SETS WITH FINITE $\mathbb{H}$-PERIMETER AND CONTROLLED NORMAL

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Abstract. In the Heisenberg group, we prove that the boundary of sets with finite $\mathbb{H}$-perimeter and having a bound on the measure theoretic normal is an $\mathbb{H}$-Lipschitz graph. Then we show that if the normal is, on the boundary, the restriction of a continuous mapping, then the boundary is an $\mathbb{H}$-regular surface.

1. Introduction

We identify the Heisenberg group $\mathbb{H}^n$, $n \geqslant 1$, with $\mathbb{C}^n \times \mathbb{R}$. A point $p \in \mathbb{H}^n$ has the coordinates $p = (z,t)$ with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$. The group law is

$$(z,t) \cdot (z',t') = \big(z + z', t + t' + 2\mathrm{Im}(z\bar{z}')\big),$$

where $\mathrm{Im}(z\bar{z}') = \mathrm{Im}(z_1\bar{z}_1' + ... + z_n'\bar{z}_n')$. A basis of left-invariant horizontal vector fields is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, ..., n, \tag{1.1}$$

where $z_j = x_j + iy_j$. We also let $X_j = Y_{j-n}$ for $j = n+1, ..., 2n$. The $\mathbb{H}$-divergence of a vector field $\psi = (\psi_1, ..., \psi_{2n}) \in \mathbf{C}^1(\mathbb{H}^n; \mathbb{R}^{2n})$ is

$$\mathrm{div}_{\mathbb{H}}\psi = \sum_{j=1}^{2n} X_j\psi_j.$$

A Lebesgue measurable set $E \subset \mathbb{H}^n$ is of finite $\mathbb{H}$-perimeter in the open set $\Omega \subset \mathbb{H}^n$ if

$$\sup\Big\{\int_E \mathrm{div}_{\mathbb{H}}\psi \, dzdt \,:\, \psi = (\psi_1, ..., \psi_{2n}) \in \mathbf{C}^1_c(\Omega; \mathbb{R}^{2n}),\ \|\psi\|_\infty \leqslant 1\Big\} < +\infty.$$

Here, $dzdt$ is the Lebesgue measure element in $\mathbb{H}^n$. The structure of sets with finite $\mathbb{H}$-perimeter is described in the fundamental paper [6]. If E has finite $\mathbb{H}$-perimeter in Ω, then by Riesz' Theorem there exist a finite Borel measure $|\partial E|_{\mathbb{H}}$ in Ω and a Borel

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mapping $\nu_E : \Omega \to \mathbb{S}^{2n-1}$, the unit sphere of $\mathbb{R}^{2n}$, such that for any $\psi \in \mathbf{C}^1_c(\Omega;\mathbb{R}^{2n})$ we have

$$\int_E \mathrm{div}_{\mathbb{H}}\psi \, dzdt = -\int_\Omega \langle \psi, \nu_E\rangle d|\partial E|_{\mathbb{H}}.$$

The mapping ν_E is called measure theoretic inward normal of E. Here and in the following, we denote by $\langle\cdot,\cdot\rangle$ the standard scalar product of $\mathbb{R}^{2n}$ and $\mathbb{H}^n = \mathbb{R}^{2n+1}$.

We are interested in the following question: which regularity for ∂E can be deduced from the regularity of the measure theoretic normal ν_E? In the setting of $\mathbb{R}^n$, the continuity of the measure theoretic normal w.r.t. the classical perimeter implies the $\mathbf{C}^1$ regularity of ∂E, the topological boundary of E, upon modifying E in a Lebesgue negligible set. Here, we obtain some results in the same spirit, and namely we prove that: 1) if one component of the measure theoretic normal ν_E is bounded away from 0, then ∂E has an intrinsic cone property, i.e. it is the intrinsic graph of an $\mathbb{H}$-Lipschitz function; 2) if ν_E is $|\partial E|_{\mathbb{H}}$-a.e. the restriction of a continuous mapping, then ∂E is an $\mathbb{H}$-regular surface.

Theorems 1.1 and 1.2 below are part of a program on the regularity of $\mathbb{H}$-perimeter minimizing sets in $\mathbb{H}^n$. It is conjectured that the measure theoretic normal of a minimizer is continuous. Indeed, the Hölder continuity of the normal is the basic step in De Giorgi's regularity theorem for perimeter minimizing sets in $\mathbb{R}^n$ (see e.g. [10]). In $\mathbb{H}^n$ the problem is still open. Theorem 1.2 can be used also to prove the full result in the isoperimetric inequality in [11]. Namely, the requirement that ∂E be an $\mathbb{H}$-regular surface made in Theorem 3.1 of [11] can be dropped.

Let us state our results in a more precise way. Define the homogeneous norm of $p = (z,t) \in \mathbb{C}^n \times \mathbb{R}$ as

$$\|p\| = \max\{|z|, |t|^{1/2}\}. \tag{1.2}$$

The ball centered at $p \in \mathbb{H}^n$ with radius $r > 0$ is denoted by $B_r(p) = \big\{q \in \mathbb{H}^n : \|p^{-1}\cdot q\| < r\big\}$. When $p = 0$ we simply let $B_r = B_r(0)$.

Let $\nu \in \mathbb{S}^{2n-1}$, i.e. $\nu \in \mathbb{R}^{2n}$ and $|\nu| = 1$. By abuse of notation, we identify $\nu = (\nu_1, ..., \nu_{2n}) \in \mathbb{R}^{2n}$, $\nu = (\nu_1 + i\nu_{n+1}, ..., \nu_n + i\nu_{2n}) \in \mathbb{C}^n$, and $\nu = (\nu, 0) \in \mathbb{H}^n$. Given $p \in \mathbb{H}^n$ we let $\nu(p) = \langle p, \nu\rangle\nu \in \mathbb{H}^n$ and we define $\nu^\perp(p) \in \mathbb{H}^n$ as the unique point such that

$$p = \nu^\perp(p)\cdot\nu(p). \tag{1.3}$$

The set $\big\{q \in \mathbb{H}^n : \|\nu^\perp(p^{-1}\cdot q)\| < \alpha\|\nu(p^{-1}\cdot q)\|\big\}$ is an "intrinsic cone" with vertex p, opening $\alpha > 0$, and axis specified by ν.

Theorem 1.1. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}$-perimeter in B_r, $r > 0$, ν_E be the measure theoretic inward normal of E, and $\nu \in \mathbb{S}^{2n-1}$. Assume there exists $k \in (0,1]$ such that $\langle \nu_E(p), \nu\rangle \leqslant -k$ for $|\partial E|_{\mathbb{H}}$-a.e. $p \in B_r$. Then there exists $\alpha > 0$ such that,* *possibly modifying E in a negligible set, we have for all $p \in \partial E \cap B_r$*

$$\big\{q \in B_r \,:\, \|\nu^{\perp}(p^{-1}\cdot q)\| < -\alpha \,\langle p^{-1}\cdot q, \nu\rangle\big\} \subset E, \tag{1.4}$$

$$\big\{q \in B_r \,:\, \|\nu^{\perp}(p^{-1}\cdot q)\| < \alpha \,\langle p^{-1}\cdot q, \nu\rangle\big\} \subset \mathbb{H}^n \setminus E. \tag{1.5}$$

Here and in the following, ∂E denotes the topological boundary of E.

The proof of (1.4) is based on the following observation: if we start from a point of $E \cap B_r$ with positive lower density and we move for a short time along a horizontal direction near ν, then we remain in the set of positive lower density of E. We can then show that for any $p \in E \cap B_r$ there is a truncated lateral cone with fixed opening that is contained in E. The construction is in two steps and it is analogous to the one used in [2]. The technical estimates are in Proposition 2.2.

The intrinsic cone property (1.4) and (1.5) is equivalent to the fact that $\partial E \cap B_r$ is the intrinsic graph of an $\mathbb{H}$-Lipschitz function. This is explained in Corollary 2.1. Intrinsic Lipschitz functions have been introduced recently by Franchi, Serapioni and Serra Cassano in the setting of Carnot groups [7] (see also [3]). In the Heisenberg group there is a Rademacher-type theorem for $\mathbb{H}$-Lipschitz functions [8].

A set $S \subset \mathbb{H}^n$ is said to be an $\mathbb{H}$-regular surface if for any $p \in S$ there exist an open neighborhood U of p and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f(p) \neq 0$ and $S \cap U = \big\{q \in U \,:\, f(q) = 0\big\}$. The vector $\nabla_{\mathbb{H}} f = \big(X_1 f, ..., X_{2n} f\big)$ is called the horizontal gradient of f. Recall that,

$$\mathbf{C}^1_{\mathbb{H}}(U) = \big\{f \in \mathbf{C}(U) \,:\, \nabla_{\mathbb{H}} f \in \mathbf{C}(U;\mathbb{R}^{2n}) \text{ exists in distributional sense}\big\}.$$

Our second result is the following

Theorem 1.2. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}$-perimeter in B_r, $r > 0$. Suppose there exists a continuous mapping $\widetilde{\nu} : B_r \to \mathbb{S}^{2n-1}$ such that $\nu_E(p) = \widetilde{\nu}(p)$ for $|\partial E|_{\mathbb{H}}$-a.e. $p \in B_r$. Then, possibly modifying E in a $\mathcal{L}^{2n+1}$-negligible set, $\partial E \cap B_r$ is an $\mathbb{H}$-regular surface.*

If ν_E is continuous in B_r, then $\partial E \cap B_r$ is locally an intrinsic graph, i.e. we can assume there exist $\nu \in \mathbb{S}^{2n-1}$, an open set ω contained in the orthogonal complement of ν and $\phi : \omega \to \mathbb{R}$ such that

$$\partial E \cap B_r = \mathrm{gr}(\phi) := \big\{p \cdot \phi(p)\nu \in \mathbb{H}^n : p \in \omega\big\}.$$

The function ϕ is $\mathbb{H}$-Lipschitz, by Theorem 1.1. Consider the case $\nu = (1, 0, \ldots, 0)$. The intrinsic gradient of ϕ is then defined as

$$\nabla^{\phi}\phi = (X_2\phi, \ldots, X_n\phi, W^{\phi}\phi, Y_2\phi, \ldots, Y_n\phi). \tag{1.6}$$

This gradient has to be understood in distributional sense. Here, $X_2, ..., X_n$ and $Y_2, ..., Y_n$ are the restrictions of the vector fields in (1.1) to $\nu^\perp = \{p = (p_1, ..., p_{2n+1}) \in \mathbb{H}^n : p_1 = 0\}$, whereas $W^\phi \phi$ is the distribution acting on $\psi \in \mathbf{C}^1_c(\omega)$ as

$$\langle W^\phi \phi, \psi\rangle = -\int_\omega \Big(\phi \frac{\partial \psi}{\partial y_1} - 2\phi^2 \frac{\partial \psi}{\partial t}\Big) d\widehat{z} dt,$$

where $d\widehat{z} = dx_2 \dots dx_n dy_1 \dots dy_n$. We prove that there exist a sequence $(\phi_\ell)_{\ell\in\mathbb{N}}$ in $\mathbf{C}^1(\omega)$ and a function $w \in \mathbf{C}(\omega; \mathbb{R}^{2n-1})$ such that:

i) $\phi_\ell \to \phi$ as $\ell \to +\infty$ locally uniformly in ω;
ii) $\nabla^{\phi_\ell} \phi_\ell \to w$ as $\ell \to +\infty$ locally uniformly in ω.

In fact, it is $\nabla^\phi \phi = w$ in distributional sense. By the characterization theorem for $\mathbb{H}$-regular surfaces in [1], it then follows that $\mathrm{gr}(\phi) = \partial E \cap B_r$ is an $\mathbb{H}$-regular surface. One technically important step in the argument is showing that the sequence $(\phi_\ell)_{\ell\in\mathbb{N}}$ is locally uniformly $\frac{1}{2}$-Hölder continuous. This is done in Lemma 3.2, whose proof is inspired by some ideas contained in [1] and [4].

The characterization of $\mathbb{H}$-regular surfaces of [1] has been generalized recently in [4] and [5]. Roughly speaking, the authors prove that, given continuous functions $\phi : \omega \to \mathbb{R}$ and $w : \omega \to \mathbb{R}^{2n-1}$, the graph $\mathrm{gr}(\phi)$ is $\mathbb{H}$-regular if and only if the system of equations $\nabla^\phi \phi = w$ is solved in the broad* sense [4], and in distributional sense [5], respectively. In [5], the authors also give a characterization of $\mathbb{H}$-Lipschitz graphs. A description of $\mathbb{H}$-regular surfaces in terms of uniform intrinsic differentiability is given in [3].

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2. Sets with a bound on the normal

In this section, we prove Theorem 1.1. First notice that for $p = (z, t) \in \mathbb{H}^n$ and $\nu \in \mathbb{S}^{2n-1}$, the point $\nu^\perp(p)$ defined through the identity (1.3) is given by

$$\nu^\perp(p) = \big(z - \langle z, \nu\rangle \nu, t - 2\langle z, \nu\rangle \mathrm{Im}(z\bar{\nu})\big). \tag{2.1}$$

We denote by $\nu^\perp = \{p = (z, t) \in \mathbb{H}^n : \langle z, \nu\rangle = 0\}$ the orthogonal complement of ν in $\mathbb{H}^n$. It is clearly $\nu^\perp(p) \in \nu^\perp$ for all $p \in \mathbb{H}^n$. We define the projection $\mathrm{pr}_\nu : \mathbb{H}^n \to \nu^\perp$ on letting $\mathrm{pr}_\nu(p) = \nu^\perp(p)$.

Proof of Theorem 1.1. Possibly modifying E in a $\mathcal{L}^{2n+1}$-negligible set, we assume that E coincides with the set of points where E has positive lower density, and precisely

$$E = \Big\{p \in \mathbb{H}^n : \liminf_{\varrho\downarrow 0} \frac{\mathcal{L}^{2n+1}(E \cap G(p, \varrho))}{\mathcal{L}^{2n+1}(G(p, \varrho))} > 0\Big\}, \tag{2.2}$$

where $G(p, \varrho)$ is the Euclidean ball centered at p having radius ϱ.

Let $\alpha > 0$ be a number, depending on k, given by Proposition 2.2 below. We show that for any $p \in E$ we have

$$\big\{q \in B_r \,:\, \|\nu^\perp(p^{-1}\cdot q)\| \leqslant -\alpha\,\langle p^{-1}\cdot q, \nu\rangle\big\} \subset E. \tag{2.3}$$

To this aim, consider the set of directions $\mathbb{S}^{2n-1}_k = \big\{\mu \in \mathbb{S}^{2n-1} \,:\, \langle \mu, \nu\rangle \leqslant -\sqrt{1-k^2}\big\}$ and the left invariant vector fields

$$Z_\mu = \mu_1 X_1 + \ldots + \mu_{2n} X_{2n}, \quad \mu \in \mathbb{S}^{2n-1}_k.$$

For any $\psi \in \mathbf{C}^1_c(B_r)$ such that $\psi \geqslant 0$ and for all $\mu \in \mathbb{S}^{2n-1}_k$, we have

$$\int_E Z_\mu \psi\, dp = -\int_{B_r} \psi \langle \mu, \nu_E\rangle d|\partial E|_{\mathbb{H}} \leqslant 0,$$

because $\langle \mu, \nu_E(p)\rangle \geqslant 0$ for $|\partial E|_{\mathbb{H}}$-a.e. $p \in B_r$. This follows from $\langle \mu, \nu\rangle \leqslant -\sqrt{1-k^2}$ and $\langle \nu_E, \nu\rangle \leqslant -k$.

By Lemma 2.1 below, it follows that if $p \in E \cap B_r$, $s > 0$ is such that $\exp sZ_\mu(p) \in B_r$, and $\varrho > 0$ is small enough, then we have

$$\mathcal{L}^{2n+1}(E \cap \exp sZ_\mu(G(p,\varrho))) \geqslant \mathcal{L}^{2n+1}(E \cap G(p,\varrho)).$$

Also using $\mathcal{L}^{2n+1}(\exp sZ_\mu(G(p,\varrho))) = \mathcal{L}^{2n+1}(G(p,\varrho))$, we deduce that

$$\liminf_{\varrho\downarrow 0} \frac{\mathcal{L}^{2n+1}(E \cap \exp sZ_\mu(G(p,\varrho)))}{\mathcal{L}^{2n+1}(\exp sZ_\mu(G(p,\varrho)))} \geqslant \liminf_{\varrho\downarrow 0} \frac{\mathcal{L}^{2n+1}(E\cap G(p,\varrho))}{\mathcal{L}^{2n+1}(G(p,\varrho))} > 0. \tag{2.4}$$

This implies that the point $q = \exp sZ_\mu(p)$ satisfies

$$\liminf_{\varrho\downarrow 0} \frac{\mathcal{L}^{2n+1}(E\cap G(q,\varrho))}{\mathcal{L}^{2n+1}(G(q,\varrho))} > 0,$$

and thus $q \in E$.

Now assume that $p = 0 \in E$ and define the truncated cone

$$\begin{aligned} A &= \big\{\exp sZ_\mu(0) \in B_r \,:\, s \geqslant 0,\ \mu \in \mathbb{S}^{2n-1}_k\big\} \\ &= \big\{(\zeta, 0) \in \mathbb{H}^n \,:\, \langle \zeta, \nu\rangle \leqslant -|\zeta|\sqrt{1-k^2},\ |\zeta| < r\big\}. \end{aligned}$$

The previous argument shows that $A \subset E$. Now consider the three conditions in (2.7) below and define the set

$$\begin{aligned} B &= \big\{\exp sZ_\mu(\zeta,0) \in B_r : s \geqslant 0,\ \mu \in \mathbb{S}^{2n-1}_k, (\zeta,0) \in A\big\} \\ &= \big\{(z,t) \in B_r : \text{there is } \zeta \in \mathbb{C}^n,\ |\zeta| < r, \text{ such that (2.7) holds}\big\}. \end{aligned}$$

The previous argument proves that $B \subset E$.

By Proposition 2.2, we have $\big\{q \in B_r \,:\, \|\nu^\perp(q)\| \leqslant -\alpha\,\langle q, \nu\rangle\big\} \subset B$, and our claim (2.3) follows in the case $p = 0$. The claim (2.3) for any $p \in E$ follows from the case $p = 0$ by a left translation.

Now consider the complement $\mathbb{H}^n \setminus E$. We have $\nu_{\mathbb{H}^n\setminus E} = -\nu_E$ in B_r. We can repeat the previous argument and obtain, for any p where $\mathbb{H}^n \setminus E$ has positive lower density,

$$\big\{q \in B_r \,:\, \|\nu^\perp(p^{-1}\cdot q)\| \leqslant \alpha \,\langle p^{-1}\cdot q, \nu\rangle\big\} \subset \mathbb{H}^n \setminus E. \tag{2.5}$$

In particular, (2.5) holds for any $p \in B_r \setminus E$ because $\mathbb{H}^n \setminus E$ has density 1 at such p.

Approximating a point $p \in \partial E \cap B_r$ with a sequence of points in $E \cap B_r$, from (2.3) we get (1.4). Approximating the point p with a sequence of points in $B_r \setminus E$, from (2.5) we get (1.5). Possibly, we have to take a smaller α. □

Let $\omega \subset \nu^\perp$ be an open set. The intrinsic graph (along $\nu \in \mathbb{S}^{2n-1}$) of a function $\phi : \omega \to \mathbb{R}$ is the set $\mathrm{gr}(\phi) = \big\{p\cdot\phi(p)\nu \in \mathbb{H}^n \,:\, p \in \omega\big\}$. The function ϕ is said to be $\mathbb{H}$-Lipschitz, if there exists a constant $0 \leqslant L < +\infty$ such that for all $p \in \mathrm{gr}(\phi)$

$$\mathrm{gr}(\phi) \cap \big\{q \in \mathbb{H}^n \,:\, \|\nu(p^{-1}\cdot q)\| > L\|\nu^\perp(p^{-1}\cdot q)\|\big\} = \emptyset.$$

Corollary 2.1. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}$-perimeter in B_r, $r > 0$, and let ν_E be the measure theoretic inward normal. Assume there exists $k \in (0,1]$ and $\nu \in \mathbb{S}^{2n-1}$ such that $\langle \nu_E(p), \nu\rangle \leqslant -k$ for $|\partial E|_{\mathbb{H}}$-a.e. $p \in B_r$. Then, possibly modifying E on a $\mathcal{L}^{2n+1}$-negligible set, the set $\partial E \cap B_r$ is the intrinsic graph of an $\mathbb{H}$-Lipschitz function.*

Proof. Consider the projection $\mathrm{pr}_\nu : \mathbb{H}^n \to \nu^\perp$. From (2.3) it follows that the set $\mathrm{pr}_\nu(E \cap B_r)$ is open in $\mathrm{pr}_\nu(B_r)$, which is relatively open in $\nu^\perp$. Consider the set

$$\omega = \big\{p \in \mathrm{pr}_\nu(E\cap B_r) : \text{there is } s \in \mathbb{R} \text{ such that } \exp sZ_\nu(p) \in B_r \setminus E\big\}.$$

From (2.3) and (2.5), it follows that ω is relatively open in $\mathrm{pr}_\nu(E \cap B_r)$, and so in $\nu^\perp$. By Theorem 1.1, the function $\phi : \omega \to \mathbb{R}$

$$\phi(p) = \sup\big\{s \in \mathbb{R} : \exp sZ_\nu(p) \in B_r \text{ and } \chi_E(\exp sZ_\nu(p)) = 1\big\}, \quad p \in \omega,$$

is $\mathbb{H}$-Lipschitz and we have $\partial E \cap B_r = \big\{p\cdot\phi(p)\nu \in \mathbb{H}^n : p \in \omega\big\}$.

□

Proposition 2.2. *Let $k \in (0,1]$ and $n \geqslant 1$. There exists $\alpha > 0$ such that for all $\nu \in \mathbb{S}^{2n-1}$, $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$ satisfying*

$$\|\nu^\perp(z,t)\| = \max\big\{|z - \langle z,\nu\rangle\nu|, |t - 2\langle z,\nu\rangle \mathrm{Im}(z\bar\nu)|^{1/2}\big\} \leqslant -\alpha\langle z,\nu\rangle, \tag{2.6}$$

there exists $\zeta \in \mathbb{C}^n$ such that

$$\langle \zeta,\nu\rangle \leqslant -\sqrt{1-k^2}|\zeta|, \quad \langle z-\zeta,\nu\rangle \leqslant -\sqrt{1-k^2}|z-\zeta|, \quad \textit{and} \quad t = 2\mathrm{Im}(\zeta\bar z). \tag{2.7}$$

Proof. We prove the case $n = 1$ first. Without loss of generality, we can assume that $\nu = (1,0) \in \mathbb{S}^1$. This can be achieved by a rotation in the plane. For $h \geqslant 0$ and $z = x + iy \in \mathbb{C}$ such that $|y| \leqslant -hx$ consider the set

$$R_z(h) = \big\{\xi + i\eta \in \mathbb{C} : |\eta| \leqslant -h\xi \text{ and } |y-\eta| \leqslant -h(x-\xi)\big\}.$$

For $z \neq 0$ and $h > 0$, the set $R_z(h)$ is a parallelogram with vertices 0, $z = x + iy$, z_1, and z_2 where

$$z_1 = \frac{y + hx}{2h}(1 + ih) \quad \text{and} \quad z_2 = \frac{hx - y}{2h}(1 - ih).$$

The function $\varphi_z : R_z(h) \to \mathbb{R}$, $\varphi_z(\zeta) = 2\mathrm{Im}(\zeta\bar{z})$, is linear and attains the maximum and the minimum on $\partial R_z(h)$, and actually at z_1 and z_2, respectively:

$$\max_{R_z(h)} \varphi_z = \varphi_z(z_1) = \frac{h^2x^2 - y^2}{h} \quad \text{and} \quad \min_{R_z(h)} \varphi_z = \varphi_z(z_2) = \frac{y^2 - h^2x^2}{h}.$$

Consider the set

$$D_h = \Big\{(x + iy, t) \in \mathbb{C} \times \mathbb{R} : |y| < -hx,\, y^2 - h^2x^2 \leqslant ht \leqslant h^2x^2 - y^2\Big\}.$$

By continuity, φ_z attains all the values between the maximum and the minimum. Then, for any $(z, t) \in D_h$ there exists $\zeta \in R_z(h)$ such that $t = 2\mathrm{Im}(\zeta\bar{z})$.

Now let $\alpha > 0$ be a number satisfying the following two conditions

$$\alpha^2 + 2\alpha \leqslant \frac{h}{2}, \quad \text{with } h = \sqrt{\frac{k^2}{2 - k^2}}, \tag{2.8}$$

$$\alpha^2 \leqslant \frac{k^2}{2 - 2k^2}. \tag{2.9}$$

From now on, h is fixed depending on k by (2.8).

Let $z = x + iy \in \mathbb{C}$ and $t \in \mathbb{R}$ be such that (2.6) holds with $n = 1$ and $\nu = (1, 0)$, i.e.

$$\max\{|y|, |t - 2xy|^{1/2}\} \leqslant -\alpha x. \tag{2.10}$$

We claim that $(z, t) \in D_h$. In fact, on the one hand it is $|y| \leqslant -\alpha x \leqslant -hx/2$. On the other hand, (2.10) also implies

$$-\alpha^2x^2 + 2xy \leqslant t \leqslant 2xy + \alpha^2x^2. \tag{2.11}$$

The last inequality in (2.11) yields $t \leqslant (2\alpha + \alpha^2)x^2 \leqslant hx^2/2$, by (2.8), and thus $ht + y^2 \leqslant h^2x^2$. The estimate $ht - y^2 \geqslant -h^2x^2$ is obtained in the same way. This proves that $(z, t) \in D_h$. Notice that, by the choice of h made in (2.8), $\zeta \in R_z(h)$ implies

$$\langle \zeta, \nu \rangle \leqslant -\sqrt{1 - k^2/2}\,|\zeta| \quad \text{and} \quad \langle z - \zeta, \nu \rangle \leqslant -\sqrt{1 - k^2/2}\,|z - \zeta|. \tag{2.12}$$

Now we prove the proposition in the general case, i.e. for any $n \geqslant 2$. We reduce the general case to the case $n = 1$.

Let $z \in \mathbb{C}^n$ be such that $z \neq 0$. We denote be π_z the complex line through z. With the notation $Jz = iz$, this complex line is $\pi_z = \{az + bJz \in \mathbb{C}^n : a, b \in \mathbb{R}\}$. We denote the orthogonal projection of ν onto π_z by

$$\pi_z\nu = \frac{1}{|z|^2}\big\{\langle z, \nu \rangle z + \langle Jz, \nu \rangle Jz\big\},$$

and we let

$$\widehat{\nu} = \frac{\pi_z \nu}{\gamma}, \quad \text{with} \quad \gamma = |\pi_z \nu| = \frac{\sqrt{\langle z, \nu\rangle^2 + \langle Jz, \nu\rangle^2}}{|z|}.$$

Notice that $\gamma \leqslant 1$. Moreover, by (2.6) we have

$$\gamma \geqslant \frac{|\langle z, \nu\rangle|}{|z|} \geqslant \frac{1}{\sqrt{1+\alpha^2}}. \tag{2.13}$$

We show that if (z, t) satisfies (2.6) relatively to ν, then $(z, \widehat{t})$ with $\widehat{t} = t/\gamma^2$ satisfies (2.6) relatively to $\widehat{\nu}$ with the same α. In fact, on the one hand we have

$$|z| \leqslant -\sqrt{1+\alpha^2}\langle z, \nu\rangle \quad \Rightarrow \quad |z| \leqslant -\gamma\sqrt{1+\alpha^2}\langle z, \widehat{\nu}\rangle \leqslant -\sqrt{1+\alpha^2}\langle z, \widehat{\nu}\rangle \tag{2.14}$$

implying $|z - \langle z, \widehat{\nu}\rangle\widehat{\nu}| \leqslant -\alpha\langle z, \widehat{\nu}\rangle$. Moreover, using the identity

$$\mathrm{Im}(z\bar{\widehat{\nu}}) = \frac{\langle Jz, \nu\rangle \mathrm{Im}(z\overline{Jz})}{\gamma|z|^2} = -\frac{\langle Jz, \nu\rangle}{\gamma} = \frac{\mathrm{Im}(z\bar{\nu})}{\gamma},$$

we get

$$|t - 2\langle z, \nu\rangle \mathrm{Im}(z\bar{\nu})|^{1/2} \leqslant -\alpha\langle z, \nu\rangle \quad \Leftrightarrow \quad |\widehat{t} - 2\langle z, \widehat{\nu}\rangle \mathrm{Im}(z\bar{\widehat{\nu}})|^{1/2} \leqslant -\alpha\langle z, \widehat{\nu}\rangle. \tag{2.15}$$

By the proof of the proposition in the $n = 1$ case, there exists $\widehat{\zeta} \in \pi_z$ such that $\widehat{t} = 2\mathrm{Im}(\widehat{\zeta}\bar{z})$ and (2.12) holds, i.e.

$$\langle \widehat{\zeta}, \widehat{\nu}\rangle \leqslant -\sqrt{1-k^2/2}\,|\widehat{\zeta}|, \quad \langle z - \widehat{\zeta}, \widehat{\nu}\rangle \leqslant -\sqrt{1-k^2/2}\,|z - \widehat{\zeta}|. \tag{2.16}$$

Then $\zeta = \gamma^2\widehat{\zeta}$ solves $t = 2\mathrm{Im}(\zeta\bar{z})$. Moreover, by (2.16), (2.13) and (2.9) we obtain

$$\langle \zeta, \nu\rangle = \langle \zeta, \pi_z\nu\rangle = \gamma\langle \zeta, \widehat{\nu}\rangle \leqslant -\gamma\sqrt{1-k^2/2}|\zeta| \leqslant -\frac{\sqrt{1-k^2/2}}{\sqrt{1+\alpha^2}}|\zeta| \leqslant -\sqrt{1-k^2}|\zeta|.$$

It remains to check the second inequality in (2.7). First notice that

$$\begin{aligned}\langle z - \zeta, \nu\rangle &= \langle z - \gamma^2\widehat{\zeta}, \nu\rangle = \gamma^2\langle z - \widehat{\zeta}, \nu\rangle + (1-\gamma^2)\langle z, \nu\rangle \\ &= \gamma^3\langle z - \widehat{\zeta}, \widehat{\nu}\rangle + (1-\gamma^2)\langle z, \nu\rangle.\end{aligned}$$

By the second inequality in (2.16), the first one in (2.14), (2.9), and the triangle inequality, we have

$$\begin{aligned}\langle z - \zeta, \nu\rangle &\leqslant -\gamma^3\sqrt{1-k^2/2}|z - \widehat{\zeta}| - (1-\gamma^2)\frac{|z|}{\sqrt{1+\alpha^2}} \\ &\leqslant -\frac{\sqrt{1-k^2/2}}{\sqrt{1+\alpha^2}}|\gamma^2 z - \zeta| - \frac{1}{\sqrt{1+\alpha^2}}|(1-\gamma^2)z|\big\} \\ &\leqslant -\sqrt{1-k^2}\big\{|\gamma^2 z - \zeta| + |(1-\gamma^2)z|\big\} \\ &\leqslant -\sqrt{1-k^2}\,|z - \zeta|.\end{aligned}$$

This finishes the proof of the proposition.

□

In the proof of Theorem 1.1, we used the following lemma.

Lemma 2.1. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}$-perimeter in B_r, $r > 0$, and let Z be a horizontal left invariant vector field such that*

$$\int_E Z\psi(p)\, dp \leqslant 0 \quad \text{for all } \psi \in \mathbf{C}^1_c(B_r) \text{ with } \psi \geqslant 0. \tag{2.17}$$

Then, for any $\mathcal{L}^{2n+1}$-measurable set $A \subset B_r$ we have $\mathcal{L}^{2n+1}(E \cap A) \leqslant \mathcal{L}^{2n+1}(E \cap \exp sZ(A))$ for all $s \geqslant 0$ such that $\exp sZ(A) \subset B_r$.

Proof. Without loss of generality, we can assume that $Z = X_1$. This can be obtained by a rotation on the space of horizontal left invariant vector fields and by multiplication with a positive scalar. The map $\Theta : \mathbb{H}^n \to \mathbb{H}^n$, $\Theta(q) = \exp q_1 X_1(0, q_2, ..., q_{2n+1})$, is a global diffeomorphism. It satisfies

$$\det J\Theta(q) = 1 \quad \text{and} \quad \Theta_*\Big(\frac{\partial}{\partial q_1}\Big) = X_1. \tag{2.18}$$

Letting $F = \Theta^{-1}(E)$ and $B = \Theta^{-1}(A)$, we have

$$\Theta(se_1 + B) = \exp sX_1(A), \quad s \in \mathbb{R}, \tag{2.19}$$

where $e_1 = (1, 0, ..., 0) \in \mathbb{H}^n$. For a given test function $\vartheta \in \mathbf{C}^1_c(\Theta^{-1}(B_r))$ with $\vartheta \geqslant 0$, define $\psi(p) = \vartheta(\Theta^{-1}(p))$. Then, by (2.17) and (2.18), we have

$$\int_F \frac{\partial \vartheta}{\partial q_1}(q)\, dq = \int_E X_1\psi(p)\, dp \leqslant 0. \tag{2.20}$$

By Fubini-Tonelli Theorem and by a standard approximation argument, from (2.20) it follows that the function $s \mapsto \chi_F(q + se_1)$ is increasing for $\mathcal{L}^{2n+1}$-a.e. $q \in \Theta^{-1}(B_r)$, as long as $q + se_1 \in \Theta^{-1}(B_r)$. Then for such an $s \geqslant 0$, we have, by Fubini-Tonelli Theorem,

$$\begin{aligned} \mathcal{L}^{2n+1}(F \cap B) &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \chi_F(q)\chi_B(q) dq_1\, dq_2 \dots dq_{2n+1} \\ &\leqslant \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \chi_F(se_1 + q)\chi_B(q) dq_1\, dq_2 \dots dq_{2n+1} \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \chi_F(q)\chi_{se_1+B}(q) dq_1\, dq_2 \dots dq_{2n+1} \\ &= \mathcal{L}^{2n+1}(F \cap (se_1 + B)). \end{aligned} \tag{2.21}$$

From (2.18), (2.19), and (2.21) we get $\mathcal{L}^{2n+1}(E \cap A) \leqslant \mathcal{L}^{2n+1}(E \cap \exp sZ(A))$. □

3. Sets with continuous normal

Proof of Theorem 1.2. Possibly modifying E in a $\mathcal{L}^{2n+1}$-negligible set, we can assume that E coincides with its set of positive lower density, as in (2.2). Possibly modifying ν_E in a $|\partial E|_{\mathbb{H}}$-negligible set, we can assume that $\nu_E(p) = \widetilde{\nu}(p)$ for all $p \in B_r$.

Let us fix a point $\bar{p} \in B_r$ and let $\nu = -\nu_E(\bar{p})$. For any $k \in (0, 1)$, by the continuity of ν_E in B_r there exists $\varrho > 0$ such that $\langle \nu_E(p), \nu \rangle \leqslant -k$ for all $p \in B_\varrho(\bar{p})$. By Corollary

2.1, the set $\partial E \cap B_\varrho(\bar p)$, if nonempty, is the intrinsic graph of an $\mathbb{H}$-Lipschitz function $\phi : \omega \to \mathbb{R}$, for some bounded open set $\omega \subset \nu^\perp$. Denote by

$$F := \big\{ p \cdot s\nu \in \mathbb{H}^n \,:\, s < \phi(p),\, p \in \omega \big\}$$

the intrinsic subgraph of ϕ.

Let $U \Subset \omega$ be an open set such that $\bar p \in U \cdot \mathbb{R}\nu$ and, for $R > 0$, consider the intrinsic cylinders

$$\begin{aligned} \Omega : &= \omega \cdot \mathbb{R}\nu \quad \text{and} \quad \Omega_R := \omega \cdot (-R, R)\nu, \\ \Upsilon : &= U \cdot \mathbb{R}\nu \quad \text{and} \quad \Upsilon_R := U \cdot (-R, R)\nu. \end{aligned}$$

Upon a localization argument, we can assume that $\partial F \cap \Omega = \partial E \cap \Omega \cap B_\varrho(\bar p)$. The normals $\nu_F = \nu_E$ are continuous on $\partial E \cap \Omega$.

Step 1: Mollification of χ_F. Without loss of generality, we can assume that $\nu = (1, 0, ..., 0) \in \mathbb{S}^{2n-1}$. This can be achieved by an orthogonal transformation. Since ϕ is $\mathbb{H}$-Lipschitz and ω is bounded, we have $M := \|\phi\|_{L^\infty(\omega)} < \infty$. Thus

$$\begin{aligned} &\chi_F(p) = 1 \qquad \text{for any } p \in \Omega \text{ with } p_1 \leqslant -M, \\ &\chi_F(p) = 0 \qquad \text{for any } p \in \Omega \text{ with } p_1 \geqslant M. \end{aligned} \tag{3.1}$$

Here, p_1 is the first coordinate of $p = (p_1, ..., p_{2n+1}) \in \mathbb{H}^n$.

For $\varepsilon > 0$, consider mollification kernels $g_\varepsilon \in \mathbf{C}^\infty_c(\mathbb{H}^n)$ such that

$$g_\varepsilon \geqslant 0, \quad g_\varepsilon > 0 \text{ in } B_\varepsilon, \quad \operatorname{spt} g_\varepsilon = \bar B_\varepsilon, \quad \int_{B_\varepsilon} g_\varepsilon(p)\, dp = 1. \tag{3.2}$$

For $0 < \varepsilon < \operatorname{dist}(\partial\Omega; \Upsilon_{3M})$, we can define the functions $f_\varepsilon : \Upsilon_{3M} \to [0, 1]$

$$f_\varepsilon(p) = \int_{B_\varepsilon} g_\varepsilon(q) \chi_F(q^{-1} \cdot p) dq. \tag{3.3}$$

If $\varepsilon > 0$ is sufficiently small, it follows from (3.1) that

$$\begin{aligned} &f_\varepsilon(p) = 1 \quad \text{for all } p \in \Upsilon_{3M} \text{ with } p_1 \leqslant -2M, \\ &f_\varepsilon(p) = 0 \quad \text{for all } p \in \Upsilon_{3M} \text{ with } p_1 \geqslant 2M. \end{aligned} \tag{3.4}$$

We can therefore extend f_ε to a smooth function defined in Υ on setting

$$f_\varepsilon(p) = 1 \text{ if } p_1 \leqslant -3M, \qquad f_\varepsilon(p) = 0 \text{ if } p_1 \geqslant 3M. \tag{3.5}$$

Clearly, we have $\nabla_{\mathbb{H}} f_\varepsilon(p) = 0$ if $|p_1| \geqslant 2M$.

Step 2: Estimates on $\nabla_{\mathbb{H}} f_\varepsilon$. Let $p \in \Upsilon$ be a point such that $0 < f_\varepsilon(p) < 1$. We claim that for all small enough $\varepsilon > 0$ we have

$$|\nabla_{\mathbb{H}} f_\varepsilon(p)| \leqslant \frac{1}{k} |X_1 f_\varepsilon(p)|. \tag{3.6}$$

To this aim, we study the behaviour of $X_j f_\varepsilon$, $j = 1, ..., 2n$, as a distributions acting on test functions $\varphi \in \mathbf{C}_c^\infty(\Upsilon_{3M})$. We have

$$\begin{aligned}\langle X_j f_\varepsilon, \varphi\rangle &= -\int_{\Upsilon_{3M}} f_\varepsilon(p') X_j \varphi(p')\, dp' \\ &= -\int_{B_\varepsilon} g_\varepsilon(p) \int_{\Upsilon_{3M}} \chi_F(p^{-1}\cdot p') X_j\varphi(p')\, dp'\, dp \\ &= -\int_{B_\varepsilon} g_\varepsilon(p) \int_{p^{-1}\cdot\Upsilon_{3M}} \chi_F(q) X_j \varphi(p\cdot q)\, dq\, dp.\end{aligned} \tag{3.7}$$

With the notation $\varphi_p(q) = \varphi(p\cdot q)$, we have $X_j\varphi(p\cdot q) = X_j\varphi_p(q)$, because X_j is left invariant. Then, by an integration by parts, we obtain from (3.7)

$$\langle X_j f_\varepsilon, \varphi\rangle = \int_{B_\varepsilon} g_\varepsilon(p) \int_{p^{-1}\cdot\Upsilon_{3M}} \nu_F^j(q)\varphi(p\cdot q)\, d|\partial F|_{\mathbb{H}}(q)\, dp. \tag{3.8}$$

As φ is compactly supported in Υ_{3M}, for all small enough $\varepsilon > 0$ we can replace the integration domain $p^{-1}\cdot\Upsilon_{3M}$ in (3.8) with Υ_{3M}. By Fubini-Tonelli theorem, a change of variable, and Fubini-Tonelli theorem again, we get

$$\begin{aligned}\langle X_j f_\varepsilon, \varphi\rangle &= \int_{\Upsilon_{3M}}\int_{\mathbb{H}^n} g_\varepsilon(p)\nu_F^j(q)\varphi(p\cdot q)\, dp\, d|\partial F|_{\mathbb{H}}(q) \\ &= \int_{\Upsilon_{3M}} \nu_F^j(q) \int_{\mathbb{H}^n} g_\varepsilon(p\cdot q^{-1})\varphi(p)\, dp\, d|\partial F|_{\mathbb{H}}(q) \\ &= \int_{\mathbb{H}^n} \varphi(p) \int_{\Upsilon_{3M}} g_\varepsilon(p\cdot q^{-1})\nu_F^j(q)\, d|\partial F|_{\mathbb{H}}(q)\, dp.\end{aligned} \tag{3.9}$$

This shows that for any $p \in \Upsilon_{3M}$ and for all small enough $\varepsilon > 0$ we have

$$X_j f_\varepsilon(p) = \int_{B_\varepsilon^R(p)} \nu_F^j(q) g_\varepsilon(p\cdot q^{-1})\, d|\partial F|_{\mathbb{H}}(q), \tag{3.10}$$

where, here and in the following, we let $B_\varepsilon^R(p) = B_\varepsilon \cdot p$.

Let $p \in \Upsilon_{3M}$ be a point such that $0 < f_\varepsilon(p) < 1$. Then we have $\mathcal{L}^{2n+1}(B_\varepsilon^R(p)\cap F) > 0$ and $\mathcal{L}^{2n+1}(B_\varepsilon^R(p) \setminus F) > 0$ and the isoperimetric inequality (see [9]) implies

$$|\partial F|_{\mathbb{H}}(B_\varepsilon^R(p)) > 0. \tag{3.11}$$

Let us introduce the quantity

$$\Delta_\varepsilon(p) := \int_{B_\varepsilon^R(p)} g_\varepsilon(p\cdot q^{-1})\, |\partial F|_{\mathbb{H}}(q). \tag{3.12}$$

By (3.11) and (3.2), we have $\Delta_\varepsilon(p) > 0$ and from (3.10) with $j = 1$ we get

$$X_1 f_\varepsilon(p) \leqslant -k\,\Delta_\varepsilon(p). \tag{3.13}$$

Letting $\widehat{\nabla}_{\mathbb{H}} f_\varepsilon := (X_2 f_\varepsilon, \dots, X_{2n} f_\varepsilon)$ and $\widehat{\nu}_F := (\nu_F^2, \dots, \nu_F^{2n})$, we have

$$
\begin{aligned}
|\widehat{\nabla}_{\mathbb{H}} f_\varepsilon(p)| &= \Big| \int_{B_\varepsilon^R(p)} \widehat{\nu}_F(q) g_\varepsilon(p\cdot q^{-1})\, |\partial F|_{\mathbb{H}}(q) \Big| \\
&\leqslant \int_{B_\varepsilon^R(p)} |\widehat{\nu}_F(q)| g_\varepsilon(p\cdot q^{-1})\, |\partial F|_{\mathbb{H}}(q) \\
&\leqslant \sqrt{1-k^2}\Delta_\varepsilon(p).
\end{aligned}
\tag{3.14}
$$

Then, by (3.13) we obtain

$$
|\widehat{\nabla}_{\mathbb{H}} f_\varepsilon(p)| \leqslant \sqrt{1-k^2}\Delta_\varepsilon(p) \leqslant \frac{\sqrt{1-k^2}}{k} |X_1 f_\varepsilon(p)|. \tag{3.15}
$$

Now (3.6) follows from (3.15).

Step 3: Approximation of ϕ. Let $F_\varepsilon := \{p \in \Upsilon : f_\varepsilon(p) > 1/2\}$. Since $X_1 f_\varepsilon(p) < 0$ for any $p \in \partial F_\varepsilon \cap \Upsilon$, F_ε is the intrinsic subgraph of a smooth function $\phi_\varepsilon : U \to [-2M, 2M]$, i.e.

$$
F_\varepsilon = \Big\{ p\cdot s\nu \in \mathbb{H}^n \,:\, s < \phi_\varepsilon(p),\, p \in U \Big\}.
$$

This follows by an Implicit Function Theorem argument as in [6, Theorem 6.5]. Recall the relation between the inner normal $\nu_{F_\varepsilon} = (\nu_{F_\varepsilon}^1, ..., \nu_{F_\varepsilon}^{2n})$ and the horizontal gradient $\nabla_{\mathbb{H}} f_\varepsilon$

$$
\nu_{F_\varepsilon}(p) = \frac{\nabla_{\mathbb{H}} f_\varepsilon(p)}{|\nabla_{\mathbb{H}} f_\varepsilon(p)|}, \quad p \in \partial F_\varepsilon \cap \Upsilon.
$$

By (3.6) and $X_1 f_\varepsilon(p) < 0$ for any $p \in \partial F_\varepsilon \cap \Upsilon$, we thus have

$$
-1 \leqslant \nu_{F_\varepsilon}^1(p) = \frac{X_1 f_\varepsilon(p)}{|\nabla_{\mathbb{H}} f_\varepsilon(p)|} \leqslant -k. \tag{3.16}
$$

By the definition of F_ε, we have

$$
f_\varepsilon - \chi_F > 1/2 \text{ in } F_\varepsilon \setminus F \quad \text{and} \quad \chi_F - f_\varepsilon \geqslant 1/2 \text{ in } F \setminus F_\varepsilon,
$$

and thus

$$
\int_\Upsilon |f_\varepsilon - \chi_F| dp \geqslant \frac{1}{2} \mathcal{L}^{2n+1}(F_\varepsilon \Delta F).
$$

Since $\lim_{\varepsilon\to 0} \|f_\varepsilon - \chi_F\|_{L^1(\Upsilon)} = 0$ we also have $\lim_{\varepsilon\to 0} \|\chi_{F_\varepsilon} - \chi_F\|_{L^1(\Upsilon)} = 0$. Straightforward computations show that

$$
\|\phi_\varepsilon - \phi\|_{L^1(U)} = \|\chi_{F_\varepsilon} - \chi_F\|_{L^1(\Upsilon)},
$$

and thus $\phi_\varepsilon \to \phi$ in $L^1(U)$.

Step 4: Local uniform convergence of ϕ_ε. The relation between ν_{F_ε}, the inner normal to ∂F_ε, and the intrinsic gradient $\nabla^{\phi_\varepsilon}\phi_\varepsilon$ (defined as in (1.6)) is

$$
\nu_{F_\varepsilon} = \Big(\frac{-1}{\sqrt{1+|\nabla^{\phi_\varepsilon}\phi_\varepsilon|^2}}, \frac{\nabla^{\phi_\varepsilon}\phi_\varepsilon}{\sqrt{1+|\nabla^{\phi_\varepsilon}\phi_\varepsilon|^2}} \Big), \tag{3.17}
$$

where the right hand side is evaluated at $p \in U$ and the left hand side is evaluated at $\Phi_\varepsilon(p) = p \cdot \phi_\varepsilon(p)\nu$. For this formula, see e.g. [1]. From equation (3.16), we deduce that

$$|\nabla^{\phi_\varepsilon}\phi_\varepsilon| \leqslant \frac{\sqrt{1-k^2}}{k} \quad \text{in } U.$$

By Lemma 3.1, for any open set $V \Subset U$ the functions ϕ_ε are $\frac{1}{2}$-Hölder continuous on V with Hölder constant independent from ε. By Ascoli-Arzelà's theorem, there exists a subsequence $(\phi_{\varepsilon_\ell})_{\ell\in\mathbb{N}}$ converging locally uniformly on U to ϕ. For the sake of simplicity, we omit the subscript ℓ and by $\varepsilon \to 0$ we mean $\ell \to \infty$.

Step 5: Local uniform convergence of $\nabla^{\phi_\varepsilon}\phi_\varepsilon$. Let us define the continuous map $w : \omega \longrightarrow \mathbb{R}^{2n-1}$

$$w := -\frac{\left(\nu_F^2 \circ \Phi, \dots, \nu_F^{2n} \circ \Phi\right)}{\nu_F^1 \circ \Phi},$$

where $\Phi(p) = p \cdot \phi(p)\nu$ for $p \in \omega$. We claim that for any $V \Subset U$ we have

$$\nabla^{\phi_\varepsilon}\phi_\varepsilon \to w \quad \text{in } L^\infty(V;\mathbb{R}^{2n-1}). \tag{3.18}$$

The (locally) uniform convergence in (3.18) implies the equality $w = \nabla^\phi\phi$ in distributional sense in U. Then, by Theorem 5.1 in [1] the intrinsic graph of ϕ is an $\mathbb{H}$-regular surface and the proof is accomplished.

We prove (3.18). To this aim, let us introduce the left and right invariant homogeneous distances

$$d^L(p,q) = \|p^{-1}\cdot q\| \quad \text{and} \quad d^R(p,q) = \|q \cdot p^{-1}\|, \quad p,q \in \mathbb{H}^n.$$

Both d^L and d^R satisfy the triangle inequality. Moreover, for any compact set $K \subset \mathbb{H}^n$ there exists a constant $C > 0$ such that for all $p,q \in K$ we have

$$d^R(p,q) \leqslant C d^L(p,q)^{1/2} \quad \text{and} \quad d^L(p,q) \leqslant C d^R(p,q)^{1/2}. \tag{3.19}$$

Consider a modulus of continuity $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ for ν_F on $\partial F \cap \Upsilon$ with respect to the metric d^R, i.e. $|\nu_F(p) - \nu_F(q)| \leqslant \beta(d^R(p,q))$ for $p,q \in \partial F \cap \Upsilon$ and $\beta(s) \to 0$ as $s \to 0$.

Fix $\varepsilon = \varepsilon_\ell$ and $v \in V$. Let $p_\varepsilon = \Phi_\varepsilon(v) = v \cdot \phi_\varepsilon(v)\nu$ and $p = \Phi(v) = v \cdot \phi(v)\nu$. By the argument in (3.11) and (3.12), we have $\Delta_\varepsilon(p_\varepsilon) > 0$. By the triangle inequality and by (3.19), we obtain for any $q \in B^R_\varepsilon(p_\varepsilon)$

$$d^R(q,p) \leqslant d^R(q,p_\varepsilon) + d^R(p_\varepsilon,p) \leqslant d^R(q,p_\varepsilon) + C d^L(p_\varepsilon,p)^{1/2} \leqslant \varepsilon + C\|\phi_\varepsilon - \phi\|^{1/2}_{L^\infty(V)},$$

that implies, for $q \in \partial F \cap B^R_\varepsilon(p_\varepsilon) \cap \Upsilon$,

$$|\nu_F(q) - \nu_F(p)| \leqslant \beta\big(\varepsilon + C\|\phi_\varepsilon - \phi\|^{1/2}_{L^\infty(V)}\big).$$

From (3.10), we thus get

$$\begin{aligned}\big|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon) - \nu_F(p)\Delta_\varepsilon(p_\varepsilon)\big| &= \left|\int_{B^R_\varepsilon(p_\varepsilon)} \big(\nu_F(q) - \nu_F(p)\big) g_\varepsilon(p_\varepsilon \cdot q^{-1})\, d|\partial F|_{\mathbb{H}}(q)\right| \\ &\leqslant \beta\big(\varepsilon + C\|\phi_\varepsilon - \phi\|^{1/2}_{L^\infty(V)}\big)\, \Delta_\varepsilon(p_\varepsilon)\end{aligned} \tag{3.20}$$

In particular, we have

$$|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)| = (1 + o(1))\Delta_\varepsilon(p_\varepsilon), \tag{3.21}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in $v \in V$, and thus

$$\big|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon) - \nu_F(p)\Delta_\varepsilon(p_\varepsilon)\big| \leqslant 2\beta\big(\varepsilon + C\|\phi_\varepsilon - \phi\|^{1/2}_{L^\infty(V)}\big)\, |\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)|. \tag{3.22}$$

Starting from

$$\begin{aligned}\big|\nu_{F_\varepsilon}(p_\varepsilon) - \nu_F(p)\big| &= \left|\frac{\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)}{|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)|} - \nu_F(p)\right| \\ &\leqslant \left|\frac{\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon) - \nu_F(p)\Delta_\varepsilon(p_\varepsilon)}{|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)|}\right| + \left|\nu_F(p)\frac{\Delta_\varepsilon(p_\varepsilon) - |\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)|}{|\nabla_{\mathbb{H}} f_\varepsilon(p_\varepsilon)|}\right|,\end{aligned} \tag{3.23}$$

and using (3.20), (3.21), and (3.22), we deduce that

$$\nu_{F_\varepsilon} \circ \Phi_\varepsilon \to \nu_F \circ \Phi \qquad \text{as } \varepsilon \to 0, \text{ uniformly on } V. \tag{3.24}$$

Finally, from $\nu^1_{F_\varepsilon} \leqslant -k$ and recalling (3.17), we get

$$\nabla^{\phi_\varepsilon}\phi_\varepsilon = -\frac{\big(\nu^2_{F_\varepsilon} \circ \Phi_\varepsilon, \dots, \nu^{2n}_{F_\varepsilon} \circ \Phi_\varepsilon\big)}{\nu^1_{F_\varepsilon} \circ \Phi_\varepsilon} \to w \qquad \text{as } \varepsilon \to 0, \text{ uniformly on } V.$$

This is our claim (3.18), and the proof of the Theorem is concluded. □

The following Lemma 3.1 has been used in the proof of Theorem 1.2. It can be proved by means of Lemma 3.2 and of a standard compactness argument. In both Lemmata, we identify $\mathbb{R}^{2n}$ with the orthogonal complement of $(1, 0, \dots, 0)$ in $\mathbb{H}^n = \mathbb{R}^{2n+1}$.

Lemma 3.1. *Let $U \subset \mathbb{R}^{2n}$ be an open set and let $\phi : U \to \mathbb{R}$ be a function of class $\mathbf{C}^1$ such that $|\phi| \leqslant M < +\infty$ and $|\nabla^\phi\phi| \leqslant N < +\infty$ on U, and let $V \Subset U$ be an open set. Then there exists a constant $L = L(N, M, U, V)$ such that*

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leqslant L \qquad \textit{for any } A, B \in V.$$

Lemma 3.2. *Let $I \subset \mathbb{R}^{2n}$ be a bounded open rectangle and $\phi \in \mathbf{C}^1(\bar{I})$ be such that $|\nabla^\phi\phi| \leqslant N$ on I. Then for any rectangle $J \Subset I$ there exists a constant $L = L(N, \|\phi\|_{L^\infty(I)}, I, J)$ such that*

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leqslant L \qquad \textit{for all } A, B \in J. \tag{3.25}$$

Proof. Since the proof can be easily adapted to the case $n = 1$, we discuss only the case $n \geqslant 2$. Let

$$K := \sup_{A \in I} |A| \quad \text{and} \quad M := \|\phi\|_{L^\infty(I)},$$

and fix two open rectangles I', I'' such that $J \Subset I' \Subset I'' \Subset I$.

In $\mathbb{R}^{2n}$ we use the following coordinates: $(y, v, t) \in \mathbb{R} \times \mathbb{R}^{2(n-1)} \times \mathbb{R}$ with $v = (v_2, ..., v_n, v_{n+2}, ..., v_{2n})$. The point $(y, v, t) \in \mathbb{R}^{2n}$ is also identified with $(iy, v_2 + iv_{n+2}, \dots, v_n + iv_{2n}, t) \in \mathbb{H}^n$.

Let W^ϕ be the vector field in I

$$W^\phi = \frac{\partial}{\partial y} - 4\phi \frac{\partial}{\partial t},$$

and for a point $A = (y, v, t) \in I''$ let $\gamma_A \in \mathbf{C}^1([y - \varepsilon, y + \varepsilon], I)$ be the solution of the Cauchy problem

$$\begin{cases} \dot\gamma_A(s) = W^\phi(\gamma_A(s)) \\ \gamma_A(y) = A. \end{cases}$$

By standard considerations, we may assume that $\varepsilon > 0$ depends only on I, I'', and M. We may also assume that $\gamma_A([y - \varepsilon, y + \varepsilon]) \subset I''$ for all $A \in I'$. The curve γ_A is of the form $\gamma_A(s) = \big(y + s, v, t_A(s)\big)$, where

$$\frac{d^2}{ds^2} t_A(s) = -4 \frac{d}{ds} \phi(\gamma_A(s)) = -4 W^\phi \phi(\gamma_A(s)). \tag{3.26}$$

Step 1. We claim that if $A = (y, v, t), B = (y, v, t') \in I''$ differ only in the last coordinate, then we have

$$\frac{|\phi(A) - \phi(B)|}{|t - t'|^{1/2}} \leqslant \delta := \max\left\{ \frac{(2K)^{1/2}}{\varepsilon}, \frac{2N}{\sqrt{3}} \right\}. \tag{3.27}$$

Without loss of generality we assume $t > t'$. Consider the curves γ_A and γ_B. By (3.26), we have for $s \in [y - \varepsilon, y + \varepsilon]$

$$\begin{aligned} t_A(s) - t_B(s) &= t - t' + \int_y^s \Big\{ \dot t_A(y) - \dot t_B(y) + \int_y^r \big[\ddot t_A(\sigma) - \ddot t_B(\sigma)\big] d\sigma \Big\} dr \\ &= t - t' - 4(s - y)\big[\phi(A) - \phi(B)\big] + \\ &\qquad - 4 \int_y^s \int_y^r \big[W^\phi \phi(\gamma_A(\sigma)) - W^\phi \phi(\gamma_B(\sigma))\big] d\sigma\, dr \\ &\leqslant t - t' - 4(s - y)\big[\phi(A) - \phi(B)\big] + 4N(s - y)^2. \end{aligned}$$

We are going to evaluate the previous inequality at the point

$$s := \begin{cases} y + (t - t')^{1/2}/\delta, & \text{if } \phi(A) - \phi(B) > 0, \\ y - (t - t')^{1/2}/\delta, & \text{otherwise.} \end{cases}$$

Notice that $\gamma_A(s)$ and $\gamma_B(s) \in I$ are well defined because $|s-y| = (t-t')^{1/2}/\delta \leqslant (2K)^{1/2}/\delta \leqslant \varepsilon$. With this choice of s we obtain

$$\begin{aligned} t_A(s) - t_B(s) &\leqslant (t-t') - 4\frac{(t-t')^{1/2}}{\delta}\big|\phi(A)-\phi(B)\big| + 4N\frac{t-t'}{\delta^2} \\ &= (t-t')\left[1 - \frac{4}{\delta}\,\frac{|\phi(A)-\phi(B)|}{|t-t'|^{1/2}} + \frac{4N}{\delta^2}\right]. \end{aligned}$$

Since $t_A(y) = t > t' = t_B(y)$, the uniqueness of the solutions to the Cauchy problem implies that $t_A(s) - t_B(s) > 0$, i.e.

$$1 - \frac{4}{\delta}\,\frac{|\phi(A)-\phi(B)|}{|t-t'|^{1/2}} + \frac{4N}{\delta^2} > 0,$$

and in turn

$$\frac{|\phi(A)-\phi(B)|}{|t-t'|^{1/2}} < \frac{\delta}{4}\left(1 + \frac{4N}{\delta^2}\right) \leqslant \delta,$$

the latter inequality following from $\frac{4N}{\delta^2} \leqslant 3$.

Step 2. Now we consider the case when $A = (y, v, t)$ and $B = (y', v, t)$ are points in I' differing only in the coordinate y. We will prove that

$$\frac{|\phi(A)-\phi(B)|}{|y-y'|^{1/2}} \leqslant \eta := 2\delta\sqrt{M} + N\sqrt{\varepsilon},$$

whenever $|y-y'| < \varepsilon$. This will be sufficient to show that

$$\frac{|\phi(A)-\phi(B)|}{|y-y'|^{1/2}} \leqslant \vartheta = \vartheta(K,\eta) \tag{3.28}$$

for all $A, B \in I'$ differing only in the coordinate y. Since $|y-y'| < \varepsilon$, the point $C := \gamma_B(y) = (y, v, t'')$ is well defined and belongs to I''. Therefore

$$|\phi(B)-\phi(C)| = \left|\int_{y'}^{y} W^\phi\phi(\gamma_B(s))ds\right| \leqslant N|y-y'|.$$

Moreover, since $A, C \in I''$ differ only in the last coordinate, we have by (3.27)

$$|\phi(A)-\phi(C)| \leqslant \delta|t''-t'|^{1/2} = \delta\left|4\int_{y'}^{y} \phi(\gamma_B(s))\,ds\right|^{1/2} \leqslant 2\delta\sqrt{M}|y-y'|^{1/2}.$$

It follows that

$$\begin{aligned} |\phi(A)-\phi(B)| &\leqslant |\phi(A)-\phi(C)| + |\phi(B)-\phi(C)| \\ &\leqslant 2\delta\sqrt{M}|y-y'|^{1/2} + N|y-y'| \\ &\leqslant \left(2\delta\sqrt{M} + N\sqrt{\varepsilon}\right)|y-y'|^{1/2}, \end{aligned}$$

as claimed.

Step 3. Thanks to (3.27) and (3.28), for any $A = (y, v, t)$, $B = (y', v, t') \in I'$ differing only in the coordinates y, t, we have

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leqslant \frac{|\phi(A) - \phi(C)|}{|y - y'|^{1/2}} + \frac{|\phi(C) - \phi(B)|}{|t - t'|^{1/2}} \leqslant \delta + \vartheta, \tag{3.29}$$

where $C := (y', v, t)$.

Step 4. Finally, in order to prove (3.25), let us consider two points $A = (y, v, t), B = (y', v', t') \in J$. We use the following notation. The point $v = (v_2, ..., v_n, v_{n+2}, ..., v_{2n}) \in \mathbb{R}^{2(n-1)}$ is identified with $v = (v_2 + iv_{n+2}, ..., v_n + iv_{2n}) \in \mathbb{C}^{n-1}$. Let $C := (y, v', t'')$ with $t'' = t + 2\mathrm{Im}(v\bar{v}')$. Notice that

$$C = \exp\Big(\sum_{j=2}^{n}(v'_j - v_j)X_j + (v'_{j+n} - v_{j+n})Y_j\Big)(A). \tag{3.30}$$

The points C and B differ only in the coordinates y, t and moreover

$$|t'' - t'| \leqslant |t - t'| + 2|\mathrm{Im}((v - v')\bar{v}')| \leqslant |t - t'| + 2K|v - v'| \leqslant C_K|A - B|,$$

where we let $C_K = \sqrt{2}(2K + 1)$. Notice that we have $C \in I'$ provided that $|v - v'| < c = c(K, J, I')$ is sufficiently small. If this is the case, we deduce from (3.30) that

$$|\phi(A) - \phi(C)| \leqslant N|v - v'| \leqslant N|A - B| \leqslant N\sqrt{2K}|A - B|^{1/2}, \tag{3.31}$$

and by (3.31) and (3.29) we can conclude

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leqslant \frac{|\phi(A) - \phi(C)|}{|A - B|^{1/2}} + \sqrt{C_K}\frac{|\phi(C) - \phi(B)|}{|t'' - t'|^{1/2}} \leqslant \sqrt{2K}N + \sqrt{C_K}(\delta + \vartheta).$$

The general case, i.e. without the assumption $|v - v'| < c$, can be easily deduced from the previous inequality.

□

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