BLOW-UP ESTIMATES AT HORIZONTAL POINTS AND APPLICATIONS

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ABSTRACT. Horizontal points of smooth submanifolds in stratified groups play the role of singular points with respect to the Carnot-Carathéodory distance. When we consider hypersurfaces, they coincide with the well known characteristic points. In two step groups, we obtain pointwise estimates for the Riemannian surface measure at all horizontal points of $C^{1,1}$ smooth submanifolds. As an application, we establish an integral formula to compute the spherical Hausdorff measure of any $C^{1,1}$ submanifold. Our technique also shows that C^2 smooth submanifolds everywhere admit an intrinsic blow-up and that the limit set is an intrinsically homogeneous algebraic variety.

CONTENTS

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1. Introduction

Metric properties of hypersurfaces in stratified groups have been extensively investigated under different perspectives in different areas of Analysis and Geometry, with a vast and always growing literature, [2], [7], [8], [12], [13], [15], [16], [19], [31], [32]. At the same time, first papers on the geometry of higher codimensional sets start to appear, [4], [6], [20], [28], [29].

This work can be thought of as a continuation of the project started in [27], [28], [29] pertaining to the intrinsic measure of arbitrary submanifolds in stratified groups. Higher codimensional submanifolds somehow possess an elusive nature, due to the higher degree of freedom of their tangent spaces. The classical Frobenius theorem implies that a smooth hypersurface must have points that are transversal to the horizontal subbundle of the group and this implies in turn a precise Hausdorff dimension of the submanifold, along with an area type formula, [27]. However, this theorem does not suffice to tackle higher codimensional surfaces, whose Hausdorff dimension depends on an interesting interaction between their tangent bundle and the grading of the group, according to Gromov's formula in 0.6 B of [22].

Using the notion of "degree", this interaction has been made more suitable for computations in [28], where the density of their spherical Hausdorff measure has been computed under a "negligibility condition". In broad terms, the degree of a submanifold Σ at a fixed point x is a certain weight $d_\Sigma(x)$ assigned to the tangent space $T_x\Sigma$, that depends on its intersections with the flag generated by the grading, see Section 2 and Remark 4.2. The maximum over all pointwise degrees is the degree $d(\Sigma)$ of the submanifold. If d is the degree of a submanifold, then the negligibility condition amounts to the $\mathcal{S}^d$-negligibility of points of degree lower than d. The spherical Hausdorff measure $\mathcal{S}^d$ is constructed with respect to a fixed homogeneous distance of the group, that is equivalent to and generalizes the so-called the Carnot-Carathéodory distance. This condition holds for C^1 non-horizontal k-codimensional submanifolds, along with an area-type formula for their $\mathcal{S}^{Q-k}$-measure, where Q is the group's Hausdorff dimension, see [27], [29]. One of the main results of the paper is the negligibility condition for all submanfolds in two step groups. This is a consequence of a more general result, namely, the following blow-up estimates.

Theorem 1.1. *Let $\mathbb{G}$ be a two step stratified group and let $\Sigma \subset \mathbb{G}$ be a p-dimensional $C^{1,1}$ smooth submanifold. Then for every $x \in \Sigma$ there exists a neighbourhood U of x and positive constants c_1, c_2 and r_0 depending on $U \cap \Sigma$ such that*

$$c_1\, r^{d_\Sigma(x)} \le \tilde{\mu}_p(\Sigma \cap B_{z,r}) \le c_2\, r^{d_\Sigma(x)} \tag{1}$$

for every $z \in \Sigma \cap U$ with $d_\Sigma(z) = d_\Sigma(x)$ and every $0 < r < r_0$.

We have denoted by $\tilde{\mu}_p$ the Riemannian surface measure induced on the p-dimensional submanifold Σ by a fixed Riemannian metric $\tilde{g}$ on the group. Notice that $\tilde{g}$ here has an auxiliary role and need not be left invariant. Before presenting some applications of (1), we wish to discuss these estimates with more details.

First of all, the degree of x in Theorem 1.1 plays an important role. In fact, in the special case x is non-horizontal, namely, it has degree equal to $Q-k$, where k is the codimension of Σ, then from results of [29], adopting the same approach of this paper, one achieves (1) with C^1 smoothness of Σ. Similarly, if x has maximum degree, then results of [28] apply and lead to (1). The point of Theorem 1.1 is that we do not assume any restriction on the nature of x. Thus, our estimates also hold at horizontal points, that play the role of characteristic points in higher codimension and actually coincide with them in codimension one, as discussed in [29].

To compare these estimates with the existing literature in codimension one, we first notice that the degree of characteristic points in two step groups is exactly $Q-2$, see Remark 2.7. Then our blow-up estimates include (20) of [26] and extend it to arbitrary submanifolds. Other interesting estimates have been recently obtained in [11], especially for the perimeter measure, in connection with boundary regularity of domains for the Dirichlet problem in CC-spaces. Next, we discuss a series of applications of (1).

Corollary 1.2. *Let Σ be a p-dimensional $C^{1,1}$ smooth submanifold of degree d in a two step stratified group. Then we have*

$$\mathcal{S}^d\left(\left\{\, x\in\Sigma \mid\ d_\Sigma(x)<d\,\right\}\right)=0 \tag{2}$$

and the following formula holds

$$\int_\Sigma \theta(\tau_\Sigma^d(x))\, d\mathcal{S}^d(x)=\int_\Sigma |\tau_\Sigma^d(x)|\, d\tilde{\mu}_p(x). \tag{3}$$

The area-type formula (3) is an immediate consequence of (2), after the blow-up at points of maximum degree. For a proof and a discussion of this formula, we address the reader to [28], where the negligibility condition (2) was announced. Furthermore, Theorem 1.1 allows us to distinguish different types of horizontal points, to get estimates on their Hausdorff dimension depending on their degree.

Corollary 1.3. *Let Σ be a $C^{1,1}$ submanifold of degree d in a two step stratified group and let δ be a positive integer. Then the subset $Z_\delta=\left\{x\in\Sigma \mid\ d_\Sigma(x)\le\delta\,\right\}$ is countably $\mathcal{H}^\delta$-finite. In particular, its Hausdorff dimension is less than or equal to δ.*

Remark 4.1 shows how Corollary 1.2 easily implies that the degree of a $C^{1,1}$ submanifold coincides with its Hausdorff dimension. Then Corollary 1.3 gives new information if $\delta<d(\Sigma)$. Again, taking into account Remark 2.7, this extends (26) of [26] to arbitrary submanifolds. Our observation on the Hausdorff dimension of submanifolds in two step groups fits into the Gromov's formula pertaining to the Hausdorff dimension of a smooth submanifolds in equiregular Carnot-Carathéodory spaces, provided that the functions $D'(x)$ and $D_H(\Sigma)$ introduced in 0.6 B of [22] coincide with the pointwise degree $d_\Sigma(x)$ and the degree $d(\Sigma)$, respectively. Although this is not difficult to check, in Remark 4.2 we show this fact.

It is then natural searching all possible Hausdorff dimensions of submanifolds whose topological dimension is fixed. This is the so-called "Gromov's dimension comparison problem", recently raised in [6], where the authors solve an interesting variant of this problem, replacing the topological dimension of a submanifold with the Euclidean Hausdorff dimension of a set. Finding the Hausdorff dimension of a submanifold can be turned to the problem of finding its degree. On the other hand, this is not necessarily a simpler problem, since finding a submanifold of given degree corresponds to solve a system of partial differential equations and could be tackled with P.D.R. techniques, as it is mentioned in 0.5 C and 0.6 B of [22]. Some computations in this vein can be found in Section 4 of [28], to find submanifolds of given degree in the Engel group. The extension of (2) to higher step groups is an intriguing open question, where also regularity of the submanifold is expected to play a role. In the Engel group, using some ad hoc arguments, this negligibility condition has been recently achieved for arbitrary submanifolds, solving the Gromov's dimension comparison problem in this group, [25]. It is worth to stress that also in this case $C^{1,1}$ regularity of submanifolds suffices, hence it would be interesting to understand whether this regularity suffices for all higher step groups. If we strengthen our regularity assumptions to C^2 smoothness, then we also get the existence of the blow-up set, that corresponds to the limit of the rescaled manifold.

Theorem 1.4. *Let $\mathbb{G}$ be a two step stratified group and let $\Sigma \subset \mathbb{G}$ be a C^2 smooth submanifold. Then for every $x \in \Sigma$ the rescaled set $\delta_{1/r} l_{x^{-1}} \Sigma$ locally converges with respect to the Hausdorff distance of sets to an algebraic variety, that is the graph of a homogeneous polynomial function.*

Notice that at points of maximum degree the intrinsic blow-up set is precisely a subgroup and $C^{1,1}$ regularity suffices, [28]. On the other hand, it is easy to find simple cases where this limit set is not a linear subspace, see for instance Example 4.3. It is rather surprising observing that Theorem 1.4 holds only in two step groups. In fact, in the Engel group it is already possible to find a point of low degree in a 2-dimensional submanifold whose blow-up is a half plane with boundary, see Remark 4.5 of [28]. This case marks how even smooth submanifolds, when embedded in higher step groups allow for points with more intricate intrinsic singularities.

This naturally leads us to discuss our assumptions on the step of the ambient space. In fact, the previous example is not the only case that shows the significant dichotomy between the geometry in step less than or equal to two and that in higher step. For instance, in two step groups, $C^{1,1}$ domains satisfy the Sobolev-Poincaré inequality and are furthermore NTA domains with respect to the Carnot-Carathéodory distance, but this does not hold in higher step groups, see [10], [21], [23], [24], [30] and the references therein. In two step stratified groups the classical De Giorgi rectifiability theorem holds, [1], [19], but the fine technology of its proof does not apply in higher step, [19]. A substantial progress has been recently obtained in [2], where the difficulty arising in higher step is also emphasized.

Finally, we point out that Theorem 1.1 precisely holds in the larger class of two step graded groups. In fact, our proofs do not use the Lie bracket generating condition of the first layer. Then one can extend our results to all nilpotent groups of step two, since they admit infinitely many gradings that make them graded groups, according to Remark 2.1.

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2. Basic notions

We begin this section defining class of groups with which we shall be dealing in this paper. These are connected, simply connected, real Lie groups of finite dimension. If one of these groups $\mathbb{G}$ has a graded Lie algebra $\mathcal{G}$, then we will $\mathbb{G}$ is a *graded group.* The algebra $\mathcal{G}$ is said to be graded if it can be written as direct sum of subspaces $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, with the property $[V_i, V_j] \subset V_{i+j}$, for any $i, j \geq 1$ and $V_i = \{0\}$ iff $i > \iota$.

Remark 2.1. Let $\mathfrak{g}$ be a two step nilpotent algebra and let $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$, that is a non-null subspace. Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{h} \oplus \mathfrak{g}^2 = \mathfrak{g}$. Then setting $V_1 = \mathfrak{h}$ and $V_2 = \mathfrak{g}^2$ makes $\mathfrak{g}$ a graded group. There are infinitely many subspaces $\mathfrak{h}$'s that are complementary to $\mathfrak{g}^2$.

The stronger assumption on the subspaces $[V_1, V_j] = V_{j+1}$ implies that the group $\mathbb{G}$ is *stratified*, see [18] for more information. The subspace V_1 of $\mathcal{G}$ defines at any point $x \in \mathbb{G}$ the horizontal subspace

$$H_x\mathbb{G} = \{X(x) \mid X \in V_1\}.$$

Left translations of the group are denoted by $l_x : \mathbb{G} \longrightarrow \mathbb{G}$, $l_x(y) = xy$. The graded structure defines a one parameter group of dilations $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$, where $r > 0$. Precisely, we have

$$\delta_r\Big(\sum_{j=1}^{\iota} v_j\Big) = \sum_{j=1}^{\iota} r^j v_j \,,$$

where $\sum_{j=1}^{\iota} v_j = v$ and $v_j \in V_j$ for each $j = 1, \ldots \iota$. To any element of V_j we associate the integer j, which is called the *degree* of the vector. Under our assumptions we have that $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is an analytic diffeomorphism, hence there is a canonical way to transpose dilations from $\mathcal{G}$ to $\mathbb{G}$. We will use the same symbol to denote dilations of the group. We have the standard properties

(1) $\delta_r(x \cdot y) = \delta_r x \cdot \delta_r y$ for any $x, y \in \mathbb{G}$ and $r > 0$,
(2) $\delta_r(\delta_s x) = \delta_{rs} x$ for any $r, s > 0$ and $x \in \mathbb{G}$.

To provide a metric structure on the group, we will fix a *graded metric* on $\mathbb{G}$, namely a left invariant metric such that all the subspaces V_j of the Lie algebra $\mathcal{G}$ are orthogonal each other. The *sub-Riemannian structure* of graded groups is given by the *homogeneous distance*, that is a continuous distance $\rho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$ that satisfies the properties

(1) $\rho(x, y) = \rho(ux, uy)$ for every $u, x, y \in \mathbb{G}$,
(2) $\rho(\delta_r x, \delta_r y) = r\,\rho(x, y)$ for every $r > 0$.

In fact, if the group is stratified, then the well known Carnot-Carathèodory distance provides the foremost example of homogeneous distance. When the group is graded it is still possible to introduce a homogeneous distance, see the appendix of [19]. However, the group equipped with this distance might not be connected by rectifiable curves.

Example 2.2. We consider the Heisenberg group $\mathbb{H}^1$ expressed in coordinates (x, y, t), satisfying the group law $(x, y, t)(x', y', t') = (x + x', y + y', t + t' + xy' - yx')$ and having dilations acting as $\delta_r(x, y, t) = (rx, ry, r^2 t)$. Then we define the vertical subgroup Π of $\mathbb{H}^1$ corresponding to the subspace $\{(0, y, t) \mid y, t \in \mathbb{R}\}$. It is immediate to realize that $d(y, t) = |y| + |t|^{1/2}$ is a homogeneous distance on the graded group $\Pi \subset \mathbb{H}^1$. In fact, the group law restricted to Π is the commutative some of vectors.

On the other hand, all homogeneous distances are bi-Lipschitz equivalent and induce the topology of $\mathbb{G}$. This is an easy consequence of properties (1) and (2), following the classical argument for norms of finite dimensional vector spaces.

Definition 2.3. We define the subsets $B_{x,r}$ and $D_{x,r}$ of $\mathbb{G}$ as the open and closed ball, respectively, of center x and radius $r > 0$ with respect to a homogeneous distance. We will omit the center of the ball if it coincides with the unit element of the group.

Definition 2.4 (Graded coordinates)**.** Let us set $m_j = \dim V_j$ for any $j = 1, \ldots, \iota$, $n_0 = 0$ and $n_i = \sum_{j=1}^{i} m_j$ for any $i = 1, \ldots \iota$. Let $(W_1, \ldots, W_q)$ of $\mathcal{G}$ be an orthonormal basis such that

$$(W_{n_{j-1}+1}, W_{n_{j-1}+2}, \ldots, W_{n_j})$$

is a basis of V_j for any $j = 1, \ldots \iota$ and consider the mapping $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ defined by

$$F(y) = \exp\Big(\sum_{i=1}^{q} y_i W_i\Big)\,.$$

We say that $(W_1, \ldots, W_q)$ is a *graded basis* and that F is a system of *graded coordinates*. The *degree* of y_i is set as $d_i = j$ if $W_i \in V_j$.

Definition 2.5 (Degree of p-vectors)**.** Let $(W_1, W_2, \dots, W_q)$ be an adapted basis of $\mathcal{G}$. The *degree* of the simple p-vector

$$W_J := W_{j_1} \wedge \cdots \wedge W_{j_p}$$

in $\Lambda_p\mathcal{G}$, with $J = (j_1, j_2, \dots, j_p)$ and $1 \le j_1 < j_2 < \cdots < j_p \le q$ is the sum $d_{j_1} + \cdots + d_{j_p}$. We denote this integer by d_J. Now, let $\tau \in \Lambda_p(\mathcal{G})$ be a simple p-vector and let $1 \le r \le Q$ be an integer. Let $\tau = \sum_J \tau_J\, W_J$, $\tau_J \in \mathbb{R}$, be represented with respect to the fixed adapted basis. The projection of τ with degree r is defined as $(\tau)_r = \sum_{d(J)=r} \tau_J\, W_J$ and the *degree* of τ is defined as the integer

$$d(\tau) = \max\{d_J \mid \text{such that } \tau_J \neq 0\}\,.$$

Definition 2.6 (Degree of manifolds)**.** Let Σ be a C^1 smooth p-dimensional submanifold and let $\tau_\Sigma(x)$ be a tangent p-vector of Σ at $x \in \Sigma$. All of them are proportional. Then the *degree* of Σ at x is the positive integer

$$d_\Sigma(x) = d\big(\tau_\Sigma(x)\big)\,.$$

The *degree* of Σ is the number $d(\Sigma) = \max_{x\in\Sigma} d_\Sigma(x)$.

Remark 2.7. Let Σ be a p-dimensional C^1 smooth submanifold and let $h = \dim(T_x\Sigma \cap H_x\mathbb{G})$. If $\mathbb{G}$ is of step two, then the pointwise degree of Σ at x is given by

$$d_\Sigma(x) = 2p - h\,.$$

In fact, one can take a horizontal basis $X_1, \dots, X_h$ of $T_x\Sigma \cap H_x\mathbb{G}$ and linearly independent vectors $T_1, \dots, T_{p-h}$ such that $(X_1, \dots, X_h, T_1, \dots, T_{p-h})$ is a basis of $T_x\Sigma$. Then it is easy to observe that $d_\Sigma(x) = h + 2(p-h)$. In particular, if Σ has codimension one, then

$$d_\Sigma(x) = \begin{cases} Q-1 & \text{if } x \text{ is not characteristic} \\ Q-2 & \text{otherwise} \end{cases}, \tag{4}$$

where $Q = m_1 + 2m_2$ is the Hausdorff dimension of $\mathbb{G}$, also called homogeneous dimension.

Definition 2.8 (Polynomial mappings)**.** Let G be a connected and simply connected nilpotent Lie group. A mapping of vector spaces is polynomial if it has polynomial components when it is expressed with respect to some bases both in the domain and in the range. A mapping $P : G \longrightarrow \mathbb{R}^k$ is said to be *polynomial* if so is the composition $P \circ \exp : \mathcal{G} \longrightarrow \mathbb{R}^k$.

Remark 2.9. The previous definitions make sense, since the exponential mapping $\exp : \mathcal{G} \longrightarrow G$ is an analytic diffeomorphism when G is connected, simply connected and nilpotent. The notion of polynomial mapping of vector spaces does not depend on the fixed bases to represent the mapping, see for [14] for more details.

Definition 2.10 (Homogeneous algebraic varieties)**.** Let $\mathbb{G}$ be a graded group. Then the set of zeros of a polynomial mapping $P : G \longrightarrow \mathbb{R}^k$ for some $k \ge 1$ defines an *algebraic variety* in G. We say that an algebraic variety $\mathcal{A}$ is *homogeneous* if $\delta_r\mathcal{A} \subset \mathcal{A}$ for every $r > 0$.

Notice that an algebraic variety defined by a homogeneous polynomial mapping $P : \mathbb{G} \longrightarrow \mathbb{R}$ is clearly a homogeneous algebraic variety.

Example 2.11. The set $\Sigma = \{(x, y, xy) \mid x, y \in \mathbb{R}\}$ is a homogeneous algebraic variety of the Heisenberg group $\mathbb{H}^1$ equipped with the standard coordinates (x, y, t) of Example 2.2. In these coordinates, the mapping $P : \mathbb{H}^1 \longrightarrow \mathbb{R}$ is given by $(x, y, t) \longrightarrow t - xy$.

Of course homogeneous algebraic varieties need not be regular. It suffices to consider the zero level set of $P(x, y, t) = x^2 - y^2$ in the Heisenberg group with coordinates defined in Example 2.2.

3. Blow-up estimates and blow-ups

This section is devoted to the proof of all our main results.

Proof of Theorem 1.1. We first choose a neighbourhood V of x and a function $f : V \longrightarrow \mathbb{R}^k$ such that $\Sigma \cap V = f^{-1}(0)$, the differential $df(s)$ is surjective whenever $s \in f^{-1}(0)$ and the kernel of $df(x)_{|H_s\mathbb{G}} : H_x\mathbb{G} \longrightarrow \mathbb{R}^k$ has dimension $h = \dim(T_x\Sigma \cap H_x\mathbb{G})$. Let $\kappa = m - h$ and notice that x is non-horizontal if and only if $\kappa = k$, namely $df(x)_{|H_s\mathbb{G}}$ is surjective. In this case, the following proof holds and become even simpler. Then it suffices to consider the interesting case $\kappa < k$, namely, the case when x is horizontal. Let $(v_{\kappa+1}, \dots, v_m)$ be an orthonormal basis of $T_x\Sigma \cap H_x\mathbb{G}$, . We choose an orthonormal basis $(v_1, \dots, v_\kappa)$ of $\big(T_x\Sigma \cap H_x\mathbb{G}\big)^\perp \cap H_x\mathbb{G}$ and define the unique left invariant orthonormal vector fields $(Y_1, \dots, Y_m)$ of V_1 such that $Y_j(x) = v_j$ for every $j = 1, \dots, m$. As a consequence, we get

$$Y_j f^i(x) = 0 \qquad \text{whenever} \qquad j = \kappa+1, \dots, m \quad \text{and} \quad i = 1, \dots, k. \tag{5}$$

Our hypothesis on the step of $\mathbb{G}$ implies that $T_x\mathbb{G} = H_x\mathbb{G} \oplus H_x^2\mathbb{G}$, where

$$H_x^2\mathbb{G} = \{U(x) \mid U \in V_2\}.$$

In view of the surjectivity of $df(p)$, there exist orthonormal vectors $v_{m+1}, \dots, v_{m+l} \in H_x^2\mathbb{G}$, with $l = k - \kappa$, such that

$$\dim\Big(\mathrm{span}\Big\{df(x)(v_1), \dots, df(x)(v_\kappa), df(x)(v_{m+1}), \dots, df(x)(v_{m+l})\Big\}\Big) = k\,. \tag{6}$$

As a consequence, we choose a graded basis $(Y_1, \dots, Y_m, Y_{m+1}, \dots, Y_q)$ of $\mathcal{G}$ such that

$$Y_j(x) = v_j \qquad \text{for every } j \in \{1, \dots, \kappa\} \cup \{m+1, \dots, m+l\}.$$

We fix a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ defined by $F(y) = \exp\Big(\sum_{j=1}^q y^j\, Y_j\Big)$ and set $F_x(y) = l_x F(y)$. We introduce the function $\tilde f(y) = f(F_x(y))$ and notice that

$$\partial_{y_j}\tilde f(0) = Y_j f(p).$$

Then in a neighbourhood $V' \subset V$ of x we get

$$\partial_{y_1}\tilde f \wedge \partial_{y_2}\tilde f \wedge \cdots \wedge \partial_{y_\kappa}\tilde f \wedge \partial_{y_{m+1}}\tilde f \wedge \cdots \partial_{y_{m+l}}\tilde f \neq 0$$

on $F_x^{-1}(V') \subset \mathbb{R}^q$. From the implicit function theorem there exists

$$\psi(\xi) = (\varphi^1(\xi), \dots, \varphi^\kappa(\xi), \xi_{\kappa+1}, \dots, \xi_m, \varphi^{m+1}(\xi), \dots, \varphi^{m+l}(\xi), \xi_{m+l+1}, \dots, \xi_q) \tag{7}$$

such that

$$\tilde f(\varphi^1(\xi), \dots, \varphi^\kappa(\xi), \xi_{\kappa+1}, \dots, \xi_m, \varphi^{m+1}(\xi), \dots, \varphi^{m+l}(\xi), \xi_{m+l+1}, \dots, \xi_q) = 0.$$

Setting $\Phi(\xi) = F_x(\psi(\xi))$ with $\xi = (\xi_{\kappa+1}, \dots, \xi_m, \xi_{m+l+1}, \dots, \xi_q)$, we clearly have $f \circ \Phi = 0$. We wish to compute the limit of $r^{-[h+2(p-h)]}\,\mathrm{vol}_{\tilde g}(\Sigma \cap B_{x,r})$ as $r \to 0^+$. This quotient equals

$$r^{-[h+2(p-h)]} \int_{\Phi^{-1}(B_{x,r})} \left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(\xi) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_m}(\xi) \wedge \frac{\partial\Phi}{\partial\xi_{m+l+1}}(\xi) \wedge \cdots \wedge \frac{\partial\Phi}{\partial\xi_q}(\xi)\right|_{\tilde g} d\xi\,. \tag{8}$$

We restrict dilations to the subspace

$$\Pi = \{\xi \in \mathbb{R}^q \mid \xi_1 = \xi_2 = \cdots = \xi_\kappa = \xi_{m+1} = \cdots = \xi_{m+l} = 0\}$$

and perform the change of variable

$$\xi=\sum_{j=\kappa+1}^{m} r\eta_j e_j+\sum_{j=m+l+1}^{q} r^2\eta_j e_j=\tilde\delta_r\eta. \tag{9}$$

Observing that the jacobian of $\tilde\delta_r$ restricted to Π is

$$m-\kappa+2(q-m-l)=h+2(p-h),$$

the quotient (8) becomes

$$\int_{\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})}\left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(\tilde\delta_r\eta)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_m}(\tilde\delta_r\eta)\wedge\frac{\partial\Phi}{\partial\xi_{m+l+1}}(\tilde\delta_r\eta)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_q}(\tilde\delta_r\eta)\right|_{\tilde g}\,d\eta,$$

where $\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})$ equals the set of points $\xi\in\mathbb{R}^p$ such that

$$\left(\frac{\varphi^1(\tilde\delta_r\xi)}{r},\ldots,\frac{\varphi^\kappa(\tilde\delta_r\xi)}{r},\xi_{\kappa+1},\ldots,\xi_m,\frac{\varphi^{m+1}(\tilde\delta_r\xi)}{r^2},\ldots,\frac{\varphi^{m+l}(\tilde\delta_r\xi)}{r^2},\xi_{m+l+1},\ldots,\xi_q\right) \tag{10}$$

belongs to $F^{-1}(B_1)$. Differentiating the equality $f(\Phi(\xi))=\tilde f(\psi(\xi))=0$, we get

$$\frac{\partial\tilde f}{\partial\xi_i}(\psi)+\sum_{j=1}^{\kappa}\frac{\partial\tilde f}{\partial\xi^j}(\psi)\,\frac{\partial\varphi^j}{\partial\xi^i}+\sum_{j=m+1}^{m+l}\frac{\partial\tilde f}{\partial\xi^j}(\psi)\,\frac{\partial\varphi^j}{\partial\xi^i}=0$$

whenever $i=\kappa+1,\ldots,m$. As a consequence of the previous equality, from (5) we get

$$\left[\frac{\partial\tilde f}{\partial\xi_1}\cdots\frac{\partial\tilde f}{\partial\xi_\kappa}\ \frac{\partial\tilde f}{\partial\xi_{m+1}}\ \cdots\frac{\partial\tilde f}{\partial\xi_{m+l}}\right]_{|\xi=0}\left[\frac{\partial\varphi}{\partial\xi_i}\right]_{|\xi'=0}=0,$$

hence (6) yields

$$\frac{\partial\varphi^j}{\partial\xi^i}(0)=0\quad\text{whenever}\quad\left\{\begin{array}{l} j\in\{1,\ldots,\kappa\}\cup\{m+1,\ldots,m+l\}\\ i\in\{\kappa+1,\ldots,m\}\end{array}\right.. \tag{11}$$

As a consequence, Taylor expansion yields

$$\varphi^j(\tilde\delta_r\xi)=r^2\sum_{m+l<i\le q}\frac{\partial\varphi^j}{\partial\xi_i}(0)\,\xi_i+O^j(|\delta_r\xi|^2), \tag{12}$$

where $|O^j(|y|)|\le L|y|^2$ and L is the Lipschitz constant of $\nabla\varphi^j$. Thus, there exists $M>0$, depending on L, such that

$$\left|r^{-2}\varphi^j(\tilde\delta_r\xi)\right|\le M|\xi|$$

for $r>0$ sufficiently small. Let us choose a product of open intervals

$$J=I_1\times I_2\times\cdots I_q\subset F^{-1}(B_1)$$

such that $0\in J$. Now we define the open set A_L formed by those points $\xi\in\mathbb{R}^p$ such that

$$\begin{gathered}\big(M|\xi|,\ldots,M|\xi|,\xi_{\kappa+1},\ldots,\xi_m,M|\xi|,\ldots,M|\xi|,\xi_{m+l+1},\ldots,\xi_q\big)\in J\quad\text{and}\\ \big(-M|\xi|,\ldots,-M|\xi|,\xi_{\kappa+1},\ldots,\xi_m,-M|\xi|,\ldots,-M|\xi|,\xi_{m+l+1},\ldots,\xi_q\big)\in J\,.\end{gathered}$$

Clearly, the size of A_L depends on L and we have

$$A_L\subset\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})$$

for r sufficiently small. It follows that

$$\frac{\mathrm{vol}_{\tilde g}(\Sigma \cap B_{x,r})}{r^{h+2(p-h)}} \geq \mathcal{H}^d_{|\cdot|}\left(A_L \cap F^{-1}(B_1)\right)\gamma(d\Phi) \tag{13}$$

where

$$\gamma(d\Phi) = \inf_{y\in U}\left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(y)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_m}(y)\wedge\frac{\partial\Phi}{\partial\xi_{m+l+1}}(y)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_q}(y)\right|_{\tilde g}.$$

and U is a suitable neighbourhood of the origin. From Hadamard's inequality we get

$$\left|\frac{\partial\Phi}{\partial\xi_{\kappa+1}}(y)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_m}(y)\wedge\frac{\partial\Phi}{\partial\xi_{m+l+1}}(y)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_q}(y)\right|_{\tilde g} \tag{14}$$

$$\leq \prod_{i\in K}\sqrt{\sum_{j\in K}\left\langle\frac{\partial\Phi}{\partial\xi_i}(y),\frac{\partial\Phi}{\partial\xi_j}(y)\right\rangle^2_{\tilde g}} \leq \left(\sum_{j\in K}\left|\frac{\partial\Phi}{\partial\xi_j}(y)\right|^2_{\tilde g}\right)^p \leq p^p\,\|d\Phi(y)\|^{2p}, \tag{15}$$

where $K=\{\kappa+1,\ldots,m\}\cup\{m+l+1,\ldots,q\}$. As a consequence, observing that we can find a bounded set $S\subset\mathbb{R}^p$ containing $\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})$ for every $r>0$, we obtain

$$\frac{\mathrm{vol}_{\tilde g}(\Sigma \cap B_{x,r})}{r^{h+2(p-h)}} \leq \mathcal{H}^p_{|\cdot|}\left(S \cap F^{-1}(B_1)\right)\,p^p\,L_0^{2p} \tag{16}$$

where L_0 is the Lipschitz constant of the local parametrization Φ of the submanifold, the estimate (1) follows, taking into account Remark 2.7. Now, we have to study the dependence of our constants in (13) and (16) on the point x when it varies in the subset of points having the same degree. Let U be a neighbourhood of x such that (6) holds replacing x with any point of U. If $U\cap\{z\in\Sigma\mid d_\Sigma(z)=d_\Sigma(x)\}$ coincides with x there is nothing to prove, then assume that

$$U\cap\{z\in\Sigma\mid d_\Sigma(z)=d_\Sigma(x)\}\setminus\{x\}\neq\emptyset\,.$$

Since the degree $\Sigma\ni z\longrightarrow d_\Sigma(z)$ is lower semicontinuous, then the set

$$A_x=\{z\in\Sigma\mid d_\Sigma(z)>d_\Sigma(x)-1\}$$

is an open neighbourhood of $S_x=\{z\in\Sigma\mid d_\Sigma(z)=d_\Sigma(x)\}$. Furthermore, S_x is closed in A_x, since

$$S_x=A_x\cap\{z\in\Sigma\mid d_\Sigma(z)\leq d_\Sigma(x)\}\,.$$

Since the degree is constant on $S'_x=S_x\cap U$, then we can find locally Lipschitz continuous vector fields $T_1,\ldots,T_h$ on S'_x such that

$$\mathrm{span}\{T_1(z),\ldots,T_h(z)\}=H_z\mathbb{G}\cap T_z\Sigma$$

for every $z\in S'_x$, see Remark 3.1. Then one can repeat the proof of the estimates (13) and (16), replacing x with $z\in S'_x$ and the basis $(v_{\kappa+1},\ldots,v_m)$ of $H_x\mathbb{G}\cap T_x\Sigma$ with $\big(T_1(z),\ldots,T_h(z)\big)$. It is not restrictive assuming that $T_j(x)=Y_j(x)$. The key point is that (5) is replaced by

$$T_jf^i(z)=0 \qquad \text{whenever} \qquad j=1,\ldots,h,\ i=1,\ldots,k \quad \text{and} \quad z\in S'_x. \tag{17}$$

Furthermore, the vectors

$$\big(Y_1(z),\ldots,Y_\kappa(z),T_1(z),\ldots,T_h(z),Y_{m+1}(z),\ldots,Y_q(z)\big)$$

are linearly independent and locally Lipschitz continuous on S'_x. Thus, the corresponding local graph centered at z

$$\psi_z(\xi)=\big(\varphi^1_z(\xi),\ldots,\varphi^\kappa_z(\xi),\xi_{\kappa+1},\ldots,\xi_m,\varphi^{m+1}_z(\xi),\ldots,\varphi^{m+l}_z(\xi),\xi_{m+l+1},\ldots,\xi_q\big) \tag{18}$$

is a Lipschitz deformation of (7) and it coincides with it for $z = x$. Then the constants appearing in (13) and (16) can be taken to be independent og z as it varies in a compact neighbourhood $S''_x \subset S'_x$ of x in the relatively open set A_x. This concludes the proof. □

Remark 3.1. Let Σ be a C^1 manifold with a countable basis for its topology and consider a closed subset F of Σ. Let $F \ni x \longrightarrow C(x) \in M_n(\mathbb{R})$ be a Lipschitz continuous matrix having constant rank equal to $s < n$ on all points of $F \subset \mathbb{R}^q$. Then one can find locally Lipschitz continuous vector fields $T_1, \ldots, T_h$ on F, with $h = n - s$, such that

$$\operatorname{span}\{T_1(x), \ldots, T_h(x)\} = \operatorname{Ker} C(x) \quad \text{for every} \quad x \in F\,.$$

Since Σ admits a C^1 partition of unity, [5], that can be eventually restricted to F, it suffices to prove that T_j can be found in any neighbourhood of a point of F. Then it is not restrictive assuming that for instance the first s columns $C_1, \ldots, C_s$ of C are linearly independent on a relatively open subset A of F. Then there exists a projection $\pi : \mathbb{R}^q \longrightarrow \mathbb{R}^s$, such that $\tilde{C} = \big[\pi C_1 \cdots \pi C_s\big]$ is an invertible matrix of $M_s(\mathbb{R})$ on a possibly smaller open subset A' and for every $x \in A'$ we have that

$$\eta_\xi(x) = \tilde{C}(x)^{-1}\big[\pi C_{s+1}(x) \cdots \pi C_n(x)\big]\xi$$

is locally Lipschitz continuous for every $\xi \in \mathbb{R}^h$. Taking $\xi = e_j$ for every $j = 1, \ldots, h$, where (e_j) is the canonical basis of $\mathbb{R}^h$ and identifying this basis with a basis of $\{0\} \times \mathbb{R}^h \subset \mathbb{R}^n$, we get

$$T_j(x) = \big(\eta^1_{e_j}(x), \ldots, \eta^s_{e_j}(x), 0, \ldots, 0\big) + e_j$$

that are clearly linearly independent and locally Lipschitz continuous on A'.

Proof of Theorem 1.4. We will use notation of the proof of Theorem 1.1. If x is non-horizontal, namely, $\kappa = k$, then the following proof holds and becomes simpler and in this special case, our claim is already explicitly shown in [29]. Then we consider the case $\kappa < k$, namely, x is horizontal, and we argue exactly as in the proof of Theorem 1.1, until we have obtained the local parametrization $\Phi = l_x \circ F \circ \psi$ of Σ around $x \in \Sigma$, where ψ is defined in (7).

As a consequence of C^2 regularity, for every $j \in \{1, \ldots, \kappa\} \cup \{m+1, \ldots, m+l\}$, the expansion (12) can be written more precisely as follows

$$\begin{aligned} \varphi^j(\delta_r\xi) &= \sum_{m+l<i\le q} \varphi^j_{\xi_i}(0) r^2 \xi_i + \frac{1}{2} \sum_{\kappa<i,j\le m} \varphi^j_{\xi_i\xi_j}(0) r^2 \xi_i\xi_j + o(|\delta_r\xi|^2) \\ &= r^2\, Q_j(\xi_{m+l+1}, \ldots, \xi_q) + o(|\delta_r\xi|^2)\,, \end{aligned} \tag{19}$$

where dilations $\tilde{\delta}_r$ are defined in (9) and Q_j are homogeneous polynomials. Now we consider the variables

$$\xi = \sum_{j=\kappa+1}^{m} \xi_j e_j + \sum_{j=m+l+1}^{q} \xi_j e_j \quad \text{and} \quad \xi' = \sum_{j=m+l+1}^{q} \xi_j e_j$$

that vary in a p-dimensional subspace of $\mathbb{R}^p$ and in a $(q-m-l)$-dimensional subspace of $\mathbb{R}^q$, respectively. Then the polynomial mapping

$$T(\xi) = (0, \ldots, 0, \xi_{\kappa+1}, \ldots, \xi_m, Q_{m+1}(\xi'), \ldots, Q_{m+l}(\xi'), \xi_{m+l+1}, \ldots, \xi_q) \tag{20}$$

defines the d-dimensional homogeneous algebraic variety

$$\mathcal{A} = (F \circ T)(\mathbb{R}^p) \subset \mathbb{G}$$

that has no singular points, since it is the graph of a polynomial mapping. We recall that $F:\mathbb{R}^q\longrightarrow\mathbb{G}$ is the system of graded coordinates defined in the proof of Theorem 1.1.

The variety $\mathcal{A}$ is the zero set of the homogeneous polynomial mapping $P:\mathbb{G}\longrightarrow\mathbb{R}^k$, where

$$P\circ F(y)=\big(y_1,\ldots,y_\kappa,y_{m+1}-Q_{m+1}(y_{m+l+1},\ldots,y_q),\ldots,y_{m+l}-Q_{m+l}(y_{m+l+1},\ldots,y_q)\big)\,.$$

Notice that $P\circ F$ has exactly k components, since $\kappa+l=k$, by the definition of l, according to notation in the proof of Theorem 1.1.

To show that $\delta_{1/r}l_{x^{-1}}\Sigma\cap D_R$ converges to $\mathcal{A}\cap D_R$ in the Kuratowski sense, as $r\to0^+$, we will use Proposition 4.5.5 of [3]. Then we fix an infinitesimal sequence $r_n>0$ and show that

(i) if $z=\lim_{n\to\infty}z_n$ for some sequence $\{z_n\}$ such that $z_n\in\big(\delta_{1/r_n}l_{x^{-1}}\Sigma\big)\cap D_R$ and $r_n\to0$, then $z\in\mathcal{A}\cap D_R$;

(ii) if $z\in\mathcal{A}\cap D_R$, then there exist $z_n\in\big(\delta_{1/r_n}l_{x^{-1}}\Sigma\big)\cap D_R$ such that $z_n\to z$.

Consider the sequence

$$z_n\in\big(\delta_{1/r_n}l_{x^{-1}}\Sigma\big)\cap D_R\,,$$

converging to z, we have that $z_n=\big(F\circ\delta_{1/r_n}\circ\psi\circ\tilde\delta_{r_n}\big)(\xi_n)$, for some ξ_n. Due to formula (10), that constitutes the explicit formula of $\big(\delta_{1/r_n}\circ\psi\circ\tilde\delta_{r_n}\big)(\xi_n)$, the boundedness of z_n implies the boundedness of ξ_n. Then we get a subsequence ξ_ν of ξ_n converging to $\overline{\xi}\in D_R$. As a consequence of (19) and (10), it follows that $z_\nu\to F\circ T(\overline{\xi})=z\in\mathcal{A}\cap D_R$. This shows the validity of (i).

Now, we set $U=F\circ T$ and fix $z\in\mathcal{A}\cap D_R$, hence we have ξ such that $U(\xi)=z$. Let us define $\Psi_n=F\circ\delta_{1/r_n}\circ\psi\circ\delta_{r_n}$ and set $\Sigma_n=\delta_{1/r_n}l_{x^{-1}}\Sigma$. If we show that

$$\operatorname{dist}\big(\xi,\Psi_n^{-1}(D_R)\big)\longrightarrow0, \tag{21}$$

with respect to the Euclidean norm $|\cdot|$ on $\mathbb{R}^q$, then we get a sequence ξ_n that is bounded, due to (10), and such that

$$D_R\ni\Psi_n(\xi_n)\longrightarrow U(\xi)=z\,,$$

since Ψ_n uniformly converges to U on compact sets. Taking into account that for n large

$$\Sigma_n\cap D_R=\operatorname{Im}(\Psi_n)\cap D_R\,,$$

it follows that $\Psi_n(\xi_n)\in\Sigma_n\cap D_R$ and (ii) follows. We are left to show (21). By contradiction, if this limit were not true, then possibly taking a subsequence, we would get

$$\operatorname{dist}\big(\xi,\Psi_\nu^{-1}(D_R)\big)\ge\varepsilon_0>0.$$

We fix an arbitrary $0<h<1$. Since Ψ_n uniformly convergences to U on compact sets, then

$$U^{-1}(D_{hR})\subset\Psi_n^{-1}(D_R)$$

whenever $n\ge n_0$, for some n_0 depending on h. In fact, $U^{-1}(D_{hR})$ is a compact set, due to formula (20). Observing that $\delta_h(U(\xi))\in D_{hR}$ and taking into account the homogeneity $U\circ\delta_h=\delta_h\circ U$, for $\nu\ge n_0$, we achieve

$$\varepsilon_0\le\operatorname{dist}\big(\xi,\Psi_\nu^{-1}(D_R)\big)\le\operatorname{dist}\big(\xi,U^{-1}(D_{hR})\big)\le|\xi-\tilde\delta_h\xi|.$$

The arbitrary choice of h yields a contradiction and concludes the proof of (21). □

Remark 3.2. According to the previous proof, formula (20) yields the limit set $\mathcal{A}$ as the image of T in some system of graded coordinates. Since T is a homogeneous polynomial mapping, then $\mathcal{A}$ is clearly a homogeneous algebraic variety. Furthermore, this variety is everywhere analytic, since it is the graph of a polynomial mapping with respect to some system of graded coordinates. In particular, $\mathcal{A}$ is an analytic manifold without boundary.

Proof of Corollary 1.3. For each $j = 1, \ldots, \delta$, we define the subsets

$$\Sigma_j = \big\{ x \in \Sigma \mid \text{estimates (1) hold with } d_\Sigma(x) = j \big\}.$$

Then $Z_\delta = \cup^m_{1 \le j \le \delta} \Sigma_j$. Note that some Σ_j might be empty. This is always true for instance if $\delta > Q - k$, where k is the codimension of Σ and Q is the Hausdorf dimension of the group. From estimates (1) and standard differentiability theorems for measures, see for instance 2.10.17(2) and 2.10.18(1) of [17], taking into account that Riemannian surface measure $\tilde{\mu}_p$ is countalbly finites, then Σ_j is countably $\mathcal{H}^j$-finite. In particular, Z_δ is $\mathcal{H}^\delta$-countably finite. $\Box$

Proof of Corollary 1.2. Let $d = d(\Sigma)$ be the degree of Σ. In view of Corollary 1.3 the subset Z_{d-1} of points in Σ with degree less than d is countably $\mathcal{H}^{d-1}$-finite. Then in particular, (2) holds. At points of degree d the blow-up limit exists. Precisely, Theorem 1.2 of [28] holds, along with the negligibility condition (2), hence formula (1.4) of [28] also holds. This formula coincides with (3). $\Box$

4. Some remarks

In this section we add remarks complementary to our main results.

Remark 4.1. The Hausdorff dimension of a $C^{1,1}$ submanifold Σ in a two step stratified group coincides with its degree $d = d(\Sigma)$. In fact, from the lower semicontinuity of $x \longrightarrow d_\Sigma(x)$ on Σ, it follows that the subset Σ_d of degree d points is an open subset of Σ. Thus, formula (3) yields

$$\int_{\Sigma_d} \theta(\tau^d_\Sigma(x))\, d\mathcal{S}^d(x) = \int_{\Sigma_d} |\tau^d_\Sigma(x)|\, d\tilde{\mu}_p(x) > 0\,.$$

From definition of $\tau^d_\Sigma(x))$ given in (2.17) of [28], using the simple argument of Remark 2.18 in [27], one easily finds two positive constants c_1 and c_2, such that

$$c_1 \le \theta(\tau) \le c_2 \quad \text{for every} \quad \tau \in \Lambda_p(\mathcal{G})\,.$$

This implies that $\mathcal{S}^d(\Sigma) > 0$ and taking into account that $\mathcal{S}^d \llcorner \Sigma$ is finite on compact sets. In particular, the Hausdorff dimension of Σ is d.

Remark 4.2. Let Σ be a p-dimensional submanifold of a graded group. We consider the flag

$$H'_j(x) = \big(H^1_x\mathbb{G} + \cdots + H^j_x\mathbb{G} \big) \cap T_x\Sigma,$$

on $T_x\Sigma$, where $H^j_x\mathbb{G} = \{X(x) \mid X \in V_j\}$. Let $(t^1_1, \ldots, t^1_{m'_1(x)})$ be a basis of $H'_j(x)$ and complete this basis with $(t^2_1(x), \ldots, t^2_{m'_2(x)})$ to a basis $(t^1_1, \ldots, t^1_{m'_1(x)}, t^2_1, \ldots, t^2_{m'_2(x)})$ of $H'_2(x)$. We iterate this procedure to construct bases

$$(t^1_1, \ldots, t^1_{m'_1(x)}, \ldots, t^j_1, \ldots, t^j_{m'_j(x)}) \quad \text{of subspaces} \quad H'_j(x)\,.$$

It is immediate to observe that the dimensions $\dim\big(H'_j(x)/H'_{j-1}(x)\big)$ coincide with $m'_j(x)$. In 0.6 B of [22], Gromov defines the number

$$D'(x) = \sum_{j=1}^{\iota} j\, m'_j(x)\,.$$

Let us define the canonical projection $\pi_j : T_x\mathbb{G} \longrightarrow H^j_x\mathbb{G}$ and observe that

$$\big(\pi_j(t^j_1), \ldots, \pi_j(t^j_{m'_j(x)})\big)$$

must be linearly independent. If this were not the case, then we could find a nontrivial linear combination of t_i^j belonging to $H'_{j-1}(x)$, but that it cannot be a linear combination of elements of this space and this is a contradiction. Setting $\tau_i^j = \pi_j(t_i^j)$ and taking into account that $\tau_j \in H_x^j\mathbb{G}$, it follows that

$$(\tau_1^1, \dots, \tau^1_{m_1'(x)}, \dots, \tau_1^\iota, \dots, \tau^\iota_{m_\iota'(x)}) \quad \text{are linearly independent.} \tag{22}$$

Since (t_j^i) is a basis of $T_x\Sigma$, we have

$$d_\Sigma(x) = d\big(t_1^1 \wedge \cdots \wedge t^1_{m_1'(x)} \wedge \cdots \wedge t_1^\iota \cdots \wedge t^\iota_{m_\iota'(x)}\big)\,. \tag{23}$$

It is understood that some m_{j_0} may vanish for some j_0, this would mean that there are no vectors $t_i^{j_0}$ both in (22) and (23). We have

$$t_1^1 \wedge \cdots \wedge t^1_{m_1'(x)} \wedge \cdots \wedge t_1^\iota \cdots \wedge t^\iota_{m_\iota'(x)} = \tau_1^1 \wedge \cdots \wedge \tau^1_{m_1'(x)} \wedge \cdots \wedge \tau_1^\iota \cdots \wedge \tau^\iota_{m_\iota'(x)} + r\,,$$

where the first addend is nonvanishing due to (22) and $d(r) < d(\tau)$, where we have set

$$\tau = \tau_1^1 \wedge \cdots \wedge \tau^1_{m_1'(x)} \wedge \cdots \wedge \tau_1^\iota \cdots \wedge \tau^\iota_{m_\iota'(x)}\,.$$

An immediate verification shows that $d(\tau) = \sum_{j=1}^\iota j\, m_j'(x)$. This shows that

$$d_\Sigma(x) = D'(x)\,.$$

Notice that we have computed the degree $d_\Sigma(x)$ by choosing an arbitrary adapted basis. Then the previous formula also shows that the degree does not depend on the choice of the adapted basis. Since $D_H(\Sigma) = \max_{x\in\Sigma} D'(x)$, according to the definition given in 0.6 B of [22], then we obviously have $D_H(\Sigma) = d(\Sigma)$.

Example 4.3. Consider the two step stratified group $\mathbb{H}^1 \times \mathbb{H}^1$, that we represent by graded coordinates $(x, y, t) = (x_1, y_1, x_2, y_2, t_1, t_2) \in \mathbb{R}^6$, satisfying the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t') + (0, 0, x_1y_1' - y_1x_1', x_2y_2' - y_2x_2')\,.$$

Let Σ be a 3-dimensional submanifold of $\mathbb{H}^1 \times \mathbb{H}^1$ passing through the origin, that is locally given by the zero level set of $f : \mathbb{R}^6 \longrightarrow \mathbb{R}^3$,

$$f(x, y, t) = \big(t_1 + t_1^3 - x_1^2 - y_1^2 - x_2^3, x_1^2 - t_2 + x_2^4, x_2 - x_1^3\big)\,.$$

It is easy to see that around the origin Σ can be then written as follows

$$\Phi(x_1, y_1, y_2) = \big(x_1, y_1, x_1^3, y_2, \varphi(x_1^2 + y_1^2 + x_2^3), x_1^2\big)\,,$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is the inverse function of $t \longrightarrow t + t^3$. To determine the blow-up set at the origin we consider converging sequences $(x^n, y^n, t^n) \to (x, y, t)$ and infinitesimal sequences $r_n \to 0$, hence the blow-up sequence

$$\delta_{1/r_n}\big(\Phi(r_nx_1^n, r_ny_1^n, r_ny_2^n)\big) = \big(x_1^n, y_1^n, r_n^2(x_1^n)^3, y_2^n, r_n^{-2}\varphi\big(r_n^2\big((x_1^n)^2 + (y_1^n)^2 + r_n(x_2^n)^3\big)\big), r_n(x_1^n)^2\big)\,,$$

taking into account that $\varphi'(0) = 1$, converges to

$$(x_1, y_1, 0, y_2, x_1^2 + y_1^2, 0)$$

that defines the blow-up set as $x_1, y_1, y_2 \in \mathbb{R}$. This set is clearly a homogeneous algebraic variety, according to Theorem 1.4. Notice that the tangent space of Σ at the origin is

$$\{(x_1, y_1, 0, y_2, 0, 0) \mid x_1, y_1, y_2 \in \mathbb{R}\} \quad \text{and} \quad d_\Sigma(0) = 3\,.$$

Since blow-up at maximum degree points is a subgroup and in particular a subspace, [28], it follows that the degree of Σ must be higher than three, as one can also check by direct computation. Incidentally, this agrees with the fact that in $\mathbb{H}^1 \times \mathbb{H}^1$ there do not exist smooth 3-dimensional submannifolds everywhere tangent to the horizontal subbundle.

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