Minimal Surface Equation and Bernstein Property on RCD spaces

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Abstract

We show that if $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space and $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ is a solution of the minimal surface equation, then u is harmonic on its graph (which has a natural metric measure space structure). If $K = 0$ this allows to obtain an Harnack inequality for u, which in turn implies the Bernstein property, meaning that any positive solution to the minimal surface equation must be constant. As an application, we obtain oscillation estimates and a Bernstein Theorem for minimal graphs in products $\mathsf{M} \times \mathbb{R}$, where M is a smooth manifold (possibly weighted and with boundary) with non-negative Ricci curvature.

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1 Introduction

In [12] Bombieri, De Giorgi and Miranda showed that the only entire positive solutions of the minimal surface equation in Euclidean space are the constant functions. If we replace the Euclidean space with a Riemannian manifold the validity of the aforementioned result depends on the geometry of the manifold. For this reason the following definition was introduced in [19].

Definition. We say that a Riemannian manifold (M, g) has the Bernstein Property if the only entire positive solutions of the minimal surface equation are the constant functions.

For example, positive non constant solutions of the minimal surface equation in the hyperbolic plane were constructed in [43], while the Bernstein Property holds for manifold with non-negative Ricci curvature and a lower sectional curvature bound thanks to [46]. Recently this was improved in [19] to manifolds with non-negative Ricci curvature and no sectional curvature constraint (see also [18] for an even stronger result in the same fashion), while Ding proved in [21] that the Bernstein Property holds in manifolds that are doubling and support a Poincaré inequality.

The fact that manifolds with non-negative Ricci curvature have the Bernstein Property and the recent generalization of some properties of minimal surfaces to RCD spaces (i.e. metric measure spaces with a notion of lower bound on the Ricci curvature) suggest that an analogue of the Bernstein Property might hold in this setting as well. We recall that an $\mathsf{RCD}(K,N)$ space is a metric measure space where $K \in \mathbb{R}$ plays the role of a lower bound on the Ricci curvature, while $N \in [1,+\infty)$ plays the role of an upper bound on the dimension. This class includes measured Gromov-Hausdorff limits of smooth manifolds of fixed dimension with uniform Ricci curvature lower bounds and finite dimensional Alexandrov spaces with sectional curvature bounded from below. With this in mind we can state the main result of this note.

Theorem 1. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space and let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ be an entire solution of the minimal surface equation. If u is positive, then it is constant.*

The previous theorem, as anticipated, is part of a wider class of recent results that aim at generalizing to the non smooth setting properties of minimal surfaces ([10], [25], [14], [42], etc.). Moreover, specializing Theorem 1 to the smooth category, we obtain that the Bernstein Property holds for certain weighted manifolds with boundary. This is the content of Theorem 2. Given a manifold (M,g) we denote by $\mathfrak{m}_g$ its volume measure and by d_g its distance. If $V : \mathsf{M} \to \mathbb{R}$ is a smooth function, we say that the metric measure space $(\mathsf{M}^n,\mathsf{d}_g,e^{-V}\mathfrak{m}_g)$ is a weighted manifold. Given $\Omega \subset \mathsf{M}$, we say that a function $u \in C^\infty(\Omega)$ is a solution of the weighted minimal surface equation on $\Omega \setminus \partial\mathsf{M}$ if

$$\mathsf{div}\Big(\frac{e^{-V}\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0 \quad \text{on } \Omega \setminus \partial\mathsf{M}.$$

We say that the boundary of a manifold with boundary is convex if its second fundamental form w.r.t. the inward pointing unit normal is positive.

Theorem 2. *Let $(\mathsf{M}^n,\mathsf{d}_g,e^{-V}\mathfrak{m}_g)$ be a weighted manifold with convex boundary such that there exists $N > n$ satisfying*

$$\mathsf{Ric}_{\mathsf{M}} + \mathsf{Hess}_V - \frac{\nabla V \otimes \nabla V}{N-n} \geq 0 \qquad \textit{on } \mathsf{M} \setminus \partial\mathsf{M}. \tag{1}$$

If $u \in C^\infty(\mathsf{M})$ is a positive solution of the weighted minimal surface equation on $\mathsf{M} \setminus \partial\mathsf{M}$ whose gradient vanishes on $\partial\mathsf{M}$, then u is constant.

The previous result is new (to the best of our knowledge) both in the boundaryless weighted setting and in the framework of unweighted manifolds with boundary.

A second consequence of Theorem 1 is that the oscillation of minimal graphs in an appropriate class of pointed manifolds grows with a uniform rate as one moves away from the base point in each manifold. This is stated precisely in Theorem 3 below. Given a pointed metric space $(\mathsf{M},\mathsf{d},x)$, $r > 0$ and $f : B_r(x) \to \mathbb{R}$, we define

$$\mathsf{Osc}_{x,r}(f) := \sup\{|f(y) - f(x)| : y \in B_r(x)\}.$$

Theorem 3. *Let $n \in \mathbb{N}$ be fixed. For every $T,t,r > 0$ there exists $R > 0$ such that if (M^n,g,x) is a pointed manifold with non-negative Ricci curvature and $u \in C^\infty(B_R(x))$ is a solution of the minimal surface equation such that $\mathsf{Osc}_{x,r}(u) \geq t$, then $\mathsf{Osc}_{x,R}(u) \geq T$.*

In Section 5 we actually prove a more general result involving weighted manifolds with boundary and a stronger notion of oscillation (see Corollary 5.13) but we preferred to state Theorem 3 in this form for simplicity. We remark that while Theorem 3 follows combining the Harnack inequality for minimal graphs given in [21] with a compactness argument where an RCD space arises as limit of manifolds, the stronger version given by Corollary 5.13 truly relies on Theorem 1 (and the same compactness argument).

We now turn our attention to the proof of Theorem 1. To this aim let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space and let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ be a solution of the minimal surface equation. The proof that we give follows the one in [21] to prove the analogous result for manifolds supporting a doubling volume measure and a Poincaré inequality. In that work Ding shows first that Sobolev and Poincaré inequalities hold on the graph of u (see also [22]) and then uses a Moser-type iteration argument, which relies on the fact that u is harmonic on its graph, to obtain the Harnack inequality for u and the Bernstein Property.

The first obstacle for adapting the previously outlined strategy is that the graph of u in our setting has very little structure, and it is not clear what it means for u to be harmonic on its graph. A key result in order to give the graph of u a metric measure space structure is Theorem 4 (which holds for $\mathsf{RCD}(K,N)$ spaces). We denote by $\mathsf{Epi}(u)$ the epigraph of u.

Theorem 4. *Let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$, then the following conditions are equivalent.*

1. *$\mathsf{Epi}(u)$ is locally perimeter minimizing in $\mathsf{X} \times \mathbb{R}$.*
2. *$\mathsf{Epi}(u)$ is perimeter minimizing in $\mathsf{X} \times \mathbb{R}$.*
3. *u solves the minimal surface equation on X.*

Remark. Thanks to Theorem 4, if we assume that the spaces in question satisfy appropriate parabolicity constraints, then Theorems 1, 2 and 3 follow from the fact that parabolic $\mathsf{RCD}(0,N)$ spaces have the Half Space Property (see [20]). We stress that the proofs of these theorems in our setting (i.e. without any parabolicity assumption) use completely different techniques from the ones of [20].

Thanks to the previous theorem, if $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ is a solution of the minimal surface equation, we can consider the closed representative of $\mathsf{Epi}(u)$ (which exists since this is a perimeter minimizing set), and we can define a complete separable metric measure space which plays the role of the graph of u. More precisely, we consider $(\mathsf{G}(u), \mathsf{d}_g, \mathfrak{m}_g)$, where $\mathsf{G}(u) := \partial\mathsf{Epi}(u)$, d_g is the restriction of the product distance in $\mathsf{X} \times \mathbb{R}$ to $\mathsf{G}(u)$ and $\mathfrak{m}_g$ is the restriction of the perimeter measure of $\mathsf{Epi}(u)$ to $\mathsf{G}(u)$. We also denote by $u_g : \mathsf{G}(u) \to \mathbb{R}$ the height function on $\mathsf{G}(u)$. With this definition of graph space, the Sobolev and Poincaré inequalities for u_g on $\mathsf{G}(u)$ follow mimicking the proofs in [21] (with the due modifications), so that the problem reduces to proving that u_g is harmonic on $\mathsf{G}(u)$ in a generalized sense that allows to repeat the iteration argument in [21].

The key results in this sense are given by Theorems 5 and 6 (which hold for $\mathsf{RCD}(K,N)$ spaces). Given $f : \mathsf{G}(u) \to \mathbb{R}$ and $x \in \mathsf{G}(u)$, we denote by $\mathsf{lip}_g(f)(x)$ the local Lipschitz constant of f at x. At every point of $\mathsf{G}(u)$ where it makes sense, given a second function $g : \mathsf{G}(u) \to \mathbb{R}$, we define

$$\mathsf{lip}_g(f) \cdot \mathsf{lip}_g(g) := \frac{1}{4}(\mathsf{lip}_g(f+g)^2 - \mathsf{lip}_g(f-g)^2).$$

One then shows that there exists a sufficiently large family $\mathcal{A}^g$ of functions on $\mathsf{G}(u)$ (and its compactly supported version $\mathcal{A}^g_c$) where the previously defined product behaves according to the usual rules of products of gradients. This is the content of Theorem 5.

Theorem 5. *Let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ be a solution of the minimal surface equation. The function $\cdot : \mathcal{A}^g \times \mathcal{A}^g \to \mathsf{L}^1_{loc}(\mathsf{G}(u))$ given by*

$$(\phi_1, \phi_2) \mapsto \mathsf{lip}_g(\phi_1) \cdot \mathsf{lip}_g(\phi_2)$$

is symmetric, bilinear, it satisfies the chain rule and the Leibniz rule in both entries and

$$\mathsf{lip}_g(\phi_1) \cdot \mathsf{lip}_g(\phi_2) \leq \mathsf{lip}_g(\phi_1)\mathsf{lip}_g(\phi_2).$$

Notably, the class $\mathcal{A}^g_c$ contains the cut off functions on $\mathsf{G}(u)$ that are needed to repeat the iteration argument given in [21]. Theorem 6 is then the analogue in our setting of the fact that in the smooth category u_g would be harmonic on its graph. This matches the usual definition of harmonicity in distributional sense if u is assumed to be locally Lipschitz, as the Corollary of Theorem 6 shows.

Theorem 6. *Let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ be a solution of the minimal surface equation. If $\phi \in \mathcal{A}^g_c$, then $\mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g) \in \mathsf{L}^1(\mathsf{G}(u))$ and*

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g = 0.$$

Corollary. *Let $u \in \mathsf{Lip}_{loc}(\mathsf{X})$ be a solution of the minimal surface equation, then for every $\phi \in \mathsf{Lip}_c(\mathsf{G}(u))$ we have*

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g = 0.$$

Finally, we outline the main ideas in the proofs of Theorems 5 and 6. We will assume for simplicity that u is locally Lipschitz on X and we will denote by ∇ the relaxed gradient on X (for Lipschitz functions this coincides with the local Lipschitz constant lip thanks to [17]). Both theorems can be easily obtained if one can show that given $\phi \in \mathsf{Lip}(\mathsf{X})$ and the projection on the graph $i : \mathsf{X} \to \mathsf{G}(u)$, then the Local Lipschitz constant of $\phi \circ i^{-1}$ (i.e. ϕ seen as a function on the graph $\mathsf{G}(u)$) satisfies

$$\mathsf{lip}_g(\phi \circ i^{-1})^2 \circ i = |\nabla \phi|^2 - \frac{(\nabla \phi \cdot \nabla u)^2}{1 + |\nabla u|^2} \quad \mathfrak{m}\text{-a.e. on } \mathsf{X}.$$

To obtain such an identity (which is what we would get on a Riemannian manifold) we use a technical blow-up argument so that we reduce the problem to the Euclidean case.

To outline such argument we first need to recall a blow-up property of Lipschitz functions on metric measure spaces due to Cheeger (see [17]). Let $(\mathsf{Y}, \mathsf{d}_y, \mathfrak{m}_y)$ be a space with a locally doubling measure which supports a local Poincaré inequality (i.e. a PI space) such that for $\mathfrak{m}$-a.e. $y \in \mathsf{Y}$ the blow-up of Y at y is a Euclidean space $\mathbb{R}^k$. Given a point $y \in \mathsf{Y}$ of the previous type, we denote by $\psi_n : B_1^{\mathbb{R}^k}(0) \to (B^{\mathsf{Y}}_{1/n}(x), n\mathsf{d})$ the Gromov Hausdorff maps realizing the blow-up (see Definition 2.13). Thanks to [17] we have that for every $\theta \in \mathsf{Lip}(\mathsf{Y})$, for $\mathfrak{m}$-a.e. $y \in \mathsf{Y}$ there exists a linear function $\theta^\infty : \mathbb{R}^k \to \mathbb{R}$, called the blow-up of θ at y, such that

$$\|n(\theta \circ \psi_n - \theta(y)) - \theta^\infty\|_{\mathsf{L}^\infty(B_1^{\mathbb{R}^k}(0))} \to 0.$$

Moreover, we have that the relaxed gradient $\nabla_y \theta(y)$ of θ in y (which coincides with the local Lipschitz constant $\mathsf{lip}(\theta)(y)$) and the local Lipschitz constant $\mathsf{lip}(\theta^\infty)(0)$ of θ^∞ in 0 coincide.

In particular, this blow-up property of Lipschitz functions holds for X and $\mathsf{G}(u)$ as these are both PI spaces (since u is now assumed to be Lipschitz). So we fix $\phi \in \mathsf{Lip}(\mathsf{X})$ and we pick a point $x \in \mathsf{X}$ where the blow-up of X is realized by the Gromov Hausdorff maps $\psi_n : B_1^{\mathbb{R}^k}(0) \to (B^{\mathsf{X}}_{1/n}(x), n\mathsf{d})$ and the functions u and ϕ admit blow-ups u^∞ and ϕ^∞ respectively. Let then $j : \mathbb{R}^k \to \mathsf{Graph}(u^\infty)$ be the projection on the graph. It turns out that the maps

$$\psi'_n := i \circ \psi_n \circ j^{-1} : j(B_1^{\mathbb{R}^k}(0)) \to (i(B^{\mathsf{X}}_{1/n}(x)), n\mathsf{d}_g)$$

are Gromov Hausdorff maps realizing the blow-up of $\mathsf{G}(u)$ at $(x, u(x))$. In particular this implies that the blow-up of $\phi \circ i^{-1}$ on $\mathsf{G}(u)$ is $\phi^\infty \circ j^{-1}$. From this and the fact that blow-ups of functions preserve the Lipschitz constant, we obtain that

$$\mathsf{lip}_g(\phi \circ i^{-1})^2(i(x)) = \mathsf{lip}(\phi^\infty \circ j^{-1})^2 = \mathsf{lip}(\phi^\infty)^2 - \frac{(\mathsf{lip}(\phi^\infty) \cdot \mathsf{lip}(u^\infty))^2}{1 + \mathsf{lip}(u^\infty)^2} = \Big(|\nabla \phi|^2 - \frac{(\nabla \phi \cdot \nabla u)^2}{1 + |\nabla u|^2}\Big)(x).$$

If u is not Lipschitz, the above strategy needs to be modified to ensure both existence of a blow-up of u on points in X and existence of blow-ups of functions in $\mathsf{Lip}(\mathsf{G}(u))$ on points in $\mathsf{G}(u)$. Note in addition that in both cases we need the Lipschitz constant of the blow-up to coincide with the local Lipschitz constant of the initial function. The existence of blow-ups for u is addressed by using a geometric property of perimeter minimizers, i.e. the fact that these admit tangent balls to their boundary (see [42] for the proof of this property in the non-collapsed case).

To deal with blow-ups of functions on $\mathsf{G}(u)$ we consider the notion of "strong blow-up" (see Definition 4.2). The Lipschitz constant of a strong blow-up trivially coincides with the local Lipschitz constant of the initial function but it might be harder to prove that such a blow-up exists. For this reason we introduce the class $\mathcal{A}^g$ (appearing in the statements of Theorems 5 and 6) of functions admitting strong blow-ups on sufficiently many points and we conclude by showing that this class is large enough to deal with the iteration argument of the proof of the Bernstein Property later on.

The note is organized as follows: Section 2 contains preliminaries, Section 3 contains the proof of Theorem 4, Section 4 contains the proofs of Theorems 5 and 6, while the final section contains the proofs of Theorems 1, 2 and 3.

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2 Preliminaries

Throughout the note will work on metric measure spaces $(\mathsf{X},\mathsf{d},\mathfrak{m})$, where (X,d) is a separable complete metric space where balls are precompact and $\mathfrak{m}$ is a non-negative Borel measure on X which is finite on bounded sets and whose support is the whole X. Given an open set $\Omega\subset\mathsf{X}$ we denote by $\mathsf{Lip}(\Omega)$, $\mathsf{Lip}_{loc}(\Omega)$ and $\mathsf{Lip}_c(\Omega)$ respectively Lipschitz functions, locally Lipschitz and Lipschitz functions with compact support in Ω. If $f\in\mathsf{Lip}_{loc}(\Omega)$ and $x\in\Omega$ we define

$$\mathsf{lip}(f)(x):=\limsup_{y\to x}\frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}\quad\text{and}\quad \mathsf{L}(f):=\sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}.$$

Given a closed interval $I\subset\mathbb{R}$, we say that a rectifiable curve $\gamma:I\to\mathsf{X}$ is a geodesic if its length coincides with the distance between its endpoints. Unless otherwise specified we assume that geodesics have constant unit speed. Throughout the note d_e will be the Euclidean distance in any dimension. In many proofs we will use the notation c_1, c_2, etc. for constants that are independent of the other quantities appearing in the statement that we are proving.

2.1 Sobolev spaces and Gromov Hausdorff convergence

We now recall some basic notions about Sobolev spaces in the setting of metric measure spaces, the main references being [17], [5], [6] and [26].

Definition 2.1. Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, $\Omega\subset\mathsf{X}$ an open set and let $p>1$. A function $f\in\mathsf{L}^p(\Omega)$ is said to be in the Sobolev space $\mathsf{W}^{1,p}(\Omega)$ if there exists a sequence of locally Lipschitz functions $\{f_i\}_{i\in\mathbb{N}}\subset\mathsf{Lip}_{loc}(\Omega)$ converging to f in $\mathsf{L}^p(\Omega)$ such that

$$\limsup_{i\to+\infty}\int_\Omega \mathsf{lip}(f_i)^p\,d\mathfrak{m}<+\infty.$$

A function $f\in\mathsf{L}^p_{loc}(\Omega)$ is said to be in the Sobolev space $\mathsf{W}^{1,p}_{loc}(\Omega)$ if for every $\eta\in\mathsf{Lip}_c(\Omega)$ we have $f\eta\in\mathsf{W}^{1,p}(\Omega)$.

For any $f\in\mathsf{W}^{1,p}(\Omega)$ one can define an object $|\nabla f|$ (a priori depending on p, but independent of the exponent in the spaces that we will work on) such that for every open set $A\subset\Omega$ we have

$$\int_A|\nabla f|^p\,d\mathfrak{m}=\inf\Big\{\liminf_{n\to+\infty}\int_A \mathsf{lip}(f_n)^p\,d\mathfrak{m}\Big|(f_n)_n\subset\mathsf{L}^p(A)\cap\mathsf{Lip}_{loc}(A),\|f_n-f\|_{\mathsf{L}^p(A)}\to 0\Big\}.$$

The quantity in the previous expression will be called p-Cheeger energy and denoted by $\mathsf{Ch}_p(f)$ while $|\nabla f|$ will be called relaxed gradient. Later we will often write Ch in place of Ch_2 for simplicity of notation. We define $\|f\|_{\mathsf{W}^{1,p}(\Omega)}:=\|f\|_{\mathsf{L}^p(\Omega)}+\mathsf{Ch}_p(f)$. One can check that with this norm the space $\mathsf{W}^{1,p}(\Omega)$ is Banach. We now introduce functions of bounded variation following [39] (see also [3]).

Definition 2.2. Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $\Omega\subset\mathsf{X}$ an open set. A function $f\in\mathsf{L}^1(\Omega)$ is said to be of bounded variation if there exists a sequence of of locally Lipschitz functions $\{f_i\}_{i\in\mathbb{N}}\subset\mathsf{Lip}_{loc}(\Omega)$ converging to f in $\mathsf{L}^1(\Omega)$ such that

$$\limsup_{i\to+\infty}\int_\Omega \mathsf{lip}(f_i)\,d\mathfrak{m}<+\infty.$$

The space of such functions is denoted $\mathsf{BV}(\Omega)$. A function $f\in\mathsf{L}^1_{loc}(\Omega)$ is said to be in $\mathsf{BV}_{loc}(\Omega)$ if for every $\eta\in\mathsf{Lip}_c(\Omega)$ we have $f\eta\in\mathsf{BV}(\Omega)$.

For any $f\in\mathsf{BV}(\Omega)$ and any open set $A\subset\Omega$ we define

$$|Df|(A)=\inf\Big\{\liminf_{n\to+\infty}\int_A \mathsf{lip}(f_n)\,d\mathfrak{m}\Big|(f_n)_n\subset\mathsf{L}^1(A)\cap\mathsf{Lip}_{loc}(A),\|f_n-f\|_{\mathsf{L}^1(A)}\to 0\Big\}.$$

One can check that the quantity in the previous expression is the restriction to the open subsets of Ω of a finite measure. We define $\|f\|_{\mathsf{BV}(\Omega)}:=\|f\|_{\mathsf{L}^1(\Omega)}+|Df|(\Omega)$. One can check that with this norm the space $\mathsf{BV}(\mathsf{X})$ is Banach. A function f belongs to $\mathsf{W}^{1,1}(\Omega)$ if $f\in\mathsf{BV}(\Omega)$ and $|Df|\ll\mathfrak{m}$. In this case we denote by $|\nabla f|$ the density of $|Df|$ with respect to $\mathfrak{m}$.

Definition 2.3. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $\Omega \subset \mathsf{X}$ be an open set. For every $p \geq 1$ we denote by $\mathsf{W}_0^{1,p}(\Omega)$ the closure in $\mathsf{W}^{1,p}(\Omega)$ of $\mathsf{Lip}_c(\Omega)$.

Definition 2.4. A metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if the space $\mathsf{W}^{1,2}(\mathsf{X})$ is a Hilbert space.

If $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is Infinitesimally Hilbertian and $\Omega \subset \mathsf{X}$ an open set, for every $f, g \in \mathsf{W}^{1,1}_{loc}(\Omega)$ we define the measurable function $\nabla f \cdot \nabla g : \Omega \to \mathbb{R}$ by

$$\nabla f \cdot \nabla g := \frac{|\nabla(f+g)|^2 - |\nabla(f-g)|^2}{4}.$$

As a consequence of the infinitesimal Hilbertianity assumption, the previously defined product of gradients is bilinear in both entries. We then define the Laplacian in the metric setting.

Definition 2.5. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and let $\Omega \subset \mathsf{X}$ be an open set. Let $f \in \mathsf{W}^{1,2}(\Omega)$. We say that $f \in D(\Delta, \Omega)$ if there exists a function $h \in \mathsf{L}^2(\Omega)$ such that

$$\int_\Omega gh \, d\mathfrak{m} = -\int_\Omega \nabla g \cdot \nabla f \, d\mathfrak{m} \quad \text{for any } g \in \mathsf{W}_0^{1,2}(\Omega).$$

In this case we say that $\Delta f = h$ in Ω.

We also have the following more general definition.

Definition 2.6. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and let $\Omega \subset \mathsf{X}$ be an open set. Let $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$ and let μ be a Radon measure on Ω. We say that $\Delta f = \mu$ in Ω in distributional sense if

$$\int_\Omega h \, d\mu = -\int_\Omega \nabla g \cdot \nabla f \, d\mathfrak{m} \quad \text{for any } g \in \mathsf{Lip}_c(\Omega).$$

Similarly we define what it means to be a solution of the minimal surface equation.

Definition 2.7. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian and let $\Omega \subset \mathsf{X}$ be an open set. We say that $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$ solves the minimal surface equation on Ω if for every $\phi \in \mathsf{Lip}_c(\Omega)$ we have

$$\int_\Omega \frac{\nabla f \cdot \nabla \phi}{\sqrt{1 + |\nabla f|^2}} \, d\mathfrak{m} = 0.$$

We now recall the main definitions concerning Gromov Hausdorff convergence, referring to [49] and [28] for an overview on the subject.

Definition 2.8. Let $(\mathsf{X}, \mathsf{d}_x, x)$ and $(\mathsf{Y}, \mathsf{d}_y, y)$ be pointed metric spaces and let $\delta > 0$. we say that a map $f : \mathsf{X} \to \mathsf{Y}$ is a δ-GH map if

- $f(x) = y$.
- $\sup_{a,b \in \mathsf{X}} \Big| \mathsf{d}_x(a, b) - \mathsf{d}_y(f(a), f(b)) \Big| \leq \delta$.
- The image of f is a δ-net in Y.

Lemma 2.9. *Let $(\mathsf{Y}, \mathsf{d}, y)$ be a metric space and let $A \subset \mathbb{R}^k$ be either an open set or a closed non-trivial ball. Fix $\delta > 0$ and let $f : (A, \mathsf{d}_e, x) \subset \mathbb{R}^k \to \mathsf{Y}$ be a δ-GH map. Let $\{y_i\}_{i=1}^m \subset \mathsf{Y}$. There exists a 4δ-GH map $g : A \subset \mathbb{R}^k \to \mathsf{Y}$ such that $g \neq f$ on at most m points and $\{y_i\}_{i=1}^m \subset \mathsf{Im}(g)$.*

Proof. For every i let y_i' be in $f(A)$ and δ-close to y_i. Let x_i be an element such that $f(x_i) = y_i'$ and if some of them coincide, replace them with sufficiently close points, in such a way that $f(x_i)$ is 2δ-close to y_i and the points $\{x_i\}_{i=1}^m$ are all distinct. Consider the map

$$g(a) := \begin{cases} f(a) & a \neq x_n \\ y_n & a = x_n. \end{cases}$$

It is easy to check that g has the desired properties. $\square$

Definition 2.10. We say that a sequence of pointed metric spaces $(\mathsf{X}_n, \mathsf{d}_n, x_n)$ converges to $(\mathsf{X}, \mathsf{d}, x)$ in pointed Gromov Hausdorff convergence if for every $\delta, R > 0$ there exists N such that for every $n \geq N$ there exists a δ-*GH* map $f_n^{\epsilon,R} : \bar{B}_R(x_n) \to \bar{B}_R(x)$.

Definition 2.11. Let (X, d) be a metric space and $x \in \mathsf{X}$. We denote by $\mathsf{Tan}_x(\mathsf{X})$ the (possibly empty) collection of (isometry classes of) metric spaces that are pointed Gromov Hausdorff limits as $r \downarrow 0$ of the family $(\mathsf{X}, r^{-1}\mathsf{d}, x)$.

Definition 2.12. We say that a sequence of pointed metric measure spaces $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converges to $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ in pointed measured Gromov Hausdorff convergence if it converges in pointed Gromov Hausdorff sense and the maps $f_n^{\epsilon,R} : \bar{B}_R(x_n) \to \bar{B}_R(x)$ given by Definition 2.10 satisfy $(f_n^{\epsilon,R})_\#(\mathfrak{m}_n \llcorner \bar{B}_R(x_n)) \to \mathfrak{m} \llcorner \bar{B}_R(x)$ weakly in duality with continuous boundedly supported functions on X.

Definition 2.13. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space such that for $\mathfrak{m}$-a.e. $x \in \mathsf{X}$ we have $\mathsf{Tan}_x(\mathsf{X}) = \{(\mathbb{R}^k, d_e)\}$. Given a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ decreasing to zero we say that (ϵ_n, ψ_n) is a blow-up of X at x if there exists a sequence δ_n decreasing to zero such that $\psi_n : \bar{B}_1(0) \subset \mathbb{R}^k \to (\bar{B}_{\epsilon_n}(x), \epsilon_n^{-1}\mathsf{d})$ is a δ_n-*GH* map.

We recall that in the case of a sequence of uniformly locally doubling metric measure spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathsf{x}_i)$ (as in the case of $\mathsf{RCD}(K, N)$ spaces), pointed measured Gromov-Hausdorff convergence to $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathsf{x})$ can be equivalently characterized asking for the existence of a proper metric space $(\mathsf{Z}, \mathsf{d}_z)$ such that all the metric spaces $(\mathsf{X}_i, \mathsf{d}_i)$ are isometrically embedded into $(\mathsf{Z}, \mathsf{d}_z)$, $\mathsf{x}_i \to \mathsf{x}$ and $\mathfrak{m}_i \to \mathfrak{m}$ weakly in duality with continuous boundedly supported functions in Z (see [28]).

2.2 General properties of $\mathsf{RCD}(K, N)$ spaces

We now recall some properties of $\mathsf{RCD}(K, N)$ spaces, i.e. infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, +\infty)$ in synthetic sense.

The Riemannian Curvature Dimension condition $\mathsf{RCD}(K, \infty)$ was introduced in [6] (see also [26, 9]) coupling the Curvature Dimension condition $\mathsf{CD}(K, \infty)$, previously pioneered in [47, 48] and independently in [38], with the infinitesimal Hilbertianity assumption. The finite dimensional counterpart $\mathsf{RCD}(K, N)$ is then obtained coupling the finite dimensional Curvature Dimension condition $\mathsf{CD}(K, N)$ with the infinitesimal Hilbertianity assumption and was proposed in [26]. For a complete introduction to the topic we refer to the survey [1] and the references therein. Let us mention that in the literature one can find also the (a priori weaker) $\mathsf{RCD}^*(K, N)$. It was proved in [23, 7], that $\mathsf{RCD}^*(K, N)$ is equivalent to the dimensional Bochner inequality. Moreover, [16] (see also [37]) proved that $\mathsf{RCD}^*(K, N)$ and $\mathsf{RCD}(K, N)$ coincide. We now recall the properties that we will use later on in the note.

The $\mathsf{RCD}(K, N)$ condition implies that the measure is locally doubling (see [47]) and the validity of a Poincaré inequality (see [45]). In particular if f is a locally Lipschitz function on a $\mathsf{RCD}(K, N)$ space, its relaxed gradient coincides with its local Lipschitz constant $\mathsf{lip}(f)$ (see [34, Theorem 12.5.1] after [17]). The aforementioned properties are recalled in the next two propositions.

Proposition 2.14. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space. For every* $R > 0$ *there exists* $C(R) > 0$ *such that for every* $x \in \mathsf{X}$ *and* $0 < r < R$ *we have*

$$\mathfrak{m}(B_{2r}(x)) \leq C(R)\mathfrak{m}(B_r(x)).$$

For $\mathsf{RCD}(0, N)$ *spaces the function* C *can be taken to be the constant function with value* 2^N.

Proposition 2.15. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space, then for every* $f \in \mathsf{W}^{1,2}(\mathsf{X})$, $x \in \mathsf{X}$ *and* $r > 0$ *the following Poincaré inequality holds:*

$$\int_{B_r(x)} |f - f_{x,r}| \, d\mathfrak{m} \leq 4re^{|K|r^2} \int_{B_{2r}(x)} |\nabla f| \, d\mathfrak{m}.$$

The next theorem can be found in [24, Theorem 6.12]. The proof in the $\mathsf{RCD}(K, N)$ setting is analogous.

Theorem 2.16. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space and let* $f \in \mathsf{BV}(\mathsf{X})$. *Then for every* ϵ *there exists a Lipschitz function* f_ϵ *such that* $\mathfrak{m}(\{f \neq f_\epsilon\}) \leq \epsilon$.

The content of the next theorem can be found at the very end of [27].

Theorem 2.17. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space and* $p \in [1, +\infty)$. *Then Lipschitz functions are dense in* $\mathsf{W}^{1,p}(\mathsf{X})$.

The next theorem follows from [41, 15].

Theorem 2.18. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space. There exists* $k \in \mathbb{N} \cap [1, N]$*, called essential dimension of* X*, such that for* $\mathfrak{m}$*-a.e.* $x \in \mathsf{X}$ *we have* $\mathsf{Tan}_x(\mathsf{X}) = \{(\mathbb{R}^k, \mathsf{d}_e)\}$*. Any such point will be called a regular point for* X*.*

A compactness argument due to Gromov and the stability of the $\mathsf{RCD}(K, N)$ condition under Gromov-Hausdorff convergence (see [48] and [28]) give the following result.

Theorem 2.19. *Let* $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ *be a a sequence of pointed normalized* $\mathsf{RCD}(K, N)$ *spaces. Then, modulo passing to a subsequence, they converge in pointed measured Gromov Hausdorff sense to an* $\mathsf{RCD}(K, N)$ *space.*

We now consider a Sobolev type inequality when $K = 0$. Given a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, a function $f : \mathsf{X} \to \mathbb{R}$, $p \in (0, +\infty)$ and $B \subset \mathsf{X}$ Borel we define

$$\|f\|_{p,B} := \Big(\fint_B |f|^p \, d\mathfrak{m} \Big)^{1/p}.$$

The proof of the next two theorems can be adapted from the one in the smooth case found in [44, Theorem 7.1.15, Theorem 7.1.13] (see also [31]).

Theorem 2.20. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0, N)$ *space and let* $f \in \mathsf{Lip}(\mathsf{X})$*, then there exists* $C(N) > 0$ *such that for every* $t > 0$

$$t^{\frac{N}{N-1}} \mathfrak{m}(\{|u - u_{B_r(x)}| > t\} \le C r^{\frac{N}{N-1}} \mathfrak{m}(B_r(x)) \|\mathsf{lip}(f)\|^{\frac{N}{N-1}}_{1,B_r(x)}.$$

Theorem 2.21. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0, N)$ *space and let* $f \in \mathsf{Lip}(\mathsf{X})$*, then there exists* $C(N) > 0$ *such that for every* $\nu \in [1, \frac{N}{N-1}]$*,* $x \in \mathsf{X}$ *and* $r > 0$

$$\|f\|_{\nu,B_r(x)} \le C r \|\mathsf{lip}(f)\|_{1,B_r(x)}.$$

We now recall some properties of the heat flow in the RCD setting, referring to [9, 6] for the proofs of these results. Given an $\mathsf{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, the heat flow $P_t : \mathsf{L}^2(\mathsf{X}) \to \mathsf{L}^2(\mathsf{X})$ is the $\mathsf{L}^2(\mathsf{X})$-gradient flow of the Cheeger energy Ch. It turns out that one can obtain a stochastically complete heat kernel $p_t : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$, so that the definition of $P_t(f)$ can be then extended to L^∞ functions by setting

$$P_t(f)(x) := \int_{\mathsf{X}} f(y) p_t(x, y) \, d\mathfrak{m}(y).$$

The heat flow has good approximation properties, in particular if $f \in \mathsf{W}^{1,2}(\mathsf{X})$, then $P_t(f) \to f$ in $\mathsf{W}^{1,2}(\mathsf{X})$; while if $f \in \mathsf{L}^\infty(\mathsf{X})$, then $P_t f \in \mathsf{Lip}(\mathsf{X})$ for every $t > 0$.

The next proposition follows combining the contractivity estimates for the heat flow of [27] with a standard lower semicontinuity argument.

Proposition 2.22. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space, let* $\Omega \subset \mathsf{X}$ *be an open set and let* $f \in \mathsf{BV}(\mathsf{X})$*. If* $|Df|(\partial\Omega) = 0$*, then*

$$\lim_{t \to 0} |DP_t(f)|(\Omega) = |Df|(\Omega).$$

2.3 Sets of finite perimeter and minimal sets

For the results of this section, unless otherwise specified, we will implicitly assume that we are working on a fixed $\mathsf{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

Definition 2.23. Let $E \subset \mathsf{X}$. We say that E has locally finite perimeter if $1_E \in \mathsf{BV}_{loc}(\mathsf{X})$. For every Borel subset $B \subset \mathsf{X}$ We denote $|D1_E|(B)$ by $P(E, B)$.

When considering the perimeter as a measure under an integral sign we will use the notation $\mathsf{Per}(E, \cdot)$ instead of $P(E, \cdot)$.

Definition 2.24. Let $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathsf{x}_i)$ be a sequence of $\mathsf{RCD}(K,N)$ spaces converging in pmGH sense to $(\mathsf{Y}, \mathsf{d}, \mathfrak{m}, \mathsf{y})$. We say that the Borel sets $E_i \subset \mathsf{X}_i$ of finite measure converge in L^1 sense to a set $E \subset \mathsf{Y}$ of finite measure if $\mathfrak{m}_i(E_i) \to \mathfrak{m}(E)$ and $1_{E_i}\mathfrak{m}_i \to 1_F\mathfrak{m}$ weakly in duality w.r.t. continuous compactly supported functions in the space $(\mathsf{Z}, \mathsf{d}_z)$ realizing the pmGH convergence.

We say that the Borel sets $E_i \subset \mathsf{X}_i$ converge in L^1_{loc} sense to a set $E \subset \mathsf{Y}$ if $E_i \cap B_r(x_i) \to E \cap B_r(y)$ in L^1 sense for every $r > 0$.

The next proposition is taken from [2, Corollary 3.4].

Proposition 2.25. *Let $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathsf{x}_i)$ be a sequence of $\mathsf{RCD}(K,N)$ spaces converging in pmGH sense to $(\mathsf{Y}, \mathsf{d}, \mathfrak{m}, \mathsf{y})$. Let $E_i \subset \mathsf{X}_i$ be Borel sets such that*

$$\sup_{i\in\mathbb{N}} P(E_i, B_r(x_i)) < +\infty \quad \textit{for every } r > 0.$$

Then there exists a (non relabeled) subsequence and a Borel set $F \subset \mathsf{Y}$ such that $E_i \to F$ in L^1_{loc}.

Definition 2.26. Let $E \subset \mathsf{X}$ be a set of finite perimeter. We say that a regular point $x \in \mathsf{X}$ such that $x \in \partial E$ is in the reduced boundary $\mathcal{F}E$ of E if for every sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ decreasing to zero, the sets $E_i := E$ in the rescaled spaces $(\mathsf{X}, \epsilon_i^{-1}\mathsf{d}, \mathfrak{m}(B_{\epsilon_i}(x))^{-1}\mathfrak{m}, x)$ converge in L^1_{loc} sense to a half space in $\mathbb{R}^k$.

The next proposition is taken from [14, Corollary 3.15].

Proposition 2.27. *Let $E \subset \mathsf{X}$ be a set of finite perimeter. Then the perimeter measure is concentrated on $\mathcal{F}E$.*

The next proposition is taken from [13, Theorem 5.2 and Proposition 6.1]. We first introduce some notation. Let $A \subset\subset \mathsf{X}$ be a set of finite perimeter and let $g \in \mathsf{Lip}_{loc}(\mathsf{X})$. We denote by $A^{(1)}$ the set of points where A has density 1. Assume that g has distributional Laplacian which is a finite Radon measure. Then there exists measures $\mu_1, \mu_2 \ll |D1_A|$ such that as $t \to 0$ we have

$$1_A \nabla P_t(1_A) \cdot \nabla g \to \mu_1 \quad \text{and} \quad 1_{^cA} \nabla P_t(1_A) \cdot \nabla g \to \mu_2$$

in weak sense testing against functions in $\mathsf{Lip}_c(\mathsf{X})$. We denote the density of μ_1 and μ_2 w.r.t. $|D1_A|$ respectively by

$$(\nabla g \cdot \nu_E)_{int} \quad \text{and} \quad (\nabla g \cdot \nu_E)_{ext}.$$

Proposition 2.28. *Let $A \subset\subset \mathsf{X}$ be a set of finite perimeter and let $g \in \mathsf{Lip}_{loc}(\mathsf{X})$. Assume that g has distributional Laplacian which is a finite Radon measure. Then for any $f \in \mathsf{Lip}_c(\mathsf{X})$ we have*

$$\int_{A^{(1)}} f\, d\Delta g + \int_A \nabla f \cdot \nabla g\, d\mathfrak{m} = -\int_{\mathcal{F}A} f(\nabla g \cdot \nu_E)_{int}\, d\mathsf{Per}$$

and

$$\int_{A^{(1)}\cup\mathcal{F}A} f\, d\Delta g + \int_A \nabla f \cdot \nabla g\, d\mathfrak{m} = -\int_{\mathcal{F}A} f(\nabla g \cdot \nu_E)_{ext}\, d\mathsf{Per}.$$

Proposition 2.29. *Let $\Omega \subset\subset \Omega' \subset \mathsf{X}$ be open domains. Let $\phi : \Omega' \to \mathbb{R}$ be a 1-Lipschitz function such that $|\nabla\phi| = 1$ $\mathfrak{m}$-a.e. on Ω' and ϕ has bounded Laplacian in distributional sense on Ω'. Then for λ^1-a.e. $t \in \mathbb{R}$ such that $\{\phi = t\} \cap \Omega \neq \emptyset$ the set $\{\phi < t\}$ has locally finite perimeter in Ω and*

$$(\nabla\phi \cdot \nu_{\{\phi<t\}})_{int} = -1.$$

Next we consider a variant of Theorem 2.20 concerning sets of finite perimeter in $\mathsf{RCD}(0,N)$ spaces.

Proposition 2.30. *There exists $C > 0$ such that if $E \subset \mathsf{X}$ has finite perimeter, $r > 0$ and $x \in \mathsf{X}$, then*

$$\mathfrak{m}(B_r(x))^{\frac{1}{N}} \min\{\mathfrak{m}(E \cap B_r(x)), \mathfrak{m}(B_r(x) \setminus E)\}^{\frac{N}{N-1}} \le CrP(E, B_r(x)).$$

Proof. By approximation it is easy to check that Theorem 2.20 holds also when $f \in \mathsf{BV}(\mathsf{X})$. Applying that theorem to $f = 1_E$ with $t = 1/2$ gives the desired inequality. $\square$

We now turn our attention to minimal sets.

Definition 2.31. Let $\Omega \subset \mathsf{X}$ be an open set. Let $E \subset \Omega$ be a set of locally finite perimeter. We say that E is perimeter minimizing in Ω if for every $x \in \Omega$, $r > 0$ and $F \subset \Omega$ such that $F \Delta E \subset\subset B_r(x) \cap \Omega$ we have that $P(E, B_r(x) \cap \Omega) \leq P(F, B_r(x) \cap \Omega)$. If we say that E is perimeter minimizing we implicitly mean that $\Omega = \mathsf{X}$.

Definition 2.32. Let $\Omega \subset \mathsf{X}$ be an open set. Let $E \subset \Omega$ be a set of locally finite perimeter. We say that E is locally perimeter minimizing in Ω if for every $x \in \Omega$ there exists $r > 0$ such that for every $F \subset \Omega$ such that $F \Delta E \subset\subset B_r(x) \cap \Omega$ we have $P(E, B_r(x) \cap \Omega) \leq P(F, B_r(x) \cap \Omega)$. If we say that E is locally perimeter minimizing we implicitly mean that $\Omega = \mathsf{X}$.

The next theorem comes from [36, Theorem 4.2 and Lemma 5.1].

Theorem 2.33. *There exist $C, \gamma_0 > 0$ depending only on K and N such that the following hold. If $E \subset \mathsf{X}$ is a set minimizing the perimeter in $\Omega \subset \mathsf{X}$, then, up to modifying E on an $\mathfrak{m}$-negligible set, for any $x \in \partial E$ and $r > 0$ such that $B_{2r}(x) \subset \Omega$ we have*

$$\frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} > \gamma_0, \quad \frac{\mathfrak{m}(B_r(x) \setminus E)}{\mathfrak{m}(B_r(x))} > \gamma_0$$

and

$$\frac{\mathfrak{m}(B_r(x))}{Cr} \leq P(E, B_r(x)) \leq \frac{C\mathfrak{m}(B_r(x))}{r}.$$

From the previous result one deduces that locally perimeter minimizing sets admit both a closed and an open representative, and these have the same boundary which in addition is $\mathfrak{m}$-negligible. Whenever we consider the boundary of a locally perimeter minimizing set, we will implicitly be referring to the boundary of its closed (or open) representative.

Corollary 2.34. *For every $R > 0$ there exists $\gamma > 0$ depending only on K, N and R such that the following happens. Let $E \subset \mathsf{X}$ be the closed representative of a set minimizing the perimeter in $\Omega \subset \mathsf{X}$. For every $x \in E$ and $0 < r < R$ such that $B_{2r} \subset \Omega$ we have*

$$\frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} \geq \gamma.$$

Proof. Let x and r be as in the statement. If $B_{r/2}(x) \subset E$ then by the local doubling property

$$\frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} \geq \frac{\mathfrak{m}(B_{r/2}(x))}{\mathfrak{m}(B_r(x))} \geq C(R).$$

Otherwise let $y \in B_{r/2}(x) \cap \partial E$ and note that $B_{2r}(y) \supset B_r(x) \supset B_{r/2}(y)$. Moreover by the local doubling property $\mathfrak{m}(B_{r/2}(y)) \geq C(R)^{-2}\mathfrak{m}(B_{2r}(y)) \geq C(R)^{-2}\mathfrak{m}(B_r(x))$. Putting these facts together and using Theorem 2.33 we obtain

$$\frac{\mathfrak{m}(E \cap B_r(x))}{\mathfrak{m}(B_r(x))} \geq C(R)^{-2} \frac{\mathfrak{m}(E \cap B_{r/2}(y))}{\mathfrak{m}(B_{r/2}(y))} \geq C(R)^{-2}\gamma_0.$$

□

The next proposition is taken from [42, Theorem 2.43].

Proposition 2.35. *Let $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathsf{x}_i)$ be a sequence of $\mathsf{RCD}(K, N)$ spaces converging in pmGH sense to $(\mathsf{Y}, \mathsf{d}, \mathfrak{m}, \mathsf{y})$. Let $E_i \subset \mathsf{X}_i$ be a sequence of Borel sets converging in L^1_{loc} sense to $E \subset \mathsf{Y}$. Assume that each E_i is perimeter minimizing in $B_{r_i}(\mathsf{x}_i)$ and that $r_i \uparrow +\infty$. Then E is perimeter minimizing and in the metric space realizing the convergence we have that $\partial E_i \to \partial F$ in Kuratowski sense.*

The next proposition can be found in [8, Theorem 5.1]. Given $\Omega \subset \mathsf{X}$ and $f : \Omega \to \mathbb{R}$ we denote by $\mathsf{Epi}(f)$ the set $\{(x,t) \in \Omega \times \mathbb{R} : t > f(x)\}$ and by $\mathsf{Epi}'(f)$ the set $\{(x,t) \in \Omega \times \mathbb{R} : t < f(x)\}$.

Proposition 2.36. *Let $f \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$. For every Borel set $B \subset \mathsf{X}$ we have*

$$P(\mathsf{Epi}(f), B) = \int_B \sqrt{1 + |\nabla f|^2}\, d\mathfrak{m}.$$

Given $f \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ we will use the notation $W_f := \sqrt{1+|\nabla f|^2}$.

Proposition 2.37. *Let $\Omega \subset \mathsf{X}$ be open and let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be continuous in x_0 and such that $\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$, then there exists an open set $A \subset \Omega$ containing x_0 such that for every $\phi \in \mathsf{W}^{1,1}(A) \cap \mathsf{L}^\infty(A)$ compactly supported in A we have*

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u}\, d\mathfrak{m} = 0. \tag{2}$$

Proof. Since $\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$ there exists a cilynder $B_\delta(x_0) \times (u(x_0) - \epsilon, u(x_0) + \epsilon) \subset \Omega \times \mathbb{R}$ where $\mathsf{Epi}(u)$ minimizes the perimeter. Moreover, modulo decreasing δ, we may assume that $|u - u(x_0)| < \epsilon/2$ in $B_\delta(x_0)$ by the continuity hypothesis on u. We then set $A := B_\delta(x_0)$. Note that if ϕ is as in the statement, then the graph of the function $u + t\phi$ restricted to A, for $t \in \mathbb{R}$ small enough, will be contained in the cylinder $B_\delta(x_0) \times (u(x_0) - \epsilon, u(x_0) + \epsilon)$. This implies that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := P(\mathsf{Epi}(u + t\phi), A \times \mathbb{R})$$

has a minimum in zero. By Proposition 2.36 we can write f as

$$f(t) = \int_A \sqrt{1 + |\nabla(u + t\phi)|^2}\, d\mathfrak{m},$$

so that by standard arguments involving Dominated Convergence Theorem we have that f is smooth and that

$$f'(0) = \int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u}\, d\mathfrak{m}.$$

Since f is smooth and has a minimum in zero we deduce that $f'(0) = 0$ as desired. □

Remark 2.38. The same argument of the previous proposition combined with a truncation argument shows that if $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ is bounded and such that $\mathsf{Epi}(u)$ is perimeter minimizing in $\Omega \times \mathbb{R}$, then for every compactly supported $\phi \in \mathsf{W}^{1,1}(\Omega)$ we have

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u}\, d\mathfrak{m} = 0.$$

2.4 Existence of tangent balls to perimeter minimizers

In this section we sketch the main steps to prove Theorem 2.44, which will be of crucial importance later on. In [42, Proposition 6.44] the same fact is proved assuming that the space is non collapsing, so we will only highlight the points in such proof where changes need to be made for the general case. For simplicity we will work under the assumption that $E \subset \mathsf{X}$ is a perimeter minimizer, although Theorem 2.44 is stated for local perimeter minimizers in an open set. For the results of this section we will implicitly assume that we are working on a fixed $\mathsf{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. The next result is taken from [29, Theorem 5.2].

Proposition 2.39. *Let $E \subset \mathsf{X}$ be a perimeter minimizing set and consider the distance function $\mathsf{d}_{\bar{E}} : \mathsf{X} \to \mathbb{R}$. There exists $t_{K,N} \in C^\infty((0, +\infty))$ such that in $\mathsf{X} \setminus \bar{E}$ we have $\Delta \mathsf{d}_{\bar{E}} \le t_{K,N} \circ \mathsf{d}_{\bar{E}}$ in distributional sense. If $K = 0$ we can take $t_{K,N} \equiv 0$.*

The proof of the next result follows immediately from the one in [42, Lemma 2.41] while the subsequent corollary is obtained via the Coarea Formula.

Proposition 2.40. *Let $E \subset \mathsf{X}$ be a perimeter minimizing set. There exist constants $C, r_0 > 0$ depending on K and N such that for every $t > 0$ and $r \in (0, tr_0)$ we have*

$$\mathfrak{m}(\{y \in B_t(x) : \mathsf{d}_{\bar{E}}(y) \le r\}) \le CrP(E, B_{2t}(x)).$$

Corollary 2.41. *Let $E \subset \mathsf{X}$ be a perimeter minimizing set. There exist constants $C, r_0 > 0$ depending on K and N such that the following happens. For every set of full measure $A \subset (0, r_0)$ and every sequence $\{t_k\}_{k \in \mathbb{N}} \subset (0, +\infty)$ there exists a sequence $\{r_i\}_{i \in \mathbb{N}} \subset A$ decreasing to zero such that for every $x \in \partial E$ and for every k fixed, if i is sufficiently large then*

$$P(\{\mathsf{d}_{\bar{E}}(y) > r_i\}, B_{t_k}(x)) \le CP(E, B_{2t_k}(x)).$$

The proof of the next result mimics the one of [42, Proposition 5.4] and is only sketched.

Proposition 2.42. *Let $E \subset \mathsf{X}$ be a perimeter minimizing set and consider the distance function $\mathsf{d}_{\bar{E}} : \mathsf{X} \to \mathbb{R}$. Then $\Delta \mathsf{d}_{\bar{E}}$ is a Radon measure in X and we have $\Delta \mathsf{d}_{\bar{E}} = \mathsf{Per}(E, \cdot) + \Delta \mathsf{d}_{\bar{E}} \llcorner (\mathsf{X} \setminus \bar{E})$ in distributional sense.*

Proof. We do the proof assuming that $K = 0$, the general case requiring only a slight modification. Thanks to Propositions 2.28 and 2.29 for λ^1-a.e. $r > 0$ and for every positive $\phi \in \mathsf{Lip}_c(\mathsf{X})$ we have

$$\int_{\mathsf{d}_{\bar{E}} > r} \phi \, d\Delta \mathsf{d}_{\bar{E}} + \int_{\mathsf{d}_{\bar{E}} > r} \nabla \phi \cdot \nabla \mathsf{d}_{\bar{E}} \, d\mathfrak{m} = - \int_{\mathsf{X}} \phi \, d\mathsf{Per}(\{\mathsf{d}_{\bar{E}} > r\}). \tag{3}$$

Now let $\{t_k\}$ be a sequence containing both a subsequence going to $+\infty$ and one going to zero. We consider the sequence r_i given by Corollary 2.41 and, modulo passing to a subsequence, we obtain a Radon measure μ which is the weak limit of the Radon measures $\mathsf{Per}(\{\mathsf{d}_{\bar{E}} > r_i\})$ in duality with compactly supported continuous functions (this exists using the subsequence of $\{t_k\}$ that goes to $+\infty$). We claim that $\mu \ll \mathsf{Per}(E, \cdot)$. It is clear that μ is supported on ∂E, so it is sufficient to prove that there exists a constant c such that for every $x \in \partial E$, considering the (non relabeled) subsequence of t_k that goes to zero, we have

$$\limsup_{t_k \to 0} \frac{\mu(B_{t_k}(x))}{\mathsf{Per}(E, B_{t_k}(x))} \leq c.$$

So fix t_k and note that

$$\mu(B_{t_k}(x)) \leq \liminf_{i \to +\infty} \mathsf{Per}(\{\mathsf{d}_{\bar{E}} > r_i\}, B_{t_k}(x)).$$

By Corollary 2.41 the r.h.s. is controlled by $CP(E, B_{2t_k}(x))$, which by Theorem 2.33 is in turn controlled by $C'P(E, B_{t_k}(x))$. summing up we get

$$\mu(B_{t_k}(x)) \leq C'P(E, B_{t_k}(x)),$$

as desired.

With this in mind we pass to the limit in (3) along the sequence r_i and obtain

$$\lim_{i \to +\infty} \int_{\mathsf{d}_{\bar{E}} > r_i} \phi \, d\Delta \mathsf{d}_{\bar{E}} + \int_{\mathsf{X}} \nabla \phi \cdot \nabla \mathsf{d}_{\bar{E}} \, d\mathfrak{m} = - \int_{\mathsf{X}} \phi \, d\mu.$$

The first addendum in the l.h.s. of the previous equation is a linear functional on the continuous compactly supported functions. Moreover it has negative sign because of Proposition 2.39, so it is represented by a negative Radon measure ν. This implies, by the same equation, that $\mathsf{d}_{\bar{E}}$ has measure valued Laplacian and that $\Delta \mathsf{d}_{\bar{E}} \llcorner (\mathsf{X} \setminus \bar{E}) = \nu$, $\Delta \mathsf{d}_{\bar{E}} \llcorner \partial E = \mu$ and $\Delta \mathsf{d}_{\bar{E}} \llcorner E = 0$. To conclude we only need to prove that $\mu = \mathsf{Per}(E)$. Since we know that $\mu \ll \mathsf{Per}(E)$ it is enough to show that for Per-a.e. $x \in \partial E$ we have

$$\lim_{t_k \to 0} \frac{\mu(B_{t_k}(x))}{\mathsf{Per}(E, B_{t_k}(x))} = 1.$$

This follows by the same blow-up argument of [42, Proposition 5.4]. □

Proposition 2.43. *Let $E \subset \mathsf{X}$ be a perimeter minimizing set and consider the distance function $\mathsf{d}_{\bar{E}} : \mathsf{X} \setminus \bar{E} \to \mathbb{R}$. Then for every $\phi \in \mathsf{Lip}_c(\mathsf{X})$ and λ^1-a.e. $r \in \mathbb{R}$ we have*

$$\int_{\mathsf{X}} \phi \, d\mathsf{Per}(\{\mathsf{d}_{\bar{E}} > r\}) - \int_{\mathsf{X}} \phi \, d\mathsf{Per}(E) \leq \int_{0 < \mathsf{d}_{\bar{E}} < r} \nabla \phi \cdot \nabla \mathsf{d}_{\bar{E}} \, d\mathfrak{m}.$$

Proof. For every $r > 0$ define $E_r := \{\mathsf{d}_{\bar{E}} < r\}$. Applying Proposition 2.28 with $A := E_r$ and $g = \mathsf{d}_{\bar{E}}$ (this can be done thanks to proposition 2.42) we get that for every $\phi \in \mathsf{Lip}_{loc}(\mathsf{X})$ we have

$$\begin{aligned} \int_{\partial E} \phi \, d\Delta \mathsf{d}_{\bar{E}} + \int_{E_r \setminus \bar{E}} \phi \, d\Delta \mathsf{d}_{\bar{E}} + \int_{0 < \mathsf{d}_{\bar{E}} < r} \nabla \phi \cdot \nabla \mathsf{d}_{\bar{E}} \, d\mathfrak{m} \\ = - \int_{\mathsf{X}} \phi (\nabla \mathsf{d}_{\bar{E}} \cdot \nu_{E_r})_{int} \, d\mathsf{Per}(E_r). \end{aligned}$$

Applying Propositions 2.42 and 2.29 we get that for λ^1-a.e. $r > 0$ we have

$$\int_{\mathsf{X}} \phi \, d\mathsf{Per}(E) + \int_{0<\mathsf{d}_{\bar{E}}<r} \nabla\phi \cdot \nabla \mathsf{d}_{\bar{E}} \, d\mathfrak{m} \geq \int_{\mathsf{X}} \phi \, d\mathsf{Per}(E_r).$$

□

The previous proposition replaces Lemma 6.12 in the proof of the analogue of Theorem 2.44 given in [42, Proposition 6.44]. With this replacement, the argument can be carried out in the same way.

Theorem 2.44. *Let $\Omega \subset \mathsf{X}$ be an open set and let $E \subset \Omega$ be a set locally minimizing the perimeter in Ω. For Per-a.e. $x \in \partial E \cap \Omega$ there exists balls $B_1 \subset E$ and $B_2 \subset \Omega \setminus E$ such that $\partial B_1 \cap \partial B_2 = \{x\}$. These balls are called tangent balls to E at x.*

3 Minimal surface equation and perimeter minimizers

In this section $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a fixed $\mathsf{RCD}(K, N)$ space and $\Omega \subset \mathsf{X}$ is an open set. We will denote by $\mathsf{d}_\times$ and $\mathfrak{m}_\times$ respectively the product distance and the product measure in $\mathsf{X} \times \mathbb{R}$. We recall that given $\Omega \subset \mathsf{X}$ and $f : \Omega \to \mathbb{R}$ we denote by $\mathsf{Epi}(f)$ the set $\{(x, t) \in \Omega \times \mathbb{R} : t > f(x)\}$ and by $\mathsf{Epi}'(f)$ the set $\{(x, t) \in \Omega \times \mathbb{R} : t < f(x)\}$. The goal of this section is to prove the following theorem (which coincides with Theorem 4 in the Introduction when $\Omega = \mathsf{X}$).

Theorem 3.1. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$. The following conditions are equivalent.*

1. *$\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$.*
2. *$\mathsf{Epi}(u)$ is perimeter minimizing in $\Omega \times \mathbb{R}$.*
3. *u solves the minimal surface equation on Ω.*

In Subsection 3.1 we show that 1 implies 3, while in Subsection 3.2 we will show that 3 implies 2.

3.1 Locally perimeter minimizing implies MSE

The proof of the next result is inspired from [30, Theorem 14.10]. We recall that whenever we refer to a locally perimeter minimizing set we implicitly mean its open representative. Moreover whenever we refer to the pointwise behavior of u we mean its precise representative defined by

$$u(x) := \limsup_{r \to 0} \fint_{B_r(x)} u \, d\mathfrak{m}. \tag{4}$$

Proposition 3.2. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be such that $\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$, then u is continuous in its Lebesgue points.*

Proof. Let $x \in \Omega$ be a Lebesgue point for u and suppose that $u(x) = 0$. Modulo a vertical translation we always trace back to this case. Since $\mathsf{Epi}(u)$ minimizes the perimeter locally, there exists an open set of the form $B_\epsilon(x) \times (-\epsilon, \epsilon) \subset \Omega \times \mathbb{R}$ where $\mathsf{Epi}(u)$ is perimeter minimizing. Now consider $r > 0$ such that $\epsilon > 4r$ and $y \in B_r(x)$. Suppose for now that $u(y) \geq 0$.

For every $i \in \mathbb{N}$ such that $2ir \in [0, \min\{u(y), \epsilon/4\}]$ we have that $(y, 2ir)$ is in the closure of $\mathsf{Epi}'(u)$ and $B_r(y, 2ir) \subset \mathsf{X} \times \mathbb{R}$ is contained in $B_{\epsilon/2} \times (-\epsilon/2, \epsilon/2)$. Hence, applying Corollary 2.34 in the product space $(\mathsf{X} \times \mathbb{R}, \mathsf{d}_\times, \mathfrak{m}_\times)$, we obtain that

$$\mathfrak{m}_\times(B_r(y, 2ir) \cap \mathsf{Epi}'(u)) \geq c_1 \mathfrak{m}_\times(B_r(y, 2ir)) \geq c_2 r \mathfrak{m}(B_r(y)).$$

In particular we have

$$\begin{aligned}\int_{B_{2r}(x)} |u| \, d\mathfrak{m} &\geq \sum_{i \in \mathbb{N} \cap [0, \min\{u(y)/2r, \epsilon/8r\}]} \mathfrak{m}_\times(B_r(y, 2ir) \cap \mathsf{Epi}'(u)) \\ &\geq c_3 r(\min\{u(y)/2r, \epsilon/8r\} - 2)\mathfrak{m}(B_r(y)).\end{aligned}$$

Using the doubling property we obtain from the previous chain of inequalities that

$$c_3 r + \fint_{B_{2r}(x)} |u| \, d\mathfrak{m} \geq c_5 \min\{u(y), \epsilon\}.$$

In particular, since $u(y)$ was assumed to be positive, we deduce that

$$\lim_{r \to 0} \sup_{B_r(x)} u \leq 0,$$

while considering the case when u is negative we obtain with an analogous argument that

$$\lim_{r \to 0} \inf_{B_r(x)} u \geq 0,$$

proving the continuity of u in x. □

Definition 3.3. Given an $\mathsf{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ we define the codimension 1 spherical Hausdorff measure to be the measure obtained with the Carathéodory construction using coverings made by balls and gauge function

$$B_r(x) \mapsto \mathfrak{m}(B_r(x))/r.$$

We denote such measure by H^h.

The next proposition corresponds to [35, Theorem 4.1].

Proposition 3.4. *Let $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$. Then H^h-a.e. $x \in \Omega$ is a Lebesgue point of f.*

The next proposition is an intermediate step to prove that 3 implies 1 in Theorem 3.1.

Proposition 3.5. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be such that $\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$. There exists an open set $A \subset \Omega$ with $\mathsf{H}^h(\Omega \setminus A) = 0$ such that for every $\phi \in \mathsf{Lip}_c(A)$ we have*

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u} \, d\mathfrak{m} = 0.$$

Proof. Combining Propositions 3.2 and 3.4 we get that u is continuous H^h-almost everywhere. For every continuity point x of u consider the set $A_x \subset \Omega$ given by Proposition 2.37. And denote with A the union of these sets. It is clear that $\mathsf{H}^h(\Omega \setminus A) = 0$. Now let $\phi \in \mathsf{Lip}_c(A)$ and let $\{A_i\}_{i=1}^m$ be a finite subcover of the support of ϕ. Let $\{\eta_i\}_{i=1}^m$ be Lipschitz functions such that their sum is equal to 1 on the support of ϕ, while each η_i is compactly supported in A_i. It is easy to check that such functions exist. We get

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u} \, d\mathfrak{m} = \sum_{i=1}^m \int_{A_i} \frac{\nabla u \cdot \nabla(\eta_i \phi)}{W_u} \, d\mathfrak{m} = 0.$$

□

Proposition 3.6. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be such that $\mathsf{Epi}(u)$ is locally perimeter minimizing in $\Omega \times \mathbb{R}$. For every $\phi \in \mathsf{Lip}_c(\Omega)$ we have*

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u} \, d\mathfrak{m} = 0.$$

Proof. Let A be the set given by Proposition 3.5 and call $C := \Omega \setminus A$. We claim that we can construct a sequence $\{\eta_i\}_{i \in \mathbb{N}}$ of Lipschitz compactly supported functions in Ω that are equal to 1 in a neighbourhood of $C \cap \mathsf{supp}(\phi)$ and such that $\|\eta_i\|_{\mathsf{W}^{1,1}(\Omega)} \to 0$ as i goes to $+\infty$.

Assume for the moment that the claim holds. In this case we get

$$\begin{aligned}
\int_\Omega \frac{\nabla u \cdot \nabla \phi}{W_u} \, d\mathfrak{m} &= \int_\Omega \frac{\nabla u \cdot \nabla(\eta_i \phi)}{W_u} \, d\mathfrak{m} + \int_\Omega \frac{\nabla u \cdot \nabla((1-\eta_i)\phi)}{W_u} \, d\mathfrak{m} \\
&= \int_\Omega \frac{\phi \nabla u \cdot \nabla \eta_i}{W_u} \, d\mathfrak{m} + \int_\Omega \frac{\eta_i \nabla u \cdot \nabla \phi}{W_u} \, d\mathfrak{m}
\end{aligned}$$

and since $|\nabla u|/W_u \leq 1$ the last expression tends to zero as i goes to $+\infty$ since $\|\eta_i\|_{\mathsf{W}^{1,1}(\Omega)} \to 0$.

So we only need to prove our previous claim. We have that $C \cap \mathsf{supp}(\phi)$ is compact and $\mathsf{H}^h(C) = 0$. So for every $\epsilon > 0$ there exists a finite collection $\{B_{r_i^\epsilon}(x_i^\epsilon)\}_{i=1}^{m_\epsilon}$ of balls with radii in $(0, \epsilon)$ and centers in $C \cap \mathsf{supp}(\phi)$, whose union covers $C \cap \mathsf{supp}(\phi)$, and such that

$$\sum_{i=1}^{m_\epsilon} \frac{\mathfrak{m}(B_{r_i^\epsilon}(x_i^\epsilon))}{r_i^\epsilon} < \epsilon.$$

For every such ball define $\eta_i^\epsilon : \mathsf{X} \to \mathbb{R}$ by

$$\eta_i^\epsilon(x) = \Big(1 - \frac{\mathsf{d}(B_{r_i^\epsilon}(x_i^\epsilon), x)}{r_i^\epsilon}\Big) \vee 0.$$

It is clear that $\|\eta_i^\epsilon\|_{\mathsf{L}^1(\mathsf{X})} \leq \mathfrak{m}(B_{2r_i^\epsilon}(x_i^\epsilon)) \leq c_1 \mathfrak{m}(B_{r_i^\epsilon}(x_i^\epsilon))$, while

$$\|\nabla \eta_i^\epsilon\|_{\mathsf{L}^1(\mathsf{X})} \leq \frac{\mathfrak{m}(B_{2r_i^\epsilon}(x_i^\epsilon))}{r_i^\epsilon} \leq c_1 \frac{\mathfrak{m}(B_{r_i^\epsilon}(x_i^\epsilon))}{r_i^\epsilon}.$$

In particular defining $\eta_\epsilon := (\sum_{i=1}^{m_\epsilon} \eta_i^\epsilon) \wedge 1$ we obtain a Lipschitz function compactly supported in Ω (if ϵ is small enough) which is equal to 1 on a neighbourhood of $C \cap \mathsf{supp}(\phi)$ and with $\|\eta_\epsilon\|_{\mathsf{W}^{1,1}(\mathsf{X})} \leq 2c_1\epsilon$. This concludes the proof. □

Remark 3.7. The result of Proposition 3.6 holds also in the setting of doubling spaces supporting a $(1,1)$ Poincaré inequality (without any curvature assumption). This follows taking into account that the density estimates for perimeter minimizers (that were used in Proposition 3.2) hold in this weaker setting as well.

3.2 MSE implies globally perimeter minimizing

So far we have seen that if $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ and its epigraph is locally perimeter minimizing in $\Omega \times \mathbb{R}$, then u solves the minimal surface equation. In this section we show that if u solves the aforementioned equation, then its epigraph minimizes the perimeter.

Proposition 3.8. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be a solution of the minimal surface equation. Then u minimizes the area functional among $\mathsf{W}^{1,1}_{loc}(\Omega)$ functions that coincide with u out of a compact set of Ω.*

Proof. Let $\phi \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be a function that coincides with u out of an open precompact set $A \subset\subset \Omega$. Then the function $f(t) = \int_A \sqrt{1 + |\nabla(u + t\phi)|^2}\, d\mathfrak{m}$ is smooth, it satisfies $f'(0) = 0$ and it is convex, so that it has a minimum in 0. The statement follows. □

The previous proposition implies that the theory of De Giorgi Classes can be applied to u. The next proposition follows by mimicking the proof in the Euclidean setting given for example in [11, Theorem 3.9] (and it can be also obtained combining the results from [32] and [8]).

Proposition 3.9. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be a solution of the minimal surface equation, then u is locally bounded.*

In the next proposition we consider functions on $\Omega \times \mathbb{R}$. Given such a function f we will denote by $\mathsf{lip}_t(f)(x,s)$ the local Lipschitz constant in (x,s) of the restriction of f to $\{x\} \times \mathbb{R}$. Similarly, $\mathsf{lip}_x(f)(x,s)$ will be the local Lipschitz constant in (x,s) of the restriction of f to $\Omega \times \{s\}$. Finally, $\mathsf{lip}_{\times}(f)$ will be the local Lipschitz constant of f.

Proposition 3.10. *Let $f \in \mathsf{Lip}(\Omega \times \mathbb{R})$ and let $B \subset \Omega$ be an open set. Suppose that there exist $\epsilon, s_1, s_2 \in \mathbb{R}$ such that $1/2 > \epsilon > 0$, $s_2 > s_1$, $f \geq 1 - \epsilon$ on $B \times \{s_1\}$ and $f \leq \epsilon$ on $B \times \{s_2\}$. Then setting*

$$w(f)(x) := \int_{s_1}^{s_2} f(x,t)\, dt.$$

we have that $w(f)$ is Lipschitz and

$$\int_{B \times (s_1, s_2)} \mathsf{lip}_{\times}(f)\, d\mathfrak{m}_{\times} \geq \int_B \sqrt{(1-2\epsilon)^2 + |\nabla w(f)|^2}\, d\mathfrak{m}.$$

Proof. By the tensorization property of the energy (see [8]) we have

$$\int_{B\times(s_1,s_2)} \mathsf{lip}_\times(f)\,d\mathfrak{m}_\times = \int_{B\times(s_1,s_2)} \sqrt{\mathsf{lip}_t(f)^2+\mathsf{lip}_x(f)^2}\,d\mathfrak{m}_\times$$

and for every $(a,b)\in C(\Omega)\times C(\Omega)$ such that $a^2+b^2\le 1$ we get

$$\int_{B\times(s_1,s_2)} \sqrt{\mathsf{lip}_t(f)^2+\mathsf{lip}_x(f)^2}\,d\mathfrak{m}_\times \ge \int_{B\times(s_1,s_2)} a\,\mathsf{lip}_t(f)+b\,\mathsf{lip}_x(f)\,d\mathfrak{m}_\times$$

and by the condition on f this is greater than or equal to

$$\int_B \left[a(1-2\epsilon)+b\int_{(s_1,s_2)} \mathsf{lip}_x(f)\,dt\right] d\mathfrak{m}.$$

We claim that $\mathsf{lip}(w(f)) = |\nabla w(f)| \le \int_{(s_1,s_2)} \mathsf{lip}_x(f)\,dt$ almost everywhere w.r.t. $\mathfrak{m}$ in Ω. If the claim holds, passing to the supremum with respect to (a,b) in the previous expression we conclude.

Observe that for every $x,y\in B$ we have

$$|w(f)(x)-w(f)(y)| \le \int_{s_1}^{s_2} |f(x,t)-f(y,t)|\,dt \le (s_2-s_1)\mathsf{L}(f)\mathsf{d}(x,y)$$

so that $w(f)$ is Lipschitz in B and hence for $\mathfrak{m}$-a.e. $x\in B$ we have $|\nabla w(f)| = \mathsf{lip}(w(f))$. Moreover by DCT we have

$$\mathsf{lip}(w(f))(x) = \lim_{x_n\to x} \frac{|w(f)(x)-w(f)(x_n)|}{\mathsf{d}(x_n,x)} \le \int_{(s_1,s_2)} \mathsf{lip}_x(f)\,dt,$$

as desired. □

Lemma 3.11. *Let $f,g\in \mathsf{W}^{1,1}_{loc}(\Omega)$ and $B\subset\subset\Omega$ be a Borel set, then $|P(\mathsf{Epi}(f),B\times\mathbb{R}) - P(\mathsf{Epi}(g),B\times\mathbb{R})| \le |D(f-g)|(B)$.*

Proof. The proof is immediate and follows from the fact that

$$\begin{aligned}|P(\mathsf{Epi}(f),B\times\mathbb{R}) - P(\mathsf{Epi}(g),B\times\mathbb{R})| &= \left|\int_B \sqrt{1+|\nabla f|^2}\,d\mathfrak{m} - \int_B \sqrt{1+|\nabla g|^2}\,d\mathfrak{m}\right| \\ &\le \int_B \big||\nabla f|-|\nabla g|\big|\,d\mathfrak{m} \le \int_B |\nabla(f-g)|\,d\mathfrak{m}.\end{aligned}$$

□

The next proposition contains a technical approximation result that will be crucial for the remaining part of the section.

Proposition 3.12. *Let $\Omega'\subset\subset\Omega''\subset\subset\Omega$ be open sets and let $f\in\mathsf{W}^{1,1}_{loc}(\Omega)$ and $\phi\in\mathsf{BV}_c(\Omega')$ be locally bounded. There exist functions $f_t\in\mathsf{Lip}(\Omega'')$ converging in $\mathsf{L}^1(\Omega'')$ to $f+\phi$ such that*

$$\limsup_{t\to 0} P(\mathsf{Epi}(f_t),\Omega''\times\mathbb{R}) = P(\mathsf{Epi}(f+\phi),\Omega''\times\mathbb{R})$$

and

$$|D(f-f_t)|(\Omega''\setminus\Omega')\to 0.$$

Proof. Since the statement concerns precompact sets of Ω, modulo using a cut off, we suppose that $f\in\mathsf{W}^{1,1}(\Omega)$ with compact support in Ω. Moreover, modulo enlarging Ω'', we may also suppose that $\mathfrak{m}(\partial\Omega'')=0$. Whenever we refer to f as a function on X we implicitly mean its extension to zero. Let B be an open set such that $\mathsf{supp}(\phi)\subset\subset B\subset\subset\Omega'$. We claim that for every $\epsilon>0$ there exists a function $f'\in\mathsf{W}^{1,1}(\Omega)$ with compact support whose restriction to $\Omega\setminus B$ is Lipschitz such that

$$|P(\mathsf{Epi}(f+\phi),\Omega''\times\mathbb{R}) - P(\mathsf{Epi}(f'+\phi),\Omega''\times\mathbb{R})| < \epsilon$$

and

$$|D(f-f')(\Omega''\setminus\Omega')|<\epsilon.$$

To prove the claim consider an open set A such that

$$\mathsf{supp}(\phi)\subset\subset A\subset\subset B$$

and let $\delta>0$ and $v\in\mathsf{Lip}_c(\Omega)$ be such that $\|f-v\|_{\mathsf{W}^{1,1}(\Omega)}<\delta$. Let then $\eta\in\mathsf{Lip}_c(B)$ be a positive function, taking value less than or equal to 1 and identically equal to 1 on A. We define $f':=\eta f+(1-\eta)v$. This function trivially satisfies $f'\in\mathsf{W}^{1,1}(\Omega)$, it has compact support, its restriction belongs to $\mathsf{Lip}(\Omega\setminus B)$, $|D(f-f')(\Omega''\setminus\Omega')|<\delta$, and

$$|P(\mathsf{Epi}(f+\phi),\Omega''\times\mathbb{R})-P(\mathsf{Epi}(f'+\phi),\Omega''\times\mathbb{R})|$$

$$=|P(\mathsf{Epi}(f),{}^cA\times\mathbb{R})-P(\mathsf{Epi}(f'),{}^cA\times\mathbb{R})|.$$

By Lemma 3.11 this last quantity is controlled by

$$|D(f'-f)|({}^cA)\le|D(1-\eta)(v-f)|(\Omega)\le c(\eta)\delta.$$

In particular choosing δ small enough we have proved the claim. Taking this into account, it is sufficient to prove the statement of the theorem under the additional assumption that the restriction of f to $\Omega\setminus B$ is Lipschitz (and we still have that $f\in\mathsf{W}^{1,1}(\Omega)$ and has compact support).

To this aim we define $f_t:=P_t(f+\phi)$ and we claim that these functions restricted to Ω'' have the right properties. First of all the functions f_t are Lipschitz by The L^∞ to Lipschitz property of the heat flow. Moreover $f_t\to f+\phi$ in $\mathsf{L}^1(\Omega'')$ since we have convergence in $\mathsf{L}^2(\mathsf{X})$.

We will now show that

$$|D(f-f_t)|(\Omega''\setminus\Omega')\to 0.$$

Let $\tau\in\mathsf{Lip}_c(\Omega')$ be positive, taking value less than or equal to 1 and equal to 1 on B. Note that

$$f_t=P_t((1-\tau)f)+P_t(\tau f+\phi),$$

and since $(1-\tau)f\in\mathsf{Lip}(\mathsf{X})\subset\mathsf{W}^{1,2}(\mathsf{X})$ we have $P_t((1-\tau)f)\to(1-\tau)f$ in $\mathsf{W}^{1,2}(\mathsf{X})$ which implies convergence in $\mathsf{W}^{1,1}(\Omega'')$ since Ω'' is bounded. In particular when we restrict these functions to $\Omega''\setminus\Omega'$ we get

$$|D(f-P_t((1-\tau)f))|(\Omega''\setminus\Omega')\to 0.$$

So to conclude the proof of the second property in the statement it is sufficient to note that by Proposition 2.22 we have

$$|D(P_t(\tau f+\phi))|(\Omega''\setminus\Omega')\to|D(\tau f+\phi)|(\Omega''\setminus\Omega')=0.$$

We now turn our attention to the perimeter condition. Modulo a vertical translation we can suppose that $f+\phi>0$ on Ω''. Consider then the function $F_t:\mathsf{X}\times\mathbb{R}\to\mathbb{R}$ given by

$$F_t(x,s):=P_t^{\mathsf{X}\times\mathbb{R}}(1_{\mathsf{Epi}'(f+\phi)})(x,s).$$

By Proposition 2.22 and the fact that $\mathfrak{m}(\partial\Omega'')=0$, we get

$$\lim_{t\to 0}|D^{\mathsf{X}\times\mathbb{R}}F_t|(\Omega''\times\mathbb{R})=P(\mathsf{Epi}'(f+\phi),\Omega''\times\mathbb{R}).\tag{5}$$

At the same time we have that $p_t^{\mathsf{X}\times\mathbb{R}}((x,r)(y,s))=p_t^{\mathsf{X}}(x,y)p_t^{\mathbb{R}}(r,s)$ and that $\int_{\mathbb{R}}p_t(r,s)\,ds=1$. We claim that this implies $\|\nabla(w(F_t)-f_t)\|_{\mathsf{L}^1(\Omega'')}\to 0$ as $t\to 0$, where

$$w(F_t)(x):=\int_I F_t(x,s)\,ds$$

and I is any finite open interval containing 0 and the supremum of $u+\phi$ in Ω''.

Indeed

$$w(F_t)(x)-f_t(x)$$

$$= \int_I \int_{\mathsf{X}\times\mathbb{R}} p_t^{\mathsf{X}}(x,y) p_t^{\mathbb{R}}(r,s) 1_{\mathsf{Epi}'(f+\phi)}(y,s)\, d\mathfrak{m}(y)\, ds\, dr - \int_{\mathbb{R}} \int_{\mathsf{X}\times\mathbb{R}_+} p_t^{\mathsf{X}}(x,y) p_t^{\mathbb{R}}(r,s) 1_{\mathsf{Epi}'(f+\phi)}(y,s)\, d\mathfrak{m}(y)\, ds\, dr$$
$$= \int_I \int_{\mathsf{X}\times\mathbb{R}_-} p_t^{\mathsf{X}}(x,y) p_t^{\mathbb{R}}(r,s)\, d\mathfrak{m}(y)\, ds\, dr - \int_{{}^cI} \int_{\mathsf{X}\times\mathbb{R}} p_t^{\mathsf{X}}(x,y) p_t^{\mathbb{R}}(r,s) 1_{\mathsf{Epi}'(f+\phi)\cap \mathsf{X}\times\mathbb{R}_+}(y,s)\, d\mathfrak{m}(y)\, ds\, dr.$$

Note now that the first addendum in the previous expression is a constant, so that to prove our claim it is sufficient to show that

$$\int_{\Omega''} \mathsf{lip}_x \int_{{}^cI} P_t(1_{\mathsf{Epi}'(f+\phi)\cap\mathsf{X}\times\mathbb{R}_+})(x,r)\, dr\, d\mathfrak{m}(x) \to 0.$$

This follows passing the Lipschitz constant inside the second integral and observing first that lip_x is controlled from above by the Lipschitz constant in the product space, and then that Proposition 2.22 guarantees that

$$\lim_{t\to 0} |D^{\mathsf{X}\times\mathbb{R}} P_t(1_{\mathsf{Epi}'(f+\phi)\cap\mathsf{X}\times\mathbb{R}_+})|(\Omega'' \times {}^cI) = 0.$$

This concludes the proof that $\|\nabla(w(F_t) - f_t)\|_{\mathsf{L}^1(\Omega'')} \to 0$. By Lemma 3.11, this implies that

$$|P(\mathsf{Epi}(w(F_t)), \Omega'') - P(\mathsf{Epi}(f_t), \Omega'')| \to 0.$$

In conclusion if we fix $\epsilon > 0$, for t small enough we have that F_t satisfies the hypotheses of Proposition 3.10 and in addition $|P(\mathsf{Epi}(w(F_t)), \Omega'') - P(\mathsf{Epi}(f_t), \Omega'')| < \epsilon$, so that

$$|DF_t|(\Omega''\times\mathbb{R}) \geq P(\mathsf{Epi}(w(F_t)), \Omega''\times\mathbb{R}) - 2\epsilon\mathfrak{m}(\Omega'') \geq P(\mathsf{Epi}(f_t), \Omega''\times\mathbb{R}) - \epsilon(1 + 2\mathfrak{m}(\Omega'')).$$

Finally, passing to the limits in the previous expression and taking into account (5), the arbitrariness of ϵ, and the lower semicontinuity of perimeters we conclude. □

Proposition 3.13. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be a solution of the minimal surface equation. Then the epigraph of u minimizes the perimeter among bounded competitors in $\mathsf{BV}_{loc}(\Omega)$ that coincide with u out of a compact set.*

Proof. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be open sets and let $\epsilon > 0$. Let $\phi \in \mathsf{BV}_c(\Omega')$ be bounded. It will be sufficient to show that for a constant C independent of ϵ we have

$$P(\mathsf{Epi}(\phi + u), \Omega''\times\mathbb{R}) \geq P(\mathsf{Epi}(u), \Omega''\times\mathbb{R}) - C\epsilon.$$

By Proposition 3.12 there exists $f \in \mathsf{Lip}(\Omega'')$ such that

$$\|f - u\|_{\mathsf{W}^{1,1}(\Omega''\setminus\bar\Omega')} + \|f - (u+\phi)\|_{\mathsf{L}^1(\Omega'')} < \epsilon$$

and

$$P(\mathsf{Epi}(f), \Omega''\times\mathbb{R}) \leq P(\mathsf{Epi}(\phi+u), \Omega''\times\mathbb{R}) + \epsilon.$$

Let $\eta \in \mathsf{Lip}_c(\Omega'')$ be positive, taking value less than or equal to 1 and equal to 1 on a neighbourhood of $\bar\Omega'$. Note that $f\eta + (1-\eta)u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ and differs from u on a precompact set of Ω'', so that

$$P(\mathsf{Epi}(u), \Omega''\times\mathbb{R}) \leq P(\mathsf{Epi}(f\eta + (1-\eta)u), \Omega''\times\mathbb{R}).$$

At the same time

$$|P(\mathsf{Epi}(f\eta + (1-\eta)u), \Omega''\times\mathbb{R}) - P(\mathsf{Epi}(f), \Omega''\times\mathbb{R})| \leq |D(1-\eta)(f-u)|(\Omega''\setminus\bar\Omega') \leq C(\eta)\epsilon.$$

Putting these inequalities together we have

$$P(\mathsf{Epi}(\phi+u), \Omega''\times\mathbb{R}) \geq P(\mathsf{Epi}(f), \Omega''\times\mathbb{R}) - \epsilon$$
$$\geq P(\mathsf{Epi}(f\eta + (1-\eta)u), \Omega''\times\mathbb{R}) - (C(\eta)+1)\epsilon$$
$$\geq P(\mathsf{Epi}(u), \Omega''\times\mathbb{R}) - (C(\eta)+1))\epsilon,$$

concluding the proof. □

Theorem 3.14. *Let $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ be a solution of the minimal surface equation. Then $\mathsf{Epi}(u)$ is a perimeter minimizing set in $\Omega \times \mathbb{R}$.*

Proof. Let $F \subset \Omega \times \mathbb{R}$ be a set such that $\mathsf{Epi}'(u)\Delta F \subset\subset \Omega \times \mathbb{R}$. Let $A \subset \Omega$ be an open precompact set such that $\mathsf{Epi}'(u)\Delta F \subset\subset A \times \mathbb{R}$. Modulo translating vertically, since u is bounded in A, we can suppose that $u > 0$ and $\mathsf{Epi}'(u)\Delta F \subset\subset A \times (0 + \infty)$.

Now consider a sequence $\{f_i\}_{i\in\mathbb{N}} \subset \mathsf{Lip}(A \times \mathbb{R})$ of functions converging in $\mathsf{L}^1(A \times \mathbb{R})$ to F and such that

$$P(F, A \times \mathbb{R}) = \lim_{i\to+\infty} \int_{A\times\mathbb{R}} \mathsf{lip}_\times(f_i)\, d\mathfrak{m}_\times .$$

It is clear that we can assume that $f_i = 1$ on $A \times (-\infty, 0)$ and that there exists $c > 0$ such that $f = 0$ on $A \times (c, +\infty)$ (if this is not the case we multiply by a cutoff and obtain a better approximation).

Using the notation introduced in Proposition 3.10, we obtain by the same proposition that

$$P(F, A \times \mathbb{R}) \geq \limsup_{i\to+\infty} \int_A \sqrt{1 + |\nabla w(f_i)|^2}\, d\mathfrak{m} = \limsup_{i\to+\infty} P(\mathsf{Epi}(w(f_i)), A).$$

Moreover the L^1 convergence of $\mathsf{Epi}(w(f_i))$ to $\mathsf{Epi}(w(1_F))$ in $A \times \mathbb{R}$ and the lower semicontinuity of the perimeter gives, passing to the limit in the previous expression, that

$$P(F, A \times \mathbb{R}) \geq P(\mathsf{Epi}(w(1_F)), A).$$

Moreover since each $w(f_i)$ is Lipschitz we deduce that $w(1_F) \in \mathsf{BV}(A)$. By definition $w(1_F) \neq u$ on a precompact subset of A, so that by Proposition 3.13 we deduce that

$$P(F, A \times \mathbb{R}) \geq P(\mathsf{Epi}(w(1_F)), A) \geq P(\mathsf{Epi}(u), A),$$

as desired. □

4 Harmonicity of u on its graph

In this section we prove Theorems 5 and 6 from the Introduction. This is the central and most technical section of the note. We first introduce the abstract machinery of strong blow-ups and then we apply it to u and its graph. This can be done thanks to the refined blow-up properties of u analyzed in Section 4.2.

4.1 Blow-ups of functions

The idea of considering blow-ups of functions comes from [17], and in this section we apply the results of the aforementioned work to define strong blow-ups. The key results are Corollaries 4.7 and 4.9, as they show that a function and its strong blow-up share the same local Lipschitz constant (note that this may not happen for blow-ups, in general). In the rest of the note when we consider the L^∞ norm of a function we mean its supremum. Moreover, if we are working with functions that are defined modulo negligible sets, we will be implicitly fixing a representative.

Definition 4.1. Let (X, d) be a metric space and let $f : \mathsf{X} \to \mathbb{R}$ be a function. Given a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ decreasing to zero, we say that a triple $(\epsilon_n, \psi_n, f^\infty)$ is a blow-up of f at x if (ϵ_n, ψ_n) is a blow-up of X at x and $f^\infty : \mathbb{R}^k \to \mathbb{R}$ is a linear function such that

$$\|\epsilon_n^{-1}(f \circ \psi_n - f(x)) - f^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^k}(0))} \to 0.$$

Definition 4.2. Let (X, d) be a metric space. We say that a triple $(\epsilon_n, \psi_n, f^\infty)$ is a strong blow-up of f at x if it is a blow-up and the following holds. For every $\{\delta'_n\}_{n\in\mathbb{N}}$ going to zero and every sequence $\{\psi'_n\}_{n\in\mathbb{N}}$ of δ'_n-GH maps

$$\psi'_n : \bar{B}_1^{\mathbb{R}^k}(0) \to (\bar{B}_{\epsilon_n}(x), \epsilon_n^{-1}\mathsf{d})$$

such that $\psi'_n \neq \psi_n$ for at most a finite number of points we have that $(\epsilon_n, \psi'_n, f^\infty)$ is a blow-up of f at x.

The next theorem follows from [17, Theorem 10.2].

Theorem 4.3. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, let $\Omega \subset \mathsf{X}$ be an open set and let $f \in \mathsf{Lip}(\Omega)$. Then for $\mathfrak{m}$-a.e. $x \in \Omega$, for every blow-up (ϵ_n, ψ_n) of X at x there exists a (non relabeled) subsequence such that $(\epsilon_n, \psi_n, f^\infty)$ is a blow-up of f at x and $|\nabla f|(x) = \mathsf{lip}(f)(x) = \mathsf{lip}(f^\infty)(0)$.*

Corollary 4.4. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, let $\Omega \subset \mathsf{X}$ be an open set and let $f \in \mathsf{Lip}(\Omega)$, then for $\mathfrak{m}$-a.e. $x \in \Omega$ for every blow-up (ϵ_n, ψ_n) of X at x there exists (modulo passing to a subsequence) a strong blow-up $(\epsilon_n, \psi_n, f^\infty)$ of f in x. We say that f is differentiable in any such point.*

Proof. Let $x \in \Omega$ be a regular point of X. Since f is Lipschitz Theorem 4.3 guarantees that for every blow-up (ϵ_n, ψ_n) of X at x there exists (modulo passing to a subsequence and asking that x lies out of an appropriate $\mathfrak{m}$-negligible set) a blow-up $(\epsilon_n, \psi_n, f^\infty)$ of f at x. Let now (ϵ_n, ψ'_n) be another blow-up of X at x such that for every $n \in \mathbb{N}$ we have $\psi'_n \neq \psi_n$ on at most a finite number of points. By the previous theorem, modulo passing to another subsequence, there exists a blow-up $(\epsilon_n, \psi'_n, f'^\infty)$. It is easy to check that f^∞ and f'^∞ coincide out of a countable set so that, being linear, they coincide everywhere, concluding the proof. □

Proposition 4.5. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function and let $\phi_1, ..., \phi_m : \Omega \to \mathbb{R}$ be functions admitting blow-ups $\{(\epsilon_n, \psi_n, \phi_i^\infty)\}_{i=1}^m$ in x. Then $f(\phi_1, ..., \phi_m)$ has blow-up $(\epsilon_n, \psi_n, f^\infty)$, where*

$$f^\infty := \nabla f((\phi_1(x), ..., \phi_m(x))) \cdot (\phi_1^\infty, ..., \phi_m^\infty).$$

The same statement holds replacing blow-ups with strong blow-ups.

Proof. We will suppose w.l.o.g. that $(\phi_1, ..., \phi_m)(x) = 0$ and that $f(0) = 0$. To lighten the notation instead of writing $\phi_i \circ \psi_n$ we will write simply ϕ_i. Since f is smooth we have

$$\begin{aligned}
&|\epsilon_n^{-1} f(\phi_1, ..., \phi_m) - f^\infty| \\
&= |\epsilon_n^{-1} \nabla f(\phi_1, ..., \phi_m)(x) \cdot (\phi_1, ..., \phi_m) + \epsilon_n^{-1} o(|(\phi_1, ..., \phi_m)|) - f^\infty| \\
&\leq |\epsilon_n^{-1} \nabla f(\phi_1, ..., \phi_m)(x) \cdot (\phi_1, ..., \phi_m) - f^\infty| + \frac{o(|(\phi_1, ..., \phi_m)|)}{|(\phi_1, ..., \phi_m)|} \frac{|(\phi_1, ..., \phi_m)|}{\epsilon_n}.
\end{aligned}$$

The first addendum goes to zero uniformly in $B_1(0)$ by definition of f^∞. Concerning the second, the quantity $(\phi_1, ..., \phi_m)/\epsilon_n$ is bounded in modulus by the existence of the linear blow-ups, so that the second addendum goes to zero uniformly in $B_1(0)$, concluding the proof of the first claim. The statement on strong blow-ups follows similarly. □

Proposition 4.6. *Consider a metric space $(\mathsf{Y}, \mathsf{d}_y)$, a point $y \in \mathsf{Y}$, and a function $f : \mathsf{Y} \to \mathbb{R}$. Suppose that*

$$\mathsf{lip}(f)(y) = \lim_{n \to +\infty} \frac{|f(y_n) - f(y)|}{\mathsf{d}_y(y_n, y)}$$

and set $\epsilon_n := \mathsf{d}_y(y, y_n)$. Let $\mathbb{R}^k \supset A \supset \bar{B}_1(0)$ be either an open set or a closed ball and let $\mathsf{Y} \supset A_n \supset \bar{B}_{\epsilon_n}(y)$. Assume that $\{\delta_n\}_{n \in \mathbb{N}}$ decreases to zero and that we have δ_n-GH approximations $\psi_n : A \to (A_n, \epsilon_n^{-1} \mathsf{d}_y)$ such that there exists a linear function $f^\infty : \mathbb{R}^k \to \mathbb{R}$ such that

$$\|\epsilon_n^{-1}(f \circ \psi_n - f(y)) - f^\infty\|_{\mathsf{L}^\infty(A)} \to 0.$$

Assume moreover that if $\{\delta'_n\}_{n \in \mathbb{N}}$ decreases to zero and $\{\psi'_n\}$ is another sequence of δ'_n-GH approximations between the same sets such that ψ'_n differs from ψ_n on at most a finite number of points we have that

$$\|\epsilon_n^{-1}(f \circ \psi'_n - f(y)) - f^\infty\|_{\mathsf{L}^\infty(A)} \to 0.$$

Then $\mathsf{lip}(f)(y) = \mathsf{lip}(f^\infty)(0)$.

Proof. We first show that $\mathsf{lip}(f)(y) \leq \mathsf{lip}(f^\infty)(0)$. By Lemma 2.9 for every $n \in \mathbb{N}$ there exists a $4\delta_n$-GH map $\psi'_n : A \to A_n$ such that $\psi'_n \neq \psi_n$ on at most one point and $y_n \in \mathsf{Im}(\psi'_n)$ with $\psi'_n(x_n) = y_n$. By hypothesis we then have

$$\Big|\frac{|f(y_n) - f(y)|}{\mathsf{d}_y(y_n, y)} - f^\infty(x_n)\Big| \to 0.$$

Moreover since f^∞ is linear and $|x_n|$ tends to 1 as n increases we have

$$\mathsf{lip}(f^\infty)(0) \geq \limsup_{n\to+\infty} \frac{|f^\infty(x_n)|}{|x_n|} \geq \limsup_{n\to+\infty} |f^\infty(x_n)|.$$

Summing up we obtain

$$\mathsf{lip}(f)(y) = \lim_{n\to+\infty} \frac{|f(y_n)-f(y)|}{\mathsf{d}_y(y_n,y)} = \lim_{n\to+\infty} |f^\infty(x_n)| \leq \mathsf{lip}(f^\infty)(0),$$

proving the first inequality.

We now show that $\mathsf{lip}(f^\infty)(0) \leq \mathsf{lip}(f)(y)$. Since f^∞ is linear there exists $x_\infty \in \mathbb{R}^n$ such that $|x_\infty| = 1$ and $\mathsf{lip}(f^\infty)(0) = f^\infty(x_\infty)$. By hypothesis we then get that

$$|{\epsilon_n}^{-1}|f(\psi_n(x_\infty)) - f(y)| - f^\infty(x_\infty)| \to 0,$$

with $\mathsf{d}_y(\psi_n(x_\infty), y) \leq \epsilon_n(1+\delta_n)$. Therefore

$$\mathsf{lip}(f^\infty)(0) = f^\infty(x_\infty) = \lim_{n\to+\infty} {\epsilon_n}^{-1}|f(\psi_n(x_\infty)) - f(y)|$$

$$\leq \limsup_{n\to+\infty} \frac{(1+\delta_n)|f(\psi_n(x_\infty)) - f(y)|}{\mathsf{d}_y(\psi_n(x_\infty),y)} \leq \mathsf{lip}(f)(y),$$

concluding the proof. □

Corollary 4.7. *Consider a metric space $(\mathsf{Y},\mathsf{d}_y)$, a point $y \in \mathsf{Y}$, and a function $f : \mathsf{Y} \to \mathbb{R}$. Suppose that*

$$\mathsf{lip}(f)(y) = \lim_{n\to+\infty} \frac{|f(y_n)-f(y)|}{\mathsf{d}_y(y_n,y)}$$

and set $\epsilon_n := \mathsf{d}_y(y, y_n)$. If f admits strong blow-up $(\epsilon_n, \psi_n, f^\infty)$ then $\mathsf{lip}(f^\infty)(0) = \mathsf{lip}(f)(y)$.

As remarked at the beginning of the section, in the rest of the note, when we consider blow-ups of Sobolev functions we assume that we are working with a fixed representative.

Proposition 4.8. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, let $\Omega \subset \mathsf{X}$ an open set and let $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$. For $\mathfrak{m}$-a.e. $x \in \mathsf{X}$, if there exists a strong blow-up $(\epsilon_n, \psi_n, f^\infty)$ of f at x, then $|\nabla f|(x) = \mathsf{lip}(f^\infty)(0)$.*

Proof. Fix $\epsilon > 0$ and consider the Lipschitz map f_ϵ such that $\mathfrak{m}(\{f \neq f_\epsilon\}) \leq \epsilon$ given by Theorem 2.16. By the locality of relaxed gradients for $\mathfrak{m}$-a.e. $x \in \{f = f_\epsilon\}$ we have

$$|\nabla f|(x) = |\nabla f_\epsilon|(x) = \mathsf{lip}(f_\epsilon)(x).$$

So to conclude we only need to show that for $\mathfrak{m}$-a.e. $x \in \{f = f_\epsilon\}$ for every strong blow-up $(\epsilon_n, \psi_n, f^\infty)$ of f at x it holds

$$\mathsf{lip}(f_\epsilon)(x) = \mathsf{lip}(f^\infty)(0). \tag{6}$$

We claim that in every point $x \in \{f = f_\epsilon\}$ where $\{f = f_\epsilon\}$ has density 1 and f_ϵ is differentiable the previous equality holds. To prove the claim, given the strong blow-up $(\epsilon_n, \psi_n, f^\infty)$, we consider (modulo passing to a subsequence) the strong blow-up $(\epsilon_n, \psi_n, f_\epsilon^\infty)$, which exists by our choice of x.

Then we fix $1/8 > \delta > 0$, a basis $\{e_i\}_{i=1}^k$ of $\mathbb{R}^k$ and we consider points

$$x_i^n \in \bar{B}_{\delta\epsilon_n}(\psi_n(e_i)) \cap \{f = f_\epsilon\} \cap \bar{B}_{\epsilon_n}(x).$$

Such points always exist if n is large enough by the density condition of $\{f = f_\epsilon\}$ in x. By Lemma 2.9 for every n there exists a $4\delta_n$-GH map ψ_n' with same domain and codomain as ψ_n, differing from ψ_n on at most k points and such that $\{x_i^n\}_{i=1}^k \subset \mathsf{Im}(\psi_n')$. Now let $\{y_i^n\}_{i=1}^k \subset \bar{B}_1(0)$ be points such that $\psi_n'(y_i^n) = x_i^n$ and, modulo passing to a subsequence, let $y_i \in \bar{B}_1(0)$ be the limit of y_i^n as n increases to $+\infty$. Observe that $\{y_i\}_{i=1}^k$ is a basis for $\mathbb{R}^k$ since for every i we have $\mathsf{d}_e(y_i, e_i) \leq \delta$. As $(\epsilon_n, \psi_n, f^\infty)$ and $(\epsilon_n, \psi_n, f_\epsilon^\infty)$ are strong blow-ups we have that $(\epsilon_n, \psi'_n, f^\infty)$ and $(\epsilon_n, \psi'_n, f_\epsilon^\infty)$ are blow-ups as well.

In particular (assuming for simplicity that $f(x) = 0$) for every $\delta_1 > 0$ and i there exists $n \in \mathbb{N}$ such that

$$|f^\infty(y_i) - f_\epsilon^\infty(y_i)| \leq |f^\infty(y_i^n) - f_\epsilon^\infty(y_i^n)| + \delta_1$$

and

$$\|f \circ \psi'_n - f^\infty\|_{\mathsf{L}^\infty(\bar{B}_1(0))} + \|f_\epsilon \circ \psi'_n - f_\epsilon^\infty\|_{\mathsf{L}^\infty(\bar{B}_1(0))} < \delta_1.$$

So we get

$$\begin{aligned}|f^\infty(y_i) - f_\epsilon^\infty(y_i)| &\leq |f^\infty(y_i^n) - f_\epsilon^\infty(y_i^n)| + \delta_1 \\ &\leq |f(\psi'_n(y_i^n)) - f_\epsilon(\psi'_n(y_i^n))| + 2\delta_1 = 2\delta_1.\end{aligned}$$

In particular as δ_1 was arbitrary we get that for every i we have $f^\infty(y_i) = f_\epsilon^\infty(y_i)$ and since these functions are linear and $\{y_i\}_{i=1}^k$ is a basis for $\mathbb{R}^k$ we deduce that they coincide. Equation (6) then follows by Theorem 4.3. □

Corollary 4.9. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space, let $\Omega \subset \mathsf{X}$ be an open set and let $f \in \mathsf{W}^{1,1}_{loc}(\Omega)$. For $\mathfrak{m}$-a.e. $x \in \Omega$, if for every sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$, modulo passing to a subsequence, there exists a strong blow-up $(\epsilon_n, \psi_n, f^\infty)$, then $|\nabla f|(x) = \mathsf{lip}(f^\infty)(0) = \mathsf{lip}(f)(x)$.*

Proof. It follows combining Corollary 4.7 and Proposition 4.8. □

Note that the previous corollary fails if we replace strong blow-ups with normal blow-ups. To see this consider the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. It is easy to see that for every $\epsilon_n \to 0$ there exists a blow-up (ϵ_n, ψ_n) of $\mathbb{R}$ at 0 such that the function f admits $(\epsilon_n, \psi_n, 0)$ as blow-up (but not as strong blow-up), but $\mathsf{lip}(0) = 0$ while $\mathsf{lip}(f)(0) = +\infty$.

4.2 Refined blow-up properties of $\mathsf{Epi}(u)$

In the remaining part of Section 4, $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ will be a fixed $\mathsf{RCD}(K, N)$ space with $K \leq 0$, $\Omega \subset \mathsf{X}$ will be an open set and $u \in \mathsf{W}^{1,1}_{loc}(\Omega)$ will be a function satisfying one of the equivalent conditions in Theorem 3.1. When writing $\mathsf{Epi}(u)$ we will always implicitly refer to its open representative while W will denote the quantity $\sqrt{1 + |\nabla u|^2}$.

We recall that whenever we refer to the pointwise behavior of u, we implicitly assume that we are considering it is precise representative (see (4)). In particular this means that the graph of u is contained in $\partial\mathsf{Epi}(u)$. We will denote by $v : \Omega \to \mathbb{R}$ a generic function such that

$$\mathsf{Graph}(v) \subset \partial\mathsf{Epi}(u). \tag{7}$$

The results of this section hold for any v of the previous type, as they depend on the geometric properties of $\mathsf{Epi}(u)$, rather than the specific representative that we choose. The key result of this section is Theorem 4.15, showing that each function v of the previous type admits a strong blow-up at $\mathfrak{m}$-almost every $x \in \mathsf{X}$. Let k be the essential dimension of X.

Remark 4.10. Let $B_1, B_2 \subset \Omega \times \mathbb{R}$ be tangent balls to $\mathsf{Epi}(u)$ in $(x, u(x))$ and let $u(x) = 0$. Modulo replacing one of the balls with a smaller one we can suppose that they have the same radius. In this case the real coordinates of their centers are one the opposite of the other one.

We say that a geodesic $\gamma : I \to \mathsf{X} \times \mathbb{R}$ is horizontal if it is contained in a set of the form $\mathsf{X} \times \{t\}$ for some $t \in \mathbb{R}$.

Lemma 4.11. *For $\mathfrak{m}$-a.e. $x \in \Omega$ there exist interior and exterior tangent balls to $\mathsf{Epi}(u)$ in $(x, u(x))$, and the geodesic connecting their centers is not horizontal.*

Proof. By Proposition 3.2, u is continuous in its Lebesgue points, so that for every such point $x \in \Omega$ we have $\{x\} \times \mathbb{R} \cap \partial\mathsf{Epi}(u) = \{(x, u(x))\}$.

Suppose now by contradiction that there is a Borel set $B \subset \Omega$ of positive measure $\mathfrak{m}(B) > 0$ such that for every $x \in B$ the set $\mathsf{Epi}(u)$ does not admit a pair of tangent balls at any point in $\{x\} \times \mathbb{R}$. Because of Proposition 2.36 we would get that the set $\{(x, t) \subset \partial\mathsf{Epi}(u) : x \in B\}$ has positive perimeter in $\partial\mathsf{Epi}(u)$ and its points never admit tangent balls, contradicting Theorem 2.44.

In particular for $\mathfrak{m}$-a.e. $x \in \Omega$ there exists a point $(x,t) \in \partial\mathsf{Epi}(u)$ where $\mathsf{Epi}(u)$ has tangent balls, and by our initial remark on Lebesgue points we have that for $\mathfrak{m}$-a.e. $x \in \Omega$ there exist tangent balls in $(x, u(x))$.

We will now prove that for $\mathfrak{m}$-a.e. point of the previous type the geodesic connecting the centers of the tangent balls is not horizontal. To this aim let $x_0 \in \Omega$ be a Lebesgue point for u and $|\nabla u|$ such that $\mathsf{Epi}(u)$ admits a pair of tangent balls in $(x_0, u(x_0))$ and assume for simplicity that $u(x_0) = 0$. We claim that in every such point the aforementioned geodesic cannot be horizontal. Suppose by contradiction that this is not the case.

Consider for every $n \in \mathbb{N}$ the dilated function

$$nu : (\Omega, n\mathsf{d}, C_n\mathfrak{m}) \to \mathbb{R},$$

where $C_n := \mathfrak{m}(B_{n^{-1}}(x_0))^{-1}$. We will denote by $B^n_s(y)$ balls in the dilated space $(\mathsf{X}, n\mathsf{d}, C_n\mathfrak{m})$ and by $|\nabla^n|$ the gradient; it is easy to check that $|\nabla^n| = |\nabla|/n$. Observe moreover that the space $(\mathsf{X}, n\mathsf{d}, C_n\mathfrak{m})$ is an $\mathsf{RCD}(K/n^2, N)$ space, so that these spaces all admit a $(1,1)$-Poincaré inequality with the same constants by Proposition 2.15. We obtain

$$\int_{B^n_1(x_0)} |\nabla^n nu| C_n \, d\mathfrak{m} = \fint_{B_{1/n}(x_0)} |\nabla u| \, d\mathfrak{m} \to c, \tag{8}$$

where the last limit is due to our initial assumption on x_0. Moreover for every $n \in \mathbb{N}$ there exists a constant c_n such that

$$\int_{B^n_{1/2}(x_0)} |nu - c_n| C_n \, d\mathfrak{m} \le 4e^{|K|} \int_{B^n_1(x_0)} |\nabla^n nu| C_n \, d\mathfrak{m},$$

which together with (8) gives that for every $n \in \mathbb{N}$ sufficiently large

$$\int_{B^n_{1/2}(x_0)} |nu - c_n| C_n \, d\mathfrak{m} < 4e^{|K|}(c+1). \tag{9}$$

Consider now the tangent balls $B_r((p,0) \subset \mathsf{Epi}(u)$ and $B_r((q,0)) \subset \mathsf{Epi}'(u)$ in $(x_0, 0)$ which, without loss of generality, we suppose to have the same radius r. For every $n \in \mathbb{N}$ sufficiently large let $p_n \in \Omega$ be the point on the geodesic connecting p and x_0 which is distant $\frac{3}{4n}$ from x_0. Observe that $B^n_{1/4}(p_n) \subset B^n_1(x_0)$, and since $C_n\mathfrak{m}(B^n_1(x_0)) = 1$, from the doubling property we deduce that there exists a constant C independent of n such that $C_n\mathfrak{m}(B^n_{1/4}(p_n)) \ge C$. Repeating the exact same construction with respect to q we obtain q_n such that $B^n_{1/4}(q_n) \subset B^n_1(x_0)$ and $C_n\mathfrak{m}(B^n_{1/4}(q_n)) \ge C$.

Our goal is now to show that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ increasing to $+\infty$ such that $nu \ge a_n$ on $B^n_{1/4}(q_n)$, while $nu \le -a_n$ on $B^n_{1/4}(p_n)$. If we are able to prove this, keeping in mind that $C_n\mathfrak{m}(B^n_{1/4}(p_n)) \ge C$ and $C_n\mathfrak{m}(B^n_{1/4}(q_n)) \ge C$, we contradict (9).

The desired sequence can be constructed observing that on $B^n_{1/4}(q_n)$ the graph of u lies above the ball $B_r((q,0))$, and so in particular

$$nu \ge n\sqrt{r^2 - (r - 1/(2n))^2} = \sqrt{nr - 1/4},$$

while an analogous argument gives $nu \le -\sqrt{nr - 1/4}$ on $B^n_{1/4}(p_n)$, as desired. $\square$

Let $\gamma : [0, L(\gamma)] \to \mathsf{X} \times \mathbb{R}$ be a geodesic, where $L(\gamma)$ denotes its length. We denote by $\gamma(t)_{\mathbb{R}}$ and $\gamma(t)_{\mathsf{X}}$ respectively the real and the X component of $\gamma(t)$. Observe that both $\gamma_{\mathbb{R}}$ and γ_{X} are geodesics in the respective spaces.

Definition 4.12. Let $\gamma : [0, L(\gamma)] \to \mathsf{X} \times \mathbb{R}$ be a geodesic, where $L(\gamma)$ denotes its length. We define the slope of γ as

$$s(\gamma) := \frac{|\gamma(0)_{\mathbb{R}} - \gamma(L(\gamma))_{\mathbb{R}}|}{L(\gamma)}.$$

In the next lemma we will denote by e_1 a fixed element of modulo 1 in $\mathbb{R}^k$. We recall that we denote by v a generic function whose graph is contained in $\partial\mathsf{Epi}(u)$.

Lemma 4.13. *Let $x \in \Omega$ be a regular point for X where $\mathsf{Epi}(u)$ admits interior and exterior tangent balls of radius r at $(x, u(x))$ and suppose that the the geodesic γ connecting their centers is not horizontal.*

Consider $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $n\epsilon_n \le r\sqrt{1 - s(\gamma)^2}$ and suppose that we have $\{\delta_n\}_{n\in\mathbb{N}}$ decreasing to zero and the δ_n-GH maps

$$\psi_n : \bar{B}^{\mathbb{R}^k}_n(0) \to (\bar{B}^{\mathsf{X}}_{n\epsilon_n}(x), \epsilon_n^{-1}\mathsf{d}).$$

Let p_n be the point on the portion of γ_{X} connecting p and x at distance $n\epsilon_n$ from x and let q_n be the analogous point replacing p with q. Suppose moreover that $\psi_n(ne_1) = q_n$, $\psi_n(-ne_1) = p_n$ and $\psi_n : \bar{B}_1^{\mathbb{R}^k}(0) \subset \bar{B}_{\epsilon_n}^{\mathsf{X}}(x)$. Then we have that

$$\|\epsilon_n^{-1}(v(\psi_n) - v(x)) - u^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^n}(0))} \to 0, \tag{10}$$

where $u^\infty : \mathbb{R}^k \to \mathbb{R}$ is the linear function whose graph is perpendicular to the line connecting $(e_1, s(\gamma)/\sqrt{1 - s(\gamma)^2}))$ and $(-e_1, -s(\gamma)/\sqrt{1 - s(\gamma)^2}))$.

Proof. Assume for simplicity that $u(x) = 0$. Consider the maps $(\psi_n, \epsilon_n \mathsf{id}) : \bar{B}_n^{\mathbb{R}^k}(0) \times \mathbb{R} \to (\bar{B}_{n\epsilon_n}^{\mathsf{X}}(x) \times \mathbb{R}, \epsilon_n^{-1}\mathsf{d}_\times)$ given by

$$(\psi_n, \epsilon_n \mathsf{id})(y, t) := (\psi_n(y), \epsilon_n t)$$

and note that these are δ_n-GH maps. We claim that

$$(\psi_n, \epsilon_n \mathsf{id})(B_{n-2\delta_n}(ne_1, ns(\gamma)/\sqrt{1 - s(\gamma)^2})) \subset B_{(n-\delta_n)\epsilon_n}(q_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2}) \tag{11}$$

and

$$(\psi_n, \epsilon_n \mathsf{id})(B_{n-2\delta_n}(-ne_1, -ns(\gamma)/\sqrt{1 - s(\gamma)^2})) \subset B_{(n-\delta_n)\epsilon_n}(p_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2}), \tag{12}$$

where we implicitly assume that the balls appearing in the left hand sides of these inequalities have been restricted to the domain of $(\psi_n, \epsilon_n \mathsf{id})$. We only prove (11) since (12) is analogous. To prove the claim we note that by construction

$$(\psi_n, \epsilon_n \mathsf{id})((ne_1, ns(\gamma)/\sqrt{1 - s(\gamma)^2}) = (q_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2}),$$

so that (11) follows because the maps $(\psi_n, \epsilon_n \mathsf{id})$ are δ_n-GH maps as noted earlier.

We now show that the graph of $\epsilon_n^{-1}(v(\psi_n))$ cannot intersect the balls

$$B_{n-2\delta_n}(ne_1, ns(\gamma)/\sqrt{1 - s(\gamma)^2})$$

and

$$B_{n-2\delta_n}(-ne_1, -ns(\gamma)/\sqrt{1 - s(\gamma)^2}).$$

Indeed if $(y, \epsilon_n^{-1}v(\psi_n(y))$ belongs to the first ball (resp. the second), then by (11) its image

$$(\psi_n, \epsilon_n \mathsf{id})(y, \epsilon_n^{-1}v(\psi_n(y)) = (z, v(z))$$

belongs to

$$B_{(n-\delta_n)\epsilon_n}(q_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2})$$

(resp. $B_{(n-\delta_n)\epsilon_n}(p_n, -n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2})$), and this is a contradiction since we will see that this ball is contained in the tangent ball to $\mathsf{Epi}(u)$. Indeed the point $(q_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2})$ lies between $(x, 0)$ and the center of one of the tangent balls to $\mathsf{Epi}(u)$ since $n\epsilon_n \leq r\sqrt{1 - s(\gamma)^2}$. Moreover $n\epsilon_n$ is less than the distance between $(q_n, n\epsilon_n s(\gamma)/\sqrt{1 - s(\gamma)^2})$ and $(x, 0)$, so that the desired inclusion follows. □

In the next lemma we use the notation p_n and q_n introduced previously.

Lemma 4.14. *Let $x \in \Omega$ be a regular point for X such that $\mathsf{Epi}(u)$ admits interior and exterior tangent balls of radius r at $(x, u(x))$ such that the the geodesic γ connecting their centers is not horizontal. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be decreasing to zero. We can pass to a subsequence such that $n\epsilon_n \leq r\sqrt{1 - s(\gamma)^2}$ and there exist $\{\delta_n\}_{n\in\mathbb{N}}$ decreasing to zero and the δ_n-GH maps*

$$\psi_n : \bar{B}_n^{\mathbb{R}^k}(0) \to (\bar{B}_{n\epsilon_n}^{\mathsf{X}}(x), \epsilon_n^{-1}\mathsf{d})$$

such that

- $\psi_n(ne_1) = q_n$, $\psi_n(-ne_1) = p_n$,
- $\psi_n(\bar{B}_1^{\mathbb{R}^k}(0)) \subset \bar{B}_{\epsilon_n}^{\mathsf{X}}(x)$.

Proof. Let $i \in \mathbb{N}$ be fixed. Modulo passing to a (non relabeled) subsequence of $\{\epsilon_n\}_{n\in\mathbb{N}}$, there exists $\{\delta_n\}_{n\in\mathbb{N}}$ decreasing to zero and δ_n-GH maps

$$\psi'_n : \bar{B}_i^{\mathbb{R}^k}(0) \to (\bar{B}^{\mathsf{X}}_{i\epsilon_n}(x_0), {\epsilon_n}^{-1}\mathsf{d}).$$

We will denote by $p_{i,n}$ the point on the portion of the geodesic γ_X between p and x_0 at distance $i\epsilon_n$ from x_0, while $q_{i,n}$ will be the analogous point replacing p with q.

We claim that replacing ψ'_n with $\psi_n := \psi'_n \circ \rho$, where ρ is an appropriate isometry of $\bar{B}_i^{\mathbb{R}^k}(0)$, we can suppose that there exists another sequence $\{\delta'_n\}_{n\in\mathbb{N}}$ decreasing to zero such that

$$\epsilon_n^{-1}\mathsf{d}(\psi_n(ie_1), q_{i,n}) \le \delta'_n$$

and

$$\epsilon_n^{-1}\mathsf{d}(\psi_n(-ie_1), p_{i,n}) \le \delta'_n.$$

This visually corresponds to asking that the maps ψ_n approximately send the line between ie_1 and $-ie_1$ in the portion of γ_X contained in $B_{i\epsilon_n}(x_0)$.

To prove the claim we note that there exist points x_1, x_2 in the image of ψ'_n that are δ_n-close in the dilated distance $\epsilon_n^{-1}\mathsf{d}$ to $q_{i,n}$ and $p_{i,n}$ respectively, so that their preimages a_1, a_2 satisfy $\mathsf{d}_e(a_1, a_2) \ge 2i - \delta_n$, meaning that they are almost antipodal. In particular there exist antipodal points $b_1, b_2 \in \partial B_1(0)$ such that $\mathsf{d}_e(b_1, a_1) < \delta_n$ and $\mathsf{d}_e(b_2, a_2) < \delta_n$.

We then get that $\psi'_n(b_1)$ is $2\delta_n$ close to $\psi'_n(a_1)$, and hence is also $3\delta_n$ close to $q_{i,n}$. Similarly $\psi'_n(b_2)$ is $3\delta_n$ close to $p_{i,n}$. We then consider a rotation ρ sending ie_1 in b_1 and $-ie_1$ in b_2 and we define define $\psi_n := \psi'_n \circ \rho$. We then get

$$\epsilon_n^{-1}\mathsf{d}(\psi_n(ie_1), q_{i,n}) = \epsilon_n^{-1}\mathsf{d}(\psi'_n(b_1), q_{i,n}) \le 3\delta_n$$

and

$$\epsilon_n^{-1}\mathsf{d}(\psi_n(-ie_1), p_{i,n}) = \epsilon_n^{-1}\mathsf{d}(\psi'_n(b_2), p_{i,n}) \le 3\delta_n.$$

In conclusion setting $\delta'_n := 3\delta_n$ we conclude the proof of the claim.

From this it follows that the maps $\psi''_n : \bar{B}_i^{\mathbb{R}^k}(0) \to (\bar{B}^{\mathsf{X}}_{i\epsilon_n}(x), \epsilon_n^{-1}\mathsf{d})$, given by

$$\psi''_n(y) = \begin{cases} \psi'_n(y) & x \notin \{ie_1, -ie_1\} \\ q_{i,n} & y = ie_1 \\ p_{i,n} & y = -ie_1 \end{cases}$$

are $\delta''_n := (\delta_n + \delta'_n)$-$GH$ maps.

Finally define the maps $\psi_n : \bar{B}_i^{\mathbb{R}^k}(0) \to \bar{B}^{\mathsf{X}}_{i\epsilon_n}(x)$ by $\psi_n(y) = \psi''_n(y)$ if $y \notin B_1(0)$ or $\psi''_n(y) \in B_{\epsilon_n}(x)$, while in any other case $\psi_n(y)$ is set to be the point on the geodesic connecting x and $\psi''_n(y)$ at distance $\epsilon_n(1 - \delta''_n)$ from x. It is easy to check that these maps are $5\delta''_n$-GH maps such that $\psi_n(ie_1) = q_{i,n}$, $\psi_n(ie_1) = p_{i,n}$ and $\psi_n(\bar{B}_1^{\mathbb{R}^k}(0)) \subset \bar{B}^{\mathsf{X}}_{\epsilon_n}(x)$. So modulo passing to nested subsequences as i increases and using a diagonal argument the proof is concluded. □

Theorem 4.15. *Let $x \in \Omega$ be a regular point for X such that $\mathsf{Epi}(u)$ admits interior and exterior tangent balls of radius r at $(x, u(x))$ such that the the geodesic γ connecting their centers is not horizontal. Then there exists $u^\infty : \mathbb{R}^k \to \mathbb{R}$ such that for every $\{\epsilon_n\}_{n\in\mathbb{N}}$, modulo passing to a subsequence, there exists a blow-up (ϵ_n, ψ_n) of X in x such that every v as in (7), modulo passing to another subsequence, has $(\epsilon_n, \psi_n, u^\infty)$ as strong blow-up at x.*

Proof. Modulo passing to a subsequence we can assume that $n\epsilon_n \le r\sqrt{1 - s(\gamma)^2}$. We will use the notation p_i, q_i and u^∞ as in the statement of Lemma 4.13. Thanks to Lemma 4.13 and 4.14 there exists a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ decreasing to zero and δ_i-GH maps

$$\psi_i : \bar{B}_i^{\mathbb{R}^k}(0) \to (\bar{B}^{\mathsf{X}}_{i\epsilon_i}(x), \epsilon_i^{-1}\mathsf{d})$$

satisfying $\psi_i(ie_1) = q_i$, $\psi_i(-ie_1) = p_i$ and $\psi_i(\bar{B}_1^{\mathbb{R}^k}(0)) \subset \bar{B}^{\mathsf{X}}_{\epsilon_i}(x)$ and such that (assuming as usual for simplicity that $u(x) = 0$)

$$\|\epsilon_i^{-1}v(\psi_i) - u^\infty\|_{\mathsf{L}^\infty(B_1^{\mathbb{R}^k}(0))} \to 0.$$

To prove that we have a strong blow-up we need to consider a sequence $\{\delta'_i\}_{i\in\mathbb{N}}$ decreasing to zero and δ'_i-GH maps $\psi'_i : \bar{B}_1^{\mathbb{R}^k}(0) \to \bar{B}_{\epsilon_i}^{\mathsf{X}}(x)$ each of which differs from ψ_i on at most a finite number of points and we need to prove that

$$\|\epsilon_i^{-1} v(\psi'_i) - u^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^k}(0))} \to 0.$$

To this aim define the maps $\psi''_i : \bar{B}_i^{\mathbb{R}^k}(0) \to (\bar{B}_{i\epsilon_i}^{\mathsf{X}}(x), \epsilon_i^{-1}\mathsf{d})$

$$\psi''_i(y) := \begin{cases} \psi'_n(y) & y \in \bar{B}_1(0) \\ \psi_n(y) & y \notin \bar{B}_1(0). \end{cases}$$

It is clear that these maps are δ''_n-GH maps between the respective domains for a sequence $\{\delta''_n\}_{n\in\mathbb{N}}$ converging to zero. Moreover they satisfy the conditions of Lemma 4.13, so that we get

$$\|\epsilon_n^{-1} v(\psi''_n) - u^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^k}(0))} \to 0,$$

concluding the proof that u^∞ is a strong blow-up. □

Corollary 4.16. *For $\mathfrak{m}$-a.e. $x \in \Omega$ we have $|\nabla u(x)| = \mathsf{lip}(u)(x)$.*

Proof. It follows by Corollary 4.9 and Theorem 4.15. □

4.3 A first definition of Graph space

In this section we combine the machinery of strong blow-ups with the refined blow-ups properties of $\mathsf{Epi}(u)$ to obtain preliminary versions of Theorems 5 and 6 in the Introduction (i.e. Corollary 4.23 and Theorem 4.26). To this aim we follow the strategy outlined in the Introduction under the additional hypothesis that u is Lipschitz.

Definition 4.17. We define the metric measure space $(\tilde{\Omega}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}})$, where $\tilde{\Omega} := \Omega$, for every $x, y \in \tilde{\Omega}$ the distance $\tilde{\mathsf{d}}$ is defined by $\tilde{\mathsf{d}}(x, y) := \mathsf{d}_\times((x, u(x)), (y, u(y))$ and for every Borel subset $B \subset \Omega$ (w.r.t. d) the measure $\tilde{\mathfrak{m}}$ is given by

$$\tilde{\mathfrak{m}}(B) := P(\mathsf{Epi}(u), B \times \mathbb{R}).$$

Note that this space is not complete if u is not continuous. For a function $f : \tilde{\Omega} \to \mathbb{R}$ we will denote the local Lipschitz constant w.r.t. the distance $\tilde{\mathsf{d}}$ at a point $x \in \tilde{\Omega}$ by $\widetilde{\mathsf{lip}}(f)(x)$. We denote by $i : \Omega \to \tilde{\Omega}$ the identification map.

Proposition 4.18. *Let $x \in \Omega$ be a point as in Theorem 4.15 and let $(\epsilon_n, \psi_n, u^\infty)$ be the strong blow-up of u at x given by such theorem. Let $j : \mathbb{R}^k \to \mathsf{Graph}(u^\infty) \subset \mathbb{R}^k \times \mathbb{R}$ be the projection on the graph, i.e. $j(x) := (x, u^\infty(x))$. Then there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ decreasing to zero such that the maps*

$$\psi'_n := i \circ \psi_n \circ j^{-1} : j(\bar{B}_1^{\mathbb{R}^k}(0)) \to (i(\bar{B}_{\epsilon_n}^{\mathsf{X}}(x)), \epsilon_n^{-1}\tilde{\mathsf{d}})$$

are δ_n-GH maps.

Proof. We first prove that for every $\delta > 0$, if n is sufficiently large $\psi'_n(j(\bar{B}_1^{\mathbb{R}^k}(0)))$ is a δ-net in $(i(\bar{B}_{\epsilon_n}^{\mathsf{X}}(x)), \epsilon_n^{-1}\tilde{\mathsf{d}})$. To this aim observe that $\psi'_n(j(\bar{B}_1^{\mathbb{R}^k}(0))) = i(\psi_n(\bar{B}_1^{\mathbb{R}^k}(0)))$, so that since (ϵ_n, ψ_n) is a blow-up of X in x we only need to show that the map

$$i : (\bar{B}_{\epsilon_n}(x), \epsilon_n^{-1}\mathsf{d}) \to (i(\bar{B}_{\epsilon_n}(x)), \epsilon_n^{-1}\tilde{\mathsf{d}})$$

sends δ'-nets to δ-nets for δ' sufficiently small and n sufficiently large.

In particular it is sufficient to prove that if $a, b \in \bar{B}_{\epsilon_n}^{\mathsf{X}}(x)$ and $\epsilon_n^{-1}\mathsf{d}(a, b) < \delta'$ then

$$\epsilon_n^{-1}\tilde{\mathsf{d}}(i(a), i(b)) \le \epsilon_n^{-1}\mathsf{d}(a, b) + 2\mathsf{lip}(u^\infty)\delta'.$$

To this aim suppose by contradiction that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \bar{B}_{\epsilon_n}(x)$ such that $\epsilon_n^{-1}\mathsf{d}(a, b) < \delta'$ and

$$|\epsilon_n^{-1}\tilde{\mathsf{d}}(i(a_n), i(b_n)) - \epsilon_n^{-1}\mathsf{d}(a_n, b_n)| > 2\mathsf{lip}(u^\infty)\delta'.$$

By definition of $\tilde{\mathsf{d}}$ we have that

$$|\epsilon_n^{-1}\tilde{\mathsf{d}}(i(a_n), i(b_n)) - \epsilon_n^{-1}\mathsf{d}(a_n, b_n)| \le \epsilon_n^{-1}|u(a_n) - u(b_n)|,$$

and since u has strong blow-up $(\epsilon_n, \psi_n, u^\infty)$ in x we can assume (modulo modifying the GH maps ψ_n) that $a_n, b_n \in \mathsf{Im}(\psi_n)$, so that $\epsilon_n^{-1}|u(a_n) - u(b_n)|$ can be made arbitrarily close to $|u^\infty(a'_n) - u^\infty(b'_n)|$ where $\psi_n(a'_n) = a_n$ and $\psi_n(b'_n) = b_n$. But if n is large enough ψ_n will be a δ'-GH map so that

$$\begin{aligned}|u^\infty(a'_n) - u^\infty(b'_n)| &\le \mathsf{lip}(u^\infty)\mathsf{d}_e(a'_n, b'_n)\\ &\le \mathsf{lip}(u^\infty)(\epsilon_n^{-1}\mathsf{d}(a_n, b_n) + \delta') \le 2\mathsf{lip}(u^\infty)\delta',\end{aligned}$$

a contradiction. This concludes the proof that for every $\delta > 0$, if n is sufficiently large $\psi'_n(j(\bar{B}_1^{\mathbb{R}^k}(0)))$ is a δ-net in $(i(\bar{B}_{\epsilon_n}^{\mathsf{X}}(x)), \epsilon_n^{-1}\tilde{\mathsf{d}})$

Now we need to show that ψ'_n almost preserves distances. We pick two points $(x_1, u^\infty(x_1))$ and $(x_2, u^\infty(x_2))$ in the domain of ψ'_n and we need to compute

$$\epsilon_n^{-1}\tilde{\mathsf{d}}(\psi'_n((x_1, u^\infty(x_1))), \psi'_n((x_2, u^\infty(x_2)))).$$

The previous expression is equal to

$$\begin{aligned}&\epsilon_n^{-1}\mathsf{d}_\times((\psi_n(x_1), u(\psi_n(x_1))), (\psi_n(x_2), u(\psi_n(x_2))))\\ &= \sqrt{\epsilon_n^{-2}\mathsf{d}(\psi_n(x_1), \psi_n(x_2))^2 + \epsilon_n^{-2}|u(\psi_n(x_1)) - u(\psi_n(x_2))|^2},\end{aligned}$$

so that using the inequality $|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}| \le |a-c| + |b-d|$ we obtain

$$\begin{aligned}&|\epsilon_n^{-1}\tilde{\mathsf{d}}(\psi'_n((x_1, u^\infty(x_1))), \psi'_n((x_2, u^\infty(x_2)))) - \mathsf{d}_\times((x_1, u^\infty(x_1)), (x_2, u^\infty(x_2)))|\\ &\le |\epsilon_n^{-1}\mathsf{d}(\psi_n(x_1), \psi_n(x_2)) - \mathsf{d}_e(x_1, x_2)| + |\epsilon_n^{-1}u(\psi_n(x_1)) - \epsilon_n^{-1}u(\psi_n(x_2)) - u^\infty(x_1) + u^\infty(x_2)|\end{aligned}$$

and the last term is going uniformly to 0 as n increases, concluding the proof. □

Definition 4.19. Let (Y, d) be a metric space and $f, g : \mathsf{Y} \to \mathbb{R}$ be functions with finite local Lipschitz constant in $y \in \mathsf{Y}$. We define

$$\mathsf{lip}(f) \cdot \mathsf{lip}(g) := \frac{1}{4}(\mathsf{lip}(f+g)^2 - \mathsf{lip}(f-g)^2).$$

Observe that the object defined in the previous definition may fail to be a quadratic form on generic metric spaces. This additional regularity, in our setting, will be a consequence of Corollary 4.23.

Proposition 4.20. *Let $\theta \in \mathsf{W}^{1,1}_{loc}(\Omega)$. For $\mathfrak{m}$-a.e. $x \in \Omega$ such that for every $(\epsilon_n, \psi_n, u^\infty)$ strong linear blow-up of u at x, modulo passing to a subsequence, θ has strong linear blow-up $(\epsilon_n, \psi_n, \theta^\infty)$, we have*

$$\widetilde{\mathsf{lip}}(\theta \circ i^{-1})^2 \circ i = |\nabla\theta|^2 - \frac{(\nabla\theta \cdot \nabla u)^2}{W^2}.$$

Proof. Let x be a point as in the statement and assume in addition that $\mathsf{Epi}(u)$ admits tangent balls at $(x, u(x))$. Let $\{y_n\}_{n\in\mathbb{N}} \subset \tilde{\Omega}$ be such that $\tilde{\mathsf{d}}(i(x), y_n) \to 0$ and

$$\widetilde{\mathsf{lip}}(\theta \circ i^{-1})(i(x)) = \lim_{n\to+\infty} \frac{|\theta(y_n) - \theta(i(x))|}{\tilde{\mathsf{d}}(i(x), y_n)},$$

and define $\epsilon_n := \tilde{\mathsf{d}}(i(x), y_n)$. Let $(\epsilon_n, \psi_n, u^\infty)$ be the strong linear blow-up of u at x given by Theorem 4.15 so that, modulo passing to a subsequence also θ admits strong blow-up $(\epsilon_n, \psi_n, \theta^\infty)$. Moreover thanks to Proposition 4.5 $\theta + u$ and $\theta - u$ admit strong blow-ups given respectively by $(\epsilon_n, \psi_n, \theta^\infty + u^\infty)$ and $(\epsilon_n, \psi_n, \theta^\infty - u^\infty)$.

In particular by Proposition 4.8, modulo asking that x is out of an appropriate $\mathfrak{m}$-negligible set, we have that

$$|\nabla\theta|(x) = \mathsf{lip}(\theta^\infty), \quad |\nabla u|(x) = \mathsf{lip}(u^\infty),$$

$$|\nabla(\theta+u)|(x) = \mathsf{lip}(\theta^\infty + u^\infty), \quad \text{and} \quad |\nabla(\theta-u)|(x) = \mathsf{lip}(\theta^\infty - u^\infty).$$

This implies in particular that $(\nabla\theta\cdot\nabla u)(x) = \mathsf{lip}(\theta^\infty)\cdot\mathsf{lip}(u^\infty)$. Let j be the map given in the statement of Proposition 4.18. We claim that $\theta\circ i^{-1} : i(\Omega)\to\mathbb{R}$ admits strong blow-up $(\epsilon_n, i\circ\psi_n\circ j^{-1}, \theta^\infty\circ j^{-1})$ at $i(x)$. Assume for the moment that the claim holds. Then, thanks to the choice of ϵ_n, using Corollary 4.7 we obtain that

$$\widetilde{\mathsf{lip}}(\theta\circ i^{-1})^2(i(x)) = \mathsf{lip}(\theta^\infty\circ j^{-1})^2.$$

By doing a simple computation in $\mathbb{R}^k$ one realizes that

$$\mathsf{lip}(\theta^\infty\circ j^{-1})^2 = \mathsf{lip}(\theta^\infty)^2 - \frac{(\mathsf{lip}(\theta^\infty)\cdot\mathsf{lip}(u^\infty))^2}{\mathsf{lip}(u^\infty)^2}$$

and this last expression, thanks to our previous remarks, coincides with

$$|\nabla\theta|^2 - \frac{(\nabla\theta\cdot\nabla u)^2}{W^2},$$

evaluated in x, concluding the proof.

To prove the claim we observe that by the previous proposition the maps $i\circ\psi_n\circ j^{-1} : j(\bar{B}_1(0)) \to i^{-1}(\bar{B}_{\epsilon_n}(x))$ are GH approximations and that

$$\begin{aligned}&\|\epsilon_n^{-1}(\theta\circ i^{-1}\circ i\circ\psi_n\circ j^{-1} - \theta(x)) - \theta^\infty\circ j^{-1}\|_{\mathsf{L}^\infty(j(\bar{B}_1^{\mathbb{R}^k}(0)))}\\ &= \|\epsilon_n^{-1}(\theta\circ\psi_n - \theta(x)) - \theta^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^k}(0))} \to 0.\end{aligned}$$

On the other hand if $\{\widetilde{\psi}_n\}_{n\in\mathbb{N}}$ is another sequence of GH maps between $j(\bar{B}_1(0))$ and $i^{-1}(\bar{B}_{\epsilon_n}(x))$ such that each $\widetilde{\psi}_n$ differs from $i\circ\psi_n\circ j^{-1}$ on a finite number of points, then we can write $\widetilde{\psi}_n = i\circ\psi'_n\circ j^{-1}$, where ψ'_n differs from ψ_n on a finite number of points as well. As a consequence we have

$$\begin{aligned}&\|\epsilon_n^{-1}(\theta\circ i^{-1}\circ\widetilde{\psi}_n - \theta(x)) - \theta^\infty\circ j^{-1}\|_{\mathsf{L}^\infty(j(\bar{B}_1^{\mathbb{R}^k}(0)))}\\ &= \|\epsilon_n^{-1}(\theta\circ\psi'_n - \theta(x)) - \theta^\infty\|_{\mathsf{L}^\infty(\bar{B}_1^{\mathbb{R}^k}(0))} \to 0,\end{aligned}$$

proving the claim. To be precise what we proved is not existence of a strong blow-up because $j(\bar{B}_1^{\mathbb{R}^k}(0))$ is not a ball w.r.t. $\mathsf{d}_\times$ on the graph $\mathsf{Graph}(u^\infty)$, but this is still sufficient to conclude applying Proposition 4.6 instead of Corollary 4.7. ☐

Proposition 4.21. *Let $\theta_1,\theta_2\in\mathsf{W}^{1,1}_{loc}(\Omega)$. For $\mathfrak{m}$-a.e. $x\in\Omega$ such that for every $(\epsilon_n,\psi_n,u^\infty)$ strong linear blow-up of u at x, modulo passing to a subsequence, there exist strong linear blow-ups $(\epsilon_n,\psi_n,\theta_1^\infty)$ and $(\epsilon_n,\psi_n,\theta_2^\infty)$, we have*

$$\left(\widetilde{\mathsf{lip}}(\theta_1\circ i^{-1})\cdot\widetilde{\mathsf{lip}}(\theta_2\circ i^{-1})\right)\circ i = \nabla\theta_1\cdot\nabla\theta_2 - \frac{1}{W^2}(\nabla\theta_1\cdot\nabla u)(\nabla\theta_2\cdot\nabla u).$$

Proof. By Proposition 4.20 for $\mathfrak{m}$-a.e. point as in the statement we have

$$\begin{aligned}\left(\widetilde{\mathsf{lip}}(\theta_1\circ i^{-1})\cdot\widetilde{\mathsf{lip}}(\theta_2\circ i^{-1})\right)\circ i &= \frac{1}{4}\Big((\widetilde{\mathsf{lip}}((\theta_1+\theta_2)\circ i^{-1})^2 - (\widetilde{\mathsf{lip}}((\theta_1-\theta_2)\circ i^{-1}))^2)\Big)\circ i\\ &= \frac{1}{4}\Big[|\nabla(\theta_1+\theta_2)|^2 - \Big(\nabla(\theta_1+\theta_2)\cdot\frac{\nabla u}{W}\Big)^2 - |\nabla(\theta_1+\theta_2)|^2 + \Big(\nabla(\theta_1-\theta_2)\cdot\frac{\nabla u}{W}\Big)^2\Big]\\ &= \nabla\theta_1\cdot\nabla\theta_2 - \frac{1}{W^2}(\nabla\theta_1\cdot\nabla u)(\nabla\theta_2\cdot\nabla u).\end{aligned}$$

☐

Corollary 4.22. *Let $A\subset\Omega$ be a Borel set, then*

$$\int_{i(A)}\widetilde{\mathsf{lip}}(u\circ i^{-1})\,d\tilde{\mathfrak{m}} = \int_A|\nabla u|\,d\mathfrak{m}.$$

Proof. From the previous proposition we obtain that $\widetilde{\mathsf{lip}}(u \circ i^{-1}) \circ i = |\nabla u|/W$ for $\mathfrak{m}$-a.e. $x \in \Omega$ and since W is also the density of $\tilde{\mathfrak{m}}$ w.r.t. $\mathfrak{m}$ we conclude. $\square$

We will denote by $\mathcal{A}$ the set of functions $\phi \in \mathsf{W}^{1,1}_{loc}(\Omega)$ such that for $\mathfrak{m}$-a.e. $x \in \Omega$ for every strong linear blow-up $(\epsilon_n, \psi_n, u^\infty)$ of u at x there exists, modulo passing to a subsequence, a strong blow-up $(\epsilon_n, \psi_n, \phi^\infty)$ of ϕ at x. Observe that $\mathcal{A}$ contains the algebra generated by $\mathsf{Lip}_{loc}(\Omega)$, if $f \in \mathcal{A}$ and $g \in \mathsf{Lip}_{loc}(\Omega)$ then $fg \in \mathcal{A}$, and $\mathcal{A}$ is closed under compositions with smooth functions. We denote by $\mathcal{A}_c$ the set of functions in $\mathcal{A}$ having compact support in Ω. We will also denote by $\mathsf{L}^0(\Omega)$ the space of $\mathfrak{m}$-a.e. real valued measurable functions on Ω.

Corollary 4.23. *The function* $\cdot : \mathcal{A} \times \mathcal{A} \to \mathsf{L}^0(\Omega)$ *given by* $(\phi_1, \phi_2) \mapsto \widetilde{\mathsf{lip}}(\phi_1) \cdot \widetilde{\mathsf{lip}}(\phi_2)$ *is symmetric, bilinear, it satisfies the chain rule and the Leibniz rule in both entries and* $\widetilde{\mathsf{lip}}(\phi_1) \cdot \widetilde{\mathsf{lip}}(\phi_2) \leq \widetilde{\mathsf{lip}}(\phi_1)\widetilde{\mathsf{lip}}(\phi_2)$.

Proof. The fact that $\widetilde{\mathsf{lip}}(\phi_1) \cdot \widetilde{\mathsf{lip}}(\phi_2) \leq \widetilde{\mathsf{lip}}(\phi_1)\widetilde{\mathsf{lip}}(\phi_2)$ follows from the fact that $\widetilde{\mathsf{lip}}(\phi_1 + \phi_2) \leq \widetilde{\mathsf{lip}}(\phi_1) + \widetilde{\mathsf{lip}}(\phi_2)$ and $\widetilde{\mathsf{lip}}(\phi_1) \leq \widetilde{\mathsf{lip}}(\phi_1 - \phi_2) + \widetilde{\mathsf{lip}}(\phi_2)$. All the other properties follow from the representation given by Proposition 4.21. $\square$

Definition 4.24. We define $\eta_{x_0} : \Omega \to \mathbb{R}$ by

$$\eta_{x_0}(x) := \mathsf{d}(x, x_0) + |u(x) - u(x_0)|$$

and $\eta^{R,r}_{x_0} : \Omega \to \mathbb{R}$ by

$$\eta^{R,r}_{x_0}(x) := 1 \wedge \Big(\frac{R}{R-r} - \frac{1}{R-r}\eta_{x_0}(x) \Big) \vee 0.$$

Proposition 4.25. *We have that* $\eta_{x_0} \in \mathcal{A}$ *and if* $R < \mathsf{d}(\partial\Omega, x_0)$ *then* $\eta^{R,r}_{x_0} \in \mathcal{A}_c$.

Proof. We only show that $\eta_{x_0} \in \mathcal{A}$, as $\eta^{R,r}_{x_0} \in \mathcal{A}_c$ then follows trivially.

For $\mathfrak{m}$-a.e. $x \in \Omega$ the function u is continuous. So suppose that we are in such a continuity point x and that $u(x) > u(x_0)$. Let $(\epsilon_n, \psi_n, u^\infty)$ be a strong linear blow-up of u at x. By Corollary 4.4 (modulo passing to a subsequence and assuming that x is out of an appropriate $\mathfrak{m}$-negligible set) the distance function $\mathsf{d}(\cdot, x_0)$ admits a strong blow-up $(\epsilon_n, \psi_n, \mathsf{d}^\infty)$, while $|u - u(x_0)| = u - u(x_0)$ locally so that it has strong blow-up $(\epsilon_n, \psi_n, u^\infty)$ by the choice of x. By Proposition 4.5 η_{x_0} has strong blow-up $(\epsilon_n, \psi_n, \mathsf{d}^\infty + u^\infty)$.

If we are in a point where $u(x) < u(x_0)$ the argument is the same, so that we are left with the points where $u(x) = u(x_0)$. Let $(\epsilon_n, \psi_n, u^\infty)$ be a strong linear blow-up of u at x. Existence of the blow-up for $\mathsf{d}(\cdot, x_0)$ follows in the same way as before, while by locality and Proposition 4.8 in $\mathfrak{m}$-a.e. such point $|u - u(x_0)|$ has strong blow-up $(\epsilon_n, \psi_n, 0)$ so that the same argument we used for the previous case allows to conclude in the same way. $\square$

Theorem 4.26. *Let* $\phi \in \mathcal{A}_c$*, then* $\widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1}) \in \mathsf{L}^1(\tilde{\Omega})$ *and*

$$\int_{\tilde{\Omega}} \widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1})\, d\tilde{\mathfrak{m}} = 0.$$

Proof. By Proposition 4.21 we get

$$\Big(\widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1})\Big) \circ i = \frac{\nabla\phi \cdot \nabla u}{W^2} \quad \mathfrak{m}\text{-a.e.},$$

so that

$$\int_{\tilde{\Omega}} |\widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1})|\, \mathsf{d}\tilde{\mathfrak{m}} \leq \int_\Omega \frac{|\nabla\phi \cdot \nabla u|}{W}\, d\mathfrak{m} \leq \int_\Omega |\nabla\phi|\, d\mathfrak{m} < +\infty,$$

giving that $\widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1}) \in \mathsf{L}^1(\tilde{\Omega})$. Hence

$$\int_{\tilde{\Omega}} \widetilde{\mathsf{lip}}(\phi \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1})\, d\tilde{\mathfrak{m}} = \int_\Omega \frac{\nabla\phi \cdot \nabla u}{W}\, d\mathfrak{m} = 0.$$

$\square$

4.4 The Graph of u

In this section we define the Graph Space of u as anticipated in the introduction and we derive Theorems 5 and 6 from their analogues (i.e. Corollary 4.23 and Theorem 4.26) in the previous section.

Definition 4.27. Let $\mathsf{G}(u) \subset \Omega \times \mathbb{R}$ be the boundary of $\mathsf{Epi}(u)$. We define the metric measure space $(\mathsf{G}(u), \mathsf{d}_g, \mathfrak{m}_g)$, where d_g is the product distance of $\Omega \times \mathbb{R}$ restricted to $\mathsf{G}(u) \times \mathsf{G}(u)$ and $\mathfrak{m}_g$ is the perimeter measure induced by $\mathsf{Epi}(u)$ on the Borel subsets of $\mathsf{G}(u)$. We denote by lip_g the local Lipschitz constant of a function $f : \mathsf{G}(u) \to \mathbb{R}$.

Remark 4.28. The function $i_g : \Omega \to G(u)$ given by $i_g(x) := (x, u(x))$ preserves distances, measures and its image has full measure in $\mathsf{G}(u)$. Nevertheless the spaces $\tilde{\Omega}$ and $\mathsf{G}(u)$ are not identifiable as the former may not be complete. On the other hand if u is continuous the two spaces coincide.

Lemma 4.29. $\mathfrak{m}_g(\mathsf{G}(u) \setminus \mathsf{Graph}(u)) = 0$.

Proof. Let $C \subset \Omega$ be the set of continuity points of u and for every $A \subset \Omega$ set $G_A := \{(x, u(x)) \in \mathsf{G}(u) : x \in A\}$. Observe that since $P(\mathsf{Epi}(u), \cdot)$ is concentrated on $\mathsf{G}(u)$ and $\mathsf{G}(u) \cap (A \cap C) \times \mathbb{R} = G_{A\cap C}$ we have

$$P(\mathsf{Epi}(u), G_{A\cap C}) = P(\mathsf{Epi}(u), (A \cap C) \times \mathbb{R}).$$

As a consequence, given an open precompact set $A \subset \Omega$, we have

$$\begin{aligned} P(\mathsf{Epi}(u), A \times \mathbb{R}) - P(\mathsf{Epi}(u), G_{A\cap C}) &= P(\mathsf{Epi}(u), A \times \mathbb{R}) - P(\mathsf{Epi}(u), (A \cap C) \times \mathbb{R}) \\ &= P(\mathsf{Epi}(u), (A \setminus C) \times \mathbb{R}) = \int_{A\setminus C} \sqrt{1 + |\nabla u|^2}\, d\mathfrak{m} = 0. \end{aligned}$$

This implies that $P(\mathsf{Epi}(u), {}^cG_C) = 0$, which in turn implies that $\mathfrak{m}_g(\mathsf{G}(u) \setminus \mathsf{Graph}(u)) = 0$. $\square$

Lemma 4.30. *Let $\phi : \mathsf{G}(u) \to \mathbb{R}$ be a Lipschitz function (w.r.t. d_g). For $\mathfrak{m}$-a.e. $x \in \Omega$ we have $\mathsf{lip}_g(\phi)(i_g(x)) = \widetilde{\mathsf{lip}}(\phi \circ i_g \circ i^{-1})(i(x))$.*

Proof. For every x the inequality

$$\mathsf{lip}_g(\phi)(i_g(x)) \geq \widetilde{\mathsf{lip}}(\phi \circ i_g \circ i^{-1})(i(x))$$

is trivial so we only need to prove the opposite one.

To this aim fix a point x where u admits strong linear blow-up $(\epsilon_n, \psi_n, u^\infty)$ as in Theorem 4.15 and let $\{(x_n, t_n)\}_{n\in\mathbb{N}} \subset \mathsf{G}(u)$ be a sequence of points such that $(x_n, t_n) \to (x, u(x))$ and

$$\mathsf{lip}_g(\phi)(i_g(x)) = \lim_{n\to+\infty} \frac{|\phi(x_n, t_n) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, t_n), (x, u(x))}.$$

We claim that, modulo passing to a subsequence,

$$\lim_{n\to+\infty} \frac{\mathsf{d}_g((x_n, t_n), (x, u(x))}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))} = 1. \tag{13}$$

To prove the claim we define $\epsilon_n := \mathsf{d}(x_n, x)$ and note that

$$\frac{\epsilon_n^{-1}\mathsf{d}_g((x_n, t_n), (x, u(x))}{\epsilon_n^{-1}\mathsf{d}_g((x_n, u(x_n)), (x, u(x))} = \frac{\sqrt{1 + \epsilon_n^{-2}|t_n - u(x)|^2}}{\sqrt{1 + \epsilon_n^{-2}|u(x_n) - u(x)|^2}}.$$

Let now $v : \Omega \to \mathbb{R}$ be given by

$$v(y) := \begin{cases} u(y) & y \notin \{x_n\}_{n\in\mathbb{N}} \\ t_n & y = x_n. \end{cases}$$

It is clear that v is a representative of u in the sense of Section 4.2, so that by Theorem 4.15 both v and u admit strong blow-up $(\epsilon_n, \psi_n, u^\infty)$ at x.

By Lemma 2.9, modulo modifying the maps ψ_n, we can also suppose that $x_n \in \mathsf{Im}(\psi_n)$. Let $y_n \in \bar{B}_1^{\mathbb{R}^k}(0)$ be such that $\psi_n(y_n) = x_n$ and, modulo passing to a subsequence, let $y_n \to y$. We then get that $\epsilon_n^{-1}|u(x_n) - u(x)| \to u^\infty(y)$ and $\epsilon_n^{-1}|t_n - u(x)| \to u^\infty(y)$ as well, proving our claim.

We now claim that

$$\lim_{n\to+\infty} \frac{|\phi(x_n, t_n) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))} = \lim_{n\to+\infty} \frac{|\phi(x_n, u(x_n)) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))}. \tag{14}$$

Since ϕ is Lipschitz and $\mathsf{d}_g((x_n, u(x_n)), (x, u(x)) \geq \mathsf{d}(x_n, x)$ we have that

$$\begin{aligned} \frac{|\phi(x_n, u(x_n)) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))|} - \frac{\mathsf{lip}_g(\phi)|t_n - u(x_n)|}{\mathsf{d}(x_n, x)} & \\ \leq \frac{|\phi(x_n, t_n) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))} & \\ \leq \frac{|\phi(x_n, u(x_n)) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))|} + \frac{\mathsf{lip}_g(\phi)|t_n - u(x_n)|}{\mathsf{d}(x_n, x)}&. \end{aligned}$$

The same argument of the previous claim allows then to show that

$$\frac{|t_n - u(x_n)|}{\mathsf{d}(x_n, x)} \to 0$$

so that (14) is verified. From (13) and (14) we deduce

$$\mathsf{lip}_g(\phi)(i_g(x)) = \lim_{n\to+\infty} \frac{|\phi(x_n, u(x_n)) - \phi(x, u(x))|}{\mathsf{d}_g((x_n, u(x_n)), (x, u(x))} \leq \widetilde{\mathsf{lip}}(\phi \circ i_g \circ i^{-1})(i(x)),$$

as desired. □

We define $\mathcal{A}^g := \{\phi \in \mathsf{Lip}_{loc}(\mathsf{G}(u)) : \phi \circ i_g \in \mathcal{A}\}$ and $\mathcal{A}^g_c := \{\phi \in \mathsf{Lip}_{loc}(\mathsf{G}(u)) : \phi \circ i_g \in \mathcal{A}_c\}$. Note that $\mathcal{A}^g_c$ is the set of functions in $\mathcal{A}^g$ that have compact support in $\mathsf{G}(u)$. These families are closed under the same operations as the family $\mathcal{A}$.

Theorem 5. *The function* $\cdot : \mathcal{A}^g \times \mathcal{A}^g \to \mathsf{L}^1_{loc}(\mathsf{G}(u))$ *given by*

$$(\phi_1, \phi_2) \mapsto \mathsf{lip}_g(\phi_1) \cdot \mathsf{lip}_g(\phi_2)$$

is symmetric, bilinear, it satisfies the chain rule and the Leibniz rule in both entries and $\mathsf{lip}_g(\phi_1) \cdot \mathsf{lip}_g(\phi_2) \leq \mathsf{lip}_g(\phi_1)\mathsf{lip}_g(\phi_2)$.

Proof. Combine Lemma 4.30 and Corollary 4.23. □

The property of the previous theorem is related to the notion of Lipschitz-infinitesimally Hilbertian space introduced in [40]. Using that if $u \in \mathsf{Lip}_{loc}(\Omega)$, then $\mathsf{Lip}_c(\mathsf{G}(u)) \subset \mathcal{A}^g_c$, one can check that under the local Lipschitzianity assumption on u the space $\mathsf{G}(u)$ is Lipschitz-infinitesimally Hilbertian.

We now give some definitions that will be very useful also for the next section

Definition 4.31. Let $(x, t), (y, s) \in \Omega \times \mathbb{R}$ and define $\rho_{(x,t)}((y, s)) := \mathsf{d}(x, y) + |t - s|$. Given $\bar{x} \in \Omega \times \mathbb{R}$ we then set $D_{\bar{x},r} = \{\bar{y} \in \Omega \times \mathbb{R} : \rho_{\bar{x}}(\bar{y}) < r\}$ and

$$A_{\bar{x},s} := \{x \in \Omega : \{x\} \times \mathbb{R} \cap \mathsf{G}(u) \cap D_{\bar{x},s} \neq \emptyset\}.$$

Given $R > r > 0$ real numbers and $x_0 \in \Omega$ we set $\eta^{R,r}_{x_0,g} : \mathsf{G}(u) \to \mathbb{R}$

$$\eta^{R,r}_{x_0,g}(\bar{x}) := 1 \wedge \Big(\frac{R}{R-r} - \frac{1}{R-r}\rho_{(x_0,u(x_0))}(\bar{x})\Big) \vee 0.$$

Observe that $\eta^{R,r}_{x_0,g} \circ i_g = \eta^{R,r}_{x_0}$ so that $\eta^{R,r}_{x_0,g} \in \mathcal{A}^g_c$ if $R < \mathsf{d}(\partial\Omega, x_0)$. Finally we define $u_g : \mathsf{G}(u) \to \mathbb{R}$ as $u_g(x, t) = t$. Note that $u_g \circ i_g = u$.

Theorem 6. *Let $\phi \in \mathcal{A}_c^g$, then $\mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g) \in \mathsf{L}^1(\mathsf{G}(u))$ and*

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g = 0.$$

Proof. Since $\phi \in \mathsf{Lip}_c(\mathsf{G}(u))$ and $u_g \in \mathsf{Lip}(\mathsf{G}(u))$, then $\mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g) \in \mathsf{L}^1(\mathsf{G}(u))$ trivially. By Lemmas 4.29 and 4.30 we have

$$\begin{aligned}\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g &= \int_{\mathsf{Graph}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g \\ &= \int_{\tilde{\Omega}} \widetilde{\mathsf{lip}}(\phi \circ i_g \circ i^{-1}) \cdot \widetilde{\mathsf{lip}}(u \circ i^{-1})\, d\tilde{\mathfrak{m}}\end{aligned}$$

and by Theorem 4.26 this last term equals zero. □

Corollary. *Let $u \in \mathsf{Lip}_{loc}(\Omega)$ be a solution of the minimal surface equation, then for every $\phi \in \mathsf{Lip}_c(\mathsf{G}(u))$ we have*

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(u_g)\, d\mathfrak{m}_g = 0.$$

Proof. Since $u \in \mathsf{Lip}_{loc}(\Omega)$ we have that $\mathsf{Lip}_c(\mathsf{G}(u)) \subset \mathcal{A}_c^g$, concluding the proof. □

Using the chain rule, the Leibniz rule, the fact that $\eta_{x_0,g}^{R,r} \in \mathcal{A}_c^g$ and applying the previous theorem we also get the following corollary. This is the version of 'integration by parts' that we will use repeatedly in the next section.

Corollary 4.32. *Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial and let $f(x,y) := xp(x,y)$ and $\phi := f(\eta_{x_0,g}^{R,r}, u_g)$ with $R < \mathsf{d}(\partial\Omega, x_0)$. Let $h = g \circ u_g$ with $g : \mathbb{R} \to \mathbb{R}$ smooth. Then*

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(\phi) \cdot \mathsf{lip}_g(h)\, d\mathfrak{m}_g = -\int_{\mathsf{G}(u)} \mathsf{lip}_g(u_g)^2 h''(u_g)\phi\, d\mathfrak{m}_g.$$

5 Bernstein Property

In this section, we work on a fixed $\mathsf{RCD}(0,N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. The first subsection contains the proof of Theorem 1 (in a stronger version, see Theorem 5.9), while the second one contains the proofs of Theorems 2 and 3 (as anticipated we prove the stronger version of Theorem 3 given by Corollary 5.13).

5.1 Harnack inequality on $\mathsf{G}(u)$

In this section we assume that $u \in \mathsf{W}^{1,1}_{loc}(B_R(p))$ is a solution of the minimal surface equation. The goal is to mimic the strategy used in [21] to prove the Harnack inequality for u_g on $\mathsf{G}(u)$ i.e. Theorem 5.8. The challenge in adapting the aforementioned strategy is to prove that u_g is harmonic on $\mathsf{G}(u)$ and this was done in Section 4.4. Besides harmonicity on the graph, the other key steps are the validity of Poincaré and Sobolev inequalities on $\mathsf{G}(u)$ i.e. Theorems 5.6 and 5.7.

These theorems can be obtained in our setting with the same ideas used in [21]. For this reason we only give a detailed proof of Theorem 5.6, so that one sees what changes need to be made from the corresponding proof in [21]; the same exact changes allow then to obtain also the Sobolev inequality from the analogous result in the aforementioned work. Once the harmonicity of u on $\mathsf{G}(u)$ and the Poincaré and Sobolev inequalities are established, the Harnack inequality follows formally repeating the same argument of [21] and for this reason the proof of Theorem 5.8 is only sketched.

We will often use the notation $\bar{x} = (x, t_x) \in \mathsf{X} \times \mathbb{R}$ when considering points in the product space. Moreover, we will use extensively the notation introduced in Definition 4.31 concerning sets of the type $D_{\bar{x},r}$ and $A_{\bar{x},s}$.

Lemma 5.1. *There exists a constant $C > 0$ depending only on N such that for every $\bar{x} \in \mathsf{G}(u)$ and $0 < t, r, s < R - \mathsf{d}(x,p)$ with $r > s$ we have*

$$\mathfrak{m}_g(D_{\bar{x},s}) \geq C\mathfrak{m}_g(D_{\bar{x},r})\frac{s^N}{r^N}$$

and

$$\frac{1}{C}\mathfrak{m}(B_t(x)) \leq \mathfrak{m}_g(D_{\bar{x},t}) \leq C\mathfrak{m}(B_t(x)).$$

Proof. From Theorem 2.33, the fact that $\mathfrak{m}_\times$ is doubling and the fact that the ball $B_s(\bar{x}) \subset \mathsf{X} \times \mathbb{R}$ contains the rectangle $B_{s/2}(x) \times (t_x - s/2, t_x + s/2)$ we get

$$\mathfrak{m}_g(D_{\bar{x},s}) \geq \mathfrak{m}_g(B_{s/2}(\bar{x})) \geq c_1 \frac{2\mathfrak{m}_\times(B_{s/2}(\bar{x}))}{s} \geq c_2 \frac{\mathfrak{m}_\times(B_s(\bar{x}))}{s}$$
$$\geq c_2 \frac{\mathfrak{m}(B_{s/2}(x))s/2}{s} \geq c_3 \mathfrak{m}(B_s(x)).$$

Since $\mathfrak{m}$ is doubling, by a standard argument (see [21, Equation (2.1)]), we obtain

$$c_3 \mathfrak{m}(B_s(x)) \geq c_4 \mathfrak{m}(B_r(x)) \frac{s^N}{r^N}.$$

Using the same tools as in the first part of the proof we then obtain that

$$c_4 \mathfrak{m}(B_r(x)) \frac{s^N}{r^N} \geq c_5 \mathfrak{m}(B_{2r}(x)) \frac{s^N}{r^N} \geq c_6 \frac{\mathfrak{m}_\times(B_r(\bar{x}))}{r} \frac{s^N}{r^N} \geq c_7 \mathfrak{m}_g(D_{\bar{x},r}) \frac{s^N}{r^N},$$

proving the first inequality. Replacing s with t in the first and last chain of inequalities we obtain the second one. □

Definition 5.2. For every $t \in \mathbb{R}$ and $\bar{x} \in \mathsf{G}(u)$ we define

$$E_{\bar{x},t} := \{\bar{y} \in \mathsf{G}(u) : u_g(\bar{y}) > t + u_g(\bar{x})\} \quad \text{and} \quad E'_{\bar{x},t} := \{\bar{y} \in \mathsf{G}(u) : u_g(\bar{y}) < t + u_g(\bar{x})\}.$$

Lemma 5.3. *There exists a constant $C > 0$ depending only on N such that for every $\bar{x} \in \mathsf{G}(u)$ and $r > 0$ with $B_{3r}(x) \subset B_R(p)$ and for λ^1-a.e. $t \in (-r, r)$ we have*

$$\mathfrak{m}(\{u > u_g(\bar{x}) + t\} \cap B_{r-t}(x)) \geq C\, \mathfrak{m}_g(E_t \cap D_{\bar{x},r}). \tag{15}$$

Moreover if

$$\mathfrak{m}_g(E_t \cap D_{\bar{x},r}) \leq \frac{1}{2} \mathfrak{m}_g(D_{\bar{x},r}), \tag{16}$$

then

$$\mathfrak{m}(\{u < u_g(\bar{x}) + t\} \cap B_{r+|t|}(x)) \geq C\, \mathfrak{m}(B_r(x)). \tag{17}$$

Analogous inequalities hold replacing E_t with E'_t and reversing the inequality signs appearing in the left hand sides of inequalities (15) *and* (17).

Proof. We first prove (17). We denote $\bar{x}_t := (x, t_x + t)$ and we consider the compact set $V \subset \mathsf{X} \times \mathbb{R}$ given by the closure of

$$\{\bar{y} \in D_{\bar{x}_t, r+|t|} : u(y) < t_x + t,\, t_y \in (u(y), t_x + t)\},$$

and we define the competitor $C := \mathsf{Epi}'(u) \cup V$. Since $\mathsf{Epi}'(u)$ minimizes the perimeter we have

$$P(\mathsf{Epi}'(u), D_{\bar{x},3r}) \leq P(C, D_{\bar{x},3r}). \tag{18}$$

We now claim that

$$P(\mathsf{Epi}'(u), (\mathsf{G}(u) \setminus \partial V) \cap D_{\bar{x},3r}) = P(C, (\partial C \setminus \partial V) \cap D_{\bar{x},3r}). \tag{19}$$

To prove the claim observe that by the definition of V we have $\partial C \setminus \partial V = \mathsf{G}(u) \setminus \partial V$ and, if $q \in \mathsf{G}(u) \setminus \partial V$, then in a small ball centered at q we have that $C = \mathsf{Epi}'(u)$, which implies the claim.

Subtracting (19) from (18) we deduce that

$$P(\mathsf{Epi}'(u), \partial V \cap \mathsf{G}(u) \cap D_{\bar{x},3r}) \leq P(C, \partial V \cap \partial C \cap D_{\bar{x},3r}).$$

Using first the definition of V, then (16) and finally Lemma 5.1 we get

$$P(\mathsf{Epi}'(u), \partial V \cap \mathsf{G}(u) \cap D_{\bar{x},3r}) \geq \mathfrak{m}_g(E'_t \cap D_{\bar{x},r})$$
$$\geq c_1 \mathfrak{m}_g(D_{\bar{x},r}) \geq c_2 \mathfrak{m}(B_r(x)),$$

so that to prove (17) it is enough to show that

$$P(C, \partial V \cap \partial C \cap D_{\bar{x},3r}) \leq c_3 \mathfrak{m}(\{u < u_g(\bar{x}) + t\} \cap B_{r+|t|}(x)).$$

To this aim observe that $\partial V \cap \partial C$ is the disjoint union of

$$A := \mathsf{X} \times \{t_x + t\} \cap \mathsf{Epi}(u) \cap D_{\bar{x}_t, r+|t|},$$

$$B := \mathsf{X} \times (-\infty, t_x + t) \cap \mathsf{Epi}(u) \cap \partial D_{\bar{x}_t, r+|t|},$$

and

$$D := (\bar{A} \setminus A) \cup (\bar{B} \setminus B).$$

If $q \in A$, then in a sufficiently small ball centered at q the set C coincides with the subgraph of the constant function $t + t_x$. This, together with the area formula given by Proposition 2.36, implies that

$$P(C, A) \leq \mathfrak{m}(\{u < t_x + t\} \cap B_{r+|t|}(x)).$$

Similarly if $q \in B$, then in a sufficiently small ball centered at q the set C coincides with the epigraph of a function with Lipschitz constant 1. Moreover if $q \in B$, then its projection on $\mathsf{X} \times \{t + t_x\}$ belongs to A (this is clear from the picture, and depends on the fact that we defined V using $D_{\bar{x}_t, r+|t|}$ and not $D_{\bar{x}, r+|t|}$). This implies that $P(C, B) \leq c_4 P(C, A)$.

Finally for λ^1-a.e. $t \in (-r, r)$ we have that $\mathfrak{m}_g(\mathsf{X} \times \{t_x + t\}) = 0$ and $\mathfrak{m}_g(\partial D_{\bar{x}, r+|t|}) = 0$. Using the area formula given by Proposition 2.36 we get that for every such t we have $\mathfrak{m}(\pi_{\mathsf{X}}(\bar{A} \setminus A)) = \mathfrak{m}(\pi_{\mathsf{X}}(\bar{B} \setminus B)) = 0$, which in turn implies that $P(C, D) = 0$.

In particular

$$P(C, \partial V \cap \partial C) = P(C, A \cup B) \leq c_5 \mathfrak{m}(\{u < t_x + t\} \cap B_{r+|t|}(x)),$$

as desired.

The proof of the other identity follows an identical argument replacing V with the closure of

$$\{\bar{y} \in D_{\bar{x}_t, r-t} : u(y) > t_x + t,\, t_y \in (t_x + t, u(y))\}.$$

In this case since since the r.h.s. is $\mathfrak{m}_g(E_t)$ we don't need to use the condition (16), which was needed to say that $\mathfrak{m}_g(E'_t) \geq c_1 \mathfrak{m}(B_r(x))$. □

The next lemma corresponds to Lemma 3.3 in [21]. Our formulation is slightly different and, later on, it will allow us to avoid the use of the Coarea formula on $\mathsf{G}(u)$, since this tool is a priori not available.

Lemma 5.4. *There exists a constant $C > 0$ depending only on N such that for every $\bar{x} \in \mathsf{G}(u)$ and $r > 0$ with $B_{3r}(x) \subset B_R(p)$ and for λ^1-a.e. $t \in (-r, r)$ if*

$$\mathfrak{m}_g(E_t \cap D_{\bar{x},r}) \leq \frac{1}{2} \mathfrak{m}_g(D_{\bar{x},r}),$$

then

$$\mathfrak{m}_g(E_t \cap D_{\bar{x},r}) \leq C r P(\{u > u_g(\bar{x}) + t\}, A_{\bar{x},3r}).$$

The same statement holds replacing E_t with E'_t and reversing the inequality sign appearing in the right hand side of the last inequality.

Proof. We have

$$3rP(\{u > t_x + t\}, B_{r+|t|}(x)) \geq (r + |t|) P(\{u > t_x + t\}, B_{r+|t|}(x))$$

and thanks to Proposition 2.30 this last quantity is greater than or equal to

$$c_1 \min \{\mathfrak{m}(\{u > t_x + t\} \cap B_{r+|t|}(x)), \mathfrak{m}(\{u \leq t_x + t\} \cap B_{r+|t|}(x))\}. \tag{20}$$

Because of (17) in Lemma 5.3, for λ^1-a.e. $t \in (-r, r)$ the quantity in (20) is greater than or equal to $c_2 \mathfrak{m}(\{u > t + t_x\} \cap B_{r+|t|}(x))$, which is then greater than or equal to $c_3 \mathfrak{m}_g(E_t \cap D_{\bar{x},r})$ by equation (15) in the same lemma. Summing up we proved that

$$\mathfrak{m}_g(E_t \cap D_{\bar{x},r}) \leq c_4 r P(\{u > t_x + t\}, B_{r+|t|}(x)),$$

so that to conclude we only need to show that

$$P(\{u > t_x + t\}, B_{r+|t|}(x)) \le P(\{u > t_x + t\}, A_{\bar{x},3r}).$$

To this aim we will show that $\partial\{u > t_x + t\} \cap B_{r+|t|}(x) \subset A_{\bar{x},3r}$, from which the desired inequality will follow immediately. Let $y \in \partial\{u > t_x + t\} \cap B_{r+|t|}(x)$ and assume by contradiction that $(y, t_x + t) \notin \mathsf{G}(u)$. As $\mathsf{G}(u)$ is closed there exists a small ball centered in $(y, t_x + t)$ which is either fully contained in the interior of $\mathsf{Epi}(u)$ or in the interior of its complement. We only consider the first case, the other one being analogous. Call B the aforementioned ball, and let $\epsilon > 0$ be small enough such that $B_\epsilon(y) \times [t_x + t - \epsilon, t_x + t + \epsilon] \subset B$, so that we deduce that $u < t_x + t - \epsilon$ in $B_\epsilon(y)$, contradicting the fact that $y \in \partial\{u > t_x + t\}$.

We deduced that $(y, t_x + t) \in \mathsf{G}(u)$, so that to conclude we only need to note that $(y, t_x + t) \in D_{\bar{x},3r}$ since $t \in (-r, r)$. □

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth monotone function and define $\phi : \mathsf{G}(u) \to \mathbb{R}$ as $\phi := f \circ u_g$. Let $\bar{x} \in \mathsf{G}(u)$ and $r > 0$ be such that $B_r(x) \subset B_R(p)$, then*

$$\int_{A_{\bar{x},r}} |\nabla(\phi \circ i_g)| \, d\mathfrak{m} = \int_{D_{\bar{x},r}} \mathsf{lip}_g(\phi) \, d\mathfrak{m}_g.$$

Proof. By Lemma 4.29 we know that

$$\mathfrak{m}_g(D_{\bar{x},r} \setminus i_g(i_g^{-1}(D_{\bar{x},r}))) = 0,$$

so that

$$\int_{D_{\bar{x},r}} \mathsf{lip}_g(\phi) \, d\mathfrak{m}_g = \int_{i_g^{-1}(D_{\bar{x},r})} \mathsf{lip}_g(\phi) \circ i_g \, d(\mathfrak{m}_g \circ i_g).$$

Since $\mathfrak{m}_g \circ i_g = \tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{m}}(A_{\bar{x},r} \, \Delta \, i_g^{-1}(D_{\bar{x},r})) = 0$ (since u is continuous $\mathfrak{m}$-a.e.) we get that the previous expression can be rewritten as

$$\int_{A_{\bar{x},r}} \mathsf{lip}_g(\phi) \circ i_g \, d\tilde{\mathfrak{m}}.$$

Thanks to Lemmas 4.30 and Corollary 4.22 this is equivalent to

$$\int_{A_{\bar{x},r}} \widetilde{\mathsf{lip}}(\phi \circ i_g) \, d\tilde{\mathfrak{m}} = \int_{A_{\bar{x},r}} |\nabla(\phi \circ i_g)| \, d\mathfrak{m},$$

concluding the proof. □

Given a function $f \in \mathsf{L}^1(\mathsf{G}(u))$ we use the notation

$$f_{\bar{x},r} := \fint_{D_{\bar{x},r}} f \, d\mathfrak{m}_g.$$

Theorem 5.6. *There exists a constant $C > 0$ depending only on N such that for every $\bar{x} \in \mathsf{G}(u)$ and $r > 0$ with $B_{3r}(x) \subset B_R(p)$ and for every smooth monotone function $f : \mathbb{R} \to \mathbb{R}$, defining $\phi : \mathsf{G}(u) \to \mathbb{R}$ as $\phi := f \circ u_g$, we have*

$$\int_{D_{\bar{x},r}} |\phi - \phi_{\bar{x},r}| \, d\mathfrak{m}_g \le Cr \int_{D_{\bar{x},3r}} \mathsf{lip}_g(\phi) \, d\mathfrak{m}_g.$$

Proof. Observe that since f is smooth and monotone, for every $s \in \mathbb{R}$ we have that $\{\phi > s\} = \{u_g > f^{-1}(s)\}$ (or $\{u_g < f^{-1}(s)\}$ if f is decreasing) and if $A \subset \mathbb{R}$ has full λ^1 measure then also $f^{-1}(A)$ has this property. This will allow us to use Lemma 5.4 in what follows. We assume for simplicity that f is increasing, as the other case is analogous.

Now suppose again for simplicity (the other case being again analogous) that

$$\mathfrak{m}_g(\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r}) \le \mathfrak{m}_g(\{\phi < \phi_{\bar{x},r}\} \cap D_{\bar{x},r}).$$

In this case for every $t \ge 0$ we have

$$\mathfrak{m}_g(\{\phi > \phi_{\bar{x},r} + t\} \cap D_{\bar{x},r}) \le \mathfrak{m}_g(\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r})$$

$$\leq \mathfrak{m}_g(\{\phi < \phi_{\bar{x},r}\} \cap D_{\bar{x},r}) \leq \mathfrak{m}_g(\{\phi < \phi_{\bar{x},r} + t\} \cap D_{\bar{x},r}),$$

so that in particular

$$\mathfrak{m}_g(\{\phi > \phi_{\bar{x},r} + t\} \cap D_{\bar{x},r}) \leq \frac{1}{2}\mathfrak{m}_g(D_{\bar{x},r}).$$

Hence, using first a variation of Fubini's Theorem (see [4, Proposition 1.78]) and then Lemma 5.4 we get

$$\begin{aligned}\int_{\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r}} (\phi - \phi_{\bar{x},r})\, d\mathfrak{m}_g &= \int_0^{+\infty} \mathfrak{m}_g(\{\phi > t + \phi_{\bar{x},r}\} \cap D_{\bar{x},r})\, dt, \\ &\leq Cr \int_0^{+\infty} P(\{\phi \circ i_g > t + \phi_{\bar{x},r}\}, A_{\bar{x},3r})\, dt.\end{aligned}$$

This last expression, by the Coarea formula in X and Lemma 5.5, is less than or equal to

$$Cr \int_{A_{\bar{x},3r}} |\nabla(\phi \circ i_g)|\, d\mathfrak{m} = \int_{D_{\bar{x},3r}} \mathsf{lip}_g(\phi)\, d\mathfrak{m}_g.$$

Summing up, we proved that

$$\int_{\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r}} (\phi - \phi_{\bar{x},r})\, d\mathfrak{m}_g \leq Cr \int_{D_{\bar{x},3r}} \mathsf{lip}_g(\phi)\, d\mathfrak{m}_g.$$

To conclude we simply observe that

$$\begin{aligned}&\int_{D_{\bar{x},r}} |\phi - \phi_{\bar{x},r}|\, d\mathfrak{m}_g \\ &\quad= \int_{\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r}} (\phi - \phi_{\bar{x},r})\, d\mathfrak{m}_g + \int_{\{\phi < \phi_{\bar{x},r}\} \cap D_{\bar{x},r}} (\phi - \phi_{\bar{x},r})\, d\mathfrak{m}_g \\ &\quad= 2\int_{\{\phi > \phi_{\bar{x},r}\} \cap D_{\bar{x},r}} (\phi - \phi_{\bar{x},r})\, d\mathfrak{m}_g \leq 2Cr \int_{D_{\bar{x},3r}} \mathsf{lip}_g(\phi)\, d\mathfrak{m}_g.\end{aligned}$$

□

We now state the Sobolev isoperimetric inequality.

Theorem 5.7. *There exists a constant $C > 0$ depending only on N such that for every $\bar{x} \in \mathsf{G}(u)$ and $r \geq \tau > 0$ with $B_{2r}(x) \subset B_R(p)$ and for every smooth monotone function $f : \mathbb{R} \to \mathbb{R}$, defining $\phi : \mathsf{G}(u) \to \mathbb{R}$ as $\phi := f \circ u_g$, we have*

$$\mathfrak{m}_g(D_{\bar{x},r})^{1/N} \Big(\int_{D_{\bar{x},r}} \phi^{\frac{N}{N-1}}\, d\mathfrak{m}_g \Big)^{\frac{N-1}{N}} \leq Cr \Big(\int_{D_{\bar{x},r+\tau}} \mathsf{lip}_g(\phi)\, d\mathfrak{m}_g + \frac{1}{\tau} \int_{D_{\bar{x},r}} \phi\, d\mathfrak{m}_g \Big)$$

and

$$\mathfrak{m}_g(D_{\bar{x},r})^{1/N} \Big(\int_{D_{\bar{x},r}} \phi^{\frac{2N}{N-1}}\, d\mathfrak{m}_g \Big)^{\frac{N-1}{N}} \leq C\Big(r^2 \int_{D_{\bar{x},r+\tau}} \mathsf{lip}_g(\phi)^2\, d\mathfrak{m}_g + \frac{2r}{\tau} \int_{D_{\bar{x},r+\tau}} \phi^2\, d\mathfrak{m}_g \Big).$$

We only sketch the proof of Theorem 5.8, so that one sees how the machinery of the previous section replaces integration by parts in the smooth setting. The part of the proof that we omit is formally the same as the one in [21].

Theorem 5.8. *There exists a constant $C > 0$ depending only on N such that if $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(0, N)$ space and $u \in \mathsf{W}^{1,1}_{loc}(B_R(p))$ is positive and satisfies one of the equivalent conditions in Theorem 3.1, setting $\bar{p} := (p, u(p))$, we have*

$$\sup_{\mathsf{G}(u) \cap D_{\bar{p},R/2}} u_g \leq C \inf_{\mathsf{G}(u) \cap D_{\bar{p},R/2}} u_g.$$

Proof. Let $0 < s < R$ and define $w : \mathsf{G}(u) \to \mathbb{R}$ by

$$w := \log u_g - \fint_{D_{\bar{p},s}} \log u_g \, d\mathfrak{m}_g.$$

Let $\eta : \mathsf{G}(u) \to \mathbb{R}$ be function of the form $\eta_{p,g}^{r_1,r_2}$ (see Definition 4.31) which is supported in $D_{\bar{p},R}$ and let $q \in [0,+\infty)$. Because of Corollary 4.32 we have

$$\begin{aligned}
\int_{\mathsf{G}(u)} \mathsf{lip}_g(w)^2 \eta^2 |w|^q \, d\mathfrak{m}_g &= \int_{\mathsf{G}(u)} \mathsf{lip}_g(w) \cdot \mathsf{lip}_g(\eta^2|w|^q) \, d\mathfrak{m}_g \\
&= 2 \int_{\mathsf{G}(u)} \eta |w|^q \mathsf{lip}_g(\eta) \cdot \mathsf{lip}_g(w) \, d\mathfrak{m}_g + q \int_{\mathsf{G}(u)} \eta^2 |w|^{q-1} w \mathsf{lip}_g(w)^2 \, d\mathfrak{m}_g \\
&\leq \frac{1}{2} \int_{\mathsf{G}(u)} \mathsf{lip}_g(w)^2 \eta^2 |w|^q \, d\mathfrak{m}_g + 2 \int_{\mathsf{G}(u)} \mathsf{lip}_g(\eta)^2 |w|^q \, d\mathfrak{m}_g + q \int_{G(u)} \eta^2 |w|^{q-1} \mathsf{lip}_g(w)^2 \, d\mathfrak{m}_g.
\end{aligned}$$

The previous inequalities imply

$$\int_{\mathsf{G}(u)} \mathsf{lip}_g(w)^2 \eta^2 |w|^q \, d\mathfrak{m}_g \leq 4 \int_{\mathsf{G}(u)} \mathsf{lip}_g(\eta)^2 |w|^q \, d\mathfrak{m}_g + 2q \int_{\mathsf{G}(u)} \eta^2 |w|^{q-1} \mathsf{lip}_g(w)^2 \, d\mathfrak{m}_g,$$

so that choosing η appropriately we get for every $r \leq R/4$ the inequality

$$\int_{D_{\bar{p},3r}} \mathsf{lip}_g(w)^2 \, d\mathfrak{m}_g \leq \frac{8}{r^2} \mathfrak{m}_g(D_{\bar{p},4r}).$$

This corresponds to equation (4.21) in [21] and the remaining part of the proof can be carried out formally repeating the same argument of [21], replacing the smooth objects with the appropriate ones in our setting. $\square$

As an immediate application of the Harnack inequality we get the following result, which implies Theorem 1.

Theorem 5.9. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(0, N)$ space and let $u \in \mathsf{W}^{1,1}_{loc}(\mathsf{X})$ be a function satisfying one of the equivalent conditions of Theorem 3.1. If u is positive then it is constant.*

5.2 Applications to the smooth setting

In this section we prove Theorems 2 and 3 from the Introduction. Given a manifold (M, g) we denote by $\mathfrak{m}_g$ its volume measure and by d_g its distance. If $V : \mathsf{M} \to \mathbb{R}$ is a smooth function we say that the metric measure space $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} \mathfrak{m}_g)$ is a weighted manifold. Given an open set $\Omega \subset \mathsf{M}$ we say that a function $u \in C^\infty(\Omega)$ is a solution of the weighted minimal surface equation on $\Omega \setminus \partial\mathsf{M}$ if

$$\mathsf{div}\Big(\frac{e^{-V} \nabla u}{\sqrt{1 + |\nabla u|^2}} \Big) = 0 \quad \text{on } \Omega \setminus \partial\mathsf{M}.$$

We say that the boundary of a manifold with boundary is convex if its second fundamental form w.r.t. the inward pointing unit normal is positive. The next proposition can be obtained repeating an argument in [33, Theorem 2.4].

Proposition 5.10. *Let $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ be a weighted manifold with convex boundary such that for a number $N \geq n$ we have*

$$\mathsf{Ric} + \mathsf{Hess}_V - \frac{\nabla V \otimes \nabla V}{N - n} \geq 0 \qquad \text{on } \mathsf{M} \setminus \partial\mathsf{M}, \tag{21}$$

with the convention that if $N = n$ only constant weights are allowed, then $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ is an $\mathsf{RCD}(0, N)$ space.

Given a weighted manifold with boundary $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ and a smooth vector field $A \in \mathsf{TM}$ we define the pointwise divergence in the weighted manifold by $\mathsf{div}_V A := \mathsf{div} A - \nabla V \cdot \nabla A$.

Proposition 5.11. *Let $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ be a weighted manifold with convex boundary satisfying condition (21) for $N > 0$ and let $u \in C^\infty(B_R(x))$ be a solution of the weighted minimal surface equation on $B_R(x) \setminus \partial\mathsf{M}$ whose gradient vanishes on $\partial\mathsf{M} \cap B_R(x)$. Then u solves the minimal surface equation on $B_R(x)$ in distributional sense.*

Proof. We first check the statement testing against $C_c^\infty(\mathsf{M})$ functions. Let $\phi \in C_c^\infty(\mathsf{M})$ and let ν be the outward unit normal on $\partial\mathsf{M}$. Integrating by parts we obtain

$$\int_{\mathsf{M}} \frac{\nabla u}{W} \cdot \nabla\phi \, e^{-V} \, d\mathfrak{m}_g = -\int_{\mathsf{M}} \phi \, \mathsf{div}_V\Big(\frac{\nabla u}{W}\Big) e^{-V} \, d\mathfrak{m}_g + \int_{\partial\mathsf{M}} \frac{\phi}{W} \nabla u \cdot \nu e^{-V} \, d\mathsf{H}^{n-1}, \tag{22}$$

where H^{n-1} is the Hausdorff measure w.r.t. d_g. Since u solves the minimal surface equation on $\mathsf{M} \setminus \partial\mathsf{M}$ we have that $\mathsf{div}_V\Big(\frac{\nabla u}{W}\Big) = 0$ pointwise, while the second addendum in (22) is zero by the condition on ∇u. The same is true for $\phi \in \mathsf{Lip}_c(\mathsf{M})$ via a standard approximation argument. $\square$

Theorem 2. *Let $(\mathsf{M}^n, \mathsf{d}_g, e^{-V}\mathfrak{m}_g)$ be a weighted manifold with convex boundary such that there exists $N > n$ satisfying*

$$\mathsf{Ric}_{\mathsf{M}} + \mathsf{Hess}_V - \frac{\nabla V \otimes \nabla V}{N-n} \geq 0 \quad \text{on } \mathsf{M} \setminus \partial\mathsf{M}.$$

If $u \in C^\infty(\mathsf{M})$ is a positive solution of the weighted minimal surface equation on $\mathsf{M} \setminus \partial\mathsf{M}$ whose gradient vanishes on $\partial\mathsf{M}$, then u is constant.

Proof. The weighted manifold $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ is an $\mathsf{RCD}(0, N)$ space by Proposition 5.10, while u solves the minimal surface equation in distributional sense on the weighted space by Proposition 5.11. The conclusion follows by the Theorem 1. $\square$

Let $x \in \mathsf{X}$, $r > 0$ and $f : B_r(x) \to \mathbb{R}$. We define

$$\mathsf{Osc}_{x,r}(f) := \sup\{|f(y) - f(x)| : y \in B_r(x)\},$$
$$\mathsf{Osc}^+_{x,r}(f) := \sup\{(f(y) - f(x))_+ : y \in B_r(x)\},$$
$$\mathsf{Osc}^-_{x,r}(f) := \sup\{(f(y) - f(x))_- : y \in B_r(x)\}.$$

Theorem 5.12. *Fix $N \in (1, +\infty)$. For every $T, t, r > 0$ there exists $R(N, T, t, r) > 0$ such that if $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ is a pointed $\mathsf{RCD}(0, N)$ space and $u \in \mathsf{W}^{1,1}_{loc}(B_R(x))$ is a function satisfying one of the equivalent conditions of Theorem 3.1 such that $\mathsf{Osc}_{x,r}(u) \geq t$, then*

$$\mathsf{Osc}^+_{x,R}(u) \geq T \quad \text{and} \quad \mathsf{Osc}^-_{x,R}(u) \geq T.$$

Proof. Assume by contradiction that the statement is false. Then there exist $T, t, r > 0$, a sequence $\{R_i\}_{i\in\mathbb{N}}$ increasing to $+\infty$, a sequence of $\mathsf{RCD}(0, N)$ spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and solutions $u_i \in \mathsf{W}^{1,1}_{loc}(B_{R_i}(x_i))$ of the minimal surface equation such that $\mathsf{Osc}_{x_i,r}(u_i) \geq t$ and

$$\mathsf{Osc}^+_{x_i,R_i}(u_i) \leq T \quad \text{or} \quad \mathsf{Osc}^-_{x_i,R_i}(u_i) \leq T.$$

Modulo normalizing the measures $\mathfrak{m}_i$ we may suppose that $\mathfrak{m}_i(B_1(x_i)) = 1$, while modulo translating vertically each function u_i we may suppose that $u_i(x_i) = T$. Moreover, passing to a (non relabeled) subsequence, we can suppose (the other case being analogous) that $\mathsf{Osc}^-_{x_i,R_i}(u_i) \leq T$. Under these assumptions we have that all the functions u_i are positive in their domains, so that using the Harnack inequality on their graphs we get that they are all locally uniformly bounded.

Denote then $E_i := \mathsf{Epi}(u_i) \subset B_{R_i}(x_i) \times [0, +\infty)$ and observe that E_i is perimeter minimizing in $B_{R_i}(x_i) \times \mathbb{R}$. Modulo passing to yet another (non relabeled) subsequence, the spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ converge in *pmGH* sense to an $\mathsf{RCD}(0, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$, which implies that $\mathsf{X}_i \times \mathbb{R} \to \mathsf{X} \times \mathbb{R}$ in *pmGH* sense as well. The sets E_i converge then (again modulo passing to a subsequence) in L^1_{loc} sense to a perimeter minimizing set $E \subset \mathsf{X} \times \mathbb{R}$. Moreover the Kuratowski convergence of ∂E_i to ∂E in the space realizing the convergence (see Proposition 2.35), the fact that $\mathsf{Graph}(u_i) \subset \partial E_i$ and that $\mathsf{Graph}(u_i)$ can only converge in Kuratowski sense to a graph, imply that there exists $u : \mathsf{X} \to [0, +\infty)$ such that $\mathsf{Graph}(u) \subset \partial E$ (u is real valued since the functions u_i are locally uniformly bounded).

It is easy to see that this implies that $E = \mathsf{Epi}(u)$. Observe in addition that $\mathsf{Osc}_{x,2r}(u) \geq t$ again by the Kuratowski convergence of $\mathsf{Graph}(u_i)$ to $\mathsf{Graph}(u)$ and the lower bound on the oscillation of each u_i. Finally, modulo doing an extra vertical translation on the functions u_i, we can assume that $0 \leq \inf_{\mathsf{X}} u < t/c$, where c is the constant appearing in Theorem 5.8 (in particular we can do this translation in such a way that the functions u_i are still positive on every ball of fixed radius in the respective space for i large enough).

We now claim that u satisfies an Harnack-type inequality on its graph, which will force its oscillation to be strictly less than t, a contradiction. As we did previously in the note we will denote by u_g the height function on ∂E and by u_{ig} the height function on ∂E_i. For every $\epsilon > 0$ and $s > 0$, thanks to the Kuratowski convergence of ∂E_i to ∂E we have for i large enough

$$\sup_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g \leq \sup_{\mathsf{G}(u_i)\cap D_{(x_i,T),2s}} u_{ig} + \epsilon$$

and

$$\inf_{\mathsf{G}(u_i)\cap D_{(x_i,T),2s}} u_{ig} \leq \inf_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g + \epsilon.$$

Using then the Harnack inequality given by Theorem 5.8 we find a constant $c > 0$ such that for i large enough

$$\sup_{\mathsf{G}(u_i)\cap D_{(x_i,T),2s}} u_{ig} \leq c \inf_{\mathsf{G}(u_i)\cap D_{(x_i,T),2s}} u_{ig}.$$

Putting these facts together we get

$$\sup_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g \leq c \inf_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g + \epsilon(c+1),$$

and since ϵ was arbitrary we obtain

$$\sup_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g \leq c \inf_{\mathsf{G}(u)\cap D_{(x,T),s}} u_g.$$

In particular letting s go to $+\infty$ and using the fact that $\inf_{\mathsf{X}} u < t/c$ we conclude. □

Combining with Propositions 5.10 and 5.11 we get the following corollary, which is a stronger version of Theorem 3.

Corollary 5.13. *Fix $N \in \mathbb{N}$. For every $T, t, r > 0$ there exists $R(N, T, t, r) > 0$ such that if $(\mathsf{M}^n, \mathsf{d}_g, e^{-V} d\mathfrak{m}_g)$ is a weighted manifold with convex boundary satisfying condition (21) for $n < N$ and $u \in C^\infty(B_R(x))$ is a solution of the weighted minimal surface equation on $B_R(x) \setminus \partial\mathsf{M}$ whose gradient vanishes on $\partial\mathsf{M} \cap B_R(x)$ such that $\mathsf{Osc}_{x,r}(u) \geq t$, then*

$$\mathsf{Osc}^+_{x,R}(u) \geq T \quad \textit{and} \quad \mathsf{Osc}^-_{x,R}(u) \geq T.$$

References

[1] Luigi Ambrosio. "Calculus, heat flow and curvature-dimension bounds in metric measure spaces". In: *Proceedings of the International Congress of Mathematicians, Rio de Janeiro 2018. Vol. I. Plenary lectures.* World Sci. Publ., Hackensack, NJ, 2018, pp. 301–340.

[2] Luigi Ambrosio, Elia Brué, and Daniele Semola. "Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces". In: *Geom. Funct. Anal.* 29.4 (2019), pp. 949–1001. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s00039-019-00504-5. URL: https://doi.org/10.1007/s00039-019-00504-5.

[3] Luigi Ambrosio and Simone Di Marino. "Equivalent definitions of BV space and of total variation on metric measure spaces". In: *J. Funct. Anal.* 266.7 (2014), pp. 4150–4188. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2014.02.002. URL: https://doi.org/10.1016/j.jfa.2014.02.002.

[4] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434. ISBN: 0-19-850245-1.

[5] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. "Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below". In: *Invent. Math.* 195.2 (2014), pp. 289–391. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-013-0456-1. URL: https://doi.org/10.1007/s00222-013-0456-1.

[6] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. "Metric measure spaces with Riemannian Ricci curvature bounded from below". In: *Duke Math. J.* 163.7 (2014), pp. 1405–1490. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2681605. URL: https://doi.org/10.1215/00127094-2681605.

[7] Luigi Ambrosio, Andrea Mondino, and Giuseppe Savaré. “Nonlinear Diffusion Equations and Curvature Conditions in Metric Measure Spaces”. In: *Memoirs of the American Mathematical Society* 262 (Sept. 2015). DOI: 10.1090/memo/1270.

[8] Luigi Ambrosio, Andrea Pinamonti, and Gareth Speight. “Tensorization of Cheeger energies, the space $H^{1,1}$ and the area formula for graphs”. In: *Adv. Math.* 281 (2015), pp. 1145–1177. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2015.06.004. URL: https://doi.org/10.1016/j.aim.2015.06.004.

[9] Luigi Ambrosio et al. “Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure”. In: *Trans. Amer. Math. Soc.* 367.7 (2015), pp. 4661–4701. ISSN: 0002-9947,1088-6850. DOI: 10.1090/S0002-9947-2015-06111-X. URL: https://doi.org/10.1090/S0002-9947-2015-06111-X.

[10] Gioacchino Antonelli et al. “Asymptotic isoperimetry on non collapsed spaces with lower Ricci bounds”. In: *Mathematische Annalen* (Aug. 2023), pp. 1–54. DOI: 10.1007/s00208-023-02674-y.

[11] Lisa Beck. *Elliptic regularity theory.* Vol. 19. Lecture Notes of the Unione Matematica Italiana. A first course. Springer, Cham; Unione Matematica Italiana, Bologna, 2016, pp. xii+201. DOI: 10.1007/978-3-319-27485-0. URL: https://doi.org/10.1007/978-3-319-27485-0.

[12] Enrico Bombieri, Ennio De Giorgi, and Mario Miranda. “Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche”. In: *Archive for Rational Mechanics and Analysis* 32 (1969), pp. 255–267. URL: https://api.semanticscholar.org/CorpusID:122065789.

[13] Elia Bruè, Enrico Pasqualetto, and Daniele Semola. “Constancy of the dimension in codimension one and locality of the unit normal on $\mathrm{RCD}(K,N)$ spaces”. In: *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* (June 2022).

[14] Elia Bruè, Enrico Pasqualetto, and Daniele Semola. “Rectifiability of the reduced boundary for sets of finite perimeter over $\mathrm{RCD}(K,N)$ spaces”. In: *Journal of the European Mathematical Society* 25 (Feb. 2022). DOI: 10.4171/JEMS/1217.

[15] Elia Brué and Daniele Semola. “Constancy of the dimension for $\mathrm{RCD}(K,N)$ spaces via regularity of Lagrangian flows”. In: *Comm. Pure Appl. Math.* 73.6 (2020), pp. 1141–1204. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.21849. URL: https://doi.org/10.1002/cpa.21849.

[16] Fabio Cavalletti and Emanuel Milman. “The globalization theorem for the curvature-dimension condition”. In: *Invent. Math.* 226.1 (2021), pp. 1–137. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-021-01040-6. URL: https://doi.org/10.1007/s00222-021-01040-6.

[17] J. Cheeger. “Differentiability of Lipschitz functions on metric measure spaces”. In: *Geom. Funct. Anal.* 9.3 (1999), pp. 428–517. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s000390050094. URL: https://doi.org/10.1007/s000390050094.

[18] Giulio Colombo, Luciano Mari, and Marco Rigoli. *On minimal graphs of sublinear growth over manifolds with non-negative Ricci curvature.* 2023. arXiv: 2310.15620 [math.DG].

[19] Giulio Colombo et al. “Bernstein and half-space properties for minimal graphs under Ricci lower bounds”. In: *Int. Math. Res. Not. IMRN* 23 (2022), pp. 18256–18290. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnab342. URL: https://doi.org/10.1093/imrn/rnab342.

[20] Alessandro Cucinotta and Andrea Mondino. *Half Space Property in RCD(0,N) spaces.* 2024. arXiv: 2402.12230 [math.DG].

[21] Qi Ding. “Liouville-type theorems for minimal graphs over manifolds”. In: *Anal. PDE* 14.6 (2021), pp. 1925–1949. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2021.14.1925. URL: https://doi.org/10.2140/apde.2021.14.1925.

[22] Qi Ding. *Poincaré inequality on minimal graphs over manifolds and applications.* 2023. arXiv: 2111.04458 [math.DG].

[23] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm. “On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces”. In: *Inventiones mathematicae* 201 (Dec. 2014). DOI: 10.1007/s00222-014-0563-7.

[24] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992, pp. viii+268. ISBN: 0-8493-7157-0.

[25] Francesco Fiorani, Andrea Mondino, and Daniele Semola. *Monotonicity formula and stratification of the singular set of perimeter minimizers in RCD spaces.* 2023. arXiv: `2307.06205 [math.DG]`.

[26] Nicola Gigli. "On the differential structure of metric measure spaces and applications". In: *Mem. Amer. Math. Soc.* 236.1113 (2015), pp. vi+91. ISSN: 0065-9266,1947-6221. DOI: `10.1090/memo/1113`. URL: `https://doi.org/10.1090/memo/1113`.

[27] Nicola Gigli and Bang-Xian Han. "Independence on p of weak upper gradients on RCD spaces". In: *J. Funct. Anal.* 271.1 (2016), pp. 1–11. ISSN: 0022-1236,1096-0783. DOI: `10.1016/j.jfa.2016.04.014`. URL: `https://doi.org/10.1016/j.jfa.2016.04.014`.

[28] Nicola Gigli, Andrea Mondino, and Giuseppe Savaré. "Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows". In: *Proceedings of the London Mathematical Society* 111 (Nov. 2013). DOI: `10.1112/plms/pdv047`.

[29] Nicola Gigli, Andrea Mondino, and Daniele Semola. "On the notion of Laplacian bounds on RCD spaces and applications". In: *Proc. Amer. Math. Soc.* 152.2 (2024), pp. 829–841. ISSN: 0002-9939,1088-6826. DOI: `10.1090/proc/16550`. URL: `https://doi.org/10.1090/proc/16550`.

[30] Enrico Giusti. *Minimal surfaces and functions of bounded variation.* Vol. 80. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984, pp. xii+240. ISBN: 0-8176-3153-4. DOI: `10.1007/978-1-4684-9486-0`. URL: `https://doi.org/10.1007/978-1-4684-9486-0`.

[31] Piotr Hajłasz and Pekka Koskela. "Sobolev met Poincaré". In: *Mem. Amer. Math. Soc.* 145.688 (2000), pp. x+101. ISSN: 0065-9266,1947-6221. DOI: `10.1090/memo/0688`. URL: `https://doi.org/10.1090/memo/0688`.

[32] Heikki Hakkarainen, Juha Kinnunen, and Panu Lahti. "Regularity of minimizers of the area functional in metric spaces". In: *Adv. Calc. Var.* 8.1 (2015), pp. 55–68. ISSN: 1864-8258,1864-8266. DOI: `10.1515/acv-2013-0022`. URL: `https://doi.org/10.1515/acv-2013-0022`.

[33] Bang-Xian Han. "Measure rigidity of synthetic lower Ricci curvature bound on Riemannian manifolds". In: *Adv. Math.* 373 (2020), pp. 107327, 31. ISSN: 0001-8708,1090-2082. DOI: `10.1016/j.aim.2020.107327`. URL: `https://doi.org/10.1016/j.aim.2020.107327`.

[34] Juha Heinonen et al. *Sobolev spaces on metric measure spaces: An approach based on upper gradients.* Vol. 27. New Mathematical Monographs. Cambridge University Press, Cambridge, 2015, pp. xii+434. ISBN: 978-1-107-09234-1. DOI: `10.1017/CBO9781316135914`. URL: `https://doi.org/10.1017/CBO9781316135914`.

[35] Juha Kinnunen et al. "Lebesgue points and capacities via the boxing inequality in metric spaces". In: *Indiana Univ. Math. J.* 57.1 (2008), pp. 401–430. ISSN: 0022-2518,1943-5258. DOI: `10.1512/iumj.2008.57.3168`. URL: `https://doi.org/10.1512/iumj.2008.57.3168`.

[36] Juha Kinnunen et al. "Regularity of sets with quasiminimal boundary surfaces in metric spaces". In: *J. Geom. Anal.* 23.4 (2013), pp. 1607–1640. ISSN: 1050-6926,1559-002X. DOI: `10.1007/s12220-012-9299-z`. URL: `https://doi.org/10.1007/s12220-012-9299-z`.

[37] Zhenhao Li. "The globalization theorem for $\mathrm{CD}(K,N)$ on locally finite spaces". In: *Ann. Mat. Pura Appl. (4)* 203.1 (2024), pp. 49–70. ISSN: 0373-3114,1618-1891. DOI: `10.1007/s10231-023-01352-9`. URL: `https://doi.org/10.1007/s10231-023-01352-9`.

[38] John Lott and Cédric Villani. "Ricci curvature for metric-measure spaces via optimal transport". In: *Ann. of Math. (2)* 169.3 (2009), pp. 903–991. ISSN: 0003-486X,1939-8980. DOI: `10.4007/annals.2009.169.903`. URL: `https://doi.org/10.4007/annals.2009.169.903`.

[39] Michele Miranda Jr. "Functions of bounded variation on "good" metric spaces". In: *J. Math. Pures Appl. (9)* 82.8 (2003), pp. 975–1004. ISSN: 0021-7824. DOI: `10.1016/S0021-7824(03)00036-9`. URL: `https://doi.org/10.1016/S0021-7824(03)00036-9`.

[40] Andrea Mondino. "A new notion of angle between three points in a metric space". In: *J. Reine Angew. Math.* 706 (2015), pp. 103–121. ISSN: 0075-4102,1435-5345. DOI: `10.1515/crelle-2013-0080`. URL: `https://doi.org/10.1515/crelle-2013-0080`.

[41] Andrea Mondino and Aaron Naber. "Structure theory of metric measure spaces with lower Ricci curvature bounds". In: *J. Eur. Math. Soc. (JEMS)* 21.6 (2019), pp. 1809–1854. ISSN: 1435-9855,1435-9863. DOI: `10.4171/JEMS/874`. URL: `https://doi.org/10.4171/JEMS/874`.

[42] Andrea Mondino and Daniele Semola. "Weak Laplacian bounds and minimal boundaries in non-smooth spaces with Ricci curvature lower bounds". Accepted in: Memoirs of the American Mathematical Society, Preprint arXiv: 2107.12344.

[43] Barbara Nelli and Harold Rosenberg. "Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$". In: *Bull. Braz. Math. Soc. (N.S.)* 33.2 (2002), pp. 263–292. ISSN: 1678-7544,1678-7714. DOI: 10.1007/s005740200013. URL: https://doi.org/10.1007/s005740200013.

[44] Peter Petersen. *Riemannian geometry*. Third. Vol. 171. Graduate Texts in Mathematics. Springer, Cham, 2016, pp. xviii+499. DOI: 10.1007/978-3-319-26654-1. URL: https://doi.org/10.1007/978-3-319-26654-1.

[45] Tapio Rajala. "Local Poincaré inequalities from stable curvature conditions on metric spaces". In: *Calc. Var. Partial Differential Equations* 44.3-4 (2012), pp. 477–494. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-011-0442-7. URL: https://doi.org/10.1007/s00526-011-0442-7.

[46] Harold Rosenberg, Felix Schulze, and Joel Spruck. "The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$". In: *J. Differential Geom.* 95.2 (2013), pp. 321–336. ISSN: 0022-040X,1945-743X. URL: http://projecteuclid.org/euclid.jdg/1376053449.

[47] Karl-Theodor Sturm. "On the geometry of metric measure spaces. I". In: *Acta Math.* 196.1 (2006), pp. 65–131. ISSN: 0001-5962,1871-2509. DOI: 10.1007/s11511-006-0002-8. URL: https://doi.org/10.1007/s11511-006-0002-8.

[48] Karl-Theodor Sturm. "On the geometry of metric measure spaces. II". In: *Acta Math.* 196.1 (2006), pp. 133–177. ISSN: 0001-5962,1871-2509. DOI: 10.1007/s11511-006-0003-7. URL: https://doi.org/10.1007/s11511-006-0003-7.

[49] Cédric Villani. *Optimal transport*. Vol. 338. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Springer-Verlag, Berlin, 2009, pp. xxii+973. ISBN: 978-3-540-71049-3. DOI: 10.1007/978-3-540-71050-9. URL: https://doi.org/10.1007/978-3-540-71050-9.