POINTWISE CONSTRAINTS FOR SCALAR CONSERVATION LAWS WITH POSITIVE WAVE VELOCITY

NICOLA DE NITTI, DENIS SERRE, AND ENRIQUE ZUAZUA

Abstract. We consider a conservation law with strictly positive wave velocity and study the well-posedness of a suitable notion of solution for the associated initial value problem under a pointwise flux constraint active in the half-line $\mathbb{R}_+$.

The strict positivity of the wave velocity allows for the dynamics in the unconstrained region $\mathbb{R}_-$ to be fully determined by the restriction of the initial data to $\mathbb{R}_-$.

On the other hand, the solution in the constrained region is dictated by the assumption that the total mass of the initial datum is conserved along the evolution. We formulate the transmission condition at the interface $\{x = 0\}$ in such a way that the boundary datum for the initial boundary value problem posed on $\mathbb{R}_+$ is given by the largest incoming flux that is admissible under the constraint, while the exceeding mass is accumulated in a "buffer" (as an atomic measure concentrated at the interface).

1. Introduction and main result

1.1. Conservation law models with pointwise density constraint. We are interested in studying the effect that a pointwise constraint on the flux has on the dynamics of the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is the unknown and $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is a non-negative *flux function*.

We assume that the *wave velocity* f' is strictly positive. Several models of interest verify this hypothesis: e.g., $f(u) := (u + \varepsilon)^2/2$, provided $u \geqslant 0$ (*Burgers' flux*); $f(u) := (u + \varepsilon)^2 \operatorname{sign}(u + \varepsilon)/2$; $f(u) := (u_{\max} - u)u$ (Greenshields–LWR traffic model), provided $0 \leqslant u \leqslant u_{\max}/2$; and $f(u) := \frac{u}{u+u_{\max}}$, for $|u| \leqslant u_{\max}$ (which arises in several supply-chain models).

To introduce the constraint, we split the physical domain into two semi-lines, $\mathbb{R}_\pm$. On the negative semi-line $\mathbb{R}_-$, the flux is not limited; on the other hand, $\mathbb{R}_+$ represents a "critical path", where we impose a constraint $u \leqslant 1$ or, equivalently (since the flux is strictly monotone), $f(u) \leqslant f(1)$.

We assume that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $u_0 \geqslant 0$, and that the initial state is consistent with the constraint, i.e.,

$$u_0(x) \leqslant 1 \quad \text{for} \quad x \in \mathbb{R}_+. \tag{1.2}$$

If $u_0 \in [0, 1]$, then Kružkov's solution of (1.1) over $[0, +\infty) \times \mathbb{R}$ satisfies $u \leqslant 1$ owing to the maximum principle—thus, it satisfies the constraint automatically. However, values of u_0 that exceed 1 over $\mathbb{R}_-$ may invade $\mathbb{R}_+$ under the action of Kružkov's semigroup[1] $(\mathcal{S}_t)_{t\geqslant 0}$, following the corresponding characteristics $x(t) = x_0 + tf'(u_0(x_0))$. If this is so, the constraint is violated and Kružkov's entropy solution must be discarded.

1.2. A new notion of solution under constraints. Let us follow the dynamics in order to build a suitable notion of solution.

2020 *Mathematics Subject Classification.* 35L65.

Key words and phrases. Conservation laws; positive wave velocity; pointwise constraint.

[1] That is, for every $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u(t, \cdot) = \mathcal{S}_t[u_0]$ is the unique entropy solution of (1.1) corresponding to the initial condition u_0. We refer to [25] for further information.

Because of $f' > 0$ and the lack of constraint on $\mathbb{R}_-$, the restriction u_{0-} of the initial data to $\mathbb{R}_-$ fully determines the solution u_- of the problem

$$\begin{cases} \partial_t u_-(t,x) + \partial_x f(u_-(t,x)) = 0, & t > 0,\ x \in \mathbb{R}_-, \\ u_-(0,x) = u_{0-}(x), & x \in \mathbb{R}_-. \end{cases} \tag{1.3}$$

That is, if we extend u_0 as a data v_0 over the whole line $\mathbb{R}$ and let $v(t,\cdot) := S_t[v_0]$, then the restriction of v to $[0,+\infty) \times \mathbb{R}_-$ does not depend upon the choice of the extension (cf. also the discussion around (3.4) below). We recall that $u \in L^\infty_{\mathrm{loc}}((0,\infty)\times\mathbb{R})$ is an *entropy solution* of (1.3) if, for every convex entropy $\eta \in C^2(\mathbb{R})$ with entropy-flux q (given by $q' = f'\eta'$),

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0, \qquad \eta(u(0,\cdot)) = \eta(u_0), \tag{1.4}$$

holds in the sense of distributions in $[0,+\infty)\times\mathbb{R}$. See [25] for further information.

Thanks to this observation, the entire problem is reduced to an initial boundary value problem (IBVP) in $\mathbb{R}_+$:

$$\partial_t u_+ + \partial_x f(u_+) = 0, \qquad t > 0,\ x \in \mathbb{R}_+,$$

with an initial datum yet to be determined to realize the constraint $f(u) \leqslant f(1)$. At the times $t > 0$ when the incoming boundary datum, $f(u_-(t,0))$ exceeds the constraint, there are only two options to enforce the constraint: since no "spill-back" of the density is possible (owing to $f' > 0$), either mass disappears or, if we assume that it is conserved, it must accumulate as a singular measure, $m(t)\delta_{\{x=0\}}$, a Dirac delta of mass $m(t) \geqslant 0$ concentrated at the interface $\{x = 0\}$.

Assuming the principle of *conservation of mass* yields

$$\int_{-\infty}^0 u_-(t,x)\,\mathrm{d}x + m(t) + \int_0^{+\infty} u_+(t,x)\,\mathrm{d}x = M := \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x. \tag{1.5}$$

The initial data are, respectively,

$$m(0) = 0, \quad u_\pm(0,x) = u_{0\pm}(x),$$

where u_{0+} and u_{0-} are the restrictions of u_0 over $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively[2].

It now remains to see how, after density accumulates in the buffer, what boundary data arise for the IBVP posed on $\mathbb{R}_+$. We start by observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^0 u_-(t,x)\,\mathrm{d}x = -f(u_-(t,0)), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} u_+(t,x)\,\mathrm{d}x = f(u_+(t,0)).$$

The conservation of mass in (1.5) thus reduces to the ODE

$$m'(t) = f(u_-(t,0)) - f(u_+(t,0)). \tag{1.6}$$

Notice in particular that, since both $u_\pm$ are bounded[3], the density m is Lipschitz continuous.

We now make a second crucial modellistic assumption: we suppose that flux $f(u_+(t,0))$ entering the critical path must be *as large as possible*. That is, we prescribe the following alternative:

(A) if $m(t) = 0$ (*no saturation*), then we prescribe

$$f(u_+(t,0)) = \min\{f(u_-(t,0)), f(1)\}$$

(that is, $u_+(t,0) = \min\{u_-(t,0), 1\}$);

(B) if $m(t) > 0$ (*saturated flow*), then we prescribe

$$f(u_+(t,0)) = f(1)$$

(that is, $u_+(t,0) = 1$).

---

[2] More generally, we may also consider an initial concentration, i.e., an initial datum $m(0) = m_0$, where $m_0 \geqslant 0$ is prescribed.

[3] Indeed, $\|u_-\|_{L^\infty(\mathbb{R}_-)} \leqslant \|u_{0-}\|_{L^\infty(\mathbb{R}_-)}$ and $\|u_+\|_{L^\infty(\mathbb{R}_+)} \leqslant 1$.

.

An important consequence of this modeling is that the evolution of the density m is decoupled from that of u_+: indeed, from (1.6), we deduce the variational inequality[4]

$$m \geqslant 0, \qquad m' \geqslant h, \qquad m\left(m' - h\right) = 0, \qquad m(0) = m_0, \tag{1.7}$$

where $m_0 \geqslant 0$ (in the previous discussion, we considered $m_0 = 0$, because the evolution starts with no mass accumulated at the interface, but we may indeed consider this more general situation) and

$$h := f(u_-(t,0)) - f(1) \in L^\infty\left(\mathbb{R}_+\right) \tag{1.8}$$

are given.

Once u_- is known and (1.7) is solved, we finally arrive to the formulation of the following Dirichlet boundary-value problem:

$$\begin{cases} \partial_t u_+ + \partial_x f\left(u_+\right) = 0, & t > 0,\ x \in \mathbb{R}_+, \\ u_+(0,x) = u_{0+}(x), & x \in \mathbb{R}_+, \\ f\left(u_+(t,0)\right) = f_+(t), & t > 0, \end{cases} \tag{1.9}$$

where incoming flux is defined by

$$f_+(t) := \begin{cases} \min\left\{f(u_-(t,0)),\ f(1)\right\} & \text{if } m(t) = 0, \\ f(1) & \text{if } m(t) > 0. \end{cases} \tag{1.10}$$

Since $f(0) \leqslant f_+ \leqslant f(1)$ and f is strictly increasing, the last equality in (1.9) amounts to writing

$$u_+(t,0) = b(t), \quad \text{with} \quad b(t) := f^{-1}\left(f_+(t)\right) \in [0,1].$$

This is a classical IBVP for a scalar conservation law, which falls within the well-posedness theory developed by Bardos, le Roux, and Nédélec in [11] (provided that b has finite total variation).

We emphasize that now the constraint is ensured by the maximum principle and the fact that ($b \in [0,1]$ or, equivalently, $f_+ \in [f(0), f(1)]$).

This discussion leads to formulating the following theorem.

Theorem 1.1 (Well-posedness of a conservation law with pointwise constraint)**.** *Let us consider the Cauchy problem for a scalar conservation law under constraint:*

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x) \leqslant 1 & x \in \mathbb{R}_+, \end{cases} \tag{1.11}$$

where we suppose

$$f \in W^{1,\infty}([\text{ess inf}\, u_0, \text{ess sup}\, u_0]), \qquad f'(u) \geqslant a > 0 \quad \textit{for } u \in [\text{ess inf}\, u_0, \text{ess sup}\, u_0];$$
$$u_0 \in L^1(\mathbb{R};\mathbb{R}_+) \cap L^\infty(\mathbb{R};\mathbb{R}_+), \qquad u_0(x) \leqslant 1 \quad \textit{for } x \in \mathbb{R}_+,$$
$$m_0 \geqslant 0.$$

Then there exists a solution

$$w(t,x) = u(t,x) + m(t)\,\delta_0,$$
$$\textit{with } (u,m) \in \left(L^\infty((0,+\infty)\times\mathbb{R}) \cap L^\infty((0,\infty);L^1(\mathbb{R})\right) \times \mathbb{R},$$

of (1.11) in the following sense:

- ***S-1.*** *$u_- := \mathbb{1}_{(-\infty,0)}\, u$ is the unique entropy solution of (1.3) in the sense of Kružkov;*
- ***S-2.*** *m is the unique Lipschitz continuous solution of the variational inequality (1.7)–(1.8);*
- ***S-3.*** *$u_+ = \mathbb{1}_{(0,+\infty)}\, u$ is the unique entropy solution of (1.9) in the sense of Bardos–le Roux–Nédélec.*

---

[4]This is, more precisely, the *complementarity problem* associated with a variational inequality. It can be equivalently formulated as

$$\begin{cases} \min\{m' - h,\ m\} = 0, & t > 0, \\ m(0) = m_0. \end{cases}$$

Moreover, if $(\tilde{u}, \tilde{m})$ *is such a solution of* (1.1) *with initial data* $(\tilde{u}_0, \tilde{m}_0)$ *as above, then the following inequality holds:*

$$\int_{\mathbb{R}} |u(t,x) - \tilde{u}(t,x)| \,\mathrm{d}x + |m(t) - \tilde{m}(t)| \leqslant \int_{\mathbb{R}} |u(0,x) - \tilde{u}(0,x)| \,\mathrm{d}x + |m_0 - \tilde{m}_0|, \quad \textit{for} \quad t \geqslant 0,$$

which, in particular, implies uniqueness for the Cauchy problem (1.1).

1.3. **Proof of Theorem 1.1.** The proof of Theorem 1.1 consists of three steps.

First, we note that the existence, uniqueness, and L^1-contraction property for entropy solutions of (1.3) follows from Kružkov's theory.

Second, we show that the variational inequality (1.7) has one and only one Lipschitz continuous solution. This is carried out in Proposition 2.1 of Section 2.

Finally, in Section 3, we consider the IBVP (1.9). In Proposition 3.1, we prove that the boundary datum (1.10) has finite total variation. As a consequence, we can apply BLN's result (recalled in Theorem A.1 and Appendix A below) to conclude that there exists one and only one entropy admissible solution of (1.9).

Putting these considerations together, the proof of Theorem 1.1 is complete.

1.4. **Comparison with the related literature.** Conservation laws with pointwise constraints on the flux have attracted much attention because of their applications, especially to traffic flow models (which typically feature bell-shaped fluxes). In [21], Colombo and Goatin introduced the concept of *point constraints on the flux* (i.e., constraints at a single point, $x = 0$, of the type $f(u(t,0)) \leqslant q(t)$ for $t > 0$) to the first-order model for road traffic. Their aim was to describe the effects of obstacles along the road (e.g., toll gates, traffic lights, or localized construction sites) and the flux function, $f : [0,1] \to \mathbb{R}$, featured in the model was Lipschitz continuous and bell-shaped. The key component of the proof of the well-posedness result in [21] is approximating the constrained conservation law by one with a discontinuous flux function.

This result was generalized in several directions, e.g., in [5, 3], where the authors also addressed the interpretation of this model in terms of the general theory of conservation laws with discontinuous-flux (for which we refer to [6, 23, 22]).

Assuming that the discontinuity is located at $x = 0$ and imposing the validity of Kružkov's entropy inequalities separately on $(-\infty, 0)$ and $(0, +\infty)$, it turns out that every pair u, v of L^∞ solutions satisfies

$$\int_{\mathbb{R}} |u(t,x) - v(t,x)| \,\mathrm{d}x \leqslant \int_{\mathbb{R}} |u(0,x) - v(0,x)| \,\mathrm{d}x + \int_0^t W\left(u(t,0\pm), v(t,0\pm)\right) \mathrm{d}t,$$

where W is a quantity that depends only on the traces $u(t,0\pm)$, $v(t,0\pm)$ of u and v at $x = 0$. If the aim is to build an L^1-contracting semigroup of solutions, as in the classical theory of conservation laws, then we need $W(u^\pm, v^\pm) \leqslant 0$ for every pair of solutions. In [6], Andreianov, Karlsen, and Risebro introduced the notion of L^1*-dissipative germ* to encode the condition $W \leqslant 0$: at a point of discontinuity of the flux f, a germ $\mathcal{G}$ is a set of pairs (u^-, u^+) satisfying the Rankine–Hugoniot condition

$$f(u^-) = f(u^+)$$

and the dissipation condition

$$W\left(u^\pm, v^\pm\right) \leqslant 0 \qquad \text{for all} \quad (u^-, u^+),\ (v^-, v^+) \in \mathcal{G}$$

Given a L^1-dissipative germ $\mathcal{G}$, they showed uniqueness of a $\mathcal{G}$*-entropy solution*, i.e., a solution of the conservation law satisfying Kruzkov's conditions outside the origin and whose traces at $x = 0$ belong to $\mathcal{G}$.

Various L^1-dissipative transmission conditions (i.e., several L^1-dissipative germs) have been proposed. Selecting the transmission condition is a modelistic choice: several conditions are available in the literature in order to have that $W \leqslant 0$, which lead to different physically relevant semigroups of solutions.

We complement the analysis of [6]. In fact, we are dealing the case of a discontinuous flux

$$\tilde{f}(x,u) := \begin{cases} f_\ell(u) := f(u), & \text{if } x < 0, \\ f_r(u) := \min\{f(u),\ f(1)\}, & \text{if } x > 0, \end{cases} \tag{1.12}$$

where $f' \geqslant a > 0$. In case of saturation of the $u \leqslant 1$ constraint, the analysis of [6, Section 4.5] yields the non-existence of L^∞ $\mathcal{G}$-entropy solutions that conserve mass for the conservation law driven by $\tilde{f}$ (because the Rankine–Hugoniot condition at the interface generally does not hold). Addressing this issue, we consider a different notion of transmission condition at the interface $\{x = 0\}$: instead of encoding Rankine–Hugoniot, we introduce a buffer density m that allows for the mass that is concentrated at the interface $\{x = 0\}$.

This viewpoint is somewhat similar also to the one of [9, 7, 8] where a Dirac delta is added (as a source term in the PDE though, instead of at an interface transmission) to allow for energy conservation in a fluid–structure interaction model.

We also refer to [1], where a non-classical shock arises from an "overcompressive" flux pair across the interface $\{x = 0\}$: the mass only enters the interface from both sides. This is different from our case: namely, mass in the "buffer" can only increase.

Further results on conservation laws under constraints are contained in [3, 10, 4, 31, 32, 28, 27, 13, 26].

A model for traffic flow at a junction used in [16, 17, 18] is also somewhat related to our set-up: if the flux of cars at a junction that wants to enter a certain road is larger than the given maximum flux allowed on that road, they are placed in a queue, first-in-first-out; specifically, in these three works, it is assumed that the queue can occupy a buffer of unlimited capacity, restricted to a certain size, or to a vanishing size, respectively.

Finally, we mention that, in [41, 29, 12, 38, 14, 40, 15, 39], the formulation of conservation laws with constraints on the density was treated in a different way (as a variational inequality, more closely related to parabolic obstacle problems) by using a penalization method à la Stampacchia. However, in this work, the mass of the initial datum may be lost during the evolution due to the obstacle, which is not desirable in traffic or supply-chain modeling. To address this issue, in [2], a nonlocal Lagrange multiplier was incorporated in the penalization argument; this, in turn, gave rise to a different set of challenges: the question of the uniqueness of the constructed solution remains open and the property of finite speed of propagation does not hold.

Yet another way to enforce constraints (broadly speaking) on the profile of the solution at time t is to turn the problem into a flux identification question: from an initial datum and solution satisfying some properties (e.g., one-sided bound), reconstruct a suitable flux that produces that solution. There are several papers about this type of inverse problem, for both linear and nonlinear conservation laws, e.g., [19, 36, 20, 34]. In these works, however, the given information is not just an L^∞-constraint on the solution, but more precise, i.e., the exact profile of the solution at time t in (a subregion of) the domain (or its trace on the boundary of the domain).

2. Existence and uniqueness of m

In this section, we study the variational inequality (1.7). We show that it admits a unique solution, which is explicitly characterized as the smallest function m satisfying the initial data and $m \geqslant 0$, $m' \geqslant h$.

Proposition 2.1 (Well-posedness of the variational inequality). *Let $m_0 \in \mathbb{R}_+$ and $h \in L^\infty(\mathbb{R}_+)$ be given. Then the variational inequality (1.7) admits a unique Lipschitz continuous solution, which is explicitly given by the formula*

$$\begin{aligned} &m(t) = \inf\{p(t) : p \in K_t\}, \quad \textit{for all } t \geqslant 0, \\ &\textit{where} \quad K_t := \left\{p \in C([0,t]) : \ p(0) = m_0 \textit{ and } p \geqslant 0, \ p' \geqslant h \textit{ in } [0,t]\right\}. \end{aligned} \tag{2.1}$$

Here, the differential inequality $p' \geqslant h$ is to be understood in the distributional sense: i.e.,

$$(\tau < t) \Longrightarrow \left(p(t) \geqslant p(\tau) + \int_\tau^t h(s)\,\mathrm{d}s\right).$$

Moreover, given $\tilde{m}_0 \in \mathbb{R}$ and the corresponding solution $\tilde{m}$ (with same h), we have that the map $t \mapsto |m(t) - \tilde{m}(t)|$ is non-increasing.

Remark 2.2 (A representation formula). A (non-explicit) formula for the in (2.1) is given by

$$m(t) = m_0 + \int_0^t h^+(s)\,\mathrm{d}s - \int_0^t h^-(s)\mathbb{1}_{\{m(s)>0\}}(s)\,\mathrm{d}s, \qquad t \geqslant 0, \tag{2.2}$$

where $h = h^+ - h^-$ is the splitting of h into positive and negative part.

We claim that $\bar{m} \in K_t$. First, we have that $\bar{m}$ is continuous and $\bar{m}(0) = m_0$. Second, $m \geqslant 0$. Indeed, if, by contradiction, there exists $t^* > 0$ such that $m(t^*) < 0$, then, by continuity, there's an interval $(t^* - \delta, t^* + \delta)$ where it is negative and $m(t^* - \delta) = 0$, and then $\bar{m}(t^*) = m_0 + \int_0^t h^+(s)\,\mathrm{d}x - \int_0^{t^*-\delta} h^-(s)\,\mathrm{d}s \geqslant m(t^* - \delta) = 0$, which is absurd. Finally, $m'(t) = h^+(t) - h^-(t)\mathbb{1}_{\{m(t)\geqslant 0\}} \geqslant h(t)$.

Next, we claim that, for any $p \in K_t$, $m \leqslant p$. By contradiction, let $p \in K_t$ and $t^* > 0$ such that $p(t^*), m(t^*) > 0$, and $p(t^*) < m(t^*)$. Then, there exists $\delta > 0$ such that $m > p > 0$ in $(t^* - \delta, t^* + \delta)$ and $m(t^* - \delta) = p(t^* - \delta)$. Then

$$\begin{aligned} p(t^*) &\geqslant p(t^* - \delta) + \int_{t^*-\delta}^{t^*} h(s)\,\mathrm{d}s \\ &= m(t^* - \delta) + \int_{t^*-\delta}^{t^*} h^+(s)\,\mathrm{d}s - \int_{t^*-\delta}^{t^*} h^-(s)\,\mathrm{d}s \\ &= m(t^* - \delta) + \int_{t^*-\delta}^{t^*} h^+(s)\,\mathrm{d}s - \int_{t^*-\delta}^{t^*} h^-(s)\mathbb{1}_{\{m(s)>0\}}\,\mathrm{d}s \\ &= m(t^*), \end{aligned}$$

which is a contradiction.

Proof of Theorem 1.1. Step 1. *The representation formula gives a solution.* The function m given by (2.1) is non-negative and satisfies

$$m(0) = \inf\{p(0) : p \in K_0\} = m_0$$

Moreover, by definition of K_t, it satisfies $m(t) \geqslant 0$ and $m'(t) \geqslant 0$ for all $t \geqslant 0$. It remains to prove that $m(m' - h) = 0$ holds for $t > 0$. To this end, we will show that, if $m \neq 0$ (i.e., $m > 0$), then $m' = h$, i.e., for all $t \geqslant \tau \geqslant 0$,

$$m(t) = m(\tau) + \int_\tau^t h(s)\,\mathrm{d}s.$$

Let $t \geqslant \tau \geqslant 0$ be given. On the one hand, every $p \in K_t$ satisfies

$$p(t) \geqslant p(\tau) + \int_\tau^t h(s)\,\mathrm{d}s \geqslant m(\tau) + \int_\tau^t h(s)\,\mathrm{d}s.$$

Minimizing over K_t, we infer

$$m(t) \geqslant m(\tau) + \int_\tau^t h(s)\,\mathrm{d}s. \tag{2.3}$$

On the other hand, any $p \in K_\tau$ can be continued as an element of K_t by defining

$$p(\tau') = p(\tau) + \int_\tau^{\tau'} |h(s)|\,\mathrm{d}s, \quad \text{for all} \quad s \in [\tau, t].$$

This yields the inequality

$$m(t) \leqslant p(\tau) + \int_\tau^t |h(s)|\,\mathrm{d}s.$$

Minimizing over K_t, we obtain

$$m(t) \leqslant m(\tau) + \int_\tau^t |h(s)|\,\mathrm{d}s. \tag{2.4}$$

We want to show that actually we can replace $|h|$ by h in (2.4). To this end, we observe that (2.3) and (2.4) together imply that m is Lipschitz continuous,

$$h \leqslant m' \leqslant |h|,$$

and, provided $t - \tau > 0$ is small enough,

$$m(\tau) - \int_\tau^t |h(s)|\,\mathrm{d}s > 0. \tag{2.5}$$

As a consequence, we find that, every $p \in K_\tau$ can be actually continued as an element of K_t by

$$p(\tau') = p(\tau) + \int_\tau^{\tau'} h(s)\,\mathrm{d}s, \quad \text{for all} \quad \tau' \in [\tau, t].$$

Then we have

$$m(t) \leqslant p(\tau) + \int_\tau^t h(s)\,\mathrm{d}s.$$

Minimizing over K_t, this yields (provided that $t - \tau > 0$ is small enough)

$$m(t) \leqslant m(\tau) + \int_\tau^t h(s)\,\mathrm{d}s. \tag{2.6}$$

Combining (2.3) and (2.6) yields

$$m(t) = m(\tau) + \int_\tau^t h(s)\,\mathrm{d}s.$$

Since m is continuous, this implies $m' \equiv h$ over every interval in which $m > 0$. Moreover, since the set defined by $m > 0$ is open in $\mathbb{R}_+$ (hence a union of such interval), we deduce

$$(m > 0) \Longrightarrow \big(m' = h\big).$$

As a result, m is a Lipschitz solution of (1.7).

Step 2. *Uniqueness.* Let us suppose that M is another solution of (1.7) with the same initial data. We claim that $t \mapsto \frac{1}{2}(M - m)^2$ is a non-increasing function. Since it vanishes at $t = 0$, it must be $\leqslant 0$ on $\mathbb{R}_+$, which yields $M \equiv m$.

To prove the needed monotonicity, we compute

$$\frac{1}{2}((M - m)^2)' = (M - m)(M - m)'$$

and observe that

$$M(m - M)' = Mm' - MM' \geqslant Mh - MM' = 0$$

and, likewise,

$$m(M - m)' \geqslant 0.$$

Summing up both inequalities, we deduce

$$(M - m)(M - m)' \leqslant 0.$$

Finally, we note that, by a similar argument, if $\tilde{m}_0 \in \mathbb{R}$ is another initial data and $\tilde{m}$ is the corresponding solution of (1.7), we can show that $t \mapsto |\tilde{m} - m|$ is non-increasing. ☐

In Proposition 2.1, if h is not merely a bounded function, but is actually the one in (1.8), we can show that the buffer density m decays.

Proposition 2.3 (Long-time behavior of the buffer density)**.** *Let $m_0 \geqslant 0$ and let $h \in L^\infty(\mathbb{R}_+)$ be given by (1.8). Then the solution m of the variational inequality (1.7) satisfies*

$$\lim_{t \to +\infty} m(t) = 0. \tag{2.7}$$

Proof. We write

$$0 = \iint_\Delta \left(\partial_t u_- + \partial_x f\left(u_-\right)\right)\mathrm{d}x\,\mathrm{d}t,$$

where Δ is the triangle whose basis is $(-aT, 0)$ at initial time and the right side is $(0, T)$ at the boundary $x = 0$ (here, recall that $f' \geqslant a > 0$). From Gauss–Green's formula, we deduce

$$0 = \int_0^T f\left(u_-(t, 0)\right)\,\mathrm{d}t - \int_{-aT}^0 u_{0-}(x)\,\mathrm{d}x + \int_0^T \left(f\left(u_-\right) - au_-\right)(t, a(t - T))\,\mathrm{d}t.$$

The last integral is non-negative because of $f(u) - au \geqslant 0$. Letting $T \to +\infty$, we obtain

$$\int_0^{+\infty} f\left(u_-(t, 0)\right)\,\mathrm{d}t \leqslant \|u_{0-}\|_{L^1(\mathbb{R})}$$

In particular, this gives $g := f \circ u_{0-} \in L^1((0, +\infty))$.

Let (t_1, t_2) be a maximal interval on which $m > 0$, with $t_1 \geqslant 0$, so that $m(t_1) = 0$. We claim that $t_2 < +\infty$.

First, we consider the case $t_1 > 0$. We have

$$m' = h = g - f(1) \geqslant g, \qquad t \in (t_1, t_2),$$

where we recall that $h = g - f(1), \quad g \in L^1(0, +\infty)$, and then

$$m(t_2) + (t_2 - t_1) f(1) = \int_{t_1}^{t_2} g(t)\,\mathrm{d}t \leqslant \int_{t_1}^{+\infty} g(t)\,\mathrm{d}t$$

Since the left-hand side is finite, we deduce that t_2 is finite.

Second, we consider the case $t_1 = 0$. Then we have

$$m(t_2) + (t_2 - t_1) f(1) = \int_{t_1}^{t_2} g(t)\,\mathrm{d}t + m_0 \leqslant \int_{t_1}^{+\infty} g(t)\,\mathrm{d}t + m_0,$$

and, again, we deduce that t_2 is finite.

In conclusion, since the intervals are disjoint, the length of the set $\{t : m(t) > 0\}$ mus be bounded by $f(1)(m_0 + \|g\|_{L^1((0,+\infty))}) \leqslant f(1)(m_0 + \|u_{0-}\|_{L^1((0,+\infty))})$.

In addition, we notice the estimate $m(t_2) \leqslant \int_{t_1}^{t_2} g(t)\,\mathrm{d}t$, which shows that the maxima over the intervals of positivity of m tend to zero (actually, they form a summable series). Thus

$$\lim_{t \to +\infty} m(t) = 0.$$

□

Remark 2.4. In general, we cannot say more than (2.7) regarding the long-time behavior of the buffer density unless we consider additional hypotheses on the initial datum or the flux function. Let us provide some examples.

Compactly supported data: If u_{0-} has compact support, the activation of the obstacle is a transient phenomenon. Indeed, by [30], we then have that there exists $\bar{T} > 0$ such that $u_-(t, 0) = 0$ for $t \geqslant \bar{T}$. Then we can deduce that m has compact support.

Linear flux: If the flux function is linear, i.e., $f(u) \equiv au$, then $g(t) = au_{0-}(-at)$. Even though u_{0-} is integrable, the set $\{t \geqslant 0 : g(t) > a = f(1)\}$ could be unbounded. In such a situation, the support of m is unbounded, although having finite length.

Strictly concave/convex flux: If the flux is strictly concave (or convex), in the sense that f'' does not vanish. Then, by the sharp Oleĭnik-type inequality in [24, 33],

$$\|u_-\|_{L^\infty(\mathbb{R})} = \mathcal{O}(t^{-1/2})$$

Therefore $g(t) = \mathcal{O}(t^{-1/2})$. This implies

$$h \leqslant -\frac{1}{2} f(1), \quad \forall t > T$$

for some T. Up to a translation of time, we may suppose that $h \leqslant -\frac{1}{2} f(1)$ for all time. Then the function

$$p(t) := \begin{cases} m_0 - \dfrac{t}{2} f(1), & \text{if } t < \frac{2m_0}{f(1)}, \\ 0, & \text{if } t \geqslant \frac{2m_0}{f(1)}, \end{cases}$$

belongs to the set K_t for every $t \geqslant 0$, thus is an upper bound for m. This shows that $m \equiv 0$ for all $t > T$. Hence, m is compactly supported.

3. Existence and uniqueness of u_+

As for the Cauchy problem, the notion of solution for the IBVP associated with a scalar conservation law is expressed in terms of integral inequalities involving entropies. Given a convex entropy $\eta \in C^2(\mathbb{R})$ with entropy-flux q, we define the *relative entropy*

$$\bar{\eta}(\xi \mid \zeta) := \eta(\xi) - \eta(\zeta) - \eta'(\zeta)(\xi - \zeta) \tag{3.1}$$

and the *flux of the relative entropy*

$$\bar{q}(\xi; \zeta) := q(\xi) - q(\zeta) - \eta'(\zeta)(f(\xi) - f(\zeta)). \tag{3.2}$$

(whose definition does not mimic that of $\bar\eta$, but it is tailored so that $\bar q' = \bar\eta' f'$ at fixed $\zeta \in \mathbb{R}$). We say that v is the *entropy solution* of the (1.9) *in the sense of BLN*[5] if, for every convex entropy $\eta \in C^2$ and entropy-flux q,

$$\begin{aligned}\iint_{\mathbb{R}^2_+} \Big(\bar\eta(u_+ \mid b)\partial_t\varphi + \bar q(u_+;b)\partial_x\varphi - (u_+ - b)\partial_x\eta'(b)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}_+} \bar\eta\left(u_{0+}(x) \mid b(0)\right)\varphi(0,x)\,\mathrm{d}x \geqslant 0.\end{aligned} \tag{3.4}$$

holds for every non-negative test function $\varphi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}_+)$.

For a general flux, the formulation of the IBVP given in (3.4) does not impose the equality $u_+ = b$ at the boundary. This is due to the possible formation of boundary layer effects. For instance, if $f' < 0$ (the waves travel to the left), then u_+ is completely determined by the initial data u_{0+} alone, while the boundary data b is simply ignored (as $\bar q \leqslant 0$ identically in this case). On the other hand, our assumption $f' > 0$ yields $\bar q(\xi;b) \geqslant 0$ for all $\xi \in \mathbb{R}$, for every convex entropy η. If η is strictly convex, we even have $\bar q(\xi;b) > 0$, unless $\xi = b$ holds[6]. For IBVP (1.9), the boundary condition $\bar q\left(u_+(t,0);b(t)\right) \leqslant 0$ then tells us therefore that the Dirichlet condition is satisfied in the classical sense:

$$u_+(\cdot,0) \equiv b.$$

With this, we do obtain the conservation of total mass, since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} u_+(t,x)\,\mathrm{d}x = f\left(u_+(t,0)\right) = f(b(t)).$$

In order to apply the classical well-posedness result by Bardos, le Roux, and Nédélec (recalled in Theorem A.1) to (1.9), we only need to show that $b \in \mathrm{BV}([0,+\infty))$. This turns out to be true: we will show that $\mathrm{TV}_{(0,t)}(b) \leqslant 2\,\mathrm{TV}_{(0,t)}\left(u_-(\cdot,0)\right)$; and, on the one hand, we already know that $\mathrm{TV}\left(u_-(\cdot,0)\right) \leqslant \mathrm{TV}\left(u_{0-}\right)$.

Proposition 3.1 (BV-estimate on b)**.** *For every $t \geqslant 0$, we have*

$$\mathrm{TV}_{(0,t)}(b) \leqslant 2\,\mathrm{TV}_{(0,t)}\left(u_-(\cdot,0)\right).$$

Proof. Let us recall that, by definition,

(1) if $m(t) > 0$, then $b(t) = 1$;
(2) if $m(t) = 0$, then $b(t) = \min\{u_-(t,0), 1\}$.

If $m > 0$ on an interval I, then $\mathrm{TV}_I(b) = 0$. On the other hand, if $m \equiv 0$ on I, then $\mathrm{TV}_I(b) \leqslant \mathrm{TV}_I\left(u_-(\cdot,0)\right)$ because the projection π over $[0,1]$ is a contraction and $b(t) = \pi \circ u_-(t,0)$.

There remains to control the jumps of b at the transition points τ, i.e., those where m passes from one regime to the other one. Such jumps connect the values 1 and $u_-(\tau)$, where the latter must be < 1 (instead b is continuous at τ). The derivative m' connects the values 0 and $f(u_-(\tau)) - f(1)$. The latter being negative (because f is increasing), we see that the transitional jump of b can only occur when m passes from > 0 to $\equiv 0$, and not in the opposite case.

---

[5] Another equivalent way to express the BLN boundary condition is as follows: the trace $u_+(t,0+)$ satisfies

$$f(u_+(t,0+)) = G(b(t), u_+(t,0+)),$$

where G denotes the *Godunov numerical flux associated to* f (see [35, Eq. (3.8)]), which is given by

$$G(\xi,\zeta) = \begin{cases} \min\limits_{[\xi,\zeta]} f & \text{if } \xi \leqslant \zeta, \\ \max\limits_{[\zeta,\xi]} f & \text{if } \xi \geqslant \zeta. \end{cases} \tag{3.3}$$

[6] To see this, we first remark that $Q := q \circ f^{-1}$ is a convex function: differentiation of the identity $Q \circ f = q$ gives $f'Q' \circ f = q'$, that is $Q' \circ f = \eta'$. Differentiating once more, we have $f'Q'' \circ f = \eta''$. Since $\eta'' \geqslant 0$ and $f' > 0$, we obtain $Q'' \geqslant 0$. The convexity then implies

$$Q(f(s)) - Q(f(b)) - Q'(f(b))(f(s) - f(b)) \geqslant 0,$$

which is the same as $\bar q(s;b) \geqslant 0$. If η is strictly convex over an interval J, then Q may not be affine on any sub-interval of $f(J)$, and therefore is strictly convex too.

Therefore, let us assume that $b(\tau-0)=1$ and $b(\tau+0)=u_-(\tau)<1$. Then m remains $\equiv 0$ over a time interval (τ,τ_1) until $\mathrm{TV}_{(\tau,\tau_1)}(u_-(\cdot,0)) \geqslant 1-u_-(\tau,0)$, because $u_-(\cdot,0)$ remains $\leqslant 1$ in this interval.

As a result, the sum of all the transitional jumps is bounded above by $\sum_\tau \mathrm{TV}_{(\tau,\tau_1)}(u_-(\cdot,0))$, which is less than or equal to $\mathrm{TV}_{(0,t)}(u_-(\cdot,0))$ because the intervals under consideration are disjoint. ☐

Appendix A. Review of BLN theory

In this appendix, for the sake of completeness, we recall the main well-posedness result of [11].

Theorem A.1 (Well-posedness of the IBVP)**.** *Let $f \in W^{1,\infty}(\mathbb{R})$, $v_0 \in \mathrm{BV}(\mathbb{R}_+)$, and $b \in \mathrm{BV}([0,+\infty))$ be given*[7]*. Then the IBVP*

$$\begin{cases} \partial_t v(t,x)+\partial_x f(v(t,x))=0, & t>0,\ x\in\mathbb{R}_+, \\ v(0,x)=v_0(x), & x\in\mathbb{R}_+, \\ v(t,0)=b(t), & t\geqslant 0, \end{cases} \tag{A.1}$$

has one and only one entropy solution $v\in \mathrm{BV}([0,+\infty)\times\mathbb{R}_+)$ that satisfies (3.4). Moreover, the following estimates hold: for every $t\geqslant 0$,

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} \leqslant \max\left\{\|v_0\|_{L^\infty(\mathbb{R}_+)}, \|b\|_{L^\infty((0,t))}\right\}, \tag{A.2}$$

$$\mathrm{TV}_x(v(t,\cdot)) \leqslant \mathrm{TV}_x(v_0)+\mathrm{TV}_{(0,t)}(b). \tag{A.3}$$

Let us briefly sketch the proof of Theorem A.1. The solution of the IBVP (A.1) is constructed by the vanishing viscosity method. We start from the Dirichlet problem for the parabolic conservation law

$$\begin{cases} \partial_t v^\varepsilon(t,x)+\partial_x f(v^\varepsilon(t,x))=\varepsilon\partial_x^2 v^\varepsilon(t,x), & t>0,\ x\in\mathbb{R}_+, \\ v^\varepsilon(0,x)=u_0(x), & x\in\mathbb{R}_+ \\ v^\varepsilon(t,0)=b(t), & t>0, \end{cases} \tag{A.4}$$

whose solution v^ε is smooth for $t>0$ and $x\in\mathbb{R}_+$.

In order to pass to the limit as $\varepsilon\searrow 0$ and recover the hyperbolic problem (A.1), we need some crucial a priori estimates.

First, the maximum principle gives

$$\|v^\varepsilon\|_{L^\infty(\mathbb{R}_+)} \leqslant \max\left\{\|u_0\|_{L^\infty(\mathbb{R}_+)}, \|a\|_{L^\infty(\mathbb{R}_+)}\right\}$$

Second, given a convex entropy η and the corresponding entropy flux q, we compute

$$\partial_t\eta(v^\varepsilon)+\partial_x q(v^\varepsilon)=\varepsilon\partial_x^2\eta(v^\varepsilon)-\varepsilon\eta''(v^\varepsilon)(\partial_x v^\varepsilon)^2$$

The choice of $\eta(\xi):=\xi^2$ yields a uniform bound on $\sqrt{\varepsilon}\partial_x v^\varepsilon$ in $L^2((0,T)\times\mathbb{R}_+)$, which ensures that $\varepsilon\partial_x^2\eta(v^\varepsilon)\to 0$ in the sense of distributions.

Finally, we prove a BV-estimate. Differentiating the PDE in (A.4) with respect to the x-variable, we have (denoting $z^\varepsilon := \partial_x v^\varepsilon$)

$$\partial_t z^\varepsilon+\partial_x\left(f'(v^\varepsilon)z^\varepsilon\right)=\varepsilon\partial_x^2 z^\varepsilon.$$

Multiplying by $\operatorname{sign} z^\varepsilon$ and using Kato's inequality (see [37, Lemma A]) yields

$$\partial_t|z^\varepsilon|+\partial_x\left(f'(v^\varepsilon)|z^\varepsilon|\right)\leqslant\varepsilon\partial_x^2|z^\varepsilon|.$$

[7]Notice that the assumption concerning b is slightly weaker than in [11] because of the one-dimensional context. In dimension $1+d$ with $d\geqslant 2$, we also need the tangential gradient of a to be of bounded variation.

Integrating in the x-variable, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{TV}_x\left(v^\varepsilon(t,\cdot)\right) &\leqslant f'\left(v^\varepsilon\right)\left|z^\varepsilon(t,0)\right| - \varepsilon\left(\partial_x\left|z^\varepsilon\right|\right)(t,0)\\ &= \left(\operatorname{sign} z^\varepsilon(t,0)\right)\left(f'\left(v^\varepsilon\right)\partial_x v^\varepsilon - \varepsilon\partial_x^2 v^\varepsilon\right)(t,0)\\ &= \left(\operatorname{sign} z^\varepsilon(t,0)\right)\left(\partial_x f\left(v^\varepsilon\right) - \varepsilon\partial_x^2 v^\varepsilon\right)(t,0)\\ &= -\left(\operatorname{sign} z^\varepsilon(t,0)\right)\partial_t v^\varepsilon(t,0)\\ &\leqslant \left|\partial_t b\right|,\end{aligned}$$

and conclude that

$$\mathrm{TV}_x\left(v^\varepsilon(t,\cdot)\right) \leqslant \mathrm{TV}_x\left(v_0\right) + \mathrm{TV}_{(0,t)}(b).$$

This BV-estimate and standard arguments (for which we refer, e.g., to [42]) yield the compactness of the sequence $\{v^\varepsilon\}_{\varepsilon>0}$: we may extract a subsequence converging almost everywhere to a limit point v. A careful analysis of initial and boundary conditions reveals that the limit v satisfies the entropy inequalities in the integral sense:

$$\iint_{\mathbb{R}_+^2} \left\{\bar\eta(v \mid b)\partial_t\varphi + \bar q(v;b)\partial_x\varphi - (v-b)\partial_x\eta'(b)\varphi\right\}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}_+}\bar\eta\left(v_0(x)\mid b(0)\right)\varphi(0,x)\,\mathrm{d}x \geqslant 0$$

for every (smooth) convex entropy and every non-negative test function $\varphi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}_+)$.

To see that the whole family $\{v^\varepsilon\}_{\varepsilon>0}$ converges as $\varepsilon \to 0^+$, it suffices to prove the uniqueness of the limit (by Urysohn's subsequence principle). For this, let us suppose that another entropy solution, denoted by w, exists. Since the entropy condition (3.4) contains the usual entropy inequalities

$$\partial_t|v-k| + \partial_x((f(v)-f(k))\operatorname{sign}(v-k)) \leqslant 0,$$

Kružkov's "doubling of variables" argument yields

$$\partial_t|w-v| + \partial_x((f(w)-f(v))\operatorname{sign}(w-v)) \leqslant 0.$$

Integrating in space, we infer (using the fact that w and v have bounded variation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R}_+)} \leqslant \Big((f(w)-f(v))\operatorname{sign}(w-v)\Big)\Big|_{x=0}.$$

On the other hand, (3.4) contains the boundary condition $\bar q(v(t,0)\mid b(t)) \leqslant 0$, which amounts to writing

$$(\operatorname{sign}(v-k)-\operatorname{sign}(b-k))(f(v)-f(k))\Big|_{x=0} \leqslant 0, \qquad \text{for all } k \in \mathbb{R}. \tag{A.5}$$

From (A.5), written for both w and v, it follows that

$$(f(w)-f(v))\operatorname{sign}(w-v))\Big|_{x=0} \leqslant 0. \tag{A.6}$$

Then $t \mapsto \|w(t,\cdot)-v(t,\cdot)\|_{L^1(\mathbb{R}_+)}$ is non-increasing. Since $w(0,\cdot) = v(0,\cdot) = v_0$, we deduce that $w \equiv v$.

To prove (A.6), we argue by contradiction. Let us suppose that $(f(w)-f(v))\operatorname{sign}(w-v)) > 0$ at $x=0$. Without loss of generality, we may assume $v < w$, and so $f(v) < f(w)$. Let us take $k \to b$, with $\operatorname{sign}(b-k) = -\operatorname{sign}(v-b)$. We infer

$$(f(v)-f(b))\operatorname{sign}(v-b) \leqslant 0.$$

Likewise, we have

$$(f(w)-f(b))\operatorname{sign}(w-b) \leqslant 0.$$

Choosing instead $k = w$ resp. $k = v$), we also have

$$(f(w)-f(v))\operatorname{sign}(w-v)) \leqslant (f(w)-f(v))\operatorname{sign}(w-b) \quad \text{and} \quad \leqslant (f(v)-f(w))\operatorname{sign}(v-b).$$

There follows $v \leqslant b \leqslant w$. If $b \neq v, w$, then the inequalities above mean $f(v) > f(b) > f(w)$, which yields a contradiction.

Acknowledgments

We thank B. Andreianov, G. M. Coclite, C. Donadello, X. Fernández-Real, M. Garavello, and G. Vergara-Hermosilla for helpful conversations on topics related to this work.

N. De Nitti is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). He has been partially supported by the Swiss State Secretariat for Education, Research and Innovation (SERI) under contract number MB22.00034 through the project TENSE.

E. Zuazua has been funded by the Alexander von Humboldt-Professorship program, the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, the COST Action MAT-DYN-NET, the Transregio 154 Project "Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks" of the Deutsche Forschungsgemeinschaft (DFG), grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MINECO (Spain), and by the Madrid Government – UAM Agreement for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

References

[1] A. Aggarwal, M. R. Sahoo, A. Sen, and G. Vaidya. "Solutions with concentration for conservation laws with discontinuous flux and its applications to numerical schemes for hyperbolic systems". In: *Stud. Appl. Math.* 145.2 (2020), pp. 247–290.

[2] P. Amorim, W. Neves, and J. F. Rodrigues. "The obstacle-mass constraint problem for hyperbolic conservation laws. Solvability". In: *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 34.1 (2017), pp. 221–248.

[3] B. Andreianov, C. Donadello, U. Razafison, and M. D. Rosini. "Analysis and approximation of one-dimensional scalar conservation laws with general point constraints on the flux". In: *J. Math. Pures Appl. (9)* 116 (2018), pp. 309–346.

[4] B. Andreianov, C. Donadello, and M. D. Rosini. "A second-order model for vehicular traffics with local point constraints on the flow". In: *Math. Models Methods Appl. Sci.* 26.4 (2016), pp. 751–802.

[5] B. Andreianov, P. Goatin, and N. Seguin. "Finite volume schemes for locally constrained conservation laws". In: *Numer. Math.* 115.4 (2010). With supplementary material available online, pp. 609–645.

[6] B. Andreianov, K. H. Karlsen, and N. H. Risebro. "A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux". In: *Arch. Ration. Mech. Anal.* 201.1 (2011), pp. 27–86.

[7] B. Andreianov, F. Lagoutiére, N. Seguin, and T. Takahashi. "Small solids in an inviscid fluid". In: *Netw. Heterog. Media* 5.3 (2010), pp. 385–404.

[8] B. Andreianov, F. Lagoutière, N. Seguin, and T. Takahashi. "Well-posedness for a one-dimensional fluid-particle interaction model". In: *SIAM J. Math. Anal.* 46.2 (2014), pp. 1030–1052.

[9] B. Andreianov and N. Seguin. "Analysis of a Burgers equation with singular resonant source term and convergence of well-balanced schemes". In: *Discrete Contin. Dyn. Syst.* 32.6 (2012), pp. 1939–1964.

[10] B. P. Andreianov, C. Donadello, U. Razafison, J. Y. Rolland, and M. D. Rosini. "Solutions of the Aw-Rascle-Zhang system with point constraints". In: *Netw. Heterog. Media* 11.1 (2016), pp. 29–47.

[11] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. "First order quasilinear equations with boundary conditions". In: *Comm. Partial Differential Equations* 4.9 (1979), pp. 1017–1034.

[12] L. Barthélemy. "The obstacle problem for a first-order quasilinear equation". In: *Ann. Fac. Sci. Toulouse, Math. (5)* 9.2 (1988), pp. 137–159.

[13] M. Benyahia and M. D. Rosini. "A macroscopic traffic model with phase transitions and local point constraints on the flow". In: *Netw. Heterog. Media* 12.2 (2017), pp. 297–317.

[14] F. Berthelin and F. Bouchut. "Weak solutions for a hyperbolic system with unilateral constraint and mass loss." In: *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 20.6 (2003), pp. 975–997.

[15] I. H. Biswas, Y. Tahraoui, and G. Vallet. “Obstacle problem for a stochastic conservation law and Lewy Stampacchia inequality”. In: *J. Math. Anal. Appl.* 527.1 (2023). Id/No 127356, p. 39.

[16] A. Bressan and K. Han. “Optima and equilibria for a model of traffic flow”. In: *SIAM J. Math. Anal.* 43.5 (2011), pp. 2384–2417.

[17] A. Bressan and K. T. Nguyen. “Conservation law models for traffic flow on a network of roads”. In: *Netw. Heterog. Media* 10.2 (2015), pp. 255–293.

[18] A. Bressan and A. Nordli. “The Riemann solver for traffic flow at an intersection with buffer of vanishing size”. In: *Netw. Heterog. Media* 12.2 (2017), pp. 173–189.

[19] P. Cannarsa, G. Floridia, F. Gölgeleyen, and M. Yamamoto. “Inverse coefficient problems for a transport equation by local Carleman estimate”. In: *Inverse Probl.* 35.10 (2019). Id/No 105013, p. 22.

[20] C. Castro and E. Zuazua. “Flux identification for 1-d scalar conservation laws in the presence of shocks”. In: *Math. Comput.* 80.276 (2011), pp. 2025–2070.

[21] R. M. Colombo and P. Goatin. “A well posed conservation law with a variable unilateral constraint”. In: *J. Differential Equations* 234.2 (2007), pp. 654–675.

[22] G. Crasta, V. De Cicco, and G. De Philippis. “Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux”. In: *Commun. Partial Differ. Equations* 40.4 (2015), pp. 694–726.

[23] G. Crasta, V. De Cicco, G. De Philippis, and F. Ghiraldin. “Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness”. In: *Arch. Ration. Mech. Anal.* 221.2 (2016), pp. 961–985.

[24] C. M. Dafermos. “Characteristics in hyperbolic conservation laws. A study of the structure and the asymptotic behaviour of solutions”. In: *Nonlinear analysis and mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I.* Res. Notes in Math., No. 17. Pitman, London-San Francisco, Calif.-Melbourne, 1977, pp. 1–58.

[25] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics.* Fourth. Vol. 325. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2016, pp. xxxviii+826.

[26] E. Dal Santo, M. D. Rosini, N. Dymski, and M. Benyahia. “General phase transition models for vehicular traffic with point constraints on the flow”. In: *Math. Methods Appl. Sci.* 40.18 (2017), pp. 6623–6641.

[27] M. L. Delle Monache and P. Goatin. “Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result”. In: *J. Differential Equations* 257.11 (2014), pp. 4015–4029.

[28] M. L. Delle Monache and P. Goatin. “A front tracking method for a strongly coupled PDE-ODE system with moving density constraints in traffic flow”. In: *Discrete Contin. Dyn. Syst. Ser. S* 7.3 (2014), pp. 435–447.

[29] J. I. Diaz and L. Véron. “Existence theory and qualitative properties of the solutions of some first order quasilinear variational inequalities”. In: *Indiana Univ. Math. J.* 32 (1983), pp. 319–361.

[30] C. Donadello and V. Perrollaz. “Exact controllability to trajectories for entropy solutions to scalar conservation laws in several space dimensions”. In: *C. R. Math. Acad. Sci. Paris* 357.3 (2019), pp. 263–271.

[31] M. Garavello and P. Goatin. “The Aw-Rascle traffic model with locally constrained flow”. In: *J. Math. Anal. Appl.* 378.2 (2011), pp. 634–648.

[32] M. Garavello and S. Villa. “The Cauchy problem for the Aw-Rascle-Zhang traffic model with locally constrained flow”. In: *J. Hyperbolic Differ. Equ.* 14.3 (2017), pp. 393–414.

[33] D. Hoff. “The sharp form of Oleĭnik’s entropy condition in several space variables”. In: *Trans. Amer. Math. Soc.* 276.2 (1983), pp. 707–714.

[34] H. Holden, F. S. Priuli, and N. H. Risebro. “On an inverse problem for scalar conservation laws”. In: *Inverse Probl.* 30.3 (2014). Id/No 035015, p. 35.

[35] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws.* Second. Vol. 152. Applied Mathematical Sciences. Springer, Heidelberg, 2015, pp. xiv+515.

[36] F. James and M. Sepúlveda. “Convergence results for the flux identification in a scalar conservation law”. In: *SIAM J. Control Optim.* 37.3 (1999), pp. 869–891.

[37] T. Kato. "On some Schrödinger operators with a singular complex potential". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 5.1 (1978), pp. 105–114.

[38] L. Levi. "Obstacle problems for scalar conservation laws". In: *M2AN, Math. Model. Numer. Anal.* 35.3 (2001), pp. 575–593.

[39] L. Lévi and G. Vallet. "Entropy solutions for first-order quasilinear equations related to a bilateral obstacle condition in a bounded domain". In: *Chin. Ann. Math., Ser. B* 22.1 (2001), pp. 93–114.

[40] L. Lévi. "The singular limit of a bilateral obstacle problem for a class of degenerate parabolic-hyperbolic operators". In: *Adv. Appl. Math.* 35.1 (2005), pp. 34–57.

[41] F. Mignot and J.-P. Puel. "Inéquations variationnelles et quasivariationnelles hyperboliques du premier ordre". In: *J. Math. Pures Appl. (9)* 55 (1976), pp. 353–378.

[42] J. Nečas, J. Málek, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*. Vol. 13. Applied Mathematics and Mathematical Computation. Chapman & Hall, London, 1996, pp. xii+317.

(N. De Nitti) EPFL, Institut de Mathématiques, Station 8, 1015 Lausanne, Switzerland.
Email address: nicola.denitti@epfl.ch

(D. Serre) UMPA, Ecole Normale Supérieure de Lyon, Allée d'Italie 46, 69364 Lyon Cedex 07, France.
Email address: denis.serre@ens-lyon.fr

(E. Zuazua) Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Mathematics, Chair for Dynamics, Control, Machine Learning and Numerics (Alexander von Humboldt Professorship), Cauerstr. 11, 91058 Erlangen, Germany.

Chair of Computational Mathematics, Fundación Deusto, Avenida de las Universidades, 24, 48007 Bilbao, Basque Country, Spain.

Universidad Autónoma de Madrid, Departamento de Matemáticas, Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain.
Email address: enrique.zuazua@fau.de