AN EXISTENCE RESULT FOR ACCRETIVE GROWTH IN ELASTIC SOLIDS

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Abstract. We investigate a model for the accretive growth of an elastic solid. The reference configuration of the body is accreted in its normal direction, with space- and deformation-dependent accretion rate. The time-dependent reference configuration is identified via the level sets of the unique viscosity solution of a suitable generalized eikonal equation. After proving the global-in-time well-posedness of the quasistatic equilibrium under prescribed growth, we prove the existence of a local-in-time solution for the coupled equilibrium-growth problem, where both mechanical displacement and time-evolving set are unknown. A distinctive challenge is the limited regularity of the growing body, which calls for proving a new uniform Korn inequality.

1. Introduction

Many mechanical systems experience *accretive growth*, namely, progressive growth of a body by adding mass at its boundary. This paradigm is of paramount relevance to numerous biological systems, where shape and function evolve over time: The formation of horns, teeth and seashells [32, 36], secondary growth in trees [17], and cell motility due to actin growth [21] are examples of accretive growth in nature. Furthermore, accretive growth is a key aspect in a variety of technological applications, such as, for example, metal solidification [31], crystal growth [26], and additive manufacturing [22].

The first theoretical study of accretive growth involved the analysis of thick-walled cylinders manufactured by wire winding of an initial elastic tube [33]. Within the framework of linear elasticity, one of the earliest problems addressed was that of a growing planet subject to self-gravity [9]. In [29], the author proposed a first large-deformation theory of accretion, specifically tailored for aging viscoelastic solids undergoing accretion. This work also introduced the notion of *time-of-attachment map*, which is the function θ used here, as well. The engineering literature on accretive growth is vast. Among the many contributions, we single out [1, 6] for the modeling of surface growth on deformable substrates, [16] for a description of the kinetics of boundary growth, [27, 38, 40] for studies in the setting of nonlinear elasticity, as well as [37] for a description of accretion on a hard spherical surface. A finite strain model combining accretion and ablation may also be found in [35]. For more detailed reviews, we refer to [14, 30, 34].

Compared with the extensive engineering literature, rigorous mathematical results on the mechanics of growth, whether accretive or volumetric, are sparse [8, 12]. To the best of our knowledge, the most significant mathematical efforts thus far have primarily focused either

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on performing numerical simulations or on specifying the correct modeling framework for tailored applications, see, e.g., [5, 15, 18].

In this paper, we revisit an accretive-growth model advanced by ZURLO AND TRUSKINOVSKI [40]. Accretive growth is described by specifying a set-valued time-dependent function $t \in [0,T] \mapsto \Omega(t) \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), identifying at each time t the reference configuration of the body under study. Such function is increasing in time with respect to set inclusion (growth) and has open and connected values. Assume for the time being that $\Omega(t)$ is known (note however that this will also be an unknown later on). One can equivalently describe the evolution of the time-dependent reference configuration by means of a time-of-attachment continuous function $\theta : \mathbb{R}^d \to [0,\infty)$, whose value $\theta(x)$ indicates the time at which the point x is *added* to the solid. Correspondingly, one defines $\Omega(0) := \mathrm{int}\{x \in \mathbb{R}^d \ : \ \theta(x) = 0\}$ (interior part) and

$$\Omega(t) := \{x \in \mathbb{R}^d \ : \ \theta(x) < t\} \quad \text{for all } t > 0.$$

Let us stress from the very beginning that the growth process may prevent $\Omega(t)$ from being regular at specific times, posing a challenge to the analysis. The mechanical state of the body is then described by its *displacement* $u(\cdot,t) : \Omega(t) \to \mathbb{R}^d$ from the time-dependent reference configuration $\Omega(t)$.

Growth and mechanical equilibration generally occur on very distinct time scales. The usual time frame for growth ranges between minutes and months, whereas mechanical equilibrations can take up to few milliseconds, depending on the material. This basic observation leads us to consider the equilibrium of the growing object in its quasistatic approximation, namely,

$$-\nabla \cdot \sigma(x,t) = f(x,t) \quad \text{for } x \in \Omega(t),\ t \in [0,T] \tag{1.1}$$

where σ indicated the *stress*. We assume a linear elastic response in the solid. As accretive growth is known to generate residual stresses, following [40] we postulate the constitutive relation

$$\sigma(x,t) = \mathbb{C}\big(\varepsilon(u(x,t)) - \alpha(x)\big) \quad \text{for } x \in \Omega(t),\ t \in [0,T]. \tag{1.2}$$

Here, $\mathbb{C}$ is the symmetric and positive-definite *elasticity tensor*, the symmetrized displacement gradient $\varepsilon(u) = (\nabla u + \nabla u^\top)/2$ is the *strain*, and $\alpha : \Omega(T) \to \mathbb{R}^{d\times d}$ denotes the *backstrain*, which has been accumulated during growth. By assuming that material is added at the boundary of the solid in a locally unstressed state, we would follow [40] and postulate

$$\alpha(x) = \varepsilon(u)(x,\theta(x)) \quad \text{for } x \in \Omega(T).$$

In fact, together with (1.2) this would entail that $\sigma(x,\theta(x)) = 0$. On the other hand, such position would require $\varepsilon(u)$ to admit a space-time trace at points of the form $(x,\theta(x))$, a possibility which might be impeded by the low regularity of $\Omega(t)$. We hence resort to a mollification of the above position by actually defining

$$\alpha(x) := (K\overline{\varepsilon})(u)(x,\theta(x)) \quad \text{for } x \in \Omega(T), \tag{1.3}$$

where $\overline{\varepsilon}(u)(\cdot,t)$ denotes the trivial extension of $\varepsilon(u)(\cdot,t)$ to the whole $\mathbb{R}^d$ and K is a space-time convolution operator of the form

$$(K\overline{\varepsilon}(u))(x,t) := \int_0^t \int_{\mathbb{R}^d} k(t-s)\phi(x-y)\overline{\varepsilon}(u)(y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{1.4}$$

for given time- and space-kernels $k \in W^{1,1}(0,T)$ and $\phi \in H^1(\mathbb{R}^d)$, respectively. From the modeling viewpoint, definition (1.3) links the residual growth-originated backstrain $\alpha(x)$

to the *local mean* strain state at added material points, rather than to the *pointwise* one. By choosing the supports of k and ϕ sufficiently small around 0, one has the possibility of arbitrarily localizing this effect. Still, under the action of the (trivial extension and) convolution operator one is allowed to take trace values on the manifold $(x, \theta(x))$, without further regularity restrictions.

Ideally, we would complement the equilibrium system (1.1) by traction-free boundary conditions

$$\sigma(x,t)\, n(x,t) = 0 \quad \text{for } x \in \partial\Omega(t),\ t \in [0,T].$$

Still, as the growing set $\Omega(t)$ cannot be expected to be regular for all times, a classical Sobolev trace on $\partial\Omega(t)$ might be not available for some t. Hence, the latter *natural* condition will have to be casted variationally, within a weak reformulation of (1.1)-(1.4), see (2.2) below. To this aim, in order to filter out rigid-body motions, some condition has to be added to the equilibrium system (1.1)-(1.3). As the boundary $\partial\Omega(t)$ is evolving, in order to keep notation to a minimum we ask for the *docking condition*

$$u(x,t) = 0 \quad \text{on } \ \omega \times [0,T], \tag{1.5}$$

where we have fixed the docking set $\omega \subset \Omega(0)$. Condition (1.5) bears some applicative relevance, especially for $d = 1$ or 2.

Let us now turn our attention to the accretion process. Here, we intend to model a situation where accretion results from deposition at the boundary, at a given rate. Correspondingly, a point $x(t) \in \partial\Omega(t)$ at the boundary is assumed to follow the *normal accretion* law

$$\dot{x}(t) = \gamma\, n(x(t))$$

where $n(x(t))$ indicates the outward normal to $\partial\Omega(t)$ at $x(t)$ and γ is the growth rate, which will be later assumed to be dependent on position and strain. Note that the evolution of $\Omega(t)$ depends on its intrinsic geometry via $n(x(t))$. As the level sets of the function θ correspond by definition to the sets $\Omega(t)$, one formally has that $n(x(t)) = \frac{\nabla\theta(x(t))}{|\nabla\theta(x(t))|}$. At the same time, by differentiating the equality $t = \theta(x(t))$ with respect to time, one gets

$$1 = \nabla\theta(x(t)) \cdot \dot{x}(t) = \nabla\theta(x(t)) \cdot \gamma\, n(x(t)) = \gamma\, \nabla\theta(x(t)) \cdot \frac{\nabla\theta(x(t))}{|\nabla\theta(x(t))|} = \gamma|\nabla\theta(x(t))|$$

so that θ ultimately solves the eikonal equation $\gamma|\nabla\theta| = 1$. Growth processes are known to be inhomogeneous and to be dependent on the deformation state [19]. We model this by letting the growth rate γ depend smoothly on the point $x(t)$ and the strain at $x(t)$. Note that no dependence on the stress is directly accounted for by this model. In fact, in the nonregularized case $\alpha(x(t)) = \varepsilon(u)(x(t),t)$, position (1.2) would imply $\sigma(x(t),t) = 0$ at $x(t) \in \partial\Omega(t)$. Additional dependencies of the growth rate γ on time and displacement could also be considered, at the cost of minor albeit tedious changes.

Starting from the datum $\theta = 0$ on $\Omega(0)$, the evolution of $\Omega(t)$ is hence determined by solving the generalized eikonal equation

$$\gamma(x, \alpha(x))|\nabla\theta(x)| = 1. \tag{1.6}$$

This equation is in principle to be solved on $\Omega(T)$ only. Still, as this set depends on the solution θ itself, one may conveniently solve (1.6) is some larger set containing $\Omega(T)$ (recall that α from (1.3) is actually defined everywhere in $\mathbb{R}^d$). Equation (1.6) does not admit classical solutions. Moreover, strong solutions of (1.6) are not unique. We hence resort to the viscosity-solution setting, where equation (1.6) turns out to be well-posed. Note that the continuity of $x \mapsto \gamma(x, \alpha(x))$ is needed in order to tackle problem (1.6) in the setting of viscosity solutions. Such continuity calls for some smoothness of α, which is in turn guaranteed by our positions (1.3)-(1.4).

The main result of the paper, Theorem 2.6, provides the existence of a weak local-in-time solution to the coupled equilibrium-growth problem (1.1)-(1.6). Note that our level-set formulation via θ allows us to consider the evolution problem beyond singularities, which occur as the growing body self-touches. As an intermediate step toward Theorem 2.6 we discuss the global well-posedness of the equilibrium problem (1.1)-(1.5) for given θ, see Theorem 2.5. Compared with the analysis in [40], the novelty of our result is twofold. At first, we do not assume to know the displacement at the added material point. Secondly, we do not assume the evolution $t \mapsto \Omega(t)$ to be known, but rather solve for it, taking into account mechanical couplings and the possible onset of singularities.

The paper is organized as follows: in Section 2 we specify our assumptions and state our main results. Section 3 is devoted to the proof of Theorem 2.5, whereas Theorem 2.6 is proven in Section 4. Eventually, in Section 5 we prove a uniform Korn inequality for the class of sets generated by our growth process.

2. Setting and main results

We devote this section to the specification of the problems under scrutiny. In particular, we introduce the assumptions and discuss some preliminary remarks. The statements of our main results are in Subsection 2.2 below. We start by collecting some notation which will be used throughout the paper.

Notation. Let $d \in \mathbb{N}$. We indicate by $B_r(x_0) := \{x \in \mathbb{R}^d \ : \ |x - x_0| < r\}$ the open ball in $\mathbb{R}^d$ centered in $x_0 \in \mathbb{R}^d$ with radius $r > 0$. By $C^0_b(\mathbb{R}^d; \mathbb{R}^{d\times d})$ we denote the space of bounded continuous functions on $\mathbb{R}^d$ with values in $\mathbb{R}^{d\times d}$. The d-dimensional Lebesgue measure of a measurable set Ω in $\mathbb{R}^d$ is denoted by $|\Omega|$. The symbol $a \cdot b$ classically indicates the scalar product between the two vectors a, $b \in \mathbb{R}^d$. The contraction between 2-tensors A, $B \in \mathbb{R}^{d\times d}$ is denoted by $A : B = A_{ij}B_{ij}$, where repeated indices are tacitly summed over. Given the 4-tensor $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}$, we let $(\mathbb{C} : A)_{ij} = \mathbb{C}_{ijkl}A_{kl}$. We indicate by $\mathbb{I}$ the identity 4-tensor. The distance between $x \in \mathbb{R}^d$ and the nonempty set $U \subset \mathbb{R}^d$ is denoted by $\mathrm{dist}\,(x; U) = \inf_{u\in U} |x - u|$. The same notation is also used for the Hausdorff distance between two nonempty sets U_1, $U_2 \subset \mathbb{R}^d$, namely, $\mathrm{dist}\,(U_1; U_2) = \max\{\sup_{u_1\in U_1} \mathrm{dist}\,(u_1; U_2), \sup_{u_2\in U_2} \mathrm{dist}\,(U_1; u_2), \}$. We say that $U_n \to U$ *in the Hausdorff sense* iff $\mathrm{dist}\,(U_n; U) \to 0$.

2.1. Assumptions and notion of weak solution.

In this section, we present our assumptions on data and introduce the notion of weak solution to problem (1.1)-(1.6). Let us first recall the definition of a John domain.

Definition 2.1 (John domain). *A nonempty open set $U \subset \mathbb{R}^d$ is said to be a* John domain *if there exists a specific point $x_0 \in U$ and a* John constant $C_J \in (0, 1]$ *such that for all points $x \in U$ one can find an arc-length parametrized curve $\rho : [0, L_\rho] \to U$ with $\rho(0) = x$, $\rho(L_\rho) = x_0$, and* $\mathrm{dist}\,(\rho(s); \partial U) \geq C_J s$ *for all $s \in [0, L_\rho]$.*

John domains have been introduced in [24], see Figure 1. Their name has been proposed in [28]. Note that John domains are connected and their boundary is negligible [25, Corollary 2.3]. All Lipschitz domains are John, whereas John domains may have fractal boundaries or internal cusps. External cusps are nonetheless excluded. We refer to [20] and the references therein for an overview on some important features of John domains.

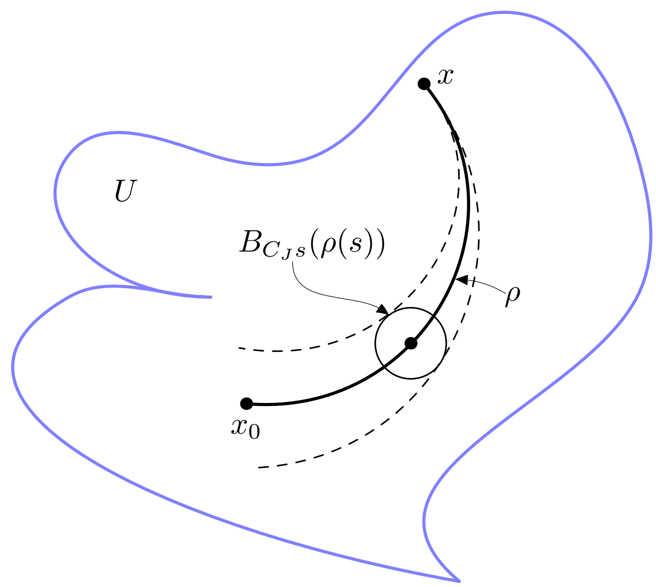

FIGURE 1. A John domain in $\mathbb{R}^2$

The following assumptions will be used throughout the paper without further mention:

$$T > 0, \tag{A1}$$

$$\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d} \text{ is symmetric with } \mathbb{C} \geq c_* \mathbb{I} \text{ for some } c_* > 0, \tag{A2}$$

$$\Omega(0) \subset \mathbb{R}^d \text{ is a bounded John domain}, \tag{A3}$$

$$\omega \subset\subset \Omega(0) \text{ is nonempty, open, and connected and } \operatorname{dist}(\omega; \partial\Omega(0)) =: \rho_0 > 0, \tag{A4}$$

$$\gamma \in C^{0,1}(\mathbb{R}^d \times \mathbb{R}^{d\times d}; \mathbb{R}) \text{ with } \gamma_* \leq \gamma(\cdot) \leq \gamma^* \text{ for some } 0 < \gamma_* < \gamma^*, \tag{A5}$$

$$k \in W^{1,1}(0,T), \quad \phi \in H^1(\mathbb{R}^d) \text{ with compact support}, \tag{A6}$$

$$f \in C^0([0,T]; L^2(\mathbb{R}^d; \mathbb{R}^d)), \tag{A7}$$

where the inequality in (A2) is meant in the sense of the Löwner order. Given the kernels k and ϕ from (A6), we define the space-time nonlocal operator K as

$$(Ke)(x,t) := \int_0^t \int_{\mathbb{R}^d} k(t-s)\phi(x-y)e(y,s)\,\mathrm{d}y\,\mathrm{d}s \qquad \text{for all } e \in L^1(\mathbb{R}^d \times (0,T); \mathbb{R}^{d\times d}). \tag{2.1}$$

For any nonempty open set $\Omega \subset\subset \mathbb{R}^d$ with $\omega \subset \Omega$ we use the notation

$$H^1_\omega(\Omega; \mathbb{R}^d) := \{u \in H^1(\Omega; \mathbb{R}^d) : u = 0 \text{ on } \omega\}.$$

Moreover, we indicate by $\overline{\varepsilon}$ the trivial extension to $\mathbb{R}^d$ of a measurable function ε defined on Ω.

The weak formulation of problem (1.1)-(1.6) reads

$$\begin{aligned}
& u(\cdot,t) \in H^1_\omega(\Omega(t);\mathbb{R}^d) \ \text{ and} \\
& \quad \int_{\Omega(t)} \mathbb{C}(\varepsilon(u(x,t)) - \alpha(x)) : \varepsilon(v(x))\,\mathrm{d}x = \int_{\Omega(t)} f(x,t)\cdot v(x)\,\mathrm{d}x \\
& \quad \text{for all } v \in H^1_\omega(\Omega(t);\mathbb{R}^d), \ \text{for a.e. } t\in(0,T),
\end{aligned} \tag{2.2}$$

$$\alpha(x) = K\overline{\varepsilon}(u)(x,\theta(x)) \quad \text{for all } x \in \mathbb{R}^d, \tag{2.3}$$

$$\gamma(x,\alpha(x))|\nabla(-\theta)(x)| = 1 \ \text{ in the viscosity sense in } \mathbb{R}^d \setminus \overline{\Omega(0)}, \tag{2.4}$$

$$\theta = 0 \quad \text{on } \Omega(0). \tag{2.5}$$

In the following, given $\alpha \in C^0_b(\mathbb{R}^d;\mathbb{R}^{d\times d})$, equation (2.4) will be solved in the following viscosity sense.

Definition 2.2 (Viscosity solution)**.** *Let $\alpha \in C^0_b(\mathbb{R}^d;\mathbb{R}^{d\times d})$ be given and $\theta : \mathbb{R}^d \to [0,\infty)$ be continuous.*

We say that θ is a viscosity subsolution *of (2.4) if for all $x_0 \in \mathbb{R}^d \setminus \overline{\Omega(0)}$ and any smooth function φ with $\varphi(x_0) = -\theta(x_0)$ and $\varphi \geq -\theta$ in a neighborhood of x_0, it holds that $\gamma(x_0,\alpha(x_0))|\nabla\varphi(x_0)| \leq 1$.*

Similarly, we say that θ is a viscosity supersolution *of (2.4) if for all $x_0 \in \mathbb{R}^d \setminus \overline{\Omega(0)}$ and any smooth function φ with $\varphi(x_0) = -\theta(x_0)$ and $\varphi \leq -\theta$ in a neighborhood of x_0, it holds that $\gamma(x_0,\alpha(x_0))|\nabla\varphi(x_0)| \geq 1$.*

Finally, θ is said to be a viscosity solution *of (2.4) if it is both a viscosity sub- and supersolution.*

Let us record the following fact.

Lemma 2.3. *For all $\theta : \mathbb{R}^d \to [0,\infty)$ continuous the set $Q_S \coloneqq \cup_{t\in(0,S)}\Omega(t)\times\{t\}$ is measurable for every $S \in (0,T]$.*

Proof. Fix $S \in (0,T]$. From the continuity, and hence the measurability of θ, it follows that the extended map $\tilde\theta : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined as $\tilde\theta(x,t) \coloneqq \theta(x)$ is measurable. Analogously, the projection $\tau : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ given by $\tau(x,t) \coloneqq t$ is continuous, and thus measurable. The measurability of Q_S follows then by observing that

$$\begin{aligned}
Q_S &= \big\{(x,t) \in \mathbb{R}^d \times (0,S) : \theta(x) - t < 0\big\} \\
&= \Big\{(x,t) \in \mathbb{R}^d \times (0,S) : (\tilde\theta - \tau)(x,t) < 0\Big\} \\
&= (\tilde\theta - \tau)^{-1}(-\infty,0) \cap (\mathbb{R}^d \times (0,S)).
\end{aligned}$$

□

Before going on, let us comment on the two subproblems (2.2)-(2.3) and (2.4)-(2.5). At first, let us discuss the eikonal problem (2.4)-(2.5) by assuming to be given $\alpha \in C^0_b(\mathbb{R}^d;\mathbb{R}^{d\times d})$. As $x \mapsto \gamma(x,\alpha(x))$ is continuous, bounded, and well-separated from 0, in view of (A5), this eikonal problem admits a unique viscosity solution θ, cf. [3, 4]. In fact, the solution θ of

(2.4)-(2.5) is Lipschitz continuous with

$$0 < \frac{1}{\gamma^*} \leq |\nabla\theta| \leq \frac{1}{\gamma_*} \ \text{ in } \ \mathbb{R}^d \setminus \overline{\Omega(0)}. \tag{2.6}$$

A first consequence of these inequalities is that $\Omega(T)$ is bounded independently of α. Indeed, one has that $\Omega(T) \subset \Omega(0) + B_{T\gamma^*}(0)$, cf. [13]. In the following, we can hence assume to be given a fixed bounded open set $D \subset \mathbb{R}^d$ such that

$$\Omega(t) \subset D \quad \text{for all } t \in [0,T] \tag{2.7}$$

for any solution of (2.4)-(2.5), namely, independently of α. As the support of ϕ is compact, see (A6), D can be assumed to be large enough that the trivial $\overline{\varepsilon}$ extension of $\varepsilon : \Omega(t) \to \mathbb{R}^{d\times d}$ can be considered to be defined on D, with no loss of generality and without introducing new notation.

Note moreover that problem (2.4)-(2.5) is stable with respect to data convergence. More precisely, if $\alpha_n \to \alpha$ locally uniformly, then $\gamma(\cdot, \alpha_n(\cdot)) \to \gamma(\cdot, \alpha(\cdot))$ locally uniformly as γ is Lipschitz continuous by (A5), and $\theta_n \to \theta$ locally uniformly, where θ_n, θ are the solutions of (2.4)-(2.5) corresponding to α_n, α, respectively. The reader is referred to [11] or to Section 4 below.

As $\Omega(0)$ is a John domain, [13, Theorem 1.1] ensures that all sublevels $\Omega(t)$ are John domains, as well. More precisely, if $\Omega(0)$ is a John domain with respect to the point $x_0 \in \mathbb{R}^d$ with John constant C_0 then all $\Omega(t)$ are John domains with respect to the same point x_0 and with John constant at least $C_J := \min\{1, C_0\}\frac{\gamma_*}{(2\gamma^*+\gamma_*)}$. In particular, one has that $\Omega(t) \in \Theta$ for all $t \in [0,T]$, where

$$\begin{aligned}\Theta := \big\{\Omega \subset\subset D \ &: \Omega \text{ is a John domain with respect to the point } x_0 \in \omega, \\ &\text{with John constant } C_J, \text{ and } \operatorname{dist}(\omega; \partial\Omega) \geq \rho_0 > 0\big\}.\end{aligned} \tag{2.8}$$

This is a crucial observation, for it entails the validity of a uniform Korn inequality. Recall that for any given John domain $\Omega \subset \mathbb{R}^d$ there exists a constant C_{Korn} such that

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq C_{\text{Korn}}\|\varepsilon(u)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \qquad \text{for all } u \in H^1_0(\Omega;\mathbb{R}^d). \tag{2.9}$$

In fact, the validity of the Korn inequality is actually equivalent to Ω being John in the special class of domains fulfilling the so-called *separation property* [2, 23]. This include simply connected planar domains [10].

Note that the constant in the Korn inequality (2.9) depends on Ω only. More precisely, $C_{\text{Korn}} = C_{\text{Korn}}(C_J, d(x_0, \partial\Omega), \operatorname{diam}(\Omega))$, see [2, Theorem 4.1]. In the following, we use the fact that the Korn constant is actually uniform on Θ. In particular, we have the following.

Proposition 2.4 (Uniform Korn inequality)**.** *Let $\omega, D \subset \mathbb{R}^d$ be open bounded domains, with $x_0 \in \omega \subset\subset D$, and $|\omega| > 0$. Define Θ as in (2.8). Then, there exists a constant $C_\Theta = C_\Theta(C_J, x_0, \omega, D, \rho_0)$ such that for every $\Omega \in \Theta$ and every $u \in H^1_\omega(\Omega;\mathbb{R}^d)$ there holds*

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq C_\Theta\|\varepsilon(u)\|_{L^2(\Omega;\mathbb{R}^{d\times d})}. \tag{2.10}$$

We prove the uniform Korn inequality (2.10) in Section 5. This inequality is paramount for studying the variational equation (2.2). By assuming to be given $t \in [0,T] \mapsto \Omega(t) \in \Theta$, as well as $\alpha \in L^2(D;\mathbb{R}^{d\times d})$, one can uniquely solve (2.2) for all times $t \in [0,T]$ by means of the standard Lax-Milgram Lemma, as the coercivity of the corresponding bilinear form follows from (2.10).

2.2. **Main results.** Recall that (A1)-(A7) from Section 2.1 are assumed throughout.

We are now in the position to state our main results.

Theorem 2.5 (Equilibrium, given the growth). *Let* $\theta : \mathbb{R}^d \to [0,\infty)$ *be given, so that the corresponding set-valued map* $t \in [0,T] \mapsto \Omega(t) := \{x \in \mathbb{R}^d \ : \ \theta(x) < t\}$ *takes values in* Θ. *Setting* $Q := \cup_{t\in(0,T)}\Omega(t) \times \{t\}$, *there exists a unique measurable function* $u : Q \to \mathbb{R}^d$ *with* $u(\cdot,t) \in H^1_\omega(\Omega(t);\mathbb{R}^d)$ *for almost every* $t \in (0,T)$ *and* $t \mapsto \|u(\cdot,t)\|_{H^1} \in L^\infty(0,T)$ *solving the equilibrium system* (2.2)-(2.3).

Theorem 2.6 (Coupled equilibrium and growth). *For* $T > 0$ *small enough there exist a Lipschitz continuous* $\theta : \mathbb{R}^d \to [0,\infty)$ *and a measurable* $u : Q \to \mathbb{R}^d$, *with* $u(\cdot,t) \in H^1(\Omega(t);\mathbb{R}^d)$ *for* $\Omega(t) := \{x \in \mathbb{R}^d \ : \ \theta(x) < t\}$, $Q := \cup_{t\in(0,T)}\Omega(t) \times \{t\}$ *for almost every* $t \in (0,T)$, *solving the coupled equilibrium and growth problem* (2.2)-(2.5).

Theorems 2.5 and 2.6 are proved in Sections 3 and 4, respectively. The choice for α in (2.3), which is inspired by [40] and assumes that material is added in a locally unstressed state, can be generalized. One can assume that material is added to the boundary of the solid at point x with some given and possibly nonvanishing stress $\widehat{\sigma}(x)$, by prescribing $\widehat{\sigma} \in L^2(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\rm sym})$. To this end, one would just need to reformulate position (2.3) as $\alpha(x) = K\overline{\varepsilon}(u)(x,\theta(x)) - \mathbb{C}^{-1}\widehat{\sigma}(x)$, where $\mathbb{C}^{-1}$ is the *compliance* tensor. The above results would still hold under this generalization.

Before moving on, let us explicitly remark that the smallness assumption on T in Theorem 2.6 is not due to the possible onset of singularities in $t \mapsto \Omega(t)$, which are here allowed. The need for restricting to small times resides in the very nature of the subsystem (1.1)–(1.3), where the backstress $\mathbb{C}\alpha$ acts as actual forcing. In absence of mollification, namely, for $K = {\rm id}$, system (1.1)–(1.3) would correspond to the quasistatic equilibrium system, with the extra forcing $\nabla\cdot\mathbb{C}\varepsilon(u)(x,\theta(x))$. This extra forcing depends on a space-time trace of the solution itself, having the same size of $\mathbb{C}\varepsilon(u)$. This represents a clear bottleneck to compactness, hence to existence. On the contrary, the action of the compactifying operator K allows *for small times* to control the size of $\mathbb{C}\alpha$, so that the extra forcing term $\nabla\cdot\mathbb{C}\alpha$ is dominated by $\mathbb{C}\varepsilon(u)$ and can be handled as a perturbation.

3. The equilibrium problem for a given growth: Proof of Theorem 2.5

As $\Omega(t) \subset D$ for all $t \in [0,T]$ we have that

$$|\Omega(T)| \leq |D|. \tag{3.1}$$

In order to find a unique solution u to (2.2)-(2.3), we start by arguing locally in time, looking for a solution on $(0,T_0)$ for some $T_0 \in (0,T]$ small. Indeed, for T_0 small enough the well-posedness of system (2.2)-(2.3) follows by a contraction argument in the function space

$$\begin{aligned} U(T_0) := \Big\{ & u : \cup_{t\in(0,T_0)}\Omega(t)\times\{t\} \to \mathbb{R}^d \text{ measurable such that} \\ & u(\cdot,t) \in H^1_\omega(\Omega(t);\mathbb{R}^d) \text{ for a.e. } t \in (0,T_0) \\ & \text{and } t \mapsto \|u(\cdot,t)\|_{H^1(\Omega(t);\mathbb{R}^d)} \in L^\infty(0,T_0)\Big\}. \end{aligned}$$

Note that $U(T_0)$ is a Banach space when endowed with the norm

$$\|u\|_{U(T_0)} := \operatorname{ess\,sup}_{t\in(0,T_0)} \|u(\cdot,t)\|_{H^1(\Omega(t);\mathbb{R}^d)}.$$

We additionally introduce the notation $E(T_0) := \{\varepsilon(u) : u \in U(T_0)\}$ for the corresponding Banach space of symmetric gradients and let

$$\|\varepsilon(u)\|_{E(T_0)} := \operatorname{ess\,sup}_{t\in(0,T_0)} \|\varepsilon(u)(\cdot,t)\|_{L^2(\Omega(t);\mathbb{R}^{d\times d})} = \operatorname{ess\,sup}_{t\in(0,T_0)} \|\overline{\varepsilon}(u)(\cdot,t)\|_{L^2(D;\mathbb{R}^{d\times d})}.$$

By combining [7, Lemma 3.1] and [7, Theorem 5.1] (see also the remark right before [7, Formula (5.3)]) we obtain existence of a constant $C_P > 0$ such that the uniform Poincaré inequality

$$\|v\|_{L^2(\Omega;\mathbb{R}^d)} \le C_P \|\nabla v\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \quad \text{for all } \Omega \in \Theta \text{ and all } v \in H^1_\omega(\Omega;\mathbb{R}^d) \tag{3.2}$$

holds. An application to $u \in U(T_0)$ gives

$$\begin{aligned}\|u\|^2_{U(T_0)} &= \operatorname{ess\,sup}_{t\in(0,T_0)} \Big(\|u(\cdot,t)\|^2_{L^2(\Omega(t);\mathbb{R}^d)} + \|\nabla u(\cdot,t)\|^2_{L^2(\Omega(t);\mathbb{R}^{d\times d})}\Big)\\ &\le (C_P^2+1)\operatorname{ess\,sup}_{t\in(0,T_0)} \Big(\|\nabla u(\cdot,t)\|^2_{L^2(\Omega(t);\mathbb{R}^{d\times d})}\Big).\end{aligned} \tag{3.3}$$

The uniform Korn inequality (2.10) then gives

$$\|u\|_{U(T_0)} \le \widehat{C}_\Theta \|\varepsilon(u)\|_{E(T_0)} \le \widehat{C}_\Theta \|u\|_{U(T_0)} \tag{3.4}$$

for $\widehat{C}_\Theta := (C_P^2+1)^{1/2} C_\Theta$.

Fix now $\tilde{u} \in U(T_0)$. As $\varepsilon(\tilde{u}) \in E(T_0)$ we have that $\overline{\varepsilon}(\tilde{u}) \in L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$ and we can define

$$\alpha(x) := K\overline{\varepsilon}(\tilde{u})(x,\theta(x)) \quad \text{for all } x \in \mathbb{R}^d. \tag{3.5}$$

We readily check that $\alpha \in L^2(\Omega(T_0);\mathbb{R}^{d\times d})$. Indeed, we have by assumption (A6) together with Young's inequality on convolutions that

$$\begin{aligned}\|\alpha\|_{L^2(\Omega(T_0);\mathbb{R}^{d\times d})} &\le |\Omega(T_0)|^{1/2} \|K\overline{\varepsilon}(\tilde{u})\|_{L^\infty(D\times(0,T_0);\mathbb{R}^{d\times d})}\\ &\overset{(3.1)}{\le} |D|^{1/2} \|k\|_{L^1(0,T_0)} \|\phi\|_{L^2(\mathbb{R}^d)} \|\varepsilon(\tilde{u})\|_{E(T_0)}.\end{aligned} \tag{3.6}$$

Correspondingly, owing again to the uniform Korn inequality (2.10), for all fixed $t \in (0,T_0)$ one finds by the Lax-Milgram Lemma a unique $u(\cdot,t) \in H^1_\omega(\Omega(t);\mathbb{R}^d)$ solving the variational equation (2.2) with α given by (3.5). By choosing $v = u(\cdot,t)$ in equation (2.2), and using assumptions (A2) and (A7), we easily check that

$$\begin{aligned}c_*\|\varepsilon(u)\|^2_{E(T_0)} &\le \|\mathbb{C}\|\,\|\alpha\|_{L^2(\Omega(T_0);\mathbb{R}^{d\times d})}\|\varepsilon(u)\|_{E(T_0)} + \|f\|_{C^0([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))}\|u\|_{U(T_0)}\\ &\le \Big(\|\mathbb{C}\|\,\|\alpha\|_{L^2(\Omega(T_0);\mathbb{R}^{d\times d})} + \widehat{C}_\Theta\|f\|_{C^0([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))}\Big)\|\varepsilon(u)\|_{E(T_0)},\end{aligned}$$

where we denoted by $\|\mathbb{C}\|$ the Frobenius norm of the elasticity tensor $\mathbb{C}$ and where we again used the uniform Korn inequality (2.10) and the uniform Poincaré inequality (3.2). This in particular ensures that $\varepsilon(u) \in E(T_0)$, hence $u \in U(T_0)$, again by the inequality (3.4).

We now show that the mapping $S(T_0) : U(T_0) \to U(T_0)$ given by $S(T_0)(\tilde{u}) = u$ is a contraction for T_0 small. To this aim, fix $\tilde{u}_1$, $\tilde{u}_2 \in U(T_0)$, let $\alpha_1(x) = K\overline{\varepsilon}(\tilde{u}_1)(x,\theta(x))$ and $\alpha_2(x) = K\overline{\varepsilon}(\tilde{u}_2)(x,\theta(x))$ for $x \in \Omega(T_0)$, and define $u_1 = S(T_0)(\tilde{u}_1)$, and $u_2 = S(T_0)(\tilde{u}_2)$.

Testing equations (2.2) written for u_1 and u_2 by $v = u_1 - u_2$ and taking their difference we deduce by assumption (A2) and by following the same arguments as in estimate (3.6) that

$$\begin{aligned} c_*\|\varepsilon(u_1-u_2)\|_{E(T_0)} &\leq \|\mathbb{C}\|\,\|\alpha_1-\alpha_2\|_{L^2(\Omega(T_0);\mathbb{R}^{d\times d})} \\ &\leq \|\mathbb{C}\|\,|D|^{1/2}\|K\overline{\varepsilon}(\tilde u_1-\tilde u_2)\|_{L^\infty(D\times(0,T_0);\mathbb{R}^{d\times d})} \\ &\leq \|\mathbb{C}\|\,|D|^{1/2}\|k\|_{L^1(0,T_0)}\|\phi\|_{L^2(\mathbb{R}^d)}\|\varepsilon(\tilde u_1-\tilde u_2)\|_{E(T_0)}. \end{aligned} \tag{3.7}$$

By using again inequality (3.4) we then conclude that

$$\|u_1-u_2\|_{U(T_0)} \leq \frac{\widehat{C}_\Theta\|\mathbb{C}\|}{c_*}|D|^{1/2}\|k\|_{L^1(0,T_0)}\|\phi\|_{L^2(\mathbb{R}^d)}\|\tilde u_1-\tilde u_2\|_{U(T_0)}.$$

Let now $T_0 = T/n$, $n \in \mathbb{N}$, be so small that

$$\frac{\widehat{C}_\Theta\|\mathbb{C}\|}{c_*}|D|^{1/2}\|k\|_{L^1((j-1)T_0,jT_0)}\|\phi\|_{L^2(\mathbb{R}^d)} < 1 \quad \text{for all } j = 1,\dots,n. \tag{3.8}$$

Note that such n exists as k is integrable, see (A6). Under condition (3.8), the mapping $S(T_0)$ is a contraction, hence admitting a unique fixed point. This proves Theorem 2.5 for small times.

We next show that one can obtain a global solution on $(0,T)$ by successively solving on $(0,T_0)$, $(0,2T_0)$, $\dots$, $(0,jT_0)$, for $j = 1,\dots,n$. Assume to have uniquely solved system (2.2)-(2.3) on $(0,(j-1)T_0)$. Indicate by $u_j \in U((j-1)T_0)$ the corresponding solution and fix

$$\tilde u \in V := \{v \in U(jT_0)\,:\, v = u_j \text{ on } (0,(j-1)T_0)\}$$

which is a closed subspace of $U(jT_0)$.

By defining α as in (3.5) with $\Omega(T_0)$ replaced by $\Omega(jT_0)$ we can reproduce bound (3.6) with jT_0 instead of T_0 so that $\alpha \in L^2(\Omega(jT_0);\mathbb{R}^{d\times d})$. One hence finds a unique solution $u = S(jT_0)(\tilde u)$ of the equilibrium system (2.2) with α given by (3.5), for almost all $t \in (0,jT_0)$. Recall that we have $u = u_j$ on $(0,(j-1)T_0)$, since u is the unique solution to (2.2) with $\alpha(x) = K\overline{\varepsilon}(u_j)(x,\theta(x))$ for $x \in \Omega((j-1)T_0)$. In particular, $u \in V$.

We conclude by checking that $S(jT_0) : V \to V$ defined by $\tilde u \mapsto u$ is a contraction. Fix $\tilde u_1$, $\tilde u_2 \in V$, let the corresponding α_1 and α_2 be given by (3.5), and define $u_1 = S(jT_0)(\tilde u_1)$, and $u_2 = S(jT_0)(\tilde u_2)$. We adapt the argument of estimate (3.7), taking into account that $\tilde u_1 = \tilde u_2$ on $(0,(j-1)T_0)$. We get

$$\begin{aligned} c_*&\|\varepsilon(u_1-u_2)\|_{E(jT_0)} \leq \|\mathbb{C}\|\,\|\alpha_1-\alpha_2\|_{L^2(D;\mathbb{R}^{d\times d})} \\ &\leq \|\mathbb{C}\|\,|D|^{1/2}\|K\overline{\varepsilon}(\tilde u_1-\tilde u_2)\|_{L^\infty(D\times(0,jT_0);\mathbb{R}^{d\times d})} \\ &= \|\mathbb{C}\|\,|D|^{1/2}\sup_{(x,t)\in D\times(0,jT_0)}\left|\int_0^t\int_{\mathbb{R}^d} k(t-s)\phi(x-y)\overline{\varepsilon}(\tilde u_1(y,s)-\tilde u_2(y,s))\,\mathrm{d}y\,\mathrm{d}s\right| \\ &\leq \|\mathbb{C}\|\,|D|^{1/2}\sup_{(x,t)\in D\times(0,jT_0)}\int_{(j-1)T_0}^t\int_{\mathbb{R}^d}|k(t-s)|\,|\phi(x-y)|\,|\overline{\varepsilon}(\tilde u_1(y,s)-\tilde u_2(y,s))|\,\mathrm{d}y\,\mathrm{d}s \\ &\leq \|\mathbb{C}\|\,|D|^{1/2}\|k\|_{L^1((j-1)T_0,jT_0)}\|\phi\|_{L^2(\mathbb{R}^d)}\|\varepsilon(\tilde u_1-\tilde u_2)\|_{E(jT_0)}. \end{aligned}$$

Using again inequality (3.4) we infer that

$$\|u_1-u_2\|_{U(jT_0)} \le \frac{\widehat{C}_\Theta \|\mathbb{C}\|}{c_*} |D|^{1/2} \|k\|_{L^1((j-1)T_0,jT_0)} \|\phi\|_{L^2(\mathbb{R}^d)} \|\tilde u_1-\tilde u_2\|_{U(jT_0)}$$

so that the smallness assumption (3.8) entails that $S(jT_0): V \to V$ is a contraction. This proves the existence of a unique solution of problem (2.2)-(2.3) almost everywhere on $(0, jT_0)$. The assertion follows by letting $j = n$.

Note that, in order to prove Theorem 2.5 one does not need to assume the differentiability of the kernels k and ϕ as in (A6) but the weaker requirements $k \in L^1(0,T)$ and $\phi \in L^2(\mathbb{R}^d)$ are indeed sufficient.

4. The coupled equilibrium-growth problem: Proof of Theorem 2.6

The existence of a solution to the equilibrium-growth coupled system (2.2)-(2.5) for $T > 0$ small follows by a fixed-point argument on the function α. Define

$$A := \{\alpha \in C^0(D;\mathbb{R}^{d\times d}) \,:\, \|\alpha\|_{W^{1,\infty}(D;\mathbb{R}^{d\times d})} \le L\}$$

where $L > 0$ depends just on the data and is specified in (4.7) below.

Given $\tilde\alpha \in A$ one has by (A5) that $x \mapsto \gamma(x, \tilde\alpha(x))$ is Lipschitz continuous and, as discussed in Subsection 2.1, there exists a unique θ solving (2.4)-(2.5) with $\gamma(x,\alpha(x))$ replaced by $\gamma(x,\tilde\alpha(x))$. With such θ one defines $t \mapsto \Omega(t) = \{x \in \mathbb{R}^d \,:\, \theta(x) < t\} \in \Theta$. As $t \mapsto \Omega(t)$ is increasing by set inclusion and $\cup_{t\in(0,T)}\Omega(t)\times\{t\} = \{(x,t) \in \mathbb{R}^d \times (0,T) \,:\, \theta(x) < t\}$ is measurable by Lemma 2.3, one uses Theorem 2.5 in order to find the unique solution u of (2.2)-(2.3) for the given $t \in [0,T] \mapsto \Omega(t)$. This in particular defines the mapping

$$S : \tilde\alpha \in A \subset C^0(D;\mathbb{R}^{d\times d}) \mapsto \alpha = K\overline{\varepsilon}(u)(\cdot,\theta(\cdot)) \in C^0(D;\mathbb{R}^{d\times d}).$$

The assertion of Theorem 2.6 follows as soon as we prove that S admits a fixed point.

To start with, let us provide an a-priori estimate on $u(\cdot,t)$. By choosing $v = u(\cdot,t)$ in (2.2) and using inequality (3.4) we get

$$c_*\|\varepsilon(u)(\cdot,t)\|_{L^2(\Omega(t);\mathbb{R}^{d\times d})} \le \|\mathbb{C}\|\,\|\alpha\|_{L^2(D;\mathbb{R}^{d\times d})} + \widehat{C}_\Theta\|f(\cdot,t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}. \tag{4.1}$$

On the other hand, using (A6) and applying Young's inequality on convolutions we can control α as

$$\|\alpha\|_{L^\infty(D;\mathbb{R}^{d\times d})} \le \|k\|_{L^1(0,T)}\|\phi\|_{L^2(\mathbb{R}^d)}\|\varepsilon(u)\|_{E(T)}. \tag{4.2}$$

We now assume that $T > 0$ is so small that

$$\frac{\|\mathbb{C}\|\,|D|^{1/2}}{c_*}\|k\|_{L^1(0,T)}\|\phi\|_{L^2(\mathbb{R}^d)} =: \eta < 1 \tag{4.3}$$

and combine (4.1)-(4.2) in order to get that

$$c_*(1-\eta)\|\varepsilon(u)\|_{E(T)} \le \widehat{C}_\Theta\|f\|_{C^0([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))}.$$

This in particular entails that

$$\|\varepsilon(u)\|_{E(T)} \le \frac{\widehat{C}_\Theta}{c_*(1-\eta)}\|f\|_{C^0([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))} =: M \tag{4.4}$$

where M depends on data only. Note that the smallness assumption (4.3) does not require the smallness of applied forces.

We now compute the gradient

$$\nabla\alpha(x) = (\nabla K\overline{\varepsilon}(u))(x,\theta(x)) + (\partial_t K\overline{\varepsilon}(u))(x,\theta(x))\,\nabla\theta(x) \qquad \text{for all } x\in D.$$

Using (4.2)-(4.4), the regularity of the kernels k and ϕ from (A6), and Young's inequality on convolutions we hence have that

$$\|\alpha\|_{L^\infty(D;\mathbb{R}^{d\times d})} \le \|k\|_{L^1(0,T)}\|\phi\|_{L^2(\mathbb{R}^d)}M, \tag{4.5}$$

$$\begin{aligned}\|\nabla\alpha\|_{L^\infty(D;\mathbb{R}^{d\times d\times d})} &\le \|k\|_{L^1(0,T)}\|\nabla\phi\|_{L^2(\mathbb{R}^d)}M\\ &\quad + \Big(\|k'\|_{L^1(0,T)} + |k(0)|\Big)\|\phi\|_{L^2(\mathbb{R}^d)}\|\nabla\theta\|_{L^\infty(D;\mathbb{R}^d)}M.\end{aligned} \tag{4.6}$$

Thus, recalling (2.6) and letting

$$L := \|k\|_{L^1(0,T)}\|\phi\|_{H^1(\mathbb{R}^d)}M + \Big(\|k'\|_{L^1(0,T)} + |k(0)|\Big)\|\phi\|_{L^2(\mathbb{R}^d)}\frac{1}{\gamma_*}M \tag{4.7}$$

one has that $\alpha = S(\tilde\alpha)$ belongs to A, as well. Note that L is bounded in terms of data only.

We now check the continuity of S with respect to the strong topology of $C^0(D;\mathbb{R}^{d\times d})$. Let $\tilde\alpha_n,\tilde\alpha\in C^0(D;\mathbb{R}^{d\times d})$ be given with $\tilde\alpha_n\to\tilde\alpha$ uniformly. As γ is Lipschitz continuous, see (A5), we have that $\gamma(\cdot,\tilde\alpha_n(\cdot))\to\gamma(\cdot,\tilde\alpha(\cdot))$ uniformly, as well. This suffices to pass to the limit in the eikonal problem (2.4)-(2.5) written for $\gamma(\cdot,\tilde\alpha_n(\cdot))$ and to find that the corresponding solutions θ_n converge uniformly to the solution θ of (2.4)-(2.5) for $\gamma(\cdot,\tilde\alpha(\cdot))$. Indeed, the functions θ_n are uniformly Lipschitz continuous with $\theta_n = 0$ on $\overline{\Omega(0)}$. They hence admit a not relabeled, locally uniformly converging subsequence $\theta_n\to\theta$ with θ Lipschitz continuous and $\theta = 0$ on $\overline{\Omega(0)}$. Let $x_0\in\mathbb{R}^d\setminus\overline{\Omega(0)}$ be given and φ be smooth with $\varphi(x_0) = -\theta(x_0)$ and $\varphi\ge-\theta$ in a neighborhood of x_0. By using the classical approximation procedure of [11, Proposition 2.4], we find $x_n\in\mathbb{R}^d\setminus\overline{\Omega(0)}$ with $x_n\to x_0$ and φ_n smooth such that $\varphi_n(x_n) = -\theta_n(x_n)$ and $\varphi_n\ge-\theta_n$ in a neighborhood of x_n, and $\nabla\varphi_n(x_n)\to\nabla\varphi(x_0)$. As θ_n are viscosity subsolutions, we have that $\gamma(x_n,\alpha_n(x_n))|\nabla\varphi_n(x_n)|\le 1$. By passing to the limit as $n\to\infty$ we obtain that $\gamma(x_0,\alpha(x_0))|\nabla\varphi(x_0)|\le 1$, so that θ is a viscosity subsolution, as well. In a similar way, we can check that θ is a viscosity supersolution, hence a viscosity solution. Given uniqueness, no extraction of subsequences was actually needed at this point.

Given such θ_n and θ we can define the corresponding $t\mapsto\Omega_n(t)$ and $t\mapsto\Omega(t)$ (both increasing by set inclusion and such that the corresponding $Q_n = \cup_{t\in[0,T]}\Omega_n(t)\times\{t\}$ and $Q = \cup_{t\in[0,T]}\Omega(t)\times\{t\}$ are measurable). As θ_n converges to θ locally uniformly and the inequalities (2.6) hold, independently of n, we have that $\Omega_n(t)\to\Omega(t)$ in the Hausdorff sense, uniformly with respect to $t\in[0,T]$. Moreover, $Q_n\to Q$ in the Hausdorff sense, as well.

Let us indicate by u_n and u the unique solutions of (2.2)-(2.3) given by Theorem 2.5. From the very definition of S let us recall that $\alpha_n = S(\tilde\alpha_n) = K\overline{\varepsilon}(u_n)(\cdot,\theta_n(\cdot))$ and $\alpha = S(\tilde\alpha) = K\overline{\varepsilon}(u)(\cdot,\theta(\cdot))$. Bounds (4.4)-(4.6) and a localization argument entail that

$$\overline{\varepsilon}(u_n) \rightharpoonup^* \overline{\varepsilon}(u) \quad \text{weakly* in} \quad L^\infty(0,T;L^2(D;\mathbb{R}^{d\times d})). \tag{4.8}$$

In fact, one has that $\overline{\varepsilon}(u_n)$ are uniformly bounded in $L^\infty(0,T;L^2(D;\mathbb{R}^{d\times d}))$, hence admitting a weak$*$ limit along some not relabeled subsequence. Denote by $\tilde\varepsilon$ such limit. Fix now $(x,t)\in Q$, as well as $\eta>0$ small enough, so that $Q_\eta = B_\eta(x)\times(t-\eta,t+\eta)\subset\subset Q$. From the convergence $Q_n\to Q$ in the Hausdorff sense we have that $Q_\eta\subset Q_n$ for all n large enough. Hence, by indicating by 1_{Q_η} the indicator function of Q_η one has that $\overline{\varepsilon}(u_n)1_{Q\eta}\to\tilde\varepsilon 1_{Q\eta}$

weakly$*$ in $L^\infty(t-\eta,t+\eta,L^2(B_\eta(x);\mathbb{R}^{d\times d}))$. At the same time $\overline{\varepsilon}(u_n)1_{Q\eta}=\varepsilon(u_n)1_{Q\eta}\to\varepsilon(u)1_{Q\eta}$ weakly$*$ in $L^\infty(t-\eta,t+\eta,L^2(B_\eta(x);\mathbb{R}^{d\times d}))$. As $(x,t)\in Q$ is arbitrary, this shows that $\tilde\varepsilon=\varepsilon(u)\equiv\overline{\varepsilon}(u)$ on Q. An analogous argument applied to $(x,t)\notin\overline{Q}$ proves that $\tilde\varepsilon=0=\overline{\varepsilon}(u)$ in $\mathbb{R}^d\times(0,T)\setminus\overline{Q}$. In order to conclude for (4.8) it hence suffices to recall that ∂Q is negligible.

Note that the whole sequence $\overline{\varepsilon}(u_n)$ converges, due to the uniqueness of the limit. Owing to the compactifying character of the nonlocal operator K we also have that

$$K\overline{\varepsilon}(u_n)\to K\overline{\varepsilon}(u)\quad\text{strongly in}\quad C^0(D\times(0,T);\mathbb{R}^{d\times d}). \tag{4.9}$$

In addition, $K\overline{\varepsilon}(u_n)$ are uniformly Lipschitz continuous in time: By following the argument leading to bounds (4.5)-(4.6), we can check that

$$\begin{aligned}\|K\overline{\varepsilon}(u_n)(\cdot,t_1)-K\overline{\varepsilon}(u_n)(\cdot,t_2)\|_{L^\infty(D;\mathbb{R}^{d\times d})}&\le\|\partial_tK\overline{\varepsilon}(u_n)\|_{L^\infty(D\times(t_1,t_2);\mathbb{R}^{d\times d})}|t_1-t_2|\\&\le\big(\|k'\|_{L^1(t_1,t_2)}+|k(0)|\big)\|\phi\|_{L^2(\mathbb{R}^d)}M\,|t_1-t_2|\end{aligned} \tag{4.10}$$

for all $0<t_1<t_2<T$. We can hence conclude that

$$\begin{aligned}&\|\alpha_n-\alpha\|_{L^\infty(D;\mathbb{R}^{d\times d})}\\&\quad=\|K\overline{\varepsilon}(u_n)(\cdot,\theta_n(\cdot))-K\overline{\varepsilon}(u)(\cdot,\theta(\cdot))\|_{L^\infty(D;\mathbb{R}^{d\times d})}\\&\quad\le\|K\overline{\varepsilon}(u_n)(\cdot,\theta_n(\cdot))-K\overline{\varepsilon}(u_n)(\cdot,\theta(\cdot))\|_{L^\infty(D;\mathbb{R}^{d\times d})}+\|K\overline{\varepsilon}(u_n)(\cdot,\theta(\cdot))-K\overline{\varepsilon}(u)(\cdot,\theta(\cdot))\|_{L^\infty(D;\mathbb{R}^{d\times d})}\\&\quad\overset{(4.10)}{\le}\big(\|k'\|_{L^1(0,T)}+|k(0)|\big)\|\phi\|_{L^2(\mathbb{R}^d)}M\|\theta_n-\theta\|_{L^\infty(D)}+\|K\overline{\varepsilon}(u_n)-K\overline{\varepsilon}(u)\|_{L^\infty(D\times(0,T);\mathbb{R}^{d\times d})}\to0,\end{aligned}$$

where we have used that $\theta_n\to\theta$ and $K\overline{\varepsilon}(u_n)\to K\overline{\varepsilon}(u)$ uniformly. This proves that $S(\tilde\alpha_n)\to S(\tilde\alpha)$ strongly in $C^0(D;\mathbb{R}^{d\times d})$, namely, that S is continuous.

Eventually, as A is convex and compact in $C^0(D;\mathbb{R}^{d\times d})$ we can apply the Schauder Fixed-Point Theorem and complete the proof of Theorem 2.6.

5. Uniform Korn inequality

We conclude this paper with a proof of the uniform Korn inequality in Proposition 2.4.

Let us argue by contradiction and assume that the statement of Proposition 2.4 is false. In particular, for every $k\in\mathbb{N}$ we assume to be given an open set $\Omega_k\in\Theta$ and a map $u_k\in H^1_\omega(\Omega_k;\mathbb{R}^d)$ such that

$$\|\nabla u_k\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})}>k\|\varepsilon(u_k)\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})}.$$

By normalizing, with no loss of generality we can assume that

$$\|\nabla u_k\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})}=1\quad\text{and}\quad\|\varepsilon(u_k)\|_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\le\frac{1}{k} \tag{5.1}$$

for every $k\in\mathbb{N}$. As all $\Omega_k\subset\subset D$ and D is bounded, one can find a not relabeled subsequence such that $\overline{\Omega}_k\to K$ in the sense of the Hausdorff convergence, where $K\subset\subset D$. Define now $\Omega_\infty:=\mathrm{int}(K)$ (interior part), so that $\overline{\Omega}_\infty=K$. From the connectedness of each set Ω_k we also infer that $\overline{\Omega}_\infty$ is connected.

Define now $S_k:=\cap_{n\ge k}\overline{\Omega}_n$, so that the sequence $\{S_k\}_{k\in\mathbb{N}}$ is increasing by set inclusion. In particular, there also holds $S_k\cap\overline{\Omega}_\infty\to\overline{\Omega}_\infty$ in the Hausdorff sense. We notice that the Korn constant $\widehat{C}_{\rm Korn}$ in the first Korn inequality is the same for all sets in Θ, cf. [2, Theorem 4.2], as such constant depends on C_J, d, ρ_0, and $\mathrm{diam}(\Omega)$ only, [2, Theorem 4.1]. Thus, we infer that

$$\begin{aligned} 1 &= \|\nabla u_k\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})} \\ &\leq \widehat{C}_{\rm Korn}\left(\|u_k\|^2_{L^2(\Omega_k;\mathbb{R}^d)} + \|\varepsilon(u_k)\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right) \\ &\leq 2\widehat{C}_{\rm Korn}\left(\|u_k\|^2_{L^2(S_k\cap\overline{\Omega}_\infty;\mathbb{R}^d)} + \|u_k\|^2_{L^2(\Omega_k\setminus(S_k\cap\overline{\Omega}_\infty);\mathbb{R}^d)} + \|\varepsilon(u_k)\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})}\right). \end{aligned} \tag{5.2}$$

In view of (5.1), we already know that the third term on the right-hand side of (5.2) converges to 0 as $k\to\infty$. In order to reach a contradiction, we proceed by showing that also the first and second contributions on the right-hand side of (5.2) are infinitesimal as $k\to\infty$.

We subdivide the remaining part of the proof into four steps.

Step 1. We first show that the set Ω_∞ is still a John domain with respect to x_0, possibly with a smaller John constant.

Let $x\in\Omega_\infty$ be fixed. First, recall that ω is connected and since $\mathrm{dist}\,(\omega;\partial\Omega_k)\geq\rho_0>0$ for every $k\in\mathbb{N}$, there holds $\omega\subset\subset\Omega_\infty$. Therefore, if $x\in\omega$, the existence of an arc-length parametrized curve ρ joining x and x_0 and with positive distance from $\partial\Omega_\infty$ is directly ensured.

Consider now the case in which $x\notin\omega$. For k big enough $x\in\Omega_k$, and there exists an arc-length parametrized curve $\rho_k:[0,L_k]\to\Omega_k$ such that $\rho_k(0)=x$, $\rho_k(L_k)=x_0$ and $\mathrm{dist}\,(\rho_k(s);\partial\Omega_k)\geq C_Js$ for all $s\in[0,L_k]$. The fact that $x\notin\omega$ implies that $L_k\geq\mathrm{dist}\,(x_0;\partial\omega)$ for all $k\in\mathbb{N}$. Without introducing new notation, we extend each curve ρ_k continuously to the whole interval $[0,\infty)$, by setting $\rho_k(s)=x_0$ for every $s>L_k$.

One has that $\sup_{k\in\mathbb{N}}\|\rho_k\|_{L^\infty(0,\infty)}<\infty$ due to the fact that $\Omega_k\subset D$ for every $k\in\mathbb{N}$ and to the boundedness of D. Moreover, $\sup_{k\in\mathbb{N}}\|\dot{\rho}_k\|_{L^\infty(0,\infty)}\leq 1$ because all curves ρ_k are parametrized by arc-length on $[0,L_k]$ and are then constant. As a result, $\{\rho_k\}_{k\in\mathbb{N}}\subset W^{1,\infty}(0,\infty)$ is uniformly bounded and there exists $L\in(0,\infty]$ and a curve $\rho:[0,\infty)\to D$ with $\rho(0)=x$, $\rho(L)=x_0$, such that, up to subsequences $L_k\to L$ and $\rho_k\to\rho$ strongly in $L^\infty(0,\infty)$ and weakly* in $W^{1,\infty}(0,\infty)$.

We proceed by showing that $\rho(s)\subset\Omega_\infty$ for all $s\geq 0$ and that $\mathrm{dist}\,(\rho(s);\partial\Omega_\infty)\geq C_Js/8$ for all $s\in[0,L]$. Indeed, fix $\overline{s}>0$ and let $\overline{k}\in\mathbb{N}$ big enough so that

$$\|\rho_k-\rho\|_{L^\infty(0,\infty)}\leq\frac{C_J\overline{s}}{8}\quad\text{for every}\quad k\geq\overline{k}.$$

Then,

$$B_{\frac{C_J\overline{s}}{8}}(\rho(\overline{s}))\subset B_{\frac{C_J\overline{s}}{8}}(\rho_k(\overline{s}))+B_{\frac{C_J\overline{s}}{8}}(0)\subset\subset B_{\frac{C_J\overline{s}}{4}}(\rho_k(\overline{s}))\subset\subset\Omega_k$$

for every $k\geq\overline{k}$. In particular, it follows that $B_{C_J\overline{s}/8}(\rho(\overline{s}))\subset\Omega_\infty$. Since the same argument holds for every $\overline{s}>0$, we deduce that $\rho(0,\infty)\subset\Omega_\infty$. From the openness of Ω_∞ we then infer that also $x=\rho(0)\in\Omega_\infty$, as well, and that Ω_∞ is a John domain with respect to x_0, with John constant at least $C_J/8$.

Step 2. In this step, we prove that the first term in the right-hand side of (5.2) is infinitesimal as $k\to\infty$.

Since $S_k \subset \Omega_k$ for all $k \in \mathbb{N}$, by (5.1) it follows that

$$\|\varepsilon(u_k)\|_{L^2(S_k\cap\Omega_\infty;\mathbb{R}^{d\times d})} \leq \frac{1}{k}$$

for all $k \in \mathbb{N}$. Let $\eta > 0$. In view of Step 1, by [39, Theorems 4.5 and 4.6], for every $\eta > 0$ there exists a John domain $\omega_\eta \subset \mathbb{R}^d$ such that $\omega \subset\subset \omega_\eta \subset\subset \Omega_\infty$, and

$$|\Omega_\infty \setminus \omega_\eta| < \eta. \tag{5.3}$$

Then, $\omega_\eta \subset\subset \Omega_k \cap \Omega_\infty$ for k big enough, and hence $\omega_\eta \subset\subset S_k$ for k big enough. Additionally, $\{u_k\}_{k\in\mathbb{N}} \subset H^1_\omega(\omega_\eta;\mathbb{R}^d)$ for k big enough and

$$\|\varepsilon(u_k)\|_{L^2(\omega_\eta;\mathbb{R}^{d\times d})} \leq \frac{1}{k}. \tag{5.4}$$

Since ω_η is a John domain, by (2.9) and in view of [7, Theorem 5.1] we infer the existence of a map $u \in H^1_\omega(\omega_\eta;\mathbb{R}^d)$ such that, up to extracting a further non-relabeled subsequence, there holds

$$u_k \rightharpoonup u \quad \text{weakly in } H^1_\omega(\omega_\eta;\mathbb{R}^d).$$

On the other hand, by (5.4) we find that $\varepsilon(u) = 0$ on ω_η. Since $u = 0$ on ω, this implies that $u \equiv 0$ on ω_η. Hence, in particular,

$$u_k \to 0 \quad \text{strongly in } L^2(\omega_\eta;\mathbb{R}^d). \tag{5.5}$$

As Ω_∞ is a John domain, the boundary $\partial\Omega_\infty$ has zero measure, cf. [25, Corollary 2.3]. Hence, we can write

$$\|u_k\|^2_{L^2(S_k\cap\overline{\Omega}_\infty;\mathbb{R}^d)} = \|u_k\|^2_{L^2(S_k\cap\Omega_\infty;\mathbb{R}^d)} = \|u_k\|^2_{L^2(\omega_\eta;\mathbb{R}^d)} + \|u_k\|^2_{L^2((S_k\cap\Omega_\infty)\setminus\omega_\eta;\mathbb{R}^d)}, \tag{5.6}$$

and prove that it converges to 0 as $k \to \infty$. Indeed, the first term in the above right-hand side is infinitesimal due to (5.5). In order to handle the second term, we first apply the Hölder inequality and then we rely again on [7, Theorem 5.1]. Let us momentarily assume that $d > 2$ (the case $d = 2$ is discussed afterwards). By letting $2^* = 2d/(d-2)$, one argues as follows

$$\begin{aligned}
\|u_k\|^2_{L^2((S_k\cap\Omega_\infty)\setminus\omega_\eta;\mathbb{R}^d)} &\leq |(S_k\cap\Omega_\infty)\setminus\omega_\eta|^{\frac{2^*-2}{2^*}} \|u_k\|^2_{L^{2^*}((S_k\cap\Omega_\infty)\setminus\omega_\eta;\mathbb{R}^d)} \\
&\leq |\Omega_\infty\setminus\omega_\eta|^{\frac{2^*-2}{2^*}} \|u_k\|^2_{L^{2^*}((S_k\cap\Omega_\infty)\setminus\omega_\eta;\mathbb{R}^d)} \\
&\leq C|\Omega_\infty\setminus\omega_\eta|^{\frac{2^*-2}{2^*}} \|\nabla u_k\|^2_{L^2(\Omega_k;\mathbb{R}^{d\times d})} \leq C\eta^{\frac{2^*-2}{2^*}},
\end{aligned} \tag{5.7}$$

where C is independent of k. From the arbitrariness of η, the decomposition (5.6) and convergences (5.5) and (5.7) entail that

$$\limsup_{k\to\infty} \|u_k\|^2_{L^2(S_k\cap\Omega_\infty;\mathbb{R}^d)} = 0. \tag{5.8}$$

We reach the same conclusion in case $d = 2$. By applying the first Hölder step in (5.7) with respect to an arbitrary exponent $p/2 > 1$ and then argue via [7, Theorem 5.1] one gets

$$\|u_k\|^2_{L^2((S_k\cap\Omega_\infty)\setminus\omega_\eta;\mathbb{R}^2)} \leq C\eta^{\frac{p-2}{p}}$$

so that (5.8) again follows.

Step 3. Here we show that the second term in the right-hand side of inequality (5.2) goes to 0 as $k \to \infty$. More precisely, we show that

$$\lim_{k\to\infty} \|u_k\|_{L^2(\Omega_k \setminus (S_k \cap \Omega_\infty);\mathbb{R}^d)} = 0. \tag{5.9}$$

Note that $\Omega_k \setminus (S_k \cap \Omega_\infty) \subset U_k$ where

$$U_k := (\cup_{n\geq k}\Omega_n) \cap (S_k \cap \Omega_\infty)^c.$$

By definition, $\{U_k\}_{k\in\mathbb{N}}$ is the intersection of two decreasing sequences of nested sets and it is thus a decreasing sequence of nested sets as well. In particular, $U_k \to \cap_{k\in\mathbb{N}} U_k \subset \partial\Omega_\infty$ both in the Hausdorff sense and in measure. As the Lebesgue measure of the boundary $\partial\Omega_\infty$ is 0, we have that $|U_k| \to 0$ as $k \to \infty$. For $d > 2$, by replicating the argument leading to (5.7), now on the sets $\Omega_k \setminus (S_k \cap \overline{\Omega}_\infty)$, one gets

$$\|u_k\|^2_{L^2(\Omega_k \setminus (S_k\cap\Omega_\infty);\mathbb{R}^d)} \leq C|U_k|^{\frac{2^*-2}{2}},$$

which implies (5.9). In the case $d = 2$, we argue analogously.

Step 4. We are now in the position of concluding the proof of Proposition 2.4. Indeed, the convergences (5.1), (5.8), and (5.9) ensure that the right-hand side of inequality (5.2) converges to 0 as $k \to \infty$, leading to a contradiction.

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