THE GEOMETRIC SIZE OF THE FUNDAMENTAL GAP

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Abstract. The fundamental gap conjecture proved by Andrews and Clutterbuck in 2011 provides the sharp lower bound for the difference between the first two Dirichlet Laplacian eigenvalues in terms of the diameter of a convex set in $\mathbb{R}^N$. The question concerning the rigidity of the inequality, raised by Yau in 1990, was left open. Going beyond rigidity, our main result strengthens Andrews-Clutterbuck inequality, by quantifying geometrically the excess of the gap compared to the diameter in terms of flatness. The proof relies on a localized, variational interpretation of the fundamental gap, allowing a dimension reduction via the use of convex partitions à la Payne-Weinberger: the result stems by combining a new sharp result for one dimensional Schrödinger eigenvalues with measure potentials, with a thorough analysis of the geometry of the partition into convex cells. As a by-product of our approach, we obtain a quantitative form of Payne-Weinberger inequality for the first nontrivial Neumann eigenvalue of a convex set in $\mathbb{R}^N$, thus proving, in a stronger version, a conjecture from 2007 by Hang-Wang.

Contents

1. Introduction and statement of the results

Given an open bounded domain $\Omega \subset \mathbb{R}^N$, the difference $\lambda_2(\Omega) - \lambda_1(\Omega)$ between its second and first Dirichlet Laplacian eigenvalues is usually referred to as the fundamental gap of Ω. It has several important implications in different areas of both mathematics and physics, e.g. heat diffusion, statistical mechanics, quantum field theory, numerical

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analysis. Finding sharp lower bounds for the fundamental gap is a problem whose history covers several decades, so that we summarize it without any attempt of completeness. In the pioneering work [54], van den Berg first observed that, for many convex domains Ω, the gap is bounded from below by $\frac{3\pi^2}{D_\Omega^2}$, where D_Ω is the diameter of Ω. The validity of such inequality for any convex domain Ω was then conjectured by Yau [55] and by Ashbaugh-Benguria [4], in the more general case of a Schrödinger operator of the form $-\Delta+V$, being V a convex potential on Ω. This more general formulation of the problem is meaningful also in the one-dimensional case, which was solved by Lavine [40] see also [4, 32]. A breakthrough in higher dimensions is due to Singer-Wong-Yau-Yau [50], who obtained the lower bound $\frac{\pi^2}{4D_\Omega^2}$, later improved into $\frac{\pi^2}{D_\Omega^2}$ by Yu-Zhong [58] and Smits [51]. These lately non-optimal lower bounds rely on the earlier fundamental result by Brascamp-Lieb [10], which states that the first Dirichlet eigenfunction is log-concave on any convex domain (different proofs were given by Korevaar [36] and Singer-Wong-Yau-Yau [50]). Let us also mention the lower bound $\frac{(\log 2)^2}{D_\Omega^2}$ obtained by Bobkov [9, inequality (2.8)] when the Lebesgue measure in the Rayleigh quotient is replaced by any absolutely continuous measure with log-concave density. In the early 2000s, the expected optimal lower bound $\frac{3\pi^2}{D_\Omega^2}$ has been obtained in some particular cases when Ω satisfies specific geometric assumptions [5, 6, 19]. An excellent survey up to that date is the paper by Ashbaugh [3], where more related references can be found. Further advances based on upper bounds for $\nabla^2 \log u_1$ were given in [57, 41]. The conjecture was finally proved in 2011 by Andrews-Clutterbuck in [2] (see also [1]): their groundbreaking new idea is the following refinement of Brascamp-Lieb result into an improved log-concavity inequality for the first Dirichlet eigenfunction

$$(1) \qquad \Big(\nabla \log u_1(y) - \nabla \log u_1(x)\Big) \cdot \frac{y-x}{\|y-x\|} \leq -2\frac{\pi}{D_\Omega} \tan\Big(\frac{\pi}{D_\Omega}\frac{\|y-x\|}{2}\Big) \quad \forall x, y \in \Omega\,.$$

This estimate is obtained by a parabolic approach and, combined with a method to control the modulus of continuity of solutions to parabolic equations, allows them to prove the conjectured lower bound $\frac{3\pi^2}{D_\Omega^2}$. Afterwards, still exploiting the improved-log-concavity estimate (1), Ni recovered the sharp control of the gap by an elliptic argument (see also the nice review by Carron in [14]). A further valuable reading, summarizing also the literature about extensions of the result to manifolds, is the paper [16]. Let us also mention that the behaviour of the fundamental gap on particular situations of collapsing domains has been investigated in [42].

The fundamental gap conjecture is then fully solved, except for the saturation of the equality case. Indeed, the strategies above left unanswered a delicate question, which was formulated in 1990 by Yau himself, see problem no. 44 in his “Open problems in geometry” paper [56]: *Is the gap inequality always strict in dimension $N \geq 2$?* In case of an affirmative answer, since the sharp lower bound is attained on a sequence of rectangular parallelepipeds converging to a line segment, the following natural question arises: *Is it possible to evaluate the excess of the gap in terms of the flatness of the convex set?* Equivalently, this amounts to investigate the validity of a quantitative form of the fundamental gap inequality, a problem which differs from the spectral quantitative

inequalities studied in the past two decades (see [30, Chapter 7]) under several aspects, including in particular the nonexistence of an optimal domain (see also [46]).
The same questions have been raised for a closely related inequality, namely the lower bound due to Payne-Weinberger [45] for the first nontrivial Neumann eigenvalue $\mu_1(\Omega)$ (see also [7, 21, 22]). Since $\mu_1(\Omega)$ can also be seen as the *Neumann* fundamental gap [3], there is an analogy with the Dirichlet fundamental gap, although the proof of the latter is much more challenging for a series of reasons which will be soon understood. In the Neumann case, the saturation question was asked by Sakai [47] (and settled for *smooth* compact Riemannian manifolds with nonnegative Ricci curvature [28, 53]), while the quantitative question has been formulated by Hang-Wang in 2007 [28], along with the conjecture of a lower bound of the type $\frac{\pi^2}{D_\Omega^2} + \overline{c}\frac{w_\Omega^2}{D_\Omega^4}$, being w_Ω the width of Ω.

Aim of our paper is to answer these questions. The strategy we adopt in the Dirichlet case allows us to solve also the Neumann one, as a simplified variant.
We start from a variational principle for the Dirichlet fundamental gap which was first observed by Thompson and Kac [52], and later has been exploited by different authors including Kirsch, Simon, Smits [35, 49, 51]. It consists in viewing the Dirichlet fundamental gap as a weighted Neumann eigenvalue: setting, for any positive weight $p \in L^1(\Omega)$,

$$(2) \qquad \mu_1(\Omega, p) := \inf\left\{ \frac{\int_\Omega |\nabla u|^2 p\, dx}{\int_\Omega u^2 p\, dx} \ : \ u \in H^1_{\rm loc}(\Omega) \cap L^2(\Omega, p\, dx)\,,\ \int_\Omega up\, dx = 0 \right\},$$

and denoting by u_1, u_2 the first two Dirichlet eigenfunctions in L^2, it holds

$$\lambda_2(\Omega) - \lambda_1(\Omega) = \mu_1(\Omega, u_1^2), \qquad \text{with eigenfunction } \overline{u} := \frac{u_2}{u_1}\,.$$

Thus the problem of bounding from below the Dirichlet fundamental gap can be seen from the perspective of Payne-Weinberger (see [45]), the main novelty and crucial difficulty being the presence of the weight u_1^2 in their partitioning method. The procedure consists in decomposing Ω as the union of n mutually disjoint convex cells of equal measure, obtained by successively "cutting" Ω by hyperplanes parallel to a fixed direction, on which the eigenfunction $\overline{u}$ has zero integral mean with respect to the measure $u_1^2\, dx$. In the limit as $n \to +\infty$, since the cells tend to become arbitrarily narrow in $(N-1)$ orthogonal directions, this operation allows to estimate from below $\mu_1(\Omega, u_1^2)$ in terms a one dimensional eigenvalue of the type $\mu_1(I, p)$, where I is a line segment contained into Ω, and $p = hu_1^2$, being h the $\mathcal{H}^{N-1}$ measure of the cell's section orthogonal to I.
In the light of the above, the challenge of determining the size of the fundamental gap consists in getting first a sharp estimate for the one dimensional eigenvalue $\mu_1(I, p)$ for $p = hu_1^2$, and then an insight about the geometric display of the cells of the partition in 2 and higher dimensions. We achieve this goal by taking into account fine properties of Dirichlet eigenfunctions: loosely speaking, the sharp one dimensional estimate is related to the improved log-concavity of the first eigenfunction, and allows to recover the optimal lower bound in terms of the diameter, whereas the analysis of the partition strongly interplays between its polygonal structure, the geometry of the cells, and the localization of the second eigenfunction, ultimately providing an extra term depending on the width.

Our main result reads:

Theorem 1. *Let $N \geq 2$. There exists a dimensional constant $\overline{c} > 0$ such that, for every open bounded convex domain Ω in $\mathbb{R}^N$ with diameter D_Ω and width w_Ω, we have*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{3\pi^2}{D_\Omega^2} + \overline{c}\frac{w_\Omega^6}{D_\Omega^8} \,. \tag{3}$$

In order to deal with the Neumann gap, we have to replace the weight u_1^2 by a constant. Then, in the same vein of Theorem 1, we obtain the following result. It encompasses Hang-Wang conjecture, as it shows that their expected lower bound holds in any space dimension with the second largest John semi-axis in place of the width. Recall that, up to a translation and rotation, for any convex domain $\Omega \subset \mathbb{R}^N$ there exists an ellipsoid $\mathcal{E} = \{\sum_{i=1}^N \frac{x_i^2}{a_i^2} < 1\}$, called *John ellipsoid*, such that $\mathcal{E} \subseteq \Omega \subseteq N\mathcal{E}$ (see e.g. [20]).

Theorem 2. *Let $N \geq 2$. There exists a dimensional constant $\overline{c} > 0$ such that, for every open bounded convex domain Ω in $\mathbb{R}^N$ with diameter D_Ω and John ellipsoid of semi-axes $a_1 \geq \cdots \geq a_N$, we have*

$$\mu_1(\Omega) \geq \frac{\pi^2}{D_\Omega^2} + \overline{c}\frac{a_2^2}{D_\Omega^4}. \tag{4}$$

Remark 3. In our proofs of Theorems 1 and 2 there is no evident loss of sharpness at any step. This leads to the power 6 for the width in (3) and to the power 2 for the second dimension of the John ellipsoid in (4). In the latter case, the power 2 is optimal: taking $\Omega_\varepsilon = (0, d) \times (0, \varepsilon)^{N-1}$, the second John axis of Ω_ε equals ε, and we have

$$\mu_1(\Omega_\varepsilon) = \frac{\pi^2}{d^2} = \frac{\pi^2}{D_{\Omega_\varepsilon}^2} + \frac{\pi^2}{D_{\Omega_\varepsilon}^4}(N-1)\varepsilon^2 + o(\varepsilon^2)\,.$$

On the other hand, in the former case, we can neither prove, nor disprove, the optimality of the power 6. The loss of sharpness in the Dirichlet case, *if true*, might be related to a possibly suboptimal knowledge of the geometry of the first Dirichlet eigenfunction near the boundary and of its localization (see [8, 24, 25, 33]). In a somewhat similar fashion, the fact that (non) localization estimates are more controllable for the first nontrivial Neumann eigenfunction than for the second Dirichlet eigenfunction is the reason why, in the Neumann case, the width (which is of the same order as the lowest John-semiaxis a_N) can be successfully replaced by the second John semi-axis a_2.

Remark 4. An explicit estimate of the constant $\overline{c}$ appearing in (4), without any attempt of optimality, might be rather easily given just by tracking it in all steps of the proof. A similar target for the constant $\overline{c}$ in (3) seems to be more delicate.

Remark 5. We point out that Theorem 1 does not hold unaltered for the Schrödinger equation with a convex potential V

$$u \in H_0^1(\Omega), \qquad -\Delta u + Vu = \lambda_k(\Omega, V)u \quad \text{in } \Omega\,.$$

Actually, while the gap inequality $\lambda_2(\Omega, V) - \lambda_1(\Omega, V) \geq \frac{3\pi^2}{D_\Omega^2}$ is still true [2], its quantitative form (3) cannot hold keeping the same positive constant $\overline{c}$ independent of Ω and V. This can be easily seen by looking at the following example in dimension $N = 2$. Let

$\Omega = \{(x,y) \in \mathbb{R}^2 : |x|+|y| < 1\}$ and $V_{\varepsilon,\delta}(x,y) = \frac{1}{\delta}\big(|y|-\varepsilon\big)^+$. We write the inequality (3) and we first pass to the limit as $\delta \to 0^+$ at fixed ε. Since $\lambda_k(\Omega, V_{\varepsilon,\delta}) \to \lambda_k(\Omega_\varepsilon)$, where $\Omega_\varepsilon := \{\Omega \cap \{(x,y) : |y| < \varepsilon\}$, we obtain $\lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) \geq \frac{3\pi^2}{4} + \frac{\overline{c}}{32}$. Then, by using the monotonicty of the eigenvalues with respect to inclusions and passing to the limit as $\varepsilon \to 0^+$, we obtain

$$\frac{3\pi^2}{4} + \frac{\overline{c}}{32} \leq \lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) \leq \frac{4\pi^2}{(2-2\varepsilon)^2} + \frac{\pi^2}{4\varepsilon^2} - \Big(\frac{\pi^2}{4} + \frac{\pi^2}{4\varepsilon^2}\Big) \to \frac{3\pi^2}{4}\,,$$

which leads to $\overline{c} = 0$.

Strategy of the proof. We give a short overview of the proof of Theorem 1. We refer to Section 6 for the specific modifications required for the proof of Theorem 2, including in particular a geometrically explicit L^∞ estimate for Neumann eigenfunctions, see Proposition 38. Our proof is developed along three main lines described below.

I. A sharp 1D lower bound stemming from the improved log-concavity of u_1. As a first delicate job we have to estimate from below a weigthed Neumann eigenvalue of the type $\mu_1(I,p)$, where I is a line segment contained into Ω, and $p = hu_1^2$, being h the $\mathcal{H}^{N-1}$ measure of the cell's section orthogonal to I. To that aim, special attention must be paid to the concavity features of the weight p.
A first basic feature is that, from the log-concavity of both u_1 and h, p itself is log-concave. This yields that $m_p := \frac{3}{4}\big(\frac{p'}{p}\big)^2 - \frac{1}{2}\frac{p''}{p}$ is a positive measure and that the inequality $\mu_1(I,p) \geq \lambda_1(I, m_p)$ holds (see [45]), being, for any positive measure q,

$$\lambda_1(I,q) := \inf\Big\{\frac{\int_I |u'|^2 dx + \int_I |u|^2\, dq}{\int_I u^2\, dx} \ : \ u \in H^1_0(I) \cap L^2(I,q)\Big\}\,.$$

But the more subtle key feature coming from the factor u_1^2 in the weight p, is that p itself satisfies the improved log-concavity estimate (1). Hence the function $\psi_p := -\frac{1}{2}(\log p)'$ belongs to the following class of functions, which are thought for convenience as functions with extended real values defined on the fixed interval $I_\pi = (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\mathcal{A}(I_\pi) := \Big\{\psi \text{ increasing, } \ \psi(y) - \psi(x) \geq 2\tan\Big(\frac{y-x}{2}\Big) \ \text{ if } x < y \text{ in } \mathrm{dom}(\psi)\Big\}\,.$$

Since the measure m_p can be written as $m_p = \psi_p' + \psi_p^2$, we arrive at the following novel $1D$ problem, which encodes in the class of competitors the *log-concavity modulus* of u_1:

$$\inf\Big\{\lambda_1(I_\pi, \psi' + \psi^2) \ : \ \psi \in \mathcal{A}(I_\pi)\Big\}\,.$$

Surprisingly, the above infimum can not only be estimated, but exactly computed: in Theorem 8 we prove that it equals 3, and it is attained uniquely at the function $\overline{\psi}(x) = \tan(x)$ (which corresponds to the weight $\overline{p} = \phi_1^2$, being $\phi_1(x) = \cos x$ the first Dirichlet eigenfunction of I_π). A noticeable feature is that the optimal function $\overline{\psi} = \tan(x)$ does not saturate pointwise the equality sign in the definition of $\mathcal{A}(I_\pi)$. The key of the proof is a new sophisticated, non standard and ad-hoc procedure of *stratified rearrangement* (see Definifion 13) which allows to handle the modulus of concavity constraint imposed on the functions in the admissible class $\mathcal{A}(I_\pi)$.
The value 3 given by Theorem 8 is clearly the good one in order to recover Andrews-Clutterbuck gap inequality for a convex domain Ω, taken for convenience of diameter

π. But, looking farther, Theorem 8 also paves the way towards the estimate of the gap excess. Indeed, it admits two distinct refinements, which are stated in Theorem 21, holding when the weight $p = hu_1^2$ enjoys some additional properties.
The first refinement has the target of handling cells of "small" diameter: for these cells the additional property of the weight is that the function $\psi = -\frac{1}{2}(\log p)'$ in $\mathcal{A}(I_\pi)$ has finiteness domain of length $d < \pi$. This leads to an improved lower bound of the following type, for an absolute constant C:

$$\lambda_1(I_\pi, \psi' + \psi^2) \geq 3 + C(\pi - d)^3 \,. \tag{5}$$

Here the power 3 is obtained via a perturbation argument. Within the class $\mathcal{A}(I_\pi)$ the power 3 is sharp: this is precisely the point leading to the power 6 of the width in Theorem 1.
The second refinement has the target of handling cells of "large" diameter. A careful analysis of the polygonal structure of the partition will reveal that, in $2D$, it is enough to analyse only such cells for which the height h in orthogonal direction to a diameter is an affine function away from the endpoints. Then, denoting by h_{min} and h_{max} the extrema of the affine function h, we obtain an improved lower bound of the following type, for an absolute constant K:

$$\lambda_1(I_d, \psi' + \psi^2) \geq 3 + K\Big(1 - \frac{h_{min}}{h_{max}}\Big)^2 \,. \tag{6}$$

II. Localized version and rigidity of Andrews-Clutterbuck inequality. Exploiting the one-dimensional estimate (5), we prove a new "localized version" of Andrews-Clutterbuck inequality, namely a lower bound for $\mu_1(\omega, p)$, where as above $p = hu_1^2$, being u_1 the first eigenfunction of Ω and h a power-concave function, but now ω is any convex subset of Ω, possibly of *lower dimension* (the Andrews-Clutterbuck inequality is recovered for $\omega = \Omega$ and $p = u_1$). Denoting by d and D the diameters of ω and Ω, the result reads (see Proposition 29)

$$\mu_1(\omega, p) \geq \frac{3\pi^2}{D^2} + C\frac{(D-d)^3}{D^5} \,. \tag{7}$$

As a consequence, in Theorem 31 we derive the rigidity of Andrews-Clutterbuck inequality. The idea is the following: if by contradiction $\lambda_2(\Omega) - \lambda_1(\Omega) = \frac{3\pi^2}{D^2}$, taking a partition of Ω into n mutually disjoint convex sets Ω_i having the mean value property $\mu_1(\Omega, u_1^2) \geq \frac{1}{n}\sum_{i=1}^n \mu_1(\Omega_i, u_1^2)$, we get

$$\frac{3\pi^2}{D^2} = \mu_1(\Omega, u_1^2) \geq \frac{1}{n}\sum_{i=1}^n \mu_1(\Omega_i, u_1^2) \geq \frac{3\pi^2}{D^2} + C\sum_{i=1}^n \frac{(D-D_i)^3}{D^5} \,,$$

where D_i denotes the diameter of Ω_i. This implies that all the D_i's are equal to D, which yields a contradiction: for $N = 2$, the contradiction comes from a geometric argument, because the equality of all the diameters forces Ω to be a circular sector, for $N \geq 3$ the same argument applies to a suitable two-dimensional section of Ω.
We stress that, in order to gain the above mentioned mean value property of the cells, we need to work with a new kind of partitions, which are *distinct* from the classical ones by Payne-Weinberger not only for the presence of the weight u_1^2, but also for the equipartition request: the measure equipartition condition $|\Omega_i| = \frac{1}{n}|\Omega|$ is replaced

by the L^2-equipartition condition $\int_{\Omega_i} \overline{u}^2 u_1^2 = \int_{\Omega_i} u_2^2 = \frac{1}{n}$. The use of such kind of L^2-equipartitions allows to prove the quantitative inequality by individuating a good proportion of cells for which some particular geometric property is fulfilled. Another relevant observation is that, thanks to the rigidity result, we are reduced to work with convex sets having arbitrarily small width.

III. A play of cells based on the assessment of their geometry. For $N = 2$, we consider a weighted L^2 equipartition of Ω of the kind described above, enjoying the mean value property $\mu_1(\Omega, u_1^2) \geq \frac{1}{n} \sum_{i=1}^{n} \mu_1(\Omega_i, u_1^2)$. This ensures that the required estimate of the excess is fulfilled as soon as, for a fixed proportion of cells, the eigenvalue $\mu_1(\Omega_i, u_1^2)$ is sufficiently large with respect to $\frac{3\pi^2}{D_\Omega^2}$, with a controlled increment. So we distinguish a list of binary crossroads in cascade, depending on different geometric features holding for a fixed proportion of cells. The main distinction is made by looking at the size of the cell's diameter: if most of the cells have "small" diameter (in the sense that the difference between the diameter of Ω and the diameter of the cell is controlled from below by the width), applying to such cells the localized inequality (7) we get the estimate of the excess. Otherwise, if most of the cells have "large" diameter, assuming that the quantitative inequality does not occur, a contradiction is obtained through a geometric argument which can be intuitively sketched as follows. Since we are dealing with the situation in which most cells are thin and long, they can be vertically piled over a diameter of Ω, and they have a profile function which is affine away from the endpoints (if this was not the case, the quantitative inequality would hold as well, by analyzing the position of vertices in the partition and the consequent presence of other cells with small diameter). Then, the one-dimensional refined inequality (6) applies to such cells, i.e., to the one dimensional problem set on their diameter and, if the extra term is small, we get the geometric information that the cells have to be "almost" rectangular (see Figure 4). At this point, we obtain a uniform control on the height of each cell, related to the non localization of the second eigenfunction (see Remark 33). This leads to the conclusion that the pile of the cells is, in a sense, "too high", because the actual diameter would be strictly larger than D_Ω, finally yielding a contradiction.
For $N \geq 3$, the result is obtained by a partial slicing procedure, reducing ourselves to a two-dimensional analysis involving a modified weight, which can be carried over by similar arguments as the ones used to treat the case $N = 2$.

The paper is organized as follows. Section 2 is devoted to the analysis of the 1D-eigenvalue problem associated with the measure potentials issued from restrictions of first eigenfunctions to line segments contained into a convex set. In Section 3 we introduce the modified Payne-Weinberger partitions and in Section 4 we prove the localized version of the Andrews-Clutterbuck inequality and establish the rigidity property of the gap inequality. Section 5 contains the proof of Theorem 1 while in Section 6 we prove Theorem 2. In the Appendix we collect some useful results about eigenfunctions associated with weighted Neumann eigenvalues.

2. The sharp one-dimensional lower bound

Let I be a one-dimensional open bounded interval. Given a positive weight p in $L^1(I)$ and a nonnegative Borel measure q possibly taking the value $+\infty$, consider the following weighted Neumann eigenvalue and Dirichlet eigenvalue with potential:

$$\mu_1(I,p) := \inf \Big\{ \frac{\int_I |v'|^2 p\,dx}{\int_I v^2 p\,dx} \ :\ v \in H^1_{\rm loc}(I) \cap L^2(I, p\,dx)\,,\ \int_I vp = 0 \Big\}$$

$$\lambda_1(I,q) := \inf \Big\{ \frac{\int_I |v'|^2 dx + \int_I |v|^2\, dq}{\int_I v^2\, dx} \ :\ v \in H^1_0(I) \cap L^2(I, q) \Big\}\,.$$

Let us mention that eigenvalues associated with potentials which are measures, such as $\lambda_1(I,q)$, have been extensively studied in the context of shape optimization in dimension $N \geq 2$, see for instance [12, Section 4.3].
If p is log-concave, we can introduce the positive measure m_p defined by

$$m_p := \Big[\frac{3}{4} \Big(\frac{p'}{p} \Big)^2 - \frac{1}{2} \frac{p''}{p} \Big] = \psi_p' + \psi_p^2\,, \qquad \text{with } \psi_p := -(\log p^{\frac12})'\,. \tag{8}$$

Here $\psi_p' = [\frac12 \Big(\frac{p'}{p} \Big)^2 - \frac12 \frac{p''}{p}]$ is the distributional derivative of the non-decreasing function ψ_p, which is a nonnegative measure thanks to the log-concavity of p, while ψ_p^2 denotes with a slight abuse of notation the nonnegative measure $\psi_p^2\, dx$.
For simplicity, also in the sequel we denote measures which are absolutely continuous simply by writing their density with respect to the Lebesgue measure.

Lemma 6. *For any positive log-concave weight $p \in L^1(I)$, if m_p is given by (8) it holds*

$$\mu_1(I,p) \geq \lambda_1(I, m_p)\,.$$

Proof. Under the additional assumptions $p \in W^{1,\infty}(I)$ and $\inf_{x\in I} p(x) > 0$, an eigenfunction v for $\mu_1(I_\pi, p)$ exists in $H^2(I)$ and satisfies

$$\begin{cases} -(pv')' = \mu_1(I,p) pv & \text{in } I \\ pv'(-\frac{\pi}{2}) = pv'(\frac{\pi}{2}) = 0\,. \end{cases}$$

Then the function $w := p^{1/2} v'$ belongs to $H^1_0(I)$ and solves

$$\begin{cases} -w'' + \Big[\frac34 \Big(\frac{p'}{p} \Big)^2 - \frac12 \frac{p''}{p} \Big] w = \mu_1(I_\pi, p) w & \text{in } I \\ w(-\frac{\pi}{2}) = w(\frac{\pi}{2}) = 0\,, \end{cases}$$

yielding the inequality $\mu_1(I,p) \geq \lambda_1(I, m_p)$.
Assume now $p \in L^1(I)$ is positive and log-concave. Letting I^ε be intervals compactly included in I and increasingly converging to I, we have $\inf_{x\in I^\varepsilon} p(x) > 0$ and $p|_{I^\varepsilon} \in W^{1,\infty}(I)$. Consequently, the Neumann eigenvalue problem $\mu_1(I^\varepsilon, p)$ is well-posed and the inequality $\mu_1(I^\varepsilon, p) \geq \lambda_1(I^\varepsilon, m_p)$ is satisfied. Then we obtain that the same inequality holds true for the interval I by observing that

$$\lambda_1(I, m_p) = \lim_{\varepsilon\to 0} \lambda_1(I^\varepsilon, m_p) \quad \text{ and } \quad \mu_1(I,p) \geq \limsup_{\varepsilon \to 0} \mu_1(I^\varepsilon, p).$$

Indeed, the first assertion follows from the inclusion $I^\varepsilon \subseteq I$ and the fact that I^ε is increasingly converging to I. The second assertion follows from the monotone convergence theorem, since, for any admissible test function for $\mu_1(I,p)$, its restriction to I^ε, corrected by a small constant so to make it orthogonal to p in $L^2(I^\varepsilon)$, becomes an admissible test function for $\mu_1(I^\varepsilon,p)$. □

By Lemma 6, we are led to deal with Dirichlet eigenvalues of the type $\lambda_1(I,m_p)$, with $p=hu_1^2$ and m_p given by (8). The corresponding function ψ_p has the form

$$\psi_p = -(\log u_1)' - \frac{1}{2}(\log h)'\,.$$

The heart of the matter is that, by the improved log-concavity estimate (1), the function ψ_p turns out to belong to the class of functions defined hereafter in (9). To formulate the problem, without any loss of generality we work on the interval

$$I_\pi := \big(-\frac{\pi}{2},\frac{\pi}{2}\big)\,,$$

and we introduce the following class of functions defined on I_π with values in $\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$ and finiteness domain $\mathrm{dom}(\psi)$

$$\mathcal{A}(I_\pi) := \Big\{\psi \text{ increasing},\ \psi(y)-\psi(x)\geq 2\tan\Big(\frac{y-x}{2}\Big)\ \text{ if } x<y \text{ in } \mathrm{dom}(\psi)\Big\}\,. \tag{9}$$

For functions $\psi\in\mathcal{A}(I_\pi)$, we tacitly extend the measures ψ^2 and ψ' to $+\infty$ in $I_\pi\backslash\mathrm{dom}(\psi)$. Then our target can be precisely expressed as the study of the minimization problem

$$\min\Big\{\lambda_1(I_\pi,q)\ :\ q=\psi'+\psi^2 \text{ for some } \psi\in\mathcal{A}(I_\pi)\Big\}\,, \tag{10}$$

The remaining of this section is entirely devoted to that goal. It is divided in two parts:

– in the first part we give the sharp lower bound for $\lambda_1(I_\pi,\psi'+\psi^2)$ for $\psi\in\mathcal{A}(I_\pi)$, namely we fully solve the minimisation problem (10), see Theorem 8;

– in the second part, for $\psi\in\mathcal{A}(I_\pi)$ with $\mathrm{dom}(\psi)=(-\frac{d}{2},\frac{d}{2})=:I_d$, being $d<\pi$, we give some lower bounds for $\lambda_1(I_d,\psi'+\psi^2)$ with extra terms involving the difference $(\pi-d)$, see Theorem 21. Here and in the sequel, for $d<\pi$, if m is a nonnegative Borel measure on I_d, we identify the eigenvalue $\lambda_1(I_d,m)$ with $\lambda_1(I_\pi,\tilde m)$, where $\tilde m$ is the measure obtained extending m to $+\infty$ on $I_\pi\setminus I_d$.

Let us start with an elementary observation:

Remark 7. The function $\overline{\psi}(x)=\tan x$ belongs to $\mathcal{A}(I_\pi)$. The corresponding weight $\overline{q}(x)=1+2\tan^2 x$ is equal to $m_{\overline{p}}$ for $\overline{p}=\phi_1^2$, being $\phi_1(x)=\cos x$ the first Dirichlet eigenvalue on I_π, and we have the following equalities for eigenvalues, all of them with eigenfunction $\cos^2 x$:

$$\begin{aligned}\lambda_1(I_\pi,2\overline{\psi}^2)&=\lambda_1(I_\pi,2\tan^2 x)=2\\ \lambda_1(I_\pi,2\overline{\psi}')&=\lambda_1(I_\pi,2(1+\tan^2 x))=4\\ \lambda_1(I_\pi,\overline{\psi}'+\overline{\psi}^2)&=\lambda_1(I_\pi,1+2\tan^2 x)=3\,.\end{aligned}$$

It is somehow natural to wonder whether the weight $\overline{q}$ associated with the one-dimensional eigenfunction ϕ_1 is optimal for the minimization problem (10). Our result below states that this exactly is the case. In addition, $\overline{q}$ is the *unique* solution.

Theorem 8. *Let $q = \psi' + \psi^2$, with $\psi \in \mathcal{A}(I_\pi)$. Then*

$$\lambda_1(I_\pi, q) \geq 3\,, \tag{11}$$

with equality if and only if $q(x) = 1 + 2\tan^2 x$ (and in this case an eigenfunction is $\cos^2 x$).

The proof of Theorem 8 is built upon the following two independent propositions.

Proposition 9. *For every $\psi \in \mathcal{A}(I_\pi)$, it holds*

$$\lambda_1(I_\pi, 2\psi^2) \geq 2\,, \tag{12}$$

with equality if and only if $\psi(x) = \tan x$ (and in this case an eigenfunction is $\cos^2 x$).

Proposition 10. *For every $\psi \in \mathcal{A}(I_\pi)$, it holds*

$$\lambda_1(I_\pi, 2\psi') \geq 4\,. \tag{13}$$

In particular, equality occurs if $\psi(x) = \tan x$ (and in this case an eigenfunction is $\cos^2 x$).

Let us first show how the above propositions imply Theorem 8, and then turn back to their proof.

Proof of Theorem 8. It is easy to check that the map $q \mapsto \lambda_1(I_\pi, q)$ is concave. Indeed, for every pair of weights q_1, q_2, every $t \in [0,1]$, and every $v \in H^1_0(I_\pi)$ we have

$$\begin{aligned}\frac{\int_{I_\pi} (v')^2 + v^2\,((1-t)dq_1 + t dq_2)}{\int_{I_\pi} u^2} &= (1-t)\frac{\int_{I_\pi} (v')^2 + v^2\,dq_1}{\int_{I_\pi} v^2} + t\frac{\int_{I_\pi} (v')^2 + v^2\,dq_2}{\int_{I_\pi} v^2} \\ &\geq (1-t)\lambda_1(I_\pi, q_1) + t\lambda_1(I_\pi, q_2)\,.\end{aligned}$$

Therefore, for every $\psi \in \mathcal{A}(I_\pi)$, we have

$$\lambda_1(I_\pi, \psi' + \psi^2) \geq \frac{\lambda_1(I_\pi, 2\psi') + \lambda_1(I_\pi, 2\psi^2)}{2}\,. \tag{14}$$

Then the inequality (11) follows immediately from Proposition 9 and Proposition 10. Concerning the equality case, when $q(x) = 1 + 2\tan^2 x$, we have $\lambda_1(I_\pi, q) = 3$, with eigenfunction $\cos^2 x$ (cf. Remark 7). Viceversa, if $\lambda_1(I_\pi, q) = 3$, for some $q = \psi' + \psi^2$ with $\psi \in \mathcal{A}(I_\pi)$, it follows from the inequalities (12), (13), and (14), that $\lambda_1(I_\pi, 2\psi') = 4$ and $\lambda_1(I_\pi, 2\psi^2) = 2$. By the last assertion in Proposition 9, we conclude that $\psi(x) = \tan x$, and hence $q(x) = 1 + 2\tan^2 x$. ☐

For later use, we state below a rigidity result for the Neumann eigenvalue $\mu_1(I_\pi, p)$, which is a straightforward by-product of Theorem 8.

Corollary 11. *If $\mu_1(I_\pi, p) = 3$ for a positive log-concave weight p such that $\psi_p \in \mathcal{A}(I_\pi)$, we have $\psi_p(x) = \tan x$, and $p(x) = k\cos^2 x$ for some positive constant k.*

Proof. If $\mu_1(I_\pi, p) = 3$, by Lemma 6 and Proposition 10, we have

$$3 = \mu_1(I_\pi, p) \geq \lambda_1(I_\pi, \psi_p' + \psi_p^2) \geq \frac{1}{2}\big(\lambda_1(I_\pi, 2\psi_p') + \lambda_1(I_\pi, 2\psi_p^2)\big) \geq 2 + \frac{1}{2}\lambda_1(I_\pi, 2\psi_p^2)\big) \geq 3\,.$$

We infer that $\lambda_1(I_\pi, 2\psi_p^2) = 2$. Therefore, by Proposition 9 we conclude that $\psi_p(x) = \tan x$, and hence $p(x) = k\cos^2 x$ for some positive constant k. ☐

We now provide the proofs of Propositions 9 and 10.

Proof of Proposition 9. We first prove the following claim: for every function $\psi \in \mathcal{A}(I_\pi)$, there exists another function $\widetilde{\psi} \in \mathcal{A}(I_\pi)$ which changes sign once in I_π, and satisfies $\lambda_1(I_\pi, \psi^2) \geq \lambda_1(I_\pi, \widetilde{\psi}^2)$. We search for $\widetilde{\psi}$ in the subclass of $\mathcal{A}(I_\pi)$ given by functions of the type $\psi + c$, for $c \in \mathbb{R}$. If v denotes an eigenfunction for $\lambda_1(I_\pi, \psi^2)$, normalized in L^2, we have

$$\lambda_1(I_\pi, \psi^2) = \int_{I_\pi} |v'|^2 dx + \int_{I_\pi} \psi^2 |v|^2 \, dx \geq \inf_{c \in \mathbb{R}} \int_{I_\pi} |v'|^2 dx + \int_{I_\pi} (\psi + c)^2 |v|^2 \, dx \,.$$

By differentiating with respect to c, we see that the above infimum is attained at $\tilde{c} = -\int_{I_\pi} \psi |v|^2$. The function $\widetilde{\psi} := \psi + \tilde{c}$ satisfies $\int_{I_\pi} \widetilde{\psi} |v|^2 = 0$, so that it changes sign at least one time, and exactly one time, because $\widetilde{\psi} \in \mathcal{A}(I_\pi)$, so that it is strictly increasing. Moreover,

$$\lambda_1(I_\pi, \psi^2) \geq \int_{I_\pi} |v'|^2 dx + \int_{I_\pi} (\psi + \tilde{c})^2 |v|^2 \, dx \geq \lambda_1(I_\pi, \widetilde{\psi}) \,.$$

In view of the claim just proved, it is not restrictive to prove the inequality (12) under the assumption that the function ψ changes sign exactly one time in I_π.
Consider the function $|\psi|$. Thanks to the assumption that ψ has exactly one zero $x_0 \in I_\pi$, we know that the function $|\psi|$ vanishes at x_0, is strictly decreasing for $x \leq x_0$, and strictly increasing for $x \geq x_0$. So, for almost every $t > 0$, the level set $\{|\psi| < t\}$ is an interval (containing x_0). We rearrange the function $|\psi|$ into the even function $|\psi|_*$ defined by

$$\{|\psi|_* < t\} := \{|\psi| < t\}^* \qquad \forall t \in (0, \|\psi\|_\infty] \,,$$

where $\{|\psi| < t\}^*$ denotes the translation of the interval $\{|\psi| < t\}$ which sends its mid-point to the origin. (Notice that this is a kind of "symmetric increasing rearrangement", which is the analogue of the classical symmetric decreasing rearrangement, just replacing super-levels with sub-levels). By construction, for every $x \in \big(0, \frac{|\mathrm{dom}(\psi)|}{2}\big)$, we have

$$|\psi|_*(x) = |\psi|(b_x) = |\psi|(a_x) \qquad x = \frac{b_x - a_x}{2}, \text{ with } a_x < x_0 < b_x \text{ and } \psi(a_x) = -\psi(b_x) \,.$$

Thus the assumption $\psi \in \mathcal{A}(I_\pi)$ be expressed as a pointwise inequality for $|\psi|_*$:

$$|\psi|_*(x) = \frac{1}{2}\big(\psi(b_x) - \psi(a_x)\big) \geq \tan\Big(\frac{b_x - a_x}{2}\Big) = \tan x \qquad \forall x \in \Big(0, \frac{|\mathrm{dom}(\psi)|}{2}\Big) \,. \tag{15}$$

Let now $v \in H^1_0(I_\pi)$. Denoting by v^* its classical symmetric decreasing rearrangement, and by $\mathrm{dom}(\psi)^*$ the interval of length $|\mathrm{dom}(\psi)|$ centred at the origin, it holds

$$\int_{\mathrm{dom}(\psi)} |v'|^2 \geq \int_{\mathrm{dom}(\psi)^*} |(v^*)'|^2 \qquad \text{and} \qquad \int_{\mathrm{dom}(\psi)} \psi^2 |v|^2 \geq \int_{\mathrm{dom}(\psi)^*} |\psi|_*^2 |v^*|^2 \,. \tag{16}$$

Indeed, the first inequality is the classical Pólya-Szegö inequality for the decreasing rearrangement (see e.g. [34, Section II.4]), while the second one follows from the classical Hardy-Littlewood inequality (see e.g. [29, Chapter 10]), the sign of the inequality being reversed because for one of the two involved functions ($|\psi|$) we take the increasing rearrangement $|\psi|_*$ defined above in place of $|\psi|^*$. Then, we have

$$\text{(17)}\qquad \frac{\int_{\text{dom}(\psi)}|v'|^2+2\psi^2v^2}{\int_{\text{dom}(\psi)}|v|^2}\geq\frac{\int_{\text{dom}(\psi)^*}|(v^*)'|^2+2|\psi|_*^2|v^*|^2}{\int_{\text{dom}(\psi)^*}|v^*|^2}\geq\frac{\int_{I_\pi}|(v^*)'|^2+2\tan^2x|v^*|^2}{\int_{I_\pi}|v^*|^2}\geq 2\,,$$

where in the second and third inequality we have used respectively the estimate (15) and Remark 7.
Concerning the equality case, if $\psi(x)=\tan x$, we have $\lambda_1(I_\pi,2\psi^2)=2$, with eigenfunction $\cos^2 x$ (cf. Remark 7). Viceversa, assume that the equality $\lambda_1(I_\pi,2\psi^2)=2$ holds for some function ψ. Then, if v is an eigenfunction for $\lambda_1(I_\pi,2\psi^2)$, all the inequalities in (16) and (17) must hold with equality sign. From the fact that the last inequality in (17) holds with equality sign, we infer that $v^*=k\cos^2 x$.
In particular, v^* does not have critical level sets of positive Lebesgue measure. Then the fact that the first inequality in (16) holds with equality sign implies that $v=v^*$ [11, Theorem 1.1] (see also [27, Theorem 4.1]).
In turn, since the second inequality in (17) holds with equality sign, we infer that $|\psi|_*^2=\tan^2 x$ a.e. By the monotonicity of ψ, we have $\psi(x)=\tan x$ for every $x\in\mathbb{R}$. $\square$

We now turn to the proof of Proposition 10, which requires some preliminaries. Given a function $\psi\in\mathcal{A}(I_\pi)$, in order to estimate from below $\lambda_1(I_\pi,2\psi')$, we need to find lower bounds for integrals of the following type, for $v\in H_0^1(I_\pi)$:

$$\int_{I_\pi} v^2\,d\nu_{\psi'}\,.$$

Here and in the sequel, we use the notation $d\nu_{\psi'}$ when writing an integral with respect to the measure ψ'. We have:

$$\text{(18)}\qquad \begin{aligned}\int_{I_\pi} v^2\,d\nu_{\psi'} &= \int_{I_\pi}\int_0^{+\infty}\chi_{\{v^2>s\}}\,ds\,d\nu_{\psi'}=\int_0^{+\infty}\int_{I_\pi}\chi_{\{v^2>s\}}\,d\nu_{\psi'}\,ds\\ &=\int_0^{+\infty}\nu_{\psi'}\big(\{v>\sqrt{s}\}\big)\,ds=\int_0^{+\infty}\nu_{\psi'}\big(\{v>t\}\big)\,2t\,dt\,.\end{aligned}$$

On the other hand, the constraint $\psi\in\mathcal{A}(I_\pi)$ can be equivalently expressed as the following inequality holding for all intervals $[a,b]\subset I_\pi\cap\text{dom}(\psi)$:

$$\text{(19)}\qquad \nu_{\psi'}([a,b])\geq 2\tan\big(\frac{b-a}{2}\big)=\int_a^b\Big(1+\tan^2\big(x-\frac{a+b}{2}\big)\Big)\,dx\,.$$

Condition (19) cannot be applied directly to estimate the integral (18), because not all level sets of u are intervals. Thus we are going to exploit condition (19) in a more subtle way, passing through the introduction of new notions of *stratified rearrangement* and *stratified potential*; they are obtained by finitely many applications of elementary constructions that we call respectively *blocked rearrangement* and *blocked potential.*
Below we introduce these definitions on a generic bounded open interval I, for v in the space $\mathcal{X}(I)$ of functions which are continuous on the closure of I, attain their global minimum at both the endpoints of I, and have finitely many local minima in I, each one attained at a single point. (Notice that $H_0^1(I)\cap\mathcal{X}(I)$ is a dense subspace of $H_0^1(I)$).
We denote by $[v^*,I]$ the symmetric decreasing rearrangement of v with respect to the mid-point x_I of I.

Definition 12 (blocked rearrangement)**.** Let $v \in \mathcal{X}(I)$. The blocked rearrangement of v in I is the function $[v^\flat, I]$ defined on I as follows (see Figure 1):

(i) If v does not have any local minimum in I,

$$[v^\flat, I] := [v^*, I]\,.$$

(ii) If v has some local minimum in I, letting ℓ be the smallest level of local minimum (so that $\{v > \ell\}$ is the union of two consecutive open intervals), and denoting respectively by x_I and x_ℓ the midpoints of I and of the closed interval $\overline{\{v > \ell\}}$,

$$[v^\flat, I](x) := \begin{cases} [v^*, I](x) & \text{if } [v^*, I](x) \leq \ell \\ v(x - x_\ell + x_I) & \text{otherwise} \end{cases}$$

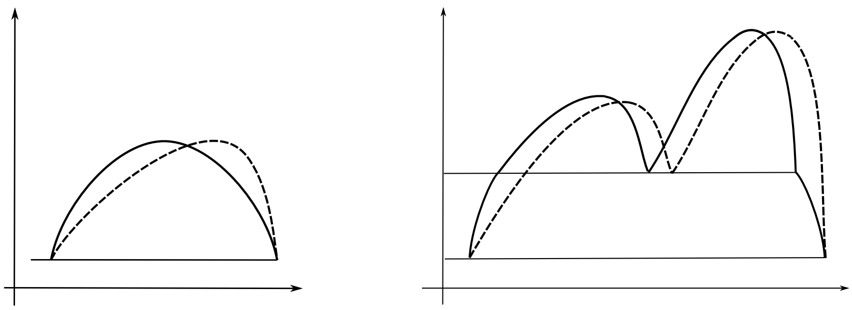

FIGURE 1. The blocked rearrangement given by Definition 12: case (i) on the left, case (ii) on the right (the graph of v is in dashed line, the graph of $[v^\flat, I]$ in continuous line).

Definition 13 (stratified rearrangement)**.** Let $v \in \mathcal{X}(I)$. Its stratified rearrangement is the function $\tilde{v}$ defined on I via a finite number of blocked rearrangements as follows:

– Set $I^1 := I$, and $v_1 := [v^\flat, I^1]$. Two cases may occur:

(i) If v does not have any local minimum in I^1, define $\tilde{v} := v_1$ in the whole interval I^1, and the definition stops here.

(ii) If v has some local minimum in I^1, letting ℓ_1 be the smallest value of local minimum for v in I^1, define $\tilde{v} := v_1$ only on the set $[v^*, I^1] \leq \ell_1$, and for the definition on its complement in I^1 go to the next step.

– Set $I^{1,j}$ the two consecutive open intervals such that $\big\{v_1 > \ell_1\big\} = I^{1,1} \cup I^{1,2}$, and $v_{1,j} := [v_1^\flat, I^{1,j}]$, for $j = 1, 2$. For a fixed $j \in \{1, 2\}$, two cases may occur:

(i) If v_1 does not have any local minimum in $I^{1,j}$, define $\tilde{v} := v_{1,j}$ in the whole interval $I^{1,j}$, and the definition on the interval $I^{1,j}$ stops here.

(ii) If v_1 has some local minimum in $I^{1,j}$, letting ℓ_2 be the smallest value of local minimum for v_1 in $I^{1,j}$, define $\tilde{v} := v_{1,j}$ only on the set $[v_1^*, I^{1,j}] \leq \ell_2$, and for the definition on its complement in $I^{1,j}$ go to the next step.

– Set $I^{1,j,1}$ and $I^{1,j,2}$ the two consecutive open intervals such that $\{v_{1,j} > \ell_2\} = I^{1,j,1} \cup I^{1,j,2}$, and proceed as in the previous steps. After finitely many steps, the procedure stops because by assumption the number of local minima is finite.

Remark 14. The open intervals constructed in Definition 13 can be labeled by a family $\mathcal{F}$ of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$, with $\alpha_1 = 1$ and $\alpha_i \in \{1, 2\}$ for $i \geq 2$, and k less than or equal to the number of levels of local minimum of v in $\overline{I}$. We denote by Γ the subfamily of $\mathcal{F}$ of multi-indices α such that I^α contains a local minimum of $\tilde{v}$. Equivalently, we set

$$\Gamma := \Big\{\alpha \in \mathcal{F} \text{ such that } (\alpha, 1) \text{ and } (\alpha, 2) \text{ belong to } \mathcal{F}\Big\}. \tag{20}$$

Then by construction the stratified rearrangement $\tilde{v}$ enjoys the following symmetry property with respect to the mid-point x_α of each interval I^α: if $\alpha \notin \Gamma$, $\widetilde{v}$ is symmetric with respect to x_α on the whole interval I^α; if $\alpha \in \Gamma$, $\widetilde{v}$ is symmetric with respect to x_α just on $I^\alpha \setminus (I^{\alpha,1} \cup I^{\alpha,2})$.

Definition 15 (blocked potential)**.** Let $v \in \mathcal{X}(I)$, and assume $|I| \leq \pi$. The blocked potential of u on I is defined as the function $1 + \tan^2(x - x_I)$, with definition domain equal to the subset of I where $[v^\flat, I] = [v^*, I]$.

Definition 16 (stratified potential)**.** Let $v \in \mathcal{X}(I)$, and assume $|I| \leq \pi$. Let I^α be the family of open intervals constructed in Definition 13, labeled as in Remark 14. The stratified potential of v is the function V defined on I by glueing all the blocked potentials of v on the intervals I^α. Equivalently, for any $x \in I$, pick the longest multiindex α such that $x \in I^\alpha$, and set

$$V(x) = 1 + \tan^2\left(\frac{x - x_\alpha}{2}\right), \qquad x_\alpha := \text{midpoint of } I_\alpha.$$

Remark 17. With the notation introduced in Remark 14, the stratified potential V can be identified as follows: if $\alpha \notin \Gamma$, then $V(x) = 1 + \tan^2(x - x_\alpha)$ on the whole interval I^α; if $\alpha \in \Gamma$, then $V(x) = 1 + \tan^2(x - x_\alpha)$ just on $I^\alpha \setminus (I^{\alpha,1} \cup I^{\alpha,2})$.

Example 18. Let $v \in \mathcal{X}(I)$ be the function whose graph is represented in Figure 2. Since the intervals I^1 and $I^{1,2}$ contain a local minimum, while the intervals $I^{1,1}$, $I^{1,2,1}$, $I^{1,2,2}$ do not, for the stratified rearrangement $\tilde{v}$ and the stratified potential V we have that:
– on $I^1 \setminus (I^{1,1} \cup I^{1,2})$, $\tilde{v}$ is symmetric with respect to x_1 and $V(x) = 1 + \tan^2\left(\frac{x - x_1}{2}\right)$;
– on $I^{1,1}$, $\tilde{v}$ is symmetric with respect to $x_{1,1}$ and $V(x) = 1 + \tan^2\left(\frac{x - x_{1,1}}{2}\right)$;
– on $I^{1,2} \setminus (I^{1,2,1} \cup I^{1,2,2})$, $\tilde{v}$ is symmetric with respect to $x_{1,2}$ and $V(x) = 1 + \tan^2\left(\frac{x - x_{1,2}}{2}\right)$;
– on $I^{1,2,1}$, $\tilde{v}$ is symmetric with respect to $x_{1,2,1}$ and $V(x) = 1 + \tan^2\left(\frac{x - x_{1,2,1}}{2}\right)$;
– on $I^{1,2,2}$, $\tilde{v}$ is symmetric with respect to $x_{1,2,2}$ and $V(x) = 1 + \tan^2\left(\frac{x - x_{1,2,2}}{2}\right)$.

The next two lemmas, relying on the definitions of stratified rearrangement and potentials, provide the intermediate results needed for the proof of Proposition 10.

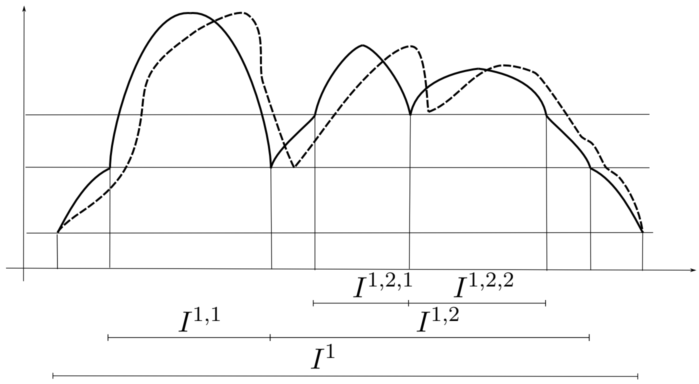

FIGURE 2. The stratified rearrangement given by Definition 13 (the graph of v is in dashed line, the graph of $\tilde{v}$ in continuous line).

Lemma 19. *Let $\psi \in \mathcal{A}(I_\pi)$ and let $v \in H^1_0(I_\pi) \cap \mathcal{X}(I_\pi)$. Denoting by $\tilde{v}$ and V respectively the stratified rearrangement and potential associated with v, it holds*

$$\int_{I_\pi} v^2(x)\, d\nu_{\psi'} \geq \int_{I_\pi} V(x)\tilde{v}^2(x)\, dx\,. \tag{21}$$

Proof. If $\mathrm{dom}(\psi)$ is strictly contained into I_π and $v \notin H^1_0(\mathrm{dom}(\psi))$, then the l.h.s. of (21) is $+\infty$, and the inequality is trivially true. So we only have to consider the situation when $v \in H^1_0(\mathrm{dom}(\psi))$. Denoting by $\nu_{\psi'}$ and ν_V the absolutely continuous measures with densities ψ' and V, and recalling (18), we have

$$\int_{I_\pi} v^2(x)\, d\nu_{\psi'} - \int_{I_\pi} V(x)\tilde{v}^2(x)\, dx = \int_0^{\|v\|_\infty} 2t \Big[\nu_{\psi'}\big(\{v > t\}\big) - \nu_V\big(\{\tilde{v} > t\}\big)\Big]\, dt$$

Let us distinguish two cases, according to whether the family of multi-indices Γ introduced in (20) is empty or not.
When $\Gamma = \emptyset$ (or equivalently, v does not have any local minimum in I_π), the set $\{v > t\}$ is an interval for every $t \in (0, \|u\|_\infty)$. Then (21) is satisfied because, by (19), it holds

$$\nu_{\psi'}\big(\{v > t\}\big) \geq 2\tan\Big(\frac{|\{v > t\}|}{2}\Big) = 2\tan\Big(\frac{|\{\tilde{v} > t\}|}{2}\Big) = \nu_V\big(\{\tilde{v} > t\}\big)\,.$$

When $\Gamma \neq \emptyset$, not all level sets $\{v > t\}$ are intervals. Therefore, in order to exploit the estimate (19), we decompose the set $\{v > t\}$ (and accordingly $\{\tilde{v} > t\}$) as a finite union of disjoint intervals $J_n(t)$ (and of their translations $\widetilde{J}_n(t)$)

$$\{v > t\} = \bigcup_{n=1}^{N(t)} J_n(t)\,, \qquad \{\tilde{v} > t\} = \bigcup_{n=1}^{N(t)} \widetilde{J}_n(t)\,.$$

We focus attention on a fixed interval $\widetilde{J}_n(t)$. Since we are working under the assumption that $\Gamma \neq \emptyset$, we have that $\widetilde{J}_n(t)$ is contained into some interval I^α with $\alpha \in \Gamma$. Among these intervals I^α containing $\widetilde{J}_n(t)$, we choose the one with multiindex $\overline{\alpha} = \overline{\alpha}(n,t)$ having the maximum number of components.
By applying the estimate (19) to the interval $J_n(t)$, we obtain

$$\nu_{\psi'}\big(J_n(t)\big) \geq 2\tan\Big(\frac{|J_n(t)|}{2}\Big) = 2\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big), \tag{22}$$

We observe that

$$\begin{aligned} 2\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big) &= 2\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) \\ &= 2\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},1}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},2}|}{2}\Big) + \mathcal{R}_{\overline{\alpha}}\,, \end{aligned}$$

where

$$\mathcal{R}_{\overline{\alpha}} := 2\tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1}|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},2}|}{2}\Big) \geq 0\,,$$

the last inequality being due to the super-additivity of the tangent function on $(0, \frac{\pi}{2})$.
In case the multiindices $(\overline{\alpha}, 1)$ and $(\overline{\alpha}, 2)$ do not belong to Γ, we have

$$\begin{aligned} &\nu_V\big(\widetilde{J}_n(t)\big) = \nu_V\big(\widetilde{J}_n(t) \setminus (I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2})\big) + \nu_V(I^{\overline{\alpha},1}) + \nu_V(I^{\overline{\alpha},2}) \\ &= 2\Big[\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big) - \tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big)\Big] + 2\tan\Big(\frac{|I^{\overline{\alpha},1}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},2}|}{2}\Big)\,. \end{aligned} \tag{23}$$

From (22) and (23) we obtain the estimate

$$\nu_{\psi'}\big(J_n(t)\big) - \nu_V\big(\widetilde{J}_n(t)\big) \geq \mathcal{R}_{\overline{\alpha}}\,.$$

Otherwise, if one or both the multiindices $(\overline{\alpha}, 1)$ and $(\overline{\alpha}, 2)$ belong to Γ, the left hand side of (23) does not correspond to the measure $\nu_V\big(\widetilde{J}_n(t)\big)$. Assume for definiteness that $(\overline{\alpha}, 1) \in \Gamma$ and $(\overline{\alpha}, 2) \notin \Gamma$. Then we split $I^{\overline{\alpha},1}$ as $I^{\overline{\alpha},1,1} \cup I^{\overline{\alpha},1,2}$, and we rewrite the left hand side of (23) as

$$\begin{aligned} &2\tan\Big(\tfrac{|\widetilde{J}_n(t)|}{2}\Big) - 2\tan\Big(\tfrac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) + 2\tan\Big(\tfrac{|I^{\overline{\alpha},1}|}{2}\Big) + 2\tan\Big(\tfrac{|I^{\overline{\alpha},2}|}{2}\Big) = \\ &2\tan\Big(\tfrac{|\widetilde{J}_n(t)|}{2}\Big) - 2\tan\Big(\tfrac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) + 2\tan\Big(\tfrac{|I^{\overline{\alpha},2}|}{2}\Big) \\ &+2\tan\Big(\tfrac{|I^{\overline{\alpha},1,1}|}{2}\Big) + 2\tan\Big(\tfrac{|I^{\overline{\alpha},1,2}|}{2}\Big) + \mathcal{R}_{\overline{\alpha},1}\,, \end{aligned} \tag{24}$$

with

$$\mathcal{R}_{\overline{\alpha},1} := 2\tan\Big(\frac{|I^{\overline{\alpha},1,1} \cup I^{\overline{\alpha},1,2}|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1,1}|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1,2}|}{2}\Big) \geq 0\,.$$

In case the multiindices $(\overline{\alpha}, 1, 1)$ and $(\overline{\alpha}, 1, 2)$ do not belong to Γ, we have

$$\begin{aligned} &2\tan\Big(\frac{|\widetilde{J}_n(t)|}{2}\Big) - 2\tan\Big(\frac{|I^{\overline{\alpha},1} \cup I^{\overline{\alpha},2}|}{2}\Big) \\ &+2\tan\Big(\frac{|I^{\overline{\alpha},2}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},1,1}|}{2}\Big) + 2\tan\Big(\frac{|I^{\overline{\alpha},1,2}|}{2}\Big) = \nu_V\big(\widetilde{J}_n(t)\big) \end{aligned}$$

Hence from (22) and (24) we obtain the estimate

$$\nu_{\psi'}\big(J_n(t)\big) - \nu_V\big(\widetilde{J}_n(t)\big) \geq \mathcal{R}_{\overline{\alpha}} + \mathcal{R}_{\overline{\alpha},1}\,.$$

Otherwise, we continue the procedure by splitting one or both the intervals $I^{\overline{\alpha},1,1}$ and $I^{\overline{\alpha},1,2}$. In a finite number of steps, we arrive at the conclusion that

$$\nu_{\psi'}\big(J_n(t)\big) - \nu_V\big(\widetilde{J}_n(t)\big) \geq \delta_n(t) := \sum_{\alpha} \mathcal{R}_\alpha \geq 0\,,$$

where the sum is extended to all indices $\alpha \in \Gamma$ of the form $(\overline{\alpha}(n,t),\dots)$. Then the inequality (21) is proved because each term $\mathcal{R}_\alpha$ is non-negative, and hence

$$\int_0^{\|v\|_\infty} 2t \sum_{n=1}^{N(t)} \delta_n(t)\,dt \geq 0\,.$$

□

Lemma 20. *Let V be the stratified potential associated with some function $v \in H^1_0(I_\pi)\cap \mathcal{X}(I_\pi)$, and let Γ denote the set of indices in* (20)*. For any $\alpha \in \Gamma$, set $I^{\alpha,1} \cup I^{\alpha,2} = (a_1^\alpha, a_2^\alpha) \cup (a_2^\alpha, a_3^\alpha)$, and consider the following eigenvalue problem with stratified potential*

$$\eta_1(I_\pi, 2V) := \inf\left\{ \frac{\int_{I_\pi} (\varphi')^2 + 2V\varphi^2}{\int_{I_\pi} \varphi^2} : \varphi \in H^1_0(I_\pi) \ s.t.,\ \forall \alpha \in \Gamma,\ \varphi(a_1^\alpha) = \varphi(a_2^\alpha) = \varphi(a_3^\alpha) \right\}.$$

Then it holds

$$\eta_1(I_\pi, 2V) \geq 4\,, \tag{25}$$

with equality if and only if $\Gamma = \emptyset$, so that $V(x) = 1 + \tan^2 x$ and an eigenfunction is $\cos^2 x$.

Proof. Throughout this proof, since we are going to work with different stratified potentials, in order to avoid any confusion we denote by Γ_V the family of multi-indices associated with a potential V according to (20). Moreover, we write for shortness that φ is an eigenfunction associated with V, if it is an eigenfunction for the eigenvalue problem $\eta_1(I_\pi, 2V)$. We argue by contradiction. Assume that the family $\mathcal{S}$ of stratified potentials such that $\eta_1(I_\pi, 2V) < 4$ is nonempty. We proceed in two steps.

Step 1. Since by assumption $\mathcal{S} \neq \emptyset$, we can select a stratified potential V for which the cardinality of Γ_V is minimal among potentials in $\mathcal{S}$. Clearly, since $V \in \mathcal{S}$, $\mathrm{card}(\Gamma_V) \geq 1$ (recall that, for $V(x) = 1 + \tan^2 x$, we have $\eta_1(I_\pi, 2V) = 4$, with eigenfunction $\cos^2 x$). By suitably perturbing V, we are going to find a potential V^ε such that

$$\eta_1(I_\pi, 2V^\varepsilon) = 4 \qquad \text{and} \qquad \mathrm{card}(\Gamma_{V^\varepsilon}) = \mathrm{card}(\Gamma_V)\,.$$

To that aim, let $-\frac{\pi}{2} = a_0 < a_1 < \dots < a_K = \frac{\pi}{2}$ denote an increasing relabelling of the family of points $\{a_i^\alpha,\ i = 1,2,3\}$ associated with V. In particular, for some $2 \leq j \leq K-2$ we have $I^{1,1} = (a_1, a_j)$, $I^{1,2} = (a_j, a_{K-1})$, and $a_{K-1} = -a_1$. We consider a one-parameter family $\{V^\varepsilon\}$ of continuous perturbations of V, given by stratified potentials such that $\mathrm{card}(\Gamma_V) = \mathrm{card}(\Gamma_{V^\varepsilon})$, in which in particular the points a_1 and a_{K-1} are replaced by $a_1 - \varepsilon$ and $a_{K-1} + \varepsilon$ (inside the interval $[a_1 - \varepsilon, a_{K-1} + \varepsilon]$, the potential V^ε can be built by rescaling the family of points a_i into the new family of points $a_i^\varepsilon = a_i \frac{a_{K-1}+\varepsilon}{a_{K-1}}$).

This operation can be carried over for $\varepsilon \in (0, \varepsilon^*)$, with $\varepsilon^* = \frac{\pi}{2} - a_{K-1}$. Clearly, there exists ε in such interval such that $\eta_1(I_\pi, 2V^\varepsilon) = 4$. Otherwise, it would be $\eta_1(I_\pi, 2V^{\varepsilon^*}) \leq 4$, which implies that an eigenfunction φ for V^{ε^*} must vanish at $a_1^{\varepsilon^*} = -\frac{\pi}{2}$, $a_{K-1}^{\varepsilon^*} = \frac{\pi}{2}$, and at $a_j^{\varepsilon^*}$. Then the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ would be disconnected into the union of the two disjoint intervals $I_*^{1,1} := (-\frac{\pi}{2}, a_j^{\varepsilon *})$ and $I_*^{1,2} := (a_j^{\varepsilon *}, \frac{\pi}{2})$. The restriction of the potential to one of the two intervals, say $I_*^{1,1}$, would give an eigenvalue less than or equal to 4. Since the length of the interval $I_*^{1,1}$ is strictly less than π, this proves the existence of a stratified potential $\widehat{V}$ on I_π with $\mathrm{card}(\Gamma_{\widehat{V}}) < \mathrm{card}(\Gamma_V)$ and $\eta_1(I_\pi, \widehat{V}) < 4$. (The stratified potential $\widehat{V}$ can be obtained by centering $I_*^{1,1}$ at the origin, and extending $\widehat{V}$ by setting it equal to $1 + \tan^2 x$ on its complement in I_π.)
We conclude that it is possible to freeze ε so that $\eta_1(I_\pi, 2V^\varepsilon) = 4$. The potential V^ε satisfies

$$\eta_1(I_\pi, 2V^\varepsilon) = 4 \qquad \text{and} \qquad \mathrm{card}(\Gamma_V) \geq \mathrm{card}(\Gamma_{V^\varepsilon}) \ \forall V \in \mathcal{S}\,. \tag{26}$$

Step 2. Let V^ε be a stratified potential satisfying (26). For simplicity of notation, in the remaining of the proof we drop the index ε and we denote it simply by V. Let φ be an eigenfunction for $\eta_1(I_\pi, 2V)$. By optimality, if $-\frac{\pi}{2} = a_0 < a_1 < \cdots < a_K = \frac{\pi}{2}$ denotes an increasing relabelling of the family of points $\{a_i^\alpha,\ i = 1, 2, 3\}$ associated with V, φ satisfies the system of equations

$$-\varphi'' + 2V\varphi = 4\varphi \qquad \text{on } (a_k, a_{k+1}) \qquad \forall k = 0, \dots K-1\,,$$

and the following global equality of the boundary terms, where $\varphi'_+(a_k)$, $\varphi'_-(a_{k+1})$ are the right and left derivatives of φ respectively at a_k and a_{k+1}:

$$\sum_{k=0}^{K-1} \big[\varphi(a_{k+1})\varphi'_-(a_{k+1}) - \varphi(a_k)\varphi'_+(a_k)\big] = 0\,, \tag{27}$$

We are now ready to reach a contradiction, by distinguishing the two cases $\mathrm{card}(\Gamma_V) = 1$ and $\mathrm{card}(\Gamma_V) > 1$.
Case a. If $\mathrm{card}(\Gamma_V) = 1$, a first eigenfunction for $\eta_1(I_\pi, 2V)$ is explicitly determined as

$$\varphi(x) = \begin{cases} C\dfrac{\cos^2 x}{\cos^2(a_1)} & \text{on } (a_0, a_1) \\ C\dfrac{\cos^2(x - (\frac{a_1+a_2}{2}))}{\cos^2(\frac{a_2-a_1}{2})} & \text{on } (a_1, a_2) \\ C\dfrac{\cos^2(x - (\frac{a_2+a_3}{2}))}{\cos^2(\frac{a_3-a_2}{2})} & \text{on } (a_2, a_3) \\ C\dfrac{\cos^2 x}{\cos^2(a_3)} & \text{on } (a_3, a_4)\,. \end{cases}$$

Then the sum in (27) can be written as

$$2C^2\Big[-\tan(a_1) - 2\tan\Big(\frac{a_2 - a_1}{2}\Big) - 2\tan\Big(\frac{a_3 - a_2}{2}\Big) - \tan(a_3)\Big]\,.$$

Using the elementary inequality

$$\tan(a_{k+1}) - \tan(a_k) > 2\tan\Big(\frac{a_{k+1}-a_k}{2}\Big) \qquad \text{for } a_{k+1} > a_k$$

we see that the equality (27) cannot hold, contradiction.

Case b. If $\mathrm{card}(\Gamma_V) > 1$, we consider the highest level among the local minima of φ, and denote by a_p the point where it is attained. Then there are two consecutive intervals, say (a_{p-1}, a_p) and (a_p, a_{p+1}) such that a first eigenfunction for $\eta_1(I_\pi, 2V)$ satisfies, for some positive constant C_p

$$\varphi(x) = \begin{cases} C_p \dfrac{\cos^2\Big(x - \big(\frac{a_{p-1}+a_p}{2}\big)\Big)}{\cos^2\big(\frac{a_p - a_{p-1}}{2}\big)} & \forall x \in (a_{p-1}, a_p) \\ C_p \dfrac{\cos^2\Big(x - \big(\frac{a_p+a_{p+1}}{2}\big)\Big)}{\cos^2\big(\frac{a_{p+1} - a_p}{2}\big)} & \forall x \in (a_p, a_{p+1})\,. \end{cases}$$

Since we are assuming that $\eta_1(I_\pi, 2V) = 4$, we have

$$\frac{\int_{I_\pi} (\varphi')^2 + 2V\varphi^2}{\int_{I_\pi} \varphi^2} = 4 \quad \text{and} \quad \sum_{k=0}^{K-1} \big[\varphi(a_{k+1})\varphi'_-(a_{k+1}) - \varphi(a_k)\varphi'_+(a_k)\big] = 0\,.$$

We then modify the potential V into a new potential $\overline{V}$ which differs from it uniquely on the interval (a_p, a_{p+1}) by setting

$$\overline{V}(x) = 1 + \tan^2\big(x - \frac{a_p + a_{p+1}}{2}\big) \qquad \forall x \in (a_p, a_{p+1})\,.$$

Accordingly, we modify the function φ uniquely on the interval (a_p, a_{p+1}) by setting

$$\overline{\varphi}(x) = C_p \frac{\cos^2\Big(x - \big(\frac{a_{p-1}+a_{p+1}}{2}\big)\Big)}{\cos^2\big(\frac{a_{p+1}-a_{p-1}}{2}\big)} \qquad \forall x \in (a_{p-1}, a_{p+1})\,.$$

Then, on each of the intervals associated with the potential $\overline{V}$, the function $\overline{\varphi}$ still satisfies the same PDE as v, namely we have

$$-\overline{\varphi}'' + 2\overline{V}\overline{\varphi} = 4\overline{\varphi} \qquad \text{on } (a_1, a_2) \cup \dots \cup (a_{p-1}, a_{p+1}) \cup \dots \cup (a_{K-1}, a_K)\,. \tag{28}$$

On the other hand, thanks to (27) and the super-additivity of the tangent function on $(0, \frac{\pi}{2})$, it holds

$$\begin{aligned} & \sum_{k=0}^{K-1} \big[\overline{\varphi}(a_{k+1})\overline{\varphi}'_-(a_{k+1}) - \overline{\varphi}(a_k)\overline{\varphi}'_+(a_k)\big] \\ = & \sum_{k=0}^{K-1} \big[\overline{\varphi}(a_{k+1})\overline{\varphi}'_-(a_{k+1}) - \overline{\varphi}(a_k)\overline{\varphi}'_-(a_k)\big] \\ & +2\tan\Big(\tfrac{a_p - a_{p-1}}{2}\Big) + 2\tan\Big(\tfrac{a_{p+1}-a_p}{2}\Big) - 2\tan\Big(\tfrac{a_{p+1}-a_{p-1}}{2}\Big) < 0\,. \end{aligned} \tag{29}$$

By combining (28) and (29), we infer that the Rayleigh quotient with potential $2\overline{V}$ of the function $\overline{\varphi}$ is strictly smaller than 4, i.e.

$$\frac{\int_{I_\pi} (\overline{\varphi}')^2 + 2\overline{V}\,\overline{\varphi}^2}{\int_{I_\pi} \overline{\varphi}^2} < 4\,.$$

We conclude that $\eta_1(I_\pi, 2\overline{V}) < 4$. Since $\mathrm{card}(\Gamma_{\overline{V}}) < \mathrm{card}(\Gamma_V)$, this contradicts condition (26) satisfied by V, and the proof of inequality (25) is achieved.

Concerning the equality case, assume that $\eta_1(I_\pi, 2V) = 4$ holds, and assume by contradiction that $\Gamma_V \geq 1$. If $\Gamma_V = 1$ by arguing as in Case a. above we obtain a contradiction; if $\Gamma_V > 1$, by arguing as in Case b. above we arrive to contradict, for another potential $\overline{V}$, the inequality $\eta_1(I, \overline{V}) \geq 4$ (that we have already proved). We conclude that $\Gamma_V = 0$, namely that $V(x) = 1 + \tan^2 x$.

□

Proof of Proposition 10. Let $\psi \in \mathcal{A}(I_\pi)$. Assume by contradiction that

$$\lambda_1(I_\pi, 2\psi') = \inf_{v \in H^1_0(I_\pi)} \frac{\int_{I_\pi} (v')^2\,dx + 2\int_{I_\pi} v^2\,d\nu_{\psi'}}{\int_{I_\pi} v^2} < 4\,.$$

By a density argument, we can find a function $v \in H^1_0(I_\pi) \cap \mathcal{X}(I_\pi)$ such that

$$\frac{\int_{I_\pi} (v')^2\,dx + 2\int_{I_\pi} v^2\,d\nu_{\psi'}}{\int_{I_\pi} v^2} < 4\,.$$

Let $\tilde{v}$ and V denote the stratified rearrangement and potential associated with v.

By exploiting, at each step of the construction of $\tilde{v}$, the well-known behaviour under decreasing rearrangement of the L^2-norm of a function and of its first derivative, we infer that

$$\int_{I_\pi} v^2 = \int_{I_\pi} \tilde{v}^2 \qquad \text{and} \qquad \int_{I_\pi} (v')^2 \geq \int_{I_\pi} (\tilde{v}')^2\,.$$

Then, by Lemma 19, we have

$$\frac{\int_{I_\pi} (\tilde{v}')^2 + 2V\tilde{v}^2}{\int_{I_\pi} \tilde{v}^2} < 4\,.$$

Therefore, for the stratified potential V, the eigenvalue $\eta_1(I_\pi, 2V)$ introduced in Lemma 20 would be strictly smaller than 4, contradicting such lemma. □

We now turn attention the problem of estimating from below $\lambda_1(I_d, q)$, where the weight is still of the form $q = \psi' + \psi^2$ for $\psi \in \mathcal{A}(I_\pi)$, and the finiteness domain of ψ is an interval $I_d = (-\frac{d}{2}, \frac{d}{2})$ with $d \leq \pi$.

Theorem 21. *Inequality* (11) *can be refined as follows:*

(i) *There exists an absolute constant $C > 0$ such that, for any $q = \psi' + \psi^2$, being $\psi \in \mathcal{A}(I_\pi)$ with* $\mathrm{dom}(\psi) = I_d$ *($d \leq \pi$), it holds*

$$\lambda_1(I_d, q) \geq 3 + C(\pi - d)^3\,. \tag{30}$$

(ii) *There exists an absolute constant K such that, for any $q = \psi' + \psi^2$, with $\psi \in \mathcal{A}(I_\pi)$ of the form $\psi = (f + \frac{g}{2})$, being $f \in \mathcal{A}(I_\pi)$ with $\mathrm{dom}(f) = I_d$ ($d \leq \pi$), and $g = -(\log h)'$ on I_d, for some $(\frac{1}{m})$-concave function h, affine on an interval $[a, b]$ with $I_{\frac{\pi}{4}} \subseteq [a, b] \subseteq I_d$, the following implication holds:*

$$\lambda_1(I_d, q) \leq 7 \ \Rightarrow\ \lambda_1(I_d, q) \geq 3 + \frac{8K}{m\pi^2}\Big[1 - \frac{\min\{h(a), h(b)\}}{\max\{h(a), h(b)\}}\Big]^2 . \tag{31}$$

Theorem 21 is obtained by an analogue strategy as Theorem 8, replacing the use of Propositions 9 and 10 by their refined versions stated respectively in Propositions 22 and 23 below. More precisely, one needs first to apply the inequality $\lambda_1(I_d, q) \geq \frac{1}{2}\big[\lambda_1(I_d, 2\psi') + \lambda_1(I_d, 2\psi^2)\big]$. Then: inequality (30) follows by using Proposition 10 to estimate $\lambda_1(I_d, 2\psi')$ and Proposition 22 below to estimate $\lambda_1(I_d, 2\psi^2)$; inequality (31) follows by using Proposition 9 to estimate $\lambda_1(I_d, 2\psi^2)$, and Propostion 23 below to estimate $\lambda_1(I_d, 2\psi')$.

Proposition 22. *There exists an absolute constant $C > 0$ such that, for any $\psi \in \mathcal{A}(I_\pi)$ with $\mathrm{dom}(\psi) = I_d$ ($d \leq \pi$), it holds*

$$\lambda_1(I_d, 2\psi^2) \geq 2 + C(\pi - d)^3 . \tag{32}$$

Proof. By following the proof of Proposition 9, we arrive at the inequality

$$\lambda_1(I_d, 2\psi^2) \geq \lambda_d := \min_{v \in H^1_0(I_d)} \frac{\int_{I_d} (v')^2 + 2(\tan^2 x)v^2}{\int_{I_d} v^2} .$$

Thus we are reduced to prove that there exists an absolute constant $C > 0$ such that $\lambda_d \geq 2 + C(\pi - d)^3$ for every $d \in (0, \pi]$. We observe that it is enough to prove that there exists $\overline{\varepsilon} > 0$ and an absolute constant $C > 0$ such that $\lambda_d \geq 2 + C(\pi - d)^3$ for every $d \in [\pi - \overline{\varepsilon}, \pi]$. Indeed in this case, since the map $d \mapsto \lambda_d$ is nonincreasing, for every $d \in (0, \pi - \overline{\varepsilon})$ we have

$$\lambda_d \geq \lambda_{\pi - \overline{\varepsilon}} \geq 2 + C\,\overline{\varepsilon}^3 = 2 + C'(\pi - d)^3 \qquad \text{with } C' = C\frac{\overline{\varepsilon}^3}{(\pi - d)^3} \geq C\frac{\overline{\varepsilon}^3}{\pi^3} ,$$

so that the required inequality is satisfied (for another absolute constant) also for $d \in (0, \pi - \overline{\varepsilon})$. Hence, in the remaining of the proof, we focus attention on the estimate of $\lambda_{\pi-\varepsilon}$, for ε sufficiently small. Denoting by $v_\varepsilon \in H^1_0(I_{\pi-\varepsilon})$ an eigenfunction for $\lambda_{\pi-\varepsilon}$, we have

$$-(\cos^2 x)'' + 2(\tan^2 x)(\cos^2 x) = 2\cos^2 x \qquad \text{in } I_\pi \tag{33}$$

$$-v_\varepsilon'' + 2(\tan^2 x)v_\varepsilon = \lambda_{\pi-\varepsilon} v_\varepsilon \qquad \text{in } I_{\pi-\varepsilon} . \tag{34}$$

We multiply (33) by v_ε (extended to 0 on $I_\pi \setminus I_{\pi-\varepsilon}$), and (34) by $\cos^2 x$, and we integrate, respectively, on I_π and on $I_{\pi-\varepsilon}$. We get:

$$\int_{I_\pi} (\cos^2 x)' v_\varepsilon' + 2\int_{I_\pi} (\sin^2 x) v_\varepsilon = 2\int_{I_\pi} (\cos^2 x) v_\varepsilon$$

$$\int_{I_{\pi-\varepsilon}} (\cos^2 x)' v_\varepsilon' - 2v_\varepsilon'\big(\frac{\pi - \varepsilon}{2}\big)\cos^2\big(\frac{\pi - \varepsilon}{2}\big) + \int_{I_{\pi-\varepsilon}} 2(\sin^2 x) v_\varepsilon = \lambda_{\pi-\varepsilon}\int_{I_{\pi-\varepsilon}} (\cos^2 x) v_\varepsilon .$$

By subtraction, we obtain

$$(\lambda_{\pi-\varepsilon}-2)\int_{I_{\pi-\varepsilon}}(\cos^2 x)v_\varepsilon = -2\big(\sin^2\frac{\varepsilon}{2}\big)v'_\varepsilon\big(\frac{\pi-\varepsilon}{2}\big)\,.$$

Since it is easily checked that v_ε converges weakly to $\cos^2 x$ in $H^1_0(I_\pi)$, we have that

$$\lim_{\varepsilon\to 0}\int_{I_{\pi-\varepsilon}}(\cos^2 x)v_\varepsilon = \int_{I_\pi}(\cos^4 x)\in(0,+\infty)\,.$$

Therefore, to prove (32), it is enough to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}v'_\varepsilon\big(\frac{\pi-\varepsilon}{2}\big)\in(0,+\infty)\,. \tag{35}$$

To that aim, we multiply equation (34) by $\frac{1}{\cos x}$, and we integrate on $I_{\pi-\varepsilon}$. We obtain

$$\int_{I_{\pi-\varepsilon}}\frac{\sin x}{\cos^2 x}v'_\varepsilon - 2\frac{v'_\varepsilon\big(\frac{\pi-\varepsilon}{2}\big)}{\cos\big(\frac{\pi-\varepsilon}{2}\big)} + 2\int_{I_{\pi-\varepsilon}}\frac{\sin^2 x}{\cos^3 x}v_\varepsilon = \lambda_{\pi-\varepsilon}\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x}$$

Since

$$\int_{I_{\pi-\varepsilon}}\frac{\sin x}{\cos^2 x}v'_\varepsilon = -\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x} - \int_{I_{\pi-\varepsilon}}2\frac{\sin^2 x}{\cos^3 x}v_\varepsilon\,,$$

we end up with

$$-2\frac{v'_\varepsilon\big(\frac{\pi-\varepsilon}{2}\big)}{\cos\big(\frac{\pi-\varepsilon}{2}\big)} = (\lambda_{\pi-\varepsilon}+1)\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x}\,. \tag{36}$$

Finally, we observe that

$$\lim_{\varepsilon\to 0}\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x}\in(0,+\infty)\,. \tag{37}$$

Indeed, since v_ε converges to $\cos^2 x$ a.e. on I_π, we have

$$\liminf_{\varepsilon\to 0}\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x}\geq\int_{I_\pi}\cos x>0\,.$$

On the other hand, by Hölder inequality we have

$$\limsup_{\varepsilon\to 0}\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon}{\cos x}\leq\pi^{\frac12}\limsup_{\varepsilon\to 0}\Big[\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon^2}{\cos^2 x}\Big]^{\frac12}<+\infty\,,$$

where the last inequality is obtained by observing that, normalizing v_ε in L^∞, and recalling from (34) that $\sup_\varepsilon\int_{I_{\pi-\varepsilon}}2(\tan^2 x)v_\varepsilon^2\leq\pi\sup_\varepsilon(\lambda_\varepsilon v_\varepsilon^2)<+\infty$, it holds

$$\begin{aligned}\int_{I_{\pi-\varepsilon}}\frac{v_\varepsilon^2}{\cos^2 x} &\leq \int_{I_{\pi-\varepsilon}\cap\{|x|\leq\frac{\pi}{4}\}}\frac{1}{\cos^2 x}+\int_{I_{\pi-\varepsilon}\cap\{|x|>\frac{\pi}{4}\}}(\tan^2 x)\frac{v_\varepsilon^2}{\sin^2 x}\\ &\leq \int_{I_{\pi-\varepsilon}\cap\{|x|\leq\frac{\pi}{4}\}}\frac{1}{\cos^2 x}+\int_{I_{\pi-\varepsilon}\cap\{|x|>\frac{\pi}{4}\}}2(\tan^2 x)v_\varepsilon^2\leq C\,.\end{aligned}$$

From (36) and (37), we see that (35) is satisfied, so that our proof is achieved. □

Proposition 23. *There exists an absolute constant K such that, for any $\psi \in \mathcal{A}(I_\pi)$ of the form $\psi = (f + \frac{g}{2})$, being $f \in \mathcal{A}(I_\pi)$ with $\mathrm{dom}(f) = I_d \subseteq I_\pi$, and $g = -(\log h)'$ on I_d, for some $(\frac{1}{m})$-concave function h, affine on an interval $[a, b]$ with $I_{\frac{\pi}{4}} \subseteq [a, b] \subseteq I_d$, the following implication holds:*

$$\lambda_1(I_d, 2\psi') \le 15 \;\Rightarrow\; \lambda_1(I_d, 2\psi') \ge 4 + \frac{16K}{m\pi^2}\Big[1 - \frac{\min\{h(a), h(b)\}}{\max\{h(a), h(b)\}}\Big]^2 . \tag{38}$$

Proof. We claim that, for ψ as in the assumptions, denoting by $\overline{v} \in H^1_0(I_d)$ an eigenfunction for $\lambda_1(I_d, 2\psi')$, normalized in $L^2(I_d)$, it holds

$$\lambda_1(I_d, 2\psi') \ge 4 + \delta_h \,, \qquad \text{with } \delta_h := \frac{1}{m}\int_{I_d} \overline{v}^2 \big(\frac{h'}{h}\big)^2 . \tag{39}$$

Indeed, we have

$$\lambda_1(I_d, 2\psi') = \int_{I_d} (\overline{v}')^2 + (2f' + g')\overline{v}^2 \ge 4 + \int_{I_d} g'\overline{v}^2 \,, \tag{40}$$

where the inequality follows by applying Proposition 10 to the function $f \in \mathcal{A}(I_\pi)$ (actually, $\overline{v}$ is an admissible test function for $\lambda_1(I_\pi, 2f')$ when extended to 0 on $I_\pi \setminus I_d$). Then the inequality (39) follows from (40) provided

$$g' \ge \frac{1}{m}\Big(\frac{h'}{h}\Big)^2 \qquad \text{on } I_d \,. \tag{41}$$

From the assumption $g = -(\log h)'$ on I_d, we have (in the sense of measures)

$$g' = \Big(\frac{h'}{h}\Big)^2 - \frac{h''}{h} \qquad \text{on } I_d \,. \tag{42}$$

The power-concavity assumption on h implies that $(h^{\frac{1}{m}})'' \le 0$. The latter inequality, by an elementary computation, implies that the right hand side of (42) is larger than or equal to the right hand side of (41). In view of (39), we have

$$\begin{aligned}
\lambda_1(I_d, 2\psi') \;&\ge 4 + \frac{1}{m}\Big[\frac{h(a) - h(b)}{a - b}\Big]^2 \frac{1}{[\max\{h(a), h(b)\}]^2}\int_{I_{\frac{\pi}{4}}} \overline{v}^2 \\
&\ge 4 + \frac{16}{m\pi^2}\Big[\frac{h(a) - h(b)}{\max\{h(a), h(b)\}}\Big]^2 \int_{I_{\frac{\pi}{4}}} \overline{v}^2 .
\end{aligned}$$

Hence, to conclude the proof of (38), it is enough to show that there exists an absolute constant $K > 0$ such that

$$\lambda_1(I_d, 2\psi') \le 15 \;\Rightarrow \int_{I_{\frac{\pi}{4}}} \overline{v}^2 \ge K .$$

Assume by contradiction this is false. Then it would be possible to find a sequence of functions ψ_n and a sequence of segments I_{d_n} as in the assumptions of the Lemma such that the eigenfunctions $\overline{v}_n \in H^1_0(I_{d_n})$ for $\lambda_1(I_{d_n}, 2\psi_n')$, normalized in $L^2(I_{d_n})$, satisfy

$$\int_{I_{\frac{\pi}{4}}} \overline{v}_n^2 \le \frac{1}{n} .$$

By the assumption $\lambda_1(I_{d_n}, 2\psi_n') \le 15$, we have $\int_{I_\pi} |\overline{v}_n'|^2 \le 15$, and hence up to subsequences $\overline{v}_n$ converges, weakly in $H^1(I_\pi)$ and strongly in $L^2(I_\pi)$, to a function $\overline{v}_\infty$ which has unit norm in $L^2(I_\pi)$ and vanishes on $I_{\frac{\pi}{4}}$. This leads to a contradiction, as

$$15 \ge \liminf_n \int_{I_\pi} |\overline{v}_n'|^2 \ge \int_{I_\pi} |\overline{v}_\infty'|^2 \ge \lambda_1(I_{\frac{\pi}{4}}) = 16\,.$$

□

3. Weighted equipartitions à la Payne-Weinberger

The key idea in the proof of Payne-Weinberger inequality in [45] is a partition procedure of the set Ω into convex cells of equal measure, such that, on each cell, the first Neumann eigenfunction has to have zero integral mean. While this procedure is still useful if adapted to a weighted Neumann problem, in order to obtain a quantitative estimate it is necessary to keep track of the L^2-norm of the eigenfunctions rather than of the measures of the cells. Consequently, we are going to work with two different types of p-weighted equipartitions, each one playing a specific role in the estimate of weighted Neumann eigenvalues in higher dimensions. Such estimate will be based the one-dimensional lower bounds given in Theorem 21. Thus we are going to handle line segments contained into $\mathbb{R}^N$: *for simplicity, in this section and in the remaining of the paper, the notation I_ℓ is adopted for any line segment of length ℓ in $\mathbb{R}^N$, say with generic direction and not necessarily centred at the origin (as it was the case in Section 2). In the few cases when we have to consider a centred interval in a fixed frame, this will be explicitly indicated, by writing e.g. $(-\frac{\ell}{2}, \frac{\ell}{2}) \times \{0\}$.*

Definition 24. Given an open bounded convex set $\omega \subset \mathbb{R}^N$, a positive weight $p \in L^1(\omega)$, and a function $u \in L^2(\omega, p\,dx)$ satisfying $\int_\omega up = 0$, we call:

- a *p-weighted measure equipartition of u in ω* a family $\mathcal{P}_n = \{\omega_1, \dots, \omega_n\}$, where ω_i are mutually disjoint convex sets such that $\omega = \omega_1 \cup \dots \cup \omega_n$ and
$$\int_{\omega_i} up = 0 \qquad \text{and} \qquad |\omega_i| = \frac{1}{n}|\omega| \qquad \forall i = 1, \dots, n\,.$$
- a *p-weighted L^2 equipartition of u in ω* a family $\mathcal{P}_n = \{\omega_1, \dots, \omega_n\}$, where ω_i are mutually disjoint convex sets such that $\omega = \omega_1 \cup \dots \cup \omega_n$ and
$$\int_{\omega_i} up = 0 \qquad \text{and} \qquad \int_{\omega_i} u^2 p = \frac{1}{n}\int_\omega u^2 p \qquad \forall i = 1, \dots, n\,.$$

Remark 25. (i) The existence of p-weighted L^2 (or measure) equipartitions of u in ω given by two cells is obtained by the analogue argument as in [45] (see also [7]). Namely, for every $\alpha \in [0, 2\pi]$, there exists a unique hyperplane with normal $(\cos\alpha_1, \sin\alpha_1, 0, \dots, 0)$ which divides ω into two subsets ω_α' and ω_α'' such that $\int_{\omega_\alpha'} u^2 p = \int_{\omega_\alpha''} u^2 p$. Since the function $\mathcal{I}(\alpha) = \int_{\omega_\alpha'} up$ is continuous and satisfies $\mathcal{I}(\alpha) = -\mathcal{I}(\alpha + \pi)$, there exists an angle $\overline{\alpha}$ such that $\mathcal{I}(\overline{\alpha}) = 0$. Applying repeatedly the above argument yields the existence of a p-weighted L^2 equipartition of u in ω given by n cells, each of them being contained into a narrow strip, determined by two hyperplanes with normal of the form

$(\cos\alpha_1, \sin\alpha_1, 0, \dots, 0)$ at infinitesimal distance from each other as $n \to +\infty$. The latter property follows from the fact that the volume of all the elements of the partition is infinitesimal as $n \to +\infty$, thanks to the assumption $p > 0$ a.e. in ω.
(ii) If the above procedure is repeated overall $(N-2)$ times, using as a last package of cutting hyperplanes those with normals of the form $(0, \dots, 0, \cos\alpha_{N-2}, \sin\alpha_{N-2}, 0)$, we obtain a p-weighted L^2 equipartition of u in ω into mutually disjoint convex cells which are narrow in $N-2$ directions, each one orthogonal to e_N. If the procedure is repeated once more, we arrive at a p-weighted L^2 equipartition of ω into mutually disjoint convex cells of one dimensional type, being narrow in $(N-1)$ orthogonal directions.

Motivated by the above remark, we state the following single-cell estimate, holding for a set $\omega_\varepsilon \subset \omega$ which is narrow in $(N-1)$ orthogonal directions.

Lemma 26. *Let $\omega \subset \mathbb{R}^N$ be an open bounded convex set, and let p be a positive uniformly continuous weight defined in ω. Given $\varepsilon > 0$, let $\omega_\varepsilon \subset \omega$ be an open bounded convex set of diameter d_ε, which satisfies, in a suitable orthogonal coordinates system and for some $\varepsilon > 0$, the inclusion*

$$\omega_\varepsilon \subset \Big\{(x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ |x_1| \leq \frac{d_\varepsilon}{2}, \ |y_j| \leq \varepsilon \ \forall j = 1, \dots, N-1\Big\} . \tag{43}$$

For every function $u \in W^{2,\infty}(\omega)$ whose restriction to ω_ε is an admissible test function for $\mu_1(\omega_\varepsilon, p)$, setting $h(x) := \mathcal{H}^{N-1}(\omega_\varepsilon \cap \{x_1 = x\})$, it holds

$$\frac{\int_{\omega_\varepsilon} |\nabla u|^2 p}{\int_{\omega_\varepsilon} |u|^2 p} \geq \Big\{\mu_1(I_{d_\varepsilon}, hp) - \frac{\alpha(\varepsilon)|\omega_\varepsilon|}{\int_{\omega_\varepsilon} |u|^2 p}\Big[1 + \mu_1(I_{d_\varepsilon}, hp)\Big(1 + \beta(\varepsilon)|\omega_\varepsilon|\Big)\Big]\Big\} , \tag{44}$$

where $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are infinitesimal as $\varepsilon \to 0$, depending only from $\|u\|_{W^{2,\infty}(\omega)}$, $\|p\|_{L^\infty(\omega)}$, and from the modulus of continuity of p at ε in ω.

Proof. Let M be a positive constant such that $\|u\|_{W^{2,\infty}(\omega)} \leq M$ and $\|p\|_{L^\infty(\omega)} \leq M$, and let $\delta_\varepsilon > 0$ be such that $|p(x) - p(y)| < \varepsilon$ for every $x, y \in \omega$ with $|x - y| < \delta_\varepsilon$.
We have

$$\Big|\int_{\omega_\varepsilon} \Big(\frac{\partial u}{\partial x_1}\Big)^2 p - \int_{I_{d_\varepsilon}} [u(x,0)']^2 hp\, dx\Big| \leq (2M^3\varepsilon + M^2\delta_\varepsilon)|\omega_\varepsilon| =: \delta_1$$

$$\Big|\int_{\omega_\varepsilon} u^2 p - \int_{I_{d_\varepsilon}} u(x,0)^2 hp\, dx\Big| \leq (2M^3\varepsilon + M^2\delta_\varepsilon)|\omega_\varepsilon| =: \delta_2 (= \delta_1)$$

$$\Big|\int_{I_{d_\varepsilon}} u(x,0) hp\, dx\Big| = \Big|\int_{\omega_\varepsilon} up - \int_{I_{d_\varepsilon}} u(x,0) hp\, dx\Big| \leq (M^2\varepsilon + M\delta_\varepsilon)|\omega_\varepsilon| =: \delta_3 .$$

Then, setting $\overline{u} := \int_{I_{d_\varepsilon}} u(x,0)hp$, we have

$$\begin{aligned}
\int_{\omega_\varepsilon} |\nabla u|^2 p \ &\geq \int_{\omega_\varepsilon} \Big(\frac{\partial u}{\partial x_1}\Big)^2 p \geq \int_{I_{d_\varepsilon}} [u(x,0)']^2 hp\,dx - \delta_1 \\
&\geq \mu_1(I_{d_\varepsilon}, hp)\Big[\int_{I_{d_\varepsilon}} u(x,0)^2 hp - \overline{u}^2\Big] - \delta_1 \\
&\geq \mu_1(I_{d_\varepsilon}, hp)\Big[\int_{\omega_\varepsilon} u^2 p\,dx - \delta_2 - \delta_3^2\Big] - \delta_1 \\
&= \mu_1(I_{d_\varepsilon}, hp)\Big[\int_{\omega_\varepsilon} u^2 p\,dx\Big] - \delta_1\Big[1 + \mu_1(I_{d_\varepsilon}, hp)\Big(1 + \frac{\delta_3^2}{\delta_1}\Big)\Big] .
\end{aligned}$$

The result follows by inserting the expressions of δ_1 and δ_3 in the above estimate. □

The next two lemmas contain lower bounds of different nature for $\mu_1(\omega, p)$ (or, more generally, for Rayleigh quotients). The first lower bound, stated in Lemma 27, is given in terms of one dimensional eigenvalues: it is in fact obtained working with measure equipartitions and applying the single cell estimate of Lemma 26. The second lower bound, stated in Lemma 28, is given in terms of the average of the eigenvalues of the cells of the partition, and is obtained working with L^2 equipartitions.

Lemma 27. *Let ω be an open bounded convex set, and let p be a positive uniformly continuous weight in ω. Let $u \in W^{2,\infty}(\omega)$ satisfy $\int_\omega up = 0$. Assume that, for every n sufficiently large, there exists a p-weighted measure equipartition $\mathcal{P}_n = \{\omega_1, \dots, \omega_n\}$ of u in ω such that $\mu_1(I_{d_i}, h_i p) \geq c$ for every $i = 1, \dots, n$, where I_{d_i} is a diameter for ω_i, and $h_i(x)$ is the $\mathcal{H}^{N-1}$-measure of the sections of ω_i as in Lemma 26. Then*

$$\frac{\int_\omega |\nabla u|^2 p}{\int_\omega |u|^2 p} \geq c\,.$$

In particular, in case ω is smooth and p is smooth and strictly positive in ω, taking u equal to a first eigenfunction for $\mu_1(\omega, p)$, we obtain that $\mu_1(\omega, p) \geq c$.

Proof. For a given $\varepsilon > 0$, for n large enough each of the sets ω_i satisfies in a suitable orthogonal coordinates system the inclusion (43) (cf. Remark 25). Moreover, the restriction of u to ω_i is an admissible test function for $\mu_1(\omega_i, p)$. Then by Lemma 26 the inequality (44) is fulfilled for every $i = 1, \dots, n$, for some infinitesimal $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ which are independent of i. (Indeed, as stated in Lemma 26, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ depend only on $\|u\|_{W^{2,\infty}(\omega)}$, $\|p\|_{L^\infty(\omega)}$, and on the the modulus of continuity of p at ε in ω).
Writing the inequality (44) for each of the sets ω_i, and recalling that $|\omega_i| = \frac{|\omega|}{n}$, we obtain

$$\int_{\omega_i} |u|^2 p \leq \frac{1}{\mu_1(I_{d_i}, h_i p)} \int_{\omega_i} |\nabla u|^2 p + \frac{1}{\mu_1(I_{d_i}, h_i p)} \Big(\alpha(\varepsilon)\frac{|\omega|}{n}\Big) + \alpha(\varepsilon)\frac{|\omega|}{n}\Big(1 + \beta(\varepsilon)\frac{|\omega|}{n}\Big) .$$

By using the assumption $\mu_1(I_{d_i}, h_i p) \geq c$ for every $i = 1, \dots, n$, and summing over $i = 1, \dots, n$, we get

$$\int_\omega |u|^2 p \leq \frac{1}{c}\int_\omega |\nabla u|^2 p + \frac{1}{c}\Big(\alpha(\varepsilon)|\omega|\Big) + \alpha(\varepsilon)\Big(1 + \beta(\varepsilon)|\omega|\Big) .$$

The statement follows by letting ε tend to 0. □

Lemma 28. *Let ω be an open bounded convex set of diameter d and let p be a positive weight in $L^1(\omega)$. Let $u \in H^1_{\rm loc}(\Omega) \cap L^2(\Omega, pdx)$ satisfy $\int_\Omega up\,dx = 0$. If $\mathcal{P}_n = \{\omega_1, \dots, \omega_n\}$ is a p-weighted L^2 equipartition of u in ω, it holds*

$$\frac{\int_\omega |\nabla u|^2 p}{\int_\omega |u|^2 p} = \frac{1}{n} \sum_{i=1}^n \frac{\int_{\omega_i} |\nabla u|^2 p}{\int_{\omega_i} |u|^2 p} \geq \frac{1}{n} \sum_{i=1}^n \mu_1(\omega_i, p)\,.$$

In particular, if u is an eigenfunction for $\mu_1(\omega, p)$, we have $\mu_1(\omega, p) \geq \frac{1}{n} \sum_{i=1}^n \mu_1(\omega_i, p)$ and, in case $\mu_1(\omega, p) = \mu_1(\omega_i, p)$ for every $i = 1, \dots, n$, u is necessarily an eigenfunction also for each $\mu_1(\omega_i, p)$.

Proof. The statement is an immediate consequence of the two facts that $\int_{\omega_i} u^2 p = \frac{1}{n} \int_\omega u^2 p$ and the restriction of u to ω_i is admissible as a test function for $\mu_1(\omega_1, p)$. In case u is an eigenfunction for $\mu_1(\omega, p)$, and $\mu_1(\omega, p) = \mu_1(\omega_i, p)$ for every $i = 1, \dots, n$, we see that none of the inequalities $\int_{\omega_i} |\nabla u|^2 p \geq \mu_1(\omega_i, p) \int_{\omega_i} |u|^2 p$ can be strict. ☐

4. Localized variational version and rigidity of Andrews-Clutterbuck inequality

In this section we establish two intermediate results which will be used in the proof of our quantitative inequality, but may have their own interest. In Proposition 29 we give a localized variational version of Andrews-Clutterbuck inequality, which consists in estimating the weighted Neumann eigenvalue $\mu_1(\omega, p)$ when $\omega \subseteq \Omega$ is a convex set (with possibly empty interior), and the weight p is associated with the first eigenfunction of Ω.

In Theorem 31 we establish the rigidity of Andrews-Clutterbuck inequality. Both results are proved exploiting the weighted equipartitions introduced in the previous section: the former is obtained via measure equipartitions, the latter via L^2 equipartitions.

For simplicity, in this section and in the remaining of the the paper, when no ambiguity arises, we omit to indicate the set Ω by writing D in place of D_Ω for its diameter (and similarly for the width).

Proposition 29. *Let $N \geq 2$. There exists an absolute constant $C > 0$ such that: if $\Omega \subset \mathbb{R}^N$ is an open bounded convex set of diameter D, Π is an affine subspace of $\mathbb{R}^N$ of dimension $k \leq N$, $\omega \subseteq (\Omega \cap \Pi)$ is a relatively open convex subset of $\Omega \cap \Pi$ of diameter $d > 0$, and $p \in L^\infty(\omega)$ is a positive weight of the form $p = hu_1^2$, being u_1 the first Dirichlet eigenfunction of Ω, and $h : \omega \to (0, +\infty)$ a $(\frac{1}{m})$-concave function (with $m \in \mathbb{N} \setminus \{0\}$), it holds:*

$$\mu_1(\omega, p) \geq \frac{3\pi^2}{D^2} + C \frac{(D-d)^3}{D^5}\,. \tag{45}$$

Proof. We first prove the inequality (45) in case ω is a smooth convex set, with $\overline{\omega} \subset \Omega$, and h is a smooth strictly positive $(\frac{1}{m})$-concave function in $\overline{\omega}$. In this case, the weight p is smooth and strictly positive in $\overline{\omega}$. Therefore, there exists a first eigenfunction for $\mu_1(\omega, p)$, of class $W^{2,\infty}(\omega)$. Then we are in a position to apply Lemma 27. Specifically, we consider a p-weighted measure equipartition $\mathcal{P}_n = \{\omega_1, \dots, \omega_n\}$ of a first eigenfunction for $\mu_1(\omega, p)$. We denote by I_{d_i} a diameter for ω_i, and by $h_i(x)$ the $\mathcal{H}^{k-1}$-measure of the sections of ω_i orthogonal to I_{d_i} as in Lemma 26. We apply Theorem 21 in order to estimate the one-dimensional eigenvalue $\mu_1(I_{d_i}, h_i p)$. To that purpose we observe that p satisfies

$$\Big(\nabla \log p^{\frac12}(y) - \nabla \log p^{\frac12}(x)\Big) \cdot \frac{y-x}{\|y-x\|} \leq -2\frac{\pi}{D} \tan\Big(\frac{\pi}{D}\frac{\|y-x\|}{2}\Big) \qquad \forall x, y \in \omega\,.$$

Therefore, each weight $h_i p$ is of the form $\psi_i' + \psi_i^2$ for the function $\psi_i \in \mathcal{A}(I_\pi)$ given by $\psi_i = -\frac12((\log p)' + (\log h_i)')$. By Lemma 6 and the inequality (30) in Theorem 21, we have

$$\mu_1(I_{d_i}, h_i p) \geq \lambda_1(I_{d_i}, \psi_i' + \psi_i^2) \geq \frac{3\pi^2}{D^2} + \frac{C}{D^2}\Big(\frac{D-d_i}{D}\Big)^3 \geq \frac{3\pi^2}{D^2} + \frac{C}{D^2}\Big(\frac{D-d}{D}\Big)^3\,.$$

By applying Lemma 27 to an eigenfunction for $\mu_1(\omega, p)$, the above inequality yields (45).

We now prove the inequality (45) for ω and p as in the statement. Let ω_n be an increasing sequence of smooth open bounded convex sets, compactly contained into ω, which converges to ω in Hausdorff distance as $n \to +\infty$. Since h is power-concave, we can approximate it by a sequence of smooth strictly positive log-concave functions h_n supported in $\overline{\omega}_n$, such that $h_n \leq h$. Then the sequence of weights $p_n := h_n u_1^2$ (where h_n are formally extended to zero on $\omega \setminus \omega_n$), converges weakly* to p in $L^\infty(\omega)$. Since we have already shown that (45) holds true for $\mu_1(\omega_n, p_n)$, to conclude our proof it is enough to observe that

$$\mu_1(\omega, p) \geq \limsup_n \mu_1(\omega_n, p_n)\,. \tag{46}$$

The argument to obtain the above inequality is similar as the one used in the proof of Lemma 6. Namely, for every $\varepsilon > 0$, we consider a function $u_\varepsilon \in H^1_{\rm loc}(\omega) \cap L^2(\omega, p\,dx)$ with $\int_\omega u_\varepsilon p = 0$, such that

$$\mu_1(\omega, p) \geq \frac{\int_\omega |\nabla u_\varepsilon|^2 p}{\int_\omega |u_\varepsilon|^2 p} - \varepsilon\,;$$

setting $c_{\varepsilon,n} := \frac{1}{|\omega_n|} u_\varepsilon p_n$, we have $\lim_n c_{\varepsilon,n} = 0$. Hence, taking $u_{\varepsilon,n} := u_\varepsilon - c_{\varepsilon,n}$ as a test function for $\mu_1(\omega_n, p_n)$, we obtain

$$\mu_1(\omega, p) \geq \frac{\int_\omega |\nabla u_\varepsilon|^2 p}{\int_\omega |u_\varepsilon|^2 p} - \varepsilon = \lim_n \frac{\int_{\omega_n} |\nabla u_{\varepsilon,n}|^2 p_n}{\int_{\omega_n} |u_{\varepsilon,n}|^2 p_n} - \varepsilon \geq \limsup_n \mu_1(\omega_n, p_n) - \varepsilon\,.$$

The inequality (46) follows by the arbitrariness of $\varepsilon > 0$. $\square$

Remark 30. Although not exploited in the sequel, let us point out that a consequence of the localized inequality (45), written in terms of the first two eigenfunctions u_1, u_2, is that, for any segment $S \subset \Omega$ such that $\int_S u_1 u_2 = 0$, it holds

$$\frac{\displaystyle\int_S \Big(u_2' - \frac{u_1'}{u_1} u_2\Big)^2}{\displaystyle\int_S u_2^2} \geq \frac{3\pi^2}{D^2} + C\frac{(D-|S|)^3}{D^5},$$

where the derivatives u_1', u_2' are taken in the direction of S.

Theorem 31 (Rigidity)**.** *Let $N \geq 2$. For every open bounded convex domain Ω in $\mathbb{R}^N$ of diameter D, we have*

$$\lambda_2(\Omega) - \lambda_1(\Omega) > \frac{3\pi^2}{D^2} \,.$$

Proof. Assume by contradiction that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an open bounded convex domain with diameter D such that $\mu_1(\Omega, u_1^2) = \frac{3\pi^2}{D^2}$ We distinguish the case of dimension $N = 2$ from the case of arbitrary dimension.

Case $N = 2$. For every $n \geq 1$, let $\mathcal{P}_n = \{\Omega_1, \dots, \Omega_n\}$ be a u_1^2-weighted L^2 equipartition of $\overline{u}$ in Ω, being $\overline{u} := \frac{u_2}{u_1}$ an eigenfunction for $\mu_1(\Omega, u_1^2)$ (cf. Proposition 41). Denoting by D_i the diameter of Ω_i, by Lemma 28 and Proposition 29 (applied with $p = u_1^2$), we have

$$\frac{3\pi^2}{D^2} = \mu_1(\Omega, u_1^2) \geq \frac{1}{n} \sum_{i=1}^n \mu_1(\Omega_i, u_1^2) \geq \frac{3\pi^2}{D^2} + C \sum_{i=1}^n \frac{(D - D_i)^3}{D^5} \,,$$

and $\overline{u}$ is an eigenfunction for each $\mu_1(\Omega_i, u_1^2)$.
We infer that

$$D = D_i \qquad \forall i = 1, \dots, n \,, \tag{47}$$

$$\mu_1(\Omega_i, u_1^2) = \tfrac{3\pi^2}{D^2} \qquad \forall i = 1, \dots, n \,. \tag{48}$$

Moreover, by the last assertion in Lemma 28, $\overline{u}$ is an eigenfunction for each $\mu_1(\Omega_i, u_1^2)$. We claim that Ω is a circular sector (and hence Ω_i are circular subsectors of the same diameter, the supremum of whose opening angles is infinitesimal as $n \to +\infty$). Indeed, consider a straight line r which determines the u_1^2-weighted L^2 equipartition $\mathcal{P}_2 = \{\Omega_1, \Omega_2\}$. Necessarily, $r \cap \Omega$ must be a diametral segment of Ω. Indeed, let I_D be a fixed diametral segment of Ω: if $I_D \neq (r \cap \Omega)$, either $I_D \cap (r \cap \Omega) = \emptyset$, or I_D is transverse to $r \cap \Omega$. In the first case, assume with no loss of generality that I_D is contained into Ω_1. Then, since also Ω_2 has diameter D, the set Ω would contain two distinct diametral segments, which is not possible since a diagonal of the quadrilateral having vertices at the endpoints of the two diametral segments would have length strictly larger than D. In the second case, since I_D is not entirely contained neither in Ω_1, nor in Ω_2, each among Ω_1 and Ω_2 would contain a segment of length D, and again this would imply the existence of a segment of length strictly larger than D in Ω. By repeating the same argument for the successive straight lines which determine the u_1^2-weighted L^2 equipartition $\mathcal{P}_n = \{\Omega_1, \dots, \Omega_n\}$ we infer that each cutting line is a diametral segment. Since, as already noticed above, there cannot be two disjoint diametral segments in Ω, we conclude that Ω contains, for every $n \geq 1$, n diametral segments having a common endpoint. By convexity, we infer that Ω contains their convex envelope, and, by passing to the limit as $n \to +\infty$, we conclude that Ω contains a circular sector. Actually, Ω must coincide with such circular sector, because any point outside the sector cannot lie into any cell of $\mathcal{P}_n$.
Once we know that Ω is a circular sector and that any cell of $\mathcal{P}_n$ a circular subsector, we exploit the fact that $\overline{u} := \frac{u_2}{u_1}$ is an eigenfunction for each $\mu_1(\Omega_i, u_1^2)$. Since this holds true for every n, we infer that the function $\overline{u}$ satisfies, on every radius of the circular sector Ω, the Neumann condition $\frac{\partial \overline{u}}{\partial \nu} = 0$, being ν the normal direction to the radius. This implies that the function $\overline{u}$ is radial on Ω. But the expression of u_2 and u_1 (which

is explicitly known for a circular sector, see for instance [23, Section 3.2]) shows that this is not the case. So we have reached a contradiction.

Step 2 (N arbitrary). We consider again a u_1^2-weighted L^2 equipartition $\mathcal{P}_n = \{\Omega_1, \dots, \Omega_n\}$ of $\overline{u}$ in Ω. By the same arguments used in case $N = 2$, all the cells of the partition satisfy (47)-(48). Moreover, by arguing as in Remark 25, we can assume that, for each each cell $\Omega_i \in \mathcal{P}_n$, it holds (in a suitable orthogonal coordinates system associated with the cell)

$$\text{(49)}\quad \Omega_i \subset \Big\{(y, x) \in \mathbb{R}^{N-2} \times \mathbb{R}^2 \ : \ |y_j| \leq \varepsilon_n \ \forall j = 1, \dots, N-2; \ |x_j| \leq \frac{D}{2} \ \forall j = 1, 2\Big\},$$

with $\varepsilon_n \to 0$ as $n \to +\infty$. This implies that, in the limit as $n \to +\infty$, any sequence of cells $\Omega_{k(n)} \in \mathcal{P}_n$ converges, up to a subsequence, to a degenerate convex set, which may be either one-dimensional or two-dimensional. Any such limit set has diameter D, because all the cells have diameter D. This allows to exclude that all the limit sets are one-dimensional. Otherwise, by fixing two distinct segments S' and S'' contained in Ω, both parallel to e_N, and considering the sequences of domains $\Omega_{k'(n)} \in \mathcal{P}_n$ and $\Omega_{k''(n)} \in \mathcal{P}_n$ which contain them, in the limit as $n \to +\infty$ we would find two parallel diametral segments contained in Ω. Let $\Omega_{k(n)} \in \mathcal{P}_n$ be a sequence which converges to a two-dimensional convex set Ω_0. We claim that, for a suitable $(\frac{1}{N-2})$-concave function h, it holds

$$\text{(50)}\qquad \liminf_{n\to+\infty} \frac{\int_{\Omega_{k(n)}} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_{k(n)}} |\overline{u}|^2 u_1^2} \geq \frac{\int_{\Omega_0} |\nabla \overline{u}|^2 h u_1^2}{\int_{\Omega_0} |\overline{u}|^2 h u_1^2}.$$

Indeed, in a coordinate system such that $\Omega_{k(n)}$ satisfies (49), with the change of variables $T_n : \mathbb{R}^{N-2} \times \mathbb{R}^2 \to \mathbb{R}^{N-2} \times \mathbb{R}^2$ defined by $T_n(y, x) = (\varepsilon_n x', x)$, we have

$$\frac{\int_{\Omega_{k(n)}} |\nabla \overline{u}|^2(y, x) u_1^2(y, x)}{\int_{\Omega_{k(n)}} |\overline{u}(y, x)|^2 u_1^2(y, x)} = \frac{\int_{T_n^{-1}(\Omega_{k(n)})} |\nabla \overline{u}|^2(\varepsilon_n x', x) u_1^2(\varepsilon_n x', x)}{\int_{T_n^{-1}(\Omega_{k(n)})} |u|^2(\varepsilon_n x', x) u_1^2(\varepsilon_n x', x)}.$$

We now pass to the limit as $n \to +\infty$: by using Fatou's lemma and denoting by $h(x)$ the $\mathcal{H}^{N-2}$-measure of the slices of the limit set of $T_n^{-1}(\Omega_{k(n)})$ at fixed $x = (x_1, x_2)$, we obtain

$$\liminf_{n\to+\infty} \int_{\Omega_{k(n)}} |\nabla \overline{u}|^2(y, x) u_1^2(y, x) \geq \int_{\Omega_0} |\nabla \overline{u}|^2(0, x) h(x) u_1^2(0, x);$$

on the other hand, recalling that $\overline{u} = \frac{u_2}{u_1}$, by dominated convergence we get

$$\lim_{n\to+\infty} \int_{T_n^{-1}(\Omega_{k(n)})} \overline{u}^2(\varepsilon_n x', x) u_1^2(\varepsilon_n x', x) = \int_{\Omega_0} |\overline{u}|^2(0, x) h(x) u_1^2(0, x).$$

Therefore, we have

$$\text{(51)}\qquad \frac{3\pi^2}{D^2} = \liminf_n \frac{\int_{\Omega_{k(n)}} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_{k(n)}} |\overline{u}|^2 u_1^2} \geq \frac{\int_{\Omega_0} |\nabla \overline{u}|^2 h u_1^2}{\int_{\Omega_0} |\overline{u}|^2 h u_1^2} \geq \mu_1(\Omega_0, h u_1^2) \geq \frac{3\pi^2}{D^2},$$

where the first equality holds by (48), the second inequality holds by (50), the third inequality holds because $\overline{u}(x, 0)$ is an admissible test function for $\mu_1(\Omega_0, u_1^2 h)$, and the fourth inequality holds by Proposition 29 (applied with $\omega = \Omega_0$ and $p = h u_1^2$).

We now focus on the two-dimensional convex set Ω_0. Since we know from (51) that $\mu_1(\Omega_0, hu_1^2) = \frac{3\pi^2}{D^2}$, with eigenfunction $\overline{u}$, we can repeat the same arguments as in Step 1 of the proof, with the weight u_1^2 replaced by hu_1^2. (More precisely, the arguments of Step 1 are repeated working with (hu_1^2)-weighted L^2 equipartitions of $\overline{u}$ in Ω_0, and applying Proposition 29 with $p = hu_1^2$).
By this way, we obtain that Ω_0 is a circular sector and, as well, all the cells of any (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in Ω_0 are circular sectors Ω_0^i, with $\mu_1(\Omega_0^i, hu_1^2) = \frac{3\pi^2}{D^2}$.
Let us fix a radius in Ω_0 which does not belong to $\partial\Omega_0$, and look at a sequence of sectors Ω_0^i, of infinitesimal opening angle, having such radius in their closure. On this radius, hereafter denote by I_D, in the limit of a large number of cells, by arguing as in the proof of (51), we obtain

$$\mu_1\big(I_D, p\big) = \frac{3\pi^2}{D^2}\,, \qquad \text{with } p(x) := \big(x + \frac{D}{2}\big)hu_1^2\,.$$

By Corollary 11 we have, for some positive constant k, $p(x) = k\cos^2\big(\frac{\pi}{D}x\big)$. This implies in particular that $p(x) \sim \big(x + \frac{D}{2}\big)^2$ as $x \to -\frac{D}{2}$, which is not possible since

$$p(x) = \big(x + \frac{D}{2}\big)hu_1^2 = o\big(x + \frac{D}{2}\big)^2 \qquad \text{as } x \to -\frac{D}{2}\,.$$

We have thus reached a contradiction and our proof is achieved. □

5. Estimate of the excess: the geometric play of cells in N dimensions

In this section we prove Theorem 1, first in dimension $N = 2$ and then in higher dimensions.

5.1. **The case $N = 2$.** We start by giving some preliminaries. For convenience, let us introduce a geometric quantity, related to the width, which will be used throughout the proof. Once fixed a diameter I_D of Ω, we define the depth η of Ω with respect to I_D as the maximum of the length of all sections orthogonal to it. We point out that, denoting by $w^\perp$ the width of Ω in direction orthogonal to I_D, it holds

$$\eta \le w^\perp \le 2\eta, \qquad \frac{\eta D}{2} \le |\Omega| \le \eta D, \qquad |\Omega| \le wD.$$

Moreover, the inequality $\frac{wD}{2} \le |\Omega|$ holds provided the width is small with respect to the diameter, precisely $w \le \frac{\sqrt{3}}{2}D$ (see [38, 48]). In particular, when the latter inequality is satisfied, the depth and the width are equivalent, in the sense that

$$\frac{w}{2} \le \eta \le 2w\,. \tag{52}$$

Next proposition ensures the existence of a dimensional constant, related to the localization of u_2, which will intervene in the proof of Theorem 1.

Proposition 32. *There exists a dimensional constant Λ such that: if $\Omega \subset \mathbb{R}^2$ is an open bounded convex set with diameter D, width $w \le \frac{\sqrt{3}}{2}D$, and Dirichlet eigenfunctions u_1, u_2, and ω is cell of a u_1^2-weighted L^2 equipartition of $\overline{u} = \frac{u_2}{u_1}$ in Ω composed by n cells,* *in an orthogonal coordinate system such that a diameter of* Ω *is the horizontal segment* $\big(-\frac{D}{2}, \frac{D}{2}\big) \times \{0\}$, *the vertical sections* $\omega_{x_1} := \omega \cap (\{x_1\} \times \mathbb{R})$ *satisfy*

$$\|\mathcal{H}^1(\omega_{x_1})\|_{L^\infty(-\frac{D}{2},\frac{D}{2})} \geq \Lambda \frac{w}{n} \,. \tag{53}$$

Proof. Let us first prove the following claim: if Ω is an open bounded convex set of diameter $(-\frac{D}{2}, \frac{D}{2}) \times \mathbb{R}$ and depth η with respect to such diameter, for any open bounded convex set $\omega \subseteq \Omega$, setting $\omega_{x_1} := \omega \cap (\{x_1\} \times \mathbb{R})$, it holds

$$\int_\omega |u|^2 \leq \eta \, \|\mathcal{H}^1(\omega_{x_1})\|_{L^\infty(-\frac{D}{2},\frac{D}{2})} \int_\Omega |\nabla u|^2 \qquad \forall u \in H^1_0(\Omega) \,.$$

It is not restrictive to prove the above inequality for $u \in \mathcal{C}^\infty_0(\Omega)$. Let $x = (x_1, x_2) \in \Omega$, and set $\Omega_{x_1} := \Omega \cap (\{x_1\} \times \mathbb{R})$. We have $u(x) = \int_{-\infty}^{x_2} \frac{\partial u}{\partial x_2}(x_1, s)\, ds$, and $\|\mathcal{H}^1(\Omega_{x_1})\|_{L^\infty(0,D)} \leq \eta$, and hence, by Hölder inequality,

$$|u(x)|^2 \leq \eta \int_{\Omega_{x_1}} \Big|\frac{\partial u}{\partial x_2}(x_1, s)\Big|^2 \, ds \qquad \forall x = (x_1, x_2) \in \omega \,.$$

Therefore,

$$\begin{aligned}
\int_\omega |u|^2 &\leq \eta \int_\omega \Big[\int_{\Omega_{x_1}} \Big|\frac{\partial u}{\partial x_2}(x_1, s)\Big|^2 \, ds \Big] \, dx_1 \, dx_2 \\
&= \eta \int_\Omega \chi_\omega(x_1, x_2) \Big[\int_{\Omega_{x_1}} \Big|\frac{\partial u}{\partial x_2}(x_1, s)\Big|^2 \, ds \Big] \, dx_1 \, dx_2 \\
&= \eta \int_{-\frac{D}{2}}^{\frac{D}{2}} \mathcal{H}^1(\omega_{x_1}) \Big[\int_{\Omega_{x_1}} \Big|\frac{\partial u}{\partial x_2}(x_1, s)\Big|^2 \, ds \Big] \, dx_1 \\
&\leq \eta \, \|\mathcal{H}^1(\omega_{x_1})\|_{L^\infty(-\frac{D}{2},\frac{D}{2})} \int_\Omega |\nabla u|^2 \,.
\end{aligned}$$

Now, we apply the claim just proved to the function u_2, assuming that ω is a cell of a u_1^2-weighted L^2 equipartition of $\overline{u}$ in Ω. Since

$$\int_\omega u_2^2 = \int_\omega |\overline{u}|^2 u_1^2 = \frac{1}{n} \int_\Omega |\overline{u}|^2 u_1^2 = \frac{1}{n} \int_\Omega u_2^2 = \frac{1}{n} \,,$$

we obtain

$$\frac{1}{n} \leq \eta \, \|\mathcal{H}^1(\omega_{x_1})\|_{L^\infty(0,D)} \int_\Omega |\nabla u_2|^2 \leq \eta \, \|\mathcal{H}^1(\omega_{x_1})\|_{L^\infty(0,D)} \lambda_2(\Omega) \,. \tag{54}$$

Denoting by ρ the inradius of Ω, for a dimensional constant Λ we have

$$\lambda_2(\Omega) \leq \frac{\Lambda}{\rho^2} \leq \frac{9\Lambda}{w^2} \,, \tag{55}$$

where the first inequality follows from the fact that Ω contains a disk of radius ρ (thanks to the monotonicity of $\lambda_2(\cdot)$ under inclusions), and the second one from the elementary inequality $w \leq 3\rho$. The conclusion follows by combining (54) and (55), and recalling that η and w satisfy (52) (thanks to the assumption $w \leq \frac{\sqrt{3}}{2} D$). $\square$

Remark 33. A consequence of Proposition 32 is the following information on the mass distribution of the second eigenfunction. Assuming that the width of the set Ω is small and the diameter of the cell ω is large, we get from (53) that

$$|\omega| \geq \Lambda' \frac{|\Omega|}{n} = \Lambda' |\Omega| \int_\omega u_2^2,$$

so that u_2^2 does not localize on ω. This means that, though in general u_2^2 may localize (for instance on thinning triangles), the concentration of mass cannot occur on cells with large diameter.

Proof of Theorem 1 in dimension $N = 2$. It is not restrictive to prove the statement under the hypotheses that Ω is smooth, $D = \pi$, and a diameter of Ω is the segment $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \{0\}$. Moreover, thanks to Theorem 31, we can assume that $w \leq w_0 := \frac{\sqrt{3}}{2}\pi$, so that (52) holds. Along the proof, the maximal admissible width will be diminished when necessary, and will be still denoted by w_0 (equivalently, we are going to indicate by η_0 a maximal admissible value for the depth η with respect to the diameter we have fixed).
By Proposition 41, we have $\lambda_2(\Omega) - \lambda_1(\Omega) = \mu_1(\Omega, u_1^2)$, with first eigenfunction $\overline{u} = \frac{u_2}{u_1}$. For every n, we let $\{\Omega_1, \dots, \Omega_n\}$ be a u_1^2-weighted L^2 equipartition of $\overline{u}$ in Ω. We set $D_i := \mathrm{diam}(\Omega_i)$ and h_i the profile function of Ω_i in direction orthogonal to a fixed diameter of Ω_i.
We let c be some number in $(0, 1)$, whose value will be diminished when necessary during the proof, its final choice being postponed to the end of the proof.
For the benefit of the reader, before giving the detailed proof, we provide below the list of cases, along with a very short heuristic description for each of them (some types of cells are represented in Figure 3).

– *Case 1*: For a sequence of integers n, there exists $\mathcal{I}_n \subset \{1, \dots, n\}$ with $\mathrm{card}(\mathcal{I}_n) \geq \frac{n}{2}$, such that, $\forall i \in \mathcal{I}_n$, the diameter of Ω_i is "small". In this case, by applying Proposition 29 to cells Ω_i for $i \in \mathcal{I}_n$, we obtain the quantitative gap inequality.

– *Case 2*: For a sequence of integers n, there exists $\mathcal{I}'_n \subset \{1, \dots, n\}$ with $\mathrm{card}(\mathcal{I}'_n) \geq \frac{n}{2}$, such that, $\forall i \in \mathcal{I}'_n$, the diameter of Ω_i "large". In this case we prove that the cells Ω_i for $i \in \mathcal{I}'_n$ must intersect the vertical walls of a strip based on the diameter of Ω, so that they can be ordered vertically and they have a polygonal boundary inside the strip.

– *Case 2.1*: For a sequence of integers n, there exists $\mathcal{J}_n \subset \mathcal{I}'_n$, with $\mathrm{card}(\mathcal{J}_n) \geq \frac{n}{4}$, such that, $\forall i \in \mathcal{J}_n$, Ω_i has a vertex in the strip. In this case we reduce ourselves back to a similar situation as in Case 1, so we prove the quantitative inequality.

– *Case 2.2*: For a sequence of integers n, there exists $\mathcal{J}'_n \subset \mathcal{I}'_n$, with $\mathrm{card}(\mathcal{J}'_n) \geq \frac{n}{4}$, such that, $\forall i \in \mathcal{J}'_n$, Ω_i has no vertex in the strip. In this case we prove that, $\forall i \in \mathcal{J}'_n$, the function h_i representing the profile of Ω_i in direction orthogonal to its diameter is affine. Then we distinguish two further sub-cases.

– *Case 2.2.1*: For a sequence of integers n, there exists $\mathcal{S}_n \subset J'_n$, with $\mathrm{card}(\mathcal{S}_n) \geq \frac{n}{8}$, such that, $\forall i \in \mathcal{S}_n$, the Rayleigh quotient of $\overline{u}$ on Ω_i with respect to the measure $u_1^2\,dx$ is "large". In this case we obtain the quantitative inequality.

– *Case 2.2.2*: For a sequence of integers n, there exists $\mathcal{S}'_n \subset J'_n$ with $\mathrm{card}(\mathcal{S}'_n) \geq \frac{n}{8}$, such that, $\forall i \in \mathcal{S}'_n$, the above mentioned Rayleigh quotient is "small". In this case we prove that, for i in a subfamily $S''_n \subset \mathcal{S}'_n$, with $\mathrm{card}(\mathcal{S}''_n) \geq \frac{n}{16}$, an extra-term $\delta_i > 0$, proportional to $(1-\frac{h_i^{max}}{h_i^{min}})^2$, can be added in our lower bound for the Neumann eigenvalue of the diameter of Ω_i with weight $h_i u_1^2$.

– *Case 2.2.2.1*: For a sequence of integers n, there exists $\mathcal{Z}_n \subset \mathcal{S}''_n$, with $\mathrm{card}(\mathcal{Z}_n) \geq \frac{n}{32}$, such that, $\forall i \in \mathcal{Z}_n$, δ_i is "large". In this case we obtain the quantitative inequality.

– *Case 2.2.2.2*: For a sequence of integers n, there exists $\mathcal{Z}'_n \subset \mathcal{S}''_n$, with $\mathrm{card}(\mathcal{Z}'_n) \geq \frac{n}{32}$, such that, $\forall i \in \mathcal{Z}'_n$, δ_i is "small". Loosely speaking, this enables us to minorate the length of the two segments of intersection between any of these cells and the vertical walls of the strip, implying that our pile is sufficiently high. And since it is composed by cells with "large" diameter, by the Pythagorean Theorem we eventually find in Ω a segment larger than its diameter, reaching a contradiction.

We now detail each of the cases above.

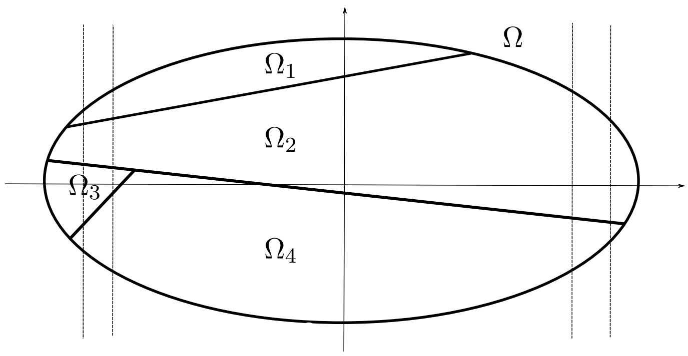

FIGURE 3. A partition of Ω as in the proof of Theorem 1, in which the indices 1 and 3 belong to $\mathcal{I}_n$, 4 belongs to $\mathcal{J}_n \subset \mathcal{I}'_n$, and 2 belongs to $\mathcal{J}'_n \subset \mathcal{I}'_n$. The dashed vertical lines are $x_1 = \pm\frac{\pi-a}{2}$ and $x_1 = \pm\frac{\pi-2a}{2}$.

Case 1. For a sequence of integers n, there exists $\mathcal{I}_n \subset \{1,\dots,n\}$ with $\mathrm{card}(\mathcal{I}_n) \geq \frac{n}{2}$, such that,

$$D_i \leq \sqrt{\pi^2 - c\eta^2} \qquad \forall i \in \mathcal{I}_n \subset \{1,\dots,n\}\,.$$

When applying Proposition 29 to cells Ω_i for $i \in \mathcal{I}_n$, the extra-term at the right hand side of inequality (45) admits the following lower bound

$$\frac{(D-D_i)^3}{D^5} \geq \frac{c^3}{8\pi^8}\eta^6\,. \tag{56}$$

Therefore we can prove the validity of the inequality (3), with $\overline{c} = C\frac{c^3}{2^{10}\pi^8}$, being C the absolute constant appearing in Proposition 29. Indeed, we have

$$\begin{aligned}\mu_1(\Omega, u_1^2) &\geq \frac{1}{n}\sum_{i=1}^{n}\mu_1(\Omega_i, u_1^2) = \frac{1}{n}\Big[\sum_{i\notin\mathcal{I}_n}\mu_1(\Omega_i, u_1^2) + \sum_{i\in\mathcal{I}_n}\mu_1(\Omega_i, u_1^2)\Big]\\ &\geq \frac{1}{n}\Big[3\,\mathrm{card}(\mathcal{I}_n^c) + \Big(3 + C\frac{c^3}{8\pi^8}\eta^6\Big)\mathrm{card}(\mathcal{I}_n)\Big]\\ &\geq 3 + C\tfrac{c^3}{16\pi^8}\eta^6 \geq 3 + C\tfrac{c^3}{2^{10}\pi^8}w^6\,,\end{aligned}$$

where the first inequality follows from Lemma 28, the second one from Proposition 29 (taking into account (56)), and the last one from the assumption $\mathrm{card}(\mathcal{I}_n) \geq \frac{n}{2}$ and (52).

Case 2: For a sequence of integers n, there exists $\mathcal{I}'_n \subset \{1, \dots, n\}$ with $\mathrm{card}(\mathcal{I}'_n) \geq \frac{n}{2}$, such that,

$$D_i \geq \sqrt{\pi^2 - c\eta^2} \qquad \forall i \in \mathcal{I}'_n\,.$$

Let a be a parameter such that $6\eta_0^2 \leq a < \frac{\pi}{4}$. We observe that for every $i \in \mathcal{I}'_n$, Ω_i must intersect the vertical lines $\{x_1 = -\frac{\pi-a}{2}\}$ and $\{x_1 = \frac{\pi-a}{2}\}$. Indeed, assume by contradiction this is not true. Since by the choice of a it holds $\pi a - \frac{a^2}{4} > 5\eta_0^2$, the length of the projection of Ω_i onto the vertical axis would be bounded from below by

$$\sqrt{D_i^2 - (\pi - \frac{a}{2})^2} \geq \sqrt{\pi^2 - c\eta^2 - \pi^2 + \pi a - \frac{a^2}{4}} > \sqrt{-c\eta_0^2 + 5\eta_0^2} \geq 2\eta_0 \geq w^\perp\,,$$

yielding a contradiction.
As a consequence of this fact, for $i \in \mathcal{I}'_n$, the cells Ω_i can be ordered in a vertical way, and inside the strip $\big(-\frac{\pi-a}{2}, \frac{\pi-a}{2}\big) \times \mathbb{R}$ none of their boundaries (except for the bottom and the top cell) can contain portions of $\partial\Omega$. Taking into account that Ω_i are convex sets obtained cutting by lines we infer that, for every $i \in \mathcal{I}'_n$ exception made for two indices, the intersection of $\overline{\Omega}_i$ with the strip $\big[-\frac{\pi-a}{2}, \frac{\pi-a}{2}\big] \times \mathbb{R}$ is a polygon.
We proceed to analyse Cases 2.1 and 2.2.

Case 2.1: For a sequence of integers n, there exists $\mathcal{J}_n \subset \mathcal{I}'_n$, with $\mathrm{card}(\mathcal{J}_n) \geq \frac{n}{4}$, such that

$$\Omega_i \text{ has a vertex in } \Big(-\frac{\pi - 2a}{2}, \frac{\pi - 2a}{2}\Big) \times \mathbb{R} \quad \forall i \in \mathcal{J}_n.$$

For $i \in \mathcal{J}_n$, denoting by V_i one of its vertices inside the strip $\big(-\frac{\pi-2a}{2}, \frac{\pi-2a}{2}\big) \times \mathbb{R}$ (meant as a vertex of the polygon $\overline{\Omega}_i \cap (\big[-\frac{\pi-2a}{2}, \frac{\pi-2a}{2}\big] \times \mathbb{R})$), there exists a neighbouring cell Ω_j such that $V_i \in \partial\Omega_i \cap \partial\Omega_j$. The diameter D_j of such Ω_j satisfies

$$D_j \leq \sqrt{(\pi - a)^2 + 4\eta^2} \leq \sqrt{\pi^2 - 26\eta_0^2}\,,$$

where the last inequality holds because we are assuming that $6\eta_0^2 \leq a < \frac{\pi}{4}$. Since the neighbouring cell Ω_j can touch at most another cell $\widetilde{\Omega}_i$ with $i \in \mathcal{J}_n$, we conclude that, in Case 2.1, for a sequence of integers n there exists $\mathcal{W}_n \subset \{1, \dots, n\}$, with $\mathrm{card}(\mathcal{W}_n) \geq \frac{n}{8}$, such that

$$D_j \leq \sqrt{\pi^2 - 26\eta_0^2} \qquad \forall j \in \mathcal{W}_n\,.$$

Then, by arguing as in Case 1, we can prove (3) (for a suitable dimensional constant $\overline{c}$).

Case 2.2: For a sequence of integers n, there exists $\mathcal{J}'_n \subset \mathcal{I}'_n$, with $\text{card}(\mathcal{J}'_n) \geq \frac{n}{4}$, such that

$$\Omega_i \text{ has no vertex in } \big(-\frac{\pi-2a}{2}, \frac{\pi-2a}{2}\big) \times \mathbb{R} \quad \forall i \in \mathcal{J}'_n.$$

For $i \in \mathcal{J}'_n$, we fix a diameter of Ω_i, and we observe that the angle α_i it forms with the horizontal axis (i.e., with the diameter of Ω) does not exceed $\arcsin(\frac{2\eta}{D_i}) \leq \arcsin(\frac{2\eta}{\sqrt{\pi^2-c\eta^2}})$. Then, if we work in a new local coordinate system in which the fixed diameter of Ω_i is the segment $\big(-\frac{D_i}{2}, \frac{D_i}{2}\big) \times \{0\}$, and we denote by h_i the profile function of Ω_i in vertical direction, the function h_i is necessarily affine on $\big(-\frac{D_i-4a}{2}, \frac{D_i-4a}{2}\big) \times \{0\}$, as soon as

$$\ell_{\eta,a} := \frac{a}{\cos(\arcsin(\frac{2\eta}{\sqrt{\pi^2-c\eta^2}}))} + \eta\frac{2\eta}{\sqrt{\pi^2-c\eta^2}} \leq 2a$$

(see Figure 4, where the marked angle is at most $\arcsin(\frac{2\eta}{\sqrt{\pi^2-c\eta^2}})$).
We remark that the above condition is satisfied provided η_0 is small enough.

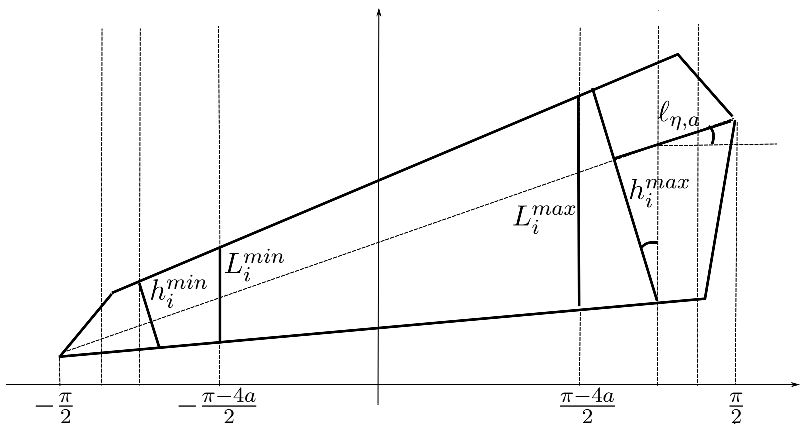

FIGURE 4. The geometry of a cell Ω_i in Case 2.2 and subsequent cases. The dashed vertical lines are $x_1 = \pm\frac{\pi-a}{2}, x_1 = \pm\frac{\pi-2a}{2}, x_1 = \pm\frac{\pi-4a}{2}$.

We proceed to analyse Cases 2.2.1 and 2.2.2.

Case 2.2.1: For a sequence of integers n, there exists $\mathcal{S}_n \subset J'_n$ with $\text{card}(\mathcal{S}_n) \geq \frac{n}{8}$, such that

$$\frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \geq 4 \quad \forall i \in \mathcal{S}_n.$$

In this case we can easily conclude by arguing in a similar way as done in Case 1. Indeed we have

$$\begin{aligned}\mu_1(\Omega,u_1^2) &= \frac{1}{n}\sum_{i=1}^n \frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2} \geq \frac{1}{n}\Big[\sum_{i\notin\mathcal{S}_n}\mu_1(\Omega_i,u_1^2)+\sum_{i\in\mathcal{S}_n}\frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2}\Big]\\ &\geq \frac{1}{n}\Big[3\,\mathrm{card}(\mathcal{S}_n^c)+4\mathrm{card}(\mathcal{S}_n)\Big]\geq 3+\frac{1}{8}\,.\end{aligned}$$

The inequality (3) follows, for a dimensional constant $\overline{c}$, provided w_0 is small enough.

– *Case 2.2.2*: For a sequence of integers n, there exists $\mathcal{S}'_n\subset J'_n$ with $\mathrm{card}(\mathcal{S}'_n)\geq\frac{n}{8}$, such that

$$\frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2}\leq 4\quad\forall i\in\mathcal{S}'_n.$$

We observe that there exists $\mathcal{S}''_n\subset\mathcal{S}'_n$, with $\mathrm{card}(\mathcal{S}''_n)\geq\frac{n}{16}$, such that

$$|\Omega_i|\leq\frac{17}{n}|\Omega|\quad\forall i\in\mathcal{S}''_n\,.$$

Indeed, otherwise it would be $|\Omega_i|\geq\frac{17}{n}$ for at least $\frac{n}{16}$ cells in $\mathcal{S}'_n$, and the union of such cells would have measure strictly larger than the measure of Ω. Let us now prove that, for every $i\in\mathcal{S}''_n$, denoting by K the absolute constant appearing in Theorem 21 (ii), and by h_i^{max} and h_i^{min} the maximum and minimum of h_i on I_{D_i-4a}, it holds

$$\mu_1(I_{D_i},h_iu_1^2)\geq 3+\frac{8K}{\pi^2}\Big(1-\frac{h_i^{max}}{h_i^{min}}\Big)^2\,. \tag{57}$$

To that aim we apply Lemma 26 with $\omega_0=\Omega_i$, $p=u_1^2$, and $v=\overline{u}$. Notice that this is possible because we are assuming that Ω is smooth, so that p is uniformly continuous in Ω_i, and $\overline{u}\in W^{2,\infty}(\Omega_i)$. We obtain

$$\begin{aligned}\frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2} &\geq\Big\{\mu_1(I_{D_i},h_iu_1^2)-\frac{\alpha(\varepsilon)|\Omega_i|}{\int_{\Omega_i}\overline{u}^2u_1^2}\Big[1+\mu_1(I_{D_i},h_iu_1^2)\Big(1+\beta(\varepsilon)|\Omega_i|\Big)\Big]\Big\}\\ &\geq\Big\{\mu_1(I_{D_i},h_iu_1^2)-\frac{\alpha(\varepsilon)|\Omega_i|}{\frac{1}{n}}\Big[1+\frac{3}{2}\mu_1(I_{D_i},h_iu_1^2)\Big]\Big\}\,,\end{aligned} \tag{58}$$

where the last inequality holds provided ε is so small that $\big(1+\beta(\varepsilon)|\Omega|\big)\leq\frac{3}{2}$ (which is true as soon as n is sufficiently large).
Taking into account that cells Ω_i for $i\in\mathcal{S}''_n$ satisfy the condition $|\Omega_i|\leq\frac{17}{n}|\Omega|$, the above inequality implies

$$\frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2}\geq\Big\{\mu_1(I_{D_i},h_iu_1^2)\Big[1-\frac{3}{2}\cdot17|\Omega|\cdot\alpha(\varepsilon)\Big]-17|\Omega|\cdot\alpha(\varepsilon)\Big\}\,.$$

We infer that, again for ε sufficiently small, it holds

$$\frac{\int_{\Omega_i}|\nabla\overline{u}|^2u_1^2}{\int_{\Omega_i}|\overline{u}|^2u_1^2}\geq\frac{6}{7}\mu_1(I_{D_i},h_iu_1^2)-2\,.$$

Recalling that, for $i \in \mathcal{S}''_n$, the left hand side of the above inequality does not exceed 4, we infer that

$$\mu_1(I_{D_i}, h_i u_1^2) \leq 7\,. \tag{59}$$

This enables us to apply Theorem 21 (ii), and conclude that (57) holds.
We proceed to analyse Cases 2.2.2.1 and 2.2.2.2.

Case 2.2.2.1: For a sequence of integers n, there exists $\mathcal{Z}_n \subset \mathcal{S}''_n$, with $\mathrm{card}(\mathcal{Z}_n) \geq \frac{n}{32}$, such that

$$\frac{8K}{\pi^2}\Big(1 - \frac{h_i^{min}}{h_i^{max}}\Big)^2 \geq c\eta^2 \qquad \forall i \in \mathcal{Z}_n.$$

By (57) and (59), for $i \in \mathcal{Z}_n$ it holds

$$\mu_1(I_{D_i}, h_i u_1^2) \geq 3 + c\eta^2 \quad \text{ and } \quad \mu_1(I_{D_i}, h_i u_1^2) \leq 7\,.$$

Then, by (58) we obtain

$$\frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \geq 3 + c\eta^2 - \frac{12\alpha(\varepsilon)|\Omega_i|}{\frac{1}{n}}\,.$$

Hence,

$$\begin{aligned}
\mu_1(\Omega, u_1^2) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \geq \frac{1}{n}\Big[\sum_{i \notin \mathcal{Z}_n} \mu_1(\Omega_i, u_1^2) + \sum_{i \in \mathcal{Z}_n} \frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2}\Big] \\
&\geq \frac{1}{n}\Big[3\,\mathrm{card}(\mathcal{Z}_n^c) + (3 + c\eta^2)\mathrm{card}(\mathcal{Z}_n)\Big] - 12\alpha(\varepsilon)|\Omega| \\
&\geq 3 + \tfrac{c}{32}\eta^2 - 12\alpha(\varepsilon)|\Omega|\,,
\end{aligned}$$

and the conclusion follows (as usual, for a suitable choice of $\overline{c}$).

Case 2.2.2.2: For a sequence of integers n, there exists $\mathcal{Z}'_n \subset \mathcal{S}''_n$, with $\mathrm{card}(\mathcal{Z}'_n) \geq \frac{n}{32}$, such that

$$\frac{8K}{\pi^2}\Big(1 - \frac{h_i^{min}}{h_i^{max}}\Big)^2 \leq c\eta^2 \qquad \forall i \in \mathcal{Z}'_n\,.$$

We shall now prove that, provided η_0 and a are small enough, and c is well-chosen (as well, small enough), this case cannot occur.
The contradiction argument relies on the following claim: denoting by $L_i^{min} \leq L_i^{max}$ the lengths of the intersections of Ω_i with the lines $\{x_1 = -\frac{\pi - 4a}{2}\}$ and $\{x_1 = \frac{\pi-4a}{2}\}$, we have

$$\text{for } \eta_0, a \ll 1\,, \quad L_i^{min} \geq \frac{\Lambda}{8}\frac{w}{n} \qquad \forall i \in \mathcal{Z}'_n\,, \tag{60}$$

where Λ is the dimensional constant appearing in Proposition 32.
We first prove (60) and then we show how it leads to a contradiction.
For $i \in \mathcal{Z}'_n$, choosing $c\eta_0^2 \leq \frac{2K}{\pi^2}$, it holds

$$\Big(1 - \frac{h_i^{min}}{h_i^{max}}\Big)^2 \leq \frac{\pi^2 c}{8K}\eta^2 \leq \frac{1}{4}\,,$$

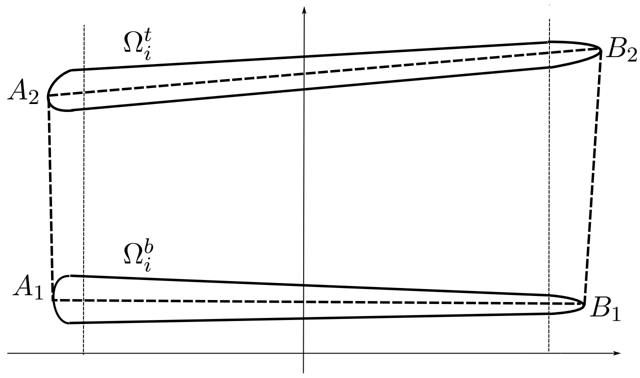

FIGURE 5. The geometry of the pile of cells in Case 2.2.2.2. The dashed vertical lines are $x_1 = \pm\frac{\pi - a}{2}$.

so that

$$\frac{h_i^{min}}{h_i^{max}} \geq \frac{1}{2}\,.$$

The above inequality, combined with the area inequality $\frac{1}{2}h_{min}D_i \leq \eta\pi$, implies that $h_{max} \leq \frac{4\eta\pi}{D_i}$. In turn this implies that, if α_i is the angle already considered in Case 2.2 formed between the diameters of Ω and Ω_i, it holds

$$h_{max} \cdot \sin\alpha_i \leq \frac{4\eta\pi}{\sqrt{\pi^2 - c\eta^2}} \frac{2\eta}{\sqrt{\pi^2 - c\eta^2}}\,.$$

Hence, for η_0 sufficiently small, we have $h_{max} \cdot \sin\alpha_i \leq a$, which ensures that the two segments of lengths L_i^{min} and L_i^{max} are interior to the trapeze with bases given by the two segments of lengths h_i^{min} and h_i^{max} and oblique sides given by portions of $\partial\Omega_i$ (see Figure 4). This implies via Thales Theorem that

$$\frac{L_i^{min}}{L_i^{max}} \geq \frac{h_i^{min}}{h_i^{max}} \geq \frac{1}{2}\,.$$

Denoting by $(\Omega_i)_{x_1}$ the intersection of Ω_i with the straight line $\{x_1\} \times \mathbb{R}$, we have

$$L_i^{min} \geq \frac{1}{2} L_i^{max} \geq \frac{1}{4} \|\mathcal{H}^1((\Omega_i)_{x_1})\|_{L^\infty(I_\pi)} \geq \frac{\Lambda}{4}\frac{w}{n}\,, \tag{61}$$

where the second inequality is satisfied provided a is small enough, and the third one holds by Proposition 32. This completes the proof of claim (60).
Eventually, let us show how (60) yields a contradiction. Let Ω_i^b and Ω_i^t be the bottom and the top cell in the family of cells Ω_i, for $i \in \mathcal{Z}_n'$. Let A_1B_1 and A_2B_2 be two diametral segments respectively for Ω_i^b and Ω_i^t, and consider the quadrilateral $A_1B_1A_2B_2$ (see Figure 5).

We can estimate from below the lengths of the segments A_1A_2 and B_1B_2 by the lengths of their orthogonal projections on the vertical lines $\{x_1 = -\frac{\pi-4a}{2}\}$ and $\{x_1 = \frac{\pi-4a}{2}\}$. Applying claim (60) to all the cells Ω_i for $i \in \mathcal{Z}'_n$ (exception made for Ω_i^b and Ω_i^t), and recalling that $\text{card}(\mathcal{Z}'_n) \geq \frac{n}{32}$ we get

$$\min\Big\{\overline{A_1A_2}, \overline{B_1B_2}\Big\} \geq \frac{\Lambda}{4}\frac{w}{n}\Big(\frac{n}{32} - 2\Big) \geq \frac{\Lambda}{256}w\,,$$

where the last inequality holds for n large enough. On the other hand, since $\mathcal{Z}'_n \subseteq \mathcal{S}''_n \subseteq \mathcal{S}'_n \subseteq \mathcal{J}'_n \subseteq \mathcal{I}'_n$, it holds

$$\min\Big\{\overline{A_1B_1}, \overline{A_2B_2}\Big\} \geq \sqrt{\pi^2 - c\eta^2}\,.$$

Then, at least one of the two diagonals of the quadrilateral $A_1B_1A_2B_2$ turns out to be larger than the diameter of Ω, yielding a contradiction. Namely, since at least one of the inner angles of the quadrilateral is larger than or equal to $\frac{\pi}{2}$, we have

$$\max\Big\{\overline{A_1B_2}, \overline{A_2B_1}\Big\} \geq \Big[(\pi^2 - c\eta^2) + \Big(\frac{\Lambda}{256}\Big)^2 w^2\Big]^{\frac{1}{2}} > \pi$$

where the last inequality follows by choosing c small enough. Having reached a contradiction, our proof is achieved. $\square$

5.2. **The case $N \geq 3$.** The main tool to prove Theorem 1 for a domain Ω in dimension $N \geq 3$ is the following quantitative estimate of a weighted two-dimensional Rayleigh quotient of $\overline{u}$ on suitable planar sections of Ω.

Theorem 34. *Let $N \geq 3$. There exists a dimensional constant $\overline{c} > 0$ such that:*
– for every open bounded convex domain Ω in $\mathbb{R}^N$ of diameter D and width w, with Dirichlet eigenfunctions u_1, u_2,
– for every planar subset U of Ω which, in a suitable orthogonal coordinate system $\{e_1, \ldots, e_N\}$ with e_N in the direction of the width of Ω, can be written as

$$U = \Big\{x \in \Omega \ : \ x_i = 0 \ \forall i = 1, \ldots, N-2\,, \ a \leq x_{N-1} \leq b\Big\}\,,$$

– for every function $h : U \to (0, 1]$ independent of x_N and $\big(\frac{1}{N-2}\big)$-concave, such that

$$\int_U |\nabla_U u_2|^2 h \leq 2\lambda_2(\Omega) \int_U |u_2|^2 h\,, \tag{62}$$

if $\overline{u} := \frac{u_2}{u_1}$ satisfies $\int_U \overline{u} h u_1^2 = 0$, and u_2 is not identically zero on U, it holds

$$\frac{\int_U |\nabla_U \overline{u}|^2 h u_1^2}{\int_U |\overline{u}|^2 h u_1^2} \geq \frac{3\pi^2}{D^2} + \overline{c}\frac{w^6}{D^8}\,.$$

For the proof of Theorem 34, we need the following preliminary results, which generalize respectively Proposition 29 and Proposition 32.

Proposition 35. *There exists an absolute constant C such that: if all the assumptions of Theorem 34 are satisfied, and ω is a cell of a (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U, of diameter d, it holds*

$$\frac{\int_\omega |\nabla_U \overline{u}|^2 h u_1^2}{\int_\omega |\overline{u}|^2 h u_1^2} \geq \frac{3\pi^2}{D^2} + C\frac{(D-d)^3}{D^5}\,. \tag{63}$$

Proof. For $\varepsilon > 0$ small, we set

$$\omega_\varepsilon := \Omega \cap \left\{ x \ : \ (x_{N-1}, x_N) \in \omega\,,\ (x_1^2 + \cdots + x_{N-2}^2)^{\frac{1}{2}} < \varepsilon \Big(\frac{h(x_{N-1})}{\gamma_{N-2}} \Big)^{\frac{1}{N-2}} \right\},$$

where γ_{N-2} is the measure of the unit ball in $\mathbb{R}^{N-2}$.
Since ω_ε is an open bounded convex subset of Ω, denoting by d_ε its diameter, by Proposition 29 we have

$$\mu_1(\omega_\varepsilon, u_1^2) \geq \frac{3\pi^2}{D^2} + C\frac{(D-d_\varepsilon)^3}{D^5}\,.$$

From this inequality, taking as a test function for $\mu_1(\omega_\varepsilon, u_1^2)$ the function $\tilde{u} - c_\varepsilon$, where

$$\tilde{u}(x) := \overline{u}(0, \dots, 0, x_{N-1}, x_N) \qquad \text{and} \qquad c_\varepsilon := \frac{\int_{\omega_\varepsilon} \tilde{u} u_1^2}{\int_{\omega_\varepsilon} u_1^2}\,,$$

we infer that

$$\frac{\frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} |\nabla \tilde{u}|^2 u_1^2}{\frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} |\tilde{u} - c_\varepsilon|^2 u_1^2} \geq \frac{3\pi^2}{D^2} + C\frac{(D-d_\varepsilon)^3}{D^5}\,. \tag{64}$$

In the limit as $\varepsilon \to 0$ we have

$$\frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} |\nabla_{x_{N-1}, x_N} \tilde{u}|^2 u_1^2 \to \int_\omega |\nabla_U \overline{u}|^2 h u_1^2\,,$$

where we have exploited in particular the fact that h is a bounded $\left(\frac{1}{N-2}\right)$-concave function on U depending only on the variable x_{N-1}, so that it is continuous on $\overline{U}$. In a similar way, we have

$$\frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} \tilde{u} u_1^2 \to \int_\omega \overline{u} h u_1^2 = 0 \qquad \text{and} \qquad \frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} u_1^2 \to \int_\omega h u_1^2 > 0\,,$$

where the equality $\int_\omega \overline{u} h u_1^2 = 0$ holds because ω is a cell of a (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U. It follows that $c_\varepsilon \to 0$, and hence

$$\frac{1}{\varepsilon^{N-2}} \int_{\omega_\varepsilon} |\tilde{u} - c_\varepsilon|^2 \to \int_\omega |\overline{u}|^2 h u_1^2\,.$$

Therefore, the inequality (63) follows from (64) by passing to the limit as $\varepsilon \to 0$. □

Proposition 36. *There exists a dimensional constant Λ such that: if all the assumptions of Theorem 34 are satisfied, and ω is a cell of a (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U composed by n cells, setting $\omega_{x_{N-1}} := \omega \cap (\{x_{N-1}\} \times \mathbb{R}e_N)$, it holds*

$$\|\mathcal{H}^1(\omega_{x_{N-1}})\|_{L^\infty(a,b)} \geq \Lambda \frac{w}{n}\,.$$

Proof. We consider the restriction of u_2 to U, and we argue in a similar way as in the proof of Proposition 32. Denoting by η_U the depth of U in direction e_N, we have $u_2(x_{N-1}, x_N) = \int_{-\infty}^{x_N} \frac{\partial u}{\partial x_N}(x_{N-1}, s)\, ds$, and $\|\mathcal{H}^1(\omega_{x_{N-1}})\|_{L^\infty(a,b)} \leq \eta_U$, and hence

$$|u_2(x_{N-1}, x_N)|^2 \leq \eta_U \int_{U_{x_{N-1}}} \big|\frac{\partial u_2}{\partial x_N}(x_{N-1}, s)\big|^2\, ds \qquad \forall (x_{N-1}, x_N) \in U\,.$$

Since h is independent of x_N, namely $h = h(x_{N-1})$, multiplying the above inequality by h we obtain the following inequality holding for every $(x_{N-1}, x_N) \in U$:

$$|u_2(x_{N-1}, x_N)|^2 h(x_{N-1}) \leq \eta_U \int_{U_{x_{N-1}}} \big|\frac{\partial u_2}{\partial x_N}(x_{N-1}, s)\big|^2 h(x_{N-1})\, ds \tag{65}$$

Integrating over ω and using (62), we obtain

$$\begin{aligned}
\int_\omega |u_2|^2 h \ &\leq \eta_U \int_\omega \Big[\int_{U_{x_{N-1}}} \big|\frac{\partial u_2}{\partial x_N}(x_{N-1}, s)\big|^2 h(x_{N-1})\, ds\Big]\, dx_{N-1}\, dx_N \\
&= \eta_U \int_U \chi_\omega(x_{N-1}, x_N)\Big[\int_{U_{x_{N-1}}} \big|\frac{\partial u_2}{\partial x_N}(x_{N-1}, s)\big|^2 h(x_{N-1})\, ds\Big]\, dx_{N-1}\, dx_N \\
&= \eta_U \int_a^b \mathcal{H}^1(\omega_{x_{N-1}})\Big[\int_{U_{x_{N-1}}} \big|\frac{\partial u_2}{\partial x_N}(x_1, s)\big|^2 h(x_{N-1})\, ds\Big]\, dx_{N-1} \\
&\leq \eta_U \cdot \|\mathcal{H}^1(\omega_{x_{N-1}})\|_{L^\infty(a,b)} \cdot 2\,\lambda_2(\Omega) \int_U |u_2|^2 h \\
&\leq \eta_U \cdot \|\mathcal{H}^1(\omega_{x_{N-1}})\|_{L^\infty(a,b)} \cdot 2\,\frac{\Lambda}{w_\Omega^2} \int_U |u_2|^2 h\,,
\end{aligned}$$

where, for the sake of clearness, we have indicated by w_Ω the width of Ω. Since ω is a cell of a (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U, we infer that

$$\|\mathcal{H}^1(\omega_{x_{N-1}})\|_{L^\infty(a,b)} \geq \frac{1}{2\Lambda}\frac{1}{n}\frac{w_\Omega^2}{\eta_U} \geq \frac{1}{4\Lambda}\frac{w_\Omega}{n}\,,$$

where the last inequality holds because $\eta_U \leq 2w_\Omega$. □

Remark 37. If in the above proposition the diameter of U has length at least $\frac{9}{10}D$, the angle it forms with the direction e_{N-1} is at most $\arcsin(\frac{10}{9D}w_\Omega)$. Then the conclusion of the proposition continues to hold, possibly with a different constant Λ, if the local system of cartesian coordinates is changed into (e'_{N-1}, e'_N), being e'_{N-1} aligned with the diameter of U.

Proof of Theorem 34. It is not restrictive to prove the statement under the hypotheses that Ω is a smooth domain with diameter π and small width. As above, for the sake of clearness, we denote by w_Ω the width of Ω, and by η_U the depth of U in direction e_N. We observe that, thanks to the assumption (62), η_U and w_Ω are comparable. Indeed, $\eta_U \leq 2w_\Omega$. To show the converse, namely that also w_Ω is controlled by η_U, we start from the pointwise inequality (65), which holds on U, and we integrate it on U. Proceeding as in the proof of Proposition 36, we arrive at

$$\int_U |u_2|^2 h \leq \eta_U \cdot \|\mathcal{H}^1(U_{x_{N-1}})\|_{L^\infty(a,b)} \cdot 2\, \frac{\Lambda}{w_\Omega^2} \int_U |u_2|^2 h \,.$$

We infer that

$$\int_U |u_2|^2 h \leq \eta_U^2 \cdot 2\, \frac{\Lambda}{w_\Omega^2} \int_U |u_2|^2 h \,,$$

which shows that

$$\eta_U \geq \frac{1}{\sqrt{2\Lambda}} w_\Omega \,. \tag{66}$$

Once we know that the quantities w_Ω and η_U are equivalent, we indicate by w_0 and η_0 respectively upper bounds for w_Ω and η_U, and we proceed by adopting the same proof line of Theorem 1. The difference is that we have to work with (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U, the unique modification with respect to the proof Theorem 1 being the presence of the extra-weight h. For every n, let $\{\Omega_1, \dots, \Omega_n\}$ be a (hu_1^2)-weighted L^2 equipartition of $\overline{u}$ in U. We set $D_i := \mathrm{diam}(\Omega_i)$ and h_i the profile function of Ω_i in direction orthogonal to a fixed diameter of Ω_i.
We denote by c some number in $(0, 1)$, and we follow step by step the same proof line of Theorem 1, distinguishing the same cases in cascade. Below we limit ourselves to indicate which are the required modifications, all the other cases being completely analogue as in Theorem 1:

- *Case 1:* In place of applying Proposition 29, apply Proposition 35.
- *Case 2.2.2:* Lemma 26 is now applied with $p = hu_1^2$ (notice that such p is still uniformly continuous on Ω_i, since h is continuous on $\overline{U}$).
- *Case 2.2.2.2:* In place of applying Proposition 32, apply Proposition 36 (taking also into account Remark 37).

□

Proof of Theorem 1 in dimension $N \geq 3$. It is not restrictive to prove the statement under the hypotheses that Ω is smooth and strictly convex, $D = \pi$, and w is small and attained in direction e_N. For every $n \in \mathbb{N}$, let $\{\Omega_1, \dots, \Omega_n\}$ be a u_1^2-weighted L^2 equipartition of $\overline{u}$ in Ω, obtained by the procedure described in Remark 25, namely using a family of hyperplanes, all parallel to e_N, with normals of the type $(\cos\alpha_1, \sin\alpha_1, 0, \dots, 0)$, $(0, \cos\alpha_2, \sin\alpha_2, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, \cos\alpha_{N-2}, \sin\alpha_{N-2}, 0)$. For n large, all the cells become narrow in $(N-2)$-directions, so that they become arbitrarily close to a convex set having at most Hausdorff dimension 2. Since by construction for every cell it holds $\int_{\Omega_i} u_2^2 = \frac{1}{n}$, for at most $\frac{n}{2}$ cells it holds $\int_{\Omega_i} |\nabla u_2|^2 \geq 2\lambda_2(\Omega) \int_{\Omega_i} u_2^2$. Equivalently,

$$\int_{\Omega_i} |\nabla u_2|^2 \leq 2\lambda_2(\Omega) \int_{\Omega_i} u_2^2 \qquad \forall i \in \mathcal{I}_n \subset \{1, \dots, n\} \text{ with } \mathrm{card}(\mathcal{I}_n) \geq \frac{n}{2} \,. \tag{67}$$

For every $i \in \mathcal{I}_n$, by same argument used to prove the inequality (66) in the proof of Theorem 34, we obtain that the depth η_{Ω_i} in direction e_N satisfies

$$\eta_{\Omega_i} \geq \frac{1}{\sqrt{2\Lambda}} w_\Omega \qquad \forall i \in \mathcal{I}_n \,. \tag{68}$$

We denote by $\mathcal{I}'_n$ the subfamily of $\mathcal{I}_n$ such that the diameter D_i of Ω_i satisfies $D_i \leq \frac{9}{10}\pi$, and we set $\mathcal{I}''_n = \mathcal{I}_n \setminus \mathcal{I}'_n$. Applying Proposition 29, we get

$$\mu_1(\Omega_i, u_1^2) \geq 3 \quad \forall i \notin \mathcal{I}_n \qquad \text{and} \qquad \mu_1(\Omega_i, u_1^2) \geq 3 + C\big(\frac{\pi}{10}\big)^3 \quad \forall i \in \mathcal{I}'_n \,.$$

By Lemma 28 it follows that

$$\begin{aligned}
\mu_1(\Omega, u_1^2) \;&= \frac{\int_\Omega |\nabla \overline{u}|^2 u_1^2}{\int_\Omega |\overline{u}|^2 u_1^2} \geq \frac{1}{n} \sum_{i=1}^n \frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \\
&\geq \frac{1}{n} \Big[\sum_{i \notin \mathcal{I}_n} \mu_1(\Omega_i, u_1^2) + \sum_{i \in \mathcal{I}'_n} \mu_1(\Omega_i, u_1^2) + \sum_{i \in \mathcal{I}''_n} \frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \Big] \\
&\geq \frac{1}{n} \Big[3\big(n - \text{card}(\mathcal{I}_n)\big) + \big(3 + C\big(\frac{\pi}{10}\big)^3\big) \text{card}(\mathcal{I}'_n) + \min_{i \in \mathcal{I}''_n} \Big[\frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \Big] \text{card}(\mathcal{I}''_n) \Big] \,.
\end{aligned}$$

Therefore, to conclude the proof we are reduced to show that

$$\liminf_{n \to +\infty} \min_{i \in \mathcal{I}''_n} \Big[\frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \Big] \geq 3 + c w_\Omega^6 \,.$$

Indeed in this case we have

$$\mu_1(\Omega, u_1^2) \geq 3 + \frac{1}{2} \Big[C\big(\frac{\pi}{10}\big)^3 \wedge c \Big] w_\Omega^6 \,.$$

Let $i_n \in \mathcal{I}''_n$ be such that

$$\min_{i \in \mathcal{I}''_n} \Big[\frac{\int_{\Omega_i} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_i} |\overline{u}|^2 u_1^2} \Big] = \frac{\int_{\Omega_{i_n}} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega_{i_n}} |\overline{u}|^2 u_1^2} \,.$$

In the sequel we write for brevity Ω^n in place of Ω_{i_n}. So our target is to show that

$$\liminf_{n \to +\infty} \frac{\int_{\Omega^n} |\nabla \overline{u}|^2 u_1^2}{\int_{\Omega^n} |\overline{u}|^2 u_1^2} \geq 3 + c w_\Omega^6 \,.$$

Let H^n be the intersection of closed halfspaces parallel to e_N such that $\overline{\Omega^n} = \overline{\Omega} \cap H^n$. Up to subsequences, and up to changing the coordinate system, we can assume that H^n converge in Hausdorff distance to the hyperplane

$$\Pi := \big\{ x \ : \ x_i = 0 \ \forall i = 1, \dots, N-2 \big\} \,.$$

In the sequel, a point $(0, \dots, 0, x_{N-1}, x_N) \in \Pi$ will be identified with the pair (x_{N-1}, x_N). Accordingly, the sequence Ω^n converge in Hausdorff distance to the set $U := \Omega \cap \Pi$, which is of the kind

$$U = \Big\{ (x_{N-1}, x_N) \ : \ x_{N-1} \in (a, b) \,, \ x_N \in \Omega \cap (x_{N-1} + \mathbb{R} e_N) \Big\} \,.$$

We remark that U is nondegenerate, namely it has positive two-dimensional measure. Indeed, the depth of U in direction e_N is strictly positive because, by (68), the depth of Ω^n in direction e_N is uniformly bounded from below. Moreover, the length of (a, b) is strictly positive, because the diameter of Ω^n is not smaller than $\frac{9\pi}{10}$, and cannot be attained in direction e_N (which is the direction of the width, assumed to be small). Moreover U cannot lie on $\partial\Omega$, thanks to our initial assumption of strict convexity on Ω.

For every $(x_{N-1}, x_N) \in \Pi$, we define the function

$$h_n(x_{N-1}) = \mathcal{H}^{N-2}\big((\Pi^\perp_{(x_{N-1},x_N)}) \cap H^n\big)\,,$$

where $\Pi^\perp_{(x_{N-1},x_N)}$ denotes the $(N-2)$-affine space passing through (x_{N-1}, x_N) and orthogonal to Π.
Up to subsequences, $\frac{h_n}{\|h_n\|_\infty}$ converges a.e. on $\mathbb{R}e_{N-1}$ to a function h such that $h = 0$ on $(-\infty, a) \cup (b, +\infty)$. Moreover, since by the Brunn-Minowski Theorem h_n is $\big(\frac{1}{N-2}\big)$-concave on (a, b), the convergence is locally uniform on (a, b), h is itself $\big(\frac{1}{N-2}\big)$-concave on (a, b), and consequenty satisfies also $\|h\|_\infty = 1$.
We claim that

$$\lim_{n\to+\infty} \frac{1}{\|h_n\|_\infty} \int_{\Omega^n} f = \int_U fh \qquad \text{for every } f \in \mathcal{C}^\infty(\overline{\Omega})\,. \tag{69}$$

and

$$u_2 \text{ is not identically zero on } U\,. \tag{70}$$

Assume by a moment these two claims hold true. Then we infer that Ω, U, and h satisfy all the assumptions of Theorem 34. Indeed, recall that Ω^n belongs to the family of cells satisfying (67), and pass to the limit as $n \to +\infty$: by using (69) with $f = |\nabla u_2|^2$ and with $f = u_2^2$, it follows that assumption (62) is fulfilled. Similarly, recalling that $\int_{\Omega^n} \overline{u} u_1^2 = \int_{\Omega^n} u_1 u_2 = 0$, and using (69) with $f = u_1 u_2$, it follows that also the assumption that $\int_U \overline{u} h u_1^2 = 0$ is satisfied. Finally, the assumption that u_2 does not vanish identically on U is satisfied by (70). Then, we have

$$\lim_{n\to+\infty} \frac{\int_{\Omega^n} |\nabla\overline{u}|^2 u_1^2}{\int_{\Omega^n} |\overline{u}|^2 u_1^2} = \frac{\int_U |\nabla\overline{u}|^2 h u_1^2}{\int_U |\overline{u}|^2 h u_1^2} \geq 3 + \overline{c} w^6\,,$$

where the first equality is obtained applying again (69) with $f = |\overline{u}|^2 u_1^2 = u_2^2$ and with $f = |\nabla\overline{u}|^2 h u_1^2$, and the second inequality follows from Theorem 34 (since $|\nabla\overline{u}|^2 \geq |\nabla_U \overline{u}|^2$).
To conclude our proof, we now give the proofs of claims (69) and (70).

- Proof of claim (69): Let δ_n denote the Hausdorff distance between Ω^n and U, and set

$$U^{-\delta_n} = \big\{x \in U \ :\ \operatorname{dist}(x, \partial U) > \delta_n\big\}\,, \qquad U^{\delta_n} = \big\{x \in \Pi \ :\ \operatorname{dist}(x, \partial U) < \delta_n\big\}\,.$$

We have

$$\int_{\Omega^n} f = \int_\Pi \int_{\Pi^\perp_{(x_{N-1},x_N)}} \chi_{\Omega^n} f = \int_{U^{-\delta_n}} \int_{\Pi^\perp_{(x_{N-1},x_N)}} \chi_{\Omega^n} f + \int_{U^{\delta_n}\setminus U^{-\delta_n}} \int_{\Pi^\perp_{(x_{N-1},x_N)}} \chi_{\Omega^n} f\,,$$

where the integrals over $\Pi^\perp_{(x_{N-1},x_N)}$ are made with respect to $x' = (x_1, \ldots, x_{N-2})$, while the integrals over $U^{-\delta_n}$, $U^{\delta_n} \setminus U^{-\delta_n}$ are made with respect to (x_{N-1}, x_N).
For $(x_{N-1}, x_N) \in U^{-\delta_n}$, setting $h = (x', 0, 0)$, we have

$$f(x', x_{N-1}, x_N) = f(0, x_{N-1}, x_N) + \nabla f(0, x_{N-1}, x_N) \cdot h + o(|h|)\,;$$

for $(x_{N-1}, x_N) \in (U^{\delta_n} \setminus U^{-\delta_n})$, if (z_{N-1}, x_N) is the projection of (x_{N-1}, x_N) onto $\overline{U}$ parallel to e_{N-1}, setting $\tilde{h} = (x', x_{N-1} - z_{N-1}, 0)$ we have

$$f(x', x_{N-1}, x_N) = f(0, z_{N-1}, x_N) + \nabla f(0, z_{N-1}, x_N) \cdot \tilde{h} + o(|\tilde{h}|)\,.$$

Accordingly, we have

$$\int_{U^{-\delta_n}} \int_{\Pi^\perp_{(x_{N-1},x_N)}} \chi_{\Omega^n} f = \int_{U^{-\delta_n}} h_n(x_{N-1})(f + O(\delta_n))\,,$$
$$\int_{U^{\delta_n}\setminus U^{-\delta_n}} \int_{\Pi^\perp_{(x_{N-1},x_N)}} \chi_{\Omega^n} f = \int_{U^{\delta_n}\setminus U^{-\delta_n}} \psi_n(x_{N-1},x_N)(f + O(\delta_n))\,,$$

where ψ_n is defined by

$$\psi_n(x_{N-1},x_N) := \mathcal{H}^{N-2}\big(\big(\Pi^\perp_{(x_{N-1},x_N)}\big) \cap \Omega^n\big)\,.$$

We now divide by $\|h_n\|_\infty$ and pass to the limit as $n \to +\infty$. Taking into account that $\psi_n \le h_n$, and that $\mathcal{H}^{N-2}\big(U^{\delta_n} \setminus U^{-\delta_n}\big)$ is infinitesimal, we conclude that

$$\lim_{n\to+\infty} \frac{1}{\|h_n\|_\infty} \int_{\Omega^n} f = \lim_{n\to+\infty} \frac{1}{\|h_n\|_\infty} \int_{U^{-\delta_n}} f h_n = \int_U f h\,.$$

• Proof of claim (70): Assume by contradiction that u_2 is identically zero on U. We are going to show that this implies

$$\frac{\int_{\Omega^n} |\nabla u_2|^2}{\int_{\Omega^n} |u_2|^2} \to +\infty \qquad \text{as } n \to +\infty\,,$$

against (67). To that aim we are going to use an argument which amounts roughly to control the value of the function by the value of its gradient, in the same spirit of the Łojasiewicz inequality [39].

For every $i \in \mathbb{N}$ let us denote by $D^{(i)}_{x_U} u_2$ the i-th order differential of u_2 computed at a point $x_U = (0,\dots,0,x_{N-1},x_N)$ of U. Let $k \in \mathbb{N}\setminus\{0\}$ be the smallest natural number such that, for some point x_U of U, $D^{(k)}_{x_U} u_2 \neq 0$. Clearly such k exists: otherwise, by the analyticity of u_2 inside Ω, u_2 would be identically zero.

In order to estimate the Rayleigh quotient of u_2 over Ω^n, we distinguish between points $x = (x',x_{N-1},x_N) \in \Omega^n$ such that $(x_{N-1},x_N) \in U$ and such that $(x_{N-1},x_N) \notin U$.

For $x = (x',x_{N-1},x_N) \in \Omega^n$ such that $(x_{N-1},x_N) \in U$, setting $x_U = (0,\dots,0,x_{N-1},x_N)$ and $\xi = (x',0,0)$, we have

$$u_2(x) = \sum_{i=0}^{k} \frac{1}{i!} D^{(i)}_{x_U} u_2[\xi^{(i)}] + o(|\xi|^k)$$
$$\nabla u_2(x) = \sum_{i=0}^{k-1} \frac{1}{i!} D^{(i)}_{x_U} \nabla u_2[\xi^{(i)}] + o(|\xi|^{k-1})\,.$$

Since

$$D^{(k)}_{x_U} u_2[\xi^{(k)}] = D^{(k-1)}_{x_U}(\nabla u_2 \cdot \xi)[\xi^{(k-1)}] = \xi \cdot \big(D^{(k-1)}_{x_U}(\nabla u_2)[h^{(k-1)}]\big)\,,$$

by applying Cauchy-Schwarz inequality we obtain

$$|D^{(k)}_{x_U} u_2[\xi^{(k)}]| \le |\xi||D^{(k-1)}_{x_U}(\nabla u_2)[\xi^{(k-1)}]|\,.$$

It follows that

$$|u_2(x)|^2 \le |\xi|^2 |\nabla u_2(x)|^2 + o(|\xi|^{2k}) \quad \forall x \in \Omega^n \ :\ (x_{N-1},x_N) \in U\,. \tag{71}$$

For $x \in \Omega^n$ such that $(x_{N-1}, x_N) \notin U$, we let (z_{N-1}, x_N) be the point introduced in the above proof of claim (69), we set $\tilde{\xi} = (x', x_{N-1} - z_{N-1}, 0)$, and we argue in a similar way as above. We obtain

$$|u_2(x)|^2 \leq |\tilde{\xi}|^2 |\nabla u_2(x)|^2 + o(|\tilde{\xi}|^{2k}) \quad \forall x \in \Omega^n \ : \ (x_{N-1}, x_N) \notin U\,. \tag{72}$$

Integrating $|u_2|^2$ over Ω^n and taking into account that, by the $(\frac{1}{N-2})$-concavity of h_n, for small δ_n we have $\int_U h_n \geq \frac{|\Omega^n|}{2}$, we get, for some positive constants C_1 and C_2,

$$C_1 |h|^{2k} \int_U h_n \geq \int_{\Omega^n} |u_2|^2 \geq \frac{1}{(k!)^2} \int_U h_n \big(|D^{(k)}_{x_U} u_2[h^{(k)}]|^2 + o(|h|^{2k}) \big) \geq C_2 |h|^{2k} \int_U h_n\,.$$

Now, summing (71)-(72) over Ω_n, we obtain

$$\frac{\delta_n^2 \int_{\Omega^n} |\nabla u_2|^2}{\int_{\Omega^n} |u_2|^2} \geq 1 + o(1)\,.$$

This is not possible since $\delta_n \to 0$ and, from (67), the ratio $\frac{\int_{\Omega^n} |\nabla u_2|^2}{\int_{\Omega^n} |u_2|^2}$ is bounded from above. As a conclusion, our assumption that u_2 is identically zero on U fails to be true, yielding (70). □

6. The Neumann gap

Through the addition of few specific new results, the approach developed in the previous sections leads to a quantitative lower bound for the first nontrivial Neumann eigenvalue $\mu_1(\Omega, \phi)$ defined according to (2), where ϕ is a generic positive weight in $L^1(\Omega)$ which is no longer related to the first eigenfunction, but is power-concave, a particular case being $\phi \equiv 1$.

Actually, the statement of Theorem 2 holds more generally with $\mu_1(\Omega)$ replaced by $\mu_1(\Omega, \phi)$, being ϕ any weight which is $(\frac{1}{m})$-concave for some $m \in \mathbb{N} \setminus \{0\}$. Such assumption is needed not only for the existence of an eigenfunction $\overline{u}$ (see Proposition 43 in Appendix), but also for the control of the constant $\overline{c}$ in Theorem 2.

The proof proceeds along the same line as Theorem 1, being considerably simpler and yet demanding some nontrivial new ingredients. The main difficulty arises from the fact that an eigenfunction $\overline{u}$ can no longer be identified with $\frac{u_2}{u_1}$. We point out that this identification was crucial to obtain Proposition 32, which in turn allowed to reach a contradiction in Case 2.2.2.2 of our proof of Theorem 1 for $N = 2$.

To overcome such difficulty, we manage to acquire a control on the Lebesgue measure of the cells of the partition in terms of the width of Ω. This will be possible thanks to the following new geometrically explicit L^∞ estimate for Neumann eigenfunctions (see [15] for global Lipschitz regularity results):

Proposition 38. *There exists a positive constant C depending only on $N + m$ such that, for every open bounded convex domain $\Omega \subset \mathbb{R}^N$ with diameter D_Ω and any positive $\frac{1}{m}$-concave weight ϕ, a first eigenfunction $\overline{u}$ associated with the Neumann eigenvalue $\mu_1(\Omega, \phi)$ satisfies*

$$\|\overline{u}\|_{L^\infty(\Omega)} \leq C \mu_1(\Omega, \phi)^{\frac{N+m}{2}} \frac{D_\Omega^{N+m}}{\big(\int_\Omega \phi\big)^{1/2}} \|\overline{u}\|_{L^2(\Omega, \phi)}. \tag{73}$$

Remark 39. (i) In the particular case $\phi \equiv 1$, the estimate (73) reads

$$\|\overline{u}\|_{L^\infty(\Omega)} \leq C\mu_1(\Omega)^{\frac{N}{2}} \frac{D_\Omega^N}{|\Omega|^{\frac{1}{2}}} \|\overline{u}\|_{L^2(\Omega)} . \tag{74}$$

(ii) As it can be seen for the proof below in case $\phi = 1$, the result continues to hold for any Neumann eigenpair.

(iii) Combined with the result proved by Maz'ya in [43, Section 2], and with the upper bound for the relative isoperimetric constant stated in [17, Theorem 1.4], the inequality (74), written for an arbitrary Neumann eigenpair $(\mu_k(\Omega), u_k)$ gives the following gradient estimate

$$\|\nabla u_k\|_{L^\infty(\Omega)} \leq C\mu_k(\Omega)^{\frac{N}{2}+1} \frac{D_\Omega^{N+1}}{|\Omega|^{\frac{1}{2}}} \|u_k\|_{L^2(\Omega)}.$$

Proof. Let us start by proving the result for $\phi \equiv 1$. Let $a_1 \geq a_2 \geq \cdots \geq a_N$ denote the semi-axes of the John ellipsoid of Ω. Since (74) is invariant under scaling, we are going to assume without loss of generality that $a_1 = 1$. In order to reduce ourselves to work with domains having the unit ball as John ellipsoid, we perform the change of variables $X = T(x)$, with

$$X_1 = x_1, \quad X_2 = \frac{x_2}{a_2}, \quad \ldots \quad , X_N = \frac{x_N}{a_N} .$$

Taking into account that a_1 is comparable to D_Ω, it is readily checked that, in terms of the function v defined on $T(\Omega)$ by

$$v(X_1, \ldots, X_N) := \overline{u}(X_1, a_2X_2, \ldots, a_NX_N),$$

the inequality we want to prove becomes:

$$\|v\|_{L^\infty(T(\Omega))} \leq C\mu_1^{\frac{N}{2}}(\Omega)\|v\|_{L^2(T(\Omega))} . \tag{75}$$

Setting for brevity $A := T(\Omega)$ and $c = (c_1, \ldots, c_N) := (\frac{1}{a_1^2}, \ldots, \frac{1}{a_N^2})$, it holds

$$\mu_1(\Omega) := \frac{\int_\Omega |\nabla \overline{u}|^2}{\int_\Omega \overline{u}^2} = \frac{\sum_{i=1}^N \int_A \left|\frac{\partial v}{\partial X_i}\right|^2 c_i \, dX}{\int_A v^2 \, dX} =: \mu_c(A) ,$$

and the optimality of $\overline{u}$ can be reformulated as the following variational property of v

$$\sum_{i=1}^N \int_A c_i \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_i} = \mu_c(A) \int_A v\varphi \qquad \forall \varphi \in H^1(A) . \tag{76}$$

We now use the Moser iteration scheme: we claim that, if a solution v to (76) belongs to $L^p(A)$, then it belongs also to $L^{\frac{pN}{N-1}}(A)$; more precisely, for a dimensional constant C, it holds

$$\Big(\int_A |v|^{\frac{pN}{N-1}}\Big)^{\frac{N-1}{N}} \leq C\mu_c(A) \frac{p^2}{2(p-1)} \int_A |v|^p . \tag{77}$$

To prove the claim we observe that, if $v \in L^p(A)$, we can take $|v|^{p-2}v$ as test function in (76). This gives

$$\sum_{i=1}^N \int_A c_i \frac{\partial v}{\partial x_i} \frac{\partial |v|^{p-2} v}{\partial x_i} = \mu_c(A) \int_A |v|^p .$$

Since $c_i \geq 1$, we infer that

$$\frac{p-1}{\left(\frac{p}{2}\right)^2} \sum_{i=1}^N \int_A \Big(\frac{\partial |v|^{\frac{p}{2}}}{\partial x_i} \Big)^2 \leq \mu_c(A) \int_A |v|^p ,$$

and hence

$$\int_A \Big| \nabla |v|^{\frac{p}{2}} \Big|^2 = \sum_{i=1}^N \int_A \Big(\frac{\partial |v|^{\frac{p}{2}}}{\partial x_i} \Big)^2 \leq \mu_c(A) \frac{p^2}{4(p-1)} \int_A |v|^p .$$

Since

$$\int_A |v|^p + \int_A |\nabla |v|^p| = \int_A |v|^p + \int_U \Big| \nabla (|v|^{\frac{p}{2}})^2 \Big| = \int_A |v|^p + 2 \int_A |v|^{\frac{p}{2}} \Big| \nabla |v|^{\frac{p}{2}} \Big| ,$$

for every positive constant $\alpha > 0$ we have

$$\begin{aligned} \int_A |v|^p + \int_A |\nabla |v|^p| \ & \leq \int_A |v|^p + \alpha \int_A |v|^p + \frac{1}{\alpha} \int_A \Big| \nabla |v|^{\frac{p}{2}} \Big|^2 \\ & \leq \Big(\alpha + 1 + \frac{1}{\alpha} \mu_c(A) \frac{p^2}{4(p-1)} \Big) \int_A |v|^p . \end{aligned} \tag{78}$$

Next we observe that there exist positive dimensional constants C' and C'' such that, for every function $w \in W^{1,1}(A)$ (extended to zero outside $\overline{A}$), it holds

$$\Big(\int_A |w|^{\frac{N}{N-1}} \Big)^{\frac{N-1}{N}} \leq C' \Big(\int_A |\nabla w| + \int_{\partial A} |w| \Big) \leq C'' \Big(\int_A |\nabla w| + \int_A |w| \Big) . \tag{79}$$

Here the first inequality is due to the continuity of the embedding of $BV(\mathbb{R}^N)$ into $L^{\frac{N}{N-1}}(\mathbb{R}^N)$, and the second one to the continuity of the trace operator from $W^{1,1}(A)$ to $L^1(\partial A)$. Notice in particular that the fact that C'' is purely dimensional is due to the condition that the John ellipsoid of A is the unit ball (so that the outradius and the inradius of A are controlled respectively from above and from below).
By applying (79) with $w = |v|^p$, we infer from (78) that

$$\Big(\int_A |v|^{\frac{pN}{N-1}} \Big)^{\frac{N-1}{N}} \leq C'' \Big(\alpha + 1 + \frac{1}{\alpha} \mu_c(A) \frac{p^2}{4(p-1)} \Big) \int_A |v|^p .$$

The validity of our claim (77) (with $C = \frac{C''}{\alpha}$) follows from the above inequality after noticing that, by Payne-Weinberger inequality, and since we have fixed $a_1 = 1$, $\mu_c(A) = \mu_1(\Omega)$ is bounded from below by a dimensional constant so that, for α below a dimensional threshold,

$$\alpha + 1 \leq \frac{1}{\alpha} \mu_c(A) \frac{p^2}{4(p-1)} .$$

We now apply (77) recursively: setting $p_0 = 2$ and $p_{k+1} = \frac{N}{N-1} p_k$ (i.e., $p_k = 2\big(\frac{N}{N-1}\big)^k$), this gives

$$\|v\|_{L^{p_{k+1}}(A)} \leq (C\mu_c(A))^{\frac{1}{p_k}} \left(\frac{p_k^2}{2(p_k-1)}\right)^{\frac{1}{p_k}} \|v\|_{L^{p_k}(A)} .$$

In the limit as $k \to +\infty$ we obtain

$$\|v\|_{L^\infty(A)} \leq \prod_{k=0}^{\infty} (C\mu_c(A))^{\frac{1}{p_k}} \Big(\frac{p_k^2}{2(p_k-1)}\Big)^{\frac{1}{p_k}} \|v\|_{L^2(A)}.$$

The validity of the required inequality (75) (for some dimensional constant C) follows by noticing that

$$\sum_{k\geq 0} \frac{1}{p_k} = \frac{1}{2} \sum_{k\geq 0} \left(\frac{N-1}{N}\right)^k = \frac{N}{2},$$

and

$$\prod_{k=0}^{\infty} \Big(\frac{p_k^2}{2(p_k-1)}\Big)^{\frac{1}{p_k}} \leq \prod_{k=0}^{\infty} p_k^{\frac{1}{p_k}} < +\infty.$$

The case of a general $(\frac{1}{m})$-concave weight ϕ is obtained by collapsing, relying on the results proved in the Appendix. Precisely, we write inequality (74) on the domain $\widetilde{\Omega}_\varepsilon \subset \mathbb{R}^{N+m}$ defined in (88), and we obtain the validity of (73) in the limit as $\varepsilon \to 0$, by using the lower semicontinuity inequality (89) from Proposition 43.

□

We are now in a position to give the proof of Theorem 2. The idea is the same as for the Dirichlet case: reduce the N-dimensional gap estimate to a two-dimensional one. However, the two-dimensional one will involve a geometric weight, and so we are going to prove it more in general replacing $\mu_1(\Omega)$ by $\mu_1(\Omega, \phi)$, where ϕ is a fixed positive weight which is $(\frac{1}{m})$-concave for some $m \in \mathbb{N} \setminus \{0\}$.
To avoid cumbersome overlaps, we are going to outline the parts which closely follow the proof of Theorem 1, focusing our attention on the differences.
As in the case of the Dirichlet fundamental gap, by the Payne-Weinberger reduction argument, estimating from below $\mu_1(\Omega, \phi)$ amounts to estimating from below a weigthed one-dimensional Neumann eigenvalue of the type $\mu_1(I, p)$. The difference must be searched in the weight: now $p = h\phi$, where ϕ is the preassigned power-concave weight (which replaces u_1^2), while h is still, as in the Dirichlet case, the power-concave function giving the $(N-1)$-dimensional measure of the cell's section orthogonal to I. As a consequence, the one dimensional part of the proof is much simpler. Indeed, by the log-concavity of p, Lemma 6 still holds, so that

$$\mu_1(I, p) \geq \lambda_1(I, m_p)\,, \qquad \text{with } m_p := \Big[\frac{3}{4}\Big(\frac{p'}{p}\Big)^2 - \frac{1}{2}\frac{p''}{p}\Big] .$$

Now, the counterpart of the sharp one dimensional lower bound for $\lambda_1(I, m_p)$ given in Theorem 8 reads simply as follows: since p is log-concave we have that m_p is a positive measure and hence, working for definiteness of the interval I_π, we have

$$\lambda_1(I_\pi, m_p) \geq \lambda_1(I_\pi) = 1 .$$

By analogy with Theorem 21, keeping the notation $I_d = (-\frac{d}{2}, \frac{d}{2})$ for any $0 < d < \pi$, the above inequality can be refined in two distinct directions:

(i) There exists an absolute constant $C > 0$ such that

$$\lambda_1(I_d, m_p) \geq \lambda_1(I_d, 0) = \frac{\pi^2}{d^2} \geq 1 + C(\pi - d)\,. \tag{80}$$

(ii) There exists an absolute constant K such that, if $p = h\phi$, with h log-concave and affine on an interval $[a, b]$ with $I_{\frac{\pi}{4}} \subseteq [a, b] \subseteq I_d$, the following implication holds:

$$\lambda_1(I_d, m_p) \leq 2 \ \Rightarrow\ \lambda_1(I_d, m_p) \geq 1 + \Big[1 - \frac{\min\{h(a), h(b)\}}{\max\{h(a), h(b)\}}\Big]^2\,. \tag{81}$$

We remark that the lower bound (80), in which the exponent 1 replaces the exponent 3 appearing in (30), is a straightforward consequence of the Taylor expansion of $\frac{\pi^2}{(\pi-\varepsilon)^2}$ as $\varepsilon \to 0^+$. On the other hand, the lower bound (81) can be proved in similar way as (31). Passing to higher dimensions, the results in Section 3 about weighted measure or L^2 equipartitions remain unaltered working with our new weight.

Then, by using measure equipartitions, as a counterpart to Proposition 29, we obtain the following localized version of the (weighted) Payne-Weinberger inequality:

$$\mu_1(\omega, \phi) \geq \frac{\pi^2}{D^2} + C\frac{(D - d)}{D^3}\,, \tag{82}$$

where $\omega \subseteq \Omega \subset \mathbb{R}^N$ $(N \geq 2)$ are open bounded convex sets of diameters $d < D$.

With this inequality at our disposal, we proceed to prove the quantitative inequality. Notice in particular that it is not necessary to prove preliminarily rigidity as in the Dirichlet case: if the quantitative form of the Payne-Weinberger inequality holds true for cells with diameter larger than $\frac{\pi}{2}$ and second John semi-axis smaller than a fixed threshold $\overline{a}_2$, rigidity follows as direct consequence. Indeed, we consider a partition of Ω into cells with second John semi-axis smaller than $\overline{a}_2$. If a cell has diameter smaller than π, we get rigidity. Otherwise, all cells have diameter equal to π, and we get as well rigidity by writing the quantitative form of the inequality for a single cell, which has a strictly positive second John semi-axis. (Notice that, in the Dirichlet case, due to the presence of the weight u_1^2, a quantitative inequality for convex sets with small width does not imply a quantitative inequality on a single cell).

The proof of Theorem 2 is carried over first in case $N = 2$ and then for $N \geq 3$. In dimension $N = 2$, we are going to use, in place of Proposition 32, the following result obtained via the L^∞ estimate given in Proposition 38.

Proposition 40. *There exists a positive constant Λ, depending only on $N + m$, such that: if $\Omega \subset \mathbb{R}^N$ is an open bounded convex set, ϕ is a positive $\left(\frac{1}{m}\right)$-concave weight in $L^1(\Omega)$ and $\overline{u}$ is a first eigenfunction for $\mu_1(\Omega, \phi)$ normalized in $L^2(\Omega, \phi)$, and ω is a cell of a ϕ-weighted L^2 equipartition of $\overline{u}$ in Ω composed by n cells, it holds*

$$|\omega| \geq \Lambda\frac{|\Omega|}{n}.$$

Proof. By Proposition 38, we have

$$\frac{1}{n\int_\omega \phi} = \frac{\int_\omega \phi \overline{u}^2}{\int_\omega \phi} \leq \|\overline{u}\|^2_{L^\infty(\Omega)} \leq C\mu_1(\Omega,\phi)^{N+m}\frac{D_\Omega^{2(N+m)}}{\int_\Omega \phi}.$$

Since $\mu_1(\Omega,\phi)D_\Omega^{2(N+m)}$ is bounded from above by a constant K depending only on $N+m$ (see [37, 31] for $\phi \equiv 1$ and Proposition 43 for the general case), we conclude that

$$\int_\omega \phi \geq \frac{K}{n}\int_\Omega \phi\,.$$

On the other hand, we have $\|\phi\|_{L^\infty(\Omega)} < +\infty$ and, assuming with no loss of generality that $0 \in \overline{\Omega}$ is a maximum point for $\phi|_\Omega$, the $\left(\frac{1}{m}\right)$-concavity of ϕ implies that $\phi(x) \geq \frac{1}{2^m}\|\phi\|_{L^\infty(\Omega)}$ for every $x \in \frac{1}{2}\Omega$. Hence

$$\int_\Omega \phi \geq \frac{1}{2^{N+m}}\|\phi\|_{L^\infty(\Omega)}|\Omega|\,.$$

We deduce that

$$|\omega|\|\phi\|_{L^\infty(\Omega)} \geq \int_\omega \phi \geq \frac{K}{n}\int_\Omega \phi \geq \frac{K}{n2^{N+m}}\|\phi\|_{L^\infty(\Omega)}|\Omega|\,,$$

and the result follows with $\Lambda = \frac{K}{2^{N+m}}$. □

Proof of Theorem 2 in dimension $N = 2$. We follow the same geometric construction as in the proof of Theorem 1. Relying on the inequalities (81) and (82), all the cases work as previously, exception made for the last one, Case 2.2.2.2. To prove that this case cannot occur, the contradiction argument is still based on the validity of claim (60). The proof of such claim proceeds unaltered up to the inequality (61). At this point we have to argue differently. Indeed, in the Dirichlet setting, the estimate $L_i^{min} \geq \frac{\Lambda}{n}$ was obtained through the control on the length of the cell's section due to Proposition 32 (whose proof is no longer valid in the Neumann setting because it is based on the identification of a first eigenfunction with $\frac{u_2}{u_1}$). However, the validity of the estimate $L_i^{min} \geq \frac{\Lambda}{n}$ can be recovered thanks to the control on the cell's area due to Proposition 40. Once we have this estimate, claim (60) follows, and the proof can be concluded as in the Dirichlet case. □

Proof of Theorem 2 in dimension $N \geq 3$. Let us prove the inequality for $\phi \equiv 1$ (the case of a general $(\frac{1}{m})$-concave weight follows by collapsing, using Proposition 43).
It is not restrictive to prove the statement under the following assumptions: $D_\Omega = \pi$, $\mu_1(\Omega) \leq \pi^2 + 1$, and Ω smooth, so that a first eigenfunction $\overline{u}$ for $\mu_1(\Omega)$ belongs to $\mathcal{C}^2(\overline{\Omega})$. We fix a coordinate system $(e_1, \dots, e_N)$ such that a_1 is aligned with e_1 and a_2 is aligned with e_N. For every $n \in \mathbb{N}$, we consider a L^2 equipartition $\{\Omega_1, \dots, \Omega_n\}$ of $\overline{u}$ in Ω, obtained by the procedure described in Remark 25, namely by using a family of cutting hyperplanes parallel to e_N, in such way that, for n large, any cell Ω_i becomes narrow in $(N-2)$-directions, i.e., arbitrarily close to a $2D$-convex section U_i. Via the

usual argument à la Payne-Weinberger (namely arguing as in Lemmas 26 and 27, except that now our cells are narrow in $(N-2)$ in place of $(N-1)$-dimensions), we obtain

$$\mu_1(\Omega_i) \geq \mu_1(U_i, \phi) + o(1) \qquad \forall i = 1, \dots, n\,,$$

where ϕ is a $(\frac{1}{N-2})$-concave weight, and $o(1)$ is an infinitesimal quantity as $n \to +\infty$. By the quantitative inequality already proved for the weighted Neumann eigenvalue in $2D$, we infer that

$$\mu_1(\Omega_i) \geq 1 + c w_{U_i}^2 + o(1) \qquad \forall i = 1, \dots, n\,.$$

We now search for a good proportion of cells such that w_{U_i} controls from above a_2. It is not restrictive to confine our search among cells whose diameter is sufficiently large (otherwise, if for a fixed proportion of cells the diameter is small, we easily obtain the quantitative inequality from (82), by arguing as in the Dirichlet case). For cells with large diameter, w_{U_i} is comparable to the width of U_i in direction e_N, which by construction is equal to the width of Ω_i in direction e_N, hereafter denoted by $w^N_{\Omega_i}$. We claim that there exists a dimensional constant K such that

$$w^N_{\Omega_i} \geq K a_2 \qquad \forall i \in \mathcal{I}_n \subset \{1, \dots, n\} \text{ with } \mathrm{card}(\mathcal{I}_n) \geq \frac{n}{2}\,. \tag{83}$$

Indeed, for $t > 0$, let us denote by $\mathcal{I}^t_n$ the family of indices $i \in \{1, \dots, n\}$ such that $w^N_{\Omega_i} < t a_2$. We have

$$|\omega_i| \sim w^N_{\Omega_i} \mathcal{H}^{N-1}(\Pi_{e_N^\perp}(\omega_i)) < t a_2 \mathcal{H}^{N-1}(\Pi_{e_N^\perp}(\omega_i)) \qquad \forall i \in \mathcal{I}^t_n\,.$$

On the other hand, for every $i = 1, \dots, n$, by Proposition 40 we have, for dimensional constants Λ, Λ'

$$|\omega_i| \geq \frac{\Lambda}{n} |\Omega| \geq \frac{\Lambda'}{n} a_1 a_2 \dots a_N\,.$$

We infer that

$$\frac{\Lambda'}{n} a_1 a_3 \dots a_N \leq t \mathcal{H}^{N-1}(\Pi_{e_N^\perp}(\omega_i)) \qquad \forall i \in \mathcal{I}^t_n\,.$$

Summing over $i \in \mathcal{I}^t_n$, we obtain

$$\mathrm{card}(\mathcal{I}^t_n) \frac{\Lambda'}{n} a_1 a_3 \dots a_N \leq t \mathcal{H}^{N-1}(\Pi_{e_N^\perp}(\Omega)) \sim t a_1 a_3 \dots a_N\,,$$

which implies

$$t - \mathrm{card}(\mathcal{I}^t_n) \frac{\Lambda'}{n} \geq 0\,.$$

Then claim (83) is proved taking $\mathcal{I}_n := \{1, \dots, n\} \setminus \mathcal{I}^t_n$, with $t := \frac{\Lambda'}{2}$.
In view of (83), our proof is easily achieved by applying in the usual way Lemma 28:

$$\begin{aligned}
\mu_1(\Omega) &= \frac{\int_\Omega |\nabla \overline{u}|^2}{\int_\Omega |\overline{u}|^2} \geq \frac{1}{n} \sum_{i=1}^n \frac{\int_{\Omega_i} |\nabla \overline{u}|^2}{\int_{\Omega_i} |\overline{u}|^2} \geq \frac{1}{n} \Big[\sum_{i \in \mathcal{I}_n} \mu_1(\Omega_i) + \sum_{i \notin \mathcal{I}_n} \mu_1(\Omega_i) \Big] \\
&\geq \frac{1}{n} \Big[(1 + K a_2^2) \mathrm{card}(\mathcal{I}_n) + (n - \mathrm{card}(\mathcal{I}_n)) \Big] \geq 1 + \frac{K}{2} a_2^2\,.
\end{aligned}$$

□

7. Appendix

In this section we fix some results about weighted Neumann eigenvalues of the type $\mu_1(\Omega, p)$, defined according to (2). First we consider the case when p equals u_1^2, and then the case when p equals a $(\frac{1}{m})$-concave function ϕ. In both cases we have that an eigenfunction exists: in the former case it can be explicitly identified with the quotient between the second and first Dirichlet eigenfunction, in the latter case it can be obtained by a collapsing procedure, namely as the limit of rescaled Neumann eigenfunctions in higher dimensional convex sets with suitable profile.

Proposition 41. *Given an open bounded convex domain Ω in $\mathbb{R}^N$, let $\lambda_1(\Omega), \lambda_2(\Omega)$ and u_1, u_2 be the first two Dirichlet eigenvalues and eigenfunctions of Ω (normalized in L^2), and let $\mu_1(\Omega, u_1^2)$ be defined according to* (2)*. Then it holds*

$$\mu_1(\Omega, u_1^2) = \lambda_2(\Omega) - \lambda_1(\Omega)\,,$$

and an eigenfunction for $\mu_1(\Omega, u_1^2)$ is given by $\overline{u} := \frac{u_2}{u_1}$.

Proof. When Ω is smooth, the result is well-known [14, Section 1.2.2]; actually, in this case we have that $\overline{u} = \frac{u_2}{u_1}$ is smooth up to the boundary of Ω, see [50, Appendix A]. When Ω is not smooth, the statement can be obtained by approximation. Consider an increasing sequence of open smooth convex domains $\Omega_\varepsilon \subset \Omega$ converging to Ω in Hausdorff distance as $\varepsilon \to 0$. For $i = 1, 2$, let $\lambda_i(\Omega_\varepsilon)$ and u_i^ε denote the first two Dirichlet eigenvalues and eigenfunctions of Ω_ε, normalized in L^2, and extended to 0 in $\Omega \setminus \Omega_\varepsilon$. We claim that, in the limit as $\varepsilon \to 0$, we have

$$\lambda_i(\Omega_\varepsilon) \to \lambda_i(\Omega)\,, \qquad u_i^\varepsilon \to u_i \text{ in } H_0^1(\Omega) \qquad i = 1, 2\,, \qquad u_1^\varepsilon \to u_1 \text{ in } L^\infty(\Omega)\,.$$

To prove this claim we observe first that, for ε small enough, the domains Ω_ε contain a fixed ball. Hence, by the decreasing monotonicity of Dirichlet eigenvalues under domain inclusion, the sequence $\lambda_i(\Omega_\varepsilon)$ is bounded. It follows that Δu_i^ε is bounded in $L^2(\Omega_\varepsilon)$, and hence u_i^ε is bounded in $H^2(\Omega_\varepsilon)$ (see e.g. [26, Theorem 3.1.2.1]), so that up to subsequences it converges weakly in $H^2(\Omega)$ and strongly in $H_0^1(\Omega)$; then, the limits of $\lambda_i(\Omega_\varepsilon)$ and u_i^ε can be identified respectively with $\lambda_i(\Omega)$ and u_i (see e.g. [12, Section 4.6]). It remains to prove that $u_1^\varepsilon \to u_1$ in $L^\infty(\Omega)$. By [18, Lemma 3.1], there exists a dimensional constant C such that $\|u_1^\varepsilon\|_\infty \leq C\lambda_1(\Omega_\varepsilon)^{N/4}$, so that the sequence u_1^ε remains bounded in $L^\infty(\Omega_\varepsilon)$.
In turn, this implies that $\sup_\varepsilon \|\nabla u_1^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} < +\infty$. Indeed, denoting by w^ε the torsion function on Ω_ε (i.e. the unique solution in $H_0^1(\Omega_\varepsilon)$ to the equation $-\Delta w = 1$ in Ω_ε), by direct computation the function

$$P_\varepsilon(x) = |\nabla u_1^\varepsilon|^2 + \lambda_1(\Omega_\varepsilon)(u_1^\varepsilon)^2 - 2\lambda_1^2(\Omega_\varepsilon)\|u_1^\varepsilon\|_\infty^2 w^\varepsilon$$

is subharmonic in Ω_ε (see also [13, Section 3]). Hence P_ε attains its maximum at the boundary. The uniform boundedness of ∇u_1^ε in $L^\infty(\Omega_\varepsilon)$ follows by taking into account that $\|w^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}$ is bounded from above by a constant depending on $|\Omega^\varepsilon|$ and that, by a classical barrier argument, $\|\nabla u_1^\varepsilon\|_{L^\infty(\partial\Omega_\varepsilon)}$ is bounded from above by $\lambda_1(\Omega^\varepsilon)\|u_1^\varepsilon\|_\infty$.
Hence the functions u_1^ε are equibounded and equicontinuous in Ω, and the uniform convergence of u_1^ε to u_1 follows from the Ascoli-Arzelà theorem.

Now we observe that the function $\overline{u}$ is admissible in the definition (2) of $\mu_1(\Omega)$, so that

$$\mu_1(\Omega, u_1^2) \leq \int_\Omega |\nabla \overline{u}|^2 u_1^2 \, dx \,. \tag{84}$$

By the strong convergences $u_i^\varepsilon \to u_i$ in $H_0^1(\Omega)$, for every compact set $K \subset \Omega$, setting $v^\varepsilon := \frac{u_2^\varepsilon}{u_1^\varepsilon}$, up to subsequences it holds $|\nabla v^\varepsilon|^2 (u_1^\varepsilon)^2 \to |\nabla \overline{u}|^2 u_1^2$ a.e. on K. Hence, by Fatou's lemma,

$$\int_K |\nabla \overline{u}|^2 u_1^2 \, dx \leq \liminf_{\varepsilon \to 0} \int_K |\nabla v^\varepsilon|^2 (u_1^\varepsilon)^2 \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 (u_1^\varepsilon)^2. \tag{85}$$

By (84) and (85), exploiting the arbitrariness of K, and using the statement for the smooth domains Ω_ε, we obtain

$$\mu_1(\Omega, u_1^2) \leq \int_\Omega |\nabla \overline{u}|^2 u_1^2 \, dx \leq \liminf_{\varepsilon \to 0} (\lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon)) = \lambda_2(\Omega) - \lambda_1(\Omega) \,. \tag{86}$$

To conclude the proof, it remains to show that the two inequalities in (86) are in fact equalities. Let $\delta > 0$ be fixed, and let v_δ be a function in $H^1_{\rm loc}(\Omega)$, with $\int_\Omega v_\delta^2 u_1^2 = 1$ and $\int_\Omega v_\delta u_1^2 = 0$, such that $\mu_1(\Omega, u_1^2) \geq \int_\Omega |\nabla v_\delta|^2 u_1^2 - \delta$. Since $\Omega_\varepsilon \subset \Omega$, we have that $v_\delta \in H^1_{\rm loc}(\Omega_\varepsilon)$. If the approximating domains Ω_ε are suitably chosen, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} v_\delta^2 (u_1^\varepsilon)^2 = \int_\Omega v_\delta^2 u_1^2 \, (=1) \,, \\ &\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} v_\delta (u_1^\varepsilon)^2 = \int_\Omega v_\delta u_1^2 \, (=0) \,, \\ &\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v_\delta|^2 (u_1^\varepsilon)^2 = \int_\Omega |\nabla v_\delta|^2 u_1^2 \,. \end{aligned} \tag{87}$$

More precisely, by the uniform convergence of u_1^ε to u_1 and Lebesgue dominated convergence theorem, the equalities in (87) are satisfied provided Ω_ε is chosen so that, for some $\eta_\varepsilon \to 0$, $u_1^\varepsilon \leq (1+\eta_\varepsilon) u_1$. Such choice is possible thanks to the following argument. Fix the origin at the maximum point of u_1, and consider a small contraction $(1-\varepsilon)\Omega$ of Ω with respect to the origin. The log-concavity of u_1 allows to order by inclusion the level sets of the functions $u_1(\frac{x}{1-\varepsilon})$ and $u_1(x)$, yielding that $u_1(\frac{x}{1-\varepsilon}) \leq u_1(x)$ for every $x \in (1-\varepsilon)\Omega$, with strict inequality except at the origin. Then, taking Ω_ε as a smooth convex approximation of $(1-\varepsilon)\Omega$, we get the required inequality $u_1^\varepsilon \leq (1+\eta_\varepsilon) u_1$, for some $\eta_\varepsilon \to 0$.

We finally notice that the convergences in (87) remain true if we replace therein the functions v_δ by their translations and normalizations $v_{\delta,\varepsilon}$, defined by

$$v_{\delta,\varepsilon} := \frac{v_\delta - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} v_\delta (u_1^\varepsilon)^2}{\| v_\delta - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} v_\delta (u_1^\varepsilon)^2 \|_{L^2(\Omega_\varepsilon, (u_1^\varepsilon)^2)}} \,.$$

Since $v_{\delta,\varepsilon}$ is an admissible test function for $\mu_1(\Omega_\varepsilon, (u_1^\varepsilon)^2)$, by using the statement for the smooth domains Ω_ε we obtain

$$\mu_1(\Omega, u_1^2) \geq \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v_{\delta,\varepsilon}|^2 (u_1^\varepsilon)^2 - \delta \geq \limsup_{\varepsilon \to 0} [\lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) - \delta] = \lambda_2(\Omega) - \lambda_1(\Omega) - \delta \,.$$

Eventually, by letting $\delta \to 0$, we conclude that the two inequalities in (86) hold with equality sign. □

We now turn attention to the case of power-concave weights. For the convenience of the reader we start with the following

Lemma 42. *Let Ω be an open bounded convex domain in $\mathbb{R}^N$, and let $\phi \in L^1(\Omega)$ be a positive weight which is $(\frac{1}{m})$-concave for some $m \in \mathbb{N} \setminus \{0\}$. Then the embedding $H^1(\Omega, \phi) \hookrightarrow L^2(\Omega, \phi)$ is compact.*

Proof. Let $\widetilde{\Omega} \subseteq \mathbb{R}^{N+m}$ be defined by

$$\widetilde{\Omega} = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^m : x \in \Omega, \ \|y\|_{\mathbb{R}^m} < \omega_m^{-1/m} \phi^{1/m}(x) \right\} .$$

Since $\phi^{1/m}$ is concave, $\widetilde{\Omega}$ is open and convex, so that $H^1(\widetilde{\Omega})$ is compactly embedded into $L^2(\widetilde{\Omega})$. Now, if $\{u_n\}$ is a bounded sequence in $H^1(\Omega, \phi)$, setting $\tilde{u}_n(x, y) := u_n(x)$ we have

$$\int_{\widetilde{\Omega}} \tilde{u}_n^2(x, y)\, dxdy = \int_\Omega u_n^2(x)\phi(x)\, dx,$$
$$\int_{\widetilde{\Omega}} |\nabla_{x,y}\tilde{u}_n|^2(x, y)\, dxdy = \int_\Omega |\nabla_x u_n|^2(x)\phi(x)\, dx.$$

Then, up to subsequences, $\tilde{u}_n$ converges weakly in $H^1(\widetilde{\Omega})$ and strongly in $L^2(\widetilde{\Omega})$ to a function $\tilde{u} \in H^1(\widetilde{\Omega})$. Since $\tilde{u}$ is constant in the y variable, the function $u(x) := \tilde{u}(x, y)$ belongs to $H^1(\Omega, \phi)$, and u_n converges strongly to u in $L^2(\Omega, \phi)$. □

Now, under the assumptions of Lemma 42, the compactness of the embedding $H^1(\Omega, \phi) \hookrightarrow L^2(\Omega, \phi)$ ensures that the operator mapping a function $f \in L^2(\Omega, \phi)$ with $\int_\Omega f\phi = 0$ into the unique solution to

$$\inf_{v \in H^1(\Omega,\phi)} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 \phi - \int_\Omega f v \phi \right)$$

is positive, self-adjoint, and compact. Then the eigenvalues of the weighted problem

$$\begin{cases} -\text{div}\,(\phi \nabla u) = \mu(\Omega, \phi)\phi u & \text{in } \Omega \\ \phi \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

can be computed by the classical min-max formula. In particular, we have that

$$\mu_1(\Omega, \phi) = \inf_{u \in H^1(\Omega,\phi)\setminus\{0\}, \int_\Omega u\phi = 0} \frac{\int_\Omega |\nabla u|^2 \phi}{\int_\Omega u^2 \phi},$$

and the infimum is attained. We now show that the above eigenvalue and a first eigenfunction associated with it can be obtained by collapsing, i.e. by a limiting procedure as as $\varepsilon \to 0^+$ starting from the convex hypographs

$$\widetilde{\Omega}_\varepsilon := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^m : x \in \Omega, \ \|y\|_{\mathbb{R}^m} < \varepsilon \omega_m^{-1/m} \phi^{1/m}(x) \right\} \subseteq \mathbb{R}^{N+m}. \tag{88}$$

Proposition 43. *Let Ω be an open bounded convex domain in $\mathbb{R}^N$, and let $\phi \in L^1(\Omega)$ be a positive weight which is $(\frac{1}{m})$-concave for some $m \in \mathbb{N} \setminus \{0\}$. Let $\tilde{u}_\varepsilon$ be L^2-normalized first eigenfunctions associated with $\mu_1(\widetilde{\Omega}_\varepsilon)$. Setting $u_\varepsilon(X,Y) := \varepsilon^{\frac{m}{2}} \tilde{u}_\varepsilon(X, \varepsilon Y)$, up to subsequences we have*

$$\mu_1(\widetilde{\Omega}_\varepsilon) \to \mu_1(\Omega, \phi) \quad \text{and} \quad u_\varepsilon \to \tilde{u} \text{ in } L^2(\widetilde{\Omega}_1),$$

where $\tilde{u}(X,Y) := u(X)$, u being a $L^2(\Omega,\phi)$-normalized eigenfunction associated with $\mu_1(\Omega,\phi)$. In particular

$$\|u\|_{L^\infty(\Omega)} \leq \liminf_{\varepsilon \to 0} \varepsilon^{\frac{m}{2}} \|\tilde{u}_\varepsilon\|_{L^\infty(\widetilde{\Omega}_\varepsilon)}. \tag{89}$$

Proof. By the change of variables $X = x, Y = \frac{1}{\varepsilon} y$, we get

$$\int_{\widetilde{\Omega}_1} |\nabla_X u_\varepsilon|^2 + \frac{1}{\varepsilon^2} |\nabla_Y u_\varepsilon|^2 dX dY = \int_{\widetilde{\Omega}_\varepsilon} |\nabla_x \tilde{u}_\varepsilon|^2 + |\nabla_y \tilde{u}_\varepsilon|^2 dx dy = \mu_1(\widetilde{\Omega}_\varepsilon)$$

$$\int_{\widetilde{\Omega}_1} u_\varepsilon^2(X,Y) dX dY = 1, \quad \int_{\widetilde{\Omega}_1} u_\varepsilon(X,Y) dX dY = 0\,.$$

From Kröger's inequality [37], we know that $\limsup \mu_1(\widetilde{\Omega}_\varepsilon) < +\infty$, so that $\{u_\varepsilon\}$ is bounded in $H^1(\widetilde{\Omega}_1)$, and possibly passing to a subsequence it converges weakly in $H^1(\widetilde{\Omega}_1)$ to some function $\tilde{u}$ with $\nabla_Y \tilde{u} = 0$ in $\widetilde{\Omega}_1$. Setting $u(X) := \tilde{u}(X,Y)$, we get that $\int_\Omega u^2 \phi = 1$, $\int_\Omega u\phi = 0$ and

$$\mu_1(\Omega,\phi) \leq \int_\Omega |\nabla u|^2 \phi \leq \liminf_{\varepsilon \to 0} \mu_1(\widetilde{\Omega}_\varepsilon).$$

To show the converse inequality, let v be a normalized first eigenfunction for $\mu_1(\Omega,\phi)$ and define

$$\tilde{v}_\varepsilon \in H^1(\widetilde{\Omega}_\varepsilon), \qquad \tilde{v}_\varepsilon(x,y) := \varepsilon^{-\frac{m}{2}} v(x).$$

We have

$$\int_{\widetilde{\Omega}_\varepsilon} \tilde{v}_\varepsilon = 0, \qquad \int_{\widetilde{\Omega}_\varepsilon} \tilde{v}_\varepsilon^2 = 1, \qquad \int_\Omega |\nabla v|^2 \phi = \int_{\widetilde{\Omega}_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2,$$

so that $\tilde{v}_\varepsilon$ is a test function for $\mu_1(\widetilde{\Omega}_\varepsilon)$ and $\mu_1(\Omega,\phi) \geq \limsup_{\varepsilon \to 0} \mu_1(\widetilde{\Omega}_\varepsilon)$. We conclude that the equality $\mu_1(\Omega,\phi) = \lim_{\varepsilon \to 0} \mu_1(\widetilde{\Omega}_\varepsilon)$ holds, and that the function u above has to be an eigenfunction for $\mu_1(\Omega,\phi)$. Together with the convergence $u_\varepsilon \to \tilde{u}$ in $L^2(\widetilde{\Omega}_1)$, this implies (89). ☐

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