STOCHASTIC HOMOGENIZATION OF NONDEGENERATE VISCOUS HJ EQUATIONS IN 1D

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Abstract. We prove homogenization for a nondegenerate viscous Hamilton-Jacobi equation in dimension one in stationary ergodic environments with a superlinear (nonconvex) Hamiltonian of fairly general type.

1. Introduction

In this paper we are concerned with the asymptotic behavior, as $\varepsilon \to 0^+$, of solutions of a viscous Hamilton-Jacobi (HJ) equation of the form

$$(\mathrm{EHJ}_\varepsilon) \qquad \partial_t u^\varepsilon = \varepsilon a\left(x/\varepsilon, \omega\right) \partial^2_{xx} u^\varepsilon + H\left(x/\varepsilon, \partial_x u^\varepsilon, \omega\right) \qquad \text{in } (0, +\infty) \times \mathbb{R},$$

where $H : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a superlinear (in the momentum) Hamiltonian of fairly general type, and $a : \mathbb{R} \times \Omega \to [0, 1]$ is Lipschitz on $\mathbb{R}$ for every fixed ω. The dependence of the equation on the random environment $(\Omega, \mathcal{F}, \mathbb{P})$ enters through the Hamiltonian $H(x, p, \omega)$ and the diffusion coefficients $a(x, \omega)$, which are assumed to be stationary with respect to shifts in x, and bounded and Lipschitz continuous on $\mathbb{R}$, for every fixed $p \in \mathbb{R}$ and $\omega \in \Omega$. The diffusion coefficient is assumed nondegenerate, i.e., $a(\cdot, \omega) > 0$ in $\mathbb{R}$ almost surely. Under these hypotheses, we prove homogenization for equation $(\mathrm{EHJ}_\varepsilon)$, see Theorem 2.2 for the precise statement. The full set of assumptions is presented in Section 2.1, here we want to stress that we do not require any convexity condition of any kind on the Hamiltonian in the momentum.

This is the first work where the homogenization of nondegenerate viscous HJ equations in 1d in random media is proved in full generality, at least as far as superlinear Hamiltonians are concerned, thus extending to this setting the results established in the 1d inviscid case in [4, 26]. With respect to previous contributions on the subject, the crucial step forward of our work consists in dealing with Hamiltonians where the dependence on x and p is not necessarily decoupled and, more important, in getting rid of any additional requirement on the pair (a, H), notably the hill and valley conditions that were assumed in [34, 17, 19].

We would also like to point out that our proof is technical simple. One could compare it with the ones available on the topic of stationary ergodic homogenization of HJ equations in 1d, both in the viscous and in the inviscid case. It relies, among other things, on the use of a novel idea which is presented in Theorem 4.2. [1] This allows us to prove, in the end, that the set of θ in $\mathbb{R}$ for which the related "cell" problem associated with $(\mathrm{EHJ}_\varepsilon)$ does not admit correctors is the possibly countable union of pairwise disjoint, bounded open intervals which correspond to flat parts of the effective Hamiltonian. This complements, in our setting, the information about the existence of correctors provided by [11] for possibly degenerate viscous HJ equations of the form (1.1) in any

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[1] This idea was first introduced by the author in [15, Theorem 4.5]. Unfortunately, the argument to prove the main homogenization result therein contained is also based on [15, Theorem 4.3], whose proof contains a non-amendable flaw.

space dimension. We refer the reader to Section 2.2 for a more detailed description of our proof strategy.

The present work, in pair with [14], implies in particular homogenization of (EHJ_ε) when the Hamiltonian is additionally assumed quasiconvex in p and the diffusion coefficient is possibly degenerate (but non null). Indeed, in [14] we have established homogenization for quasiconvex Hamiltonians belonging to the same class herein considered in the degenerate regime, i.e., when the diffusion coefficient satisfies $\min_{\mathbb{R}} a(\cdot,\omega)=0$ almost surely and $a\not\equiv 0$. The effective Hamiltonian is also shown to inherit the quasiconvexity from H. This is not to be expected in general in the nondegenerate case, in fact it does not even hold in the classical periodic case, as it was recently pointed out in [33].

1.1. **Literature review.** Equations of the form (EHJ_ε) are a subclass of general viscous stochastic HJ equations

$$\partial_t u^\varepsilon=\varepsilon\operatorname{tr}\left(A\left(\frac{x}{\varepsilon},\omega\right)D^2u^\varepsilon\right)+H\left(Du^\varepsilon,\frac{x}{\varepsilon},\omega\right)\qquad\text{in }(0,+\infty)\times\mathbb{R}^d, \tag{1.1}$$

where $A(\cdot,\omega)$ is a bounded, symmetric, and nonnegative definite $d\times d$ matrix with a Lipschitz square root. The ingredients of the equation are assumed to be stationary with respect to shifts in x. This setting encompasses the periodic and the quasiperiodic cases, for which homogenization has been proved under fairly general assumptions by showing the existence of (exact or approximate) correctors, i.e., sublinear functions that solve an associated stationary HJ equations [35, 22, 30, 37]. In the stationary ergodic setting, such solutions do not exist in general, as it was shown in [36], see also [20, 11] for inherent discussions and results. This is the main reason why the extension of the homogenization theory to random media is nontrivial and required the development of new arguments.

The first homogenization results in this framework were obtained for convex Hamiltonians in the case of inviscid equations in [40, 39] and then for their viscous counterparts in [38, 32]. By exploiting the metric character of first order HJ equations, homogenization has been extended to the case of quasiconvex Hamiltonians, first in dimension 1 [20] and then in any space dimension [2].

The topic of homogenization in stationary ergodic media for HJ equations that are nonconvex in the gradient variable remained an open problem for about fifteen years. Recently, it has been shown via counterexamples, first in the inviscid case [43, 25], then in the viscous one [24], that homogenization can fail for Hamiltonians of the form $H(x,p,\omega):=G(p)+V(x,\omega)$ whenever G has a strict saddle point. This has shut the door to the possibility of having a general qualitative homogenization theory in the stationary ergodic setting in dimension $d\geqslant 2$, at least without imposing further mixing conditions on the stochastic environment.

On the positive side, a quite general homogenization result, which includes as particular instances both the inviscid and viscous cases, has been established in [3] for Hamiltonians that are positively homogeneous of degree $\alpha\geqslant 1$ and under a finite range of dependence condition on the probability space $(\Omega,\mathcal{F},\mathbb{P})$.

In the inviscid case, homogenization has been proved in [5, 26] in dimension $d=1$ for a rather general class of coercive and nonconvex Hamiltonians, and in any space dimension for Hamiltonians of the form $H(x,p,\omega)=\left(|p|^2-1\right)^2+V(x,\omega)$, see [4].

Even though the addition of a diffusive term is not expected to prevent homogenization, the literature on viscous HJ equations in stationary ergodic media has remained rather limited until very recently. The viscous case is in fact known to present additional challenges which cannot be overcome by mere modifications of the methods used for $a\equiv 0$.

Apart from already mentioned work [1], several progresses concerning the homogenization of the viscous HJ equation (1.1) with nonconvex Hamiltonians have been recently made in [17, 42, 34, 41, 19, 16, 14]. In the joint paper [17], we have shown homogenization of (1.1) with $H(x,p,\omega)$ which are “pinned” at one or several points on the p-axis and convex in each interval in between. For example, for every $\alpha > 1$ the Hamiltonian $H(p, x, \omega) = |p|^\alpha - c(x, \omega)|p|$ is pinned at $p = 0$ (i.e. $H(0, x, \omega) \equiv \text{const}$) and convex in p on each of the two intervals $(-\infty, 0)$ and $(0, +\infty)$. Clearly, adding a non-constant potential $V(x, \omega)$ breaks the pinning property. In particular, homogenization of equation (1.1) for $d = 1$, $A \equiv \text{const} > 0$ and $H(p, x, \omega) := \frac{1}{2}|p|^2 - c(x, \omega)|p| + V(x, \omega)$ with $c(x, \omega)$ bounded and strictly positive remained an open problem even when $c(x, \omega) \equiv c > 0$. Homogenization for this kind of equations with $A \equiv 1/2$ and H as above with $c(x, \omega) \equiv c > 0$ was proved in [34] under a novel hill and valley condition,[2] that was introduced in [42] to study a sort of discrete version of this problem. The approach of [34] relies on the Hopf-Cole transformation, stochastic control representations of solutions and the Feynman-Kac formula. It is applicable to (EHJ$_\varepsilon$) with $H(x, p, \omega) := G(p) + V(x, \omega)$ and $G(p) := \frac{1}{2}|p|^2 - c|p| = \min\{\frac{1}{2}p^2 - cp, \frac{1}{2}p^2 + cp\}$ only. In the joint paper [18], we have proposed a different proof solely based on PDE methods. This new approach is flexible enough to be applied to the possible degenerate case $a(x, \omega) \geqslant 0$, and to any G which is a minimum of a finite number of convex superlinear functions G_i having the same minimum. The original hill and valley condition assumed in [42, 34] is weakened in favor of a scaled hill and valley condition. The arguments crucially rely on this condition and on the fact that all the functions G_i have the same minimum.

The PDE approach introduced in [18] was subsequently refined in [41] in order to prove homogenization for equation (EHJ$_\varepsilon$) when the function G is superlinear and quasiconvex, $a > 0$ and the pair (a, V) satisfies a scaled hill condition equivalent to the one adopted in [18]. The core of the proof consists in showing existence and uniqueness of correctors that possess stationary derivatives satisfying suitable bounds. These kind of results were obtained in [18] by proving tailored-made comparison principles and by exploiting a general result from [11] (and this was the only point where the piecewise convexity of G was used). The novelty brought in by [41] relies on the nice observation that this can be directly proved via ODE arguments, which are viable since we are in one space dimension and $a > 0$.

By reinterpreting this approach in the viscosity sense and by making a more substantial use of viscosity techniques, the author has extended in [16] to the possible degenerate case $a \geqslant 0$ the homogenization result established in [41], providing in particular a unified proof which encompasses both the inviscid and the viscous case.

A substantial improvement of the results of [18] is provided in our joint work [19], where the main novelty consists in allowing a general superlinear G without any restriction on its shape or the number of its local extrema. Our analysis crucially relies on the assumption that the pair (a, V) satisfies the scaled hill and valley condition and, differently from [18], that the diffusive coefficient a is nondegenerate, i.e., $a > 0$. The proof consists in showing existence of suitable correctors whose derivatives are confined on different branches of G, as well as on a strong induction argument which uses as base case the homogenization result established in [41].

Our recent work [14] is the first one where a homogenization result for equation (EHJ$_\varepsilon$) has been established without imposing any hill or valley-type condition of any sort on the pair (a, H). In [14] we prove homogenization of equation (EHJ$_\varepsilon$) for a quasiconvex Hamiltonian belonging to the same class considered in the present paper and with the only further assumption that the diffusion coefficient a is not identically zero and vanishes at some points or on some regions of $\mathbb{R}$, almost surely. This assumption on the diffusion coefficient is crucially exploited to obtain suitable existence, uniqueness and regularity results of correctors, as well as a quasi-convexity property of the expected effective Hamiltonian. This is not to be expected in general in the nondegenerate case. In fact, it does not even hold in the classical periodic case, as it was recently pointed out in [33], where the authors identify a class of quasiconvex functions $G(p)$ and 1-periodic potentials

[2]Such a hill and valley condition is fulfilled for a wide class of typical random environments without any restriction on their mixing properties, see [34, Example 1.3] and [42, Example 1.3]. It is however not satisfied if the potential is "rigid", for example, in the periodic case.

$V(x)$ for which the effective Hamiltonian arising in the homogenization of equation (EHJ$_\varepsilon$) with $H(x,p) := G(p) + V(x)$ and $a \equiv 1$ is not quasiconvex.

1.2. **Outline of the paper.** In Section 2.1 we present the setting and the standing assumptions and we state our homogenization result, see Theorem 2.2. In Section 2.2 we describe the strategy we will follow to prove homogenization. In Section 3 we characterize the set of admissible λ for which the corrector equation can be solved and we correspondingly show existence of (deterministic) solutions, with ω treated as a fixed parameter. The derivatives of such solutions can be put in one-to-one correspondence with solutions of a corresponding ODE. We exploit this duality to show the existence of random correctors with maximal and minimal slope, for any admissible λ. Section 4 is addressed to prove Theorem 4.2, where we provide a novel argument to suitably bridge a pair of distinct stationary solution of the same ODE. In Section 5 we give the proof of our homogenization result. The paper ends with two appendices. In the first one, we have collected some results that we repeatedly use in the paper concerning stationary sub and supersolutions of the ODEs herein considered. The second one contains some PDE results needed for our proofs.

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2. Preliminaries

2.1. **Assumptions and statement of the main result.** We will denote by $\mathrm{C}(\mathbb{R})$ and $\mathrm{C}(\mathbb{R}\times\mathbb{R})$ the Polish spaces of continuous functions on $\mathbb{R}$ and on $\mathbb{R}\times\mathbb{R}$, endowed with a metric inducing the topology of uniform convergence on compact subsets of $\mathbb{R}$ and of $\mathbb{R}\times\mathbb{R}$, respectively.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, where Ω is a Polish space, $\mathcal{F}$ is the σ-algebra of Borel subsets of Ω, and $\mathbb{P}$ is a complete probability measure on $(\Omega, \mathcal{F})$.[3] We will denote by $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$ and equip the product space $\mathbb{R}\times\Omega$ with the product σ-algebra $\mathcal{B}\otimes\mathcal{F}$.

We will assume that $\mathbb{P}$ is invariant under the action of a one-parameter group $(\tau_x)_{x\in\mathbb{R}}$ of transformations $\tau_x : \Omega \to \Omega$. More precisely, we assume that the mapping $(x,\omega) \mapsto \tau_x\omega$ from $\mathbb{R}\times\Omega$ to Ω is measurable, $\tau_0 = id$, $\tau_{x+y} = \tau_x \circ \tau_y$ for every $x, y \in \mathbb{R}$, and $\mathbb{P}\big(\tau_x(E)\big) = \mathbb{P}(E)$ for every $E \in \mathcal{F}$ and $x \in \mathbb{R}$. We will assume in addition that the action of $(\tau_x)_{x\in\mathbb{R}}$ is *ergodic*, i.e., any measurable function $\varphi : \Omega \to \mathbb{R}$ satisfying $\mathbb{P}(\varphi(\tau_x\omega) = \varphi(\omega)) = 1$ for every fixed $x \in \mathbb{R}$ is almost surely equal to a constant. If $\varphi \in L^1(\Omega)$, we write $\mathbb{E}(\varphi)$ for the mean of φ on Ω, i.e. the quantity $\int_\Omega \varphi(\omega)\, d\mathbb{P}(\omega)$.

A measurable function $f : \mathbb{R}\times\Omega \to \mathbb{R}$ is said to be *stationary* with respect to $(\tau_x)_{x\in\mathbb{R}}$ if $f(x+y,\omega) = f(x, \tau_y\omega)$ for every $x, y \in \mathbb{R}$ and $\omega \in \Omega$.

In this paper, we will consider an equation of the form

$$\partial_t u = a(x,\omega)\partial^2_{xx} u + H(x, \partial_x u, \omega), \quad \text{in } (0,+\infty)\times\mathbb{R}, \tag{2.1}$$

where $a : \mathbb{R}\times\Omega \to [0,1]$ is a stationary function satisfying the following assumptions for some constant $\kappa > 0$:

(A1) $a(\cdot,\omega) > 0$ on $\mathbb{R}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;

(A2) $\sqrt{a(\,\cdot\,,\omega)} : \mathbb{R} \to [0,1]$ is κ–Lipschitz continuous for all $\omega \in \Omega$.[4]

As for the Hamiltonian $H : \mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$, we will assume that it is stationary with respect to shifts in x variable, i.e., $H(x+y,p,\omega) = H(x,p,\tau_y\omega)$ for every $x, y \in \mathbb{R}$, $p \in \mathbb{R}$ and $\omega \in \Omega$, and that it belongs to the class $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ defined as follows.

[3] The assumption that Ω is a Polish space and $\mathbb{P}$ is a complete probability measure is used in many points of the paper, but only to show joint measurability of the random objects therein introduce, see the proofs of Lemmas 3.3 and 3.7, of Theorem 4.2 and of Proposition 5.3.

[4] Note that (A2) implies that $a(\cdot,\omega)$ is 2κ–Lipschitz in $\mathbb{R}$ for all $\omega \in \Omega$. Indeed, for all $x, y \in \mathbb{R}$ we have

$$|a(x,\omega) - a(y,\omega)| = |\sqrt{a(x,\omega)} + \sqrt{a(y,\omega)}||\sqrt{a(x,\omega)} - \sqrt{a(y,\omega)}| \leqslant 2\kappa|x-y|.$$

Definition 2.1. *A function $H : \mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$ is said to be in the class $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ if it satisfies the following conditions, for some constants $\alpha_0, \alpha_1 > 0$ and $\gamma > 1$:*

(H1) $\alpha_0|p|^\gamma - 1/\alpha_0 \leqslant H(x, p, \omega) \leqslant \alpha_1(|p|^\gamma + 1)$ *for all* $(x, p, \omega) \in \mathbb{R}\times\mathbb{R}\times\Omega$;

(H2) $|H(x, p, \omega) - H(x, q, \omega)| \leqslant \alpha_1 (|p| + |q| + 1)^{\gamma-1} |p - q|$ *for all* $(x, \omega) \in \mathbb{R}\times\Omega$ *and* $p, q \in \mathbb{R}$;

(H3) $|H(x, p, \omega) - H(y, p, \omega)| \leqslant \alpha_1(|p|^\gamma + 1)|x - y|$ *for all* $x, y, p \in \mathbb{R}$ *and* $(p, \omega) \in \mathbb{R}\times\Omega$.

We will denote by $\mathscr{H}$ the union of the families $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$, where α_0, α_1 vary in $(0, +\infty)$ and γ in $(1, +\infty)$.

Solutions, subsolutions and supersolutions of (2.1) will be always understood in the viscosity sense, see [9, 12, 7, 6], and implicitly assumed continuous, without any further specification. Assumptions (A2) and (G1)-(G3) guarantee well-posedness in $\mathrm{UC}([0, +\infty)\times\mathbb{R})$ of the Cauchy problem for the parabolic equation (2.1), as well as Lipschitz estimates for the solutions under appropriate assumptions on the initial condition, see Appendix B for more details. We stress that our results hold (with the same proofs) under any other set of assumptions apt to ensure the same kind of PDE results.

The purpose of this paper is to prove the following homogenization result.

Theorem 2.2. *Suppose a satisfies (A1)-(A2) and $H \in \mathscr{H}$. Then the viscous HJ equation* (EHJ_ε) *homogenizes, i.e., there exists a continuous and coercive function $\mathcal{H}(H) : \mathbb{R} \to \mathbb{R}$, called* effective Hamiltonian*, and a set $\hat\Omega$ of probability 1 such that, for every uniformly continuous function g on $\mathbb{R}$ and every $\omega \in \hat\Omega$, the solutions $u^\varepsilon(\cdot, \cdot, \omega)$ of* (EHJ_ε) *satisfying $u^\varepsilon(0, \cdot, \omega) = g$ converge, locally uniformly on $[0, +\infty) \times \mathbb{R}$ as $\varepsilon \to 0^+$, to the unique solution $\overline{u}$ of*

$$\begin{cases} \partial_t \overline{u} = \mathcal{H}(H)(\partial_x \overline{u}) & \text{in } (0, +\infty) \times \mathbb{R} \\ \overline{u}(0, \cdot) = g & \text{in } \mathbb{R}. \end{cases}$$

Furthermore, $\mathcal{H}(H)$ is locally Lipschitz and superlinear.

We remark that the effective Hamiltonian $\mathcal{H}(H)$ also depends on the diffusion coefficient a. Since a will remain fixed throughout the paper, we will not keep track of this in our notation.

2.2. **Description of our proof strategy.** In this section, we outline the strategy that we will follow to prove the homogenization results stated in Theorem 2.2.

Let us denote by $u_\theta(\cdot, \cdot, \omega)$ the unique Lipschitz solution to (2.1) with initial condition $u_\theta(0, x, \omega) = \theta x$ on $\mathbb{R}$, and let us introduce the following deterministic quantities, defined almost surely on Ω, see [18, Proposition 3.1]:

$$\mathcal{H}^L(H)(\theta) := \liminf_{t\to+\infty} \frac{u_\theta(t, 0, \omega)}{t} \quad \text{and} \quad \mathcal{H}^U(H)(\theta) := \limsup_{t\to+\infty} \frac{u_\theta(t, 0, \omega)}{t}. \tag{2.2}$$

In view of [17, Lemma 4.1], proving homogenization amounts to showing that $\mathcal{H}^L(H)(\theta) = \mathcal{H}^U(H)(\theta)$ for every $\theta \in \mathbb{R}$. If this occurs, their common value is denoted by $\mathcal{H}(H)(\theta)$. The function $\mathcal{H}(H) : \mathbb{R} \to \mathbb{R}$ is called the effective Hamiltonian associated with H. It has already appeared in the statement of Theorem 2.2.

The we first remark that we can reduce to prove Theorem 2.2 for stationary Hamiltonians H belonging to $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ for constants α_0, $\alpha_1 > 0$ and $\gamma > 2$. In fact, the following holds.

Proposition 2.3. *If Theorem 2.2 holds for every $\tilde H$ belonging to $\bigcup_{\alpha_0,\alpha_1>0,\gamma>2} \mathscr{H}(\alpha_0, \alpha_1, \gamma)$, then it holds for every $H \in \mathscr{H}$.*

Proof. Pick a stationary Hamiltonian H in $\mathscr{H}$. Let us fix $R > 0$. For any $\theta \in [-R, R]$, we denote by $u_\theta(\cdot,\cdot,\omega)$ the unique Lipschitz solution to (2.1) with initial condition $u_\theta(0, x, \omega) = \theta x$ on $\mathbb{R}$. According to Theorem B.4, there exists a constant K, depending on $R > 0$, such that $u_\theta(\cdot,\cdot,\omega)$ is K-Lipschitz on $[0, +\infty) \times \mathbb{R}$ for every $\omega \in \Omega$ and $|\theta| \leqslant R$. Let us now set $\tilde{H}(x, p, \omega) := \max\{H(x, p, \omega), |p|^4 - n\}$ for all $(x, p, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$, with $n \in \mathbb{N}$ chosen large enough so that $\tilde{H} \equiv H$ on $\mathbb{R} \times [-K, K] \times \Omega$. This impies that the function $u_\theta(\cdot,\cdot,\omega)$ is also the unique Lipschitz solution to (2.1) with $\tilde{H}$ in place of H and initial condition $u_\theta(0, x, \omega) = \theta x$ on $\mathbb{R}$, for every $|\theta| \leqslant R$. Clearly, $\tilde{H}$ belongs to $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ for suitable constants $\alpha_0, \alpha_1 > 0$ and $\gamma \geqslant 4$. Since by hypothesis Theorem 2.2 holds for $\tilde{H}$, in view of [17, Lemma 4.1] we have, for every $|\theta| \leqslant R$,

$$\mathcal{H}^L(H)(\theta) = \liminf_{t\to+\infty} \frac{u_\theta(t, 0, \omega)}{t} = \limsup_{t\to+\infty} \frac{u_\theta(t, 0, \omega)}{t} = \mathcal{H}^U(H)(\theta) \quad \text{almost surely.}$$

This also implies that

$$\mathcal{H}(H)(\theta) = \mathcal{H}(\tilde{H})(\theta) \qquad \text{for every } |\theta| \leqslant R. \tag{2.3}$$

By arbitrariness of $R > 0$, we derive that equation (EHJ$_\varepsilon$) homogenizes for H as well. The fact that $\mathcal{H}(H)$ is superlinear and locally Lipschitz follows from Proposition B.5. $\square$

Let us describe our strategy to prove Theorem 2.2. In order to prove that $\mathcal{H}^L(\theta) = \mathcal{H}^U(\theta)$ for all $\theta \in \mathbb{R}$, we will adopt the approach that was taken in [34, 18] and substantially developed in [41]. It consists in showing the existence of a viscosity Lipschitz solution $u(x, \omega)$ with stationary derivative for the following stationary equation associated with (2.1), namely

$$a(x, \omega)u'' + H(x, u', \omega) = \lambda \qquad \text{in } \mathbb{R} \tag{HJ$_\lambda$}$$

for every $\lambda \in \mathbb{R}$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$. With a slight abuse of terminology, we will call such solutions *(random) correctors* in the sequel for the role they play in homogenization.[5] In fact, the following holds.

Proposition 2.4. *Let $\lambda \in \mathbb{R}$ such that equation* (HJ$_\lambda$) *admits a viscosity solution u with stationary gradient. Let us set $\theta := \mathbb{E}[u'(0, \omega)]$. Then $\mathcal{H}^L(\theta) = \mathcal{H}^U(\theta) = \lambda$.*

Proof. Let us set $F_\theta(x, \omega) := u(x, \omega) - \theta x$ for all $(x, \omega) \in \mathbb{R} \times \Omega$. Then $F_\theta(\cdot, \omega)$ is sublinear for $\mathbb{P}$-a.e. $\omega \in \Omega$, see for instance [20, Theorem 3.9]. Furthermore, it is a viscosity solution of (HJ$_\lambda$) with $H(\cdot, \theta + \cdot, \cdot)$ in place of H. The assertion follows arguing as in the proof of [18, Lemma 5.6]. $\square$

The first step consists in identifying the set of $\lambda \in \mathbb{R}$ for which equation (HJ$_\lambda$) can be solved, for every fixed $\omega \in \Omega$. In Section 3 we will show that this set is equal to the half-line $[\lambda_0(\omega), +\infty)$, where $\lambda_0(\omega)$ is a critical constant suitably defined. Furthermore, we will show the existence of a Lipschitz viscosity solution $u_\lambda(\cdot, \omega)$ to (HJ$_\lambda$) for every $\lambda \geqslant \lambda_0(\omega)$. Differently from previous works on the subject [34, 18, 41, 19, 16], such a critical constant is associated to the pair (a, H) in an intrinsic way. Indeed, when $a \equiv 0$ or when H is of the form $G(p) + V(x, \omega)$ with (a, V) satisfying the scaled hill condition, λ_0 is always equal to $\sup_{x\in\mathbb{R}} \min_{p\in\mathbb{R}} H(x, p, \omega)$, while in our case λ_0 is in general strictly less than this quantity. For the results presented in Section 3, we use in a crucial way the fact that H belongs to $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ with $\gamma > 2$. This will allow us to use known Hölder regularity results for continuous supersolutions of equation (HJ$_\lambda$), see Proposition B.2 for more details. The stationarity of a and H and the ergodicity assumption imply that $\lambda_0(\omega)$ is almost surely equal to a constant λ_0.

The subsequent step consists in showing existence of random correctors associated with equation (HJ$_\lambda$) for every $\lambda \geqslant \lambda_0$. To this aim, we first remark that, due to the almost sure non degenerating

[5] The word *corrector* is usually used in literature to refer to the function $u(x, \omega) - \theta\, x$ with $\theta := \mathbb{E}[u'(0, \omega)]$, see for instance [11] for a more detailed discussion on the topic.

condition $a(\cdot,\omega)>0$ in $\mathbb{R}$, the derivatives of such correctors are in one-to-one correspondence with stationary solutions of the following ODE

$$a(x,\omega)f' + H(x,f,\omega) = \lambda \qquad \text{in } \mathbb{R}. \tag{ODE$_\lambda$}$$

We prove that the set $\mathcal{S}_\lambda$ of stationary functions $f:\mathbb{R}\times\Omega\to\mathbb{R}$ which solve (ODE$_\lambda$) almost surely is nonempty for every $\lambda\geqslant\lambda_0$. This is obtained by showing that the random functions defined as the pointwise maximum and minimum of deterministic solutions of (ODE$_\lambda$), with ω treated as a fixed parameter, are elements of $\mathcal{S}_\lambda$, see Lemma 3.7.

We then proceed by defining the set-valued map $\Theta:[\lambda_0,+\infty)\to\mathbb{R}$ as

$$\Theta(\lambda) := \{\mathbb{E}[f(0,\cdot)]\,:\, f\in\mathcal{S}_\lambda\} \qquad \text{for all } \lambda\geqslant\lambda_0.$$

The multifunction Θ is injective, locally equi–compact and upper semicontinuous, in the sense of set-valued analysis. Furthermore, its image $\mathrm{Im}\,(\Theta)$ is contained in the set $D:=\{\theta\in\mathbb{R}\,:\,\mathcal{H}^L(H)(\theta)=\mathcal{H}^U(H)(\theta)\}$, in view of Proposition 2.4. If Θ is surjective, then the effective Hamiltonian $\mathcal{H}(H)$ can be defined as the inverse of Θ, i.e. $\mathcal{H}(H)(\theta)=\lambda$, where λ is the unique real number in $[\lambda_0,+\infty)$ such that $\theta\in\Theta(\lambda)$, for every fixed $\theta\in\mathbb{R}$. Yet, Θ need not be surjective. To take care of this possibility, we first show that $\mathrm{Im}\,(\Theta)$ is a closed subset of $\mathbb{R}$ and, due to the property (v) in Proposition 5.1, any connected component of $\mathbb{R}\setminus\mathrm{Im}\,(\Theta)$ is a bounded open interval of the form (θ_1,θ_2). Next, we show that this can happen only if θ_1 and θ_2 belong to $\Theta(\lambda)$ for the same $\lambda\geqslant\lambda_0$. If $f_1<f_2$ are the stationary random functions in $\mathcal{S}_\lambda$ such that $\mathbb{E}[f_i(0,\cdot)]=\theta_i$ for $i\in\{1,2\}$, the condition that (θ_1,θ_2) is in the complement of $\mathrm{Im}\,(\Theta)$ implies that the set $\mathfrak{S}(\mu;f_1,f_2)$ of stationary functions in $\mathcal{S}_\mu$ almost surely trapped between f_1 and f_2 is empty for any $\mu\geqslant\lambda_0$, in view of Proposition 2.4. This implies that

$$\lambda\leqslant\mathcal{H}^L(H)(\theta)\leqslant\mathcal{H}^U(H)(\theta)\leqslant\lambda \qquad \text{for every } \theta\in(\theta_1,\theta_2), \tag{2.4}$$

namely, (θ_1,θ_2) corresponds to a flat part of the effective Hamiltonian $\mathcal{H}(H)$. We conclude that $D=\mathbb{R}$ and the effective Hamiltonian $\mathcal{H}(H)$ can be again defined as the inverse of Θ, as it was explained before.

In order to prove the first (respectively, last) inequality in (2.4), we aim to show that, for every fixed ω in a set of probability 1 and for every $\varepsilon>0$, there exists a C^1 function g_ε that bridges $f_2(\cdot,\omega)$ with $f_1(\cdot,\omega)$ (resp., $f_1(\cdot,\omega)$ with $f_2(\cdot,\omega)$) in such a way to be a classical supersolution (resp., subsolution) of (ODE$_\lambda$) with $\lambda-\varepsilon$ (resp., $\lambda+\varepsilon$) in place of λ, see Proposition 4.1 for more details. This argument already appeared in [18] and was implemented in [41, 19, 16] for Hamiltonians of the form $H(x,p,\omega):=G(p)+V(x,\omega)$ by showing that $\inf_{\mathbb{R}}(f_2-f_1)=0$ almost surely. The proof of this latter property crucially relies on the assumption that the pair (a,V) satisfies either a scaled hill or valley condition, see Remark 4.3 for further comments. This point is handled here with a novel idea, corresponding to Theorem 4.2. It exploits the fact that $\mathfrak{S}(\mu;f_1,f_2)=\varnothing$ for every $\mu<\lambda$ (resp., $\mu>\lambda$) to construct a lift which allows to gently descend (resp., ascend) from $f_2(\cdot,\omega)$ to $f_1(\cdot,\omega)$ (resp., from $f_1(\cdot,\omega)$ to $f_2(\cdot,\omega)$).

We end this section with a disclaimer. As already precised in the previous section, sub and supersolutions of equation (HJ$_\lambda$) are understood in the viscosity sense and will be implicitly assumed continuous, without any further specification. We want to remark that, due to the positive sign of the diffusion term a, a viscosity supersolution (repectively, subsolution) u to (HJ$_\lambda$) that is twice differentiable at x_0 will satisfy the inequality

$$a(x_0,\omega)u''(x_0,\omega)+H(x_0,u'(x_0),\omega)\leqslant\lambda. \qquad (\text{resp.,}\ \geqslant\lambda.)$$

3. Existence of correctors

In this section we will characterize the set of admissible $\lambda\in\mathbb{R}$ for which the corrector equation

$$a(x,\omega)u''+H(x,u',\omega)=\lambda \qquad \text{in } \mathbb{R} \tag{HJ$_\lambda$}$$

admits solutions. In this section we assume that H is a stationary Hamiltonian belonging to the class $\mathscr{H}(\alpha_0, \alpha_1, \gamma)$ for some fixed constants $\alpha_0, \alpha_1 > 0$ and $\gamma > 2$, for every $\omega \in \Omega$.

We will start by regarding at (HJ$_\lambda$) as a deterministic equation, where ω is treated as a fixed parameter. The first step consists in characterizing the set of real λ for which equation (HJ$_\lambda$) admits (deterministic) viscosity solutions. To this aim, for every fixed $\omega \in \Omega$, we define a critical value associated with (HJ$_\lambda$) defined as follows:

$$\lambda_0(\omega) := \inf\{\lambda \in \mathbb{R} \,:\, \text{equation (HJ}_\lambda\text{) admits a continuous viscosity supersolution}\}. \tag{3.1}$$

It can be seen as the lowest possible level under which supersolutions of equation (HJ$_\lambda$) do not exist at all. Such a kind of definition is natural and appears in literature with different names, according to the specific problem for which it is introduced: ergodic constant, Mañe critical value, generalized principal eigenvalue. In the inviscid stationary ergodic setting, it appears in this exact form in [21], see also [31, 10] for an analogous definition in the noncompact deterministic setting.

It is easily seen that $\lambda_0(\omega)$ is uniformly bounded from above on Ω. Indeed, the function $u \equiv 0$ on $\mathbb{R}$ is a classical supersolution of (HJ$_\lambda$) with $\lambda := \sup_{(x,\omega)} H(x, 0, \omega) \leqslant \alpha_1$. Hence we derive the upper bound

$$\lambda_0(\omega) \leqslant \alpha_1 \qquad \text{for all } \omega \in \Omega. \tag{3.2}$$

We proceed to show that $\lambda_0(\omega)$ is also uniformly bounded from below and that equation (HJ$_\lambda$) admits a solution for every $\omega \in \Omega$ and $\lambda \geqslant \lambda_0(\omega)$. For this, we need a preliminary lemma first.

Lemma 3.1. *Let $\omega \in \Omega$ and $\lambda > \lambda_0(\omega)$ be fixed. For every fixed $y \in \mathbb{R}$, there exists a viscosity supersolution w_y of* (HJ$_\lambda$) *satisfying $w_y(y) = 0$ and*

$$a(x, \omega)u'' + H(x, u', \omega) = \lambda \qquad \text{in } \mathbb{R} \setminus \{y\} \tag{3.3}$$

in the viscosity sense. In particular

$$|w_y(x) - w_y(z)| \leqslant K|x - z|^{\frac{\gamma-2}{\gamma-1}} \qquad \text{for all } x, z \in \mathbb{R}, \tag{3.4}$$

where $K = K(\alpha_0, \gamma, \lambda) > 0$ is given explicitly by (B.2).

Proof. Let us denote by $\overline{\mathfrak{S}}(\lambda)(\omega)$ the family of continuous viscosity supersolutions of equation (HJ$_\lambda$). This set is nonempty since $\lambda > \lambda_0(\omega)$. In view of Proposition B.2, we know that these functions satisfy (3.4) for a common constant $K = K(\alpha_0, \gamma, \lambda) > 0$ given explicitly by (B.2). Let us set

$$w_y(x) := \inf\big\{w \in \mathrm{C}(\mathbb{R}) \,:\, w \in \overline{\mathfrak{S}}(\lambda)(\omega),\ w(y) = 0\big\} \qquad \text{for all } x \in \mathbb{R}.$$

The function w_y is well defined, it satisfies (3.4) and $w_y(y) = 0$. As an infimum of viscosity supersolutions of (HJ$_\lambda$), it is itself a supersolution. Let us show that w_y solves (3.3) in the viscosity sense. If this where not the case, there would exist a C^2 strict supertangent φ to w_y at a point $x_0 \neq y$ [6] such that $a(x_0, \omega)\varphi''(x_0) + H(x_0, \varphi'(x_0), \omega) < \lambda$. By continuity, we can pick $r > 0$ small enough so that $|y - x_0| > r$ and

$$a(x, \omega)\varphi''(x) + H(x, \varphi'(x), \omega) < \lambda \qquad \text{for all } x \in (x_0 - r, x_0 + r). \tag{3.5}$$

Choose $\delta > 0$ small enough so that

$$\varphi(x_0 + r) - \delta > w_y(x_0 + r) \qquad \text{and} \qquad \varphi(x_0 - r) - \delta > w_y(x_0 - r). \tag{3.6}$$

Let us set $\tilde{w}(x) := \min\{\varphi(x) - \delta, w_y(x)\}$ for all $x \in \mathbb{R}$. The function $\tilde{w}$ is the minimum of two viscosity supersolution of (HJ$_\lambda$) in $(x_0 - r, x_0 + r)$, in view of (3.5), and it agrees with w_y in $\mathbb{R} \setminus [x_0 - \rho, x_0 + \rho]$ for a suitable $0 < \rho < r$, in view of (3.6). We infer that $\tilde{w} \in \overline{\mathfrak{S}}$ with $\tilde{w}(y) = 0$. But this contradicts the minimality of w_y since $\tilde{w}(y) = \varphi(y) - \delta < w_y(y)$. □

With the aid of Lemma 3.1, we can now prove the following existence result.

[6] i.e., $\varphi > w_y$ in $\mathbb{R} \setminus \{x_0\}$ and $\varphi(x_0) = w_y(x_0)$.

THEOREM 3.2. *Let $\omega \in \Omega$ and $\lambda \geqslant \lambda_0(\omega)$ be fixed. Then there exists a viscosity solution u to* (HJ$_\lambda$). *Furthermore u is K-Lipschitz and of class C^2 on $\mathbb{R}$, where the constant $K = K(\alpha_0, \alpha_1, \gamma, \kappa, \lambda) > 0$ is given explicitly by* (B.1). *In particular, $\lambda_0(\omega) \geqslant \min H \geqslant -1/\alpha_0$.*

Proof. Let us first assume $\lambda > \lambda_0(\omega)$. Let us pick a sequence of points $(y_n)_n$ in $\mathbb{R}$ with $\lim_n |y_n| = +\infty$ and, for each $n \in \mathbb{N}$, set $w_n(\cdot) := w_{y_n}(\cdot) - w_{y_n}(0)$ on $\mathbb{R}$, where w_{y_n} is the function provided by Lemma 3.1. Accordingly, we know that the functions w_n are equi-Hölder continuous and hence locally equi–bounded on $\mathbb{R}$ since $w_n(0) = 0$ for all $n \in \mathbb{N}$. By the Ascoli-Arzelà Theorem, up to extracting a subsequence, the functions w_n converge in $\mathrm{C}(\mathbb{R})$ to a limit function u. For every fixed bounded interval I, the functions w_n are viscosity solutions of (HJ$_\lambda$) for all $n \in \mathbb{N}$ big enough since $|y_n| \to +\infty$ as $n \to +\infty$, and so is u by the stability of the notion of viscosity solution. We can now apply Proposition B.1 to infer that u is K–Lipschitz continuous and of class C^2 on $\mathbb{R}$. In particular

$$\lambda = a(x,\omega)u'' + H(x,u',\omega) \geqslant a(x,\omega)u'' + \min H \qquad \text{for all } x \in \mathbb{R}.$$

Being u' of class C^1 and bounded, we must have $\inf_{\mathbb{R}} |u''(x)| = 0$, hence $\lambda \geqslant \min H$. This readily implies $\lambda_0(\omega) \geqslant \min H \geqslant -1/\alpha_0$.

Let us now choose a decreasing sequence of real numbers $(\lambda_n)_n$ converging to $\lambda_0(\omega)$ and, for each $n \in \mathbb{N}$, let u_n be a viscosity solution of (HJ$_\lambda$) with $u_n(0) = 0$. In view of Proposition B.1, the functions $(u_n)_n$ are equi-Lipschitz and thus locally equi–bounded on $\mathbb{R}$, hence they converge, along a subsequence, to a function u in $\mathrm{C}(\mathbb{R})$. By stability, u is a solution of (HJ$_\lambda$) with $\lambda := \lambda_0(\omega)$. The Lipschitz and regularity properties of u are again a consequence of Proposition B.1. □

The main output of the next result is that the function $\lambda_0(\cdot)$ is almost surely equal to a constant, that will be denoted by λ_0 in the sequel.

LEMMA 3.3. *The random variable $\lambda_0 : \Omega \to \mathbb{R}$ is measurable and stationary. In particular, it is almost surely constant.*

Proof. If $u \in \mathrm{C}(\mathbb{R})$ is a viscosity supersolution of (HJ$_\lambda$) for some $\omega \in \Omega$ and $\lambda \geqslant \lambda_0(\omega)$, then the functions $u(\cdot + z)$ is a viscosity supersolution of (HJ$_\lambda$) with $\tau_z\omega$ in place of ω by the stationarity of H and a. By its very definition, we infer that λ_0 is a stationary function.

Let us show that $\lambda_0 : \Omega \to \mathbb{R}$ is measurable. Since the probability measure $\mathbb{P}$ is complete on $(\Omega, \mathcal{F})$, it is enough to show that, for every fixed $\varepsilon > 0$, there exists a set $F \in \mathcal{F}$ with $\mathbb{P}(\Omega \setminus F) < \varepsilon$ such that the restriction of λ_0 to F is measurable. To this aim, we notice that the measure $\mathbb{P}$ is inner regular on $(\Omega, \mathcal{F})$, see [8, Theorem 1.3], hence it is a Radon measure. By applying Lusin's Theorem [23] to the random variables $a : \Omega \to \mathrm{C}(\mathbb{R})$ and $H : \Omega \to \mathrm{C}(\mathbb{R} \times \mathbb{R})$, we infer that there exists a closed set $F \subseteq \Omega$ with $\mathbb{P}(\Omega \setminus F) < \varepsilon$ such that $a_{|F} : F \to \mathrm{C}(\mathbb{R})$ and $H_{|F} : F \to \mathrm{C}(\mathbb{R} \times \mathbb{R})$ are continuous. We claim that $F \ni \omega \mapsto \lambda_0(\omega) \in \mathbb{R}$ is lower semicontinuous. Indeed, let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence converging to some ω_0 in F. For each $n \in \mathbb{N}$, let u_n be a solution of (HJ$_\lambda$) with ω_n and $\lambda(\omega_n)$ in place of ω and λ. Let us furthermore assume that $u_n(0) = 0$ for all $n \in \mathbb{N}$. From (3.2) and Proposition B.1 we derive that the functions u_n are equi-Lipschitz and locally equi-bounded in $\mathbb{R}$. Let us extract a subsequence such that $\liminf_n \lambda(\omega_n) = \lim_k \lambda(\omega_{n_k}) =: \tilde{\lambda}$ and $\left(u_{n_k}\right)_k$ converges to a function u in $C(\mathbb{R})$. Since $a(\cdot, \omega_n) \to a(\cdot, \omega_0)$ in $\mathrm{C}(\mathbb{R})$ and $H(\cdot,\cdot,\omega_n) \to H(\cdot,\cdot,\omega_0)$ in $\mathrm{C}(\mathbb{R} \times \mathbb{R})$, we derive by stability that u solves (HJ$_\lambda$) with $\omega := \omega_0$ and $\lambda := \tilde{\lambda}$ in the viscosity sense. By definition of $\lambda_0(\omega_0)$, we conclude that $\tilde{\lambda} \geqslant \lambda(\omega_0)$, i.e., $\liminf_n \lambda(\omega_n) \geqslant \lambda_0(\omega_0)$ as it was claimed. □

We now take advantage of the fact that the diffusion coefficient a is strictly positive to remark that the derivatives of viscosity solutions to (HJ$_\lambda$) are in one-to-one correspondence with classical solutions of the ODE

$$a(x,\omega)f' + H(x,f,\omega) = \lambda \qquad \text{in } \mathbb{R} \tag{ODE$_\lambda$}$$

The precise statement is the following.

Proposition 3.4. *Let $\omega \in \Omega$ and $\lambda \geqslant \lambda_0(\omega)$ be fixed. Then f is a C^1 solution of* (ODE_λ) *if and only if there exists a viscosity solution u of* (HJ_λ) *with $u' = f$. In particular, there exists a constant $K = K(\alpha_0, \alpha_1, \gamma, \kappa, \lambda) > 0$, only depending on $\alpha_0, \alpha_1, \gamma, \kappa, \lambda$ through* (B.1)*, such that any C^1 solution f of* (ODE_λ) *satisfies $\|f\|_\infty \leqslant K$.*

Proof. If f is a C^1 solution of (ODE_λ), then $u(x) := \int_0^x f(z)\,dz$ is a (classical) solution of (HJ_λ). Conversely, if u is a viscosity solution of (HJ_λ), then u is of class C^2 by Proposition B.1 and $f := u'$ is a C^1 solution of (ODE_λ). The remainder of the statement is a direct consequence of Proposition B.1. □

For every $\lambda \geqslant \lambda_0$ and $\omega \in \Omega$, let us set

$$\mathcal{S}_\lambda(\omega) := \left\{ f \in \mathrm{C}^1(\mathbb{R}) \,:\, f \text{ solves } (\mathrm{ODE}_\lambda) \right\},$$

where we agree that the above set is empty when $\lambda < \lambda_0(\omega)$.

Proposition 3.5. *For every $\lambda \geqslant \lambda_0$ and almost every $\omega \in \Omega$, the set $\mathcal{S}_\lambda(\omega)$ is nonempty and compact in* $\mathrm{C}(\mathbb{R})$.

Proof. The fact that $\mathcal{S}_\lambda(\omega)$ is almost surely nonempty is a direct consequence of Theorem 3.2, Lemma 3.3 and Proposition 3.4. Let us prove compactness. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_\lambda(\omega)$. This sequence is equi-bounded in view of Proposition 3.4. The fact that each f_n solves (ODE_λ) and $a(\,\cdot\,,\omega) > 0$ on $\mathbb{R}$ implies that the sequence $(f_n)_n$ is locally equi-Lipschitz in $\mathbb{R}$, hence it converges, up to subsequences, to a function f in $\mathrm{C}(\mathbb{R})$ by the Arzelá-Ascoli Theorem. Again by (ODE_λ), the derivatives $(f_n')_{n\in\mathbb{N}}$ form a Cauchy sequence in $\mathrm{C}(\mathbb{R})$, hence the functions f_n actually converge to f in the local C^1 topology and f solves (ODE_λ). □

For every $\omega \in \Omega$ and $\lambda \geqslant \lambda_0$, let us set

$$\underline{f}_\lambda(x,\omega) := \inf_{f\in\mathcal{S}_\lambda(\omega)} f(x) \quad \text{and} \quad \overline{f}_\lambda(x,\omega) := \sup_{f\in\mathcal{S}_\lambda(\omega)} f(x) \qquad \text{for all } x \in \mathbb{R}, \tag{3.7}$$

where we agree to set $\underline{f}_\lambda(\cdot,\omega) \equiv \overline{f}_\lambda(\cdot,\omega) \equiv 0$ when $\mathcal{S}_\lambda(\omega) = \varnothing$. The next lemma shows in particular that $\underline{f}_\lambda \leqslant \overline{f}_\lambda$ almost surely, with equality possibly holding only when $\lambda = \lambda_0$. In fact, the following monotonicity property holds.

Lemma 3.6. *Let $\omega \in \Omega$ be such that $\lambda_0(\omega) = \lambda_0$. Then*

$$\underline{f}_\mu(\,\cdot\,,\omega) < \underline{f}_\lambda(\,\cdot\,,\omega) \leqslant \overline{f}_\lambda(\,\cdot\,,\omega) < \overline{f}_\mu(\,\cdot\,,\omega) \quad \text{in } \mathbb{R} \qquad \text{for all } \mu > \lambda \geqslant \lambda_0.$$

Proof. Let us fix $\mu > \lambda \geqslant \lambda_0$. Choose $R > \|\overline{f}_\lambda\|_\infty$, where the latter quantity is finite due to Proposition 3.8. By the fact that $H \in \mathscr{H}(\alpha_0, \alpha_1, \gamma)$, we can find $p_1 > R$ such that $\inf_{x\in\mathbb{R}} H(x, p_1, \omega) > \mu$. We can apply Lemma A.6 with $M(\cdot) = p_1$, $m(\cdot) := \overline{f}_\lambda(\cdot,\omega)$ and $G := H - \mu$, to deduce the existence of a solution $f \in \mathrm{C}^1(\mathbb{R})$ of (ODE_λ) with μ in place of λ satisfying $\overline{f}_\lambda(\cdot,\omega) < f < p_1$ on $\mathbb{R}$. This implies that $\overline{f}^+_\lambda(\cdot,\omega) < \overline{f}^+_\mu(\cdot,\omega)$ in $\mathbb{R}$ by definition of $\overline{f}^+_\mu$. The other inequality can be proved in a similar way. □

The next lemma yields that the functions $\underline{f}_\lambda, \overline{f}_\lambda$ are stationary solutions to (ODE_λ). To prove joint measurability, we use again the fact that Ω is a Polish space, $\mathcal{F}$ is its Borel σ-algebra and $\mathbb{P}$ is a complete probability measure.

Lemma 3.7. *Let $\lambda \geqslant \lambda_0$. Then the functions $\underline{f}_\lambda, \overline{f}_\lambda : \mathbb{R} \times \Omega \to \mathbb{R}$ are jointly measurable and stationary. Moreover, $\underline{f}_\lambda(\,\cdot\,,\omega)$, $\overline{f}_\lambda(\,\cdot\,,\omega) \in \mathcal{S}_\lambda(\omega)$ for every $\omega \in \Omega$.*

Proof. The last assertion is a direct consequence of Proposition 3.5 and Lemma A.7. The stationarity property of $\underline{f}$, $\overline{f}$ is a consequence of the fact that, for every fixed $z \in \mathbb{R}$ and $\omega \in \Omega$, $f \in \mathcal{S}_\lambda(\omega)$ if and only if $f(\,\cdot + z) \in \mathcal{S}_\lambda(\tau_z\omega)$ since $a(\,\cdot\, + z, \omega) = a(\,\cdot\,, \tau_z\omega)$ and $H(\,\cdot\, + z, \cdot, \omega) = H(\,\cdot\,, \cdot, \tau_z\omega)$.

Let us prove that $\underline{f}:\mathbb{R}\times\Omega\to\mathbb{R}$ is measurable with respect to the product σ-algebra $\mathcal{B}\otimes\mathcal{F}$. This is equivalent to showing that $\Omega\ni\omega\mapsto\underline{f}(\,\cdot\,,\omega)\in \mathrm{C}(\mathbb{R})$ is a random variable from $(\Omega,\mathcal{F})$ to the Polish space $\mathrm{C}(\mathbb{R})$ endowed with its Borel σ-algebra, see for instance [20, Proposition 2.1]. Since the probability measure $\mathbb{P}$ is complete on $(\Omega,\mathcal{F})$, it is enough to show that, for every fixed $\varepsilon>0$, there exists a set $F\in\mathcal{F}$ with $\mathbb{P}(\Omega\setminus F)<\varepsilon$ such that the restriction $\underline{f}$ to F is a random variable from F to $\mathrm{C}(\mathbb{R})$. To this aim, we notice that the measure $\mathbb{P}$ is inner regular on $(\Omega,\mathcal{F})$, see [8, Theorem 1.3], hence it is a Radon measure. By applying Lusin's Theorem [23] to the random variables $a:\Omega\to\mathrm{C}(\mathbb{R})$ and $H:\Omega\to\mathrm{C}(\mathbb{R}\times\mathbb{R})$, we infer that there exists a compact set $F\subseteq\Omega$ with $\mathbb{P}(\Omega\setminus F)<\varepsilon$ such that $a_{|F},H_{|F}:F\to\mathrm{C}(\mathbb{R}\times\mathbb{R})$ are continuous. We claim that $\underline{f}:\mathbb{R}\times F\to\mathbb{R}$ is lower semicontinuous. Indeed, let $(x_n,\omega_n)_{n\in\mathbb{N}}$ be a sequence converging to some (x_0,ω_0) in $\mathbb{R}\times F$. By the continuity of a on $\mathbb{R}\times F$, we have that $\min_{J\times F}a>0$ for every compact interval $J\subset\mathbb{R}$. This implies that the functions $\underline{f}(\,\cdot\,,\omega_n)$ are locally equi-Lipschitz on $\mathbb{R}$. Since they are also equi-bounded on $\mathbb{R}$ in view of Proposition 3.4, by the Arzelà-Ascoli Theorem, we can extract a subsequence $\left(x_{n_k},\omega_{n_k}\right)_{k\in\mathbb{N}}$ such that $\liminf_n\underline{f}(x_n,\omega_n)=\lim_k\underline{f}(x_{n_k},\omega_{n_k})$ and $\underline{f}(\,\cdot\,,\omega_{n_k})$ converge to some f in $\mathrm{C}(\mathbb{R})$. Since each function $\underline{f}(\,\cdot\,,\omega_n)$ is a solution to (ODE$_\lambda$) with $\omega:=\omega_n$ and $a(\,\cdot\,,\omega_n)\to a(\,\cdot\,,\omega_0)$, $H(\,\cdot\,,\cdot,\omega_n)\to H(\,\cdot\,,\cdot,\omega_0)$ in $\mathrm{C}(\mathbb{R}\times\mathbb{R})$, an argument analogous to the one used in the proof of Proposition 3.5 shows that the functions $\underline{f}(\,\cdot\,,\omega_{n_k})$ actually converge to f in the local C^1 topology. This readily implies that $f\in\mathcal{S}_\lambda(\omega_0)$. By the definition of $\underline{f}$, we conclude that $\underline{f}(\,\cdot\,,\omega)\leqslant f$, in particular

$$\underline{f}(x_0,\omega_0)\leqslant f(x_0)=\lim_k\underline{f}(x_{n_k},\omega_{n_k})=\liminf_n\underline{f}(x_n,\omega_n),$$

proving the asserted lower semicontinuity property of $\underline{f}$. This implies that $\underline{f}_{|F}:F\to\mathrm{C}(\mathbb{R})$ is measurable (see, e.g., [20, Proposition 2.1]). Via a similar argument, one can show that $\overline{f}:\mathbb{R}\times F\to\mathbb{R}$ is upper semicontinuous. ☐

We end this section by stating a result which give a more precise description of the bounds enjoyed by stationary solutions of (ODE$_\lambda$). It is a direct consequence of Corollary A.3.

Proposition 3.8. *Let $\lambda\geqslant\lambda_0$ and let $f:\mathbb{R}\times\Omega\to\mathbb{R}$ be a stationary function such that $f(\cdot,\omega)$ is a C^1 solution of* (ODE$_\lambda$)*, for almost every ω. Then, for every ω in a set of probability 1, we have*

$$\sup_{\mu>\lambda}p_\mu^-\leqslant f(x,\omega)\leqslant\inf_{\mu>\lambda}p_\mu^+\qquad\text{for all } x\in\mathbb{R}.$$

where $p_\mu^\pm$ are constants depending on $\mu\in\mathbb{R}$ defined almost surely as follows:

$$p_\lambda^-:=\inf_{x\in\mathbb{R}}\inf\{p\in\mathbb{R}:\ H(p,x,\omega)\leqslant\lambda\}\quad\text{and}\quad p_\lambda^+:=\sup_{x\in\mathbb{R}}\sup\{p\in\mathbb{R}:\ H(p,x,\omega)\leqslant\lambda\}.$$

4. Bridging stationary solutions

This section is devoted to the proof of Theorem 4.2, where we provide a novel argument to suitably bridge a pair of distinct stationary functions $f_1<f_2$ which solve the following ODE for the same value of λ:

$$a(x,\omega)f'+H(x,f,\omega)=\lambda\qquad\text{in }\mathbb{R}.\tag{ODE$_\lambda$}$$

It exploits the absence of stationary functions trapped between f_1 and f_2 which solve (ODE$_\lambda$) for values of the parameter at the right-hand side lower (respectively, greater) than λ in order to construct a lift which allows to gently descend from f_2 to f_1 (resp., to ascend from f_1 to f_2). The absence of such stationary solutions will be used in the proof of Proposition 5.3, which can be regarded as a crucial step in order to establish our homogenization result stated in Theorem 2.2.

Most of the results of this section holds under a very general set of assumptions on H, such as conditions (G1)-(G2) in Appendix A. However, for the sake of readability and consistency with the rest of the paper, we shall assume that $H\in\mathscr{H}(\alpha_0,\alpha_1,\gamma)$ for some constants $\alpha_0,\alpha_1>0$ and $\gamma>2$.

Let us denote by $\mathcal{S}_\lambda$ the family of essentially bounded and jointly measurable stationary functions $f:\mathbb{R}\times\Omega\to\mathbb{R}$ such that $f(\cdot,\omega)$ is a (bounded) C^1 solution of (ODE$_\lambda$) almost surely. We know that this set is nonempty whenever $\lambda\geqslant\lambda_0$, in view of Proposition 3.5 and Lemmas 3.6 and 3.7.

We distill in the next statement an arguments that already appeared in [18] and which will be needed for the proof of Theorem 4.2.

Proposition 4.1. *Let f_1, f_2 be functions belonging to $\mathcal{S}_\lambda$ for some $\lambda\in\mathbb{R}$ with with $f_1<f_2$ on $\mathbb{R}$ almost surely. Let us set $\theta_i:=\mathbb{E}[f(0,\omega)]$ for $i\in\{1,2\}$.*

(i) *Let us assume that there exist $\varepsilon>0$ and a set Ω_ε of positive probability such that, for every $\omega\in\Omega_\varepsilon$, there exists a C^1 and bounded function g_ε satisfying*
$$a(x,\omega)g_\varepsilon'+H(x,g_\varepsilon,\omega)\leqslant\lambda+\varepsilon\qquad\text{in }\mathbb{R}$$
and such that $g_\varepsilon(\cdot)=f_1(\cdot,\omega)$ in $(-\infty,-L)$ and $g_\varepsilon(\cdot)=f_2(\cdot,\omega)$ in $(L,+\infty)$ for some $L>0$. Then $\mathcal{H}^U(H)(\theta)\leqslant\lambda+\varepsilon$ for every $\theta\in(\theta_1,\theta_2)$.

(ii) *Let us assume that there exist $\varepsilon>0$ and a set Ω_ε of positive probability such that, for every $\omega\in\Omega_\varepsilon$, there exists a C^1 and bounded function g_ε satisfying*
$$a(x,\omega)g_\varepsilon'+H(x,g_\varepsilon,\omega)\geqslant\lambda-\varepsilon\qquad\text{in }\mathbb{R}$$
and such that $g_\varepsilon(\cdot)=f_2(\cdot,\omega)$ in $(-\infty,-L)$ and $g_\varepsilon(\cdot)=f_1(\cdot,\omega)$ in $(L,+\infty)$ for some $L>0$. Then $\mathcal{H}^L(H)(\theta)\geqslant\lambda-\varepsilon$ for every $\theta\in(\theta_1,\theta_2)$.

Proof. The proof is based on an argument already used in [18, Lemma 5.6]. We shall only prove (i), being the proof of (ii) analogous. Let us fix $\theta\in(\theta_1,\theta_2)$ and choose a set $\hat\Omega$ of probability 1 such that, for every $\omega\in\hat\Omega$, both the limits in (2.2) and the following ones (by Birkhoff Ergodic Theorem) hold:

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x f_2(s,\omega)\,ds=\mathbb{E}[f_2(0,\omega)]>\theta \tag{4.1}$$
$$\lim_{x\to-\infty}\frac{1}{|x|}\int_x^0 f_1(s,\omega)\,ds=\mathbb{E}[f_1(0,\omega)]<\theta. \tag{4.2}$$

Let us fix $\omega\in\hat\Omega\cap\Omega_\varepsilon$ and set $w_\varepsilon(x):=\int_0^x g_\varepsilon(z)\,dz$ for all $x\in\mathbb{R}$, where g_ε is as in statement (i). Let us set
$$\tilde w_\varepsilon(t,x)=(\lambda+\varepsilon)t+w_\varepsilon(x)+M_\varepsilon\qquad\text{for all }(t,x)\in[0,+\infty)\times\mathbb{R},$$
where the constant M_ε will be chosen later to ensure that $\tilde w_\varepsilon(0,x)\geqslant\theta x$ for all $x\in\mathbb{R}$. The function $\tilde w_\varepsilon$ is a supersolution of (2.1). Indeed,
$$\partial_t\tilde w_\varepsilon=(\lambda+\varepsilon)\geqslant a(x,\omega)g_\varepsilon'+H(x,g_\varepsilon,\omega)=a(x,\omega)\partial^2_{xx}\tilde w_\varepsilon+H(x,\partial_x\tilde w_\varepsilon,\omega).$$
In view of (4.1) and (4.2), we can pick M_ε large enough so that $\tilde w_\varepsilon(0,x)\geqslant\theta x$ for all $x\in\mathbb{R}$. By the comparison principle, $\tilde w_\varepsilon(t,x)\geqslant u_\theta(t,x,\omega)$ on $[0,+\infty)\times\mathbb{R}$ and, hence,
$$\mathcal{H}^U(H)(\theta)=\limsup_{t\to+\infty}\frac{u_\theta(t,0,\omega)}{t}\leqslant\limsup_{t\to+\infty}\frac{\tilde w_\varepsilon(t,0)}{t}=\lambda+\varepsilon.$$ □

Let us now prove the main result of this section.

Theorem 4.2. *Let f_1,f_2 be functions belonging to $\mathcal{S}_\lambda$ for some $\lambda\in\mathbb{R}$. Let us assume that $f_1<f_2$ on $\mathbb{R}$ almost surely and set $\theta_i:=\mathbb{E}[f_i(0,\omega)]$ for $i\in\{1,2\}$. For any $\mu\in\mathbb{R}$, let us denote by $\mathfrak{S}(\mu;f_1,f_2)$ the family of bounded and C^1 stationary solutions f of (ODE$_\lambda$) with μ in place of λ that satisfy $f_1<f<f_2$ in $\mathbb{R}$, almost surely.*

(i) If $\mathfrak{S}(\mu;f_1,f_2)=\varnothing$ for any $\mu<\lambda$, then
$$\mathcal{H}^L(H)(\theta)\geqslant\lambda\qquad\text{for all }\theta\in(\theta_1,\theta_2).$$

(ii) If $\mathfrak{S}(\mu; f_1, f_2) = \varnothing$ *for any* $\mu > \lambda$, *then*

$$\mathcal{H}^U(H)(\theta) \leqslant \lambda \qquad \textit{for all } \theta \in (\theta_1, \theta_2).$$

Proof. (i) Let us fix $\mu < \lambda$ and $\varepsilon > 0$. In view of Proposition 4.1, it suffices to show that, for every fixed ω in a set of probability 1, there exists a C^1 and bounded function g_ε such that

$$a(x,\omega)g_\varepsilon' + H(x, g_\varepsilon, \omega) > \mu - 2\varepsilon \qquad \text{in } \mathbb{R} \tag{4.3}$$

and satisfying $g_\varepsilon(\cdot) = f_2(\cdot, \omega)$ in $(-\infty, -L]$, $g_\varepsilon(\cdot) = f_1(\cdot, \omega)$ in $[L, +\infty)$ for $L > 0$ large enough.

To this aim, pick $R > 1 + \max\{\|f_1\|_{L^\infty(\mathbb{R}\times\Omega)}, \|f_2\|_{L^\infty(\mathbb{R}\times\Omega)}\}$ and denote by C_R the Lipschitz constant of H on $\mathbb{R} \times [-R, R] \times \mathbb{R}$. Let us pick $\omega \in \Omega_0 := \{\omega \,:\, f_1(\cdot, \omega) < f_2(\cdot, \omega) \text{ in } \mathbb{R}\}$. For fixed $n \in \mathbb{N}$, let us denote by $\tilde g_n : [-n, b) \to \mathbb{R}$ the unique solution of the following ODE

$$a(x,\omega)f' + H(x, f, \omega) = \mu \qquad \text{in } [-n, b) \tag{4.4}$$

satisfying $\tilde g_n(-n) = f_2(-n, \omega)$, for some $b > -n$. The existence of such a b is guaranteed by the classical Cauchy-Lipschitz Theorem. Let us denote by I the maximal interval of the form $(-n, b)$ where such a $\tilde g_n$ is defined. It is easily seen that $\tilde g_n < f_2(\cdot, \omega)$ on I: since $\mu < \lambda$ and $a(\cdot, \omega) > 0$ on $\mathbb{R}$, we have $g_n'(x_0) < f_2'(x_0, \omega)$ at any possible point $x_0 > -n$ where $g_n(x_0) = f_2(x_0, \omega)$, so in fact this equality never occurs. Let us denote by $y_n := \sup\{y \in I \,:\, \tilde g_n > f_1(\cdot, \omega) \text{ in } (-n, y)\}$. We claim that there exists $n \in \mathbb{N}$ such that y_n is finite. If this were not the case, then $f_1(\cdot, \omega) < \tilde g_n < f_2(\cdot, \omega)$ in $(-n, +\infty)$. Being $a(\cdot, \omega) > 0$, we infer from (4.4) that the functions $\tilde g_n$ are locally equi-Lipchitz and equi-bounded on their domain of definition. Up to extracting a subsequence, we derive that the functions $\tilde g_n$ locally uniformly converge to a function $\tilde g$ on $\mathbb{R}$, and also locally in C^1 norm being each $\tilde g_n$ a solution of (4.4). Hence $\tilde g$ is a solution of (4.4) on $\mathbb{R}$ satisfying $f_1(\cdot, \omega) \leqslant \tilde g \leqslant f_2(\cdot, \omega)$ in $\mathbb{R}$. The same kind of argument employed above shows that in fact these inequalities are actually strict, namely $\tilde g$ satisfies

$$f_1(\cdot, \omega) < \tilde g < f_2(\cdot, \omega) \qquad \text{in } \mathbb{R}. \tag{4.5}$$

Let us denote by $\mathfrak{S}(\mu; f_1, f_2)(\omega)$ the set of deterministic C^1 functions satisfying (4.5) which solve equation (4.4) in $\mathbb{R}$. By the stationary character of a, H, f_1, f_2, it is easily checked that $\tilde g \in \mathfrak{S}(\mu; f_1, f_2)(\omega)$ if and only if $\tilde g(\cdot + z) \in \mathfrak{S}(\mu; f_1, f_2)(\tau_z\omega)$. By the ergodicity assumption, we derive that the set $\hat\Omega := \{\omega \in \Omega \,:\, \mathfrak{S}(\mu; f_1, f_2)(\omega) = \varnothing\}$ has either probability 0 or 1. If it has probability 0, for every $\omega \in \Omega$ we set

$$\overline{g}(x, \omega) := \sup_{\tilde g \in \mathfrak{S}(\mu; f_1, f_2)(\omega)} \tilde g(x, \omega) \qquad \text{for all } x \in \mathbb{R},$$

where we agree that $\overline{g}(\cdot, \omega) \equiv 0$ when $\omega \in \hat\Omega$. The function $\overline{g} : \mathbb{R} \times \Omega \to \mathbb{R}$ is jointly measurable and stationary, and $\overline{g}(\cdot, \omega) \in \mathfrak{S}(\mu; f_1, f_2)(\omega)$ almost surely. This follows by arguing as in the proof of Lemma 3.7. Note that the fact that $\overline{g}$ is stationary depends on the fact that the both the ingredients of equation (4.4), both the functions f_1, f_2 are stationary. The fact that $\overline{g}$ satisfies (4.5) with strict inequalities can be shown by arguing as above, or by invoking Lemma A.5. We derive that $\overline{g}$ belongs to $\mathfrak{S}(\mu; f_1, f_2)$, contradicting the assumption.

This implies that $\hat\Omega$ has probability 1, hence for every $\omega \in \hat\Omega$ there exist points $\hat x < \hat y$ in $\mathbb{R}$ and a function $g : [\hat x, \hat y] \to \mathbb{R}$ which solves equation (4.4) in $(\hat x, \hat y)$ and satisfies

$$f_1(\cdot, \omega) < g < f_2(\cdot, \omega) \quad \text{in } (\hat x, \hat y), \quad g(\hat x) = f_2(\hat x, \omega), \quad g(\hat y) = f_1(\hat y, \omega).$$

Let us extend the function g to the whole $\mathbb{R}$ by setting $g = f_2$ on $(-\infty, \hat x)$ and $g = f_1$ on $(\hat y, +\infty)$. Take a sequence of standard even convolution kernels ρ_n supported in $(-1/n, 1/n)$ and set $g_n := \rho_n * g$. Let us pick $r > 0$ and choose $n \in \mathbb{N}$ big enough so that $1/n < r$ and $\|g - g_n\|_\infty < 1$. We claim that we can choose n big enough such that

$$a(x,\omega)g_n' + H(x, g_n, \omega) > \mu - \varepsilon \qquad \text{in } (\hat x - r, \hat y + r). \tag{4.6}$$

To this aim, first observe that the map $x \mapsto H(x, g(x), \omega)$ is K-Lipschitz continuous in $(\hat{x}-2r, \hat{y}+2r)$, for some constant K, due to the fact that g is bounded and locally Lipschitz on $\mathbb{R}$. For every $x \in (\hat{x} - r, \hat{y} + r)$ and $|y| \leqslant 1/n$ we have

$$\begin{aligned} H(x, g_n(x), \omega) &\geqslant H(x, g(x), \omega) - C_R\|g - g_n\|_{L^\infty([\hat{x}-r,\hat{y}+r])} \\ &\geqslant H(x - y, g(x - y), \omega) - C_R\|g - g_n\|_{L^\infty([\hat{x}-r,\hat{y}+r])} - \frac{K}{n}, \end{aligned}$$

hence

$$H(x, g_n(x), \omega) \geqslant \int_{-1/n}^{1/n} H(x - y, g(x - y), \omega)\, \rho_n(y)\, dy - C_R\|g - g_n\|_{L^\infty([\hat{x}-r,\hat{y}+r])} - \frac{K}{n}.$$

For all $x \in (\hat{x} - r, \hat{y} + r)$ we have

$$\begin{aligned} a(x, \omega)g_n'(x) + H(x, g_n(x), \omega) &\geqslant -C_R\|g - g_n\|_{L^\infty([\hat{x}-r,\hat{y}+r])} - \frac{K + \kappa\|g'\|_{L^\infty([\hat{x}-2r,\hat{y}+2r])}}{n} \\ &+ \int_{-1/n}^{1/n} \Big(a(x - y, \omega)g'(x - y) + H(x - y, g(x - y), \omega)\Big)\, \rho_n(y)\, dy \\ &> \mu - \varepsilon \end{aligned}$$

for $n \in \mathbb{N}$ big enough, where we have used the fact that $g_n \to g$ in $C(\mathbb{R})$ as $n \to +\infty$. Let us now take $\xi \in C^1(\mathbb{R})$ such that

$$0 \leqslant \xi \leqslant 1 \quad \text{in } \mathbb{R}, \quad \xi \equiv 0 \quad \text{in } (-\infty, \hat{x} - r] \cup [\hat{y} + r, +\infty), \quad \xi \equiv 1 \quad \text{in } [\hat{x} - r/2, \hat{y} + r/2],$$

and we set $g_\varepsilon(x) := \xi(x)g_n(x) + (1 - \xi(x))g(x)$ for all $x \in \mathbb{R}$. We will show that we can choose n in $\mathbb{N} \cap (1/r, +\infty)$ big enough so that g_ε satisfies (4.3). We only need to check it in $(\hat{x} - r, \hat{y} + r)$. For notational simplicity, we momentarily suppress (x, ω) from some of the notation below and observe that

$$\begin{aligned} ag_\varepsilon' + H(x, g_\varepsilon, \omega) &= \xi\big(ag_n' + H(x, g_n, \omega)\big) + (1 - \xi)\big(ag' + H(x, g, \omega)\big) \\ &+ \xi\big(H(x, \xi g_n + (1 - \xi)g, \omega) - H(x, g_n, \omega)\big) \\ &+ (1 - \xi)\big(H(x, \xi g_n + (1 - \xi)g, \omega) - H(x, g, \omega)\big) + a\xi'(g_n - g) \\ &> \mu - \varepsilon - \big(C_R + \|\xi'\|_\infty\big)\|g - g_n\|_{L^\infty([\hat{x}-r,\hat{y}+r])}. \end{aligned}$$

By choosing n big enough we get the assertion.

(ii) We will just sketch the proof, since the argument is analogous to the one presented above. Let us fix $\mu > \lambda$ and $\varepsilon > 0$. In view of Proposition 4.1, it suffices to show that, for every fixed ω in a set of probability 1, there exists a C^1 and bounded function g_ε such that

$$a(x, \omega)g_\varepsilon' + H(x, g_\varepsilon, \omega) < \mu + 2\varepsilon \qquad \text{in } \mathbb{R} \tag{4.7}$$

and satisfying $g_\varepsilon(\cdot) = f_1(\cdot, \omega)$ in $(-\infty, -L]$, $g_\varepsilon(\cdot) = f_2(\cdot, \omega)$ in $[L, +\infty)$ for $L > 0$ large enough.

To this aim, pick $R > 1 + \max\{\|f_1\|_{L^\infty(\mathbb{R}\times\Omega)}, \|f_2\|_{L^\infty(\mathbb{R}\times\Omega)}\}$ and denote by C_R the Lipschitz constant of H on $\mathbb{R} \times [-R, R] \times \mathbb{R}$. Let us pick $\omega \in \Omega_0 := \{\omega : f_1(\cdot, \omega) < f_2(\cdot, \omega) \text{ in } \mathbb{R}\}$. For fixed $n \in \mathbb{N}$, let us denote by $\tilde{g}_n : [-n, b) \to \mathbb{R}$ the unique solution of the following ODE

$$a(x, \omega)f' + H(x, f, \omega) = \mu \qquad \text{in } [-n, b) \tag{4.8}$$

satisfying $\tilde{g}_n(-n) = f_1(-n, \omega)$, for some $b > -n$. Arguing as in item (i) and by exploiting the assumption $\mathfrak{S}(\mu; f_1, f_2) = \varnothing$, we get that there exists a set $\hat{\Omega}$ of probability 1 such that, for every $\omega \in \hat{\Omega}$, there exist points $\hat{x} < \hat{y}$ in $\mathbb{R}$ and a function $g : [\hat{x}, \hat{y}] \to \mathbb{R}$ which solves equation (4.8) in $(\hat{x}, \hat{y})$ and satisfies

$$f_1(\cdot, \omega) < g < f_2(\cdot, \omega) \quad \text{in } (\hat{x}, \hat{y}), \quad g(\hat{x}) = f_1(\hat{x}, \omega), \quad g(\hat{y}) = f_2(\hat{y}, \omega).$$

We extend the function g to the whole $\mathbb{R}$ by setting $g = f_1$ on $(-\infty, \hat{x})$ and $g = f_2$ on $(\hat{y}, +\infty)$, then we take a sequence of standard even convolution kernels ρ_n supported in $(-1/n, 1/n)$ and we set $g_n := \rho_n * g$. We choose n big enough so that

$$a(x,\omega)g_n' + H(x, g_n, \omega) < \mu + \varepsilon \qquad \text{in } (\hat{x} - r, \hat{y} + r).$$

Next, we define $g_\varepsilon(x) := \xi(x)g_n(x) + (1 - \xi(x))g(x)$ for all $x \in \mathbb{R}$, where ξ is a C^1-function on $\mathbb{R}$ such that

$$0 \leqslant \xi \leqslant 1 \quad \text{in } \mathbb{R}, \quad \xi \equiv 0 \quad \text{in } (-\infty, \hat{x} - r] \cup [\hat{y} + r, +\infty), \quad \xi \equiv 1 \quad \text{in } [\hat{x} - r/2, \hat{y} + r/2].$$

Arguing as in (i), we infer that we can choose n big enough so that g_ε satisfies (4.7). The proof is complete. □

Remark 4.3. In [41, 19, 16] the lower (respectively, upper) bound for $\mathcal{H}^L(H)$ (resp., $\mathcal{H}^U(H)$) appearing in the statement of Theorem 4.2 was proved for Hamiltonians of the form $G(p)+V(x,\omega)$ by showing that $\inf_{\mathbb{R}} (f_2 - f_1) = 0$ almost surely. The proof of this latter property crucially relies on the assumption that the pair (a, V) satisfies a scaled valley (resp., hill) condition, see [41, Lemma 4.7], [19, Lemma 4.2], [16, Lemma 4.3]. In the periodic setting, this condition is met by constant potentials only. It is worth noticing that the hypothesis $\mathfrak{S}(\mu; f_1, f_2) = \varnothing$ for every $\mu \neq \lambda$ herein assumed is always met when $\inf_{\mathbb{R}} (f_2 - f_1) = 0$ almost surely, in view of Lemma A.5.

5. The homogenization result

This section is devoted to the proof of the homogenization result stated in Theorem 2.2. To get to it, we need to prove first a series of preparatory results. Throughout the section, we will assume $G \in \mathscr{H}(\alpha_0, \alpha_1, \gamma)$ for constants $\alpha_0, \alpha_1 > 0$ and $\gamma > 2$.

We start by recalling that we have denoted by $\mathcal{S}_\lambda$ the family of essentially bounded and jointly measurable stationary functions $f : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $f(\cdot, \omega)$ is a (bounded) C^1 solution of the following ODE:

$$a(x,\omega)f' + H(x, f, \omega) = \lambda \qquad \text{in } \mathbb{R} \tag{ODE$_\lambda$}$$

This set is nonempty whenever $\lambda \geqslant \lambda_0$, according to the results presented in Section 3.

We define the set-valued map $\Theta : [\lambda_0, +\infty) \to \mathscr{P}(\mathbb{R})$ as follows:

$$\Theta(\lambda) := \{\mathbb{E}[f(0,\cdot)] \,:\, f \in \mathcal{S}_\lambda\} \qquad \text{for all } \lambda \geqslant \lambda_0.$$

The following holds.

Proposition 5.1. *The set-valued map $\Theta : [\lambda_0, +\infty) \to \mathscr{P}(\mathbb{R})$ satisfies the following properties:*

- *(i)* $\Theta(\lambda)$ *is a nonempty compact set for every* $\lambda \geqslant \lambda_0$*;*
- *(ii)* $\Theta(\cdot)$ *is locally equi-compact;*
- *(iii)* $\Theta(\cdot)$ *is upper semicontinuous;*
- *(iv)* $\Theta(\lambda_1) \cap \Theta(\lambda_2) = \varnothing \quad$ *if* $\lambda_1 \neq \lambda_2$*;*
- *(v)* $\displaystyle\lim_{\lambda \to +\infty} \max \Theta(\lambda) = +\infty, \qquad \lim_{\lambda \to -\infty} \min \Theta(\lambda) = -\infty.$

Proof. The fact that $\Theta(\lambda)$ is nonempty for every $\lambda \geqslant \lambda_0$ has been already remarked above. According to Proposition 3.8, for every $\lambda \geqslant \lambda_0$ we have

$$\sup_{\mu > \lambda} p_\mu^- \leqslant \theta \leqslant \inf_{\mu > \lambda} p_\mu^+.$$

This shows that $\Theta(\cdot)$ is locally equi-bounded. The fact that $\Theta(\cdot)$ is locally equi-compact is a consequence of the upper semicontinuity of $\Theta(\cdot)$, that we proceed to show below.

(iii) Let $\lambda_n \to \lambda$ in $[\lambda_0, +\infty)$ and $\theta_n \in \Theta(\lambda_n)$ for each $n \in \mathbb{N}$ with $\theta_n \to \theta$. We aim to show that $\theta \in \Theta(\lambda)$. Up to extracting a subsequence, if necessary, we can furthermore assume that $(\theta_n)_n$ is monotone, let us say nondecreasing for definitiveness. Then, according to Lemma A.4, we must have

$$f_n(\cdot,\omega) \leqslant f_{n+1}(\cdot,\omega) \quad \text{in } \mathbb{R} \qquad \text{a.s. in } \Omega \qquad \text{for all } n\in\mathbb{N}.$$

According to Corollary A.3, the functions $\big(f_n(\cdot,\omega)\big)_n$ are equi-bounded in $\mathbb{R}$ for every $\omega\in\Omega$, and hence they are also locally equi-Lipschitz since they solve (ODE$_\lambda$) and $a(\cdot,\omega)>0$ in $\mathbb{R}$. We conclude that

$$f_n(\cdot,\omega)\to f(\cdot,\omega)=\sup_n f_n(\cdot,\omega) \qquad \text{for all } \omega\in\Omega$$

in the local C^1 topology on $\mathbb{R}$. Since $\lambda_n\to\lambda$, it is easily seen that $f\in\mathcal{S}_\lambda$. By the Dominated Convergence Theorem, we conclude that

$$\theta=\lim_n\theta_n=\lim_n\mathbb{E}[f_n(0,\cdot)]=\mathbb{E}[f(0,\cdot)],$$

thus showing that $\theta\in\Theta(\lambda)$.

(iv) Obvious in view of Lemma A.5.

(v) Let us prove the first inequality. By the fact that $H\in\mathscr{H}(\alpha_0,\alpha_1,\gamma)$, for every fixed $R>0$ we can find $p_2>p_1>R$ and $\lambda=\lambda(R)\in\mathbb{R}$ such that $\inf_{(x,\omega)}H(x,p_2,\omega)>\lambda>\sup_{(x,\omega)}H(x,p_1,\omega)>\lambda_0$. For every fixed $\omega\in\Omega$, we can apply Lemma A.6 with $M(\cdot)=p_2$, $m(\cdot):=p_1$ and $G:=H-\lambda$, to deduce the existence of a solution $f\in\mathrm{C}^1(\mathbb{R})$ of (ODE$_\lambda$) satisfying $p_1<f(\cdot)<p_2$ on $\mathbb{R}$. This implies that $\overline{f}_\lambda(\cdot,\omega)>p_1$ in $\mathbb{R}$ almost surely, yielding $\max\Theta(\lambda)>p_1>R$. Since the function $\max\Theta(\cdot)$ is also strictly increasing on $[\lambda_0,+\infty)$ in view of Lemma 3.6, this proves the asserted coercivity property of the function $\max\Theta(\cdot)$. The proof of the second inequality is analogous and is omitted. □

With the aid of Proposition 5.1, we proceed to show the following crucial result.

Proposition 5.2. *The set* $\mathrm{Im}(\Theta):=\bigcup_{\lambda\geqslant\lambda_0}\Theta(\lambda)$ *is a closed and unbounded subset of* $\mathbb{R}$.

Proof. The fact that $\mathrm{Im}(\Theta)$ is unbounded is a direct consequence of Proposition 5.1-(v). Let us show it is closed. Pick an accumulation point θ of $\mathrm{Im}(\Theta)$ and a sequence $(\theta_n)_n\subset\mathrm{Im}(\Theta)$ such that $\theta_n\to\theta$. Let $(\lambda_n,f_n)\in[\lambda_0,+\infty)\times\mathcal{S}_{\lambda_n}$ such that $\theta_n:=\mathbb{E}[f_n(0,\cdot)]$.

Claim: $(\lambda_n)_n$ is bounded.

Up to extracting a subsequence, we can assume that the θ_n have constant sign.

Case $\theta_n\geqslant 0$ for all $n\in\mathbb{N}$.

For every $n\in\mathbb{N}$, let us set $C_n(\omega):=\{x\in\mathbb{R}\,:\,f_n(x,\omega)\geqslant\theta_n\}$. Then $C_n(\omega)$ is an almost surely nonempty, closed random stationary set, in particular

$$C_n(\omega)\cap(-\infty,-k)\neq\varnothing \quad\text{and}\quad C_n(\omega)\cap(k,+\infty)\neq\varnothing \quad \text{for all } k\in\mathbb{N} \qquad \text{almost surely,} \tag{5.1}$$

see Propositions 3.2. and 3.5 in [20], for instance.

If $f_n(\cdot,\omega)\geqslant 0$ in $\mathbb{R}$ almost surely, then, according to Lemma A.2, for every fixed ω in a set of probability 1, there exists a local minimum point $y\in\mathbb{R}$ of $f(\cdot,\omega)$ in $\mathbb{R}$ with $0\leqslant f(y,\omega)\leqslant\theta_n$. Hence

$$\lambda_n=a(y,\omega)f_n'(y,\omega)+H(y,f_n(y,\omega),\omega)=H(y,f_n(y,\omega),\omega)\leqslant\alpha_1(|f_n(y,\omega)|+1)\leqslant\alpha_1(\theta_n+1).$$

If, on the other hand, $f_n(\cdot,\omega)$ changes sign almost surely, then, according to (5.1), for every fixed ω in a set of probability 1, there exists a point $z\in\mathbb{R}$ such that $f_n(z,\omega)=0$ and $f_n'(z,\omega)\leqslant 0$. Hence

$$\lambda_n=a(z,\omega)f_n'(z,\omega)+H(z,f_n(z,\omega),\omega)\leqslant H(z,0,\omega)\leqslant\alpha_1.$$

In either case, we get the claim.

Case $\theta_n<0$ for all $n\in\mathbb{N}$.

The function $\hat f_n(x,\omega):=-f_n(-x,\omega)$ is a solution of (ODE$_\lambda$) with $\lambda:=\lambda_n$, $\hat a(x,\omega):=a(-x,\omega)$ and

$\hat{H}(x,p,\omega) := H(-x,-p,\omega)$. The assertion follows by applying the previous step to the function $\hat{f}_n$.

We have thus shown that the sequence $(\lambda_n)_n$ is bounded. Up to extracting a subsequence, we can assume that $\lambda_n \to \lambda$ in $[\lambda_0, +\infty)$. We conclude that $\theta \in \Theta(\lambda)$ by the upper semicontinuity of the set-valued map $\Theta(\cdot)$, see Proposition 5.1-(iii). □

Last, we show the following result, which is the final step to establish homogenization of equation (EHJ$_\varepsilon$).

PROPOSITION 5.3. *Let us set* $D := \{\theta \in \mathbb{R} : \mathcal{H}^L(H)(\theta) = \mathcal{H}^U(H)(\theta)\}$. *Then* $D = \mathbb{R}$.

Proof. In view of Proposition 2.4, we already know that $\mathrm{Im}(\Theta) \subseteq D$. The set $A := \mathbb{R} \setminus \mathrm{Im}(\Theta)$ is an open set in view of Proposition 5.2. Let us assume that $A \neq \varnothing$ and let us denote by I a connected component of A. Due to Proposition 5.1-(v) and the fact that $\mathbb{R}$ is locally connected, we infer that I is a bounded open interval of the form (θ_1, θ_2) with $\theta_i \in \Theta(\lambda_i)$ for $i \in \{1,2\}$ and $\lambda_1, \lambda_2 \in [\lambda_0, +\infty)$. The fact that $I \cap \mathrm{Im}(\Theta) = \varnothing$ implies, in view of Proposition 2.4, that

$$\{f \in \mathcal{S}_\mu \,:\, \mathbb{E}[f(0,\cdot)] = \theta \text{ for some } \mu \geqslant \lambda_0\} = \varnothing \qquad \text{for every fixed } \theta \in I. \tag{5.2}$$

Claim: $\lambda_1 = \lambda_2$.

Let us assume the claim false. Let $f_i \in \mathcal{S}_{\lambda_i}$ such that $\mathbb{E}[f(0,\cdot)] = \lambda_i$ for $i \in \{1,2\}$ with $\lambda_1 \neq \lambda_2$. According to Lemma A.5 and the fact that $\theta_1 < \theta_2$, we must have, almost surely, $f_1(\cdot,\omega) < f_2(\cdot,\omega)$ in $\mathbb{R}$. Pick a constant μ strictly in between λ_1 and λ_2. Let us denote by $\mathfrak{S}(\mu; f_1, f_2)(\omega)$ the set of functions $f \in C^1(\mathbb{R})$ which solve equation (ODE$_\lambda$) with μ in place of λ and satisfy $f_1(\cdot,\omega) < f < f_2(\cdot,\omega)$ in $\mathbb{R}$, for every fixed $\omega \in \Omega$. Such set is almost surely nonempty. This follows by applying Lemma A.6 with $M(\cdot) = f_2(\cdot,\omega)$, $m(\cdot) := f_1(\cdot,\omega)$ and $G := H - \mu$. Let us set

$$\overline{g}(x,\omega) := \sup\{f(x,\omega) \,:\, f \in \mathfrak{S}(\mu; f_1, f_2)(\omega)\}, \qquad (x,\omega) \in \mathbb{R} \times \Omega.$$

By arguing as in the proofs of Lemma 3.7 and Theorem 4.2, we derive that g is jointly measurable and stationary, and satisfies $\overline{g}(\cdot,\omega) \in \mathfrak{S}(\mu; f_1, f_2)(\omega)$ almost surely. This means that $\overline{g} \in S_\mu$ with $\theta := \mathbb{E}[\overline{g}(0,\cdot)] \in (\theta_1, \theta_2) \cap \mathrm{Im}(\Theta)$, in contradiction with (5.2).

We have thus proved that $\lambda_1 = \lambda_2 =: \lambda$. By exploiting (5.2) again, we infer, in view of Theorem 4.2, that

$$\lambda \leqslant \mathcal{H}^L(H)(\theta) \leqslant \mathcal{H}^U(H)(\theta) \leqslant \lambda \qquad \text{for every } \theta \in I,$$

i.e., $I \subset D$. □

REMARK 5.4. It is worth pointing out that what we have shown above is that any connected component I of $\mathbb{R} \setminus \mathrm{Im}(\Theta)$ corresponds to a flat part of the effective Hamiltonian $\mathcal{H}(H)$.

From the information gathered, we can now prove the homogenization result stated in Theorem 2.2.

Proof of Theorem 2.2. In view of Proposition 2.3, we can assume that $H \in \mathscr{H}(\alpha_0, \alpha_1, \gamma)$ for some constants $\alpha_0, \alpha_1 > 0$ and $\gamma > 2$. From Proposition 5.3 we derive that, for every fixed θ we have $\mathcal{H}^L(\theta) = \mathcal{H}^U(\theta)$. We denote this common value by $\mathcal{H}(H)(\theta)$. According to Proposition B.5, this defines a function $\mathcal{H}(H) : \mathbb{R} \to [\lambda_0, +\infty)$ which is superlinear and locally Lipschitz. Homogenization of equation (EHJ$_\varepsilon$) follows in view of [17, Lemma 4.1]. □

Appendix A. ODE results

We state here for the reader's convenience a series of results that we repeatedly used throughout the paper and which hold true for rather general continuous and coercive Hamiltonians. They had been obtained by the author for the current research and had been already shown their usefulness in the work [19]. We refer the reader to [19, Appendix A] for the proofs.

We briefly recall the setting under which such results hold. We will work with a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ and $\mathcal{F}$ denote the probability measure on Ω and the σ–algebra of $\mathbb{P}$–measurable subsets of Ω, respectively. We will assume that $\mathbb{P}$ is invariant under the action of a one-parameter group $(\tau_x)_{x\in\mathbb{R}}$ of transformations $\tau_x : \Omega \to \Omega$ and that the action of $(\tau_x)_{x\in\mathbb{R}}$ is ergodic. No topological or completeness assumptions are made on the probability space.

Let $a : \mathbb{R}\times\Omega \to [0, 1]$ and $G : \mathbb{R}\times\mathbb{R}\times\Omega \to [m_0, +\infty)$ be stationary functions with respect to the shifts in the x-variable, were m_0 is real constant such that $m_0 = \min_{\mathbb{R}\times\mathbb{R}} G(\cdot, \cdot, \omega)$ almost surely. We will assume that a is continuous in the first variable and G is continuous in the first two variables, for any fixed $\omega \in \Omega$.

We introduce the following conditions on G and specify in each statement which of them are needed for that result:

(G1) there exist two coercive functions $\alpha_G, \beta_G : [0, +\infty) \to \mathbb{R}$ such that

$$\alpha_G\left(|p|\right) \leqslant G(p, x, \omega) \leqslant \beta_G\left(|p|\right) \quad \text{for every } (p, x, \omega) \in \mathbb{R}\times\mathbb{R}\times\Omega;$$

(G2) for every fixed $R > 0$, there exists a constant $C_R > 0$ such that

$$|G(p, x, \omega) - G(q, x, \omega)| \leqslant C_R|p - q| \quad \text{for every } p, q \in [-R, R] \text{ and } (x, \omega) \in \mathbb{R}\times\Omega.$$

For our first statement, we will need the following notation: for each $\lambda \geqslant 0$,

$$p^-_\lambda := \inf_{x\in\mathbb{R}} \inf\{p \in \mathbb{R} : \ G(p, x, \omega) \leqslant \lambda\} \quad \text{and} \quad p^+_\lambda := \sup_{x\in\mathbb{R}} \sup\{p \in \mathbb{R} : \ G(p, x, \omega) \leqslant \lambda\}. \tag{A.1}$$

By the ergodicity assumption and (G1), the quantities $p^\pm_\lambda$ are a.s. constants. The functions $\lambda \mapsto p^-_\lambda$ and $\lambda \mapsto p^+_\lambda$ are, respectively, non-increasing and non-decreasing (and, in general, not continuous). Furthermore, $p^-_\lambda \to -\infty$ and $p^+_\lambda \to +\infty$ as $\lambda \to +\infty$.

Lemma A.1. *Assume that G satisfies* (G1)*. Take any $\lambda > m_0$. Let $f(x, \omega)$ be a stationary function such that, for all $\omega \in \Omega$, $f(\,\cdot\,, \omega) \in \mathrm{C}^1(\mathbb{R})$, and*

$$a(x, \omega)f'(x, \omega) + G(f(x, \omega), x, \omega) < \lambda \quad \forall x \in \mathbb{R}.$$

Then, on a set Ω_f of probability 1, $f(x, \omega) \in (p^-_\lambda, p^+_\lambda)$ for all $x \in \mathbb{R}$, where $p^\pm_\lambda$ are defined in (A.1)*.*

Lemma A.2. *Let $f(x, \omega)$ be a stationary function such that $f(\,\cdot\,, \omega) \in \mathrm{C}(\mathbb{R})$ for every $\omega \in \Omega$. Then, we have the following dichotomy:*

(i) *$\mathbb{P}(f(x, \omega) = c \ \forall x \in \mathbb{R}) = 1$ for some constant $c \in \mathbb{R}$;*
(ii) *for $\mathbb{P}$-a.e. ω, $f(\,\cdot\,, \omega)$ has infinitely many local maxima and minima.*

As a consequence of Lemma A.1, we infer

Corollary A.3. *Assume that G satisfies* (G1)*. Take any $\lambda \geqslant m_0$. Let $f(x, \omega)$ be a stationary function such that, for all $\omega \in \Omega$, $f(\,\cdot\,, \omega) \in \mathrm{C}^1(\mathbb{R})$, and*

$$a(x, \omega)f'(x, \omega) + G(f(x, \omega), x, \omega) \leqslant \lambda \quad \forall x \in \mathbb{R}.$$

Then, on a set Ω_f of probability 1, $f(x, \omega) \in [\sup_{\mu>\lambda} p^-_\mu, \inf_{\mu>\lambda} p^+_\mu]$ for all $x \in \mathbb{R}$.

The next result generalizes the fact that, under suitable conditions, two distinct solutions of an ODE do not touch each other.

LEMMA A.4. *Assume that H satisfies* (H2), *and $a(x,\omega)>0$ for all $(x,\omega)\in\mathbb{R}\times\Omega$. Let $f_1(x,\omega)$ and $f_2(x,\omega)$ be stationary processes such that, for all $\omega\in\Omega$, $f_1(\,\cdot\,,\omega), f_2(\,\cdot\,,\omega)\in \mathrm{C}^1(\mathbb{R})\cap \mathrm{C}_b(\mathbb{R})$, and*

$$a(x,\omega)f_1'(x,\omega)+H(f_1(x,\omega),x,\omega)\leqslant a(x,\omega)f_2'(x,\omega)+H(f_2(x,\omega),x,\omega)\quad \forall x\in\mathbb{R}. \tag{A.2}$$

Then, one of the following events has probability 1*:*

$$\begin{aligned}\Omega_0 &=\{\omega\in\Omega:\ (f_1-f_2)(x,\omega)=0 \text{ for all } x\in\mathbb{R}\};\\ \Omega_- &=\{\omega\in\Omega:\ (f_1-f_2)(x,\omega)<0 \text{ for all } x\in\mathbb{R}\};\\ \Omega_+ &=\{\omega\in\Omega:\ (f_1-f_2)(x,\omega)>0 \text{ for all } x\in\mathbb{R}\}.\end{aligned}$$

The next result shows that two distinct solutions of an ODE do not touch each other and gives a quantitative estimate of the distance between them.

LEMMA A.5. *Assume that G satisfies* (G1) *and* (G2). *Take any $\lambda_1,\lambda_2\in\mathbb{R}$ such that $m_0\leqslant\lambda_1<\lambda_2$. Let $f_1(x,\omega)$ and $f_2(x,\omega)$ be stationary functions such that, for all $i\in\{1,2\}$ and $\omega\in\Omega$, $f_i(\,\cdot\,,\omega)\in \mathrm{C}^1(\mathbb{R})$, and*

$$a(x,\omega)f_i'(x,\omega)+G(f_i(x,\omega),x,\omega)=\lambda_i\quad\forall x\in\mathbb{R}.$$

Then, there is a constant $\delta>0$, which depends only on λ_2 and G, such that

$$\mathbb{P}((f_1-f_2)(x,\omega)>\delta(\lambda_2-\lambda_1)\ \forall x\in\mathbb{R})=1\quad\text{or}\quad \mathbb{P}((f_2-f_1)(x,\omega)>\delta(\lambda_2-\lambda_1)\ \forall x\in\mathbb{R})=1.$$

The next statement is deterministic. One should think of $a(\,\cdot\,)$ and $G(\,\cdot\,,\,\cdot\,)$ as $a(\,\cdot\,,\omega)$ and $G(\,\cdot\,,\,\cdot\,,\omega)$ with ω fixed. It states that we can always insert a global C^1 solution between two bounded strict sub and supersolutions which do not intersect. The result is standard, a proof can be found [19, Lemma A.8]

LEMMA A.6. *Assume that $G(p,x)$ satisfies* (G1) *and* (G2), *$a(x)>0$ for all $x\in\mathbb{R}$, there exist bounded functions $m,M\in \mathrm{C}^1(\mathbb{R})$ such that $m(x)<M(x)$ for all $x\in\mathbb{R}$, and either one of the following inequality holds:*

(i) $a(x)m'(x)+G(m(x),x)<0<a(x)M'(x)+G(M(x),x)\qquad\forall x\in\mathbb{R}$;

(ii) $a(x)m'(x)+G(m(x),x)>0>a(x)M'(x)+G(M(x),x)\qquad\forall x\in\mathbb{R}$.

Then, there exists a function $f\in \mathrm{C}^1(\mathbb{R})$ that solves the equation

$$a(x)f'(x)+G(f(x),x)=0\quad\forall x\in\mathbb{R},$$

and satisfies

$$m(x)<f(x)<M(x)\quad\forall x\in\mathbb{R}.$$

We end this section with the following useful stability result, see [19, Lemma A.9] for a proof.

LEMMA A.7. *Assume $G(p,x)$ satisfies (G2) and $a(x)>0$ for all $x\in\mathbb{R}$. Let $\mathcal{S}_\lambda$ be a nonempty family of solutions to*

$$a(x)u'(x)+G(u(x),x)=\lambda,\quad x\in\mathbb{R}. \tag{A.3}$$

If $\mathcal{S}_\lambda$ is a compact subset of $\mathrm{C}(\mathbb{R})$, then the functions

$$\underline{u}(x):=\inf_{u\in\mathcal{S}_\lambda}u(x),\quad \overline{u}(x):=\sup_{u\in\mathcal{S}_\lambda}u(x),\qquad x\in\mathbb{R},$$

are in $\mathcal{S}_\lambda$.

Appendix B. PDE results

In this appendix, we collect some known PDE results that we need in the paper. Throughout this section, we will denote by $\mathrm{UC}(X)$, $\mathrm{LSC}(X)$ and $\mathrm{USC}(X)$ the space of uniformly continuous, lower semicontinuous and upper semicontinuous real functions on a metric space X, respectively. We will denote by H a continuous function defined on $\mathbb{R}\times\mathbb{R}$. If not otherwise stated, we shall assume that H belongs to the class $\mathscr{H}(\alpha_0,\alpha_1,\gamma)$ introduced in Definition 2.1, for some constants $\alpha_0,\alpha_1>0$ and $\gamma>1$.

We will assume that $a:\mathbb{R}\to[0,1]$ is a function satisfying the following assumption, for some constant $\kappa>0$:

(A) $\sqrt{a}:\mathbb{R}\to[0,1]$ is κ–Lipschitz continuous.

Note that (A) implies that a is 2κ–Lipschitz in $\mathbb{R}$.

B.1. **Stationary equations.** Let us consider a stationary viscous HJ equation of the form

$$a(x)u''(x)+H(x,u')=\lambda \qquad \text{in } \mathbb{R}, \tag{SHJ}$$

where $\lambda\in\mathbb{R}$, the nonlinearity H belongs to $\mathscr{H}(\alpha_0,\alpha_1,\gamma)$, and $a:\mathbb{R}\to[0,1]$ satisfies condition (A). The following holds.

Proposition B.1. *Let $u\in\mathrm{C}(\mathbb{R})$ be a viscosity solution of* (SHJ). *Let us assume that $H\in\mathscr{H}(\alpha_0,\alpha_1,\gamma)$ for some constants $\alpha_0,\alpha_1>0$ and $\gamma>1$. Then*

$$|u(x)-u(y)|\leqslant K|x-y| \qquad \textit{for all } x,y\in\mathbb{R},$$

where $K>0$ is given explicitly by

$$K:=C\left(\left(\kappa\frac{\sqrt{1+\alpha_1+|\lambda|}}{\alpha_0}\right)^{\frac{2}{\gamma-1}}+\left(\frac{1+\lambda\alpha_0}{\alpha_0^2}\right)^{\frac{1}{\gamma}}\right) \tag{B.1}$$

with $C>0$ depending only on γ. Furthermore, u is of class C^2 (and hence a pointwise solution of (SHJ)*) in every open interval I where $a(\cdot)$ is strictly positive.*

Proof. The Lipschitz character of u is direct consequence of [3, Theorem 3.1], to which we refer for a proof. Let us now assume that $a(\cdot)$ is strictly positive on some open interval I. Without loss of generality, we can assume that I is bounded and $\inf_I a>0$. From the Lipschitz character of u we infer that $-C\le u''\le C$ in I in the viscosity sense for some constant $C>0$, or, equivalently, in the distributional sense, in view of [29]. Hence, $u''\in L^\infty(I)$. The elliptic regularity theory, see [27, Corollary 9.18], ensures that $u\in W^{2,p}(I)$ for any $p>1$ and, hence, $u\in\mathrm{C}^{1,\sigma}(I)$ for any $0<\sigma<1$. Since u is a viscosity solution to (SHJ) in I, by Schauder theory [28, Theorem 5.20], we conclude that $u\in\mathrm{C}^{2,\sigma}(I)$ for any $0<\sigma<1$. ☐

We shall also need the following Hölder estimate for supersolutions of (SHJ).

Proposition B.2. *Let us assume that $H\in\mathscr{H}(\alpha_0,\alpha_1,\gamma)$ with $\gamma>2$. Let $u\in\mathrm{C}(\mathbb{R})$ be a supersolution of* (SHJ) *for some $\lambda\in\mathbb{R}$. Then*

$$|u(x)-u(y)|\leqslant K|x-y|^{\frac{\gamma-2}{\gamma-1}} \qquad \textit{for all } x,y\in\mathbb{R},$$

where $K>0$ is given explicitly by

$$K:=C\left(\left(\frac{1}{\alpha_0}\right)^{\frac{1}{\gamma-1}}+\left(\frac{1+\lambda\alpha_0}{\alpha_0^2}\right)^{\frac{1}{\gamma}}\right) \tag{B.2}$$

with $C>0$ depending only on γ.

Proof. The function $v(x) := -u(x)$ is a viscosity subsolution of (SHJ) with $-a(\cdot)$ in place of $a(\cdot)$ and $\check{H}(x,p) := H(x,-p)$ in place of H. By the fact that $a \leqslant 1$ and $H \in \mathscr{H}(\alpha_0, \alpha_1, \gamma)$, we derive that v satisfies the following inequality in the viscosity sense:

$$-|v''| + \alpha_0 |v'|^\gamma \leqslant \lambda + \frac{1}{\alpha_0} \quad \text{in } \mathbb{R}.$$

The conclusion follows by applying [3, Lemma 3.2]. □

B.2. **Parabolic equations.** Let us consider a parabolic PDE of the form

$$\text{(EHJ)} \qquad \partial_t u = a(x)\partial^2_{xx} u + H(x, \partial_x u), \quad (t,x) \in (0,+\infty) \times \mathbb{R}.$$

We start with a comparison principle stated in a form which is the one we need in the paper.

Proposition B.3. *Suppose a satisfies (A) and $H \in \mathrm{UC}(B_r \times \mathbb{R})$ for every $r > 0$. Let $v \in \mathrm{USC}([0,T] \times \mathbb{R})$ and $w \in \mathrm{LSC}([0,T] \times \mathbb{R})$ be, respectively, a sub and a supersolution of* (EHJ) *in $(0,T) \times \mathbb{R}$ such that*

$$\text{(B.3)} \qquad \limsup_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{v(t,x) - \theta x}{1+|x|} \leqslant 0 \leqslant \liminf_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{w(t,x)-\theta x}{1+|x|}$$

for some $\theta \in \mathbb{R}$. Let us furthermore assume that either $\partial_x v$ or $\partial_x w$ belongs to $L^\infty((0,T)\times\mathbb{R})$. Then,

$$v(t,x) - w(t,x) \leqslant \sup_{\mathbb{R}} \big(v(0,\cdot) - w(0,\cdot)\big) \quad \textit{for every } (t,x) \in (0,T) \times \mathbb{R}.$$

Proof. The functions $\tilde{v}(t,x) := v(t,x) - \theta x$ and $\tilde{w}(t,x) := w(t,x) - \theta x$ are, respectively, a subsolution and a supersolution of (EHJ) in $(0,T) \times \mathbb{R}$ with $H(\cdot, \theta + \cdot)$ in place of H. The assertion follows by applying [13, Proposition 1.4] to $\tilde{v}$ and $\tilde{w}$. □

Let us now assume that H belongs to the class $\mathscr{H}(\alpha_0,\alpha_1,\gamma)$ for some fixed constants $\alpha_0, \alpha_1 > 0$ and $\gamma > 1$. The following holds.

Theorem B.4. *Suppose a satisfies (A) and $H \in \mathscr{H}(\alpha_0,\alpha_1,\gamma)$. Then, for every $g \in \mathrm{UC}(\mathbb{R})$, there exists a unique function $u \in \mathrm{UC}([0,+\infty)\times\mathbb{R})$ that solves the equation* (EHJ) *subject to the initial condition $u(0,\cdot) = g$ on $\mathbb{R}$. If $g \in W^{2,\infty}(\mathbb{R})$, then u is Lipschitz continuous in $[0,+\infty)\times\mathbb{R}$ and satisfies*

$$\|\partial_t u\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant K \quad \textit{and} \quad \|\partial_x u\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant K$$

for some constant K that depends only on $\|g'\|_{L^\infty(\mathbb{R})}$, $\|g''\|_{L^\infty(\mathbb{R})}$, $\kappa, \alpha_0, \alpha_1$ and γ. Furthermore, the dependence of K on $\|g'\|_{L^\infty(\mathbb{R})}$ and $\|g''\|_{L^\infty(\mathbb{R})}$ is continuous.

Proof. A proof of this result when the initial datum g is furthermore assumed to be bounded is given in [13, Theorem 3.2], see also [3, Proposition 3.5]. This is enough, since we can always reduce to this case by possibly picking a function $\tilde{g} \in W^{3,\infty}(\mathbb{R}) \cap \mathrm{C}^\infty(\mathbb{R})$ such that $\|g - \tilde{g}\|_{L^\infty(\mathbb{R})} < 1$ (for instance, by mollification) and by considering equation (EHJ) with $\tilde{H}(x,p) := a(x)(\tilde{g})'' + H(x, p + (\tilde{g})')$ in place of H, and initial datum $g - \tilde{g}$. □

For every fixed $\theta \in \mathbb{R}$, we will denote by u_θ the unique solution of (EHJ) in $\mathrm{UC}([0,+\infty)\times\mathbb{R})$ satisfying $u_\theta(0,x) = \theta x$ for all $x \in \mathbb{R}$. Theorem B.4 tells us that u_θ is Lipschitz in $[0,+\infty)\times\mathbb{R}$ and that its Lipschitz constant depends continuously on θ. Let us define

$$\text{(B.4)} \qquad \mathcal{H}^L(H)(\theta) := \liminf_{t\to+\infty} \frac{u_\theta(t,0)}{t} \quad \text{and} \quad \mathcal{H}^U(H)(\theta) := \limsup_{t\to+\infty} \frac{u_\theta(t,0)}{t}.$$

By definition, we have $\mathcal{H}^L(H)(\theta) \leqslant \mathcal{H}^U(H)(\theta)$ for all $\theta \in \mathbb{R}$. Furthermore, the following holds, see [19, Proposition B.3] for a proof.

Proposition B.5. *Suppose a satisfies (A) and $H \in \mathscr{H}(\alpha_0, \alpha_1, \gamma)$. Then the functions $\mathcal{H}^L(H)$ and $\mathcal{H}^U(H)$ satisfy (H1) and are locally Lipschitz on $\mathbb{R}$.*

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