defined by

$$\widetilde{F}(y,v,z) := F\big(\mathcal{T}^{-1}(y), v + \tilde{g}(y), z\,D\mathcal{T}(\mathcal{T}^{-1}(y)) + Dg(\mathcal{T}^{-1}(y))\big)\,,$$

and proceeding as in Section 3.2 we calculate that $\widetilde{F}$ satisfies structure conditions analogous to (1.4) for new coefficients (depending on the original constants, the boundary data and p), namely that there holds for all $y, \bar{y} \in B^+_r$, $v, \bar{v} \in \mathbb{R}^N$ and $z, \bar{z} \in \mathbb{R}^{nN}$ the following estimates:

- $2^{-2p}\,\nu\,(1 + \|Dg\|_\infty)^{-p}\,\big(1 + |z|\big)^p \leq \widetilde{F}(y,v,z) \leq 2^{\frac{p}{2}}\,L\,(1 + \|Dg\|_\infty)^p\,\big(1 + |z|\big)^p$,
- $2^{-2p+2}\,\nu\,(1 + \|Dg\|_\infty)^{2-p} \int_{(0,1)^n} (1 + |z| + |D\varphi(x)|)^{p-2}|D\varphi(x)|^2\,dx$
  $\leq \int_{(0,1)^n} \big[\widetilde{F}(y,v,z + D\varphi(x)) - \widetilde{F}(y,v,z)\big]\,dx \qquad \text{for all } \varphi \in C^1_0((0,1)^n, \mathbb{R}^N)$,
- $\big|\widetilde{F}(y,v,z) - \widetilde{F}(\bar{y},\bar{v},z)\big| \leq 2^{p+1}\,L\,(1 + \|Dg\|_\infty)^p\,\big(1 + |z|\big)^p\,\big(\omega\big(|y - \bar{y}| + |v - \bar{v}|\big) + \tau\big(|y - \bar{y}|\big)\big)$,
- $\big|D_{zz}\widetilde{F}(y,v,z) - D_{zz}\widetilde{F}(y,v,\bar{z})\big| \leq 2^{\frac{p}{2}+3}\,L\,(1 + \|Dg\|_\infty)^{p-1}\,\big(1 + |z| + |\bar{z}|\big)^{p-2}\,\mu\big(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\big)$.

Step 6: Transformation of the smallness conditions and final conclusion. The last step of Theorem 1.2 is achieved as in the setting of elliptic systems. The only point where we have to proceed by a slight modifications of the arguments above is when transforming the second of the smallness assumptions since in the case of variational integrals, we have defined the Morrey-type excess $M(x_0, \rho)$ using the full derivative instead of only the normal derivative. Applying Caccioppoli's and Poincaré's inequality, $M(x_0, \rho)$ is reduced to considering only the normal part of the derivatives of $\tilde{v}$, and the result follows as in the proof of Theorem 1.1. □

Remark 4.6: Also in the case of variational integrals, we obtain for $n \leq p$ a better inclusion presented in Remark 3.4.

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