OPTIMAL TRANSPORTATION NETWORKS AS FREE DIRICHLET REGIONS FOR THE MONGE-KANTOROVICH PROBLEM

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ABSTRACT. In the paper the problem of constructing an optimal urban transportation network in a city with given densities of population and of workplaces is studied. The network is modeled by a closed connected set of assigned length, while the optimality condition consists in minimizing the Monge-Kantorovich functional representing the total transportation cost. The cost of trasporting a unit mass between two points is assumed to be proportional to the distance between them when the transportation is carried out outside of the network, and negligible when it is carried out along the network. The same problem can be also viewed as finding an optimal Dirichlet zone minimizing the Monge-Kantorovich cost of transporting the given two measures. The paper basically studies qualitative topological and geometrical properties of optimal networks. A mild regularity result for optimal networks is also provided.

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1. Introduction

Let an open bounded set $\Omega \subset \mathbb{R}^n$ stand for the city where the density of population is given by a finite Borel measure φ^+. The aim of an urban transportation network (i.e. the set of metro and street traffic routes) is to provide the easiest access of the people to their workplaces, the density of which is supposed to be a given finite Borel measure φ^-. Such a transportation network will be modeled by a closed connected set $\Sigma \subset \bar{\Omega}$. It is reasonable to suppose that getting to a workplace without using urban traffic in general would cost a single citizen an effort (estimated in time required and, at the end, also in terms of respective financial loss) proportional to the actual walking distance while the cost of using the urban traffic (the price of the ticket) is independent on the actual lenght of the traffic route used, and, moreover, for simplicity can be considered negligible with respect the cost of transportaion without using urban traffic. Under this assumption the total cost of everyday movement of the city population to their workplaces can be modeled by the functional I_Σ defined on the set of finite Borel measures over $\bar{\Omega} \times \bar{\Omega}$ by the relationship

$$I_\Sigma(\gamma) := \int_{\bar{\Omega}\times\bar{\Omega}} d_\Sigma(x,y)\, d\gamma(x,y),$$

where the distance function d_Σ in $\bar{\Omega}$ is defined by the formula

$$d_\Sigma(x,y) := d(x,y) \wedge (\mathrm{dist}\,(x,\Sigma) + \mathrm{dist}\,(y,\Sigma)),$$

d standing for the geodesic distance on $\bar{\Omega}$ (which is equal to the standard Euclidean one when Ω is convex) and $\mathrm{dist}\,(x,\Sigma)$ standing for the distance between x and Σ, i.e.

$$\mathrm{dist}\,(x,\Sigma) := \inf_{z\in\Sigma} d(x,z).$$

In other words, each resident x to reach a destination y will choose the most convenient route between avoiding the urban transportation network (i.e. "not taking any urban traffic line and going by feet"), in which case he has to walk the distance $d(x,y)$, or using the latter, in which case he has to walk the distance $\mathrm{dist}\,(x,\Sigma) + \mathrm{dist}\,(y,\Sigma)$.

The measure γ can be viewed here as the "plan of trasportation" (i.e. in a very intuitive way one can think that $\gamma(x,y)$ stands for the amount of people which moves from point x to point y) and has to satisfy the restriction

$$\pi^+_\#\gamma - \pi^-_\#\gamma = \varphi^+ - \varphi^- \text{ over } \bar{\Omega} \setminus \Sigma, \tag{1}$$

where $\pi^\pm\colon \bar{\Omega} \times \bar{\Omega} \to \bar{\Omega}$ stand for the projections on the first and the second factor respectively, namely, $\pi^\pm(x^+, x^-) := x^\pm$. The generalized Monge-Kantorovich optimal transport problem consists in fact in finding a Borel measure γ_{opt} over $\bar{\Omega} \times \bar{\Omega}$ (called *optimal transport plan*) minimizing I_Σ over all Borel measures γ in $\bar{\Omega} \times \bar{\Omega}$ satisfying (1) (called *admissible* measures). It is quite easy to show that such a problem always admits a solution. More realistic models where different pricing policies for using the transportation network are considered can be found in [6].

We will further refer to the set Σ as a Dirichlet constraint for the Monge-Kantorovich problem. The reason for this name stems from the fact that, as shown in [4], the above mass transportation problem is equivalent to the system of PDEs

$$\begin{cases} -\mathrm{div}\,(\mu\nabla_\mu u) &= \varphi^+ - \varphi^- \text{ in } \bar\Omega\setminus\Sigma,\\ |\nabla_\mu u| &= 1 \qquad \mu-\text{a.e. in } \bar\Omega,\\ \mu(\Sigma) &= 0 \end{cases} \tag{2}$$

with respect to the unknown 1-Lipschitz function u called the *Kantorovich potential* and the unknown positive Radon measure μ called the *transport density*, where ∇_μ is the tangential gradient operator with respect to μ defined in [4]. From the point of view of the model chosen, Σ stands for the zone where the cost of transporting the measure φ^+ to φ^- vanishes. In the sequel we will denote by $MK(\varphi^+,\varphi^-,\Sigma)$ both the above Monge-Kantorovich problem with the Dirichlet constraint Σ and the respective minumum value of the functional I_Σ. We will also omit the reference to Σ and write simply $MK(\varphi^+,\varphi^-)$, if $\Sigma=\emptyset$.

This paper is devoted to the problem of finding the optimal Dirichlet constraint Σ (or, in terms of the above model, the optimal urban transportation network) for the Monge-Kantorovich problem of transporting φ^+ to φ^- under the given constraint on the total length of Σ, i.e. to the problem of finding

$$\min\left\{MK(\varphi^+,\varphi^-,\Sigma)\,:\,\Sigma\subset\bar\Omega \text{ closed and connected}, \mathcal{H}^1(\Sigma)\le\alpha\right\}, \tag{3}$$

where $\alpha>0$ is given. The length constraint corresponds to the restriction on the cost of building the urban transportation network which, of course, depends on the length of the latter.

It is worth remarking that the above model does not necessarily require $\varphi^+(\bar\Omega)=\varphi^-(\bar\Omega)$. In fact, it even remains reasonable when $\varphi^-=0$ while $\varphi^+\neq 0$. In the latter case one simply has that

$$MK(\varphi^+,\varphi^-,\Sigma)=\int_{\bar\Omega}\mathrm{dist}\,(x,\Sigma)\,d\varphi^+(x)$$

is just the average distance functional, and hence the problem (3) can be interpreted as that of finding the urban transportation network Σ_{opt} which is in average most easily reachable by the city population and satisfies the given length constraint. This model situation has been considered in [5] where the reader can find also several numerical computations.

2. Preliminaries on connected spaces

We recall that a topological space Σ is called

- *connected*, if it contains no subset except Σ and $\emptyset$ which is both closed an open,
- *continuum*, if it is connected and compact,
- *metric continuum*, if it a metric space and a continuum in the topology induced by the metric,
- *locally connected*, if every open set contains a connected open subset.

A *path* (resp. *arc*) in Σ is defined as a continuous (resp. homeomorphic) image of an interval. Namely, we call a path (resp. arc) $[a,b]\subset\Sigma$ connecting the couple of points $\{a,b\}\subset\Sigma$ an image $\mathrm{Im}\,\gamma$ of some continuous (resp. continuous injective) map γ: $[0,L]\subset\mathbb{R}\to\Sigma$ satisfying $\gamma(0)=a$ and $\gamma(L)=b$ for some $L\in\mathbb{R}$. We also say in this case that the path (resp. arc) $[a,b]\subset\Sigma$ starts at a and ends at b. The map γ is called a parametrization of $[a,b]$. For the sake of brevity we will frequently abuse the notation identifying the path (resp. arc) with its parametrization, i.e.

writing $\gamma = [a, b]$ instead of $\mathrm{Im}\,\gamma = [a, b]$. A path (resp. arc) is called *Lipschitz*, if it admits Lipschitz-continuous parametrization. If γ_1 and γ_2 are two paths satisfying $\gamma_1(1) = \gamma_2(0)$, then we set $\gamma_1 \circ \gamma_2$ to be a path defined by

$$\gamma_1 \circ \gamma_2(t) := \begin{cases} \gamma_1(2t), & t \in [0, 1/2), \\ \gamma_2(2t-1), & t \in [1/2, 1]. \end{cases}$$

At last, we let (a, b) stand for the path (resp. arc) without endpoints connecting a and b, i.e. $(a, b) := [a, b] \setminus \{a, b\}$.

Further, a topological space Σ is called

- *arcwise connected*, if every couple of points $\{a, b\} \subset \Sigma$, $a \neq b$, is connected by some arc $[a, b] \subset \Sigma$.
- *locally arcwise connected*, if every open set contains an arcwise connected open subset.

Note that according to these definition formally every topological space consisting of a single point is always considered to be arcwise connected and locally arcwise connected.

It is well-known that

(i) every arcwise connected (resp. locally arcwise connected) space is connected (resp. locally connected);
(ii) the reverse implication to the above is, generally speaking, false;
(iii) nevertheless, a complete locally connected space is locally arcwise connected (Mazurkiewicz-Moore-Menger theorem II.1 from [10, § 50]);
(iv) a connected locally arcwise connected space is arcwise connected (theorem I.2 from [10, § 50]);
(v) a continuum is locally connected (hence also locally arcwise connected and arcwise connected by (iii) and (iv)), if an only if it is a path, i.e. a continuous image of an interval (Hahn-Mazurkiewicz-Sierpiński theorem II.2 from [10, § 50]);
(vi) in view of (v), every path connecting a to b, $a \neq b$, contains an arc connecting those points. In particular, a topological space Σ is arcwise connected (resp. locally arcwise connected), if and only if every couple of points in Σ is connected by some path $[a, b] \subset \Sigma$ (resp. if every open set $U \subset \Sigma$ contains an open subset $V \subset U$ such that every couple of points of the latter can be connected by a path contained in V).

We now introduce the following notions.

Definition 2.1. *Let Σ be a connected space. Then $x \in \Sigma$ is called noncut point of Σ, if $\Sigma \setminus \{x\}$ is connected. Otherwise, x is called cut point of Σ.*

The following general assertion (Moore theorem IV.5 from [10, § 47]) holds.

Proposition 2.1. *Every continuum has at least two noncut points.*

Another relevant notion is that of an order of a point and of branching points and endpoints.

Definition 2.2. *Let Σ be a topological space. We will say that the order of the point $x \in \Sigma$ does not exceed $\mathfrak{n}$, writing*

$$ord_x \Sigma \leq \mathfrak{n},$$

where $\mathfrak{n}$ is a cardinal, if for every $\varepsilon > 0$ there is an open subset $U \subset \Sigma$ such that $x \in U$, $diam(U) < \varepsilon$ and $\#\partial U \leq \mathfrak{n}$, $\#$ standing for cardinality of a set.

The order of the point $x \in \Sigma$ is said to be equal $\mathfrak{n}$, written

$$ord_x\Sigma = \mathfrak{n},$$

if $\mathfrak{n}$ is the least cardinal for which $ord_x\Sigma \leq \mathfrak{n}$.

If $ord_x\Sigma = \mathfrak{n}$, with $\mathfrak{n} \geq 3$, then x will be called branching point of Σ, while if $ord_x\Sigma = 1$, then x will be called endpoint of Σ.

We recall that according to the theorem V.1 from [10, § 51], if $\mathrm{ord}_x\Sigma \leq 1$, then x is a noncut point of Σ. In particular, every endpoint of Σ is a noncut point.

Further, the point $x \in \Sigma := \cup_{i=1}^{k}[x, a_i]$ belonging to the union Σ of k arcs $[x, a_1], \ldots, [x, a_k]$ which does not intersect elsewhere but in x, i.e. $[x, a_i] \cap [x, a_j] = \{x\}$ for all $i, j = 1, \ldots, k$, has order k in Σ. Vice versa, the following proposition (Claim 8 in [10, § 51.II]) is valid.

Proposition 2.2 (Menger n-Beinsatz)**.** *Let $x \in \Sigma$, where Σ is a locally connected continuum. Then*

$$ord_x\Sigma \geq k, \qquad k \in \mathbb{N},$$

implies the existence of at least k arcs $\{\gamma_i\}_{i=1}^{k}$, $\gamma_i \subset \Sigma$, $i = 1, \ldots, k$ starting at x which are pairwise nonintersecting except in the point x, i.e. $\gamma_i \cap \gamma_j = \{x\}$ for all $i, j = 1, \ldots, k$.

We remark that in view of the above proposition 2.2, whenever Σ is a locally connected continuum which does not contain any simple closed curve (a homeomorphic image of S^1), then every noncut point of Σ is its endpoint. In fact, if $x \in \Sigma$ is a noncut point and $\mathrm{ord}_x\Sigma \geq 2$, then there exist at least two arcs $[x, a_1] \subset \Sigma$ and $[x, a_2] \subset \Sigma$ such that $[x, a_1] \cap [x, a_2] = \{x\}$. But since $\Sigma \setminus \{x\}$ is still arcwise connected and hence there is an arc $\gamma \subset \Sigma \setminus \{x\}$ connecting a_1 and a_2. But then $[x, a_1] \cup \gamma \cup [a_2, x]$ forms a simple closed curve.

3. Existence of optimal sets

Theorem 3.1. *Problem (3) admits a solution Σ_{opt}.*

PROOF: Let

$$m := \inf\left\{MK(\varphi^+, \varphi^-, \Sigma) \,:\, \Sigma \subset \Omega \text{ closed and connected}, \mathcal{H}^1(\Sigma) \leq \alpha\right\}$$

and consider a sequence $\{\Sigma_\nu\}_{\nu=1}^{\infty}$ of closed connected subsets of $\bar{\Omega}$, satisfying $\mathcal{H}^1(\Sigma_\nu) \leq \alpha$ for all $\nu \in \mathbb{N}$ and such that

$$MK(\varphi^+, \varphi^-, \Sigma_\nu) \searrow m,$$

that is, a minimizing sequence for the functional $MK(\varphi^+, \varphi^-, \cdot)$. The space of closed subsets of $\bar{\Omega}$ endowed by the Hausdorff metric is a compact metric space according to the Blaschke theorem (theorem 4.4.6 of [2]), and therefore up to a subsequence (not relabeled) we may assume that $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff convergence, while $\Sigma \subset \bar{\Omega}$ is still closed and connected. The Golab semicontinuity theorem (theorem 4.4.7 of [2]) implies then $\mathcal{H}^1(\Sigma) \leq \alpha$. Observe now that the Hausdorff convergence implies $d(x, \Sigma_\nu) \to d(x, \Sigma)$ for all $x \in \bar{\Omega}$, and hence

$$d_{\Sigma_\nu}(x, y) \to d_\Sigma(x, y)$$

for all $(x,y) \in \bar\Omega \times \bar\Omega$. Moreover, since all the functions d_{Σ_ν} are Lipschitz continuous with the same Lipschitz constant, then $d_{\Sigma_\nu} \to d_\Sigma$ uniformly in $\bar\Omega$.

We modify the measures $\varphi^\pm$ over each Σ_ν in order to obtain new Borel measures $\varphi^\pm_\nu$ over $\bar\Omega$ satisfying

$$\begin{aligned} \varphi^\pm_\nu \llcorner \bar\Omega \setminus \Sigma_\nu &= \varphi^\pm \llcorner \bar\Omega \setminus \Sigma_\nu, \quad \varphi^\pm_\nu(\bar\Omega) \leq \varphi^+(\bar\Omega) \vee \varphi^-(\bar\Omega) \text{ and} \\ \varphi^+_\nu(\bar\Omega) &= \varphi^-_\nu(\bar\Omega). \end{aligned}$$

By lemma 3.1 below there exists an optimal transport plan γ_ν of the problem $MK(\varphi^+_\nu, \varphi^-_\nu, \Sigma_\nu)$, and hence of $MK(\varphi^+, \varphi^-, \Sigma_\nu)$, satisfying $\pi^\pm_\# \gamma_\nu = \varphi^\pm_\nu$, and hence by construction the sequence $\{\gamma_\nu(\bar\Omega \times \bar\Omega)\}_{\nu=1}^\infty$ is bounded. Therefore, up to a subsequence (again not relabeled) $\gamma_\nu \rightharpoonup \gamma$ $*$-weakly in the sense of measures, where γ is some positive Borel measure over $\bar\Omega$. Clearly then

$$\pi^+_\# \gamma - \pi^-_\# \gamma = \varphi^+ - \varphi^- \text{ on } \bar\Omega \setminus \Sigma,$$

or, in other words, γ is an admissible measure for the problem $MK(\varphi^+, \varphi^-, \Sigma)$. In fact, since $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff, then every function with compact support in $\bar\Omega \setminus \Sigma$ has also compact support in $\bar\Omega \setminus \Sigma_\nu$ for sufficiently large $\nu \in \mathbb{N}$, and therefore, for every $\psi \in C_0(\bar\Omega \setminus \Sigma)$ one has

$$\int_{\bar\Omega} \psi \, d(\pi^+_\# \gamma - \pi^-_\# \gamma) = \lim_\nu \int_{\bar\Omega} \psi \, d(\pi^+_\# \gamma_\nu - \pi^-_\# \gamma_\nu) = \int_{\bar\Omega} \psi \, d(\varphi^+ - \varphi^-).$$

At last it remains to observe that

$$MK(\varphi^+, \varphi^-, \Sigma) \leq \int_{\bar\Omega \times \bar\Omega} d_\Sigma(x,y) \, d\gamma(x,y) = \lim_\nu \int_{\bar\Omega \times \bar\Omega} d_{\Sigma_\nu}(x,y) \, d\gamma_\nu(x,y) = m,$$

which means that Σ is a minimizer. $\square$

Lemma 3.1. *Let $\varphi^+(\bar\Omega) = \varphi^-(\bar\Omega)$. Then there exists an optimal transport plan γ of the problem $MK(\varphi^+, \varphi^-, \Sigma)$ such that*

$$\pi^\pm_\# \gamma = \varphi^\pm \text{ on } \bar\Omega.$$

PROOF: Let γ' be an arbitrary optimal transport plan for $MK(\varphi^+, \varphi^-, \Sigma)$. Define the measure γ over $\bar\Omega \times \bar\Omega$ by

$$\gamma(e) := \gamma'(e) - \delta^+(\pi^+(e \cap \Delta) \cap (\bar\Omega \setminus \Sigma)) - (\delta^+ \otimes \delta^-)(e \cap (\Sigma \times \Sigma))/|\delta^-|(\Sigma),$$

where

$$\delta^\pm := \pi^\pm_\# \gamma' - \varphi^\pm$$

$\Delta := \{(x,x) : x \in \bar\Omega\}$. Since γ' is an admissible transport plan, then $\delta^+ = \delta^-$ over $\bar\Omega \setminus \Sigma$, and therefore $|\delta^-|(\Sigma) = |\delta^+|(\Sigma)$. Now, a simple calculation for a Borel set $e \subset \bar\Omega$ shows

$$\begin{aligned} \gamma(e \times \bar\Omega) &= (\pi^+_\# \gamma')(e) - (\pi^+_\# \gamma')(e \setminus \Sigma) \\ &\qquad + \varphi^+(e \setminus \Sigma) \\ &\qquad - \delta^+(e \cap \Sigma) \\ &= \varphi^+(e), \end{aligned}$$

and symmetrically, $\gamma(\bar\Omega \times e) = \varphi^-(e)$. Hence γ is still admissible for the problem $MK(\varphi^+, \varphi^-, \Sigma)$, while

$$\pi^\pm_\# \gamma = \varphi^\pm \text{ on } \bar\Omega.$$

It is enough now to observe that since the function d_Σ vanishes over the diagonal Δ and over $\Sigma\times\Sigma$, then $I_\Sigma(\gamma)=I_\Sigma(\gamma')$, that is, γ is still an optimal transport plan for $MK(\varphi^+,\varphi^-,\Sigma)$. $\square$

4. Auxiliary constructions

Consider the Monge-Kantorovich problem $MK(\varphi^+,\varphi^-,\Sigma)$ with the Dirichlet condition (i.e. with $\Sigma\neq\emptyset$). From now on we will be silently assuming that Ω is a convex set. Of course, without loss of generality we may consider $\varphi^\pm(\Sigma)=0$. Denote

$$\mathcal{R}:=\{(x,y)\in\bar\Omega\times\bar\Omega\,:\,\mathrm{dist}\,(x,\Sigma)+\mathrm{dist}\,(y,\Sigma)>d(x,y)\}$$

and observe that $d_\Sigma(x,y)=d(x,y)$ over $\mathcal{R}$. Set now

$$\varphi_r^\pm:=\pi^\pm_{\#}(\gamma\,\llcorner\mathcal{R}),\qquad \varphi_s^\pm:=\pi^\pm_{\#}(\gamma\,\llcorner(\bar\Omega\times\bar\Omega)\setminus\mathcal{R})\,\llcorner(\bar\Omega\setminus\Sigma),$$

where γ is an optimal plan for the Monge-Kantorovich problem $MK(\varphi^+,\varphi^-,\Sigma)$. It is worth noting that, since according to our assumption $\varphi^\pm$ do not charge Σ, one has $\varphi^\pm=\varphi_r^\pm+\varphi_s^\pm$. We denote $\varphi_s:=\varphi_s^++\varphi_s^-$.

Define $K\colon\bar\Omega\multimap\Sigma$ to be the set-valued projection to Σ according to the formula

$$K(x):=\{y\in\Sigma\,:\,d(x,y)=\mathrm{dist}\,(x,\Sigma)\}\,.$$

The map K admits a Borel measurable selection $k\colon\bar\Omega\to\Sigma$. Let then

$$\gamma^+(e):=\varphi_s^+(\{x\,:\,(x,k(x))\in e\}),\ \text{and}\ \gamma^-(e):=\varphi_s^-(\{y\,:\,(k(y),y)\in e\})$$

for all Borel sets $e\subset\bar\Omega\times\bar\Omega$. At last, set $\psi^\mp:=\pi^\mp_{\#}\gamma^\pm$ and $\psi:=\psi^++\psi^-$.

We also will use the dual setting for the problem $MK(\varphi^+,\varphi^-,\Sigma)$. The latter consists in finding a $u\in\mathrm{Lip}_{1,\Sigma}(\Omega,\Sigma)$ (called *Kantorovich transport potential*) which maximizes the functional

$$MK'(v):=\int_{\bar\Omega}v\,d(\varphi^+-\varphi^-).$$

over all $v\in\mathrm{Lip}_{1,\Sigma}(\Omega)$, where $\mathrm{Lip}_{1,\Sigma}(\Omega)$ stands for the set of Lipschitz continuous functions on $\bar\Omega$ vanishing on Σ and having Lipschitz constant one. In particular, if $\Sigma=\emptyset$, then $\mathrm{Lip}_{1,\Sigma}(\Omega)=\mathrm{Lip}_1(\Omega)$ is just the set of 1-Lipschitz continuous functions on $\bar\Omega$.

The following result holds.

Lemma 4.1. *Assume that there exist Borel measures $\tilde\varphi_r^\pm$ and $\tilde\varphi_s^\pm$ over $\bar\Omega$ such that $\tilde\varphi_r^+(\bar\Omega)=\tilde\varphi_r^-(\bar\Omega)$ and $\varphi^\pm=\tilde\varphi_r^\pm+\tilde\varphi_s^\pm$. Then*

$$\begin{aligned}MK(\varphi^+,\varphi^-,\Sigma)\le MK(\tilde\varphi_r^+,\tilde\varphi_r^-)+&\\ \int_{\bar\Omega}\mathrm{dist}\,(z,\Sigma)\,d\tilde\varphi_s^+(z)+\int_{\bar\Omega}\mathrm{dist}\,(z,\Sigma)\,d\tilde\varphi_s^-(z).&\end{aligned}$$

If $\tilde\varphi_r^\pm=\varphi_r^\pm$ and $\tilde\varphi_s^\pm=\varphi_s^\pm$, then the above estimate becomes an equality.

Proof: Let γ stand for an optimal transport plan of $MK(\varphi^+,\varphi^-,\Sigma)$. Denote by $\tilde\gamma_r$ an optimal transport plan for the problem $MK(\tilde\varphi_r^+,\tilde\varphi_r^-)$, and set

$$\tilde\gamma^+(e):=\tilde\varphi_s^+(\{x\,:\,(x,k(x))\in e\}),\qquad\tilde\gamma^-(e):=\tilde\varphi_s^-(\{y\,:\,(k(y),y)\in e\})$$

for all Borel sets $e\subset\bar\Omega\times\bar\Omega$. Clearly, in this case $\tilde\gamma^\pm$ are optimal transport plans for problems $MK(\tilde\varphi_s^\pm,\tilde\psi^\mp)$, where $\tilde\psi^\mp:=\pi^\mp_{\#}\tilde\gamma^\pm$. Due to the optimality of γ and to the fact that the measure $\tilde\gamma_r+\tilde\gamma^+ +\tilde\gamma^-$ satisfies the admissibility condition (1), one has

$$\begin{aligned} MK(\varphi^+,\varphi^-,\Sigma) &= \int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\gamma(x,y)\\ &\leq \int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d(\tilde\gamma_r+\tilde\gamma^+ +\tilde\gamma^-)(x,y)\\ &= \int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\tilde\gamma_r(x,y)+\int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\tilde\gamma^+(x,y)+\\ &\qquad\qquad \int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\tilde\gamma^-(x,y). \end{aligned}$$

But

$$\int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\tilde\gamma_r(x,y)\leq \int_{\bar\Omega\times\bar\Omega} d(x,y)\,d\tilde\gamma_r(x,y)=MK(\tilde\varphi_r^+,\tilde\varphi_r^-),$$

while

$$\int_{\bar\Omega\times\bar\Omega} d_\Sigma(x,y)\,d\tilde\gamma^\pm(x,y)=\int_{\bar\Omega} \mathrm{dist}\,(z,\Sigma)\,d\tilde\varphi_s^\pm(z),$$

thus showing the first part of the statement.

To prove the second part, we note that

$$\begin{aligned} MK(\varphi^+,\varphi^-,\Sigma) &= I^0_\Sigma(\gamma)+I'_\Sigma(\gamma),\ \text{where}\\ I^0_\Sigma(\gamma) &:= \int_{\mathcal R} d_\Sigma(x,y)\,d\gamma(x,y),\\ I'_\Sigma(\gamma) &:= \int_{(\bar\Omega\times\bar\Omega)\setminus\mathcal R} d_\Sigma(x,y)\,d\gamma(x,y). \end{aligned}$$

Since $d_\Sigma=d$ over $\mathcal R$, one has

$$I^0_\Sigma(\gamma)=MK(\varphi_r^+,\varphi_r^-).$$

It remains to observe that

$$\begin{aligned} I'_\Sigma(\gamma) &= \int_{\bar\Omega\times\bar\Omega} d(x,K(x))\,d\gamma_{\mathcal R^c}(x,y)+\int_{\bar\Omega\times\bar\Omega} d(y,K(y))\,d\gamma_{\mathcal R^c}(x,y)\\ &= \int_{\bar\Omega} d(x,K(x))\,d(\varphi_s^+ +\tilde\psi^+)(x)+\int_{\bar\Omega} d(y,K(y))\,d(\varphi_s^- +\tilde\psi^-)(y)\\ &= \int_{\bar\Omega\setminus\Sigma} \mathrm{dist}\,(x,\Sigma)\,d\varphi_s^+(x)+\int_{\bar\Omega\setminus\Sigma} \mathrm{dist}\,(y,\Sigma)\,d\varphi_s^-(y), \end{aligned}$$

where $\gamma_{\mathcal R^c}:=\gamma\,\llcorner(\bar\Omega\times\bar\Omega\setminus\mathcal R)$ and $\tilde\psi^\pm$ are some measures concentrated on Σ. $\square$

We will also use the following constructions from [8]. Let u stand for a Kantorovich potential of $MK(\varphi^+,\varphi^-)$. We call a *transport ray* the maximum segment $R=(l^+,l^-)$ not including endpoints l^+, l^- such that

$$u(l^+)-u(l^-)=d(l^+,l^-).$$

For every $p\in\mathbb R$ the set

$$\{x\in\Omega\,:\,u(x)=p\text{ and }\exists\nabla u(x)\neq 0\}$$

admits a Borel covering $\{S_{pi}\}_{i=1}^\infty$ such that there exist Lipschitz coordinates U: $\mathbb R^n\to\mathbb R^{n-1}$ and V: $\mathbb R^{n-1}\to\mathbb R^n$ satisfying $V(U(x))=x$ for all $x\in S_{pi}$. For each triple $(p,i,j)\in\mathbb Q\times\mathbb N^2$ one can define now the "ray clusters" T_{pij} as the union of transport rays R piercing S_{pi} in some point which is distant at least $1/j$ from both endpoints of R. Note that without loss of generality these clusters can be considered disjoint, while it is easy to prove that every transport ray R of $MK(\varphi^+,\varphi^-)$ belongs to some of these clusters. Each cluster T_{pij} then admits coordinates $G := G_{pij}$: $T_{pij} \to \mathbb{R}^{n-1} \times \mathbb{R}$, which are Lipschitz over

$$T_{pij}^{\sigma} := \left\{x \in T_{pij} \,:\, \min\{d(z,l^+), d(z,l^-)\} \geq \sigma > 0\right\},$$

where l^+ and l^- stand for the ends of the unique transport ray R_x passing through x, and admit the Lipchitz inverse $F = F_{pij}$: $G(T_{pij}) \to \mathbb{R}^n$ given by the formula

$$F(X, x_n) := V(X) + x_n \nu(V(X)),$$

where $\nu(x)$ stands for the unit vector pointng in the direction of R_x. Moreover, one has

$$G(x) = (U(z), u(x) - u(z)), \text{ where } \{z\} := R_x \cap S_{pi}.$$

In particular, all the clusters T_{pij} are Borel measurable. Moreover, the set $\mathcal{E}$ of endpoints of all transport rays is also Borel measurable and $\mathcal{L}^n(\mathcal{E}) = 0$. Hence, if $\varphi^\pm << \mathcal{L}^n$, we may consider without loss of generality the latter measures to be concentrated over the union of all the ray clusters. Further, in this case one has the mass balance relationship

$$\varphi^+(T_{pij}) = \varphi^-(T_{pij}),$$

for each ray cluster T_{pij}, as well as the relationship

$$MK(\varphi^+,\varphi^-) = \sum_{(p,i,j)\in\mathbb{Q}\times\mathbb{N}^2} MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij}). \tag{4}$$

With the use of this construction we may prove the following statement.

Lemma 4.2. *Let $\varphi^\pm << \mathcal{L}^n$, $\varphi^+ \neq \varphi^-$ and $\Sigma = \Sigma_{opt}$. Then*

$$\varphi_s(\bar{\Omega}) > 0.$$

PROOF: Assume the contrary, namely, that both $\varphi_s^+(\bar{\Omega}) = 0$ and $\varphi_s^-(\bar{\Omega}) = 0$, i.e. $\varphi^\pm = \varphi_r^\pm$. Note that this can only happen when $\varphi^+(\bar{\Omega}) = \varphi^-(\bar{\Omega})$, since by definition $\varphi_r^+(\bar{\Omega}) = \varphi_r^-(\bar{\Omega})$.

Our assumption means that $MK(\varphi^+,\varphi^-,\Sigma_{opt}) = MK(\varphi^+,\varphi^-)$. To arrive at a contradiction, it is enough therefore to find a closed connected Σ with very small length such that $MK(\varphi^+,\varphi^-,\Sigma) < MK(\varphi^+,\varphi^-)$. In order to do so, in view of (4), it is enough to show the existence of a sequence of $\varepsilon_k \to 0^+$ (further on the index k will be omitted for brevity) and of closed connected sets Σ_ε with $\mathcal{H}^1(\Sigma_\varepsilon) \leq C\varepsilon$ for some $C > 0$ such that

$$MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij}, \Sigma_\varepsilon) < MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij}) \tag{5}$$

just for some particular ray cluster T_{pij} of $MK(\varphi^+,\varphi^-)$, where the ray clusters are chosen in such a way that

$$\operatorname{diam} S_{pi} \leq \varepsilon$$

for all $(p,i) \in \mathbb{Q}\times\mathbb{N}$. In fact, one obviously has

$$MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij}, \Sigma_\varepsilon) \leq MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij})$$

for all $(p,i,j) \in \mathbb{Q}\times\mathbb{N}^2$, and

$$MK(\varphi^+,\varphi^-,\Sigma_\varepsilon) \leq \sum_{(p,i,j)\in\mathbb{Q}\times\mathbb{N}^2} MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij}, \Sigma_\varepsilon).$$

Consider for this purpose a ray cluster T_{pij} for which

$$\phi^+ := \varphi^+ \llcorner T_{pij} \neq \phi^- := \varphi^- \llcorner T_{pij}$$

(such a cluster exists since otherwise one would have $\varphi^+ = \varphi^-$ contrary to the assumption).

From now on we will concentrate on the problem $MK(\phi^+, \phi^-)$. For $c \in \mathbb{R}$ and $\varepsilon > 0$ set

$$\Gamma_{c,\varepsilon} := \{(x,y) \in \bar{\Omega} \times \bar{\Omega} \,:\, u(x) \geq c + \varepsilon/2 \text{ and } u(y) \leq c - \varepsilon/2\}.$$

Let us show now that there exists a $c \in \mathbb{R}$ and an $\varepsilon_0 > 0$ such that

$$\gamma(\Gamma_{c,\varepsilon}) > 0 \tag{6}$$

for each $\varepsilon < \varepsilon_0$, where γ stands for the optimal transport plan of the problem $MK(\phi^+, \phi^-)$. For this purpose, supposing the contrary, we have the existence for every $c \in \mathbb{R}$ of a sequence $\varepsilon_k \to 0^+$ as $k \to \infty$ such that $\gamma(\Gamma_{c,\varepsilon_k}) = 0$ for each sufficiently large $k \in \mathbb{N}$. Let now

$$\Gamma_c := \{(x,y) \in \bar{\Omega} \times \bar{\Omega} \,:\, u(x) > c\} \text{ and } \Gamma'_c := \{(x,y) \in \Gamma_c \,:\, u(y) < c\}.$$

One has then $\gamma(\Gamma'_c) = 0$ for all $c \in \mathbb{R}$. In fact, for all $(x,y) \in \Gamma'_c$ one has $u(x) \geq c + \varepsilon/2$ and $u(y) \leq c - \varepsilon/2$ for each sufficently small $\varepsilon > 0$, and hence $\Gamma'_c \subset \Gamma_{c,\varepsilon_k}$ whenever $k \in \mathbb{N}$ is sufficiently large, while the latter sets are γ-negligible. Consider now the set

$$\Gamma := \{(x,y) \in \bar{\Omega} \times \bar{\Omega} \,:\, u(x) \neq u(y)\}.$$

For every $(x,y) \in \Gamma$ there is a $c \in \mathbb{Q}$ such that $u(x) > c$ and $u(y) < c$, that is, $(x,y) \in \Gamma'_c$. Hence,

$$\Gamma = \bigcup_{c \in \mathbb{Q}} \Gamma'_c,$$

which implies $\gamma(\Gamma) = 0$. Therefore, for γ-a.e. $(x,y) \in \bar{\Omega} \times \bar{\Omega}$ one has $u(x) = u(y)$, and hence $MK(\phi^+, \phi^-) = 0$ which implies $\phi^+ = \phi^-$ contradicting the original assumption and proving the claim.

Choose now $c \in \mathbb{R}$ such that (6) is satisfied for all sufficiently small $\varepsilon > 0$. Let $\bar{Q}_\varepsilon \subset \mathbb{R}^{n-1} \times \mathbb{R}^1$ stand for for the closed parallelepiped with the base of side $\lambda_U \varepsilon$ and height ε, centered at $(0, \ldots, 0, c)$, where λ_U stands for the Lipschitz constant of U. Note that in this way the projection of $U(S_{pi})$ to the plane $\{x_n = 0\}$ is contained in $\bar{Q}_\varepsilon$. Denote by $\Sigma'_\varepsilon \subset \bar{Q}_\varepsilon$ the union of two uniform grids in the upper and lower faces of $\bar{Q}_\varepsilon$ dividing each of those faces into equal smaller $n-1$-dimensional cubes with side $\varepsilon/2(n-1)^{1/2}\lambda_F$, and a vertical segment of length ε connecting them, where λ_F stands for the Lipschitz constant of F (see figure 1). Finally, let $\Sigma_\varepsilon := F(\Sigma'_\varepsilon)$. Clearly, $\mathcal{H}^1(\Sigma_\varepsilon) \leq C\varepsilon$ for a $C > 0$ which depends only on n, λ_U and λ_F (note that the latter two constants can be considered independent of ε).

Then for γ-a.e. $(x,y) \in \Gamma_{c,\varepsilon}$ one has

$$d_{\Sigma_\varepsilon}(x,y) \leq d(x,y) - \varepsilon + \varepsilon/2 < d(x,y),$$

and hence,

$$MK(\phi^+, \phi^-, \Sigma_\varepsilon) < MK(\phi^+, \phi^-)$$

which shows (5) and thus concludes the proof. □

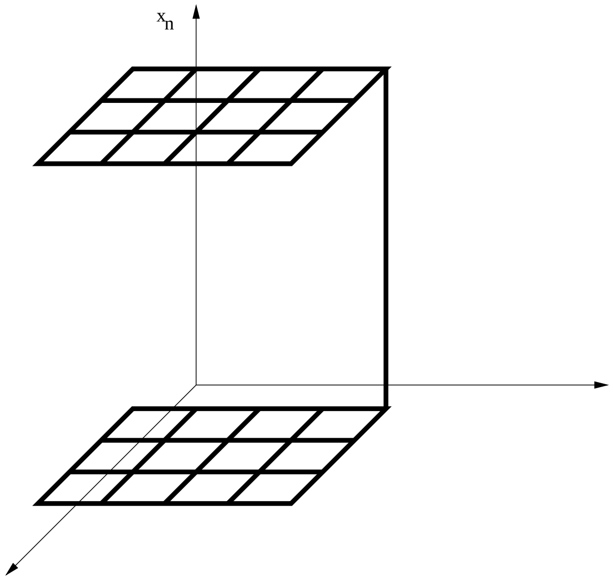

FIGURE 1. The set $\Sigma'_\varepsilon \subset \bar{Q}_\varepsilon$ (thick lines)

5. Basic properties of optimal sets

We first prove that when the measures $\varphi^\pm$ are absolutely continuous with respect to the Lebesgue measure, then the optimal Dirichlet region Σ_{opt} must have maximum length. Clearly, here as well as everywhere in the sequel, the trivial case $\varphi^+ = \varphi^-$ (which allows $\Sigma_{opt} = \emptyset$) has to be excluded. We remind that here and everywhere in the sequel we are assuming that Ω is a convex set.

Theorem 5.1. *Let $n > 1$, $\varphi^\pm << \mathcal{L}^n$, $\varphi^+ \neq \varphi^-$. Then $\mathcal{H}^1(\Sigma_{opt}) = \alpha$.*

PROOF: Suppose $\mathcal{H}^1(\Sigma_{opt}) < \alpha$. Lemma 4.2 implies that $\varphi_s \neq 0$. Then according to lemma 5.1 below one can find a closed connected $\Sigma' \subset \bar{\Omega}$ with $\Sigma_{opt} \subset \Sigma'$ and $\mathcal{H}^1(\Sigma') \leq \alpha$ such that

$$\int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma')\, d\varphi_s(x) < \int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_{opt})\, d\varphi_s(x).$$

Using lemma 4.1 one arrives then at inequalities

$$\begin{aligned} MK(\varphi^+, \varphi^-, \Sigma') &\leq MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \mathrm{dist}\,(z, \Sigma')\, d\varphi_s(z) \\ &< MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) \\ &= MK(\varphi^+, \varphi^-, \Sigma_{opt}), \end{aligned}$$

contradicting the optimality of Σ_{opt}. $\square$

Lemma 5.1. *Let φ be a nonzero finite Borel measure over $\bar{\Omega}$, $\varphi << \mathcal{L}^n$, while $\Sigma \subset \bar{\Omega}$ be a closed set satisfying $\mathcal{L}^n(\Sigma) = 0$. Then for every $\varepsilon > 0$ there is a closed segment $I_\varepsilon \subset \bar{\Omega}$ with $\mathcal{H}^1(I_\varepsilon) = \varepsilon$ such that $\Sigma \cap I_\varepsilon \neq \emptyset$ and*

$$\int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_\varepsilon)\, d\varphi(x) < \int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma)\, d\varphi(x) - L(\varepsilon),$$

where $\Sigma_\varepsilon := \Sigma \cup I_\varepsilon$ *and* $L(\varepsilon) > 0$*, while for all* $\varepsilon > 0$ *small enough one has*

$$L(\varepsilon) \geq C\varepsilon^{(n+1)/2}$$

for some constant $C > 0$ *independent of* ε*.*

PROOF: It is enough to apply lemma 5.2 below with $\Sigma_\nu := \Sigma$ and $B := \bar{\Omega}$ (note that then $T' = \bar{\Omega} \setminus \Sigma$, and thus the validity of the condition $\varphi(T') > 0$ is provided in this case by the assumptions $\mathcal{L}^n(\Sigma) = 0$ and $\varphi << \mathcal{L}^n$, $\varphi(\bar{\Omega}) > 0$). □

The lemma below is just a slightly refined version of lemma 5.1 which we will use in the sequel.

Lemma 5.2. *Let* φ *be a nonzero finite Borel measure over* $\bar{\Omega}$*,* $\varphi << \mathcal{L}^n$*, while* $\Sigma_\nu \subset \bar{\Omega}$ *be a sequence of closed sets. Assume that there exists a Borel set* $B \subset \bar{\Omega}$ *such that* $\Sigma := \Sigma_\nu \cap B$ *is independent of* ν*. Denote*

$$T' := \left\{x \in \bar{\Omega} \,:\, \exists \nu_0 = \nu_0(x) \text{ with } 0 < \operatorname{dist}(x, \Sigma) < \operatorname{dist}(x, \Sigma_\nu \setminus \Sigma) \text{ for all } \nu \geq \nu_0\right\}$$

and assume that $\varphi(T') > 0$*. Then for every* $\varepsilon > 0$ *there is a closed segment* $I_\varepsilon \subset \bar{\Omega}$ *with* $\mathcal{H}^1(I_\varepsilon) = \varepsilon$ *such that* $\Sigma \cap I_\varepsilon \neq \emptyset$ *and*

$$\int_{\bar{\Omega}} \operatorname{dist}(x, \Sigma_{\nu,\varepsilon})\, d\varphi(x) < \int_{\bar{\Omega}} \operatorname{dist}(x, \Sigma_\nu)\, d\varphi(x) - L(\varepsilon),$$

where $\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup I_\varepsilon$ *and* $L(\varepsilon) > 0$*, while there is an* $\varepsilon_0 > 0$ *independent of* ν *such that*

$$L(\varepsilon) \geq C\varepsilon^{(n+1)/2}$$

for some constant $C > 0$ *independent of both* ν *and* ε *and for all* $\varepsilon < \varepsilon_0$*.*

PROOF: Let $\Omega'_\nu \subset \bar{\Omega}$ stand for the set of points admitting a unique projection to Σ_ν. The respective projection map will be denoted $k^\nu\colon \Omega'_\nu \to \Sigma_\nu$. Clearly since Ω'_ν is the set of differentiability points of the distance function $u(\cdot) := \operatorname{dist}(\cdot, \Sigma_\nu)$, then $\mathcal{L}^n(\Omega'_\nu) = \mathcal{L}^n(\bar{\Omega})$. Our assumption $\varphi << \mathcal{L}^n$ allows hence to consider without loss of generality φ to be restricted to $\Omega' := \cap_\nu \Omega'_\nu$.

Consider an arbitrary $x \in T' \cap \Omega'$ and the transport ray R_x^ν of $MK(\varphi, k^\nu_{\#}\varphi)$ of positive length l_ν passing through x. According to the conditions of the lemma being proven, this ray ends on Σ once ν is sufficiently large, and, moreover, $R_x^\nu \subset R_x^{\nu'}$ once $\nu' \geq \nu \geq \nu_0$. Then $R_x^\nu \uparrow R_x$ in the set-theoretic sense, where R_x is some ray of positive length l ending on Σ. Clearly, $l_\nu \uparrow l$. Let $O_\nu \notin \Sigma$ stand for the upper end of R_x^ν. Then

$$\Sigma_\nu \cap B_{l_\nu}(O_\nu) = \emptyset.$$

We assume without loss of generality the origin of the coordinate system to coincide with O_ν, and, moreover, the coordinate axis x_n to be directed along R_x upwards, so that the lower endpoint A of R_x has coordinates $A = (0, \ldots, 0, -l_\nu)$. Let $A_\varepsilon := (0, \ldots, 0, -l_\nu + \varepsilon l_\nu)$, $I_\varepsilon := [A, A_\varepsilon]$ stand for the closed segment with endpoints A and A_ε and $\Sigma_{\nu,\varepsilon} := \Sigma_\nu \cup I_\varepsilon$. For each $\nu \in \mathbb{N}$ and $\varepsilon < l_\nu$ consider the set

$$\Lambda_{\nu,\varepsilon} := \left\{z = (Y, x_n) \in B_{l_\nu}(0) \,:\, x_n \leq 0,\, d(z, A_\varepsilon) - \operatorname{dist}(z, \partial B_{l_\nu}(0)) \leq -\varepsilon l_\nu/4\right\}.$$

If we are able to prove that, up to a correct choice of x, one has

$$\varphi(\Lambda_{\nu,\varepsilon}) \geq C\varepsilon^{(n-1)/2} \tag{7}$$

for all sufficiently small (independently of ν) $\varepsilon > 0$ and for some constant $C > 0$ (independent both of ν and ε), then the proof will be finished. Indeed, let $\nu_1 = \nu_1(x) \geq \nu_0 \in \mathbb{N}$ be such that

$$l_\nu \geq l/2 \text{ for all } \nu \geq \nu_1.$$

Then for $z \in \Lambda_{\nu,\varepsilon}$ one has

$$\operatorname{dist}(z, \Sigma_{\nu,\varepsilon}) \leq d(z, A_\varepsilon) \leq \operatorname{dist}(z, \partial B_{l_\nu}(0)) - \varepsilon l_\nu/4 \leq \operatorname{dist}(z, \Sigma) - \varepsilon l/8$$

for all $\nu \geq \nu_1$, and hence

$$\begin{aligned} \int_{\bar{\Omega}} \operatorname{dist}\ (z, \Sigma_{\nu,\varepsilon})\ d\varphi(z) - \int_{\bar{\Omega}} \operatorname{dist}\ (z, \Sigma)\ d\varphi(z) \quad &\leq \quad -\varepsilon l \varphi\left(\Lambda_{\nu,\varepsilon}\right)/8 \\ &\leq \quad -C\varepsilon^{(n+1)/2}. \end{aligned}$$

It remains therefore to obtain the estimate (7). For this purpose let $0 < m < 1/2$ and set (see figure 2)

$$\Pi_{m,\nu,\varepsilon} := \\ \left\{ z = (Y, x_n) \in B_{l_\nu}(0)\ :\ |x_n + l_\nu/2| \leq l_\nu(1 - 4m^2)^{1/2}/2,\ |Y| \leq m l_\nu \varepsilon^{1/2} \right\}.$$

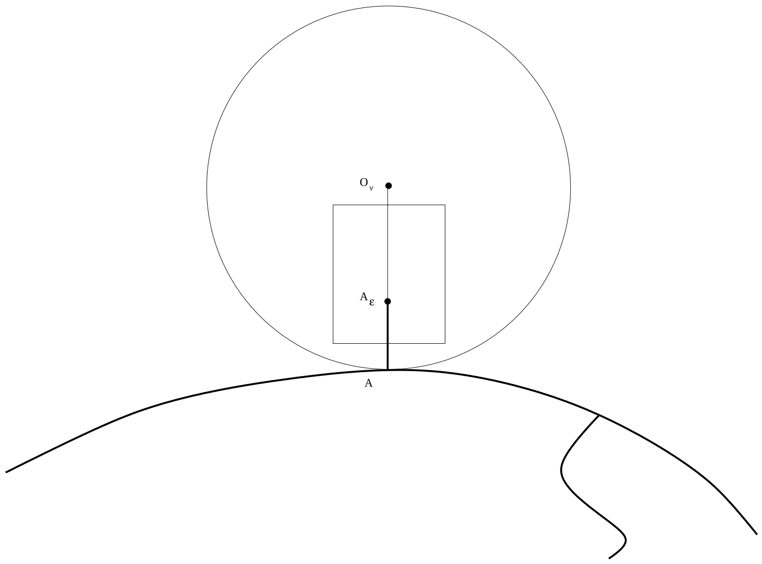

FIGURE 2. The sets $\Sigma_{\nu,\varepsilon}$ (thick lines) and $\Pi_{m,\nu,\varepsilon}$ (rectangle)

For every $z=(Y,x_n)\in\Pi_{m,\nu,\varepsilon}$ we have

$$\begin{aligned} d\left(z,A_\varepsilon\right)-\text{dist }\left(z,\partial B_{l_\nu}(0)\right) &\leq \left(|Y|^2+\left(x_n+l_\nu-\varepsilon l_\nu\right)^2\right)^{1/2}-l_\nu+ \\ &\qquad \left(x_n^2+|Y|^2\right)^{1/2} \\ &\leq \left(m^2l_\nu^2\varepsilon+\left(x_n+l_\nu-\varepsilon l_\nu\right)^2\right)^{1/2}-l_\nu+ \\ &\qquad \left(m^2l_\nu^2\varepsilon+x_n^2\right)^{1/2} \\ &\leq -l_\nu+|x_n|+m^2l_\nu^2\varepsilon/2|x_n|+\left(x_n+l_\nu\right)+ \\ &\quad \varepsilon\left(m^2l_\nu^2-2l\left(x_n+l_\nu\right)\right)/2\left(x_n+l_\nu\right)+\alpha\varepsilon^2 \\ &= \varepsilon\left(m^2l_\nu^2/\left(x_n+l_\nu\right)-2l_\nu+m^2l_\nu^2/\left|x_n\right|\right)/2+\alpha\varepsilon^2, \end{aligned}$$

where $\alpha=\alpha(m,l)>0$ is some constant independent of ν (we minded that $l_\nu\uparrow l$ when making the latter estimate). It is easy to verify that

$$m^2l_\nu^2/\left(x_n+l_\nu\right)-2l_\nu+m^2l_\nu^2/\left|x_n\right|\leq -l_\nu,$$

whenever $z\in\Pi_{m,\nu,\varepsilon}$, and hence, assuming $\nu\geq\nu_1$, one has

$$d\left(z,A_\varepsilon\right)-\text{dist }\left(z,\partial B_{l_\nu}(0)\right)\leq -\varepsilon l_\nu/4$$

whenever $\varepsilon<l/8\alpha$ (in fact, in this case $\varepsilon<l_\nu/4\alpha$ since $\nu\geq\nu_1$). This means $\Pi_{m,\nu,\varepsilon}\subset\Lambda_{\nu,\varepsilon}$ for such ε.

Assume now that x is chosen in such a way that $\mathcal{H}^1$-a.e. $z\in R_x^\nu$ is a Lebesgue point of $\frac{d\varphi}{d\mathcal{L}^n}$ (in view of lemma 5.4 from [7], this holds for φ-a.e. $x\in\bar{\Omega}$, hence for φ-a.e. $x\in T'\cap\Omega'$). Then, fixed an arbitrary $\nu\geq\nu_1$, the relationship

$$\liminf_{\varepsilon\to 0^+}\frac{\varphi(\Pi_{m,\nu,\varepsilon})}{\varepsilon^{(n-1)/2}}\geq C_m(R_x^\nu):=C\int_{l_{\nu,m}^-}^{l_{\nu,m}^+}\frac{d\varphi}{d\mathcal{L}^n}(Y,x_n)\,d\mathcal{H}^1(x_n) \tag{8}$$

holds, where $l_{\nu,m}^\pm:=l_\nu/2\pm l_\nu(1-4m^2)^{1/2}/2$, and $C>0$ is some constant independent of ν. In fact, one has

$$\begin{aligned} \varphi(\Pi_{m,\nu,\varepsilon}) &= \int_{l_{\nu,m}^-}^{l_{\nu,m}^+}d\mathcal{H}^1(x_n)\int_{|Y|<ml_\nu\sqrt{\varepsilon}}\frac{d\varphi}{d\mathcal{L}^n}(Y,x_n)\,d\mathcal{H}^{n-1}(Y) \\ &= \frac{1}{2\sqrt{\varepsilon}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}dh\int_0^l d\mathcal{H}^1(x_n)\int_{|Y|<ml_\nu\sqrt{\varepsilon}}f_\nu(Y,x_n+h)\,d\mathcal{H}^{n-1}(Y) \\ &= \int_0^l f_{\nu,\varepsilon}(x_n)\,d\mathcal{H}^1(x_n), \end{aligned}$$

where

$$\begin{aligned} f_\nu(Y,x_n) &:= \frac{d\varphi}{d\mathcal{L}^n}(Y,x_n)\cdot 1_{[l_{\nu,m}^+,l_{\nu,m}^-]}(x_n), \\ f_{\nu,\varepsilon}(x_n) &:= \frac{1}{2\sqrt{\varepsilon}}\int_{x_n-\sqrt{\varepsilon}}^{x_n+\sqrt{\varepsilon}}\int_{|Y|<ml_\nu\sqrt{\varepsilon}}f_\nu(Y,s)\,d\mathcal{H}^{n-1}(Y)d\mathcal{H}^1(s). \end{aligned}$$

But in view of our assumption on R_x, we have $\mathcal{H}^1$-a.e. on R_x the convergence

$$\frac{f_\varepsilon(x_n)}{\varepsilon^{(n-1)/2}}\to C_n(ml_\nu)^{n-1}f(0,x_n),$$

where $C_n>0$ is a constant depending only on the space dimension n. This, together with Fatou lemma and the estimate $l_\nu\geq l/2$ shows (8).

Observe now that $C_m(R_x^\nu) > 0$ for some $x \in T' \cap \Omega'$ and some m since otherwise $\varphi \llcorner T'$ would be concentrated on the set of upper and lower ray ends of $MK(\varphi, k_\#^\nu \varphi)$, contrary to our original assumption. Choose a $\bar\nu \geq \nu_1$ and an $\bar m = \bar m(\bar\nu) \in (0, 1/2)$ such that $C_{\bar m}(R_x^{\bar\nu}) > 0$. Then (8) implies

$$\varphi\left(\Pi_{\bar m,\bar\nu,\varepsilon}\right) \geq C_{\bar m}(R_x^{\bar\nu})\varepsilon^{(n-1)/2} \tag{9}$$

for all $\varepsilon < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(\bar\nu) > 0$.

We claim now that for every $\nu \geq \bar\nu$ there is an $m = m(\nu)$ such that

$$\Pi_{\bar m,\bar\nu,\varepsilon} \subset \Pi_{m,\nu,\varepsilon}. \tag{10}$$

In fact, in view of the inequality $l_\nu \geq l_{\bar\nu}$ the desired inclusion (10) is valid whenever

$$l_{\nu,m}^+ \geq l_{\bar\nu,\bar m}^+, \qquad l_{\nu,m}^- \leq l_{\bar\nu,\bar m}^- \text{ and } \bar m l_{\bar\nu} \leq m l_\nu. \tag{11}$$

Denoting

$$\delta_\nu := l_{\bar\nu}/l_\nu,$$

one immediately observes that (11) are equivalent to the system of inequalities

$$\begin{array}{rcccl} 4\bar m^2\delta_\nu^2 & \leq & 4m^2 & \leq & 1 - (\delta_\nu\sqrt{1-4\bar m^2} + 1 - \delta_\nu)^2 \\ & & & = & \delta_\nu(1 - \sqrt{1-4\bar m^2})(2 + \delta_\nu\sqrt{1-4\bar m^2} - \delta_\nu). \end{array}$$

For this system of inequalities to be solvable with respect to m it is necessary and sufficient that

$$4\bar m^2\delta_\nu \leq (1 - \sqrt{1-4\bar m^2})(2 + \delta_\nu\sqrt{1-4\bar m^2} - \delta_\nu).$$

Multiplying both sides of the latter inequality by $(1+\sqrt{1-4\bar m^2})/4\bar m^2$, one obtains the equivalent one

$$\delta_\nu(1 + \sqrt{1-4\bar m^2}) \leq 2 + \delta_\nu\sqrt{1-4\bar m^2} - \delta_\nu,$$

which is clearly equivalent to $\delta_\nu \leq 1$, and hence is always valid, since l_ν is a nondecreasing sequence. Therefore, one can also find an $m = m(\nu)$ satisfying (11) and hence also (10).

It remains to observe that (10) applied to (9) yields

$$\varphi\left(\Pi_{m,\nu,\varepsilon}\right) \geq \varphi\left(\Pi_{\bar m,\bar\nu,\varepsilon}\right) \geq C\varepsilon^{(n-1)/2} \tag{12}$$

for a $C = C(\bar\nu) > 0$ independent on ν and ε and for all $\varepsilon \leq \varepsilon_0$. At last, since $\Pi_{m,\nu,\varepsilon} \subset \Lambda_{\nu,\varepsilon}$ we get from (12) the estimate

$$\varphi(\Lambda_{\nu,\varepsilon}) \geq \varphi(\Pi_{m,\nu,\varepsilon}) \geq C\varepsilon^{(n-1)/2}$$

for the same $C > 0$ and ε as in (12), which is exactly (7), and hence the proof is concluded. $\square$

We emphasize that both the condition $\varphi^\pm << \mathcal{L}^n$ and $n > 1$ are essential for the result of theorem 5.1 to hold, as the following examples show.

Example 5.1. Let $\varphi^+ := \delta_O$, where O stands for the origin of the coordinate system, $\varphi^- := 0$, $\alpha := 1$ and $\Omega := B_1(0)$. Then clearly $\Sigma := \{O\}$ is optimal, but $\mathcal{H}^1(\Sigma) = 0 < \alpha$.

Example 5.2. Let $n = 1$, $\Omega = (0, 1)$ and $\alpha > 1$. Then, whatever are the measures $\varphi^\pm$ concentrated on $\bar\Omega$, one has that $\Sigma := [0, 1]$ is optimal, while $\mathcal{H}^1(\Sigma) = 1 < \alpha$.

Another relatively simple result regards the location of the optimal set Σ_{opt}.

Proposition 5.1. *Let $n > 1$, $\varphi^{\pm} << \mathcal{L}^n$, $\varphi^+ \neq \varphi^-$. Then*

$$\Sigma_{opt} \subset \overline{co}\,(supp\,\varphi^+ \cup supp\,\varphi^-).$$

PROOF: Denote

$$D := \overline{\mathrm{co}}\,(\mathrm{supp}\,\varphi^+ \cup \mathrm{supp}\,\varphi^-)$$

and suppose $\Sigma_{opt} \setminus D \neq \emptyset$. Let then Σ' stand for the projection of Σ_{opt} on D. Clearly, Σ' is still closed and connected, while $\mathcal{H}^1(\Sigma') < \alpha$. We claim that for all $x \in D$ one has

$$\mathrm{dist}\,(x, \Sigma') \leq \mathrm{dist}\,(x, \Sigma_{opt}). \tag{13}$$

This claim in fact leads immedately to the conclusion since it implies $d_{\Sigma'}(x, y) \leq d_{\Sigma_{opt}}(x, y)$ for all $(x, y) \in \mathrm{supp}\,\varphi^+ \times \mathrm{supp}\,\varphi^-$, and hence

$$MK(\varphi^+, \varphi^-, \Sigma') \leq MK(\varphi^+, \varphi^-, \Sigma_{opt}).$$

The latter means that Σ' is also an optimal set for the same problem, while then $\mathcal{H}^1(\Sigma') < \alpha$ contradicts theorem 5.1.

To prove (13), suppose the contrary, namely that

$$\mathrm{dist}\,(x, \Sigma') > \mathrm{dist}\,(x, \Sigma_{opt}). \tag{14}$$

for some $x \in D$. Let $x_0 \in \Sigma_{opt}$ be a point of minimum distance to x_0 in Σ_{opt}. Clearly, (14) can only hold when $x_0 \in \Sigma_{opt} \setminus D$. Let x_0' stand for the projection of x_0 on D. Since $x_0' \in \Sigma'$, then

$$\mathrm{dist}\,(x, \Sigma') \leq \mathrm{dist}\,(x, x_0').$$

But

$$\mathrm{dist}\,(x, x_0') < d(x_0, x).$$

The latter inequality follows immediately from the fact that both x_0' and x belong to the same half-space formed by a hyperplane π which is parallel to the supporting hyperplane of D at x_0' and passes through the medium point of the segment $[x_0, x_0']$: in fact, $\pi = \{z : d(x_0, z) = d(x_0', z)\}$ (see figure 3). We conclude now

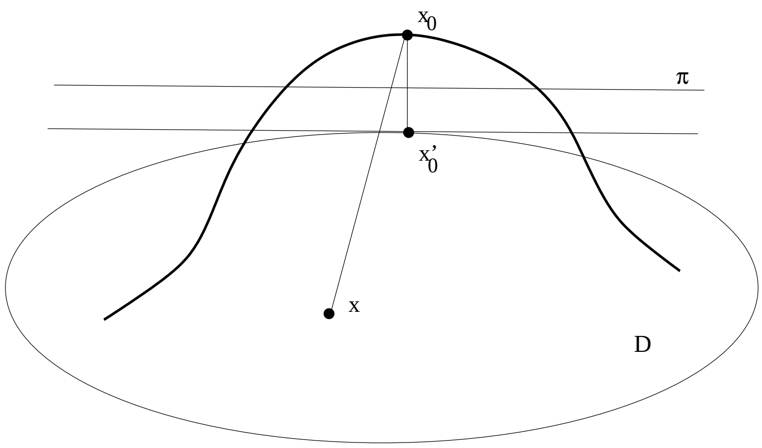

FIGURE 3. The original set Σ_{opt} (thick line) and the hyperplane π

$$\mathrm{dist}\,(x, \Sigma') \leq \mathrm{dist}\,(x, x_0') < d(x_0, x) = \mathrm{dist}\,(x, \Sigma_{opt}),$$

contradicting (14). $\square$

The condition $\varphi^\pm << \mathcal{L}^n$ is clearly essential also in the statement of proposition 5.1. In fact, in example 5.1 every closed connected Σ containing the origin and satisfying the length constraint is optimal, while for all such Σ except $\Sigma = \{O\}$ proposition 5.1 does not hold.

6. Absence of loops

The following result is valid.

Theorem 6.1. *Let $n = 2$, $\varphi^\pm << \mathcal{L}^2$, $\varphi^+ \neq \varphi^-$. Then Σ_{opt} contains no simple closed curve (a homeomorphic image of S^1). In particular, $\mathbb{R}^2 \setminus \Sigma_{opt}$ is connected.*

To prove the above theorem, we need several auxiliary constructions. We start with the following lemma.

Lemma 6.1. *Let Σ be a locally connected metric continuum consisting of more than one point and $x \in \Sigma$ be a noncut point of Σ. Then there is a sequence of open sets $D_\nu \subset \Sigma$ satisfying*

(i) *$x \in D_\nu$ for all sufficiently large ν;*
(ii) *$\Sigma \setminus D_\nu$ are connected for all ν;*
(iii) *$\operatorname{diam} D_\nu \searrow 0$ as $\nu \to \infty$;*
(iv) *D_ν are connected for all ν.*

Proof: Let $z \in \Sigma$, $z \neq x$. For every couple of points $\{y_1, y_2\} \subset \Sigma$ we will say that y_1 is connected to y_2 through a closed connected set $\Gamma \subset \Sigma$, if $\{y_1, y_2\} \subset \Gamma$. Define now

$$\begin{array}{rcl} X_\nu & := & \{y \in \Sigma : y \text{ connected to } z \text{ through a } \Gamma \subset \Sigma \setminus B_{1/\nu}(x)\}, \\ O_\nu & := & \Sigma \setminus X_\nu. \end{array}$$

We show first that O_ν are open for all ν, while

(i') $x \in O_\nu$ for all sufficiently large ν;
(ii') $\Sigma \setminus O_\nu$ are connected for all ν;
(iii') $\operatorname{diam} O_\nu \searrow 0$ as $\nu \to \infty$.

We first observe, that all X_ν are closed, hence O_ν are open. In fact, if $y_k \in X_\nu$, $y_k \to y$ as $k \to \infty$, then each y_k is connected to z through a set Γ_k satisfyng $\Gamma_k \subset \Sigma \setminus B_{1/\nu}(x)$. Then, up to a subsequence (not relabeled) $\Gamma_k \to \Gamma$ in the sense of Hausdorff convergence, while Γ is a closed connected set which connects y to z, and $\Gamma \subset \Sigma \setminus B_{1/\nu}(x)$. In other words, $y \in X_\nu$.

Now, clearly, (i') holds since otherwise one would have $z = x$. Moreover, (ii') is also immediate from the definition of X_ν. To prove (iii'), suppose the contrary, namely, that there is an $r > 0$ such that for all ν outside of $B_r(x)$ there are points $y_\nu \in O_\nu \setminus B_r(x)$. It follows that for every set Γ_ν connecting y_ν with z one has

$$\Gamma_\nu \cap \bar{B}_{1/\nu}(x) \neq \emptyset. \tag{15}$$

Consider an arbitrary accumulation point y of a sequence $\{y_\nu\}$. The local connectedness of Σ implies (see theorem I.2 from [10, § 49]) that for each sufficiently large ν there is a closed connected set $C_\nu \subset \Sigma$ satisfying $C_\nu \cap B_{r/2}(x) = \emptyset$ which connects y and y_ν. Therefore, for every set $\Gamma \subset \Sigma$ connecting y to z one has $x \in \Gamma$. In fact, otherwise minding the compactness of Γ we would have $(\Gamma \cup C_\nu) \cap B_\varepsilon(x) = \emptyset$ for some $\varepsilon > 0$ independent of ν, although $\Gamma \cup C_\nu$ connects y_ν to z, in contradiction with (15).

Mind now that $y \notin B_r(x)$ and hence $y \in \Sigma \setminus \{x\}$. But the latter space is locally connected (as an open subset of a locally connected space Σ) and, being completely metrizable (again, as an open subset of Σ), is therefore also locally arcwise connected. Since it is supposed to be connected (because x is a noncut point of Σ), then it is arcwise connected. In particular, this means that there is an arc $\gamma \subset \Sigma \setminus \{x\}$ connecting y to z. But according to what has just been proven, every arc in Σ connecting y to z must pass through x. This contradiction thus proves (iii').

Let now D_ν stand for the connected component of O_ν which contains x. Then D_ν is relatively (with respect to O_ν) open in view of local connectedness of Σ and hence is open in Σ. One immediately observes that (i) and (iii) hold in view of (i') and (iii') respectively. Since (iv) automatically follows from the construction, it remains to verify (ii). The latter, in view of (ii') will be shown once we prove that for every $y \in O_\nu \setminus D_\nu$ there is a set Γ connecting y to z such that $\Gamma \cap D_\nu = \emptyset$.

To prove the latter claim, suppose the contrary, i.e. that for some $y \in O_\nu \setminus D_\nu$ and for every Γ connecting y to z one has $\Gamma \cap D_\nu \neq \emptyset$. Let γ be an arbitrary arc connecting y to z (so that $\gamma(0) = y$ and $\gamma(1) = z$), and set

$$\bar{t} := \sup\{t \in [0,1] \,:\, \gamma(s) \notin D_\nu \text{ for all } s \in [0,t)\}, \qquad \tilde{x} := \gamma(\bar{t}).$$

Consider now the arc $[y,\tilde{x}] \subset \gamma$ which connects y to $\tilde{x}$, and let $[y,\tilde{x}) := [y,\tilde{x}] \setminus \{\tilde{x}\}$. We claim

$$[y,\tilde{x}) \subset O_\nu. \tag{16}$$

In fact, if $v \in [y,\tilde{x})$, then for every set Γ' connecting v with z there is an $x' \in \Gamma' \cap D_\nu$, since otherwise, minding that for $[y,v] \subset \gamma$ one has $[y,v] \cap D_\nu = \emptyset$ and hence, contrary to our assumption, $([y,v] \cup \Gamma') \cap D_\nu = \emptyset$ despite the fact that $[y,v] \cup \Gamma'$ connects y to z. Since $x' \in O_\nu$ then necessarily $\Gamma' \cap \bar{B}_{1/\nu}(x) \neq \emptyset$. Since the latter holds for an arbitrary closed connected Γ' connecting v to z, then by construction $v \in O_\nu$ which implies (16) in view of arbitrarity of v.

We prove now $\tilde{x} \in O_\nu$. In fact, if this is not the case, then there is a set $\tilde{\Gamma}$ connecting $\tilde{x}$ to z such that $\tilde{\Gamma} \subset X \setminus B_{1/\nu}(x)$, hence, $\tilde{\Gamma} \cap O_\nu = \emptyset$ and in particular $\tilde{\Gamma} \cap D_\nu = \emptyset$. But then $([y,\tilde{x}] \cup \tilde{\Gamma}) \cap D_\nu = \emptyset$ contrary to our assumption since $[y,\tilde{x}] \cup \tilde{\Gamma}$ connects y to z.

Combining the latter observation with (16), we get

$$[y,\tilde{x}] = \gamma([0,\bar{t}]) \subset O_\nu.$$

But since O_ν is open then there is a $t' > \bar{t}$ such that $\gamma([0,t']) \subset O_\nu$, which means that $\gamma([0,t'])$ belongs to the same connected component of O_ν. Since by definition of $\bar{t}$ one has $\gamma([0,t']) \cap D_\nu \neq \emptyset$, then this component is D_ν, and in particular, $y \in D_\nu$. This however contradicts the choice of y and thus proves the claim. □

We may now claim the following assertion.

Proposition 6.1. *Let $n = 2$, $\varphi^\pm << \mathcal{L}^2$, $\varphi^+ \neq \varphi^-$. Then $\mathcal{H}^1$-a.e. $x \in \Sigma_{opt}$ is a cut point of Σ_{opt}.*

PROOF: Suppose the contrary, namely that the set A of noncut points of Σ_{opt} has positive length, i.e. $\mathcal{H}^1(A) > 0$. We observe that since $\mathcal{H}^1(\Sigma_{opt}) < +\infty$, then Σ_{opt}

is a continuous (in fact, even Lipschitz continuous) image of the unit interval, and hence is locally connected in view of the Hahn-Mazurkiewicz-Sierpiński theorem II.2 from [10, § 50]. Hence for each $x \in A$ we may consider the sequence of sets $\{D_\nu\}$ with the properties announced in lemma 6.1. We denote $\varepsilon_\nu := \operatorname{diam} D_\nu$. Note that $\mathcal{H}^1(D_\nu) \geq \varepsilon_\nu$. For the sake of brevity of the notation, further on we omit the reference to the index ν writing always ε instead of ε_ν. The contradiction will be achieved in two steps.

Step 1. For every $x \in A$ and $\varepsilon > 0$ let $T(x, \varepsilon)$ stand for the union of transport rays of the Monge-Kantorovich problem $MK(\varphi_s, \psi)$ which end at $\Sigma_{opt} \cap B_\varepsilon(x)$. Choose an arbitrary $x \in A$ and set $\Sigma_\varepsilon(x) := \Sigma_{opt} \setminus D_\nu$. Clearly, according to our assumption, $\Sigma_\varepsilon(x)$ is still closed and connected, and, of course, satisfies the length constraint since $\mathcal{H}^1(\Sigma_\varepsilon(x)) \leq \mathcal{H}^1(\Sigma_{opt})$. Since $D_\nu \subset B_\varepsilon(x)$, the following estimate is valid

$$\begin{aligned}
&\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, d\varphi_s(z) \\
&\qquad = \int_{\bar\Omega\setminus T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, d\varphi_s(z) + \int_{T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, d\varphi_s(z) \\
&\qquad \leq \int_{\bar\Omega\setminus T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \int_{T(x,\varepsilon)} (\operatorname{dist}(z, \Sigma_{opt}) + \varepsilon)\, d\varphi_s(z) \\
&\qquad = \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\varphi_s(T(x,\varepsilon)).
\end{aligned}$$

By definition of ψ one has

$$\varphi_s(T(x,\varepsilon)) = \psi(B_\varepsilon(x)).$$

But

$$\limsup_{\varepsilon\to 0^+} \frac{\psi(B_\varepsilon(x))}{\varepsilon} < +\infty$$

for $\mathcal{H}^1$-a.e. $x \in A$, which implies

$$\psi(D_\nu) \leq \psi(B_\varepsilon(x)) \leq C\varepsilon + o(\varepsilon)$$

for some $C = C(x) > 0$ depending on x, $\mathcal{H}^1$-a.e. in A. Summing up, for $\mathcal{H}^1$-a.e. $x \in A$ one has

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, d\varphi_s(z) \leq \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + C\varepsilon^2 + o(\varepsilon^2). \tag{17}$$

Note that the constant C depends on x. Since $\mathcal{H}^1(A) > 0$ in view of our original assumption, we can choose an $x \in A$ such that the above estimate holds with some constant $C > 0$.

Step 2. Let

$$l(\varepsilon) := \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon(x)).$$

One clearly has $l(\varepsilon) \geq \varepsilon$. In view of lemma 4.2 we may apply now lemma 5.2 with $\varphi := \varphi_s$, $\Sigma_\nu := \Sigma_\varepsilon(x)$ and $B := \bar\Omega\setminus T(x, \varepsilon_0)$ (for some fixed sufficiently small $\varepsilon_0 > 0$) to find a closed connected $\Sigma' \subset \bar\Omega$ satisfying $\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma_\varepsilon) + \varepsilon$ and (using the fact that $n = 2$)

$$\int_{\bar\Omega} \operatorname{dist}(x, \Sigma')\, d\varphi(x) < \int_{\bar\Omega} \operatorname{dist}(x, \Sigma_\varepsilon(x))\, d\varphi(x) - C\varepsilon^{3/2}$$

for some $C > 0$ as $\varepsilon > 0$ is small enough. But

$$MK(\varphi^+, \varphi^-, \Sigma') \leq MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \mathrm{dist}\,(z, \Sigma')\, d\varphi_s(z)$$

in view of lemma 4.1, and hence one has

$$MK(\varphi^+, \varphi^-, \Sigma') \leq MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \mathrm{dist}\,(z, \Sigma_\varepsilon(x))\, d\varphi_s(z) - C\varepsilon^{3/2}.$$

Using (17), one arrives at the estimate

$$\begin{aligned} MK(\varphi^+, \varphi^-, \Sigma') &\leq MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) - C\varepsilon^{3/2} \\ &= MK(\varphi^+, \varphi^-, \Sigma_{opt}) - C\varepsilon^{3/2} \end{aligned}$$

as $\varepsilon > 0$ is small enough. Hence for such ε one has

$$MK(\varphi^+, \varphi^-, \Sigma') < MK(\varphi^+, \varphi^-, \Sigma_{opt})$$

contradicting the optimality of Σ_{opt}. □

Proof of theorem 6.1: Suppose there is a simple closed loop $S \subset \Sigma_{opt}$. If $x \in S$ is a cut point for Σ_{opt} then there is a continuum L_x such that $L_x \cap S = \{x\}$ and $L_x \setminus \{x\} \neq \emptyset$. One has then $L_x \cap L_y = \emptyset$ whenever $x \neq y$. But then $\mathcal{H}^1(L_x) > 0$ for only at most countable number of x, otherwise one would have $\mathcal{H}^1(\Sigma_{opt}) = +\infty$. All the other $x \in S$ are noncut for Σ_{opt} contradicting proposition 6.1.

To prove the connectedness of $\mathbb{R}^2 \setminus \Sigma_{opt}$, note that it follows from theorem II.5 of [10, § 61], that if $D \subset \mathbb{R}^2$ is a bounded connected set with locally connected boundary, then there is a simple closed curve $S \subset \partial D$. Then, if $\mathbb{R}^2 \setminus \Sigma_{opt}$ were disconnected, there would exist a bounded connected component D of $\mathbb{R}^2 \setminus \Sigma_{opt}$ such that $\partial D \subset \Sigma_{opt}$, and hence Σ_{opt} would contain a simple closed curve contrary to what has been just proven. □

7. Noncut points

The following fundamental result regarding noncut points of the optimal set is valid.

Theorem 7.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$, where $p > 4/3$, and $\varphi^+ \neq \varphi^-$. Then there is a constant $C > 0$ such that for every noncut point $x \in \Sigma_{opt}$ one has $\psi(\{x\}) > C$. In particular, the number of noncut points in Σ_{opt} is finite.*

The proof of the above theorem is just a combination of the following two statements.

Proposition 7.1. *Let $n = 2$, $\varphi^\pm << \mathcal{L}^n$, $\varphi^+ \neq \varphi^-$. Assume that there is an $y \in \Sigma_{opt}$ such that $\psi(\{y\}) > 0$. Then there is a constant $C > 0$ such that for every noncut point $x \in \Sigma_{opt}$ one has $\psi(\{x\}) \geq C$. In particular, in this case the number of noncut points in Σ_{opt} is finite.*

proof: According to lemma 4.2 one has $\varphi_s \neq 0$, hence $\psi \neq 0$. We divide the proof in two steps.

Step 1. Let $x \in \Sigma_{opt}$ be a noncut point. We prove first

$$\psi(\{x\}) \geq \sup_{y \in \Sigma_{opt}} \psi(\{y\})/2\pi. \tag{18}$$

For this purpose consider an arbitrary $y \in \Sigma_{opt}$ and let $\{D_\nu\}_{\nu\in\mathbb{N}}$ stand for the decreasing sequence of subsets of Σ_{opt} indicated by lemma 6.1, with $x \in D_\nu$. We also suppose the diameters of D_ν to be small enough so that $y \notin D_\nu$ for all $\nu \in \mathbb{N}$. We set $\varepsilon_\nu := \operatorname{diam} D_\nu$. Note that $\mathcal{H}^1(D_\nu) \geq \varepsilon_\nu$. For the sake of brevity, further on we omit the reference to the index ν writing always ε instead of ε_ν.

According to proposition 5.1 we may assume without loss of generality that

$$\overline{\mathrm{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega.$$

Setting $\Sigma_\varepsilon^1 := \Sigma_{opt} \setminus D_\nu$, while $\Sigma_\varepsilon^2 := \Sigma_\varepsilon^1 \cup \partial B_{\varepsilon/2\pi}(y)$, one has that $\Sigma_\varepsilon^2 \subset \Omega$ and is still closed, connected and satisfies the length constraint. Then

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) \leq \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\psi(D_\nu),$$

but

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) \leq \\ \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) - \frac{\varepsilon}{2\pi}\left(\psi(\{y\}) - \varphi_s(B_{\varepsilon/2\pi}(y))\right).$$

Therefore,

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) \leq \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\psi(D_\nu) - \\ \frac{\varepsilon}{2\pi}\left(\psi(\{y\}) - \varphi_s(B_{\varepsilon/2\pi}(y))\right).$$

Taking into account $\varepsilon \to 0$, $\psi(D_\nu) \to \psi(\{x\})$ and $\varphi_s(B_{\varepsilon/2\pi}(y)) \to 0$ as $\nu \to \infty$, one has that the latter inequality would contradict the optimality of Σ_{opt} unless $\psi(\{x\}) \geq \psi(\{y\})/2\pi$, which shows (18).

Step 2. Consider now two arbitrary noncut points $x_i \in \Sigma_{opt}$, $i = 1, 2$. As an immediate corollary of (18) one has

$$\psi(\{x_2\}) \leq 2\pi\psi(\{x_1\}).$$

Exchanging x_1 with x_2, one arrives then at the estimate

$$\psi(\{x_2\}) \leq 2\pi\psi(\{x_1\}) \leq 4\pi^2\psi(\{x_2\}). \tag{19}$$

Under the conditions of the proposition being proven, the estimate (18) implies $\psi(\{x\}) > 0$ for every noncut point $x \in \Sigma_{opt}$, and hence, combined with (19) proves the existence of a $C > 0$ such that $\psi(\{x\}) > C$ for every noncut point $x \in \Sigma_{opt}$, which proves the first claim. It is now immediate to conclude that the number of noncut points is finite since $\psi(\bar\Omega) < +\infty$. □

Another key ingredient of the proof of theorem 7.1 is the following auxiliary weak assertion.

Lemma 7.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$ where $p > 4/3$. Then there is an $x \in \Sigma_{opt}$ such that $\psi(\{x\}) > 0$.*

To prove the above lemma, we need some auxiliary constructions. Let Σ be a locally connected metric continuum consisting of more than one point. Take $x \in \Sigma$, $z \in \Sigma$ with $z \neq x$ and define

$$D(x,z) := \{y \in \Sigma\,;\, y \text{ is connected to } x \text{ through a path } \gamma,\ \text{such that } z \notin \gamma\}.$$

Figure 4 illustrates how $D(x,z)$ may look like when z is a nonbranching point (upper figure) and when z is a branching point (lower figure) of Σ.

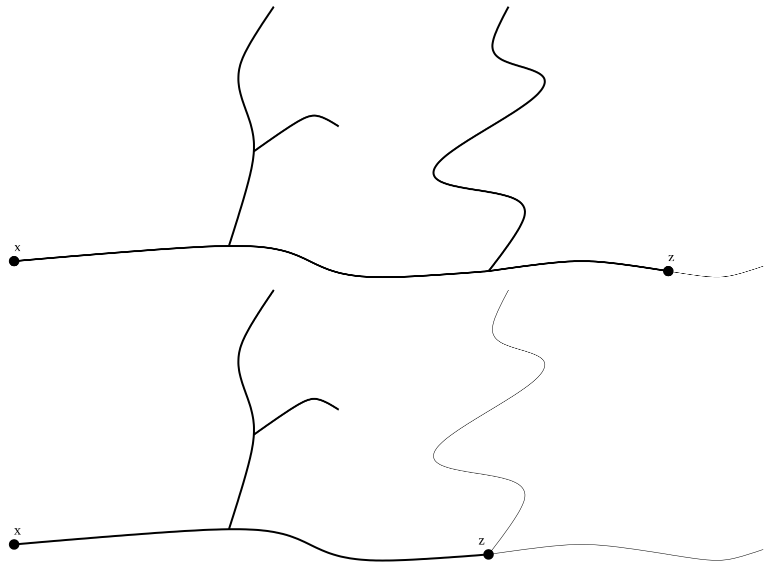

FIGURE 4. The set $D(x,z)$ (thick line)

We make an important observation: if Σ contains no simple closed curve (a homeomorphic image of S^1), then every couple of points in Σ is connected by a unique arc (corollary VI.2 in [10, § 51]). In this case, if γ is an arc of Σ starting at x and $\{z_1, z_2\} \subset \gamma$, we will write

$$z_1 \leq_{\gamma,x} z_2,$$

if $[x, z_1] \subset [x, z_2]$. The above relationship obviously defines a total order on γ. We also will write $z_1 <_{\gamma,x} z_2$, if $z_1 \leq_{\gamma,x} z_2$ and $z_1 \neq z_2$. For a sequence $\{z_\nu\}_{\nu=1}^\infty \subset \gamma$ we will say that $z_\nu \searrow_\gamma x$ as $\nu \to \infty$, if for every $z \in \gamma$ there is a $\bar{\nu} \in \mathbb{N}$ such that $z_\nu \leq_{\gamma,x} z$ for all $\nu \geq \bar{\nu}$. In particular, minding that $[x, z_\nu]$ is a unique arc connecting x to z_ν, one has that $z_\nu \searrow_\gamma x$ holds if and only if $d^\Sigma(z_\nu, x) \searrow 0$ as $\nu \to \infty$, where d^Σ stands for the geodesic distance over Σ, namely,

$$d^\Sigma(u,v) := \inf\{\mathcal{H}^1(\gamma) : \gamma \text{ is an arc in } \Sigma \text{ connecting } u \text{ and } v\}.$$

The following statement regarding $D(x,z)$ is valid.

Proposition 7.2. *Let Σ be a locally connected metric continuum consisting of more than one point and $x \in \Sigma$. Then for all $z \in \Sigma$, $z \neq x$ one has*

(i) *$D(x,z)$ is connected and $x \in D(x,z)$;*
(ii) *$\Sigma \setminus D(x,z)$ is connected and closed.*

Moreover, suppose that Σ satisfies $\mathcal{H}^1(\Sigma) < +\infty$ and contains no simple closed curve (a homeomorphic image of S^1). Then for every arc γ in Σ starting at x one has

(iii) *$D(x,z_1) \subset D(x,z_2)$ and $\mathcal{H}^1(D(x,z_1)) < \mathcal{H}^1(D(x,z_2))$ whenever $z_1 \leq_{\gamma,x} z_2$, while*
(iv) *$\cap_\nu D(x,z_\nu) = \{x\}$ whenever $z_\nu \searrow_\gamma x$ as $\nu \to \infty$ and x is an endpoint of Σ (i.e. $ord_x \Sigma = 1$).*

PROOF: The statement (i) is immediate from the definition of $D(x,z)$. To show (ii), note first that $z \notin D(x,z)$. If $\gamma' \subset \Sigma$ is an arc connecting $y \in \Sigma \setminus D(x,z)$ to z, then $\gamma' \subset \Sigma \setminus D(x,z)$. In fact, otherwise for some $v \in \gamma'$ we would have $v \in D(x,z)$, which would mean the existence of some path $\tilde{\gamma}$ connecting v to x such that $z \notin \tilde{\gamma}$. Let $[y,v] \subset \gamma'$ stand for the path connecting y and v. Then $[y,v] \circ \tilde{\gamma}$ would connect y to x without containing z, which contradicts the choice of y. Therefore, $\Sigma \setminus D(x,z)$ is connected.

To complete the proof of (ii), we need to show that the latter set is closed. For this purpose consider an arbitrary sequence $\{y_\nu\}_{\nu=1}^\infty \subset \Sigma \setminus D(x,z)$ such that $y_\nu \to y \in \Sigma$ as $\nu \to \infty$. To prove $y \notin D(x,z)$ suppose the contrary, namely that y is connected to x through some path $\gamma \subset \Sigma$ satisfying $z \notin \gamma$. We may suppose $y \neq z$ (otherwise there is nothing to prove since $z \notin D(x,z)$). Let $\varepsilon > 0$ be such that $z \notin B_\varepsilon(y)$. Since for all sufficiently large $\nu \in \mathbb{N}$ one has $y_\nu \in B_\varepsilon(y)$, then by the local arcwise connectedness of Σ for each sufficiently large ν there is an arc γ_ν connecting y_ν with y such that $\gamma_\nu \subset \Sigma \cap B_\varepsilon(y)$, hence $z \notin \gamma_\nu$. Since γ is also supposed not to touch z, then we conclude that y_ν for all suffciently large $\nu \in \mathbb{N}$ are connected to x through a path $\gamma_\nu \circ \gamma \subset \Sigma$ which does not touch z, hence contradicting the assumption $y_\nu \notin D(x,z)$ and concluding the proof of (ii).

To prove (iii), let γ be an arc in Σ passing through x and $z_1 \leq_{\gamma,x} z_2$. To show $D(x,z_1) \subset D(x,z_2)$ consider an arbitrary $y \in D(x,z_1)$. We exploit extensively the fact that every two points in Σ are connected by a unique arc. Consider the arc γ' connecting x to y. The set $\gamma \cap \gamma'$ is such that whenever $v \in \gamma \cap \gamma'$ then also for all u satisfying either $u <_{\gamma,x} v$ or $u <_{\gamma',x} v$ one has $u \in \gamma \cap \gamma'$ (otherwise there would be a couple of points which can be connected by different arcs). Hence, $\gamma \cap \gamma'$ is either the point x or an arc $[x,y'] \subset \gamma$. Obviously, $y' <_{\gamma,x} z_1$ for some $z \in \gamma$, since otherwise we would have $y \notin D(x,z_1)$ contrary to our assumption. Hence $y' <_{\gamma,x} z_2$, and therefore $z_2 \notin \gamma'$, which means $y \in D(x,z_2)$. In view of arbitrarity of y we have shown therefore that $D(x,z_1) \subset D(x,z_2)$.

Let now $\Delta := [z_1,z_2]$. Clearly, $\Delta \subset \gamma$ since every two points are connected by a unique arc, i.e. Δ is a piece of γ between z_1 and z_2. We show that $\Delta \subset D(x,z_2) \setminus D(x,z_1)$, which will imply (iii) since $\mathcal{H}^1(\Delta) > 0$. To this aim consider an arbitrary $z \in \Delta$. Clearly, $z \in D(x,z_2)$. To show $z \notin D(x,z_1)$, suppose the opposite, namely that z is connected to x through an arc γ_1 in Σ such that $z_1 \notin \gamma_1$. This comes in contradiction with the fact that z cannot be connected to x by two different arcs.

At last, to show (iv), assume that $x \in \Sigma$ is an endpoint and $z_\nu \searrow_\gamma x$ as $\nu \to \infty$. Suppose the contrary, namely that there is an $y \in \cap_\nu D(x,z_\nu)$, $y \neq x$. Let $\gamma' := [x,y]$ (mind that such an arc is uniquely defined). But $\gamma \cap \gamma' \neq \{x\}$, since in this case $\mathrm{ord}_x \Sigma \geq 2$. Hence the latter intersection is an arc, namely, $\gamma \cap \gamma' = [x,y']$ for some $y' \in \gamma$. But this implies $y \notin D(x,z_\nu)$ for all $\nu \in \mathbb{N}$ such that $z_\nu \leq_{\gamma,x} y'$, contradicting the assumption. □

It might be curious to note that proposition 7.2 provides an alternative construction to that suggested by lemma 6.1. In fact, suppose that Σ contains no simple closed curve, $x \in \Sigma$ be an endpoint and γ be an arc in Σ starting at x. One might set then $D_\nu := D(x,z_\nu)$ for an arbitrarily chosen sequence $\{z_\nu\}_{\nu=1}^\infty \subset \gamma$ satisfying $z_\nu \searrow_\gamma x$ as $\nu \to \infty$. The sets D_ν will then according to proposition 7.2 satisfy all the conditions of lemma 6.1. However, it is important to emphasize that the assertions (iii) and (iv) of proposition 7.2 are only true when Σ contains no simple closed curve (in fact, this has been used essentially in the proof). If Σ contained loops then both assertions could be false: e.g. (iii) does not hold for $\Sigma = S^1$, since in this case $D(x,z) = \Sigma \setminus \{z\}$ for all $z \in \Sigma \setminus \{x\}$.

We need now the following lemmata regarding the optimal set Σ_{opt}.

Lemma 7.2. *Let $\{x_\nu\}_{\nu=1}^\infty \subset \Sigma_{opt}$ be any sequence of noncut points, and $\{z_\nu\}_{\nu=1}^\infty \subset \Sigma_{opt}$ be such that*

$$\varepsilon_\nu := diam\, D(x_\nu, z_\nu) \to 0$$

as $\nu \to \infty$. Then for every $\sigma > (n-1)/2$ one has

$$\lim_{\nu\to\infty} \frac{\psi(D(x_\nu, z_\nu))}{\varepsilon_\nu^\sigma} = +\infty.$$

PROOF: Up to passing to a subsequence (not relabeled), we assume that $x_\nu \to x \in \Sigma_{opt}$. We abbreviate $D_\nu := D(x_\nu, z_\nu)$, while ε_ν will, as usual, be further denoted by ε omitting the index ν for brevity. Set $\Sigma_\varepsilon := \Sigma_{opt} \setminus D_\nu$ and note that the Σ_ε is still closed and connected in view of proposition 7.2. Moreover, one has

$$\int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_\varepsilon)\, d\varphi_s(z) \leq \int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\psi(D_\nu). \tag{20}$$

Observe now that there is a $r > 0$ such that

$$\psi(\bar\Omega) > \psi(B_r(x)).$$

In fact, otherwise one would have $\psi(\bar\Omega) = \psi(B_r(x))$ for all $r > 0$, and hence, taking a limit in the latter relationship as $r \to 0$, $\psi(\bar\Omega) = \psi(\{x\})$, which means that the one-point set $\{x\}$ would be optimal, contradicting theorem 5.1. Lemma 5.2 is then applicable with $B := \bar\Omega \setminus B_r(x)$, $\Sigma_\nu := \Sigma_\varepsilon$ and $\varphi := \varphi_s$. In fact, $\Sigma_\varepsilon \cap B = \Sigma_{opt} \cap B$ for all sufficiently large ν, while $k^{-1}(\Sigma_{opt} \cap B) \subset T'$ and hence

$$\varphi_s(T') = \psi(B) = \psi(\bar\Omega) - \psi(B_r(x)) > 0.$$

Hence, lemma 5.2 asserts the existence of a new Σ'_ε such that

$$\mathcal{H}^1(\Sigma'_\varepsilon) = \mathcal{H}^1(\Sigma_\varepsilon) + \mathcal{H}^1(D_\nu) = \mathcal{H}^1(\Sigma_{opt})$$

and

$$\int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma'_\varepsilon)\, d\varphi_s(z) \leq \int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_\varepsilon)\, d\varphi_s(z) - C\varepsilon^{(n+1)/2},$$

for some $C > 0$ independent of ν as $\varepsilon > 0$ is small enough (also independently of ν). We now argue by contradiction. Supposing the contrary, we get

$$\varepsilon\psi(D_\nu) = o(\varepsilon^{(n+1)/2})$$

for some subsequence of ν (not relabeled). Together with (20) and lemma 4.1 this gives

$$MK(\varphi^+, \varphi^-, \Sigma'_\varepsilon) < MK(\varphi^+, \varphi^-, \Sigma_{opt})$$

for sufficiently small ε contradicting the optimality of Σ_{opt}. $\square$

Lemma 7.3. *Let $\varphi^\pm \in L^p(\Omega)$, $p > 2n/(n+1)$, and $x \in \Sigma_{opt}$ be an endpoint, while γ be an injective arc in Σ starting at x. Then there is a $\sigma > 1$ such that $(n-1)p'\sigma/2n < 1$ and for all $z \in \gamma$ the set of $y \in D(x,z)$ for which there is a $z' \in \gamma$ such that $y \in D(x,z')$ and*

$$l_y \geq (\mathcal{H}^1(D(x,z')))^{(n-1)p'\sigma/2n},$$

where l_y stands for the maximum length of transport rays of $MK(\varphi_s, \psi)$ which end in y, has positive measure ψ.

PROOF: Due to our assumption on p one has $p' < 2n/(n-1)$ and hence, there is a $\sigma > 1$ such that $(n-1)p'\sigma/2n < 1$. Suppose that the statement of the lemma is not valid with this σ, namely, that there is a $z \in \gamma$ such that $\psi(B_z) = \psi(D(x,z))$, where

$$B_z := \{y \in D(x,z) \,:\, l_y < (\mathcal{H}^1(D(x,z')))^{(n-1)p'\sigma/2n} \text{ for all } z' \in \gamma, y \in D(x,z')\}.$$

Let $z' \in \gamma$ be an arbitrary point with $z' \in D(x,z)$ and let $y \in B_z$ and be an arbitrary point satisfying $y \in D(x,z')$ (which exists due to our assumption). For the sake of brevity we denote $\varepsilon := \mathcal{H}^1(D(x,z'))$. Note that due to the arbitrarity of the choice of z' we can make ε as small as necessary. Consider an arbitrary v such that $k(v) = y$ (in other words, v belongs to a closure of some transport ray of $MK(\varphi_s, \psi)$ which ends in y). Then

$$|v - z'| \leq |v - y| + |y - z'| \leq l_y + \varepsilon < \varepsilon^{(n-1)p'\sigma/2n} + \varepsilon.$$

Observe that one has $(n-1)p'\sigma/2n < 1$ and hence, if ε is sufficiently small, the above estimate implies $|v - z'| < 2\varepsilon^{(n-1)p'\sigma/2n}$. Therefore

$$\begin{aligned} \psi(D(x,z')) &= \psi(D(x,z') \cap B_z) \\ &= \varphi_s(\{v \,:\, k(v) \in D(x,z') \cap B_z\}) \\ &\leq \varphi_s(B_{2\varepsilon^{(n-1)p'\sigma/2n}}(z')) \end{aligned}$$

and applying to the latter expression the Hölder inequality one gets

$$\psi(D(x,z')) \leq C||\varphi_s||_p \varepsilon^{(n-1)\sigma/2}$$

for some $C > 0$, which contradicts lemma 7.2, hence concluding the proof. □

At last, we provide also the following construction which is specific to a two-dimensional case, i.e. $n = 2$. Let $T_s \subset \bar{\Omega}$ stand for the transport set (i.e. the union of transport rays without ray ends) of the Monge-Kantorovich problem $MK(\varphi_s, \psi)$ and set

$$\begin{aligned} T_s^+ &:= \{x \in T_s \,:\, \pi_1(x) > \pi_1(k(x)) \text{ or } \pi_1(x) = \pi_1(k(x)),\, \pi_2(x) > \pi_2(k(x))\}, \\ T_s^- &:= T_s \setminus T_s^+, \end{aligned}$$

where π_i: $\mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, stand for projections on the first and second coordinate axis respectively, i.e. $\pi_i(x_1, x_2) := x_i$, $i = 1, 2$, while $k(x)$ stands for the projection of x onto Σ_{opt} (note that for $x \in T_s$ such a projection is uniquely defined). Define then new Borel measures $\psi_\pm$ by setting $\psi_\pm(e) := k_\# \varphi_s \llcorner T_s^\pm$, i.e.

$$\psi_\pm(e) = \varphi_s(k^{-1}(e) \cap T_s^\pm)$$

for all Borel sets $e \subset \bar{\Omega}$. Let $y \in \Sigma_{opt}$ be such a lower endpoint of some transport ray of $MK(\varphi_s, \psi)$ that all the transport rays ending at y belong to the same line l. Let then $\theta(x) \in [0, \pi/2]$ stand for the angle between l and e_2, where e_2 stands for the second unit coordinate vector.

Lemma 7.4. *Assume $n = 2$. Let $x \in \Sigma_{opt}$ be a noncut point and $D \subset \bar{\Omega}$ be such an open set that $\theta(y)$ is well defined for ψ-a.e. $y \in \Sigma_{opt} \cap D$, while $x \notin D$ and $\psi^+(D) \geq \psi^-(D)$. Then for every $f \in C_0^1(D)$ one has*

$$\int_D f \sin\theta \, d(\psi^+ - \psi^-) \leq \psi(\{x\}).$$

PROOF: As in the proof of proposition 7.1, let $\{D_\nu\}_{\nu\in\mathbb{N}}$ stand for the decreasing sequence of subsets of Σ_{opt} indicated by lemma 6.1, with $x \in D_\nu$. We also suppose $\varepsilon_\nu := \operatorname{diam} D_\nu$ (denoted further ε omitting the index ν for brevity) to be small enough so that $D_\nu \cap D = \emptyset$ for all $\nu \in \mathbb{N}$. Let $\Sigma_\varepsilon := \Sigma_{opt} \setminus D_\nu$. Then

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon)\, d\varphi_s(z) \le \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\psi(D_\nu).$$

We further assume without loss of generality, referring to proposition 5.1, that Ω is large enough so as to satisfy

$$\overline{\mathrm{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega.$$

Now consider an arbitrary positive smooth function $f \in C_0^1(D)$ and define a diffeomorphism Φ_δ: $\bar\Omega \to \bar\Omega$ by the formula

$$\Phi_\delta(x_1, x_2) := (x_1 - \delta f(x_1, x_2), x_2),$$

for all sufficiently small $\delta > 0$. Then

$$\mathcal{H}^1(\Phi_\delta(\Sigma_\varepsilon)) \le \mathcal{H}^1(\Sigma_\varepsilon) + C_f\delta$$

for some constant $C_f > 0$ depending on f. We choose $\delta := \varepsilon/C_f$. With this choice one has for $\Sigma'_\varepsilon := \Phi_\delta(\Sigma_\varepsilon)$ the estimate

$$\mathcal{H}^1(\Sigma'_\varepsilon) \le \mathcal{H}^1(\Sigma_\varepsilon) + C_f\delta \le \mathcal{H}^1(\Sigma_{opt}) - \varepsilon + C_f\delta = \mathcal{H}^1(\Sigma_{opt}),$$

because $\mathcal{H}^1(D_\nu) \ge \varepsilon$. It follows that Σ'_ε satisfies the length constraint and is closed and connected, while

$$\begin{aligned}
\int_{\bar\Omega} \operatorname{dist}(z, \Sigma'_\varepsilon)\, d\varphi_s(z) &\le \int_{\bar\Omega} \operatorname{dist}(z, \Sigma'_\varepsilon)\, d\varphi_s(z) - C\varepsilon + o(\varepsilon) \\
&\le \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) - C\varepsilon + o(\varepsilon) + \varepsilon\psi(D_\nu),
\end{aligned}$$

where

$$C := \int_D f \sin\theta\, d(\psi_- - \psi_+).$$

Using lemma 4.1, we obtain thus

$$MK(\varphi^+, \varphi^-, \Sigma'_\varepsilon) \le MK(\varphi^+, \varphi^-, \Sigma_{opt}) + C\varepsilon + o(\varepsilon) + \varepsilon\psi(D_\nu).$$

In order not ho have a contradiction with the optimality of Σ_{opt} for sufficiently small ε, the relationship $C \le \psi(D_\nu)$ has to be satisfied. Taking a limit in the latter when $\nu \to \infty$, we obtain the desired conlusion. $\square$

We are now finally in a position to prove lemma 7.1.

PROOF OF LEMMA 7.1: Suppose the opposite, namely, $\psi(\{x\}) = 0$ for all $x \in \Sigma_{opt}$. We will then achieve a contradiction in several steps.

Step 1. Preparatory constructions. Observe that the assumption on ψ implies that θ is defined ψ-a.e. In fact, the set E where θ is undefined can be decomposed as follows:

$$E := E_0 \cup E_1,$$

where E_0 stands for the set of points of Σ_{opt} which are not endpoints of transport rays of nonzero length, while E_1 stands for the set of those points of Σ_{opt} which are endpoints of several noncollinear transport rays of positive length. Clearly, then

$$\psi(E_0) = \varphi_s(E_0) \le \varphi_s(\Sigma_{opt}) = 0,$$

while the set E_1 is at most countable, and hence the assumption $\psi(\{y\}) = 0$ for all $y \in \Sigma_{opt}$ implies also $\psi(E_1) = 0$.

We claim now

$$\psi_+ = \psi_-.$$

In fact, lemma 7.4 together with the assumption on ψ implies $\psi_{\theta,+}(e) = \psi_{\theta,-}(e)$ for all Borel $e \subset \Sigma_{opt}$, where $\psi_{\theta,\pm} := \sin\theta \cdot \psi_\pm$. This means

$$\psi_+(e \cap \{\theta \neq 0\}) = \psi_-(e \cap \{\theta \neq 0\})$$

and exchanging the coordinate axes we prove the above claim.

Step 2. Fix an arbitrary endpoint $x \in \Sigma_{opt}$ and an arbitrary $z \in \Sigma_{opt}$. Since θ is defined ψ-a.e. over $D(x,z)$, we may apply now lemma 7.3 and mind $\psi^+ = \psi^-$, thus concluding that for a set of $y \in D(x,z)$ of positive measure ψ the angle $\theta(y)$ is defined and, moreover, y is an endpoint of exactly two transport rays $R_y^+ \subset T_s^+$ and $R_y^- \subset T_s^-$ belonging to the same line l, which satisfy

$$l_y^\pm \geq (\mathcal{H}^1(D(x,z)))^{p'\sigma/4}$$

for some $\sigma > 1$, where $l_y^\pm := \mathcal{H}^1(R_y^\pm)$, while $p'\sigma/4 < 1$. Denote C_z the set of such $y \in D(x,z)$. We claim now that ψ-a.e. $y \in C_z$ are not endpoints of Σ_{opt}, once $\mathcal{H}^1(D(x,z))$ is sufficiently small.

To prove this claim, denote

$$C(y') := \{y \in D(x,z) \text{ endpoint } : [x,y] \cap [x,z] = [x,y']\}$$

for every branching point $y' \in [x,z]$. One clearly has then that every endpoint $y \in D(x,z)$ belongs to some $C(y')$ for some $y' \in [x,z]$, and, moreover, $C(y_1') \cap C(y_2') = \emptyset$ whenever $y_1' \neq y_2'$ (otherwise Σ_{opt} would contain a simple closed curve). The claim will be proven, if we show that

$$\#\{y \in C(y') \,:\, l_y^\pm \geq (\mathcal{H}^1(D(x,z')))^{p'\sigma/4} \text{ for some } z' \in [x,z], y \in D(x,z')\} \leq 1 \tag{21}$$

provided z is sufficiently close to x. In fact, in view of theorem VI.7 from [10, § 51], the set of branching points of Σ_{opt} is at most countable, since Σ_{opt} contains no simple closed curve. This, together with (21) implies that the set C_z^{end} of endpoints belonging to C_z is at most countable and hence, due to the assumption on ψ, one has $\psi(C_z^{end}) = 0$. We concentrate therefore on proving (21).

Denote

$$\delta(y) := \inf\{\mathcal{H}^1(D(x,z')) \,:\, z' \in [x,z], y \in D(x,z')\}.$$

It is easy to observe that if $y \in C(y')$, then

$$\delta(y) := \inf\{\mathcal{H}^1(D(x,z')) \,:\, y' <_{\gamma,x} z' \in [x,z]\},$$

and therefore for all $y \in C(y')$ one has $\delta(y) = const$. We denote the value of the latter constant still δ. Let $y_1 \in C(y') \cap C_z$. Without loss of generality we consider the origin of the coordinate system to coincide with y_1 and the coordinate axis x_1 to coincide with the line containing the transport rays $R_{y_1}^\pm$. Then

$$\Sigma_{opt} \cap (B_{\delta^{p'\sigma/4}}((0,\delta^{p'\sigma/4})) \cup B_{\delta^{p'\sigma/4}}((0,-\delta^{p'\sigma/4}))) = \emptyset.$$

Let $z' \geq_{\gamma,x} y'$ be such that $\mathcal{H}^1(D(x,z')) \leq 2\delta$. Then for all $\xi \in D(x,z')$ one has

$$|\xi - y_1| \leq \operatorname{diam} D(x,z') \leq \mathcal{H}^1(D(x,z')) \leq 2\delta,$$

and hence $D(x,z') \subset B_{2\delta}(y_1)$.

We assume that z is close enough to y so that $\delta \leq \mathcal{H}^1(D(x,z))$ be as small as necessary, in particular, $\delta^{p'\sigma/4} >> 2\delta$ due to our original assumption on p. Also we suppose without loss of generality that x belongs to a lower semiplane $\{x_2 < 0\}$, which implies then $D(x,z') \subset \{x_2 < 0\}$. In fact, if there were some $y \in D(x,z')$ belonging to the upper semiplane, then the arc $[x,y]$ would necessarily pass through y_1, since it has to stay in $B_{2\delta}(y_1)$ but outside of $B_{\delta^{p'\sigma/4}}((\pm\delta^{p'\sigma/4},0))$, and hence y_1 clearly would not be an endpoint of Σ_{opt}.

Suppose that there is another $y_2 \in C(y') \cap C_z$, $y_2 \neq y_1$. Then Σ_{opt} must be also outside of the pair of balls with radii $\delta^{p'\sigma/4}$ centered at $R^+_{y_2}$ and at $R^-_{y_2}$ respectively and touching y_2. Now observe that since $y_2 \in B_{2\delta}(y_1)$ then x must be inside the region bounded by thick arcs on figure 5. In fact, otherwise, the arc $[x,y_1]$ would necessarily pass through y_2 contradicting the fact that the latter is an endpoint. But then all Σ_{opt} must belong to this region, because if there were at least one point $x' \in \Sigma_{opt}$ outside of this region, then $[x,x']$ would have to pass either through y_1 or through y_2 contradicting the assumption that the latter be endpoints. It is now enough to note that the diameter of this region cannot exceed 2δ, and hence, $\operatorname{diam}\Sigma_{opt} \leq \delta$. Letting $z \searrow_\gamma x$ one has $\delta \to 0$ and therefore $\operatorname{diam}\Sigma_{opt} = 0$ which means that Σ_{opt} consists of just one point, contradicting theorem 5.1. The latter contradiction completes the proof of the claim.

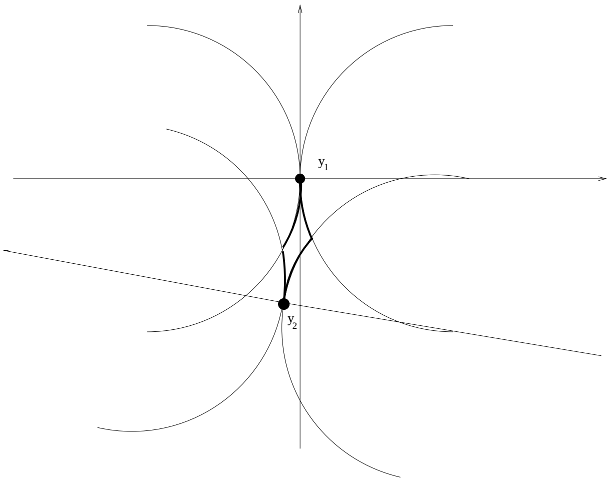

FIGURE 5

Step 3. Let $x \in \Sigma_{opt}$ be an endpoint, γ be an arc in Σ_{opt} starting at x. Using the result of Step 2, we deduce that for any sequence $\{z'_\nu\}_{\nu=1}^\infty \subset \gamma$ with $z'_\nu \searrow_\gamma x$ there is a sequence $\{y_\nu\}_{\nu=1}^\infty$ satisfying

(i) $y_\nu \in D(x,z'_\nu)$ and are not endpoints of Σ_{opt},

(ii) $\theta(y_\nu)$ is defined and, moreover, y_ν is an endpoint of exactly two transport rays belonging to the same line l_ν, both starting from outside of $B_{r_\nu}(y_\nu)$, where

$$r_\nu := (\mathcal{H}^1(D(x,z'_\nu)))^{p'\sigma/4}. \tag{22}$$

Then for every $\nu \in \mathbb{N}$ there is an endpoint x_ν of Σ_{opt} such that $D(x_\nu,y_\nu) \subset D(x,z'_\nu)$. In fact, either $y_\nu \in \gamma$, in which case we can take $x_\nu := x$, or $y_\nu \notin \gamma$. In the latter case $\gamma \cap [x, y_\nu] = [x, y'_\nu]$ for some $y'_\nu <_{\gamma,x} z'_\nu$, and we may take for x_ν any of the endpoints of Σ_{opt} belonging to $D(y_\nu, y'_\nu)$.

We further abbreviate then $D_\nu := D(x_\nu, y_\nu)$ and $\varepsilon_\nu := \operatorname{diam} D_\nu$, omitting the index ν for brevity whenever it cannot cause misunderstanding. Note that

$$\varepsilon_\nu = \operatorname{diam} D_\nu \le \mathcal{H}^1(D_\nu) \le \mathcal{H}^1(D(x, z'_\nu)) \to 0$$

as $\nu \to \infty$. Without loss of generality we consider the system of cartesian coordinates to be situated as follows: the coordinate axis x_1 coincides with l_ν, while the origin coincides with y_ν. Let

$$r_\varepsilon := \varepsilon_\nu^{p'\sigma/4} \le r_\nu$$

and consider then the points $P_\varepsilon^\pm := (\pm r_\varepsilon, 0)$. Clearly then

$$\Sigma_{opt} \cap (B_{r_\varepsilon}(P_\varepsilon^+) \cup B_{r_\varepsilon}(P_\varepsilon^-)) = \emptyset,$$

while $D_\nu \subset B_\varepsilon(y_\nu)$, and hence

$$D_\nu \subset B_\varepsilon(y_\nu) \setminus (B_{r_\varepsilon}(P_\varepsilon^+) \cup B_{r_\varepsilon}(P_\varepsilon^-)).$$

Without loss of generality we assume D_ν to be situated in the semiplane $\{x_2 < 0\}$. Consider the points $A_\varepsilon^\pm$ defined by

$$A_\varepsilon^\pm := \partial B_\varepsilon(y_\nu) \cap \partial B_{r_\varepsilon}(P_\varepsilon^\pm) \cap Q^\pm,$$

where Q^+ and Q^- are the fourth and third quadrants (coordinate angles) respectively. Then D_ν belongs to the curvilinear triangle with "vertices" y_ν, A_ε^+ and A_ε^- (see figure 6). A direct calculation shows

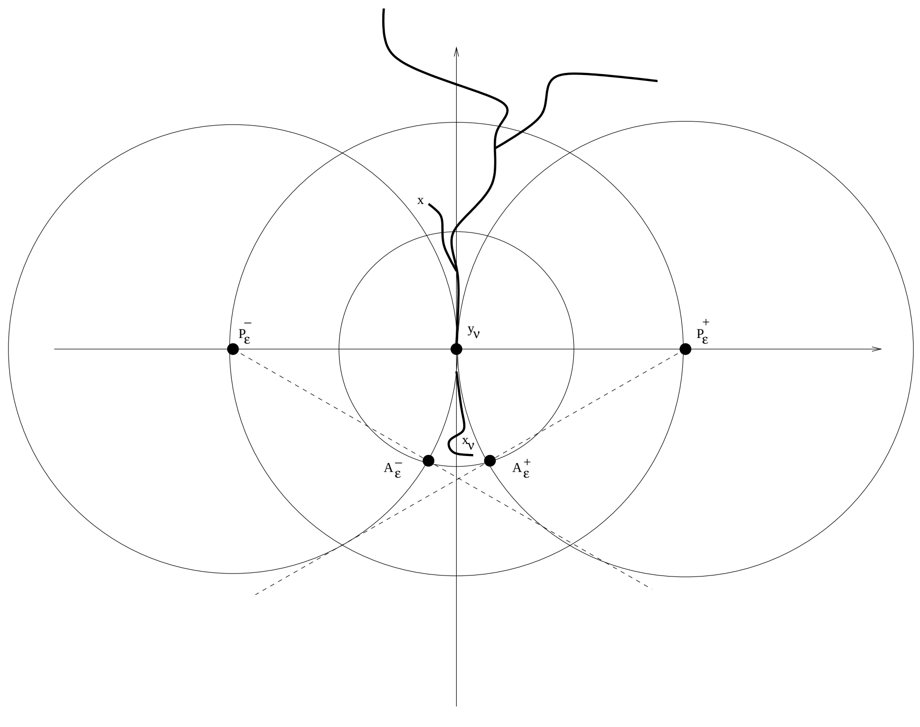

FIGURE 6. The original set Σ_{opt}

$$A_\varepsilon^\pm = (\pm\varepsilon^2/2r_\varepsilon, \varepsilon\sqrt{1-\varepsilon^2/4r_\varepsilon^2}) = (o(\varepsilon), \varepsilon + o(\varepsilon))$$

due to the choice of r_ε and to the assumption on p.

Let $R^\pm$ stand for the rays starting at $P_\varepsilon^\pm$ and passing through $A_\varepsilon^\pm$, while $C_\varepsilon^\pm$ stand for the cones with vertices $P_\varepsilon^\pm$ and formed by the x_1 axis and the rays $R^\pm$. Let also

$$D_\varepsilon^\pm := C_\varepsilon^\pm \setminus B_{r_\varepsilon}(y_\nu) \text{ and } D_\varepsilon^0 := \{x_2 < 0\} \setminus (D_\varepsilon^+ \cup D_\varepsilon^- \cup B_{r_\varepsilon}(y_\nu)).$$

Define the new measures φ_s^ε, $\varphi_{s,\pm}^\varepsilon$ by the relationships

$$\varphi_{s,\pm}^\varepsilon := \varphi_s \llcorner (k^{-1}(D_\nu) \cap T_s^\pm), \qquad \varphi_s^\varepsilon := \varphi_{s,+}^\varepsilon + \varphi_{s,-}^\varepsilon = \varphi_s \llcorner k^{-1}(D_\nu)$$

(we recall in fact that φ_s is assumed to be concentrated over the set T_s where k is injective). Disintegrating the measures $\varphi_{s,\pm}^\varepsilon$, we get

$$\varphi_{s,\pm}^\varepsilon(e) = \int_{D_\nu} \varphi_{s,\pm}^{\varepsilon\prime}(t,e)\, d\psi_\pm(t),$$

where the Borel probability valued function $t \in D_\nu \mapsto \varphi_{s,\pm}^{\varepsilon\prime}(t,\cdot)$ is measurable and the probability measures $\varphi_{s,\pm}^{\varepsilon\prime}(t,\cdot)$ are concentrated on $k^{-1}(t) \cap T_s^\pm$ for $\psi_\pm$-a.e., and hence, in view of the result of Step 1, for ψ-a.e. $t \in D_\nu$. We note then

$$\varphi_s^\varepsilon(D_\varepsilon^0) \le \varphi_s^\varepsilon(B_{r_\varepsilon}(y_\nu)). \tag{23}$$

In fact, for ψ-a.e. $t \in D_\nu$ the set $k^{-1}(t)$ is contained in a line which we will denote by l_t. It is easy to observe that if l_t passes through D_ε^0, then either of the rays $l_t^\pm := l_t \cap T_s^\pm$ intersects the horizontal segment $(P_\varepsilon^-, P_\varepsilon^+)$. Suppose, without loss of generality, that $l_t^+ \cap (P_\varepsilon^-, y_\nu] \neq \emptyset$ (the case $l_t^- \cap [y_\nu, P_\varepsilon^+)$ is completely symmetric). Then $k^{-1}(t) \cap T_s^+ \subset (P_t, t)$, where $\{P_t\} := l_t^+ \cap (P_\varepsilon^-, y_\nu]$. Since $\varphi_{s,\pm}^{\varepsilon\prime}(t,\cdot)$ are concentrated on $l_t^\pm$, then

$$1 = \varphi_{s,+}^{\varepsilon\prime}(t, l_t^+) = \varphi_{s,+}^{\varepsilon\prime}(t, (T_t, t)) = \varphi_{s,+}^{\varepsilon\prime}(t, B_{r_\varepsilon}(y_\nu)),$$

while

$$1 = \varphi_{s,-}^{\varepsilon\prime}(t, l_t^-) = \varphi_{s,-}^{\varepsilon\prime}(t, B_{r_\varepsilon}(y_\nu)) + \varphi_{s,-}^{\varepsilon\prime}(t, D_\varepsilon^0),$$

and hence $\varphi_{s,-}^{\varepsilon\prime}(t, D_\varepsilon^0) \le \varphi_{s,+}^{\varepsilon\prime}(t, B_{r_\varepsilon}(y_\nu))$. The latter implies

$$\varphi_{s,-}^\varepsilon(D_\varepsilon^0) \le \varphi_{s,+}^\varepsilon(B_{r_\varepsilon}(y_\nu))$$

and, by a symmetric argument, $\varphi_{s,+}^\varepsilon(D_\varepsilon^0) \le \varphi_{s,-}^\varepsilon(B_{r_\varepsilon}(y_\nu))$, which together imply (23).

We also remark for further use that (22) implies in view of lemma 7.2 and the Hölder inequality that

$$\varphi_s^\varepsilon(B_{r_\varepsilon}(y_\nu)) \le \varphi_s(B_{r_\varepsilon}(y_\nu)) \le C||\varphi_s||_p \varepsilon^{\sigma/2} \le \beta\psi(D_\nu) \tag{24}$$

for $\beta < 1/10$ and for sufficiently large ν (i.e. for sufficiently small $\varepsilon > 0$).

We now construct a new set Σ_ε in the following way: we cut from Σ_{opt} the set D_ν and add a segment I_ε centered at y_ν along the x_1 axis and having length ε (see figure 7). The respective set is still closed, connected and satisfies the length constraint, since $\mathcal{H}^1(D_\nu) \ge \varepsilon$. Observe now that

$$\mathrm{dist}\,(z, \Sigma_\varepsilon) \le \mathrm{dist}\,(z, \Sigma_{opt}) + \varepsilon$$

for all $z \in \bar\Omega$. On the other hand, it is a matter of straightforward though tedious calculation that

$$\mathrm{dist}\,(z, \Sigma_\varepsilon) = d(z, z_\varepsilon^\pm) \le \mathrm{dist}\,(z, D_\nu) - \varepsilon/4$$

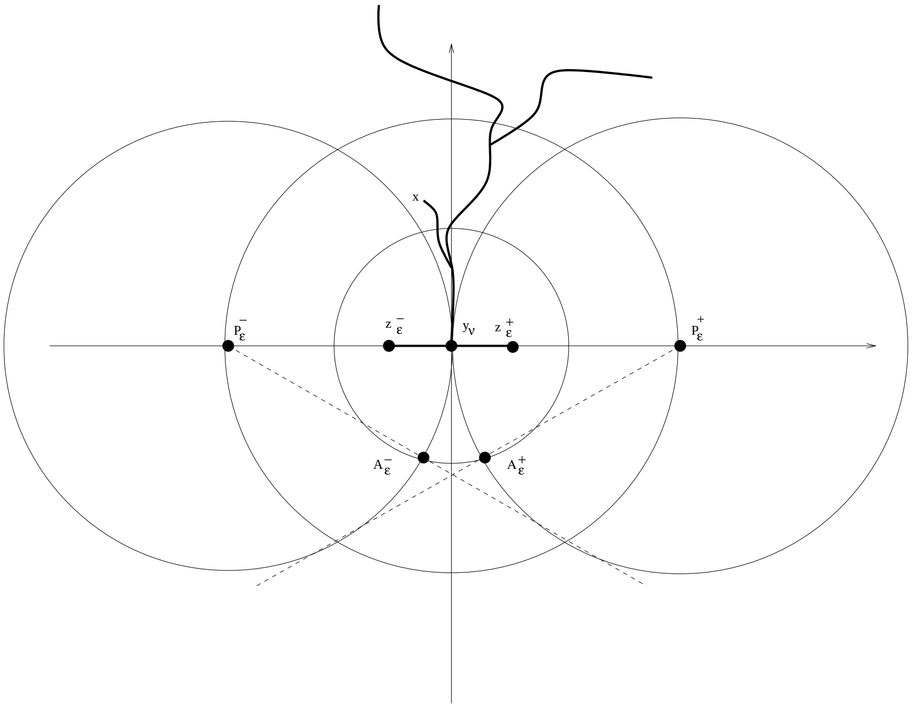

FIGURE 7. The modified set Σ_ε

for all $z \in D_\varepsilon^+ \cup D_\varepsilon^-$, where $z_\varepsilon^\pm$ are the endpoints of I_ε in Q^+ and Q^- respectively. Therefore

$$\begin{aligned}\int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_\varepsilon)\, d\varphi_s(z) \le \int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) \quad & +\varepsilon\varphi_s^\varepsilon(D_\varepsilon^0 \cup B_{r_\varepsilon}(y_\nu)) \\ & -\varepsilon\varphi_s^\varepsilon(D_\varepsilon^+ \cup D_\varepsilon^-)/4.\end{aligned}$$

But

$$\varphi_s^\varepsilon(D_\varepsilon^+ \cup D_\varepsilon^-) + \varphi_s^\varepsilon(D_\varepsilon^0 \cup B_{r_\varepsilon}(y_\nu)) = \psi(D_\nu)$$

and with the help of (23) and (24) minding $\beta < 1/10$ we get

$$\int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_\varepsilon)\, d\varphi_s(z) < \int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z).$$

The proof for this case is then concluded since the latter estimate together with lemma 4.1 gives $MK(\varphi^+, \varphi^-, \Sigma_\varepsilon) < MK(\varphi^+, \varphi^-, \Sigma_{opt})$ for sufficiently small ε contradicting the optimality of Σ_{opt}. $\square$

8. Branching points

The following result regarding branching points of the optimal set Σ_{opt} is valid.

Theorem 8.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$, $p > 4/3$ and $\varphi^+ \neq \varphi^-$. Then*

(i) *if x is a branching point of Σ_{opt}, then $ord_x \Sigma_{opt} = 3$ (in other words, all the branching points of Σ_{opt} have order three);*
(ii) *the number of branching points of Σ_{opt} is at most finite;*

In particular, Σ_{opt} is a union of a finite number of simple curves connected by at most finite number of branching points.

Before proving the theorem, we introduce another auxiliary notion which will be used both in the proof of the above theorem and further in the sequel. For a finite set of points $N \subset \partial B_r(x)$ we call its *centered Steiner connection* the set $St(N, x, r)$ providing the minimum for the length functional

$$\Sigma \mapsto \mathcal{H}^1(\Sigma)$$

among all closed connected $\Sigma \subset \bar{B}_r(x)$ which satisfy $N \cup \{x\} \subset \Sigma$ (the existence of such a set is trivial). The following simple assertion is valid.

Lemma 8.1. *Let $N = \{a, b\} \subset \partial B_1(0)$, while the minimum arc of $\partial B_1(0)$ between a and b has angle $\gamma < 2\pi/3$. Then*

$$\mathcal{H}^1(St(N, 0, 1)) \leq 2\sin(\gamma/2 + \pi/6).$$

PROOF: It is enough to observe that due to the definition

$$\mathcal{H}^1(St(N, 0, 1)) \leq \mathcal{H}^1(\Sigma),$$

where Σ stands for the union of three segments connecting a, b and x and having angle $2\pi/3$ with each other (see figure 8) and the result follows by a direct calculation of $\mathcal{H}^1(\Sigma)$. $\square$

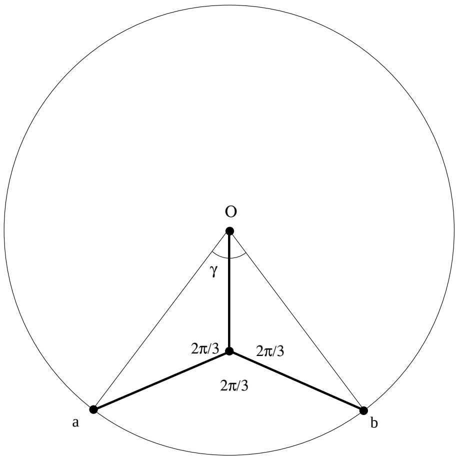

FIGURE 8. The set Σ constructed in the proof of lemma 8.1 (thick lines)

We are able now to prove theorem 8.1.

PROOF OF THEOREM 8.1: The proof will be achieved in several steps.

Step 1. We prove first (ii), i.e. that the number of branching points of Σ_{opt} is at most finite.

For the sake of completeness it is worth mentioning that actually Σ_{opt} has no points of order zero since it is connected, and hence all the endpoints have order one. Let k stand for the number of endpoints.

We will say that a point $x \in \Sigma_{opt}$ has *depth* not exceeding $\mathfrak{n}$, where $\mathfrak{n}$ is a cardinal, if there is an endpoint l and an arc connecting x to l such that the number of branching points of Σ_{opt} on this arc has cardinality $\mathfrak{n}$. Moreover, we will say that the depth of $x \in \Sigma_{opt}$ is equal to $\mathfrak{n}$, if $\mathfrak{n}$ is the least cardinal for which the depth of x does not exceed $\mathfrak{n}$.

Let us show first that the depth of each point of Σ_{opt} is at most $k-1$. Consider an arbitrary point $x_0 \in \Sigma_{opt}$ and an arc γ_0 connecting it to an arbitrary endpoint l_0. Proving the claim amounts to showing that the number of branching points on such an arc is finite. If $x_i \in \gamma_0$ is a branching point, then according to the Menger n-Beinsatz (proposition 2.2) there is an arc γ_i' starting in x_i and intersecting γ_0 only at x_i. Denote x_i' an arbitrary internal point of this arc, and consider any connected component of $\Sigma_{opt} \setminus \{x_i\}$ containing x_i'. Pick an arbitrary endpoint l_i of the latter. It is clearly also an endpoint of Σ_{opt} and cannot coincide with l_0 since otherwise Σ_{opt} would contain a simple closed curve contradicting theorem 6.1. Moreover, for the same reason when $j \neq i$, then for the same reason $l_j \neq l_i$. Therefore, to each branching point $x_i \in \gamma_0$ corresponds at least one distinct endpoint $l_i \neq l_0$ so that $l_i \neq l_j$ when $i \neq j$, which proves that there can be no more that $k-1$ distinct endpoints on γ_0 (or, in other words, the depth of x_0 doe not exceed $k-1$).

We can represent now the set $\mathcal{B}$ of branching points of Σ_{opt} in the form

$$\mathcal{B} = \bigcup_{i=0}^{k-1} \mathcal{B}_i,$$

where $\mathcal{B}_i$ stands for the set of branching points of depth i. But clearly, $\#\mathcal{B}_0 \leq k$, while $\#\mathcal{B}_{i+1} \leq \#\mathcal{B}_i$, and hence in the above union all the sets $\mathcal{B}_i$ are finite. Therefore, the set $\mathcal{B}$ is finite, which concludes the proof of (ii).

Step 2. Before showing (i), we prove a slightly weaker assertion, namely, that every branching point of Σ_{opt} has finite order. Consider an arbitrary branching point $x \in \Sigma_{opt}$. We claim that its order cannot exceed the number of endpoints k (see notation of Step 1). In fact, otherwise in view of the Menger n-Beinsatz there would exist at least $k+1$ closed arcs $\gamma_i \subset \Sigma_{opt}$, $i = 1, \ldots, k+1$, starting at x and pairwise disjoint outside of x (i.e. $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$). Taking an arbitrary internal point x_i of each of the arcs γ_i, we see that each of the connected components of the set $\Sigma_{opt} \setminus \{x\}$ containing x_i contains necessarily some endpoint l_i which is also an endpoint of Σ_{opt}. Further, $l_i \neq l_j$ when $i \neq j$ since otherwise Σ_{opt} would contain a simple closed curve in contradiction with theorem 6.1.

Step 3. At this point, we are finally able to prove (i). Let us suppose the contrary, namely that there is an $x \in \Sigma_{opt}$ with $\mathrm{ord}_x \Sigma_{opt} > 3$. Then according to the Menger n-Beinsatz, there are at least 4 closed arcs $\gamma_i \subset \Sigma_{opt}$, $i = 1, \ldots, 4$, starting at x and pairwise disjoint outside of x (i.e. $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$). We choose an $\varepsilon > 0$ small enough so that the closed ball $\bar{B}_\varepsilon(x)$ does not contain other noncut or branching points (this is possible since both the former and the latter are finite) and so as to have also

$$\gamma_i \cap \partial B_\varepsilon(x) \neq \emptyset, \qquad i = 1, \ldots, 4.$$

Let each $a_\varepsilon^i \in \gamma_i \cap \partial B_\varepsilon(x)$ stand for the first point where γ_i hits $\partial B_\varepsilon(x)$, i.e. such that for all $y \in (x, a_\varepsilon^i)$ one has $y \in B_\varepsilon(x)$, $i = 1, \ldots, 4$. Among these four points there are at least two (without loss of generality, up to renumbering, we assume that the latter are a_ε^1 and a_ε^2) such that the minimum arc of $\partial B_\varepsilon(x)$ has angle $\delta \leq \pi/2$ (see figure 9).

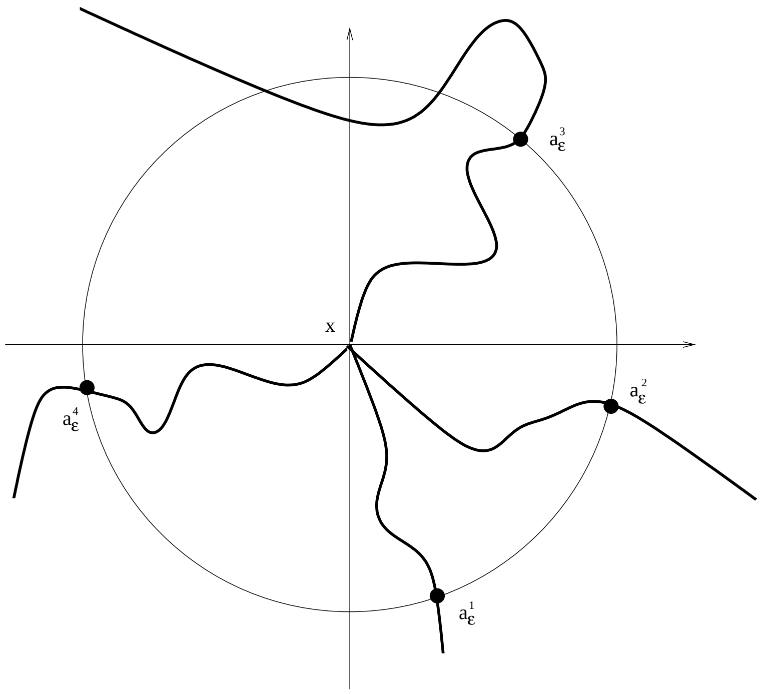

FIGURE 9. The points a_ε^i, $i = 1, \dots, 4$

Assume without loss of generality that Ω is large enough so as to satisfy

$$\overline{\mathrm{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega$$

(we may do it according to proposition 5.1), and set

$$\Sigma_\varepsilon^1 := \left(\Sigma_{opt} \setminus ((x, a_\varepsilon^1) \cup (x, a_\varepsilon^2))\right) \cup St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon).$$

Note that Σ_ε^1 is still connected, since the arcs erased did non contain any branching point of Σ_{opt}, and is obviously closed. We estimate then

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \le \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1((x, a_\varepsilon^1)) - \mathcal{H}^1((x, a_\varepsilon^2)) + \mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)).$$

Observe now that $\mathcal{H}^1((x, a_\varepsilon^i)) \ge \varepsilon$, $i = 1, 2$, while

$$\mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)) = \varepsilon \mathcal{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1)).$$

Lemma 8.1 implies then

$$\mathcal{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1) \le 2 - \beta$$

for some constant $\beta > 0$ depending only on δ. Putting the above estimates together, we get

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \le \mathcal{H}^1(\Sigma_{opt}) - \beta\varepsilon.$$

Moreover, one has

$$\int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) \le \int_{\bar\Omega} \mathrm{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) + \varepsilon\psi(B_\varepsilon(x) \setminus \{x\}). \qquad (25)$$

Let now

$$\Sigma_\varepsilon^2 := \Sigma_\varepsilon^1 \cup \partial B_{\beta\varepsilon/2\pi}(y),$$

where y is some noncut point of Σ_{opt}. Using theorem 7.1, one arrives at the estimate

$$\int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) \leq \int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) - C\varepsilon \tag{26}$$

for some $C > 0$. Since $\psi(B_\varepsilon(x) \setminus \{x\}) = o(1)$ as $\varepsilon \to 0$, then combining (25) and (26), one arrives at the estimate

$$\int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) < \int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z),$$

and hence in view of lemma 4.1, one has

$$\begin{aligned} MK(\varphi^+, \varphi^-, \Sigma_\varepsilon^2) &\leq MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) \\ &< MK(\varphi_r^+, \varphi_r^-) + \int_{\bar{\Omega}} \text{dist}\,(z, \Sigma_{opt})\, d\varphi_s(z) \\ &= MK(\varphi^+, \varphi^-, \Sigma_{opt}), \end{aligned}$$

contradicting the optimality of Σ_{opt}. $\square$

9. Triple points and curvature measure

The goal of this section is to characterize the nature of branching points of the optimal set Σ_{opt}. In fact, we expect, as the numerical results from [5] suggest, that at least when the measures $\varphi^\pm$ are sufficiently nice (i.e. they have densities with restpect to the Lebesgue measure summable with sufficiently large exponent), then all the branching points of Σ_{opt} are *regular tripods*, i.e. triple points where three smooth branches meet with angles of 120 degrees. However since we know nothing about the regularity of Σ_{opt} up to branching points, we will be only able to prove a weaker formulaton of the desired statement. For this purpose we recall the notion of a generalized mean curvature from [3]. The generalized mean curvature H_Σ of a $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ (or, in terms of [3], of the measure $\mathcal{H}^k \llcorner \Sigma$) is the vector-valued distribution defined by the relationship

$$< X, H_\Sigma >:= \int_\Sigma \text{div}^\Sigma X\, d\mathcal{H}^k$$

for all $X \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Of course, H_Σ is a (signed) Radon measure, if the quantity

$$|H_\Sigma|(D) := \sup\left\{ \int_{\Sigma\cap D} \text{div}^\Sigma X\, d\mathcal{H}^k \;:\; X \in C_0^\infty(D; \mathbb{R}^n), ||X||_\infty = 1 \right\},$$

is finite for every open set $D \subset \mathbb{R}^n$. In the sequel let H stand for the generalized mean curvature of Σ_{opt}. The following statement is valid.

Proposition 9.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$, where $p > 4/3$, and $\varphi^+ \neq \varphi^-$. Then H is a signed Radon measure satisfying $H << \psi$.*

PROOF: We show that there exists a constant $C > 0$ such that

$$|H|(D) \leq C\psi(D)$$

for all open $D \subset \bar{\Omega}$ which does not contain noncut points of Σ_{opt}, from which the statement follows immediately. For this purpose suppose $|H|(D) \neq 0$ (otherwise there is nothing to prove) and for every $\nu \in \mathbb{N}$ consider the vector field $X_\nu \in C_0^\infty(D;\mathbb{R}^2)$ such that $||X_\nu||_\infty = 1$ and

$$\left| \int_{\Sigma_{opt}\cap D} \operatorname{div}^{\Sigma_{opt}} X_\nu \, d\mathcal{H}^1 - |H|(D) \right| \leq 1/\nu.$$

For every $\varepsilon \in \mathbb{R}$ and $\nu \in \mathbb{N}$ define the diffeomorphism Φ_ν^ε by

$$\Phi_\nu^\varepsilon(x) := x - \varepsilon X_\nu(x).$$

As usual, we assume without loss of generality, referring to proposition 5.1, that Ω is large enough so as to satisfy

$$\overline{\mathrm{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega.$$

Let $\Sigma_\varepsilon := \Phi_\nu^\varepsilon(\Sigma_{opt})$. According to the theorem 7.31 of [1] one has then

$$\frac{d}{d\varepsilon}\mathcal{H}^1(\Sigma_\varepsilon) = \int_{\Sigma_{opt}\cap D} \operatorname{div}^{\Sigma_{opt}} X_\nu \, d\mathcal{H}^1.$$

Hence,

$$\delta := \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon) = \varepsilon \int_{\Sigma_{opt}\cap D} \operatorname{div}^{\Sigma_{opt}} X_\nu \, d\mathcal{H}^1 + o(\varepsilon).$$

We suppose ν to be sufficiently large. In this case $\delta \geq \varepsilon(|H|(D) - 1/\nu) + o(\varepsilon) > 0$ for small ε.

Let now $\Sigma'_\varepsilon := \Sigma_\varepsilon \cup \partial B_{\delta/2\pi}(x)$, where $x \in \Sigma_{opt}$ is some noncut point. Note that Σ'_ε is closed, connected and satisfies the length constraint. One has then

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon)\, d\varphi_s(z) \leq \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + |\varepsilon|\psi(D),$$

while on the other hand

$$\int_{\bar\Omega} \operatorname{dist}(z, \Sigma'_\varepsilon)\, d\varphi_s(z) \leq \int_{\bar\Omega} \operatorname{dist}(z, \Sigma_\varepsilon)\, d\varphi_s(z) - \delta\left(\psi(\{x\}) - \varphi_s(B_{\delta/2\pi}(x))\right)/2\pi.$$

Minding $\varphi_s(B_{\delta/2\pi}(x)) = o(\varepsilon)$ for $\varepsilon \to 0$, one has that the above estimates do not lead to a contradiction for small ε unless

$$(|H|(D) - 1/\nu)\psi(\{x\})/2\pi \leq \psi(D).$$

Minding that $\psi(\{x\}) > C$ for some $C > 0$ according to theorem 7.1, and letting $\nu \to +\infty$, we get the desired result. $\square$

With the help of the above proposition one can prove the following result regarding further characterization of branching points of Σ_{opt}.

Theorem 9.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$, where $p > 4/3$, and $\varphi^+ \neq \varphi^-$. Then every branching point $x \in \Sigma_{opt}$ is a triple point (i.e. has order three) satisfying $H(\{x\}) = 0$.*

PROOF: Let $x \in \Sigma_{opt}$ be a branching point. According to theorem 8.1 it is a triple point. The result regarding the curvature measure will follow from proposition 9.1 once we show that if $\psi(\{x\}) = 0$. We will now concentrate on proving the latter assertion.

Suppose the opposite, i.e. that $\psi(\{x\}) \neq 0$. Then there is a piece of the cone $C \subset \bar\Omega$ with vertex x and nonzero aperture angle (i.e. not reduced to a ray) such that $k(C) = x$. Note that clearly $C \cap B_\varepsilon(x) \cap \Sigma_{opt} = \{x\}$. Moreover, let C' stand for the cone with vertex x, coaxial with C and with aperture angle $\gamma + 2\pi/3$, where $\gamma > 0$ is the aperture angle of the cone C, and $C \subset C'$. Then

$$C' \cap B_\varepsilon(x) \cap \Sigma_{opt} = \{x\},$$

since for every $z \in C$ there is a $z' \in C'$ such that $d(z, z') \leq d(z, x)$ (see figure 10). We apply now a construction almost identical to the one used at Step 3 of the proof of theorem 8.1. In fact, since x is assumed to be a triple point, then according to the Menger n-Beinsatz, there are 3 closed arcs $\gamma_i \subset \Sigma_{opt}$, $i = 1, \ldots, 3$, starting at x and pairwise disjoint outside of x, i.e. $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$. We suppose $\varepsilon > 0$ to be small enough so that the closed ball $\bar{B}_\varepsilon(x)$ does not contain other branching points of Σ_{opt} (which is possible since the number of braching points is finite) and so as to have

$$\gamma_i \cap \partial B_\varepsilon(x) \neq \emptyset, \qquad i = 1, \ldots, 3.$$

Let $a_\varepsilon^i \in \gamma_i \cap \partial B_\varepsilon(x)$ stand for the first point where γ_i hits $\partial B_\varepsilon(x)$, i.e. such that for all $y \in (x, a_\varepsilon^i)$ one has $y \in B_\varepsilon(x)$, $i = 1, \ldots, 3$. But since all a_ε^i must be outside of C', then there is a couple of points among them (which without loss of generality, up to renumbering, we may assume to be $\{a_\varepsilon^1, a_\varepsilon^2\}$) such that the minimum arc of $\partial B_\varepsilon(x)$ connecting it has an angle not exceeding $2\pi/3 - \gamma/2$.

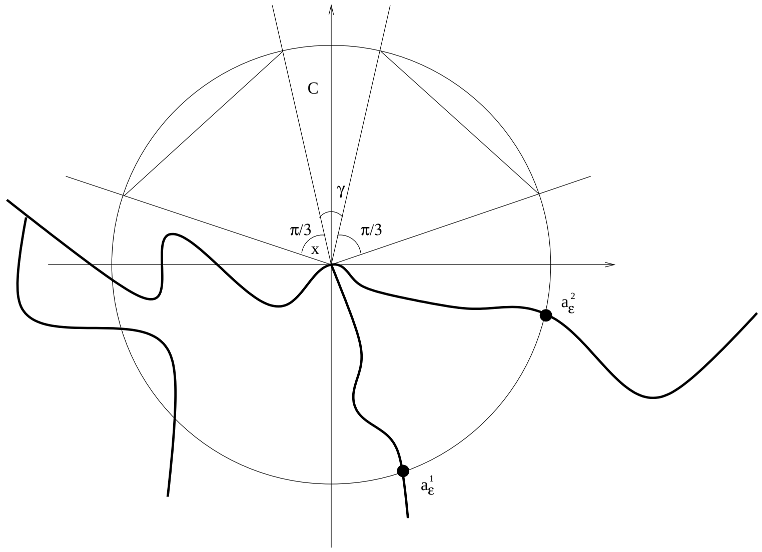

FIGURE 10. The cone C and the points a_ε^1, a_ε^2

We assume, as usual, in view of proposition 5.1, without loss of generality Ω to be large enough so as to satisfy

$$\overline{\mathrm{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega.$$

Set then

$$\Sigma_\varepsilon^1 := \left(\Sigma_{opt} \setminus ((x, a_\varepsilon^1) \cup (x, a_\varepsilon^2))\right) \cup St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon).$$

We observe that Σ_ε^1 is still connected, since the arcs erased did non contain any branching point of Σ_{opt}, and closed. The estimate

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \leq \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1((x, a_\varepsilon^1)) - \mathcal{H}^1((x, a_\varepsilon^2)) + \mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)).$$

combined with that $\mathcal{H}^1((x, a_\varepsilon^i)) \geq \varepsilon$, $i = 1, 2$, and

$$\mathcal{H}^1(St(\{a_\varepsilon^1, a_\varepsilon^2\}, x, \varepsilon)) = \varepsilon \mathcal{H}^1(St(\{(a_\varepsilon^1 - x)/\varepsilon, (a_\varepsilon^2 - x)/\varepsilon\}, 0, 1)) \geq \varepsilon(2 - \beta)$$

for some constant $\beta > 0$ depending only on γ (the latter estimate being valid in view of lemma 8.1), implies

$$\mathcal{H}^1(\Sigma_\varepsilon^1) \leq \mathcal{H}^1(\Sigma_{opt}) - \beta\varepsilon.$$

Now define

$$\Sigma_\varepsilon^2 := \Sigma_\varepsilon^1 \cup \partial B_{\beta\varepsilon/2\pi}(y),$$

where y is some noncut point of Σ_{opt}. Note that Σ_ε^2 is still closed, connected and satisfies the length constraint. Using theorem 7.1, one has

$$\int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) \leq \int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) - C\varepsilon$$

for some $C > 0$. But

$$\int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_\varepsilon^1)\, d\varphi_s(z) \leq \int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z) + \psi(B_\varepsilon(x) \setminus \{x\}),$$

and since $\psi(B_\varepsilon(x) \setminus \{x\}) = o(1)$ as $\varepsilon \to 0$, then combining the above estiates one arrives at

$$\int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_\varepsilon^2)\, d\varphi_s(z) < \int_{\bar{\Omega}} \operatorname{dist}(z, \Sigma_{opt})\, d\varphi_s(z),$$

and hence $MK(\varphi^+, \varphi^-, \Sigma_\varepsilon^2) < MK(\varphi^+, \varphi^-, \Sigma_{opt})$, in view of lemma 4.1, contradicting the optimality of Σ_{opt}. $\square$

It is worth emphasizing that the result of the above theorem 9.1 automatically implies that every branching point of Σ_{opt} is a regular tripod, provided Σ_{opt} is sufficiently regular (i.e. sufficiently smooth up to a branching point).

10. Regularity

We expect the optimal set Σ_{opt} to be rather regular away from branching points and endpoints. However, we are able to prove only the following rather weak regularity statement.

Proposition 10.1. *Let $n = 2$, $\varphi^\pm \in L^p(\Omega)$, where $p > 4/3$, and $\varphi^+ \neq \varphi^-$. Then Σ_{opt} is Ahlfors regular.*

Proof: It is enough to show the existence of an $r_0 > 0$ and a $C > 0$ such that $\mathcal{H}^1(\Sigma_{opt} \cap B_r(x)) \leq Cr$ for all $x \in \Sigma_{opt}$ and for all $r < r_0$. We show that one can take $C = 2\pi + \varepsilon$, where $\varepsilon > 0$ is an arbitrary positive number. Suppose the contrary, namely, the existence of a sequence $\{x_\nu\} \subset \Sigma_{opt}$ and of a sequence $r_\nu \to 0$ such that

$$l_\nu(x) := \mathcal{H}^1(\Sigma_{opt} \cap B_{r_\nu}(x_\nu)) > (2\pi + \varepsilon) r_\nu.$$

Up to choosing a subsequence of $\{x_\nu\}$, we assume $x_\nu \to x \in \Sigma_{opt}$. Further, we will write r instead of r_ν omitting the index ν whenever this cannot cause confusion.

According to proposition 5.1 we may assume without loss of generality that

$$\overline{\text{co}}\,(\operatorname{supp}\varphi^+ \cup \operatorname{supp}\varphi^-) \subset\subset \Omega.$$

Letting $\Sigma_r := (\Sigma_{opt} \setminus B_r(x_\nu)) \cup \partial B_r(x_\nu)$, one has that the latter is closed and connected, while still $\mathcal{H}^1(\Sigma_r) \leq \alpha - \varepsilon r$. But then

$$\int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_r)\, d\varphi_s(x) \leq \int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_{opt})\, d\varphi_s(x) + 2r\varphi_s(B_r(x_\nu)).$$

We would like to apply now lemma 5.2. For this purpose consider a $\rho > 0$ such that

$$\psi(\bar{\Omega}) > \psi(B_\rho(x)).$$

Such a ρ exists since otherwise one would have $\psi(\bar{\Omega}) = \psi(B_\rho(x))$ for all $\rho > 0$, and hence, taking a limit in the latter relationship as $\rho \to 0$, $\psi(\bar{\Omega}) = \psi(\{x\})$, which would mean that the one-point set $\{x\}$ is optimal, in contradiction with theorem 5.1. Then $\Sigma_r \cap B = \Sigma_{opt} \cap B$, where $B := \bar{\Omega} \setminus B_\rho(x)$, for all sufficiently large ν. What is more, for all $y \in k^{-1}(\Sigma_{opt} \cap B)$ (i.e. y has unique projection on Σ_{opt}, and, moreover, the latter belongs to B) one has

$$\mathrm{dist}\,(y, \Sigma_{opt} \cap B) = \mathrm{dist}\,(y, \Sigma_{opt}) < \mathrm{dist}\,(y, \Sigma_r \setminus B) \tag{27}$$

for all $\nu \geq \bar{\nu}$, where $\bar{\nu} = \bar{\nu}(x)$ is some number which may depend on x. In fact, otherwise one would have

$$\mathrm{dist}\,(y, \Sigma_{opt} \cap B) = \mathrm{dist}\,(y, \Sigma_{opt}) \geq \mathrm{dist}\,(y, \Sigma_r \setminus B) \tag{28}$$

for a subsequence of ν (not relabeled). The latter can only happen, if there is a $x'_\nu \in \partial B_{r_\nu}(x_\nu)$ such that

$$\mathrm{dist}\,(y, \Sigma_r \setminus B) = d(y, x'_\nu). \tag{29}$$

Combining the above two relationships (28) and (29), one gets

$$\mathrm{dist}\,(y, \Sigma_{opt} \cap B) = \mathrm{dist}\,(y, \Sigma_{opt}) \geq d(y, x'_\nu)$$

and taking then a limit as $\nu \to \infty$ in the resulting estimate one has,

$$\mathrm{dist}\,(y, \Sigma_{opt}) \geq d(y, x),$$

which can only happen when y has a nonunique projection on Σ_{opt}, while this case is ruled out by our construction. Hence, $k^{-1}(\Sigma_{opt} \cap B) \subset T'$ and

$$\varphi_s(T') \geq \psi(B) = \psi(\bar{\Omega}) - \psi(B_\rho(x)) > 0.$$

We may use then lemma 5.2 with $\varphi := \varphi_s$ and $\Sigma_\nu := \Sigma_r$ to find a new closed and connected set $\Sigma_{r,\varepsilon} \subset \bar{\Omega}$ satisfying $\mathcal{H}^1(\Sigma_{r,\varepsilon}) \leq \alpha$, such that

$$\int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_{r,\varepsilon})\, d\varphi_s(x) \leq \int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_r)\, d\varphi_s(x) - C_\varepsilon r^{3/2},$$

where C_ε stands for a positive constant depending on ε but not on r. Since $\varphi_s \neq 0$ in view of lemma 4.2, then using Hölder inequality one has $\varphi_s(B_r(x_\nu)) \leq Cr^{2/p'}$ for some $C > 0$. It is now enough to note that with $p > 4/3$ one has $2/p' + 1 > 3/2$ and hence

$$\int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_{r,\varepsilon})\, d\varphi_s(x) < \int_{\bar{\Omega}} \mathrm{dist}\,(x, \Sigma_{opt})\, d\varphi_s(x)$$

for sufficiently small $r > 0$. Therefore, in view of lemma 4.1, for such r one has

$$MK(\varphi^+, \varphi^-, \Sigma_{r,\varepsilon}) < MK(\varphi^+, \varphi^-, \Sigma_{opt})$$

contradicting the optimality of Σ_{opt}. $\square$

We remark that Ahlfors regularity of the optimal set Σ_{opt} implies in view of its connectedness another mild regularity condition on Σ_{opt} called *uniform rectifiability.* The latter condition, which has been extensively studied in [9], provides a lot of nice analytical properties of the set and can be considered as a kind of "quantitative rectifiability" condition as opposed to the classical "qualitative" rectifiability which is one of the basic concepts of the geometric measure theory.

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