ON A NONLOCAL REGULARIZATION OF A NON-STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS

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ABSTRACT. We revisit the proof of the existence of the (unique) admissible solution to a class of non-strictly hyperbolic 2×2 system of conservation laws with triangular structure. We show that this solution can be obtained as the limit of the one of a nonlocal system (involving a convolution term) when the kernel tends to a Dirac delta.

1. INTRODUCTION AND MAIN RESULT

We consider the following 2×2 system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad \partial_t v + \partial_x (a(u)v) = 0, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $a : \mathbb{R} \to \mathbb{R}$ are smooth functions.

In what follows, we are particularly interested in the case $a = f'$, which makes the systems hyperbolic, but not strictly hyperbolic (since the Jacobian matrix has a double real eigenvalue). We will consider non-negative initial data for the two equations (which are functions of bounded variation and bounded Borel measures, respectively) and focus on the prototypical case of the *Burgers* flux, *i.e.*, $f(u) := u^2$ and, correspondingly, $a(u) := f'(u) = 2u$. With minor changes to the assumptions and arguments presented in this paper, we could consider the *Greenshields–Lighthill–Whitham–Richards* flux, which is common in traffic flow modeling (see [21]), *i.e.*, $f(u) := u(1-u)$ and $a(u) := f'(u) = 1 - 2u$.

The system (1.1) has a particular triangular structure. For this reason, a natural approach to deal with it is to solve the first equation, which is a scalar conservation law in the unknown $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and then the linear continuity equation in $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, keeping u fixed. This method works well for smooth solutions; however, when a shock wave appears in u, the velocity $a(u)$ becomes discontinuous and several difficulties arise. Yet, the well-posedness of (a suitable notion of weak solution for) this system was established in [26, Theorems 4.1 & 4.2].

The suitable notion of solution for the conservation law is the *entropy-admissible* one, in the sense of Kružkov (see [25]).

On the other hand, for the linear continuity equation, we consider the notion of *measure solution* (introduced in [31]) or, equivalently, of *(conservative) duality solution* (as in [5])[1].

While Kružkov's entropy condition guarantees uniqueness for the conservation law (*e.g.*, with an initial condition of bounded variation), the uniqueness problem for transport equation with discontinuous velocity is more delicate and requires further assumptions. A sufficient condition for uniqueness is a one-sided Lipschitz bound (in the space variable) on the velocity for $t \geqq 0$. To achieve it, in [26, Theorems 4.2], the flux is assumed to be strongly convex[2] and a one-sided Lipschitz condition on the initial datum $u(0, \cdot)$ is

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[1] We notice that [26] actually studies both (1.1) and a non-conservative system and relies on yet another notion of solution, based on Volpert's product (see [32]), later generalized by Dal Maso, Le Floch, and Murat in [19]. However, in the case under consideration, this definition coincides with ones of [31, 5].

[2] Alternatively, one can make analogous arguments assuming the flux is strongly concave.

imposed (*cf.* assumption (1.4) in Theorem 1.1 below); under these assumptions, a one-sided Lipschitz bound, up to time $t = 0$, follows from Oleĭnik's inequality (see [30]). On the other hand, we point to [26, p. 137] for examples illustrating non-uniqueness.

In [26], it was also observed that the solution of the continuity equation in (1.1) is explicitly provided by the following formula (to be interpreted in the sense of distributions):

$$v(t,x) := \partial_x \int_{-\infty}^{\bar{x}} v_0\left(\frac{y - \xi(t,y)}{t}\right) \mathrm{d}y,$$

where $\xi : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is the function characterized by the property

$$\begin{aligned} &\min_{y\in\mathbb{R}} G(t,x,y) = G(t,x,\xi(t,x)), && (t,x) \in (0,+\infty)\times\mathbb{R},\\ &G(t,x,y) := \int_{-\infty}^{y} u_0(z)\,\mathrm{d}z + t\,g\left(\frac{x-y}{t}\right), && (t,x,y) \in (0,+\infty)\times\mathbb{R}^2, \end{aligned}$$

and g is the Legendre transform of the (convex) function f.

For (more general and possibly multi-dimensional) triangular systems of conservation laws, solutions have also been constructed in the literature via the vanishing viscosity approximation and numerical schemes of Engquist–Osher type (see, *e.g.*, [8, 9]).

The main aim of this note is to provide an alternative existence proof by introducing a *nonlocal regularization* of (1.1). We define a convolution kernel γ satisfying

$$\begin{aligned} &\gamma(x) \geqslant 0 \text{ for every } x \in \mathbb{R}, \quad \gamma(x) = 0 \text{ for every } x \in (-\infty,0), \quad \int_{\mathbb{R}} \gamma(x)\,\mathrm{d}x = 1,\\ &\gamma \in \mathrm{Lip}([0,+\infty)), \qquad \gamma(x) \leqslant -D\gamma'(x), \qquad \text{for a.e. } x \in (0,+\infty), \end{aligned} \tag{1.2}$$

for some $D > 0$, its rescaling $\gamma_\varepsilon := \varepsilon^{-1}\gamma(\cdot/\varepsilon)$, and introduce the system

$$\partial_t u_\varepsilon + \partial_x((u_\varepsilon * \gamma_\varepsilon)u_\varepsilon) = 0, \qquad \partial_t v_\varepsilon + 2\,\partial_x((u_\varepsilon * \gamma_\varepsilon)v_\varepsilon) = 0, \tag{1.3}$$

for a given $\varepsilon > 0$. The assumptions in (1.2) are analogous to those used in [14].

When the flux f of (1.1) is given by $f(u) := u\,V(u)$ and V is monotone non-decreasing, and the convolution kernel is supported and non-increasing on the positive axis $[0,+\infty)$, strong analytic results are available for the nonlocal conservation law in (1.3) with non-negative initial condition, including global well-posedness and a maximum principle (see Lemma 2.1 below and also, *e.g.*, [23, 12]). These assumptions on the kernel express the fact that the velocity is adjusted on the basis of the density upstream. The situation is similar when V is monotone non-increasing and the convolution kernel is supported and non-decreasing on the negative axis $(-\infty,0]$, *i.e.*, when the velocity is adjusted based on the density downstream, which is the typical setting in traffic flow models.

We will show that the (unique) weak solution of (1.3) converges to the admissible one of (1.1) as $\varepsilon \to 0$, *i.e.*, as the convolution kernel converges to a Dirac delta.

Theorem 1.1 (Nonlocal–to–local convergence)**.** *Let us assume that the convolution kernel γ_ε satisfies (1.2) and let us consider $u_0 \in L^\infty(\mathbb{R};\mathbb{R}_+) \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$ and $v_0 \in L^1(\mathbb{R})$ such that*

$$u_0'(x) \leqslant \kappa_0 \tag{1.4}$$

holds in the sense of distributions.

Let $(u_\varepsilon, v_\varepsilon) \in L^\infty((0,+\infty);\mathrm{BV}(\mathbb{R};\mathbb{R}_+)) \times L^\infty((0,+\infty);L^1(\mathbb{R}))$ be the unique weak solution of the nonlocal system

$$\begin{cases} \partial_t u_\varepsilon + \partial_x((u_\varepsilon * \gamma_\varepsilon)u_\varepsilon) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R},\\ \partial_t v_\varepsilon + 2\,\partial_x((u_\varepsilon * \gamma_\varepsilon)v_\varepsilon) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R},\\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R},\\ v_\varepsilon(0,x) = v_{0,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $u_{0,\varepsilon} := \min\{u_0 * \rho_{\varepsilon^{2/3}},\ c_0\,\varepsilon^{1/3}\}$, $v_{0,\varepsilon} := v_0 * \rho_\varepsilon$, $\rho_\varepsilon := \varepsilon^{-1}\rho(\cdot/\varepsilon)$, $\rho : \mathbb{R} \to \mathbb{R}_+$ *is a fixed smooth convolution kernel, and* c_0 *is a suitable positive constant.*

Then $(u_\varepsilon, v_\varepsilon)$ *converges to the unique admissible solution* $(u, v) \in L^\infty((0, +\infty); \mathrm{BV}(\mathbb{R}; \mathbb{R})) \times L^\infty((0, +\infty); \mathcal{M}_1(\mathbb{R}))$ *of the local system*

$$
\begin{cases}
\partial_t u + \partial_x(u^2) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R},\\
\partial_t v + 2\,\partial_x(u\,v) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R},\\
u(0, x) = u_0(x), & x \in \mathbb{R},\\
v(0, x) = v_0(x), & x \in \mathbb{R}.
\end{cases}
\tag{1.6}
$$

Here, $\mathcal{M}_1(\mathbb{R})$ *denotes the space of bounded Borel measures on* $\mathbb{R}$.

The study of nonlocal–to–local singular limits for scalar conservation laws has received much attention recently (we mention, in particular, [24, 14, 11, 15, 10, 13]). On the other hand, for systems, the only available result is contained in [7], which, however, only studies a system with a very weak coupling—namely, in a (well-behaved) source term instead of in the fluxes—modeling multi-lane traffic. A different issue is the convergence of the (incompressible) α-Euler system to the classical Euler equations, which has been extensively studied (see, *e.g.*, [28, 1, 29, 6]).

In this paper, we study the nonlocal–to–local limit in the presence of a coupling in the advective terms. The system under consideration enjoys a very particular structure that allows for a (relatively) straightforward analysis. However, because of the significance of such special systems and the directness of the approach, we feel it worthwhile to put forward this contribution and defer the examination of more general models to a forthcoming work.

1.1. **Strategy of the proof.** To prove the main result, we take advantage of the triangular structure of the system and split our analysis into two parts.

First, we study the conservation law. Since, owing to the local theory recalled above, we need to consider initial data satisfying $u_0' \leqslant \kappa_0$, we can exploit the Oleĭnik-type one-sided Lipschitz estimate on u_ε (uniform with respect to ε) proven in [14, Theorem 3] (*cf.* Lemma 2.2 below) to gain the (strong) convergence of the family $\{u_\varepsilon\}_{\varepsilon>0}$ to the (unique) entropy-admissible solution.

Second, we analyze the linear continuity equation. Since the velocity field satisfies a (uniform) one-sided Lipschitz bound, we can apply a stability result obtained by Poupaud and Rascle in [31, Theorem 3.6] to deduce the convergence of the family $\{v_\varepsilon\}_{\varepsilon>0}$ to the (unique) measure solution.

1.2. **Generalizations of Theorem 1.1.** We stress that, as it emerges from an analysis of the key ingredients of the argument, we can actually prove Theorem 1.1 in a more general setting: namely, considering

$$
\begin{cases}
\partial_t u_\varepsilon + \partial_x(V(u_\varepsilon * \gamma_\varepsilon)u_\varepsilon) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R},\\
\partial_t v_\varepsilon + \partial_x(a(u_\varepsilon * \gamma_\varepsilon)v_\varepsilon) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R},\\
u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R},\\
v_\varepsilon(0, x) = v_{0,\varepsilon}(x), & x \in \mathbb{R},
\end{cases}
\tag{1.7}
$$

where

(1) $u \mapsto f(u) := u\,V(u)$ strongly convex (in order for Oleĭnik's estimate to hold for (1.1)) and satisfying the hypotheses needed in Lemma 2.2 to deduce a one-sided Lipschitz bound for the nonlocal conservation law[3];

[3]Alternatively, the flux could be taken strongly concave, up to suitably changing the assumptions on V and γ (see [14]).

(2) $u \mapsto a(u)$ locally (one-sided) Lipschitz continuous and monotone (in order for the velocity of the transport equation to satisfy a one-sided Lipschitz bound, recalling the chain rule for BV functions from [32, 2, 19, 27]). This is automatically verified if $a = f'$ and $f \in C^2$ is strongly convex.

2. Proof of the main result

We start by recalling a well-posedness result for the Cauchy problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x((u_\varepsilon * \gamma_\varepsilon)u_\varepsilon) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u_\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

which is essentially contained, up to minor changes, in [13, Proposition 2.1 & Corollary 2.2] and [14, Proposition 8].

Lemma 2.1 (Well-posedness of the nonlocal conservation law)**.** *Let us suppose that* $u_0 \in L^\infty(\mathbb{R};\mathbb{R}_+) \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$*. Then, for every* $\varepsilon > 0$*, there exists a unique weak solution* $u_\varepsilon \in C\big([0,+\infty); L^1(\mathbb{R})\big) \cap L^\infty((0,+\infty); L^\infty(\mathbb{R}))$ *of* (2.1) *and the following maximum principle holds:*

$$\operatorname{ess\,inf}_{x\in\mathbb{R}} u_0(x) \leqslant u_\varepsilon(t,x) \leqslant \operatorname{ess\,sup}_{x\in\mathbb{R}} u_0(x), \qquad (t,x) \in [0,+\infty) \times \mathbb{R}. \tag{2.2}$$

Moreover, if $u_0 \in C^k(\mathbb{R})$*, then* $u_\varepsilon \in C^k((0,+\infty)\times\mathbb{R})$ *for* $k \geqslant 0$*. Finally,*

$$w_\varepsilon := u_\varepsilon * \gamma_\varepsilon \in W^{1,\infty}\left((0,+\infty)\times\mathbb{R}\right)$$

and

$$\operatorname{ess\,inf}_{x\in\mathbb{R}} u_0(x) \leqslant w_\varepsilon(t,x) \leqslant \operatorname{ess\,sup}_{x\in\mathbb{R}} u_0(x), \qquad (t,x) \in [0,+\infty) \times \mathbb{R}. \tag{2.3}$$

As a consequence of Lemma 2.1, since w_ε is Lipschitz continuous, we note that the solution v_ε of the linear continuity equation in (1.5) can be defined classically (*i.e.*, by relying on the Cauchy–Lipschitz theory; see [17, Chapter 1, Section 2]).

Second, we present a small modification of the Oleĭnik-type estimate in [14, Theorem 3 & Corollary 4].

Lemma 2.2 (One-sided Lipschitz bound)**.** *Let us consider a convolution kernel* γ_ε *satisfying* (1.2)*; an initial datum* $u_0 \in L^\infty(\mathbb{R};\mathbb{R}_+) \cap \mathrm{BV}(\mathbb{R};\mathbb{R}_+)$ *satisfying* $\operatorname{ess\,inf} u_0 > 0$ *and* (1.4)*; and a velocity* $V \in C^2(\mathbb{R})$ *such that*

$$V'(\xi) \geqslant \delta > 0, \qquad V''(\xi) \geqslant 0, \qquad \textit{for all } \xi \in [\operatorname{ess\,inf} u_0, \operatorname{ess\,sup} u_0]. \tag{2.4}$$

Then, if

$$\varepsilon < \frac{\operatorname{ess\,inf} u_0}{2D\kappa_0}, \tag{2.5}$$

the solution u_ε *of* (2.1) *satisfies the one-sided Lipschitz bound*

$$\frac{u_\varepsilon(t,x) - u_\varepsilon(t,y)}{x-y} \leqslant \frac{\kappa_0}{2\kappa_0 t + 1} < \frac{1}{2t}, \qquad t > 0,\ x,\, y \in \mathbb{R}, \quad \textit{with } x \neq y.$$

Proof. We can additionally assume that $u_0 \in C^2(\mathbb{R})$. Then, by (a small modification of) the stability result in [13, Proposition 3.1], we can deduce the general claim.

By differentiating the PDE in (2.1) with respect to the x variable, we have

$$\begin{aligned} &\partial^2_{tx} u_\varepsilon + \partial^2_{xx} u_\varepsilon\, V(u_\varepsilon * \gamma_\varepsilon) + 2\,\partial_x u_\varepsilon\,(\partial_x u_\varepsilon * \gamma_\varepsilon)\, V'(u_\varepsilon * \gamma_\varepsilon) \\ &\qquad + u_\varepsilon\,(\partial_x u_\varepsilon * \gamma_\varepsilon)^2\, V''(u_\varepsilon * \gamma_\varepsilon) + u_\varepsilon\,(\partial^2_{xx} u_\varepsilon * \gamma_\varepsilon)\, V'(u_\varepsilon * \gamma_\varepsilon) = 0. \end{aligned} \tag{2.6}$$

For $t \geqslant 0$, let $\bar{x} \in \mathbb{R}$ such that

$$\max_{x\in\mathbb{R}} \partial_x u_\varepsilon(t,x) = \partial_x u_\varepsilon(t,\bar{x}) =: m(t).$$

We can assume that $m(t) \geqslant 0$ (otherwise the proof is done).

Since $V'' \geqslant 0$, $\partial^2_{xx} u_\varepsilon(t,\bar{x}) = 0$, and $u_\varepsilon \geqslant 0$, from (2.6), we deduce (arguing as in [16, Theorem 2.1])

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} m(t) &\leqslant -V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\Big(2\,m(t)\,(\partial_x u_\varepsilon * \gamma_\varepsilon(t,\bar{x})) + u_\varepsilon(t,\bar{x})\,(\partial^2_{xx} u_\varepsilon * \gamma(t,\bar{x}))\Big)\\
&= -V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\left(2\,m(t)\,\frac{1}{\varepsilon}\int_{-\infty}^{\bar{x}} \partial_y u(t,y)\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right)\,\mathrm{d}y\right.\\
&\qquad\qquad\left. + u_\varepsilon(t,\bar{x})\frac{1}{\varepsilon}\int_{-\infty}^{\bar{x}} \partial^2_{yy} u_\varepsilon(t,y)\,\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right)\,\mathrm{d}y\right)
\end{aligned}$$

and, integrating by parts,

$$\begin{aligned}
&= -V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\left(2\,m(t)\,\frac{1}{\varepsilon}\int_{-\infty}^{\bar{x}} \partial_y u(t,y)\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right)\,\mathrm{d}y\right.\\
&\qquad\left. + \frac{1}{\varepsilon}u_\varepsilon(t,\bar{x})m(t)\gamma(0) + u_\varepsilon(t,\bar{x})\frac{1}{\varepsilon^2}\int_{-\infty}^{\bar{x}} \partial_y u_\varepsilon(t,y)\,\gamma'\left(\frac{\bar{x}-y}{\varepsilon}\right)\,\mathrm{d}y\right)\\
&= -V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\left(u_\varepsilon(t,\bar{x})m(t)\gamma(0)\right.\\
&\qquad\left. + \frac{1}{\varepsilon}\int_{-\infty}^{\bar{x}} \partial_y u(t,y)\underbrace{\left(2\,m(t)\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right) + \frac{1}{\varepsilon}u_\varepsilon(t,\bar{x})\,\gamma'\left(\frac{\bar{x}-y}{\varepsilon}\right)\right)}_{=:I(t,y)}\,\mathrm{d}y\right).
\end{aligned}$$

We observe that $I(0,y) \leqslant 0$. Indeed, thanks to (1.2), (1.4), and (2.5),

$$\begin{aligned}
I(0,y) &= 2\partial_x u_0(\bar{x})\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right) + \frac{1}{\varepsilon}u_0(\bar{x})\gamma'\left(\frac{\bar{x}-y}{\varepsilon}\right)\\
&\leqslant \left(2\kappa_0 - \frac{1}{\varepsilon D}\operatorname{ess\,inf} u_0\right)\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right)\\
&\leqslant 0.
\end{aligned}$$

Provided that, for some $T > 0$,

$$m(t) \leqslant \kappa_0, \qquad \text{for } t \in [0,T]. \tag{2.7}$$

holds, then we can prove (arguing in the same way as above and using (2.2)) that

$$I(t,y) \leqslant 0, \qquad \text{for } (t,y) \in [0,T]\times\mathbb{R}. \tag{2.8}$$

Now, if (2.8) holds, we deduce, for $t \in [0,T]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} m(t) &\leqslant -V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\left(u_\varepsilon(t,\bar{x})m(t)\gamma(0)\right.\\
&\qquad\left. + \frac{1}{\varepsilon}\int_{-\infty}^{\bar{x}} m(t)\left(2\,m(t)\gamma\left(\frac{\bar{x}-y}{\varepsilon}\right) + \frac{1}{\varepsilon}u_\varepsilon(t,\bar{x})\,\gamma'\left(\frac{\bar{x}-y}{\varepsilon}\right)\right)\,\mathrm{d}y\right)\\
&= -2\,V'(u_\varepsilon * \gamma_\varepsilon(t,\bar{x}))\,m^2(t)\\
&\leqslant -2\delta\,m^2(t),
\end{aligned}$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \leqslant -2\delta\,m^2(t), \qquad \text{for } t \in [0,T]. \tag{2.9}$$

Owing to the comparison principle for the Riccati-type ODE above, we conclude that

$$m(t) \leqslant \frac{\kappa_0}{2\delta\kappa_0 t + 1}, \qquad \text{for } t \in [0, T]. \tag{2.10}$$

As a consequence,

$$m(T) \leqslant \frac{\kappa_0}{2\delta\kappa_0 T + 1} < \kappa_0.$$

By continuity, we have that

$$m(t) \leqslant \kappa_0, \qquad \text{for } t \in [0, +\infty);$$

therefore

$$I(t, y) \leqslant 0, \qquad \text{for } (t, y) \in [0, +\infty) \times \mathbb{R}.$$

As a consequence, (2.9) and (2.10) holds for every $t \geqslant 0$. □

We are now ready to prove the main result.

Proof of Theorem 1.1. We need to prove that $\{u_\varepsilon\}_{\varepsilon>0}$ converges to the (unique) entropy-admissible solution of the conservation law in (1.6) and $\{v_\varepsilon\}_{\varepsilon>0}$ converges to the (unique) measure solution of the (conservative) transport equation in (1.6).

By Lemma 2.1, for $\varepsilon > 0$, there exists a unique weak solution $u_\varepsilon \in C([0, +\infty); L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0, +\infty); L^\infty(\mathbb{R})) \cap L^\infty((0, +\infty); \mathrm{BV}(\mathbb{R}))$ of the Cauchy problem associated with the nonlocal conservation law in (1.5) and the uniform bound (2.2) holds.

As in [14, Remark 5], the particular regularization of the initial data chosen in the statement of Theorem 1.1 is motivated by the assumption (2.5) in Lemma 2.2. Due to Lemma 2.2, we have that

$$\frac{u_\varepsilon(t, x) - u_\varepsilon(t, y)}{x - y} \leqslant \frac{\kappa_0}{2\kappa_0 t + 1} < \frac{1}{2t}, \qquad t > 0, \; x, \, y \in \mathbb{R}, \quad \text{with } x \neq y, \tag{2.11}$$

holds for a sufficiently small $\varepsilon > 0$. As a consequence, we deduce that the family[4] $\{u_\varepsilon\}_{\varepsilon>0}$ converges strongly in L^1 to the (unique) entropy-admissible solution of the scalar conservation law in (1.6), which, in particular, satisfies Oleĭnik's entropy condition (since $f(u) := u^2$ is strictly convex):

$$\frac{u(t, x) - u(t, y)}{x - y} \leqslant \frac{\kappa_0}{2\kappa_0 t + 1} < \frac{1}{2t}, \qquad t > 0, \; x, \, y \in \mathbb{R}, \quad \text{with } x \neq y. \tag{2.12}$$

Here, by using [18, Lemma 1.3.3], we could assume, without loss of generality, that the functions $t \mapsto u_\varepsilon(t, \cdot)$ is continuous from $\mathbb{R}_+$ to $L^\infty(\mathbb{R})$ endowed with the L^∞-weak-$*$ and the strong L^1_{loc} topology, respectively.

To conclude the proof of Theorem 1.1, it remains to show that $\{v_\varepsilon\}_{\varepsilon>0}$ converges to the solution of the transport equation in (1.6). By Lemma 2.1, we notice that w_ε belongs to $W^{1,\infty}((0, +\infty) \times \mathbb{R}; \mathbb{R}_+)$ and satisfies (2.3). Moreover, owing to (2.11), we have

$$\frac{w_\varepsilon(t, x) - w_\varepsilon(t, y)}{x - y} \leqslant \frac{\kappa_0}{2\kappa_0 t + 1}, \qquad t > 0, \; x, \, y \in \mathbb{R}, \quad \text{with } x \neq y. \tag{2.13}$$

Owing to the bounds (2.3) and (2.13), we can apply [31, Theorem 3.6] and deduce that there exists a sub-sequence of $v_{\varepsilon_k}(t, \cdot) = X_{\varepsilon_k}(t)_\# v_{0,\varepsilon_k}$ that converges in $C([0, +\infty); \mathcal{M}_1(\mathbb{R}))$ to the unique measure solution $v(t, \cdot) = X(t)_\# v_0$ of the transport equation in (1.6). Here, X denotes the unique Filippov flow associated with u. Such flow X and the corresponding

[4] By Urysohn's sub-sequence principle, owing to the uniqueness of entropy solutions for the scalar conservation law in (1.6), the whole family converges, not just up to sub-sequences.

solution v are indeed unique[5], owing to [31, Theorems 2.2, 3.1, & 3.2], because u satisfies the one-sided Lipschitz bound in (2.12).

This concludes the proof. □

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