HORIZONTAL SEMICONCAVITY FOR THE SQUARE OF CARNOT-CARATHÉODORY DISTANCE ON IDEAL CARNOT GROUPS AND APPLICATIONS TO HAMILTON-JACOBI EQUATIONS

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Abstract. We show that the square of Carnot-Carathéodory distance from the origin, in ideal Carnot groups, enjoys the horizontal semiconcavity (h-semiconcavity) everywhere in the group including the origin. We apply this property to show h-semiconcavity for the solutions of a class of non-coercive evolutive Hamilton-Jacobi equations, by using the associated Hopf-Lax solutions.

1. Introduction.

Semiconvexity and semiconcavity properties are key regularity properties for functions, related to bounds for the second derivatives, and applied in many different contexts. We refer to the monograph by Cannarsa and Sinestrari [16] for an overview on the topic. One of the most interesting functions where one can apply this property is the distance function. It is easily seen that the standard Euclidean distance is convex and thus semiconvex everywhere but not semiconcave. However, the squared Euclidean distance to a given point is both semiconvex and semiconcave, since its Hessian is a constant nonnegative matrix. This leads to many interesting consequences, and it is somehow behind the successful use of the squared distance to prove many results in PDEs. This opens to question the relation between semiconvexity/semiconcavity and distance functions in different geometrical settings.

In the case of Carnot groups, there is a vast literature investigating the notion of convexity (or concavity) associated to their sub-Riemannian structure. Later in the paper we will review several known notions that can be defined in these spaces. Let us just quickly recall the notion of horizontal convexity (h-convexity for short) introduced by Lu-Mandredi-Stroffolini in the Heisenberg group [30]; see also [25] for extension to more general Carnot groups. At the same time, the notion of h-convexity is also studied independently by Danielli-Garofalo-Nhieu in [18] by adapting the standard convexity definition to the algebraic structure of Carnot groups. The notion of h-concavity can be symmetrically defined; see Definition 2.14 for a precise definition. Later such a notion was generalised by Bardi and the first author in [10] with a more geometrical approach that does not require any underlying Lie group structure and cover sub-Riemannian structures up to the case of Carnot-type Hörmander vector fields.

While many results are known for h-convex or h-concave functions, less research has been conducted on their semiconvex/semiconcave counterparts, which can be defined by easily

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adapting the concept of h-convexity/h-concavity. As an analogue of the Euclidean case, in [10] it is shown that these notions are equivalent to bounds in the viscosity sense for the intrinsic Hessian, namely the horizontal Hessian as defined in (2.16). This equivalence demonstrates that horizontal semiconvexity/semiconcavity serves as a natural sub-Riemannian generalization, elucidating why proving h-semiconvexity/h-semiconcavity properties is exceptionally useful in studying degenerate PDEs associated with Carnot groups.

In this paper we are interested in exploring such semiconcavity for functions related to the metric of a class of sub-Riemannian manifolds. In the setting of Carnot groups, various notions of metric can be considered and they turn out to be all locally equivalent. In this work, we discussed the Carnot-Carathéodory distance (CC distance for short), which is intrinsic and therefore the most important distance. It is also the geodesic distance, which can be defined as the minimal length among admissible curves joining two given points. A precise definition is given in Definition 2.6. It has many important geometric and metric properties; for example it is the only distance solving the eikonal equation in this geometrical setting (consult e.g. [19, 32]).

Regarding the sub-Riemiannian manifolds in this work, we shall focus on the so-called ideal Carnot groups, a subclass of step 2 Carnot groups that includes the Heisenberg group and general H-type groups as simple yet significant examples. An important property of such type of Carnot groups is that the abnormal set, which consists of endpoints of abnormal minimizing geodesics starting from the group identity, contains only the identity itself. Under certain assumptions for a general Carnot group, in [15] it is shown that the CC distance from the identity is locally Euclidean semiconcave, and therefore locally h-semiconcave, away from the abnormal set. See also [22, Theorem 5.9] for this result. However, despite many potential applications in nonlinear PDEs, the behaviour near the identity has not been well understood in the literature. For ideal Carnot groups, we aim to clarify the regularity of CC distance in the whole space, including any neighborhood of the identity.

Our main result of this paper is the following theorem.

Theorem 1.1 (H-semiconcavity of square of CC distance)**.** *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Then $d^2(\cdot, 0)$ is h-semiconcave in $\mathbb{G}$.*

We emphasize that our result complements [15, 22], extending the h-semiconcavity of d_0^2 to the identity, and thus obtaining the regularity globally. To prove this property we combine general results from Lie groups, in particular the fact that, the abnormal set is simple enough so that the missing local estimate can be obtained by a comparison of the homogeneous norm, with viscosity theory techniques. The assumption that $\mathbb{G}$ is ideal is essential. In fact, as shown in [33], the squared CC distance fails to be h-semiconcave in the Engel group, which is a non-ideal Carnot group; see Proposition 3.5.

The consequence of this surprising regularity property for the squared CC distance allows various applications to the study of degenerate nonlinear PDEs in ideal Carnot groups. In Section 4, for a class of time-dependent convex Hamilton-Jacobi equations, we show the spatial h-semiconcavity of viscosity solution that is given by the Hopf-Lax formula. The h-semiconcavity constant we obtained depends on $t > 0$ but is independent of the space variables. Our result provides a sub-Riemannian generalization of the Euclidean counterpart; see for example [16, Theorem 1.6.1] for the spatial semiconcavity of the Hopf-Lax solution to Hamilton-Jacobi equations in the Euclidean space.

Although we have proved that the square of the CC distance is h-semiconcave, its regularity turns out to be still very different from the Euclidean case. It is obvious that the squared Euclidean distance from the origin is convex in the space. In contrast, we will show that, the squared CC distance fails to be h-semiconvex in the Heisenberg group; see Proposition 3.8. More related discussions will be given in a forthcoming paper [28].

The paper is organized as follows: In Section 2.1 we go over some basics about Carnot groups including the group multiplication, the dilation and the CC distance. We recall some known (local) inclusions between the CC balls and the Euclidean balls, and between the CC distance and the homogeneous distance. The notions and properties of the endpoint map, normal and abnormal geodesics, and ideal Carnot groups are reviewed in Section 2.2. Some related details for the special case of the Heisenberg group as well as the notions of the horizontal gradient and the horizontal Laplacian (or sub-Laplacian) are provided in these two sections as well. In Section 2.3 we go over the notions and basic properties of h-concave and h-semiconcave functions.

Section 3 is devoted to the proof of our main result of the paper, Theorem 1.1. We also mention some related consequences and disprove the h-convexity of the squared CC distance. We provide our applications to Hamilton-Jacobi equations in Section 4, showing the h-semiconcavity of Hopf-Lax solutions in space under suitable assumptions for the Hamiltonian.

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2. Preliminaries

2.1. **Carnot groups.** We begin with some basic facts about Carnot groups. For more details, we refer to [13].

Definition 2.1 (Carnot group). A *Carnot group* is a connected and simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ has a stratification $\mathfrak{g} = \bigoplus_{j=1}^{s} \mathfrak{g}_j$, that is, a linear splitting $\mathfrak{g} = \bigoplus_{j=1}^{s} \mathfrak{g}_j$ where $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$ for $j = 1, \dots, s-1$ and $[\mathfrak{g}_1, \mathfrak{g}_s] = \{0\}$. If $\mathfrak{g}_s \neq \{0\}$, the number s is called the *step* of $\mathbb{G}$.

Note that the case $s = 1$ coincides with the standard Euclidean space, therefore, here we will always consider the case $s \geq 2$.

By using the exponential map, we can always identify a Carnot group $\mathbb{G}$ with its Lie algebra $\mathfrak{g}$ with the group law given by the so-called Baker-Campbell-Dynkin-Hausdorff formula (see [13, § 15] for more details). Furthermore, by choosing a suitable basis of $\mathfrak{g}$ consisting of bases of $\mathfrak{g}_j$, it can be further identified with $(\mathbb{R}^n, \cdot)$ with $\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}$, where $\cdot$ is a non commutative operation, and $n = n_1 + \cdots + n_s$, with n denoting the topological dimension of $\mathbb{G}$ as a manifold, while n_j the dimension of $\mathfrak{g}_j$. After this identification, the group identity becomes 0 and $p^{-1} = -p$. For $r > 0$, writing $p = (p^{(1)}, \ldots, p^{(s)}) \in \mathbb{R}^n \cong \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}$, we can define the *dilation* δ_r on $(\mathbb{R}^n, \cdot)$ by

$$\delta_r(p^{(1)}, \ldots, p^{(s)}) := (rp^{(1)}, \ldots, r^s p^{(s)}),$$

which is an automorphism of $(\mathbb{R}^n, \cdot)$. Note that the dilations defined above are anisotropic; for a more formal definition of the dilations defined on Carnot groups, we refer to [13].

Moreover, the group multiplication satisfies

$$p \cdot q = p + q + \mathcal{R}(p, q), \qquad \forall\, p, q \in \mathbb{R}^n, \tag{2.1}$$

with $\mathcal{R} = (\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(s)}) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}$, $\mathcal{R}^{(j)}$ a polynomial depending only on the first $n_1 + \ldots + n_{j-1}$ variables of p and q, i.e. the variables associated via the exponential map with the first $j-1$ layer of the Lie algebra. In particular, when the step $s = 2$, we have

$$\mathcal{R}(p, q) = (0, \mathcal{B}(p^{(1)}, q^{(1)})) \in \mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \qquad \forall\, p, q \in \mathbb{R}^n, \tag{2.2}$$

for some skew-symmetric bilinear form $\mathcal{B} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$. For this identification, more details can be found in [13, Proposition 2.2.22 and § 3.2].

Remark 2.2. Given $\mathbb{G}$ step two Carnot group, let us consider $\mathcal{R}(p, q)$ introduced in (2.2), there exists a constant $C_0 > 0$ such that

$$|\mathcal{R}(p, q)| = |\mathcal{B}(p^{(1)}, q^{(1)})| \le C_0 |p^{(1)}||q^{(1)}| \le C_0 |p||q|, \quad \forall\, p, q \in \mathbb{G}, \tag{2.3}$$

where $|\cdot|$ is the norm on $\mathbb{G} \cong \mathbb{R}^n$.

For the sake of simplicity, we use $m := n_1 = \dim \mathfrak{g}_1$ and write $p^{(1)} = (p_1, \ldots, p_m)$.

Definition 2.3. For $1 \le i \le m$, we use $\mathrm{X}_i \in \mathfrak{g}_1$ to denote the left-invariant vector field on $\mathbb{G}$ which coincides with $\frac{\partial}{\partial p_i}$ at the identity. Note that $\{\mathrm{X}_1, \ldots, \mathrm{X}_m\}$ forms a basis of $\mathfrak{g}_1$.

Example 2.4. The *Heisenberg group* $\mathbb{H} = \mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ is the simplest Carnot group whose group law (2.1) is given by

$$(x, y, z) \cdot (\tilde{x}, \tilde{y}, \tilde{z}) = \left(x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + \frac{1}{2}(x\tilde{y} - \tilde{x}y) \right),$$

which means that $\mathcal{B}((x, y), (\tilde{x}, \tilde{y})) = \frac{1}{2}(x\tilde{y} - \tilde{x}y)$ in (2.2). From the group multiplication defined above, it is easy to see that the center of $\mathbb{H}$ (the set of elements which can commute with all the other elements) is $\{0\} \times \mathbb{R}$. The basis of $\mathfrak{g}_1$ is given by

$$\mathrm{X}_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad \mathrm{X}_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}. \tag{2.4}$$

Let $\mathrm{Y} = \frac{\partial}{\partial z}$. The Lie algebra $\mathfrak{g}$ of $\mathbb{H}$ is given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with

$$\mathfrak{g}_1 = \mathrm{span}\{\mathrm{X}_1, \mathrm{X}_2\}, \qquad \mathfrak{g}_2 = \mathrm{span}\{\mathrm{Y}\}.$$

and the only nontrivial bracket relation of $\mathfrak{g}$ is $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{Y}$. In this particular case of the Heisenberg group, one can show easily that (2.3) holds with $C_0 = 1$. In fact, we have

$$|\mathcal{B}(p^{(1)}, q^{(1)})| = \frac{1}{2}|x\tilde{y} - \tilde{x}y| \le \frac{1}{2}\big(|x||\tilde{y}| + |\tilde{x}||y|\big) \le |p^{(1)}||q^{(1)}|,$$

for $p^{(1)} = (x, y)$ and $q^{(1)} = (\tilde{x}, \tilde{y})$.

Carnot groups are Lie groups, therefore they have also a manifold structure. Next we briefly introduce it and we refer to [34, § 4.5] for more details.

Definition 2.5 (Sub-Riemannian structure)**.** On a Carnot group $\mathbb{G}$, the canonical left-invariant sub-Riemannian structure $(\mathcal{D}, g)$ is defined as follows: the horizontal distribution $\mathcal{D}$ (a sub-bundle of the tangent bundle $T\mathbb{G}$ satisfying the bracket generating condition) is generated by $\mathfrak{g}_1$ and the metric g on $\mathcal{D}$ is determined by $\{\mathrm{X}_1, \dots, \mathrm{X}_m\}$. To be more precise, we have

$$\mathcal{D}_p := \mathrm{span}\{\mathrm{X}_1(p), \dots, \mathrm{X}_m(p)\},$$

and $\{\mathrm{X}_1(p), \dots, \mathrm{X}_m(p)\}$ forms an orthonormal basis at every point $p \in \mathbb{G}$.

An *horizontal path* is an absolutely continuous map $\gamma : [0, 1] \to \mathbb{G}$ such that $\dot\gamma(\tau) \in \mathcal{D}_{\gamma(\tau)}$ for a.e. τ, whose *length* can be calculated by

$$\ell(\gamma) := \int_0^1 \sqrt{g(\dot\gamma(\tau), \dot\gamma(\tau))}\, d\tau. \tag{2.5}$$

Definition 2.6 (Carnot-Carathéodory distance)**.** The *Carnot-Carathéodory distance* (or in short *CC distance*) between two points $p, q \in \mathbb{G}$ is defined as

$$d(p, q) := \inf\{\ell(\gamma) \mid \gamma \text{ horizontal}, \gamma(0) = p, \gamma(1) = q\}.$$

By the celebrated Chow-Rashevsky Theorem (see for example [2, 34]), d is a well-defined finite distance and its induced topology is the same as the manifold topology; in particular, in Carnot groups, this means that d is continuous w.r.t. the standard Euclidean topology. Therefore $(\mathbb{G}, d)$ is a metric space. We call the sub-Riemannian structure $(\mathcal{D}, g)$ *complete* if it is complete as a metric space. A *(length) minimizing geodesic* is a horizontal path γ such that $\ell(\gamma) = d(\gamma(0), \gamma(1))$.

In addition, the following properties of the CC distance hold (cf. [13, Proposition 5.2.6]):

$$\begin{aligned} &d(\delta_r(p), \delta_r(q)) = r d(p, q), &&\forall\, r > 0, p, q \in \mathbb{G},\\ &d(p, q) = d(q^{-1} \cdot p, 0) = d(p^{-1} \cdot q, 0), &&\forall\, r > 0, p, q \in \mathbb{G}. \end{aligned} \tag{2.6}$$

In the following we use $B_E(p, r)$ and $B_{CC}(p, r)$ to denote, respectively, the Euclidean ball and the CC ball centred at $p \in \mathbb{G}$ with radius $r > 0$, i.e.

$$B_E(p, r) := \{q \in \mathbb{G} \mid |p - q| < r\}, \quad B_{CC}(p, r) := \{q \in \mathbb{G} \mid d(p, q) < r\}. \tag{2.7}$$

A well-known relation between the Euclidean distance and the CC distance is as follows.

Proposition 2.7 ([3], Proposition 1.1)**.** *On a Carnot group $\mathbb{G}$ with step s, for every compact set $K \subset \mathbb{G}$, there exists a constant $C(K) > 0$ such that*

$$\frac{1}{C(K)}|p - q| \le d(p, q) \le C(K)|p - q|^{\frac{1}{s}}, \quad \forall\, p, q \in K.$$

It immediately implies the following local inclusions between the Euclidean balls and the CC balls, i.e. given any compact set $K \subset \mathbb{G}$, if $B_{CC}(p, r) \subset K$, then

$$B_E(p, C(K)^{-s} r^s) \subset B_{CC}(p, r) \subset B_E(p, C(K) r). \tag{2.8}$$

In order to simplify our notation, we denote by $d_0(p)$ the CC distance between the point p and the group identity 0, i.e. $d_0(p) = d(p, 0)$. It is worth mentioning the following equivalence between the CC distance from the identity and the homogeneous norm.

Proposition 2.8 ([13], Proposition 5.1.4)**.** *Let d be the CC distance of a Carnot group and $d_0 = d(\cdot, 0)$. Then, there exists a constant $C \geq 1$ such that*

$$C^{-1}|p|_{\mathbb{G}} \leq d_0(p) \leq C|p|_{\mathbb{G}}, \quad \forall\, p \in \mathbb{G},$$

where the homogeneous norm $|\cdot|_{\mathbb{G}}$ is defined by

$$|p|_{\mathbb{G}} = \left(\sum_{i=1}^{s} |p^{(i)}|^{\frac{2s!}{i}}\right)^{\frac{1}{2s!}}, \qquad \forall\, p = (p^{(1)}, \ldots, p^{(s)}) \in \mathbb{G}, \tag{2.9}$$

with s the step of the group $\mathbb{G}$ and $p^{(i)}$ associated by the exponential map to the i-layer $\mathfrak{g}_i$.

We remark that in the special case of Heisenberg group $\mathbb{H}$ (see Example 2.4), the homogeneous norm is

$$|(x, y, z)|_{\mathbb{H}} = \left((x^2 + y^2)^2 + z^2\right)^{\frac{1}{4}}, \qquad \forall\, (x, y, z) \in \mathbb{H},$$

which corresponds to (2.9) with $s = 2$, $p = (x, y, z)$, $p^{(1)} = (x, y)$ and $p^{(2)} = z$.

2.2. **Endpoint map and ideal Carnot groups.** Next we introduce the endpoint and ideal Carnot groups. More details on this part can be found in [2, § 8] and [34, 39]. We recall that it is usually more convenient to minimize the energy J as below rather than the original length ℓ defined in (2.5):

$$J(\gamma) := \frac{1}{2}\int_0^1 g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))\, d\tau. \tag{2.10}$$

In fact, we have the following relation (see e.g. [35, Theorem 1.1.7] or [39, Proposition 2.1]):

$$\frac{1}{2}d^2(p, q) = \inf\{J(\gamma) \mid \gamma : [0, 1] \to \mathbb{G},\ \text{horizontal}, \gamma(0) = p, \gamma(1) = q\}. \tag{2.11}$$

For a fixed $p \in \mathbb{G}$ and any control $u \in L^2([0, 1], \mathbb{R}^m)$, let γ_u be the unique maximal solution of the following Cauchy problem:

$$\dot{\gamma}_u(\tau) = \sum_{j=1}^{m} u_j(\tau)\mathrm{X}_j(\gamma_u(\tau)), \qquad \gamma(0) = p. \tag{2.12}$$

Definition 2.9 (Endpoint map)**.** We use $\mathcal{U}_p$ to denote the set of $u \in L^2([0, 1], \mathbb{R}^m)$ such that the corresponding trajectories γ_u solving (2.12) starting at p are defined on the interval $[0, 1]$. $\mathcal{U}_p$ is an open set in $L^2([0, 1], \mathbb{R}^m)$. The *endpoint map based at p* is the map $\mathcal{E}_p : \mathcal{U}_p \to \mathbb{G}$ defined as

$$\mathcal{E}_p(u) := \gamma_u(1).$$

We then obtain an *energy functional* on $\mathcal{U}_p$ given by

$$\mathcal{J}(u) := J(\gamma_u) = \frac{1}{2}\int_0^1 |u(\tau)|^2 d\tau.$$

Note that $\mathcal{E}_p$ is a smooth function on $\mathcal{U}_p$ (cf. [34, Appendix D]) and a length minimizing geodesic joining p and q is just γ_u with u minimizing $\mathcal{J}$ under the constraint $\mathcal{E}_p(u) = q$. Thus, by the method of Lagrange multipliers (see e.g. [39, Theorem B.2] or [2, § 8.2]), for such u, there exists a non-trivial pair (λ, ν), such that

$$\lambda \circ D_u \mathcal{E}_p = \nu D_u \mathcal{J}, \qquad \lambda \in T_q^*\mathbb{G},\ \nu \in \{0,1\}, \tag{2.13}$$

here $\circ$ denotes the composition and D_u the differential with respect to u.

Definition 2.10 (Normal and abnormal minimizing geodesic)**.** Given ν introduced in (2.13), the length minimizing geodesic γ_u is called *normal* if $\nu = 1$ and *abnormal* if $\nu = 0$. We remark that a minimizing geodesic could be both normal and abnormal at the same time since the pair (λ, ν) is not necessarily unique (see [34, § 5.3.3]). Finally we call an abnormal minimizing geodesic *trivial* if it is a constant curve.

The next definition introduces the groups under consideration in this paper; we refer to [38, 39] for more details.

Definition 2.11 (Ideal Carnot group)**.** Given a Carnot group $\mathbb{G}$, we say that $\mathbb{G}$ is *ideal* if the sub-Riemannian structure $(\mathcal{D}, g)$, introduced in Definition 2.5, is ideal, which by definition means that it is complete and it does not admit non-trivial abnormal minimizing geodesics.

The following notion of fatness can help us check whether a Carnot group $\mathbb{G}$ is ideal or not. Recall that $\mathbb{G}$ is called *fat* if for every $p \in \mathbb{G}$ and $\mathrm{X} \in \mathcal{D}$ with $\mathrm{X}(p) \neq 0$, we have

$$\mathcal{D}_p + [\mathcal{D}, \mathrm{X}]_p = T_p\mathbb{G}.$$

Thanks to the left invariance of the sub-Riemannian structure $(\mathcal{D}, g)$ on $\mathbb{G}$, the property above is equivalent to saying that for every $\mathrm{X} \in \mathfrak{g}_1 \setminus \{0\}$, the following holds true:

$$\mathfrak{g}_1 + [\mathfrak{g}_1, \mathrm{X}] = \mathfrak{g}. \tag{2.14}$$

By [40, Theorem 10], a Carnot group is ideal if and only if it is fat, which trivially implies that $\mathbb{G}$ is step two, i.e., $s = 2$.

Example 2.12. The Heisenberg group $\mathbb{H}$ appearing in Example 2.4 is ideal. The simplest non-ideal Carnot group is $\mathbb{R} \times \mathbb{H}$. To be more precise, suppose that T is a nonzero constant vector field on $\mathbb{R}$ and the Lie algebra of $\mathbb{R} \times \mathbb{H}$ is given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with

$$\mathfrak{g}_1 = \mathrm{span}\{\mathrm{T}, \mathrm{X}_1, \mathrm{X}_2\}, \qquad \mathfrak{g}_2 = \mathrm{span}\{\mathrm{Y}\}.$$

Since T commutes with $\mathrm{X}_1, \mathrm{X}_2, \mathrm{T}$, (2.14) fails for $\mathrm{X} = \mathrm{T}$, which implies that this Carnot group is not ideal.

Carnot groups of step ≥ 3 are never ideal Carnot groups because they are not fat.

Example 2.13. The *Engel group* $\mathbb{E} = \mathbb{R}^4 \cong \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, which is the simplest step three Carnot group, is not ideal. To be more precise, the Engel group can be identified with $\mathbb{R}^4$ with the following multiplication: given $p = (x, y, z, s), \tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{s}) \in \mathbb{E}$

$$p \cdot \tilde{p} = \left(x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + \frac{1}{2}(x\tilde{y} - \tilde{x}y), s + \tilde{s} + \frac{1}{2}(x\tilde{z} - \tilde{x}z) + \frac{1}{12}(x - \tilde{x})(x\tilde{y} - \tilde{x}y)\right).$$

Defining

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \Big(\frac{xy}{12} + \frac{z}{2}\Big)\frac{\partial}{\partial s}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2}{12}\frac{\partial}{\partial s},$$

$$Y = \frac{\partial}{\partial z} + \frac{x}{2}\frac{\partial}{\partial s}, \qquad Z = \frac{\partial}{\partial s}.$$

The Lie algebra $\mathfrak{g}$ of $\mathbb{E}$ is given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ with

$$\mathfrak{g}_1 = \mathrm{span}\{X_1, X_2\}, \qquad \mathfrak{g}_2 = \mathrm{span}\{Y\}, \qquad \mathfrak{g}_3 = \mathrm{span}\{Z\}.$$

Here the nontrivial bracket relations of $\mathfrak{g}$ are $[X_1, X_2] = Y$ and $[X_1, Y] = Z$. For more details on the Engel group, we refer to [1, 5, 6, 7] and the references therein.

We conclude this subsection with the definition of several differential operators in Carnot groups, which utilizes the derivatives along the vector fields introduced in Definition 2.3. The *horizontal gradient* of $\varphi \in C^1(\mathbb{G})$ at the point $p \in \mathbb{G}$, denoted by $\nabla_H\varphi(p)$, is defined by

$$\nabla_H\varphi(p) = (X_1\varphi(p), \dots, X_m\varphi(p)) \in \mathbb{R}^m.$$

The *horizontal plane* $\mathcal{H}_0$ *passing through the identity* 0 is a subspace of $\mathbb{G} \cong \mathbb{R}^n$ defined by $\mathbb{R}^m \times \{0\} \times \ldots \times \{0\}$. It is clear that $\mathcal{H}_0$ is isomorphic to $\mathbb{R}^m$ and thus from now on we will not distinguish $\mathcal{H}_0$ from $\mathbb{R}^m$. If we use $\langle \cdot, \cdot \rangle$ to denote the inner product on the Euclidean space $\mathbb{G} \cong \mathbb{R}^n$ inducing the norm $|\cdot|$, with these notations, it is not difficult to see that

$$\langle \nabla_H\varphi(p), h\rangle = \frac{d}{d\tau}\Big|_{\tau=0} \varphi(p \cdot \tau h), \quad \forall\, h \in \mathcal{H}_0. \tag{2.15}$$

Moreover, similar to the definition of the horizontal gradient, the (symmetrized) *horizontal Hessian* of $\varphi \in C^2(\mathbb{G})$ at the point $p \in \mathbb{G}$, denoted by $(\nabla_H^2\varphi(p))^*$, is the unique $m \times m$ symmetric matrix satisfying the following formula:

$$\langle (\nabla_H^2\varphi(p))^* h, h\rangle = \frac{d^2}{d\tau^2}\Big|_{\tau=0} \varphi(p \cdot \tau h), \quad \forall\, h \in \mathcal{H}_0. \tag{2.16}$$

The *(canonical) sub-Laplacian* at $p \in \mathbb{G}$ is thus defined by $\Delta_H u(p) = \sum_{i=1}^m X_i^2 u(p)$.

2.3. **H-concavity and h-semiconcavity.** On Carnot groups it is possible to introduce several notions of convexity and, respectively, concavity. However, not all the notions behave well in this sub-Riemannian setting. For example, in [36], the authors showed that geodetical convexity is not a suitable notion on Heisenberg group in the sense that the family of geodetically convex sets only consist of the whole group, the empty set and the geodesics, and as a consequence the only geodetically convex functions are the constant functions. Furthermore, strong H-convexity is considered in [18, 14], and it turns out that it is still a too restrictive notion. A notion of horizontal convexity, shortly h-convexity, was introduced, independently, by Lu, Manfredi and Stroffolini [30], for the Heisenberg group, and by Danielli, Garofalo and Nhieu [18] in more general Carnot groups. Later the notion has been discussed in various papers e.g. [9, 23, 31, 37, 42, 25]. In more recent years h-convexity has been applied to study properties of solutions for certain classes of PDEs [27, 29, 28]. See also [20] for an overview on how h-convexity can be applied to sets. The authors of [10, 11] extended the notion of the h-concavity the setting of vector fields and further studied the notion of h-semiconcavity, which is the key topic of the present work.

For the purpose of this paper, we introduce directly the notion of h-concavity, holding the usual relation that u is h-convex if and only if $-u$ is h-concave.

Definition 2.14 (H-concavity)**.** Given an open set $\Omega \subset \mathbb{G}$, a function $u \in LSC(\Omega)$ (i.e. the function is lower semicontinuous in Ω) is called *h-concave* if and only if

$$\begin{aligned} u(p \cdot h) + &u(p \cdot h^{-1}) - 2u(p) \leq 0, \\ &\forall\, p \in \Omega, h \in \mathcal{H}_0 \text{ such that } [p \cdot h^{-1}, p \cdot h] := \{p \cdot \tau h \,|\, \tau \in [-1, 1]\} \subset \Omega. \end{aligned} \tag{2.17}$$

Remark 2.15. Note that for Definition 2.14 to hold true, one could relax the assumption of lower semicontinuity. Nevertheless, since later we use the viscosity characterization of h-concave functions, we ask such regularity directly in the definition. This definition follows from [30] and [18], where the notion is stated in a slightly different form, but these formulations coincide with ours for LSC functions; see [25] for details.

Similarly to the standard Euclidean characterization established first by Alvarez, Lasry and Lions in 1997 [4], the h-concavity of a function can be characterized by the sign of its horizontal Hessian in the viscosity sense: this characterization was first introduced for the Heisenberg group in [30] and later proved in Carnot groups in [43, 25]. See also [10] for the more general case of Carnot-type Hörmander vectors fields.

Combining the results from the above mentioned papers, we have

$$u \text{ is h-concave in } \Omega, \text{ if and only if, } -(\nabla^2_H u)^* \geq 0 \text{ in } \Omega \text{ holds in the viscosity sense.} \tag{2.18}$$

To be more precise, the viscosity inequality means that $-(\nabla^2_H \varphi(p))^* \geq 0$ whenever there exist $\varphi \in C^2(\Omega)$ and $p \in \Omega$ such that $u - \varphi$ has a local minimum at p.

Example 2.16. On the Heisenberg group $\mathbb{H}$ in Example 2.4, we have $\mathrm{X}_1 = \partial_x - \frac{y}{2}\partial_z$, $\mathrm{X}_2 = \partial_y + \frac{x}{2}\partial_z$, and the horizontal Hessian can be represented by

$$(\nabla^2_H \varphi)^* = \begin{pmatrix} \mathrm{X}_1^2\varphi & \frac{1}{2}(\mathrm{X}_1\mathrm{X}_2\varphi + \mathrm{X}_2\mathrm{X}_1\varphi) \\ \frac{1}{2}(\mathrm{X}_1\mathrm{X}_2\varphi + \mathrm{X}_2\mathrm{X}_1\varphi) & \mathrm{X}_2^2\varphi \end{pmatrix}.$$

In this case, one can easily verify that every Euclidean concave function in $\mathbb{R}^3$ is also h-concave in $\mathbb{H}$. The reverse however is not true. See [18, 20].

Next we recall the property under investigation in this paper.

Definition 2.17 (H-semiconcavity)**.** Given an open set $\Omega \subset \mathbb{G}$, we call a function $u \in LSC(\Omega)$ *h-semiconcave* if there exists a constant $C \geq 0$ such that

$$u(p \cdot h) + u(p \cdot h^{-1}) - 2u(p) \leq C|h|^2, \quad \forall\, p \in \mathbb{G}, h \in \mathcal{H}_0 \text{ such that } [p \cdot h^{-1}, p \cdot h] \subset \Omega, \tag{2.19}$$

where we recall that $|\cdot|$ is the Euclidean norm on $\mathbb{G} \cong \mathbb{R}^n$. The constant C is called *h-semiconcavity constant.*

This is a generalization of the notion of semiconcave functions in the Euclidean space, for which we refer to [16]. A function u is called *h-semiconvex* it $-u$ is h-semiconcave. Similarly to the characterization in (2.18) for h-concave functions, the notion of h-semiconcavity can be characterized by a bound for the horizontal Hessian in the viscosity sense.

Theorem 2.18 (Proposition 5.1 of [10]). *Given an open set $\Omega \subset \mathbb{G}$ and $u \in LSC(\Omega)$, the follow statements are equivalent:*

(1) u is h-semiconcave in Ω with h-semiconcavity constant $C \geq 0$.

(2) We have

$$-\left(\nabla_H^2 u\right)^* \geq -C\, \mathrm{Id}_m \quad \text{in } \Omega \text{ in the viscosity sense,} \tag{2.20}$$

which means that $-(\nabla_H^2 \varphi(p))^* \geq -C\, \mathrm{Id}_m$ whenever there exist $\varphi \in C^2(\Omega)$ and $p \in \Omega$ such that $u-\varphi$ has a local minimum at p. Here Id_m denotes the $m \times m$ identity matrix.

The result below is a direct consequence of Theorem 2.18 and the stability of viscosity supersolutions with respect to the infimum. It will be useful in our later applications. Below we denote by $LSC(A)$ the set of lower semicontinuous functions in a set A of a metric space.

Proposition 2.19. *Let Ω be an open set of a Carnot group $\mathbb{G}$. Let $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of h-semiconcave functions on Ω. Assume that for every $\alpha \in \mathcal{A}$, the function u_α is h-semiconcave functions in Ω with h-semiconcavity constant $C \geq 0$ independent of α. Suppose that*

$$u(p) := \inf_{\alpha\in\mathcal{A}} u_\alpha(p) > -\infty \quad \text{for all } p \in \Omega.$$

If $u \in LSC(\Omega)$, then u is also h-semiconcave in Ω with the same h-semiconcavity constant $C \geq 0$.

3. H-semiconcavity of square of CC distance

In this section, we present our main theorem, which states that the square of CC distance in ideal Carnot groups is h-semiconcave. One of the tools we will use is the local (Euclidean) semiconcavity studied in [15]. Recall that a function u is *locally semiconcave* in an open set Ω if for every compact convex set $K \subset \Omega$, there exists a constant $C(K) \geq 0$ such that the following holds:

$$\lambda u(p) + (1-\lambda)u(q) - u(\lambda p + (1-\lambda)q) \leq \lambda(1-\lambda)C(K)|p-q|^2, \quad \forall\, p, q \in K, \lambda \in [0,1]. \tag{3.21}$$

Here the constant $2C(K)$ is called the *semiconcavity constant on compact set K*. Note that the definition is independent of the choice of the norm $|\cdot|$, since, in the Euclidean setting, different norms are equivalent up to a multiplicative constant. Hereafter let us just use the standard norm on $\mathbb{R}^n$. By definition an ideal Carnot group only possesses trivial abnormal minimizing geodesics. Consequently, the abnormal set of the identity 0, which is just the set of endpoints of abnormal minimizing geodesics starting from the identity 0, must be $\{0\}$. It follows from [15, Theorem 1] (and also [22, Theorem 5.9]) that $d^2(\cdot, 0)$ is locally (Euclidean) semiconcave on $\mathbb{G} \setminus \{0\}$. The following lemma, which will be useful in the proof of our main result, is a direct consequence.

Lemma 3.1. *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Then, for $d_0 = d(\cdot, 0)$, there exist two constants $C \geq 0$ and $c > 0$ such that*

$$d_0^2(p+v) + d_0^2(p-v) - 2d_0^2(p) \leq C|v|^2, \quad \forall\, p \in \partial B_{CC}(0,1),\ |v| \leq c.$$

Proof. Set $S = \partial B_{CC}(0,1)$ and $\eta = \inf_{q\in S} |q|$, that is, η is the Euclidean distance between the origin and the boundary of the unit CC ball. By (2.8) with $K = \overline{B_{CC}(0,1)}$, it is easy to see that $\eta > 0$. Thus, for every $q \in S$, the compact set $\overline{B_E(q,\eta/2)} \subset \mathbb{G} \setminus \{0\}$; see Figure 1.

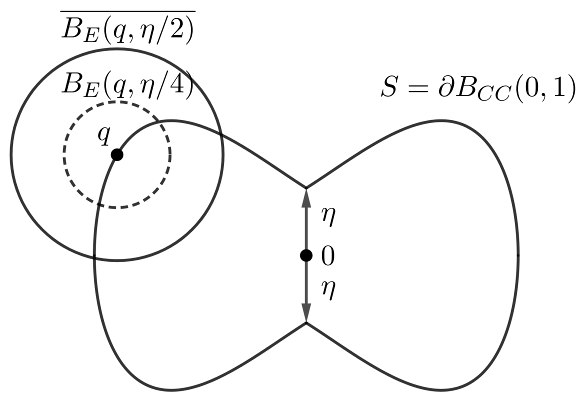

FIGURE 1. Relation between the CC ball and the Euclidean ball in the proof of Lemma 3.1.

It follows from the local (Euclidean) semiconcavity of d_0^2 on $\mathbb{G} \setminus \{0\}$ (see [15, Theorem 1]) that there exists a $C(q,\eta) \geq 0$ such that

$$\begin{aligned} \lambda d_0^2(p_+) + (1-\lambda)d_0^2(p_-) - d_0^2(\lambda p_+ + (1-\lambda)p_-) \leq \lambda(1-\lambda)C(q,\eta)|p_+ - p_-|^2, \\ \forall\, p_+, p_- \in \overline{B_E(q,\eta/2)}, \lambda \in [0,1]. \end{aligned} \tag{3.22}$$

As a result, choosing $p_+ = p + v$, $p_- = p - v$, and $\lambda = \frac{1}{2}$, we apply (3.22) to deduce

$$d_0^2(p+v) + d_0^2(p-v) - 2d_0^2(p) \leq 2C(q,\eta)|v|^2, \quad \forall\, p \in B_E(q,\eta/4), |v| \leq \eta/4.$$

See Figure 2.

Since $\{B_E(q,\eta/4)\}_{q\in S}$ is an open cover of S, by compactness, there exists a finite cover $\{B_E(q_i,\eta/4)\}_{1\leq i\leq N}$ with $N < +\infty$. Now we introduce

$$C := 2 \max_{1\leq i\leq N} C(q_i,\eta) \geq 0,$$

then, for every $p \in S$, there exists an $i \in \{1,\dots,N\}$ such that $p \in B_E(q_i,\eta/4)$, which yields

$$d_0^2(p+v) + d_0^2(p-v) - 2d_0^2(p) \leq 2C(q_i,\eta)|v|^2 \leq C|v|^2, \quad \forall\, |v| \leq \eta/4.$$

By taking $c := \eta/4 > 0$, we complete the proof of the lemma. □

We stress that both constants $C, c > 0$ in Lemma 3.1 depend only on $S = \partial B_{CC}(0,1)$, therefore they are both universal constants.

Let us now prove our main result, Theorem 1.1.

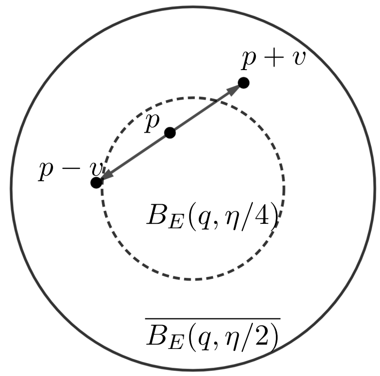

FIGURE 2. Relation between the points and balls in the proof of Lemma 3.1.

Proof of Theorem 1.1. We continue using d_0 to denote the CC distance from the identity. Theorefore we have $d_0^2(p) = d^2(p,0)$ for $p \in \mathbb{G}$. Moreover, since Carnot groups satisfy the Hörmander condition, meaning the distribution associated to Carnot groups is bracket generating, the CC distance d is continuous (see [34, Theorem 2.2 and Theorem 2.3]). Therefore it is clear that the condition $d_0^2 \in LSC(\mathbb{G})$.

We first pose the following **claim:**

$$\text{For every } p \in \mathbb{G}, \text{ there exist } C > 0 \text{ (independent of } p\text{) and } c(p) > 0 \text{ such that} \\ d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) \le C|h|^2, \quad \forall\, h \in \mathcal{H}_0, |h| \le c(p). \tag{3.23}$$

Assuming that (3.23) holds, we can use the viscosity characterization for h-semiconcave functions given in Theorem 2.18 to easily conclude. In fact, take $\varphi \in C^2(\mathbb{G})$ such that $d_0^2 - \varphi$ has a local minimum at some $p \in \mathbb{G}$. Without loss of generality, by standard viscosity theory techniques, we can assume that the local minimum is equal to 0, i.e. $\varphi(p) = d_0^2(p)$ (see e.g. [26, Proposition 2.2]). Then claim (3.23) implies that for $h \in \mathcal{H}_0$ with $|h|$ small enough we get

$$\varphi(p \cdot h) + \varphi(p \cdot h^{-1}) - 2\varphi(p) \le d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) \le C|h|^2. \tag{3.24}$$

However, recalling the horizontal differential operators introduced in (2.15) and (2.16) and by applying the Taylor expansion in the group (see e.g. [13, § 20]), we can write

$$\begin{aligned} \varphi(p \cdot h) &= \varphi(p) + \langle \nabla_H \varphi(p), h\rangle + \frac{1}{2}\langle (\nabla_H^2 \varphi(p))^* h, h\rangle + o(|h|^2), \\ \varphi(p \cdot h^{-1}) &= \varphi(p) - \langle \nabla_H \varphi(p), h\rangle + \frac{1}{2}\langle (\nabla_H^2 \varphi(p))^* h, h\rangle + o(|h|^2). \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we obtain

$$\langle (\nabla_H^2 \varphi(p))^* h, h\rangle + o(|h|^2) \le C\,|h|^2,$$

for all $h \in \mathcal{H}_0$ with $|h|$ small enough. Dividing this inequality by $|h|^2$ and letting $|h| \to 0^+$, we deduce that $(\nabla_H^2 \varphi(p))^* \le C\,\mathrm{Id}_m$ and thus $-(\nabla_H^2 d_0^2)^* \ge -C\,\mathrm{Id}_m$ in the viscosity sense, which concludes the result by Theorem 2.18.

Let us now prove **claim** (3.23). We fix the constants $C \geq 0$ and $c > 0$ appearing in Lemma 3.1, and split the proof of (3.23) into three cases:

Case 1: $p \in S = \{q \in \mathbb{G} \,|\, d(q,0) = 1\}$. We would like to use Lemma 3.1. Since $\mathbb{G}$ is an ideal Carnot group, it is fat and thus of step 2. (The case of step 1 Carnot group can be reduced to the known Euclidean case.) It follows from from (2.1) and (2.2) that

$$p \cdot h = p + h + \mathcal{R}(p,h), \quad p \cdot h^{-1} = p - h - \mathcal{R}(p,h), \quad \forall\, p, h \in \mathbb{G}. \tag{3.26}$$

Furthermore, we apply inequality (2.3) to $p \in S = \partial B_{CC}(0,1)$ to get $|\mathcal{R}(p,h)| \leq C_1|h|$, where $C_1 = C_0 \sup_{p\in S} |p| \in (0, +\infty)$. As a result, we have

$$|v_{p,h}| \leq |h| + C_1|h|, \quad \forall\, p \in S, h \in \mathbb{G}, \quad \text{where } v_{p,h} := h + \mathcal{R}(p,h).$$

Then for $c_1 = \frac{c}{1+C_1}$, whenever $p \in S$ and $h \in \mathcal{H}_0$ such that $|h| \leq c_1$, we deduce $|v_{p,h}| \leq c$. Combining this with Lemma 3.1 (with constant $\overline{C}$ in place of C) and (3.26), we obtain

$$\begin{aligned} &d_0^2(p\cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) = d_0^2(p + v_{p,h}) + d_0^2(p - v_{p,h}) - 2d_0^2(p) \\ &\leq \overline{C}|v_{p,h}|^2 \leq \overline{C}\,(1+C_1)^2\,|h|^2, \quad \forall\, p \in S, h \in \mathcal{H}_0, |h| \leq c_1. \end{aligned}$$

This proves **claim** (3.23) with $C = \overline{C}\,(1+C_1)^2$.

Case 2: $p \in \mathbb{G}$ and $p \neq 0$. In this case, we use the properties of the CC distance as in (2.6). In fact, for every $p \neq 0$ we can define $\widetilde{p} = \delta_{1/r}(p)$ with $r = d_0(p)$ (i.e. $p = \delta_r(\widetilde{p})$) so that $\widetilde{p} \in S$. Indeed, we have

$$d_0(\widetilde{p}) = d_0\left(\delta_{1/r}(p)\right) = \frac{1}{r}\, d_0(p) = \frac{1}{r}\, r = 1.$$

Then we can adopt Case 1 for $\widetilde{p}$ with $\widetilde{h} = \delta_{1/r}(h)$ for all $h \in \mathcal{H}_0$ such that $|h| \leq c_1 r$, where c_1 is the constant determined in Case 1. It is worth pointing out that, due to the condition $h \in \mathcal{H}_0$, we have

$$\delta_{1/r}(h) = \frac{|h|}{r}, \quad |\widetilde{h}| = \frac{|h|}{r} \leq \frac{c_1\, r}{r} = c_1.$$

Hence by our result in Case 1 we can write

$$\begin{aligned} &d_0^2(p\cdot h) + d_0^2(p\cdot h^{-1}) - 2d_0^2(p) = d_0^2(\delta_r(\widetilde{p})\cdot h) + d_0^2(\delta_r(\widetilde{p})\cdot h^{-1}) - 2d_0^2(\delta_r(\widetilde{p})) \\ =&\, r^2[d_0^2(\widetilde{p}\cdot \delta_{1/r}(h)) + d_0^2(\widetilde{p}\cdot \delta_{1/r}(h)^{-1}) - 2d_0^2(\widetilde{p})] = r^2\left[d_0^2(\widetilde{p}\cdot \widetilde{h}) + d_0^2(\widetilde{p}\cdot \widetilde{h}^{-1}) - 2d_0^2(\widetilde{p})\right] \\ \leq&\, r^2 C|\,\widetilde{h}\,|^2 \leq r^2 C\frac{|h|^2}{r^2} = C|h|^2, \end{aligned}$$

where we take $C = \overline{C}\,(1+C_1)^2$ as in Case 1 and $c(p) = c_1 d_0(p)$. This proves **claim** (3.23) for the current case with $C = \overline{C}\,(1+C_1)^2$ and $c(p) = c_1 d_0(p)$.

Case 3: $p = 0$. It remains to prove the claim in the case $p = 0$. To this end, we use the equivalence of the CC distance and the homogeneous norm introduced in (2.9). Noticing that, for all $h \in \mathcal{H}_0$ we have that $|h|_{\mathbb{G}} = |h|$, hence for $p = 0$ we obtain

$$d_0^2(p\cdot h) + d_0^2(p\cdot h^{-1}) - 2d_0^2(p) = d_0^2(h) + d_0^2(-h) \leq 2\,C_2^2\,|h|^2,$$

where $C_2 \geq 1$ is the constant given in Proposition 2.8. This proves **claim** (3.23) for the point $p = 0$ with the constants $C = 2\,C_2^2$ and $c(p) = 1$.

To sum up, considering all of the cases discussed above, we have shown that **claim** (3.23) holds for all $p \in \mathbb{G}$ and $h \in \mathcal{H}_0$ such that $|h| \leq c(p)$, with

$$C = \max\left(\overline{C}\,(1+C_1)^2\,, 2C_2^2\right) > 0, \quad c(p) = \begin{cases} c_1 d_0(p), & \text{if } p \neq 0,\\ 1, & \text{if } p = 0.\end{cases}$$

□

Applying Theorem 2.18 to the function $d_0^2 = d^2(\cdot, 0)$, a direct consequence of Theorem 1.1 is the following corollary.

Corollary 3.2. *Let d be the CC distance of an ideal Carnot group $\mathbb{G}$. For $d_0 = d(\cdot, 0)$, there exists a constant $C > 0$ such that*

$$-(\nabla_H^2[d_0^2])^* \geq -C\,\mathrm{Id}_m \quad \textit{in } \mathbb{G} \textit{ holds in the viscosity sense.}$$

In particular, $-\Delta_H[d_0^2] \geq -mC$ in $\mathbb{G}$ holds in the viscosity sense, where m denotes the dimension of the first layer of the Lie algebra of $\mathbb{G}$. Here, the viscosity inequalities mean that $-(\nabla_H^2\varphi(p))^* \geq -C\,\mathrm{Id}_m$ and $-\Delta_H\varphi(p) \geq -mC$ hold for any $\varphi \in C^2(\mathbb{G})$ and $p \in \mathbb{G}$ such that $d_0^2 - \varphi$ attains a local minimum at p.

Proof. From Theorem 1.1 we know that d_0^2 is h-semiconcave. Then by (ii) of Theorem 2.18, there exists a constant $C \geq 0$ such that for any $\varphi \in C^2(\mathbb{G})$ and $p \in \mathbb{G}$ with the property that $d_0^2 - \varphi$ has a local minimum at p, we have $-(\nabla_H^2\varphi(p))^* \geq -C\,\mathrm{Id}_m$. Taking the trace on both sides, we obtain $-\Delta_H\varphi(p) + mC \geq 0$, which concludes the proof. □

Corollary 3.3. *Let d be the CC distance of an ideal Carnot group $\mathbb{G}$. For $d_0 = d(\cdot, 0)$, there exists a constant $C > 0$ such that*

$$(\nabla_H^2[d_0^2])^* \leq C\,\mathrm{Id}_m,$$

and

$$\Delta_H[d_0^2] \leq mC$$

hold almost everywhere in $\mathbb{G}$.

Proof. We first define the *cut locus (of the identity* 0*)* as follows:

$$\mathrm{Cut}_0 := \{p \in \mathbb{G} \,|\, d_0^2 \text{ is not smooth in a neighbourhood of } p\}.$$

It is known [38, Proposition 15] that the cut locus has measure zero if the step of a Carnot group is two. From Corollary 3.2 we know that $-(\nabla_H^2[d_0^2])^* \geq -C\,\mathrm{Id}_m$ and $-\Delta_H[d_0^2] \geq -mC$ hold in the viscosity sense. Therefore, by standard techniques of viscosity solution theory, $-(\nabla_H^2[d_0^2](p))^* \geq -C\,\mathrm{Id}_m$ and $-\Delta_H[d_0^2](p) \geq -mC$ hold at all points p where d_0^2 is smooth including all $p \in \mathbb{G} \setminus \mathrm{Cut}_0$. Then the conclusion of the corollary follows. □

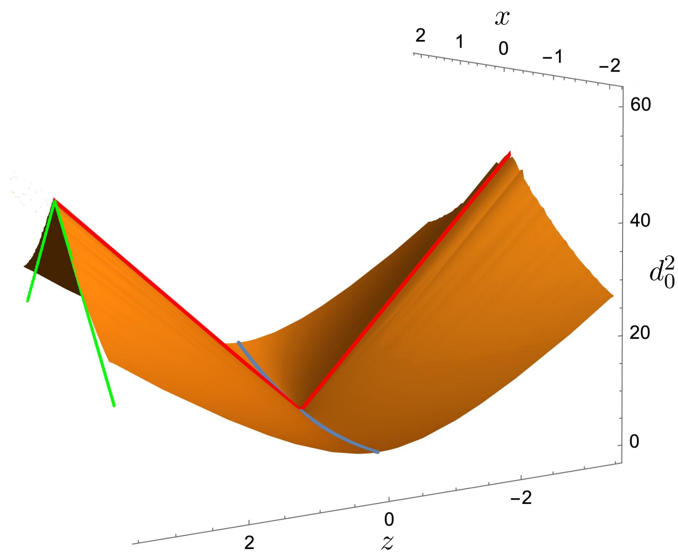

FIGURE 3. Graph of $d_0^2 = d^2(\cdot, 0)$ in the Heisenberg group $\mathbb{H}$.

Remark 3.4. We include Figure 3 to illustrate the significant difference between the h-semiconcavity on Heisenberg group $\mathbb{H}$ and the usual Euclidean semiconcavity. The following explicit expression of the CC distance in $\mathbb{H}$ is obtained in [12, Theorem 1.36]:

$$d_0^2(x, y, z) = \begin{cases} \left(\dfrac{\theta}{\sin\theta}\right)^2 (x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \text{ and } \theta = \mu^{-1}\left(\frac{4|z|}{x^2+y^2}\right), \\ 4\pi|z|, & \text{if } (x, y) = (0, 0), \end{cases} \tag{3.27}$$

where $\mu : (-\pi, \pi) \to \mathbb{R}$ given by

$$\mu(s) := \frac{2s - \sin(2s)}{2\sin^2 s} \tag{3.28}$$

is an increasing diffeomorphism (cf. [24, Lemme 3, p. 112]). Since d_0^2 is rotational symmetric in the coordinates x and y, we only draw the graph of d_0^2 on the set $\{(x, 0, z) : x, z \in \mathbb{R}\}$. It can be seen from the red curve that a corner-like singularity occurs at the identity 0 in the direction z (i.e. the forbidden direction). As a result, it is not Euclidean semiconcave at 0, which by definition means that d_0^2 is not locally Euclidean semiconcave in any neighborhood of the identity. This observation might explain why the result [15, Theorem 1] or [22, Theorem 5.9] did not touch the identity on $\mathbb{H}$.

Our result confirms that such singularity actually does not affect the horizontal semiconcavity of d_0^2 at the identity 0. As a matter of fact, to investigate the definition of h-semiconcavity, we should restrict the function to the horizontal plane, which gives a smooth function

$$d_0^2(x, y, 0) = x^2 + y^2.$$

The graph of this function is plotted as the blue curve in Figure 3.

The assumption of ideal Carnot groups in Theorem 1.1 is essential. The h-semiconcavity of the square of the CC distance actually fails to hold on the Engel group introduced in

Example 2.13. We refer to [33, Theorem 1.2] for this observation. To be more precise, the following can be found in the proof in [33, § 4].

Proposition 3.5 ([33]). *On the Engel group $\mathbb{E}$, there exists a nonzero element p in $\mathbb{E}$ (in the abnormal set) such that the following limit holds:*

$$\lim_{h\in\mathbb{R}\setminus\{0\},h\to 0}\frac{d_0(p\cdot he_1)-d_0(p)}{h^2}=+\infty,$$

where $e_1:=(1,0,0,0)\in\mathcal{H}_0\subset\mathbb{E}$. In particular,

$$\lim_{h>0,h\to 0^+}\frac{d_0^2(p\cdot he_1)+d_0^2(p\cdot(he_1)^{-1})-2d_0^2(p)}{h^2}=+\infty.$$

It is possible to generalize our result in Theorem 1.1 for other functions related to the CC distance. We present the following generalization in a bounded open set of an ideal Carnot group.

Corollary 3.6. *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Let Ω be a bounded open set of $\mathbb{G}$. Assume that $\Psi:[0,+\infty)\to[0,+\infty)$ is an increasing function such that its even extension $\widetilde{\Psi}:\mathbb{R}\to[0,+\infty)$ is a locally semiconcave function in $\mathbb{R}$, as defined in (3.21). Then, $\Psi(d(\cdot,0))$ is h-semiconcave in Ω.*

Proof. We still use the notation $d_0=d(\cdot,0)$ in $\mathbb{G}$. We first assume that $\widetilde{\Psi}\in C^2(\mathbb{R})$. Since $\widetilde{\Psi}$ is even, it is easy to obtain $\widetilde{\Psi}'(0)=0$. Now let $T(\Omega):=\sup_{p\in\Omega}d(p)<+\infty$. On the compact set $[0,T(\Omega)]$, by the local semiconcavity there exists a constant $C(\Omega)>0$ such that $\widetilde{\Psi}''\leq C(\Omega)$. Consequently this implies

$$0\leq\widetilde{\Psi}'(\tau)=\widetilde{\Psi}'(\tau)-\widetilde{\Psi}'(0)\leq C(\Omega)\tau,\qquad\forall\,\tau\in[0,T(\Omega)]. \tag{3.29}$$

Moreover, for $p\in\Omega\setminus\{0\}$ and $h\in\mathcal{H}_0$ such that $[p\cdot h^{-1},p\cdot h]\subset\Omega$, by Taylor expansion we have

$$\Psi(d_0(p\cdot h))-\Psi(d_0(p))\leq\widetilde{\Psi}'(d_0(p))(d_0(p\cdot h)-d_0(p))+\frac{C(\Omega)}{2}(d_0(p\cdot h)-d_0(p))^2. \tag{3.30}$$

Notice that

$$\widetilde{\Psi}'(d_0(p))(d_0(p\cdot h)-d_0(p))\leq\frac{\widetilde{\Psi}'(d_0(p))}{2d_0(p)}(d_0^2(p\cdot h)-d_0^2(p)),$$

and, also by the fact $h\in\mathcal{H}_0$, we have

$$|d_0(p\cdot h)-d_0(p)|\leq d(p\cdot h,p)=d_0(h)=|h|,$$

since it is easy to see that $t\to th, t\in[0,1]$ is a length minimizing geodesic. Inserting these two estimates into (3.30), we obtain

$$\Psi(d_0(p\cdot h))-\Psi(d_0(p))\leq\frac{\widetilde{\Psi}'(d_0(p))}{2d_0(p)}(d_0^2(p\cdot h)-d_0^2(p))+\frac{C(\Omega)}{2}|h|^2.$$

Similarly we have

$$\Psi(d_0(p\cdot h^{-1}))-\Psi(d_0(p))\leq\frac{\widetilde{\Psi}'(d_0(p))}{2d_0(p)}(d_0^2(p\cdot h^{-1})-d_0^2(p))+\frac{C(\Omega)}{2}|h|^2.$$

Adding them together and apply Theorem 1.1 as well as (3.29), we have

$$\begin{aligned}&\Psi(d_0(p\cdot h))+\Psi(d_0(p\cdot h^{-1}))-2\Psi(d_0(p))\\ \leq&\frac{\widetilde{\Psi}'(d_0(p))}{2d_0(p)}(d_0^2(p\cdot h)+d_0^2(p\cdot h^{-1})-2d_0^2(p))+C(\Omega)|h|^2\\ \leq&\frac{\widetilde{\Psi}'(d_0(p))}{2d_0(p)}C|h|^2+C(\Omega)|h|^2\leq(C/2+1)C(\Omega)|h|^2,\end{aligned}$$

under the assumption that $p\in\Omega\setminus\{0\}$ and $h\in\mathcal{H}_0$ such that $[p\cdot h^{-1},p\cdot h]\subset\Omega$, where C is the h-semiconcavity constant of d_0^2. This estimate still holds for $p=0$, since $\widetilde{\Psi}'(d_0(p))=\widetilde{\Psi}'(0)=0$ in (3.30). This ends the proof for the case $\widetilde{\Psi}\in C^2(\mathbb{R})$.

For general $\widetilde{\Psi}$, it is sufficient to approximate by standard mollification. Take $\phi_\varepsilon(\cdot)=\varepsilon^{-1}\phi(\cdot/\varepsilon)$, where ϕ is an even, nonnegative, smooth function with compact support such that $\int_{\mathbb{R}}\phi\,dx=1$ and decreasing on $[0,+\infty)$. For every $\varepsilon\in(0,1)$, we can show that $\phi_\varepsilon*\widetilde{\Psi}$ is smooth, even, increasing on $[0,+\infty)$, and locally semiconcave with the semiconcavity constant on any compact set independent of $\varepsilon\in(0,1)$. Since $\phi_\varepsilon*\widetilde{\Psi}\to\widetilde{\Psi}$ locally uniformly as $\varepsilon\to 0$, we can apply the standard stability argument for viscosity solutions to conclude our proof. □

Remark 3.7. Typical examples of the function Ψ satisfying the assumptions of Corollary 3.6 are $\Psi(\tau)=C\tau^\gamma$ with $C>0$ and $\gamma\geq 2$.

While we have proved in Theorem 1.1 that the squared CC distance d_0^2 in an ideal Carnot group is h-semiconcave, it however fails to be an h-semiconvex function even in the Heisenberg group $\mathbb{H}$, the simplest example of ideal Carnot groups. To see this, we present the following result, which suggests a corner-like singularity at every nonzero point in the center; see the green curve in Figure 3. See the forthcoming paper [28] for more discussions about this property.

Proposition 3.8 ([28], Proposition 2)**.** *Let d be the CC distance of the Heisenberg group $\mathbb{H}$ and $d_0=d(\cdot,0)$ in $\mathbb{H}$. For every $p=(0,0,\tau)\in\mathbb{H}$ with $\tau\neq 0$, the following holds:*

$$\lim_{h\in\mathcal{H}_0\setminus\{0\},h\to 0}\frac{d_0^2(p\cdot h)+d_0^2(p\cdot h^{-1})-2d_0^2(p)}{|h|}=-4d_0(p). \tag{3.31}$$

In particular,

$$\lim_{h\in\mathcal{H}_0\setminus\{0\},h\to 0}\frac{d_0^2(p\cdot h)+d_0^2(p\cdot h^{-1})-2d_0^2(p)}{|h|^2}=-\infty.$$

Proof. The key to the proof is the expression of the squared CC distance given in (3.27). Since d_0^2 is symmetric with respect to the xy-plane and rotationally symmetric about the z-axis, without loss of generality, we may assume that $h_1>0$, $h_2=0$, and $z>0$. Moreover, in view of the 1-homogeneity of the CC distance with respect to the group dilation as shown in (2.6), it suffices to prove (3.31) for $e_3=(0,0,1)$. In fact, for a generic $p=(0,0,\tau)=\delta_\tau(e_3)$, we have, as $\mathcal{H}_0\ni h\to 0$,

$$\begin{aligned}&\frac{d_0^2(p\cdot h)+d_0^2(p\cdot h^{-1})-2d_0^2(p)}{|h|}=\frac{\tau^2}{|h|}\left(d_0^2\big(e_3\cdot\delta_{1/\tau}(h)\big)+d_0^2\big(e_3\cdot\delta_{1/\tau}(h)^{-1}\big)-2d_0^2(e_3)\right)\\ =&\frac{\tau}{|\widetilde{h}|}\left(d_0^2\big(e_3\cdot\widetilde{h}\big)+d_0^2\big(e_3\cdot\widetilde{h}^{-1}\big)-2d_0^2(e_3)\right)\to-4\tau d_0(e_3)=-4d_0(p),\end{aligned}$$

where we applied (3.31) at $p = e_3$ with $\widetilde{h} = \delta_{1/\tau}(h) = h/\tau$.

Let us now prove (3.31) at $p = e_3$. By symmetry, we may further take $h = (h_1, 0, 0)$ with $h_1 > 0$. In this case, we have $p \cdot h = (h_1, 0, 1)$ and $p \cdot h^{-1} = (-h_1, 0, 1)$. Our goal is then to show

$$\lim_{h_1 \to 0+} \frac{1}{h_1} \left(d_0^2(h_1, 0, 1) + d_0^2(-h_1, 0, 1) - 2d_0^2(0, 0, 1) \right) = -4d_0(0, 0, 1). \tag{3.32}$$

We use the expression of the squared CC distance given by (3.27), which yields

$$d_0^2(0, 0, 1) = 4\pi, \qquad d_0^2(h_1, 0, 1) = d_0^2(-h_1, 0, 1) = \left(\frac{\theta}{\sin\theta} \right)^2 h_1^2,$$

where $\theta = \theta(h_1) = \mu^{-1}\left(\frac{4}{h_1^2}\right) \to \pi-$ as $h_1 \to 0+$. The equation for θ also gives

$$\frac{4}{h_1^2} = \mu(\theta) = \frac{2\theta - \sin(2\theta)}{2\sin^2(\theta)}, \tag{3.33}$$

which yields

$$\frac{h_1}{\pi - \theta} \to \frac{2}{\sqrt{\pi}}, \quad \text{as } h_1 \to 0+. \tag{3.34}$$

Then using (3.33), combined with (3.28), we get

$$\begin{aligned} \frac{d_0^2(h_1, 0, 1) + d_0^2(-h_1, 0, 1) - 2d_0^2(0, 0, 1)}{h_1} &= 2h_1 \frac{d_0^2(h_1, 0, 1) - d_0^2(0, 0, 1)}{h_1^2} \\ = 2h_1 \left[\left(\frac{\theta}{\sin\theta} \right)^2 - \pi\mu(\theta) \right] &= 2h_1 \frac{\theta(\theta - \pi) + \pi \sin\theta \cos\theta}{\sin^2\theta}. \end{aligned}$$

By (3.34), we can pass to the limit of the relation above as $h_1 \to 0+$ to obtain

$$\lim_{h_1 \to 0+} \frac{d_0^2(h_1, 0, 1) + d_0^2(-h_1, 0, 1) - 2d_0^2(0, 0, 1)}{h_1} = \lim_{h_1 \to 0+} -\frac{2h_1(\theta + \pi)}{\pi - \theta} = -8\sqrt{\pi}.$$

Our proof of (3.32) is complete, since $-8\sqrt{\pi} = -4d_0(0, 0, 1)$. □

4. Applications to Hamilton-Jacobi equations

In this section, we study h-semiconcavity of the viscosity solutions for a class of time-dependent Hamilton-Jacobi equations of the form:

$$\begin{cases} u_t + \Phi\big(|\nabla_H u|\big) = 0, & p \in \mathbb{G}, \ t > 0, \\ u(0, p) = g(p), & p \in \mathbb{G}, \end{cases} \tag{4.35}$$

where u_t denotes the time derivative of u, $\nabla_H u$ is the horizontal gradient in an ideal Carnot group $\mathbb{G}$, and $\Phi : [0, +\infty) \to [0, +\infty)$ is a continuous, convex, non-decreasing function such that $\Phi(0) = 0$. We assume throughout this section that $g \in LSC(\mathbb{G})$, where we recall that $LSC(A)$ denotes the set of lower semicontinuous functions in a set A. In [19] (see also [32] for the case of the Heisenberg group) it was proved that, if $g \in LSC(\mathbb{G})$ and

$$\exists C > 0 \quad \text{such that} \quad g(p) \geq -C(1 + d_0(p)) \quad \text{holds for all } p \in \mathbb{G}, \tag{4.36}$$

then the viscosity solution $u \in LSC([0,+\infty)\times\mathbb{G})$ of the Cauchy problem (4.35) can be obtained by the (metric) Hopf-Lax formula

$$u(t,p) = \inf_{q\in\mathbb{G}} \left[g(q) + t\Phi^*\left(\frac{d(p,q)}{t}\right)\right], \quad (t,p) \in [0,+\infty)\times\mathbb{G}, \tag{4.37}$$

where Φ^* is the Legendre-Fenchel function associated to Φ, that is defined by

$$\Phi^*(s) := \sup_{\tau\geq 0} \{s\tau - \Phi(\tau)\}, \quad s \geq 0. \tag{4.38}$$

For the uniqueness, we refer to [17], where comparison principles are proved for the Cauchy problem with continuous initial data. Using Theorem 1.1, we prove that, under suitable conditions on Φ^*, for all $t > 0$, the viscosity solution of problem (4.35) given by (4.37) is h-semiconcave in space. For the convenience of the reader, we will first show the result in the easiest case when $\Phi(s) = s^2/2$ for $s \geq 0$ and then study the case of a more general Φ.

Let us first recall another known result for the Hopf-Lax function.

Lemma 4.1 ([19])**.** *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Let $d_0 = d(\cdot,0)$ in $\mathbb{G}$. Assume that $g \in LSC(\mathbb{G})$ and satisfies* (4.36)*. Then $u \in LSC([0,+\infty)\times\mathbb{G})$ and there exists a constant $C' > 0$ such that*

$$u(t,p) \geq -C'(1 + d_0(p) + t), \quad \forall\, p \in \mathbb{G}, t > 0. \tag{4.39}$$

Moreover, if $g \in LSC(\mathbb{G})$ is bounded, then the infimum in (4.35) *is actually a minimum and it is attained in a CC ball centred at the point p with radius depending only on Φ and t.*

Our first result is the following.

Theorem 4.2. *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Let $d_0 = d(\cdot,0)$ in $\mathbb{G}$. Assume that $g \in LSC(\mathbb{G})$ satisfies* (4.36)*. Let $\Phi(s) = s^2/2$ for $s \geq 0$ and u be defined as in* (4.37)*. Then $u(t,\cdot)$ is h-semiconcave in $\mathbb{G}$, for every $t > 0$.*

Proof. Since $d(p,q) = d(q^{-1}\cdot p, 0) = d_0(q^{-1}\cdot p)$ holds for all $p,q \in \mathbb{G}$, given the choice $\Phi(s) = s^2/2$ for $s \geq 0$, we have $\Phi^*(s) = s^2/2$ by (4.38) and the function u in (4.37) reduces to

$$u(t,p) = \inf_{q\in\mathbb{G}} \left\{ g(q) + \frac{d_0^2(q^{-1}\cdot p)}{2t}\right\}, \quad (t,p) \in [0,+\infty)\times\mathbb{G}.$$

By Theorem 1.1, for every $q \in \mathbb{G}$ and $t > 0$, the function

$$p \mapsto g(q) + \frac{d_0^2(q^{-1}\cdot p)}{2t}$$

is h-semiconcave with h-semiconcavity constant $C/(2t)$, where $C > 0$ is the h-semiconcavity constant of d_0^2. In view of (4.39), we have $u(t,p) > -\infty$ for any $t > 0$ and $p \in \mathbb{G}$. Then the h-semiconcavity of $u(t,\cdot)$ in $\mathbb{G}$ follows from Proposition 2.19. □

Remark 4.3. Theorem 4.2 implies that $u(t,\cdot)$ defined by (4.37) is also locally Lipschitz continuous with respect to the CC distance, which in turn yields a local Hölder continuity with respect to the Euclidean distance.

We next generalize the previous result for more general Hamilton-Jacobi equations but under additional boundedness assumption on g and locally strong convexity of the Hamiltonian. Here we say that a function f is *locally strongly convex* in an open set $\Omega \subset \mathbb{R}^n$ if for every compact convex set $K \subset \mathbb{R}$, there exists a constant $C(K) > 0$ such that

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \geq \lambda(1-\lambda)C(K)|x-y|^2, \quad \forall\, x, y \in K, \lambda \in [0,1].$$

Theorem 4.4. *Let $\mathbb{G}$ be an ideal Carnot group with CC distance d. Assume that $g \in LSC(\mathbb{G})$ is bounded. Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a continuous, convex, non-decreasing function with $\Phi(0) = 0$. Assume in addition that Φ is coercive in the sense that*

$$\frac{\Phi(\tau)}{\tau} \to +\infty \quad \text{as } \tau \to +\infty, \tag{4.40}$$

and the even extension $\widetilde{\Phi} : \mathbb{R} \to [0,+\infty)$ of Φ is locally strongly convex in $\mathbb{R}$. Let u be the viscosity solution of (4.35) *defined as in* (4.37)*. Then, $u(t,\cdot)$ is h-semiconcave in $\mathbb{G}$ for every $t > 0$.*

Proof. It is well known that the Legendre-Fenchel transform of a coercive, strongly convex function is semiconcave in $\mathbb{R}$; see for example [21, Lemma 4 in Chapter 3.4]. Using (4.40), one can localize this property to prove the Legendre-Fenchel transform of a coercive, locally strongly convex function is locally semiconcave in $\mathbb{R}$. Since $\widetilde{\Phi}$ is locally strongly convex in $\mathbb{R}$, we thus obtain the local semiconcavity of $(\widetilde{\Phi})^*$ in $\mathbb{R}$.

On the other hand, as $\Psi = \Phi^*$ in $[0,+\infty)$, we have $(\widetilde{\Phi})^*\big|_{[0,+\infty)} = \Psi$ under current assumptions on Φ; in other words, Ψ has a semiconcave even extension in $\mathbb{R}$. Now applying Corollary 3.6 with $\Psi = \Phi^*$, we obtain the local h-semiconcavity of $\Phi^*(d_0)$, that is $\Phi^*(d_0)$ is h-semiconcave in any bounded open set $\Omega \subset \mathbb{G}$.

Let us fix $t > 0$ arbitrarily. For any $p \in \mathbb{G}$, we can use the boundedness of g to deduce that

$$g(q) + t\Phi^*\left(\frac{d_0(q^{-1}\cdot p)}{t}\right) \to +\infty \quad \text{as } d_0(q) \to +\infty.$$

By (4.37), it then follows that there exists $\hat{q} \in \mathbb{G}$ depending on p such that

$$u(t,p) = g(\hat{q}) + t\Phi^*\left(\frac{d_0(\hat{q}^{-1}\cdot p)}{t}\right) \leq g(p),$$

which yields

$$t\Phi^*\left(\frac{d_0(\hat{q}^{-1}\cdot p)}{t}\right) \leq g(p) - g(\hat{q}).$$

Applying the boundedness of g again, we are led to $d_0(\hat{q}^{-1}\cdot p) < M$ for some $M > 0$ depending on Φ, t but independent of $p, \hat{q}$. In other words, for every fixed $t > 0$ and $p \in \mathbb{G}$, we have

$$u(t,p) = \inf_{\Omega}\left\{g(q) + t\Phi^*\left(\frac{d_0(q^{-1}\cdot p)}{t}\right)\right\}, \tag{4.41}$$

for any open set $\Omega \subset \mathbb{G}$ satisfying $B_{CC}(p,M) \subset \Omega$.

By the local h-semiconcavity of $\Phi^*(d_0)$, for any $p_0 \in \mathbb{G}$, we see that

$$p \mapsto g(q) + t\Phi^*\left(\frac{d_0(q^{-1}\cdot p)}{t}\right)$$

is h-semiconcave in $B_{CC}(p_0, 1)$ for all $q \in B_{CC}(p_0, M+1)$ with h-semiconcavity constant depending on M. In view of Proposition 2.19, we obtain the h-semiconcavity of $u(t,\cdot)$ in

$B_{CC}(p_0, 1)$ with the same h-semiconcavity constant by taking infimum of the function above over $q \in B_{CC}(p_0, M+1)$ and noticing that

$$u(t, \cdot) = \inf_{q \in B_{CC}(p_0, M+1)} \left\{ g(q) + t\Phi^* \left(\frac{d_0(q^{-1} \cdot p)}{t} \right) \right\},$$

thanks to (4.41) with $\Omega = B_{CC}(p_0, M+1) \supset B_{CC}(p, M)$.

Since the h-semiconcavity constant of $u(t, \cdot)$ in $B_{CC}(p_0, 1)$ is independent of $p_0 \in \mathbb{G}$, we thus obtain the h-semiconcavity of $u(t, \cdot)$ in $\mathbb{G}$. □

Our h-semiconcavity result above can be applied to some particular Hamilton-Jacobi equations such as

$$u_t + \frac{1}{\alpha} |\nabla_H u|^\alpha = 0 \quad \text{in } (0, +\infty) \times \mathbb{G},$$

with a bounded initial value $g \in LSC(\mathbb{G})$ and $1 < \alpha \leq 2$. The case $\alpha = 1$ is not covered by Theorem 4.4, but if $g \in LSC(\mathbb{G})$ is assumed to be bounded and h-semiconcave in $\mathbb{G}$, then we have preservation of the spatial h-semiconcavity of the viscosity solution given by the optimal control formula

$$u(t, p) = \inf_{q \in B_{CC}(p, t)} g(q), \quad t > 0, \ p \in \mathbb{G}.$$

The proof is simply a straightforward application of Proposition 2.19.

It is not our intention to study in detail stationary PDE problems in this paper, but one possible simple application of Theorem 1.1 in this direction is for the eikonal equation

$$|\nabla_H u| = 1 \quad \text{in } \Omega, \tag{4.42}$$

where $\Omega \subset \mathbb{G}$ is a given open set. Let $\mathbf{S} = \mathbb{G} \setminus \Omega$, we define the *CC distance from the set* $\mathbf{S}$ by

$$d_{\mathbf{S}}(p) := \min_{q \in \mathbf{S}} d(p, q) = \min_{q \in \mathbf{S}} d_0(q^{-1} \cdot p).$$

Note that $d_{\mathbf{S}}$ is continuous, and when $\mathbf{S} = \{0\}$, $d_{\mathbf{S}} = d_0$, which is exactly the CC distance from the group identity. It is well known that $u = d_{\mathbf{S}}$ is a viscosity solution of (4.42) satisfying the boundary condition $u = 0$ on $\partial\Omega$. We can use Theorem 1.1 to prove easily that its square is h-semiconcave in Ω.

Proposition 4.5. *Let* $\mathbf{S} \subset \mathbb{G}$ *be a nonempty closed set on an ideal Carnot group* $\mathbb{G}$*. Then* $d_{\mathbf{S}}^2$ *is an h-semiconcave function in* $\mathbb{G} \setminus \mathbf{S}$.

Proof. Observe that

$$d_{\mathbf{S}}^2(p) = \min_{q \in \mathbf{S}} d^2(p, q) = \min_{q \in \mathbf{S}} d_0^2(q^{-1} \cdot p),$$

and for every $q \in \mathbf{S}$, the function $p \mapsto d_0^2(q^{-1} \cdot p)$ is h-semiconcave with h-semiconcavity constant the same as the one of d_0^2. As a result, the proof follows from Proposition 2.19. □

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