

On the equivalence of distributional and synthetic Ricci curvature lower bounds

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Abstract

The goal of the paper is to prove the equivalence of distributional and synthetic Ricci curvature lower bounds for a weighted Riemannian manifold with continuous metric tensor having Christoffel symbols in L^2_{loc} , and with weight in $C^0 \cap W^{1,2}_{\text{loc}}$. The regularity assumptions are sharp, in the sense that they are minimal in order to define the distributional Ricci curvature tensor.

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1 Introduction

Let (M, g) be a smooth n -dimensional Riemannian manifold. One can locally compute the coefficients of the Ricci curvature tensor in terms of the metric tensor g and its first two derivatives. More precisely, denoting by Γ_{ij}^k the Christoffel symbols of g

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_j g_{li} + \partial_i g_{jl} - \partial_l g_{ij}), \quad i, j, k = 1, \dots, n,$$

the Ricci curvature can be written locally as

$$\text{Ric}_{ij} = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{ji}^k \Gamma_{lk}^l - \Gamma_{jl}^k \Gamma_{ik}^l, \quad i, j, k, l = 1, \dots, n, \quad (1.1)$$

where ∂_i is the shorthand notation for $\frac{\partial}{\partial x^i}$, and where we used the Einstein convention that repeated indices are summed.

Given a smooth function V on M , one can define the weighted measure $d\mu = e^{-V} d\text{vol}_g$. Given $N \in [n, \infty]$, the Bakry-Émery N -Ricci curvature tensor of the weighted Riemannian manifold (M, g, μ) is defined by

$$\text{Ric}_{\mu, N} := \text{Ric} + \text{Hess}V - \frac{1}{N-n} \nabla V \otimes \nabla V. \quad (1.2)$$

In case $N = n$, we adopt the convention that V must be constant and that $\text{Ric}_{\text{vol}_g, n} = \text{Ric}$. We also adopt the standard convention that $1/\infty = 0$, so that for $N = \infty$ the last adding term in the right hand side of (1.2) disappears. For $K \in \mathbb{R}$, we say that the Bakry-Émery N -Ricci curvature tensor is bounded below by K , if $\text{Ric}_{\mu, N}(X, X) \geq Kg(X, X)$ for all smooth vector fields X .

Since the local expression of the Ricci tensor involves two derivatives of the metric tensor g , and the Bakry-Émery N -Ricci curvature tensor involves the Ricci tensor and the Hessian of the weight function V , some care is needed if g or V are not twice differentiable. We will work with two approaches to Ricci curvature lower bounds for weighted manifold of regularity below C^2 : the distributional and the synthetic approaches.

Via the distribution theory on smooth manifolds (cf. [22, 18, 17]), one can define a distributional Ricci curvature and generalise the notion of Ricci curvature lower bounds. This approach is suitable to handle the case of a smooth manifold M endowed with a continuous metric g having Christoffel symbols in L_{loc}^2 , as it is apparent from (1.1). Moreover, assuming $V \in C^0 \cap W_{\text{loc}}^{1,2}$ suffices to define a distributional Bakry-Émery N -Ricci curvature tensor of the corresponding weighted space, as one can check using (1.2). Note that this approach assumes the space to be smooth: all the non-smoothness is encoded in the metric tensor and, possibly, in the weighted measure. Let us also mention [23] for a notion of Ricci curvature lower bounds for continuous metrics conformal to smooth ones, based on the concept of viscosity solutions for non-linear elliptic partial differential equations.

A different approach is to drop also the smoothness assumption of the underlying space and consider the general framework of metric measure spaces. A metric measure space is a triplet (X, d, \mathfrak{m}) , where (X, d) is a complete and separable metric space and \mathfrak{m} is a non-negative σ -finite Borel measure (playing the role of reference volume measure). By analysing convexity properties of suitable entropy functionals on the space of probability measures endowed with the Kantorovich-Wasserstein quadratic transportation distance W_2 , Sturm [29, 30] and Lott-Villani [26, 25] devised synthetic notions of Ricci curvature bounded below by some constant $K \in \mathbb{R}$ and dimension bounded above by some $N \in [1, \infty]$. The metric measure spaces satisfying such a synthetic notion of Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$ are called $\text{CD}(K, N)$ spaces. Such a synthetic notion is consistent with the smooth definitions, i.e. a smooth (resp. weighted) Riemannian manifold is a $\text{CD}(K, N)$ space if and only if it has dimension bounded above by N and Ricci curvature bounded below by K (resp. it has Bakry-Émery N -Ricci curvature bounded below by K). Moreover, the class of $\text{CD}(K, N)$ spaces is stable under pointed measure Gromov-Hausdorff convergence [29, 30, 26, 31], and satisfies several geometric and analytic properties such as Brunn-Minkowski and Bishop-Gromov inequalities [30], Poincaré inequality [25, 28], Lévy-Gromov isoperimetric inequality [10] (under an essentially non-branching assumption on geodesics).

Since the conditions $\text{CD}(K, \infty)$ and $\text{CD}(K, N)$ make sense of Ricci curvature lower bounds for manifolds with metrics of regularity below C^2 , a natural question to ask is whether distributional and synthetic Ricci curvature lower bounds continue to agree on metrics with regularity below C^2 . Using regularisations of the metric tensor, Kunzinger, Oberguggenberger and Vickers [20] proved that distributional Ricci curvature bounded below by K implies the $\text{CD}(K, \infty)$ -condition in case M is a compact manifold endowed with a C^1 Riemannian metric, and established the reverse implication under the stronger assumption that g is a $C^{1,1}$ metric tensor satisfying an additional convergence condition on its regularisations.

In this paper, we will prove the following equivalence result:

Theorem 1 (see Theorem 7.2). *Let M be a smooth manifold, g a continuous Riemannian metric with Christoffel symbols in L^2_{loc} , and $V \in C^0 \cap W^{1,2}_{\text{loc}}(M)$ a positive function on M . Define the weighted measure μ as $d\mu := e^{-V} \text{dvol}_g$. Let $N \in [n, \infty]$ and $K \in \mathbb{R}$. The following are equivalent:*

- (i) (M, d_g, μ) is a $\text{CD}(K, N)$ space.
- (ii) The distributional Bakry-Émery N -Ricci curvature tensor is bounded below by K and μ has at most exponential volume growth in the sense of (4.1).

Remark 1.1 (On the smoothness assumption on M). *Note that, for C^0 -Riemannian metrics, the natural class of differentiability of the manifolds is C^1 . However, a C^1 -manifold always possesses a C^∞ -sub-atlas, and one can choose some such sub-atlas whenever convenient.*

Remark 1.2 (Sharpness of the regularity assumptions). *The regularity assumptions on the metric g (resp. on the weight V) in Theorem 1 are the minimal ones in order to define the distributional Ricci curvature tensor (resp. the distributional Bakry-Émery N -Ricci curvature tensor). In this sense, the result is sharp.*

Remark 1.3 (On the volume growth assumption). *The volume growth assumption (4.1) is satisfied by any $\text{CD}(K, N)$ metric measure space, $N \in [1, \infty]$, as proved in [29, Thm. 4.24]. Thus it is necessary for the implication from distributional to synthetic Ricci lower bounds to hold. In Proposition 7.3, we show that (4.1) is satisfied for a smooth manifold endowed with a C^1 -Riemannian metric and a C^1 -weighted measure satisfying a distributional lower bound on the N -Bakry-Émery Ricci tensor.*

Main ingredients in the proof and organisation of the paper. A first important observation is that a smooth manifold endowed with a continuous Riemannian metric and a continuous weight on the volume measure is an infinitesimally Hilbertian metric measure space (in the sense of [3, 14]) satisfying the Sobolev-to-Lipschitz property (see Cor. 4.27). Thus it is a $\text{CD}(K, N)$ space if and only if it is a $\text{RCD}(K, N)$ space. Now, a deep result by Cavalletti-Milman [9] (see also [24] for the extension to the case of σ -finite measures) is that the $\text{RCD}(K, N)$ condition is equivalent to the $\text{RCD}^*(K, N)$ condition.

The main ingredient to prove Theorem 1, is the equivalence of the $\text{RCD}^*(K, N)$ condition (corresponding to a Lagrangian formulation of the curvature-dimension condition) and the Bakry-Émery condition $\text{BE}(K, N)$, which roughly corresponds to the Bochner inequality (and provides a Eulerian formulation of the curvature-dimension condition). Such an equivalence is a deep result that was first established in the case $N = \infty$ by Ambrosio-Gigli-Savaré [3, 4] (after the work [16] in Alexandrov spaces) and later for $N < \infty$ by Erbar-Kuwada-Sturm [13] and Ambrosio-Mondino-Savaré [6]. We will prove that the $\text{BE}(K, N)$ condition is equivalent to having the distributional N -Bakry-Émery Ricci tensor bounded below by K . Most of the work consists in comparing the test objects used in metric measure setting for the $\text{BE}(K, N)$ condition, and the smooth functions and vector fields, which are the test objects in the distributional formulation. More precisely, we will make use of the second order calculus developed in [15] in the metric measure framework. Such a non-smooth approach requires to work with gradient vector fields, while the distributional lower Ricci bounds are classically tested against arbitrary smooth vector fields. To this regard, a key technical tool will be an approximation result of smooth vector fields by gradient vector fields (see Lemma 6.6). Also, since we will work with metrics of low regularity, we will use the parabolic version of the De Giorgi-Nash-Moser regularity (see Proposition 6.9). The proof of the implication from RCD to distributional Ricci lower bounds will be obtained via a contradiction argument involving suitable coverings by balls centred in Lebesgue points of the test objects and using the Besicovitch covering theorem and the Hahn decomposition theorem for signed Radon measures. Such an implication will be established in Theorem 6.11 for the case $N = \infty$

and in Theorem 6.20 for the case $N \in [n, \infty)$. Let us mention that the case $N \in [n, \infty)$ is slightly more subtle, as the distributional N -Bakry-Émery Ricci tensor involves a quadratic non-linearity in the weight (term which is not present in the case $N = \infty$). The converse implication will be obtained in Theorem 7.1.

Along the way, in Sec. 4, we will compare the theory of classical Sobolev spaces on manifolds (see for instance [19]) with the generalised notion of differentiability introduced by Cheeger [11] and further investigated by Ambrosio, Gigli and Savaré [2]; the framework will be a smooth manifold endowed with a continuous Riemannian metric g and a positive continuous weight on the volume measure.

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Notation

Consider a smooth manifold M . When equipped with a Riemannian metric, we will always denote the metric by g . Moreover, we denote by d_g the distance and by vol_g the volume form induced by g .

In local coordinates g_{ij} will denote the coefficients of g as a matrix, g^{ij} the coefficients of the inverse matrix, and $|g|$ the determinant of $(g_{ij})_{ij}$. We will write Γ for the Levi-Civita connection of g and denote its coefficients by Γ_{ij}^k .

The covariant derivative with respect to Γ will be denoted by either ∇ or ∇_c , with the only exception that $\nabla f = g^{ij} \partial_i f \partial_{x^j} = (df)^\sharp$ always denotes the gradient field of f and never its covariant derivative (which in that particular case equals the differential df).

For a vector field V and a differentiable function f , we define $V(f) := df(V)$ to be the action of V on f . For a vector field V and a k -form $\omega \in \Omega^k(M)$ on M , where $k \geq 1$, we denote by $\iota_V \omega \in \Omega^{k-1}(M)$ the contraction of the first entry of ω with V .

Given a topological Hausdorff space X , we denote by $\text{Meas}(X)$ the space of Radon measures on X equipped with the total variation norm $\|\cdot\|_{TV}$. Given a finite dimensional vector space V with an inner product $\langle \cdot, \cdot \rangle$, we denote by $|\cdot|_{HS}$ the Hilbert-Schmidt norm on the space of linear operators on V .

2 Distributional calculus with a Riemannian metric of low regularity

In this section, we recall some elements of the theory of distributions on manifolds, including the distributional Riemann tensor; for more details, we refer to [17], [18], [20], and [22].

Let M be a smooth real manifold of dimension n , and let (E, M, π) be a vector bundle over M . To keep notation short, we will simply write E instead of (E, M, π) . For $0 \leq k \leq \infty$, we denote by $\Gamma^k(M, E)$ (resp. $\Gamma_c^k(M, E)$) the space of C^k -sections (resp. with compact support). In the case $k = \infty$, we often drop the superscript to simplify notation. We denote $\mathcal{T}_s^r(M) := (\otimes_{i=1}^r TM) \otimes (\otimes_{i=1}^s T^*M)$ and $\mathcal{T}_s^r := \Gamma(M, \mathcal{T}_s^r(M))$. We will also write $\mathfrak{X}(M)$ for $\mathcal{T}_0^1(M)$ and $\mathfrak{X}^*(M)$ for $\mathcal{T}_1^0(M)$. For some vector bundle F and $V \subset M$ open, we say that a sequence ω_j converges to ω in $\Gamma_c^k(V, F)$ if there exists a compact set $K \subset V$ such that $\text{supp } \omega_j, \text{supp } \omega \subset K$ and $\nabla^l \omega_j \rightarrow \nabla^l \omega$ uniformly on K for all $l \leq k$.

We denote by $\text{Vol}(M)$ the vector bundle of 1-densities, i.e. the one dimensional vector bundle with transition functions $\Psi_{\alpha\beta}(x) = |\det D(\psi_\beta \circ \psi_\alpha^{-1})(x)|$, where ψ_α, ψ_β denote local charts into \mathbb{R}^n . A section $\omega \in \Gamma_c^k(M, \text{Vol}(M))$ is called a *test volume*. The *space of distributions of order k on M* is defined as the topological dual of $\Gamma_c^k(M, \text{Vol}(M))$ ([18], Sec. 3.1), i.e.

$$\mathcal{D}'^{(k)}(M) := \Gamma_c^k(M, \text{Vol}(M))'.$$

The space of distributional (r, s) -tensor fields of order k is given by

$$\mathcal{D}'^{(k)}\mathcal{T}_s^r(M) := \Gamma_c^k(M, T_r^s(M) \otimes \text{Vol}(M))'.$$

It is known that

$$\mathcal{D}'^{(k)}(\mathcal{T}_s^r)_{C^k}(M) \cong \mathcal{D}'(M) \otimes_{C^k(M)} \mathcal{T}_s^r(M) \cong L_{C^k(M)}(\mathfrak{X}^*(M)_{C^k}^r \times \mathfrak{X}(M)_{C^k}^s; \mathcal{D}'(M)),$$

where the latter denotes the $C^k(M)$ -module of $C^k(M)$ -multilinear maps from $\mathfrak{X}^*(M)_{C^k}^r \times \mathfrak{X}(M)_{C^k}^s$ to $\mathcal{D}'^{(k)}(M)$ (cf. [18]).

Definition 2.1 (Action of a vector field on a distribution, [22]). *For a scalar distribution $T \in \mathcal{D}'(M)$ and a vector field X , we define the action of X on T via*

$$\langle X(T), \omega \rangle_{\mathcal{D}', \mathcal{D}} := -\langle T, \mathcal{L}_X \omega \rangle_{\mathcal{D}', \mathcal{D}},$$

where ω is a test volume and $\mathcal{L}_X \omega$ denotes the Lie derivative of ω in the direction of X .

Definition 2.2 ([22], Def. 4.1). *A distribution $g \in \mathcal{D}'\mathcal{T}_2^0(M)$ is called a generalised Riemannian metric, if it satisfies*

$$\begin{aligned} g(X, Y) &= g(Y, X) \quad \text{in } \mathcal{D}'(M) \quad \text{and} \\ g(X, Y) &= 0 \quad \forall Y \in \mathfrak{X}(M) \quad \implies \quad X = 0. \end{aligned}$$

Using partition of unity, it is clear that one can localise and reduce the computations to distributions on \mathbb{R}^n .

In order to generalise the Riemann curvature tensor, it is appropriate to consider a Riemannian metric that is locally given as a positive definite matrix such that the coefficients g_{ij} and the coefficients of the inverse matrix g^{ij} are both in $L_{\text{loc}}^\infty \cap H_{\text{loc}}^1$ (cf. [18], 5.2.1). This condition is automatically met, if $g \in C^0(M)$ and admits L_{loc}^2 -Christoffel symbols (see Lemma 3.8).

In order to define a distributional covariant derivative, one generalises the first Koszul formula. For smooth vector fields X, Y, Z one can define $\nabla_X^b Y \in \mathcal{D}'\mathcal{T}_1^0(M)$ as:

$$\langle \nabla_X^b Y, Z \rangle := \frac{1}{2} (X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]))$$

in $\mathcal{D}'(M)$. Raising the index gives the distributional covariant derivative $\nabla_X Y := (\nabla_X^b Y)^\sharp$ and a computation in local coordinates yields that indeed

$$(\nabla_X Y)_s = \delta_i^s (X^k \partial_k Y^i) + \Gamma_{ik}^s X^i Y^k \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

For a 1-form θ , the distributional covariant derivative is defined via

$$\langle \nabla_X \theta, Z \rangle := X(\langle \theta, Z \rangle) - \langle \theta, \nabla_X Z \rangle \quad \text{in } \mathcal{D}'(M),$$

which, when spelled out in local coordinates, coincides with the classical covariant derivative of a 1-form. We have all the ingredients to define the distributional Riemann tensor. Here, it is crucial that the coefficients of the covariant derivatives are in L_{loc}^2 , as the expression of the Riemann tensor involves a quadratic term in the Christoffel symbols. Following the notation of [22, Definition 3.3], we have that

$$\langle \text{Riem}(X, Y)Z, \theta \rangle = X\langle \nabla_Y Z, \theta \rangle - Y\langle \nabla_X Z, \theta \rangle - \langle \nabla_Y Z, \nabla_X \theta \rangle_{L^2} + \langle \nabla_X Z, \nabla_Y \theta \rangle_{L^2} - \langle \nabla_{[X, Y]} Z, \theta \rangle$$

in $\mathcal{D}'(M)$. All the quantities in the right hand side can be computed locally. Thus, in local coordinates, the Riemann tensor can be written as

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{kj}^s \Gamma_{is}^l - \Gamma_{ki}^s \Gamma_{js}^l, \quad (2.1)$$

in the sense of distributions on \mathbb{R}^n . Note that in local coordinates, the components of X, Y, Z and θ “are part of the test function”. Note that, if g is a smooth Riemannian metric, the expression (2.1) coincides with the classical Riemann tensor (cf. [27]). For a C^0 -metric g admitting L^2_{loc} -Christoffel symbols, let e_i be a $C^0 \cap W^{1,2}_{\text{loc}}$ -local frame in $\mathfrak{X}_{C^0}(M)$. Note that this exists by Lemma 3.8. For two smooth vector fields X, Y , the distributional Ricci tensor of g is then given by

$$\text{Ric}(X, Y) = \sum_i g(e_i, \text{Riem}(e_i, X)Y).$$

It is immediate to check that Ric is a symmetric bilinear form that can be expressed in local coordinates as:

$$\text{Ric}_{jk} = R^p_{pj k} = \partial_p \Gamma^p_{jk} - \partial_j \Gamma^p_{pk} + \Gamma^s_{kj} \Gamma^p_{ps} - \Gamma^s_{kp} \Gamma^p_{js}, \quad (2.2)$$

which coincides with the classical expression in the smooth case. We next define distributional lower bounds of the Ricci curvature. We say that a one-density is non-negative if it can be written as $\phi \text{vol}_g \in \text{Vol}(M)$ for some non-negative $\phi \in C^\infty(M)$.

Definition 2.3. *Let $K \in \mathbb{R}$. We say that a C^0 -Riemannian metric g with L^2_{loc} Christoffel symbols satisfies $\text{Ric} \geq K$ in the distributional sense if, for all $X \in \mathfrak{X}(M)$, it holds $\text{Ric}(X, X) \geq Kg(X, X)$ in $\mathcal{D}'^{(1)}(M)$, i.e. for all non-negative test volumes ω it holds,*

$$\int \text{Ric}(X, X)\omega \geq \int Kg(X, X)\omega.$$

We conclude this section by recalling how to perform convolutions on a manifold. Fix a function $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\text{supp } \rho \subset B_1(0)$, $\rho \geq 0$, and $\int_{\mathbb{R}^n} \rho = 1$. Define $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon})$. For a smooth manifold M , fix an atlas (U_α, ψ_α) such that $(U_\alpha)_\alpha$ is locally finite and each U_α is relatively compact, and fix a partition of unity η_α subordinate to the atlas. Choose a family of smooth cutoff functions χ_α such that $\text{supp } \chi_\alpha \subset U_\alpha$ and $\chi_\alpha = 1$ on $\text{supp } \eta_\alpha$. For any tensor $T \in L^1_{\text{loc}}(T^r_s(M))$, define

$$T * \rho_\varepsilon(x) := \sum_\alpha \chi_\alpha(x) \psi_\alpha^*((\psi_{\alpha*}(\eta_\alpha T)) * \rho_\varepsilon)(x). \quad (2.3)$$

3 A weak formulation of Bochner’s formula for metrics of low regularity

3.1 Smooth calculus on manifolds

In this subsection, we will recall some basic calculus concepts from smooth Riemannian geometry. More details can be found in [27].

Lemma 3.1. *Let g be a smooth Riemannian metric on M and f, h be smooth functions on M . The following equations hold in local coordinates:*

- (i) $\nabla f = g^{ik}(\partial_i f)\partial_{x^k}$.
- (ii) $\text{vol}_g = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$.
- (iii) $\Delta f = \partial_j g^{ij} \partial_i f + g^{ij} \partial_{ij} f + \frac{1}{2} \text{tr}(g^{-1} \partial_i g) g^{ij} \partial_j f = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f)$.
- (iv) $\langle \nabla f, \nabla h \rangle_g = \partial_i f g^{ij} \partial_j h$.

Note that these expressions still hold for $g \in C^0(M)$, $\Gamma^k_{ij} \in L^2_{\text{loc}}$. The Hessian is given by

$$\text{Hess}(\partial_{x^i}, \partial_{x^j}) = \partial_{ij} f - \Gamma^s_{ij} \partial_s f.$$

For all smooth functions ϕ, h , and f , it holds

$$2\text{Hess}f(\nabla\phi, \nabla h) = \langle \nabla\phi, \langle \nabla f, \nabla h \rangle_g \rangle_g + \langle \nabla h, \langle \nabla f, \nabla\phi \rangle_g \rangle_g - \langle \nabla f, \langle \nabla h, \nabla\phi \rangle_g \rangle_g. \quad (3.1)$$

Now let X be a smooth vector field and ω a smooth 1-form. Then the musical isomorphisms $\sharp : T^*M \rightarrow TM$ and $\flat : TM \rightarrow T^*M$ are defined via

$$X^\flat(Y) = \langle X, Y \rangle_g, \text{ and } \omega^\sharp(Y) = \langle \omega^\sharp, Y \rangle_g,$$

for all vector fields Y . Note that the musical isomorphisms come from the smooth setting, but also make sense for C^0 -Riemannian metrics.

3.2 Bochner's formula for vector fields on smooth weighted manifolds

In this subsection we briefly recall the generalisation of Bochner's formula to the case of a smooth weighted manifold. Denote by Δ_H the Hodge-Laplacian and recall that for $f \in C^\infty(M)$ it holds $\Delta_H f = -\Delta f$. Then:

Proposition 3.2. *For a smooth manifold M with a smooth Riemannian metric g and any smooth vector field X , it holds*

$$-\frac{1}{2}\Delta_H |X|^2 = -\langle (\Delta_H X^\flat)^\sharp, X \rangle + |\nabla X|_{\text{HS}}^2 + \text{Ric}(X, X). \quad (3.2)$$

Now let (M, g) be a smooth Riemannian manifold and let μ be a measure on M such that $d\mu = h^2 d\text{vol}_g$ for a smooth and positive function h . As usually, we write for a C^1 -function u , that $(\nabla u)^i = g^{ij} \partial_j u$. We can now define the associated divergence operator div_μ by

$$\text{div}_\mu(X) = \frac{1}{h^2} \text{div}(h^2 X) = \frac{1}{h^2} \left(h^2 \partial_i X^i + 2h \partial_i h X^i + \frac{1}{2} h^2 X^i \text{tr}(g^{-1} \partial_i g) \right) \quad (3.3)$$

and the weighted Laplace operator Δ_μ

$$\Delta_\mu := \text{div}_\mu \circ \nabla = \frac{1}{h^2} \text{div}(h^2 \nabla) = \Delta + \frac{2\langle \nabla h, \nabla \rangle}{h}.$$

The divergence operator satisfies

$$\int_M f \text{div}_\mu(X) d\mu = - \int_M X(f) d\mu, \quad (3.4)$$

for any function $f \in C_c^\infty(M)$ and any smooth vector field X . It then follows that for $u, v \in C^\infty(M)$ such that u or v are compactly supported, we have that

$$\int_M u \Delta_\mu v d\mu = - \int_M \langle \nabla u, \nabla v \rangle_g d\mu = \int_M \Delta_\mu uv d\mu.$$

Sometimes it is notationally convenient to write $h^2 = e^{-V}$ for some smooth function V .

Definition 3.3. *Let $N \geq n$ and V, h as above. The generalised Bakry-Émery N -Ricci tensor is defined by*

$$\text{Ric}_{\mu, N} := \text{Ric} + \nabla^2 V - \frac{1}{N-n} \nabla V \otimes \nabla V.$$

In the case $n = N$, we use the convention that the only admissible V is constant. For $N = \infty$, we get

$$\text{Ric}_{\mu, \infty} := \text{Ric} + \nabla^2 V = \text{Ric} + \text{Hess}[-2 \log h].$$

We will now generalise the Hodge star operator and the codifferential to this setting. Recall the Hodge star operator $*$: $\Omega^p(M) \rightarrow \Omega^{n-p}(M)$ in a Riemannian manifold without weight, where $*\beta$ is defined via $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}_g$ for all $\alpha \in \Omega^p(M)$. This shall motivate our next definition.

Definition 3.4. *The weighted Hodge star operator $*_\mu : \Omega^p(M) \rightarrow \Omega^{n-p}(M)$ is defined as h^2* , where $*$ denotes the classical Hodge star operator. It satisfies*

$$\alpha \wedge *_\mu \beta = \langle \alpha, \beta \rangle_g \mu,$$

where we interpret the n -form $\alpha \wedge *_\mu \beta$ as a measure on M .

It directly follows that for all $p \in \{0, \dots, n\}$ and $\beta \in \Omega^p(M)$, it holds

$$*_\mu^{-1} \beta = *^{-1} \left(\frac{1}{h^2} \beta \right).$$

As in the unweighted case, we get that $\delta_\mu := (-1)^p *_\mu^{-1} d *_\mu$ is adjoint to d in the $L^2(M, \mu)$ -inner product, i.e.

$$\int_M \langle d\alpha, \beta \rangle_g d\mu = (-1)^p \int_M \langle \alpha, *_\mu^{-1} d *_\mu \beta \rangle_g d\mu,$$

for all compactly supported $(p-1)$ -form α and p -form β .

Definition 3.5. *The above operator $\delta_\mu : \Omega^\bullet(M) \rightarrow \Omega^{\bullet-1}(M)$ is called the weighted codifferential. We denote by $\Delta_{\mu, H} : \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$ the weighted Laplace-Beltrami operator, which is defined by*

$$\Delta_{\mu, H} := d\delta_\mu + \delta_\mu d.$$

We now aim to generalise (3.2) to the weighted case. In local normal coordinates, we can compute that

$$-\frac{1}{2} \Delta_{\mu, H} |\alpha|^2 = \frac{1}{2} \Delta_\mu |\alpha|^2 = \frac{1}{2} \Delta |\alpha|^2 + \frac{1}{h} \langle \nabla h, \nabla |\alpha|^2 \rangle = -\frac{1}{2} \Delta_H |\alpha|^2 + \frac{2}{h} \partial_j h \alpha_k \partial_j \alpha_k$$

and

$$\text{Ric}_{\mu, \infty}(\alpha^\sharp, \alpha^\sharp) = \text{Ric}(\alpha^\sharp, \alpha^\sharp) + \text{Hess}[-2 \log h](\alpha^\sharp, \alpha^\sharp) = \text{Ric}(\alpha^\sharp, \alpha^\sharp) - 2\alpha_j \alpha_k \left(\frac{h \partial_{jk}^2 h - \partial_j h \partial_k h}{h^2} \right). \quad (3.5)$$

Finally, we get that

$$\begin{aligned} -\langle \alpha, \Delta_{\mu, H} \alpha \rangle &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h} (\partial_j h \partial_j \alpha_k \alpha_k - \partial_k h \partial_j \alpha_k \alpha_j) + \frac{2}{h} \partial_j \alpha_k \partial_k h \alpha_j + \partial_j \left(\frac{2\partial_k h}{h} \right) \alpha_k \alpha_j \\ &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h} \partial_j h \partial_j \alpha_k \alpha_k - \frac{2}{h} \partial_k h \partial_j \alpha_k \alpha_j + \frac{2}{h} \partial_j \alpha_k \partial_k h \alpha_j + \alpha_k \alpha_j \text{Hess}[2 \log h]_{kj} \\ &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h} \partial_j h \partial_j \alpha_k \alpha_k + \alpha_k \alpha_j \text{Hess}[2 \log h]_{kj}. \end{aligned}$$

Recalling (3.2), this yields

Proposition 3.6. *Let (M, g, μ) be a weighted Riemannian manifold with smooth metric and smooth weight. Then, for every smooth 1-form α , it holds*

$$-\frac{1}{2} \Delta_{\mu, H} |\alpha|^2 = -\langle \alpha, \Delta_{\mu, H} \alpha \rangle + \text{Ric}_{\mu, \infty}(\alpha^\sharp, \alpha^\sharp) + |\nabla \alpha|_{HS}^2.$$

In other terms, for every smooth vector field X , it holds

$$-\frac{1}{2} \Delta_{\mu, H} |X|^2 = -\langle X, (\Delta_{\mu, H} X^\flat)^\sharp \rangle + \text{Ric}_{\mu, \infty}(X, X) + |\nabla X|_{HS}^2. \quad (3.6)$$

3.3 A Bochner's formula for weighted manifolds of lower regularity

We can generalise Definition 3.3 to the distributional case:

Definition 3.7. *Let M be a smooth manifold and $N \in [n, \infty]$. Let g be a C^0 -Riemannian metric with L^2_{loc} Christoffel symbols and let $h \in C^0(M, (0, \infty)) \cap W^{1,2}_{\text{loc}}(M)$, $V \in C^0 \cap W^{1,2}_{\text{loc}}(M)$ such that $h^2 = e^{-V}$. We define the distributional Bakry-Émery N -Ricci-curvature tensor of the weighted manifold $(M, g, h^2 \text{vol}_g)$ as*

$$\text{Ric}_{\mu, N} = \text{Ric} + \nabla_g^2 V - \frac{1}{N-n} \nabla V \otimes \nabla V \in \mathcal{D}'\mathcal{T}_2^0. \quad (3.7)$$

The case $N = \infty$ can also be rewritten as:

$$\text{Ric}_{\mu, \infty} := \text{Ric} - 2\text{Hess}[\log h] \in \mathcal{D}'\mathcal{T}_2^0. \quad (3.8)$$

In the following, we will integrate by parts, to get a version of Bochner's formula that only involves first derivatives of g and h . First we make the following useful observation: Assume g to be a smooth metric on the manifold M . Then the Christoffel symbols (of the second kind) are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{il} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

We can recover the Christoffel symbols of the first kind by multiplication with g , i.e.

$$\Gamma_{ij,l} = \frac{1}{2} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = g_{lk} \Gamma_{ij}^k = g(\nabla_{\partial_i} \partial_j, \partial_l). \quad (3.9)$$

Now note that

$$(\Gamma_{il,j} + \Gamma_{jl,i}) = \partial_l g_{ij}. \quad (3.10)$$

This follows directly from the definition of the Levi-Civita connection, hence the last identity holds in a weak sense for the distributional Levi-Civita connection for g of lower regularity. More precisely:

Lemma 3.8. *Let g be a C^0 -Riemannian metric such that g admits L^2_{loc} -Christoffel symbols (of the second kind). Then $g \in W^{1,2}_{\text{loc}}(M)$.*

Proof. As g is in C^0 , it follows from (3.9), that the Christoffel symbols of the first kind are in L^2_{loc} as well. Then using (3.10) integration by parts and the linearity of the Christoffel symbols of the first kind in terms of g , we get that all partial derivatives of g are in L^2_{loc} . \square

Recall that (3.6) holds on a smooth manifold. Using that for functions $\Delta_\mu = -\Delta_{\mu, H}$ and testing with a function $\phi \in C_c^\infty(M)$, we obtain

$$\int_M \left(-\frac{1}{2} \Delta_\mu |X|^2 - \langle (\Delta_{\mu, H} X^b)^\sharp, X \rangle + |\nabla X|^2 + \text{Ric}_{\mu, \infty}(X, X) \right) \phi \, d\mu = 0.$$

We will again integrate by parts, where necessary, to spell this out in local coordinates. By potentially using a partition of unity, we can assume that ϕ is supported in one coordinate patch. We have that

$$\int \frac{1}{2} \langle \nabla |X|^2, \nabla \phi \rangle_g \, d\mu = \int -\frac{1}{2} \Delta_\mu |X|^2 \phi \, d\mu$$

and

$$\begin{aligned}
\int_M \text{Ric}_{\mu, \infty}(X, X) \phi \, d\mu &= \int X^j X^k (\partial_p \Gamma_{jk}^p - \partial_j \Gamma_{pk}^p + \Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \phi h^2 \sqrt{|g|} \, dx^1 \dots dx^n \\
&\quad + 2 \int X^j X^k (\partial_j h \partial_k h - h \partial_{jk}^2 h + h \partial_s h \Gamma_{kj}^s) \phi \sqrt{|g|} \, dx^1 \dots dx^n \\
&= \int X^j X^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \phi h^2 \sqrt{|g|} \, dx^1 \dots dx^n \\
&\quad - \int \Gamma_{jk}^p \partial_p (X^j X^k \phi h^2 \sqrt{|g|}) \, dx^1 \dots dx^n + \int \Gamma_{pk}^p \partial_j (X^j X^k \phi h^2 \sqrt{|g|}) \, dx^1 \dots dx^n \\
&\quad + 2 \int X^j X^k (\partial_j h \partial_k h + h \partial_s h \Gamma_{kj}^s) \phi \sqrt{|g|} \, dx^1 \dots dx^n \\
&\quad + 2 \int \partial_j h \partial_k (X^j X^k h \phi \sqrt{|g|}) \, dx^1 \dots dx^n.
\end{aligned}$$

The above holds for any smooth g and h . Using that each function $h \in C^0 \cap W_{\text{loc}}^{1,2}$ and each metric $g \in C^0$ with L_{loc}^2 -Christoffel symbols can be approximated in the $C^0 \cap W_{\text{loc}}^{1,2}$ topology by smooth objects, we get:

Proposition 3.9. *Let M be a smooth manifold, g a C^0 -Riemannian metric with L_{loc}^2 Christoffel symbols and $h \in C^0(M, (0, \infty)) \cap W_{\text{loc}}^{1,2}(M)$, $V \in C^0 \cap W_{\text{loc}}^{1,2}(M)$ such that $h^2 = e^{-V}$.*

Then, for every smooth vector field X , it holds:

$$-\frac{1}{2} \Delta_\mu |X|^2 - \langle (\Delta_H X^b)^\sharp, X \rangle + |\nabla X|^2 + \text{Ric}_{\mu, \infty}(X, X) = 0 \quad \text{in } \mathcal{D}'(M). \quad (3.11)$$

For what will come next, it is convenient to investigate the other terms of the formula more closely:

$$\langle (\Delta_H X^b)^\sharp, X \rangle = |dX^b|^2 + |\delta_\mu X^b|^2 = |dX^b|^2 + |\text{div}_\mu(X)|^2.$$

Note that if locally $X = X^i \partial_{x^i}$ then $X^b = g_{ij} X^i dx^j$ and hence

$$dX^b = d(g_{ij} X^i dx^j) = \partial_k (g_{ij} X^i) dx^k \wedge dx^j = \sum_{j < k} (\partial_k (g_{ij} X^i) - \partial_j (g_{ik} X^i)) dx^k \wedge dx^j.$$

It follows that

$$|dX^b|^2 = \sum_{j < k} \sum_{l < m} g^{jl} g^{km} (\partial_k (g_{ij} X^i) - \partial_j (g_{ik} X^i)) (\partial_m (g_{pl} X^p) - \partial_l (g_{pm} X^p)).$$

We also know that

$$|\text{div}_\mu(X)|^2 = \frac{1}{h^4} \left(h^2 \partial_i X^i + 2h \partial_i h X^i + \frac{1}{2} h^2 X^i \text{tr}(g^{-1} \partial_i g) \right)^2.$$

Recalling Lemma 3.8 and arguing by density, we get that the following formula holds for all $g \in C^0$ with

L^2_{loc} Christoffel symbols, all $\phi \in C^0 \cap W^{1,2}_{\text{loc}}(M)$ and all positive $h \in C^0 \cap W^{1,2}_{\text{loc}}(M)$:

$$\begin{aligned}
& \int X^j X^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \phi h^2 \sqrt{|g|} dx^1 \dots dx^n - \int \Gamma_{jk}^p \partial_p (X^j X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
& + \int \Gamma_{pk}^p \partial_j (X^j X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
& + 2 \int X^j X^k (\partial_j h \partial_k h + h \partial_s h \Gamma_{kj}^s) \phi \sqrt{|g|} dx^1 \dots dx^n + 2 \int \partial_j h \partial_k (X^j X^k h \phi \sqrt{|g|}) dx^1 \dots dx^n \\
& = -\frac{1}{2} \int \langle \nabla |X|^2, \nabla \phi \rangle_g h^2 \sqrt{|g|} dx^1 \dots dx^n + \int \left(\partial_i X_i + X_i \frac{1}{2} \text{tr}(g^{-1} \partial_i g) + \frac{2}{h} \partial_i h X^i \right)^2 h^2 \phi \sqrt{|g|} dx^1 \dots dx^n \\
& + \int \sum_{j < k} \sum_{l < m} g^{jl} g^{km} (\partial_k (g_{ij} X^i) - \partial_j (g_{ik} X^i)) (\partial_m (g_{pl} X^p) - \partial_l (g_{pm} X^p)) \phi h^2 \sqrt{|g|} dx^1 \dots dx^n \\
& - \int \left(\sum_{i,j,r,s} (\partial_i X^s + \Gamma_{ip}^r X^p) (\partial_j X^r + \Gamma_{jq}^s X^q) g^{ij} g_{sr} \right) \phi h^2 \sqrt{|g|} dx^1 \dots dx^n. \tag{3.12}
\end{aligned}$$

4 First order calculus

In this section we recall the notion of Cheeger energy and the associated Sobolev space $W_w^{1,2}$, which provide a well settled first order calculus for general metric measure spaces. On a smooth manifold endowed with a continuous metric and a continuous weight on the measure, one can define the first order quadratic Sobolev space H_1^2 also using local charts and distributional weak derivatives. After recalling the two approaches, we will show that they actually coincide, in the latter framework. The results will be useful in the establishing the equivalence of distributional and synthetic Ricci curvature lower bounds.

4.1 The Cheeger energy

In this subsection we recall a generalised notion of modulus of gradients and Sobolev functions in metric measure spaces introduced by Cheeger in [11] and further analysed by Ambrosio, Gigli and Savaré [2].

Throughout the section, (X, d) is a complete and separable metric space. The slope (or local Lipschitz constant) of a real valued function $F : X \rightarrow \mathbb{R}$ is defined by

$$|DF|(x) := \limsup_{y \rightarrow x} \frac{|F(x) - F(y)|}{d(x, y)},$$

if $\{x\}$ is not isolated, and 0 otherwise.

We endow (X, d) with a non-negative σ -finite Borel measure, obtaining the metric measure space (X, d, \mathbf{m}) . Throughout the rest of this work, we assume that there exists a bounded Borel Lipschitz map $V : X \rightarrow [0, \infty)$ such that (cf. [2, Ch. 4])

V is bounded on each compact set $K \subset X$ and

$$\int_X e^{-V^2} d\mathbf{m} \leq 1. \tag{4.1}$$

The metric speed of a curve $\gamma : [0, 1] \rightarrow X$ is given by $|\dot{\gamma}_t| := \lim_{h \rightarrow 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|}$. The curve γ is said to be absolutely continuous if $t \rightarrow |\dot{\gamma}_t| \in L^1((0, 1))$. We next recall the notion of test plan and weak upper gradient. We will use the conventions of [15], after [2].

Definition 4.1. Let $\pi \in \mathcal{P}(C([0, 1], X))$. We say that π is a test plan if there exists a constant $C(\pi) > 0$ such that

$$(e_t)_\# \pi \leq C(\pi) \mathbf{m} \text{ for all } t \in [0, 1]$$

and

$$\int \int_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) < \infty.$$

We use the convention that if γ is not absolutely continuous, then $\int_0^1 |\dot{\gamma}_t|^2 dt = \infty$.

Definition 4.2. Given $f : X \rightarrow \mathbb{R}$ a \mathbf{m} -measurable function, then a \mathbf{m} -measurable function $G : X \rightarrow [0, \infty]$ is called a weak upper gradient of f , if

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq \int \int_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma) < \infty, \quad \text{for all test plans } \pi. \quad (4.2)$$

We say that f is in the Sobolev class $S^2(X, \mathbf{d}, \mathbf{m})$, if there exists $G \in L^2(\mathbf{m})$ such that (4.2) holds.

The discussion in [2], Prop. 5.7 and Def. 5.9 shows the existence of a weak upper gradient $|Df|_w$ such that $|Df|_w \leq G$ for all other weak upper gradients G . We will call it the *minimal weak upper gradient* of f .

Definition 4.3. The Sobolev space $W_w^{1,2}(X, \mathbf{d}, \mathbf{m})$ is defined as $L^2 \cap S^2(X)$ and becomes a Banach space with the norm

$$\|f\|_{W_w^{1,2}(X)}^2 = \|f\|_{L^2}^2 + \||Df|_w\|_{L^2}^2.$$

Definition 4.4. The Cheeger energy is defined in the class of \mathbf{m} -measurable functions by

$$\text{Ch}(f) := \begin{cases} \frac{1}{2} \int_X |Df|_w^2 d\mathbf{m} & \text{if } f \text{ has a weak upper gradient in } L^2(X, \mathbf{m}), \\ \infty & \text{otherwise,} \end{cases}$$

with proper domain $D(\text{Ch}) = \{f : X \rightarrow [0, \infty], \mathbf{m}\text{-measurable}, \text{Ch}(f) < \infty\} = S^2(X, \mathbf{d}, \mathbf{m})$.

A metric measure space is said *infinitesimally Hilbertian* if Ch is a quadratic form or, equivalently, if $W_w^{1,2}(X, \mathbf{d}, \mathbf{m})$ is a Hilbert space [3, 14]. In this case, we can define the associated Dirichlet form $\mathcal{E} : D(\text{Ch})^2 \rightarrow \mathbb{R}$, via $\mathcal{E}(f, f) = 2\text{Ch}(f)$ [3, Ch. 4.3].

Proposition 4.5 ([3], Def. 4.12 and Prop. 4.14). For any $f, g \in D(\text{Ch})$, it holds

$$\mathcal{E}(f, g) = \int_X G(f, g) d\mathbf{m},$$

where

$$G(f, g) = \lim_{\varepsilon \searrow 0} \frac{|D(f + \varepsilon g)|_w^2 - |Df|_w^2}{2\varepsilon} \in L^1(X, \mathbf{m}).$$

By the theory of Dirichlet forms (cf. [8]), there exists an associated Laplace operator $\Delta : D(\Delta) \rightarrow L^2(X, \mathbf{m})$, given by

$$-\int_X \Delta f h d\mathbf{m} = \int_X G(f, h) d\mathbf{m}.$$

Here $D(\Delta) = \{f \in L^2(\mathbf{m}) : \Delta f \text{ exists, } \Delta f \in L^2\}$ is dense in L^2 . Moreover, it induces a linear semi-group $(H_t)_{t \geq 0}$, $H_t : L^2(X) \rightarrow L^2(X)$ such that $H_{t+s} = H_s H_t$, for all $f \in L^2$, we have that $\lim_{t \rightarrow 0} H_t f = f$, and for all $t > 0$, we have that $H_t f \in D(\Delta)$ and $\frac{d}{dt} H_t f = \Delta H_t f$. $(H_t)_{t \geq 0}$ is called *heat flow* semi-group. Write f_t for $H_t(f_0)$. By [2, (4.26)], we have that

$$\text{Ch}(f_t) \leq \inf \left\{ \text{Ch}(g) + \frac{1}{2t} \int_X |g - f_0|^2 d\mathbf{m} : g \in D(\text{Ch}) \right\}.$$

Another useful fact is the maximum principle for the heat flow: [3, Prop. 2.14]: if $f \in L^2$, $f \leq C$, ($f \geq C$) \mathbf{m} -a.e. then $H_t f \leq C$ ($H_t f \geq C$) \mathbf{m} -a.e. in X .

4.2 Optimal transport and curvature dimension conditions

In this subsection, we recall the notion of an $\text{RCD}(K, \infty)$ -space, starting by some elements of optimal transport theory. Throughout the section, (X, d) is a complete and separable metric space. Denote by $\mathcal{P}(X)$ the set of Borel probability measures on X and $\mathcal{P}_2(X)$ the space of probability measures with finite second moment:

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) : \int d^2(x, x_0) d\mu(x) < \infty \text{ for some } x_0 \in X \right\},$$

Given a Borel map $T : X \rightarrow Y$, $\mu \in \mathcal{P}(X)$, the measure $T_{\#}\mu$, called the push-forward of μ by T , is defined by

$$T_{\#}\mu(E) = \mu(T^{-1}(E)).$$

For $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, we define the set $\Gamma(\mu, \nu)$ as the set of all transport plans $\gamma \in \mathcal{P}(X \times Y)$, i.e. the set of Borel probability measures on $X \times Y$ such that $\pi_{\#}^X \gamma = \mu$ and $\pi_{\#}^Y \gamma = \nu$. Here π^X and π^Y denote the natural projections from $X \times Y$ onto X and Y respectively.

For $\mu, \nu \in \mathcal{P}_2(X)$, the quadratic Kantorovich-Wasserstein distance W_2 is defined as

$$W_2(\mu, \nu) := \sqrt{\inf_{\gamma \in \Gamma(\mu, \nu)} \int d^2(x, y) d\gamma(x, y)}.$$

The Wasserstein distance W_2 is a distance on $\mathcal{P}_2(X)$ and turns it into a complete metric space. Moreover, if (X, d) is a geodesic space, then so is $(\mathcal{P}_2(X), W_2)$.

Given a metric measure space (X, d, \mathbf{m}) , we denote by $\mathcal{P}_2^a(X) \subset \mathcal{P}_2(X)$ the set of measures with finite second moment that are absolutely continuous with respect to \mathbf{m} .

In order to introduce curvature dimension conditions, we will need the following distortion coefficients. For $\kappa \in \mathbb{R}$ and $\theta \geq 0$, let

$$\mathfrak{s}_{\kappa}(\theta) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ 0 & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa < 0. \end{cases}$$

Moreover, for $t \in [0, 1]$, set

$$\sigma_{\kappa}^{(t)}(\theta) := \begin{cases} \frac{\mathfrak{s}_{\kappa}(t\theta)}{\mathfrak{s}_{\kappa}(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2. \end{cases}$$

Define the convex and continuous function U_{∞} on $[0, \infty)$ as

$$U_{\infty}(z) := z \log(z).$$

The Boltzmann-Shannon entropy functional $\mathcal{E}_{\infty} : \mathcal{P}(X) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{E}_{\infty}(\mu) := \begin{cases} \int_X U_{\infty}(\rho) d\mathbf{m}, & \text{if } \mu \ll \mathbf{m}, \text{ with } \mu = \rho\mathbf{m} \text{ and } U_{\infty}(\rho) \in L^1(X, \mathbf{m}) \\ \infty, & \text{else.} \end{cases}$$

Next we recall the synthetic notion of Ricci curvature bounded below for a metric measure space, pioneered by Sturm [29, 30] and Lott-Villani [26]. For the finite dimensional case, we will use the (reduced) curvature dimension condition of Bacher-Sturm [7].

Definition 4.6. We say that a metric measure space $(X, \mathbf{d}, \mathbf{m})$ has Ricci curvature bounded from below by $K \in \mathbb{R}$ provided the functional $\mathcal{E}_\infty : \mathcal{P}(X) \rightarrow \mathbb{R} \cup \{+\infty\}$ is (weakly) K -geodesically convex on $(\mathcal{P}_2^a(X), W_2)$, that means that for any $\mu_0, \mu_1 \in \mathcal{P}_2^a(X)$, there exists a Wasserstein geodesic $(\mu_t)_{t \in [0,1]}$ such that for all $t \in [0, 1]$,

$$\mathcal{E}_\infty(\mu_t) \leq (1-t)\mathcal{E}_\infty(\mu_0) + t\mathcal{E}_\infty(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1).$$

In this case we say that $(X, \mathbf{d}, \mathbf{m})$ is a $\text{CD}(K, \infty)$ -space.

Definition 4.7. Given two numbers $K \in \mathbb{R}$ and $N \in [1, \infty)$, we say that a metric measure space $(X, \mathbf{d}, \mathbf{m})$ satisfies the (reduced) curvature-dimension condition $\text{CD}^*(K, N)$ if for each pair $\mu_0, \mu_1 \in \mathcal{P}_2^a(X)$ with bounded supports there exists an optimal transport coupling γ and a Wasserstein geodesic $(\mu_t = \rho_t \mathbf{m})_{t \in [0,1]} \subset \mathcal{P}_2^a(X)$ connecting $\mu_0 = \rho_0 \mathbf{m}$ and $\mu_1 = \rho_1 \mathbf{m}$ such that for all $t \in [0, 1]$, $N' \geq N$, it holds

$$\int_X \rho_t^{-\frac{1}{N'}} d\mu_t \geq \int_{X \times X} \left[\sigma_{K/N'}^{(1-t)}(\mathbf{d}(x_0, x_1)) \rho_0(x_0)^{-\frac{1}{N'}} + \sigma_{K/N'}^{(t)}(\mathbf{d}(x_0, x_1)) \rho_1(x_1)^{-\frac{1}{N'}} \right] d\gamma(x_0, x_1).$$

In order to single out the ‘‘Riemannian’’ like structures out of the ‘‘possibly Finslerian’’ CD spaces, Ambrosio-Gigli-Savaré [3] (see also [14, 1]) in the $N = \infty$ case and Gigli [14] in the $N < \infty$ case introduced the Riemannian curvature-dimension condition.

Definition 4.8. We say that $(X, \mathbf{d}, \mathbf{m})$ satisfies the Riemannian curvature-dimension condition $\text{RCD}(K, \infty)$ if it is infinitesimally Hilbertian and satisfies the $\text{CD}(K, \infty)$ -condition. For $N \in [1, \infty)$, we say that $(X, \mathbf{d}, \mathbf{m})$ satisfies the Riemannian curvature-dimension condition $\text{RCD}^*(K, N)$ if it is infinitesimally Hilbertian and satisfies the $\text{CD}^*(K, N)$ -condition.

Let us recall the following useful results:

Lemma 4.9 ([13], Lem. 3.2 and Thm. 3.17). *If $(X, \mathbf{d}, \mathbf{m})$ satisfies the $\text{RCD}^*(K, N)$ -condition then it also satisfies the $\text{RCD}^*(K', N')$ -condition for any $K' \leq K$ and $N' \geq N$. Moreover, it satisfies the $\text{RCD}(K, \infty)$ -condition.*

Theorem 4.10 ([3], Thm. 6.2). *Let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{RCD}(K, \infty)$ space and $f \in W_w^{1,2}(X, \mathbf{d}, \mathbf{m})$. Then*

$$|D(H_t f)|_w^2 \leq e^{-2Kt} (H_t |Df|_w^2).$$

Remark 4.11. *By [29, Thm. 4.24], the condition (4.1) is automatically satisfied in every $\text{CD}(K, \infty)$ space, by choosing the function $V : X \rightarrow \mathbb{R}$, $V(x) := C \mathbf{d}(x, x_0)$ for a suitable $C > 0$ and any base point $x_0 \in X$.*

4.3 L^p and Sobolev spaces on weighted Riemannian manifolds of low regularity

In this subsection, we recall the language of Sobolev spaces on manifolds. We will summarise facts from [19, Ch. 2], adapting to the case of weighted manifolds of low regularity. Let (M, g) be a Riemannian manifold (with g a smooth metric for now). Let $h : M \rightarrow (0, \infty)$ be a C^0 -function and define the measure μ via $d\mu := h^2 \text{dvol}_g$. For an integer $k \geq 0$ and $u \in C^\infty(M)$ define $\nabla_c^k u$ to be the k -th covariant derivative ($\nabla_c^0 u := u$). Notice that $\nabla_c^k u$ is a $(0, k)$ -tensor. We will only be interested in the cases $k \leq 2$, so we write down the explicit formulas for the covariant derivatives in local coordinates:

$$(\nabla_c^1 u)_i = (du)_i = \partial_i u \quad \text{and} \quad (\nabla_c^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u.$$

By definition, we have that

$$|\nabla_c^k u|^2 = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla_c^k u)_{i_1 \dots i_k} (\nabla_c^k u)_{j_1 \dots j_k},$$

where $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. Let $p \in [1, \infty)$. Denote by $C_k^p(M, \mu)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla_c^j u| \in L^p(M, \mu)$ for any $j = 0, \dots, k$.

Definition 4.12. The Sobolev space $H_k^p(M, \mu)$ is the completion of $C_k^p(M, \mu)$ with respect to the norm

$$\|u\|_{H_k^p(M, \mu)} = \sum_{j=0}^k \left(\int_M |\nabla_c^j u|^p d\mu \right)^{\frac{1}{p}}.$$

Proposition 4.13. $H_k^2(M, \mu)$ is a Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{m=0}^k \int_M (g^{i_1 j_1} \dots g^{i_m j_m} (\nabla_c^m u)_{i_1 \dots i_m} (\nabla_c^m v)_{j_1 \dots j_m}) h^2 d\mu.$$

Note that the inner product induces an equivalent norm on $H_k^2(M, \mu)$. In the case $p = 2$, we will work with the norm induced by the inner product. We can generalise these definitions to the following:

Definition 4.14. Let $s, r \geq 0$ and $p \geq 1$. Define $C_k^p(T_s^r(M), \mu)$ the space of smooth sections $u \in \mathcal{T}_s^r(M)$ such that $|\nabla_c^j u|_g \in L^p(M, \mu)$ for $j = 0, \dots, k$. The Sobolev space $H_k^p(T_s^r(M), \mu)$ is defined to be the closure of $C_k^p(T_s^r(M), \mu)$ under the norm

$$\|u\|_{H_k^p(T_s^r(M), \mu)} := \sum_{j=0}^k \left(\int_M |\nabla_c^j u|^p d\mu \right)^{\frac{1}{p}}.$$

Moreover, for all such s, r , we define the space $\mathring{H}_k^p(T_s^r(M), \mu)$ as the closure of the set of smooth compactly supported (r, s) -tensors under the $H_k^p(T_s^r(M), \mu)$ -norm.

We notice that these definitions make sense for $k \leq 2$ and $p = 2$ if $g \in C^0(M)$ with L_{loc}^2 -Christoffel symbols. The following criterion for weak convergence will be useful later.

Lemma 4.15. Let (M, g, μ) be a weighted Riemannian manifold with a C^0 -Riemannian metric g with L_{loc}^2 -Christoffel symbols and endowed with a measure μ defined via $d\mu = h^2 d\text{vol}_g$ for a function $h \in C^0(M, (0, \infty))$.

Let $(V_\alpha, \psi_\alpha)_\alpha$ be an atlas for M such that $\overline{V_\alpha}$ is compact, for all α .

Let f_i be a sequence in $L^2(M, \mu)$ and let $f \in L^2(M, \mu)$.

Then $f_i \rightharpoonup f$ in $L^2(M, \mu)$ if and only if $(\psi_\alpha)_* f_i \rightharpoonup (\psi_\alpha)_* f$ in $L^2(\psi_\alpha(V_\alpha), (\psi_\alpha)_\# \mu)$, for all α .

If $f, f_i \in H_k^2(M, \mu)$ ($k \leq 2$) and $\partial^\beta f_i \rightharpoonup \partial^\beta f$ in $L^2(\psi_\alpha(V_\alpha), (\psi_\alpha)_\# \mu)$ for all α and all $\beta \in \mathbb{N}^n$, $|\beta| \leq k$, then $f_i \rightharpoonup f$ in $H_k^2(M, \mu)$.

Proof. For the first part, note that it suffices to prove convergence of the inner product on a dense subset of $L^2(M, \mu)$. Using Lemma 4.17, we can pick a function $\varphi \in C_c(M)$. Let $(V_j)_{j=1}^l$ be a finite cover of $\text{supp } \varphi$ and pick a partition of unity (ρ_j) subordinate to that cover. Now

$$\int_M \varphi f_i h^2 d\text{vol}_g = \int_M \sum_{j=1}^l \rho_j \varphi f_i h^2 d\text{vol}_g = \sum_{j=1}^l \int_{\psi_j(V_j)} (\psi_j)_* (\rho_j \varphi f_i) h^2 \sqrt{|g|} d\mathcal{L}^n.$$

If $(\psi_j)_* f_i \rightharpoonup (\psi_j)_* f$, it is easy to see that the above term converges. For the other direction it suffices to note that every function $\varphi \in L^2(\psi_j(V_j), (\psi_j)_\# \mu)$ for some j can be extended to a function $\Phi \in L^2(M, \mu)$ via pullback and extension by 0.

For the second part, we simply use the first part and the definition of the $H_k^2(M, \mu)$ -inner product. \square

Now, we can generalise the validity of the identity (3.12) to Sobolev vector fields:

Lemma 4.16 (Distributional Ricci curvature for $H_1^2(TM, \mu)$ -fields). Let g be a C^0 -Riemannian metric whose Christoffel symbols are in L_{loc}^2 and μ be the measure defined by $d\mu = h^2 d\text{vol}_g$ for a positive $h \in C^0(M, (0, \infty)) \cap W_{\text{loc}}^{1,2}(M)$. Let $X \in H_1^2(TM, \mu)$. Then (3.12) holds.

Proof. This follows from the density of $C_1^2(TM, \mu)$ in $H_1^2(TM, \mu)$ and the fact that $\phi, |\nabla \phi|_g \in L^1 \cap L^\infty(M)$ have compact support. \square

We conclude the subsection with the following elementary result.

Lemma 4.17. *Let M be a smooth manifold, g be a continuous Riemannian metric and h a continuous positive function. Moreover let $p \in [1, \infty)$. Define the measure μ by $\mu := h^2 \text{dvol}_g$ and assume that μ is a regular Borel measure. Then $C_c^\infty(M)$ is dense in $L^p(M, \mu)$.*

4.4 Cheeger energy on a manifold with continuous Riemannian metric and weight

In this subsection, we examine the Cheeger energy in the setting of a smooth manifold M endowed with a continuous Riemannian metric g and a continuous weight h^2 on the volume form. The metric induced by g is given by

$$d_g(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}_t|_g dt : \gamma \text{ piecewise } C^\infty, \gamma_0 = x, \gamma_1 = y \right\}. \quad (4.3)$$

We also consider the Riemannian volume form given locally by $\sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$ and the associated volume measure $\text{dvol}_g = \sqrt{|g|} d\mathcal{L}^n$. We will consider the metric measure space (M, d_g, μ) , where $d\mu = h^2 \text{dvol}_g$ for a positive and continuous h . It is easily checked that μ is a σ -finite Borel measure. We assume that (4.1) holds, as we will work with $\text{CD}(K, \infty)$ -manifolds (cf. Rem. 4.11).

We will now state an important property of the metric:

Proposition 4.18 (cf. [27], Thm. 5.3.8). *Let f be a Lipschitz function on (M, d_g) and $(V_\alpha, \psi_\alpha)_\alpha$ be an atlas for M . Then for each α , $f \circ (\psi_\alpha)^{-1}$ is locally Lipschitz.*

Proof. We show that $|\cdot|_{euc}$ and d_g are locally equivalent. Pick a point $p \in M$, α such that $p \in V_\alpha$ and fix an $\varepsilon > 0$ such that $\overline{B_\varepsilon^{euc}(\psi_\alpha(p))} \subset \psi_\alpha(V_\alpha)$. From now on, we will drop the index α . Take two arbitrary x, y such that $\psi(x), \psi(y) \in B_{\varepsilon/2}^{euc}(\psi(p))$ and let $c : [0, 1] \rightarrow M$ be a piecewise C^∞ -curve from x to y . We note that we can find a $\lambda \geq 1$ such that $\frac{1}{\lambda} \|v\|_{euc} \leq g(v, v) \leq \lambda \|v\|_{euc}$ for all $q \in \psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ and all $v \in T_q M$. We now distinguish three cases:

1. c is a straight line in the Euclidean metric. Then

$$|x - y|_{euc} = L_{euc}(c) = \int_0^1 |\dot{c}|_{euc} dt \geq \frac{1}{\lambda} \int_0^1 |\dot{c}|_g dt \geq \frac{1}{\lambda} d_g(x, y).$$

2. If c is a general curve that lies in $\psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ then

$$L_g(c) = \int_0^1 |\dot{c}|_g dt \geq \lambda \int_0^1 |\dot{c}|_{euc} dt \geq \lambda |x - y|_{euc}.$$

3. If c leaves $\psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ then there is a minimal t_0 such that $c(t_0) \notin \psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ and hence

$$L_g(c) \geq \int_0^{t_0} |\dot{c}|_g dt \geq \lambda \int_0^{t_0} |\dot{c}|_{euc} dt \geq \lambda \frac{\varepsilon}{2} \geq \frac{\lambda}{2} |x - y|_{euc}.$$

This yields that the two metrics are equivalent in $\psi^{-1}(B_{\varepsilon/2}^{euc}(\psi(p))) =: U_p$. Hence, for a d_g -Lipschitz function f and any $p \in M$, there exists a constant L_{U_p} such that f is locally L_{U_p} -Lipschitz with respect to the induced Euclidean metric via ψ on U_p . \square

A mollification argument shows:

Proposition 4.19. *The metric (4.3) turns M into a length space.*

The Hopf-Rinow theorem directly implies:

Corollary 4.20. *If M is compact, (M, d_g) is a geodesic space.*

Next we state two useful facts, whose proofs are omitted since standard:

Proposition 4.21. *Let $f \in C^1(M)$. Then*

$$|Df|(x) = |\nabla f|_g(x), \quad \text{for every } x.$$

Proposition 4.22. *Let $\gamma : [0, 1] \rightarrow M$ be an absolutely continuous curve. Then the metric speed $|\dot{\gamma}_t|$ coincides a.e. with $|\dot{\gamma}_t|_g = \sqrt{\langle \dot{\gamma}_t, \dot{\gamma}_t \rangle_g}$, where $\dot{\gamma}_t$ denotes the (a.e. existing) derivative.*

For brevity we omit the proof of Proposition 4.22, as it involves similar arguments as the proof of Proposition 4.24. We will show that if $f \in C^1$, then $|\nabla f|_g$ is the minimal weak upper gradient. We know that it is a weak upper gradient because for any absolutely continuous curve γ , we have

$$|f(\gamma_1) - f(\gamma_0)| = \left| \int_0^1 \langle \nabla f, \dot{\gamma}_t \rangle_g |_{\gamma_t} dt \right| \leq \int_0^1 |\nabla f|_g(\gamma_t) |\dot{\gamma}_t|_g dt.$$

The next lemma will be used in the proof of Proposition 4.24.

Lemma 4.23. *Let M be a smooth manifold equipped with a C^0 -Riemannian metric g and a measure μ defined via $d\mu = h^2 d\text{vol}_g$, for some $h \in C^0(M, (0, \infty))$. Let $k \in L^1_{\text{loc}}(M, \mu)$ and fix some coordinate patch (V_m, ψ_m) . Let $x \in \psi_m(V_m)$ and $\theta > 0$, such that $B_{4\theta}^{\text{euc}}(x) \subset \psi_m(V_m)$. For $\delta \in (0, \theta) \cap \mathbb{Q}$, $\dot{\gamma} \in \mathbb{Q}^n \cap B_\theta^{\text{euc}}(0)$, we define*

$$F_{\dot{\gamma}, \delta, x} : t \rightarrow \frac{1}{\mathcal{L}^n(B_\delta(x))} \int_{B_\delta(x)} |\dot{\gamma}|_{g(y+t\dot{\gamma})} k(y+t\dot{\gamma}) d\mathcal{L}^n(y), \quad \forall t, |t| < 1.$$

Then for \mathcal{L}^n -a.e. $x \in \psi_m(V_m)$ and all $\delta \in (0, 1) \cap \mathbb{Q}$, $\dot{\gamma} \in \mathbb{Q}^n \cap B_\theta^{\text{euc}}(0)$, we have that $t = 0$ is a Lebesgue point of $F_{\dot{\gamma}, \delta, x}$.

Proof. As g and h are continuous, we get that $(\psi_m)_* k \in L^1_{\text{loc}}(\psi_m(V_m), \mathcal{L}^n)$. Fix $w \in \psi_m(V_m)$, and choose $\theta > 0$ accordingly. Moreover, fix δ and $\dot{\gamma}$. For any $z \in B_\theta^{\text{euc}}(w)$, the Lebesgue differentiation theorem yields that \mathcal{L}^1 -almost every $t \in (-1, 1)$ is a Lebesgue point of $F_{\dot{\gamma}, \delta, z}$. In other words that means that for \mathcal{L}^1 -almost every y on the line $\{z + t\dot{\gamma}, |t| < 1\}$, $t = 0$ is a Lebesgue point of $F_{\dot{\gamma}, \delta, y}$. The set of non-Lebesgue points on that line shall be denoted by $N_{\dot{\gamma}, \delta, z}$, and it holds that $\mathcal{L}^1(N_{\dot{\gamma}, \delta, z}) = 0$. Let $H_{\dot{\gamma}, w}$ be the $(n-1)$ -dimensional hyperplane through w that is orthogonal to $\dot{\gamma}$, and intersected with $B_\theta^{\text{euc}}(w)$. Then, by Fubini's Theorem, it holds

$$\mathcal{L}^n \left(\bigcup_{z \in H_{\dot{\gamma}, w}} N_{\dot{\gamma}, \delta, z} \right) = \int_{H_{\dot{\gamma}, w}} \mathcal{L}^1(N_{\dot{\gamma}, \delta, z}) d\mathcal{L}^{n-1}(z) = 0.$$

Hence for \mathcal{L}^n -almost every $y \in B_\theta^{\text{euc}}(w)$, $t = 0$ is a Lebesgue point of $F_{\dot{\gamma}, \delta, y}$. Note that, by construction, $\dot{\gamma}$ and δ vary in a countable set. This shows the claim in an open neighbourhood around w . As w was arbitrary, the proof is complete. \square

Proposition 4.24. *Let M be a smooth manifold, g a C^0 -Riemannian metric on M and μ a measure on M defined by $d\mu = h^2 d\text{vol}_g$, where $h \in C^0(M, (0, \infty))$. Let f be a C^1 -function. Then*

$$|Df|_w(x) = |\nabla f|_g(x), \quad \text{for } \mu\text{-a.e. } x.$$

Proof. We argue by contradiction. Suppose that $|\nabla f|_g$ is not minimal. Then there exists an $\varepsilon \in (0, \frac{1}{4})$ such that $\mu(\{|\nabla f|_g - |Df|_w > \varepsilon\}) > 0$. We can then find an open set V such that \bar{V} is compact, is contained in one coordinate patch and $\mu(\{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V) > 0$. We denote by $\psi : V \rightarrow \mathbb{R}^n$ the associated coordinate chart. By the continuity of g and h , there exists a constant $C = C(g, h, V)$ such that for each Borel set $E \subset V$, $\frac{1}{C} \mathcal{L}^n(\psi(E)) \leq \mu(E) \leq C \mathcal{L}^n(\psi(E))$. For almost all $x \in \psi(V)$ take $\theta, \dot{\gamma}, \delta$ as in Lemma 4.23.

It still holds $\mathcal{L}^n(\{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V) > 0$. By the Lebesgue differentiation theorem and Lemma 4.23 with $k = |Df|_w$, we get that there exists an $x \in \{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V$ such that x is a Lebesgue point of $|Df|_w$, and for all $\dot{\gamma}, \delta$ as above, $t = 0$ is a Lebesgue point of $F_{\dot{\gamma}, \delta, x}$. Set

$$\eta = \frac{1}{101} \min \left(1, \frac{1}{\|\nabla f\|_{L^\infty}}, \frac{1}{2n^3 \|g\|_{L^\infty}} \right)^2.$$

Choose $\dot{\gamma} \in \mathbb{Q}^n$ such that $\frac{99}{100} \leq |\dot{\gamma}|_g(x) \leq \frac{101}{100}$ and $|\langle \nabla f, \dot{\gamma} \rangle_g(x) - |\nabla f|_g(x) |\dot{\gamma}|_g(x)| \leq \frac{1}{4} \eta \varepsilon$. Note that if $t = 0$ is a Lebesgue-point of $F_{x, \dot{\gamma}, \delta}$ then it is a Lebesgue-point of $F_{x, \alpha \dot{\gamma}, \delta}$ for all $\alpha \in \mathbb{Q}$. Moreover, take $q \in \mathbb{Q} \cap (0, \theta)$ such that for all $\delta \in (0, q)$, it holds

$$\begin{aligned} \frac{1}{\mathcal{L}^n(B_\delta(x))} \int_{B_\delta(x)} (|Df|_w(y) - |Df|_w(x)) &< \eta \varepsilon, \\ |\nabla f|_g(y) - |\nabla f|_g(z) &\leq \frac{1}{2} \eta \varepsilon, \quad \forall y, z \in B_{3\delta}(x), \\ \|\nabla f|_g |\dot{\gamma}|_g(y) - \nabla f|_g |\dot{\gamma}|_g(z)\| &\leq \frac{1}{2} \eta \varepsilon, \quad \forall y, z \in B_{3\delta}(x), \quad \text{and} \\ |g_{ij}(y) - g_{ij}(z)| &\leq \frac{1}{2} \eta \varepsilon, \quad \forall 1 \leq i, j \leq n, \quad y, z \in B_{3\delta}(x). \end{aligned}$$

We can now compute that for $s \leq q$ and $y \in B_q(x)$,

$$|f(y - s\dot{\gamma}) - f(y + s\dot{\gamma})| = \int_{-s}^s \langle \nabla f, \dot{\gamma} \rangle_g(y + t\dot{\gamma}) dt.$$

For $t \in [-s, s]$, we get under the above assumptions that

$$\begin{aligned} |\langle \nabla f, \dot{\gamma} \rangle_g(y + t\dot{\gamma}) - |\nabla f|_g |\dot{\gamma}|_g(x)| &\leq |\langle \nabla f(y + t\dot{\gamma}), \dot{\gamma} \rangle_{g(y+t\dot{\gamma})-g(x)}| + |\langle \nabla f(y + t\dot{\gamma}) - \nabla f(x), \dot{\gamma} \rangle_{g(x)}| \\ &+ |\langle \nabla f, \dot{\gamma} \rangle_g(x) - |\nabla f|_g(x) |\dot{\gamma}|_g(x)| \leq \frac{3\varepsilon}{100}. \end{aligned}$$

So, for all $y \in B_\delta(x)$,

$$|(|f(y - s\dot{\gamma}) - f(y + s\dot{\gamma})|) - (2s |\nabla f|_g(x) |\dot{\gamma}|_g(x))| \leq \frac{6s\varepsilon}{100}. \quad (4.4)$$

Moreover, we get that

$$\left| \frac{1}{|B_q(x)|} \int_{B_q(x)} |\nabla f|_g(y) d\mathcal{L}^n(y) - |\nabla f|_g(x) \right| \leq \frac{1}{100} \varepsilon, \quad (4.5)$$

$$\left| \frac{1}{|B_q(x)|} \int_{B_q(x)} |\nabla f|_g |\dot{\gamma}|_g(y) d\mathcal{L}^n(y) - |\nabla f|_g |\dot{\gamma}|_g(x) \right| \leq \frac{1}{100} \varepsilon, \quad (4.6)$$

and

$$\left| \frac{1}{|B_q(x)|} \int_{B_q(x)} |Df|_w(y) d\mathcal{L}^n(y) - |Df|_w(x) \right| \leq \frac{1}{100} \varepsilon, \quad (4.7)$$

thus,

$$\frac{1}{|B_q(x)|} \int_{B_q(x)} (|\nabla f|_g - |Df|_w)(y) d\mathcal{L}^n(y) > \frac{9\varepsilon}{10}. \quad (4.8)$$

Using that $t = 0$ is a Lebesgue point of $F_{\dot{\gamma}, q, x}$, and the fact that g and $|\nabla f|_g$ are continuous, we can choose $s \in (0, q)$ small enough such that

$$\left| \frac{1}{|B_q(x)|} \int_{-s}^s \int_{B_q(x)} |\nabla f|_g(y + t\dot{\gamma}) |\dot{\gamma}|_{g(y+t\dot{\gamma})} d\mathcal{L}^n(y) dt - 2s \frac{1}{|B_q(x)|} \int_{B_q(x)} |\nabla f|_g(y) |\dot{\gamma}|_g(y) d\mathcal{L}^n(y) \right| \leq \frac{2s\varepsilon}{100}, \quad (4.9)$$

and

$$\left| \frac{1}{|B_q(x)|} \int_{-s}^s \int_{B_q(x)} |Df|_w(y + t\dot{\gamma}) |\dot{\gamma}|_{g(y+t\dot{\gamma})} d\mathcal{L}^n(y) dt - 2s \frac{1}{|B_q(x)|} \int_{B_q(x)} |Df|_w(y) |\dot{\gamma}|_g(y) d\mathcal{L}^n(y) \right| \leq \frac{2s\varepsilon}{100}. \quad (4.10)$$

Now, we have all the ingredients to construct a test plan that yields a contradiction. For $y \in B_q(x)$ define $\gamma_y : [0, 1] \rightarrow \psi_m(V_m)$, $t \mapsto y + 2(t - \frac{1}{2})s\dot{\gamma}$ and define a test plan π as

$$d\pi(\gamma) := \begin{cases} \frac{1}{|B_q(x)|} d\mathcal{L}^n(y) & \text{if } \gamma = \gamma_y, \text{ for some } y \in B_q(x), \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can directly compute that

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) = \frac{1}{|B_q(x)|} \int_{B_q(x)} |f(y - s\dot{\gamma}) - f(y + s\dot{\gamma})| d\mathcal{L}^n(y). \quad (4.11)$$

Moreover,

$$\begin{aligned} & \int \int_0^1 |\nabla f|_g(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) \\ &= \frac{1}{|B_q(x)|} \int_{B_q(x)} \int_0^1 |\nabla f|_g\left(y + 2\left(t - \frac{1}{2}\right)s\dot{\gamma}\right) 2s |\dot{\gamma}|_{g((\gamma_y)_t)} dt d\mathcal{L}^n(y) \\ &= \frac{1}{|B_q(x)|} \int_{B_q(x)} \int_{-s}^s |\nabla f|_g(y + t\dot{\gamma}) |\dot{\gamma}|_{g((\gamma_y)_t)} dt d\mathcal{L}^n(y), \end{aligned} \quad (4.12)$$

and similarly

$$\begin{aligned} & \int \int_0^1 |Df|_w(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) \\ &= \frac{1}{|B_q(x)|} \int_{B_q(x)} \int_{-s}^s |Df|_w(y + t\dot{\gamma}) |\dot{\gamma}|_{g((\gamma_y)_t)} dt d\mathcal{L}^n(y). \end{aligned}$$

We then have that

$$\begin{aligned} & \int \int_0^1 |\nabla f|_g(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) - \int \int_0^1 |Df|_w(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) \\ &= \frac{1}{|B_q(x)|} \int_{B_q(x)} \int_{-s}^s (|\nabla f|_g - |Df|_w)(y + t\dot{\gamma}) |\dot{\gamma}|_{g(y+t\dot{\gamma})} dt d\mathcal{L}^n(y) \\ &\stackrel{(4.10), (4.9)}{\geq} 2s \frac{1}{|B_q(x)|} \int_{B_q(x)} (|\nabla f|_g - |Df|_w) |\dot{\gamma}|_g(y) d\mathcal{L}^n(y) - \frac{4s\varepsilon}{100} \\ &\stackrel{(4.8)}{>} \frac{99}{100} \frac{9s\varepsilon}{10} - \frac{4s\varepsilon}{100} \geq \frac{3s\varepsilon}{2} > 0. \end{aligned} \quad (4.13)$$

The combination of the above estimates gives:

$$\begin{aligned}
\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) &\stackrel{(4.11),(4.4)}{\geq} 2s |\nabla f|_g(x) |\dot{\gamma}|_g(x) - \frac{6s\varepsilon}{100} \\
&\stackrel{(4.6)}{\geq} \frac{2s}{|B_q(x)|} \int_{B_q(x)} |\nabla f|_g |\dot{\gamma}|_g(y) d\mathcal{L}^n(y) - \frac{s\varepsilon}{10} \\
&\stackrel{(4.9)}{\geq} \frac{1}{|B_q(x)|} \int_{-s}^s \int_{B_q(x)} |\nabla f|_g(y + t\dot{\gamma}) |\dot{\gamma}|_{g(y+t\dot{\gamma})} d\mathcal{L}^n(y) dt - \frac{3s\varepsilon}{20} \\
&\stackrel{(4.12)}{=} \iint_0^1 |\nabla f|_g(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) - \frac{3s\varepsilon}{20} \\
&\stackrel{(4.13)}{>} \iint_0^1 |Df|_w(\gamma_t) |\dot{\gamma}_t|_{g(\gamma_t)} dt d\pi(\gamma) - \frac{3s\varepsilon}{20} + \frac{3s\varepsilon}{2}.
\end{aligned}$$

This yields a contradiction, as $|Df|_w$ fails to satisfy the condition of a weak upper gradient for the chosen test plan π . \square

In the following, we will investigate $W_w^{1,2}(X, \mathbf{d}, \mathbf{m})$ for $(X, \mathbf{d}, \mathbf{m}) = (M, \mathbf{d}_g, \mu)$, with $d\mu = h^2 \text{dvol}_g$.

A mollification argument together with Rademacher's theorem and Proposition 4.24 shows that:

Proposition 4.25. *Let f be a locally \mathbf{d}_g -Lipschitz function. Then f is differentiable μ -almost everywhere. If moreover $f, |\nabla f|_g \in L^2(M, \mu)$ then $|Df|_w = |\nabla f|_g$ almost everywhere, $f \in W_w^{1,2}(M, \mu) \cap H_1^2(M, \mu)$, and $\|f\|_{W_w^{1,2}(M, \mu)} = \|f\|_{H_1^2(M, \mu)}$.*

By [2, Rem. 5.5], the slope of a $|Df|$ of a \mathbf{d}_g -Lipschitz function f is a weak upper gradient. Thus, if $f \in L^2(M, \mu)$ is a \mathbf{d}_g -Lipschitz function such that $|Df| \in L^2(M, \mu)$, then Proposition 4.25 implies that:

$$|Df|_w = |\nabla f|_g \leq |Df|. \quad (4.14)$$

This observation is crucial for the following lemma.

Lemma 4.26. *Let M be a smooth manifold, g a C^0 -Riemannian metric on M , \mathbf{d}_g the distance induced by g , vol_g the volume form, and h be a positive continuous function. Define the measure μ by $d\mu = h^2 \text{dvol}_g$. Then $W_w^{1,2}(M, \mu) \subset H_1^2(M, \mu)$ and for each $f \in W_w^{1,2}(M, \mu)$, it holds $|Df|_w = |\nabla f|_g$.*

Proof. Let $f \in W_w^{1,2}(M, \mu)$. By [3, (2.22)], we can find a sequence of Lipschitz functions $(f_l)_{l=1}^\infty$ such that $f_l \rightarrow f$ in L^2 and $|Df_l| \rightarrow |Df|_w$ in L^2 . Using (4.14), we get that $(f_l)_{l=1}^\infty$ is a bounded sequence in $H_1^2(M, \mu)$. As this space is reflexive, we can find a subsequence that converges weakly in $H_1^2(M, \mu)$. As $f_l \rightarrow f$ strongly in L^2 , we get that $f_l \rightharpoonup f$ in $H_1^2(M, \mu)$. As we now know that $f \in H_1^2(M, \mu)$, we can approximate it with a different sequence $f_k \in H_1^2(M, \mu) \cap C_1^2(M, \mu)$, i.e. $\lim_{k \rightarrow \infty} \|f_k - f\|_{H_1^2(M, \mu)} = 0$. Then f_k is a Cauchy sequence in $H_1^2(M, \mu)$. As for $C_1^2(M, \mu)$ -functions $\|\cdot\|_{W_w^{1,2}(M, \mu)} = \|\cdot\|_{H_1^2(M, \mu)}$ (by Proposition 4.24), it is a Cauchy sequence in $W_w^{1,2}$. Hence $|\nabla f_k|_g = |Df_k|_w \rightarrow |Df|_w$ in L^2 , which in particular shows that $|Df|_w = |\nabla f|_g$ almost everywhere. \square

We can now finally identify the space $W_w^{1,2}$ with the classical Sobolev space H_1^2 :

Corollary 4.27. *Let M be a smooth manifold with a C^0 -Riemannian metric g . Consider the metric measure space (M, \mathbf{d}_g, μ) , where $d\mu = h^2 \text{dvol}_g$ for a continuous positive function h . Then $W_w^{1,2}(M, \mu) = H_1^2(M, \mu)$ and the Dirichlet form associated to the Cheeger energy on (M, \mathbf{d}_g, μ) is a quadratic form given by the $L^2(M, \mu)$ -inner product of the gradients. Moreover, every $f \in D(\text{Ch})$ with $|Df|_w \leq 1$ μ -a.e. admits a 1-Lipschitz representative μ -a.e. with respect to \mathbf{d}_g .*

Proof. We prove that $H_1^2(M, \mu) \subset W_w^{1,2}(M, \mu)$. At first we notice that by Proposition 4.24, $C_1^2(M, \mu) \subset W_w^{1,2}(M, \mu)$ and for each $f \in C_1^2(M, \mu)$, we have that $\|f\|_{H_1^2(M, \mu)} = \|f\|_{W_w^{1,2}(M, \mu)}$. Now, fix $f \in H_1^2(M, \mu)$

and approximate it with a sequence $f_k \in C_1^2(M, \mu)$. This is then a Cauchy sequence in $H_1^2(M, \mu)$ and hence in $W_w^{1,2}(M, \mu)$, by the norm equality we proved in Lemma 4.26. Hence, f_k converges in $W_w^{1,2}(M, \mu)$ to some \tilde{f} . It follows that $f_k \rightarrow \tilde{f}$ in $L^2(M, \mu)$, hence, $\tilde{f} = f \in W_w^{1,2}(M, \mu)$. We get $W_w^{1,2}(M) = H_1^2(M, \mu)$ from Lemma 4.26.

For the second part, we use that by Proposition 4.5, the associated Dirichlet form \mathcal{E} is defined on $(H_1^2(M, \mu))^2$ and is of the form

$$\mathcal{E}(f, k) = \int_M G(f, k) \, d\mu,$$

where

$$G(f, k) = \lim_{\varepsilon \searrow 0} \frac{|D(f + \varepsilon k)|_w^2 - |Df|_w^2}{2\varepsilon} = \langle \nabla f, \nabla k \rangle_g \in L^1(X, \mathfrak{m}). \quad (4.15)$$

For the last part, without loss of generality by localising to a suitable coordinate patch and by using partition of unity, we can assume that f has compact support in an open set $U \subset \mathbb{R}^n$ endowed with a continuous Riemannian metric g such that for every $x \in U$ it holds that $C^{-1} \text{Id}_n \leq g \leq C \text{Id}_n$, for some constant $C \geq 1$ uniform on U . It follows that f lies in $W^{1,\infty}(\mathbb{R}^n)$ and thus, by the classical result in \mathbb{R}^n , f has a Lipschitz representative \mathcal{L}^n -a.e. that we identify with f . Hence, f is Lipschitz also in (M, g) and, by Proposition 4.25 and the assumption on f , it holds that $|\nabla f|_g = |Df|_w \leq 1$ μ -a.e. Let $\rho_\delta(x) := \frac{1}{\delta^n} \rho(\frac{x}{\delta})$, $\delta > 0$, be standard mollifiers in \mathbb{R}^n and consider $f_\delta := f \star \rho_\delta$. It is easily seen that

$$|\nabla f_\delta|_g \leq 1 + \theta(\delta) \quad \text{and} \quad f_\delta \rightarrow f \quad \text{uniformly as } \delta \rightarrow 0. \quad (4.16)$$

Here $\lim_{\delta \rightarrow 0} \theta(\delta) = 0$. Let $x, y \in M$, $x \neq y$. By Proposition 4.19, for every $\varepsilon \in (0, d_g(x, y)/2)$ there exists a rectifiable curve $\gamma : [0, 1] \rightarrow M$ parametrised by arc-length such that

$$\text{length}_g(\gamma) \leq d_g(x, y) + \varepsilon, \quad |\dot{\gamma}| = \text{length}_g(\dot{\gamma}), \quad \mathcal{L}^1\text{-a.e. on } [0, 1]. \quad (4.17)$$

Then

$$\begin{aligned} |f_\delta(x) - f_\delta(y)| &= \left| \int_0^1 \frac{d}{dt} f_\delta(\gamma_t) \, dt \right| \leq \int_0^1 |\dot{\gamma}_t|_g |\nabla f_\delta|_g \, dt \\ &\stackrel{(4.16), (4.17)}{\leq} (1 + \theta(\delta))(d_g(x, y) + \varepsilon). \end{aligned}$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get that f_δ is $1 + \theta(\delta)$ -Lipschitz with respect to d_g . Passing further to the limit as $\delta \rightarrow 0$ and recalling the second in (4.16) we conclude that f is 1-Lipschitz with respect to d_g . \square

5 Second order calculus

In this section, we will specialise the second order calculus developed in [15, Ch. 3] to the setting of a smooth manifold with a C^0 -Riemannian metric with L_{loc}^2 -Christoffel symbols and $C^0 \cap W_{\text{loc}}^{1,2}$ -weight on the measure.

5.1 Some elements of the theory for general $\text{RCD}(K, \infty)$ -spaces

Let (M, d, \mathfrak{m}) be a metric measure space that satisfies the $\text{RCD}(K, \infty)$ -condition. Recall that

$$D(\Delta) := \{f \in L^2(\mathfrak{m}) : \Delta f \in L^2(\mathfrak{m})\} \subset W_w^{1,2}(M, \mathfrak{m}).$$

Following the notation of [15, Ch. 3], we define the space of test functions

$$\text{TestF}(M) := \{f \in D(\Delta) \cap L^\infty(M, \mathfrak{m}) : |\nabla f| \in L^\infty(M, \mathfrak{m}) \text{ and } \Delta f \in W_w^{1,2}(M, \mathfrak{m})\},$$

test vector fields

$$\text{TestV}(M) := \left\{ \sum_{i=1}^n h_i \nabla f_i : n \in \mathbb{N}, f_i, h_i \in \text{TestF}(M), i = 1, \dots, n \right\},$$

and test forms

$$\text{TestForm}_k(M) := \{\text{linear combinations of forms of the kind } f_0 df_1 \wedge \dots \wedge df_k : f_0, \dots, f_k \in \text{TestF}(M)\},$$

for $0 \leq k \leq n$. For the notion of gradients and differentials in metric measure spaces, we refer to [15, Ch. 2]. We will apply this theory to weighted manifolds with continuous metrics and weights. In that case, the non-smooth notions of [15] coincide with the classical gradients and differentials on manifolds.

All expressions are well-defined, because $f \in \text{TestF}(M)$ implies that $f \in W_w^{1,2}$, so the differential and the gradient are well-defined. We have that if $f \in L^2 \cap L^\infty(M, \mathfrak{m})$ then for all $t > 0$, $f_t := H_t f \in \text{TestF}(M)$. In the next proposition we recall two important density results proved in [15, Prop. 2.2.5] for general metric measure spaces. In the setting of a smooth manifold with a continuous metric and a continuous weight, we will prove an even stronger result in Lemma 6.10.

Proposition 5.1. *TestForm₁(M) is dense in $L^2(T^*M)$ and TestV(M) is dense in $L^2(TM)$.*

In [15] it is shown that for $f, g \in \text{TestF}(M)$, then $\langle \nabla f, \nabla g \rangle \in W_w^{1,2}(M, \mathfrak{m})$, which allows to define a notion of Hessian $H[f] : [\text{TestF}(M)]^2 \rightarrow L^2(M)$ of a function $f \in \text{TestF}(M)$ as

$$H[f](g, h) = \frac{1}{2} (\langle \nabla \langle \nabla f, \nabla g \rangle, \nabla h \rangle + \langle \nabla \langle \nabla f, \nabla h \rangle, \nabla g \rangle - \langle \nabla \langle \nabla g, \nabla h \rangle, \nabla f \rangle).$$

Definition 5.2. *The space $D(\Delta) \subset W_w^{1,2}(M, \mathfrak{m})$ is the space of functions $f \in W_w^{1,2}(M, \mathfrak{m})$ such that there exists a measure $\nu \in \text{Meas}(M)$ satisfying*

$$\int h \, d\nu = - \int \langle \nabla h, \nabla f \rangle \, d\mathfrak{m},$$

for all $h : M \rightarrow \mathbb{R}$ Lipschitz with bounded support. In this case the measure ν is unique and denoted by Δf .

By [15, Lemma 3.2.6], we have that if $X, Y \in \text{TestV}(M)$, then $\langle X, Y \rangle \in D(\Delta)$. Define

$$D_{W_w^{1,2}}(\Delta) = \{f \in W_w^{1,2}(M), \Delta f \in W_w^{1,2}(M)\}.$$

We are now able to define the measure valued operator $\Gamma_2 : [D_{W_w^{1,2}}(\Delta)]^2 \rightarrow \text{Meas}(M)$ given by

$$\Gamma_2(f, g) := \frac{1}{2} \Delta \langle \nabla f, \nabla g \rangle - \frac{1}{2} (\langle \nabla \Delta f, \nabla g \rangle + \langle \nabla f, \nabla \Delta g \rangle).$$

In the next definition we recall the Bakry-Émery condition $\text{BE}(K, N)$, the reader is referred to [4, 5, 13] for more details.

Definition 5.3. *Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. We say that (M, d, \mathfrak{m}) satisfies the $\text{BE}(K, N)$ condition if for every $f \in D_{W_w^{1,2}}(\Delta)$, it holds*

$$\Gamma_2(f, f) \geq \left(K |\nabla f|^2 + \frac{1}{N} (\Delta f)^2 \right) \mathfrak{m}.$$

The following important fact was proved for $N = \infty$ in [3, 4] (see also [16, 1]) and for $N \in [1, \infty)$ in [13, Sec. 4] and [6, Sec. 12]:

Theorem 5.4. *Let $(X, \mathbf{d}, \mathbf{m})$ be a metric measure space, let $K \in \mathbb{R}$ and $N \in [1, \infty]$. The following are equivalent:*

- (i) $(X, \mathbf{d}, \mathbf{m})$ is a $\text{RCD}^*(K, N)$ -space or, in case $N = \infty$, $(X, \mathbf{d}, \mathbf{m})$ is a $\text{RCD}(K, \infty)$ -space.
- (ii) $(X, \mathbf{d}, \mathbf{m})$ satisfies the $\text{BE}(K, N)$ -condition and (4.1). Moreover, every $f \in D(\text{Ch})$ with $|Df|_w \leq 1$ \mathbf{m} -a.e. admits a 1-Lipschitz representative \mathbf{m} -a.e. with respect to \mathbf{d} .

5.2 Application to smooth manifolds with lower regularity Riemannian metrics and weights

Let M be a smooth manifold. Take a Riemannian metric $g \in C^0$ that admits L^2_{loc} -Christoffel symbols and induces the distance \mathbf{d}_g . Define the measure μ via $d\mu := h^2 \text{dvol}_g$ for a $h \in C^0(M; (0, \infty)) \cap W^{1,2}_{\text{loc}}(M, \mu)$. Moreover, we assume that (M, \mathbf{d}_g, μ) is a $\text{CD}(K, \infty)$ -space. Thus (4.1) is satisfied (see Remark 4.11) and by Corollary 4.27, (M, \mathbf{d}_g, μ) is an $\text{RCD}(K, \infty)$ -space.

We will now specialise some concepts from [15] to this particular case. Using integration by parts, [3, Prop. 2.14 (iv)] and Lemma 4.15 yields:

Proposition 5.5. *Let $f \in W^{1,2}_w(M, \mu)$ and H_t denote the heat flow of the Cheeger energy. Then $f_t := H_t f \rightarrow f$ in $W^{1,2}_w(M, \mu)$ as $t \rightarrow 0$.*

Proposition 5.6. *The space $L^2(T^*M)$ (as defined in [15, Def. 2.2.1]) is isometric to $H^2_0(T^*M, \mu)$ and the space $L^2(TM)$ (as defined in [15, Def. 2.3.1]) is isometric to $H^2_0(TM, \mu)$.*

We will not give a proof as it directly follows from the (quite technical) definitions and from [15, Prop. 2.2.5]; in the following we will identify the isomorphic spaces with each other.

We will now turn to a generalisation of the **divergence**. For a compactly supported $X \in H^1_1(TM, \mu)$ we have that $\text{div}(X) = (\partial_i X^i + X^i \frac{2}{h} \partial_i h + X^i \frac{1}{2} \text{tr}(g^{-1} \partial_i g)) \in L^2(M, \mu)$ satisfies (3.4). Hence, any compactly supported and smooth vector field X is in $D(\text{div})$, according to [15, Def. 2.3.11].

Next, we look at a generalisation of the **Hessian**. The identity (3.1) motivates the following definition of the space $W^{2,2}(M)$.

Definition 5.7 ([15] Def. 3.3.1). *The space $W^{2,2}(M, \mu) \subset W^{1,2}_w(M, \mu)$ is the space of all functions $f \in W^{1,2}_w(M, \mu)$ with the following property: There exists $A \in L^2((T^*)^{\otimes 2}M)$ such that for any $h_1, h_2, \phi \in \text{TestF}(M)$ it holds*

$$\begin{aligned} & 2 \int \phi A(\nabla h_1, \nabla h_2) \, d\mu \\ &= - \int \langle \nabla f, \nabla h_1 \rangle \text{div}(\phi \nabla h_2) + \langle \nabla f, \nabla h_2 \rangle \text{div}(\phi \nabla h_1) + \phi \langle \nabla f, \nabla \langle \nabla h_1, \nabla h_2 \rangle \rangle \, d\mu. \end{aligned}$$

We will call A the Hessian of f and denote it as $\text{Hess}(f)$. The space $W^{2,2}(M)$ is endowed with the norm $\|\cdot\|_{W^{2,2}(M)}$ defined via

$$\|f\|_{W^{2,2}(M)}^2 = \|f\|_{L^2(M)}^2 + \|df\|_{L^2(T^*M)}^2 + \|\text{Hess}f\|_{L^2((T^*)^{\otimes 2}M)}^2.$$

We next investigate such a space $W^{2,2}(M, \mu)$ in case of a smooth manifold M with a C^0 -Riemannian metric g with L^2_{loc} -Christoffel symbols and a C^0 weighted measure μ . If f is a $C^2_2(M, \mu)$ -function, we know that the Hessian exists. By density, we know that this also holds for $H^2_2(M, \mu)$ -functions. More precisely:

Proposition 5.8. $H^2_2(M, \mu) \subset W^{2,2}(M, \mu)$ and for all $f \in H^2_2(M, \mu)$, it holds $\|f\|_{H^2_2(M, \mu)} = \|f\|_{W^{2,2}(M)}$.

Moreover, it holds that $\text{TestF}(M) \subset W^{2,2}(M, \mu)$, see [15, Thm. 3.3.8].

A consequence of Proposition 5.5, Lemma 4.15, and Mazur's lemma is:

Proposition 5.9. *Let M be a smooth manifold, g a C^0 -Riemannian metric that admits L^2_{loc} -Christoffel symbols and μ a measure on M defined by $d\mu = h^2 \text{dvol}_g$ for a $h \in C^0(M, (0, \infty)) \cap W^{1,2}_{\text{loc}}(M)$. Then $\text{TestF}(M)$ is $H^2_2(M, \mu)$ -dense in $L^\infty \cap H^2_2(M, \mu)$.*

Definition 5.10 ([15], Def. 3.3.17). *The space $H^{2,2}(M)$ is defined as the $W^{2,2}$ -closure of $\text{TestF}(M)$.*

By Proposition 3.3.18 in [15], we have that $H^{2,2}(M)$ coincides with the $W^{2,2}$ -closure of $D(\Delta_\mu)$, so we get:

Proposition 5.11. $H^{2,2}(M) = \overline{H^2_2(M, \mu) \cap L^\infty(M, \mu)}^{H^2_2(M, \mu)}$ and it coincides with the $W^{2,2}$ -closure of $D(\Delta_\mu)$.

Now, we can turn to the abstract definition of the **covariant derivative**:

Definition 5.12 ([15] Def. 3.4.1). *The Sobolev space $W^{1,2}_C(TM) \subset L^2(TM)$ is defined as the space of all $X \in L^2(TM)$ such that there exists $T \in L^2(T^{\otimes 2}M)$ such that for every $h_1, h_2, \varphi \in \text{TestF}(M)$ it holds*

$$\int \varphi T : (\nabla h_1 \otimes \nabla h_2) d\mu = - \int \langle X, \nabla h_2 \rangle \text{div}(\varphi \nabla h_1) - \varphi \text{Hess}(h_2)(X, \nabla h_1) d\mu.$$

In this case we call the tensor T the covariant derivative of X and denote it by ∇X . We endow $W^{1,2}_C(TM)$ with the norm $\|\cdot\|_{W^{1,2}_C(TM)}$ defined by

$$\|X\|_{W^{1,2}_C(TM)}^2 := \|X\|_{L^2(TM)}^2 + \|\nabla X\|_{L^2(T^{\otimes 2}M)}^2.$$

We denote by $H^{1,2}_C(TM)$ the closure of $\text{TestV}(M)$ in $W^{1,2}_C(TM)$.

This definition makes sense, as in [15] it is proved that test vector fields are indeed in $W^{1,2}_C(TM)$. We note that by our previous computations and the local density of gradients in $C^0(TM)$, we get that for $X \in C^2_1(TM, \mu)$, $\nabla X = (\nabla_c X)^\sharp$. It follows that $H^2_1(TM, \mu) \subset W^{1,2}_C(TM)$ and the norms are equal on their intersection. Similarly, we get that $\nabla f \in H^{1,2}_C(TM)$ for all $f \in H^2_2(M, \mu)$.

Proposition 5.13. $\text{TestV}(M) \subset H^2_1(TM, \mu)$, and hence $H^{1,2}_C(TM)$ is a subspace of $H^2_1(TM, \mu)$.

For $X \in W^{1,2}_C(TM)$ and $Z \in L^2(TM)$, we define $\nabla_Z X$ via

$$\langle \nabla_Z X, Y \rangle_g = \nabla X : (Z \otimes Y).$$

A computation shows that in the case of smooth vector fields, this coincides with the smooth covariant derivative. As for C^1 -vector fields, we have that $\nabla_Z X = Z^i \partial_i X^s + \Gamma^s_{ij} X_j Z_i$, this holds by density for all vector fields $X \in H^{1,2}_C(TM)$ and motivates the following definition of **Lie bracket**:

Definition 5.14 ([15], Def. 3.4.8). *For $X, Y \in H^{1,2}_C(TM)$, we define $[X, Y] \in L^1(TM)$ via*

$$[X, Y] := \nabla_X Y - \nabla_Y X.$$

Note that this again coincides with the Lie bracket in the smooth case and the local expressions carry over by density. The generalised **differential** is defined as follows:

Definition 5.15 ([15], Def. 3.5.1, 3.5.5). *The space $W^{1,2}_d(\Lambda^k T^*M) \subset L^2(\Lambda^k T^*M)$ is the space of k -forms $\omega \in L^2(\Lambda^k T^*M)$ such that there exists a $(k+1)$ -form $\eta \in L^2(\Lambda^{k+1} T^*M)$ for which the identity*

$$\begin{aligned} \int \eta(X_0, \dots, X_k) d\mu &= \int \sum_i (-1)^{i+1} \omega(X_0, \dots, \hat{X}_i, \dots, X_k) d\mu \\ &\quad + \int \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) d\mu, \end{aligned}$$

*holds for any $X_0, \dots, X_k \in \text{TestV}(M)$. In this case we call η the exterior differential of ω and we will denote it as $d\omega$. We endow $W^{1,2}_d(\Lambda^k T^*M)$ with the norm $\|\cdot\|_{W^{1,2}_d(\Lambda^k T^*M)}$ given by*

$$\|\omega\|_{W^{1,2}_d(\Lambda^k T^*M)}^2 := \|\omega\|_{L^2(\Lambda^k T^*M)}^2 + \|d\omega\|_{L^2(\Lambda^{k+1} T^*M)}^2.$$

*Moreover, we define $H^{1,2}_d(\Lambda^k T^*M)$ as the $W^{1,2}_d$ -closure of $\text{TestForm}_k(M)$.*

Again, this definition makes sense as in [15] it is proved that test forms are contained in $W_d^{1,2}(\Lambda^k T^*M)$. If we first assume that X_0, \dots, X_k are C^1 , we notice that for a smooth k -form ω the classical differential $d\omega$ satisfies the condition above. By density, this also holds for $X_0, \dots, X_k \in \text{TestV}(M)$, so the coefficients coincide with the ones from the classical differential. It follows that $H_1^2(M, \mu) \subset W_d^{1,2}(\Lambda^0 T^*M)$ and $C_c^1(T^*M) \subset W_d^{1,2}(T^*M)$. By similar arguments as above, we get that $H_d^{2,1}(M, \mu) \subset H_1^2(M, \mu) \subset W_d^{1,2}(M, \mu)$ and the norms are equivalent on the intersection. The cases of $k = 0, 1$ will be of particular interest.

Next, we recall the definition of the **codifferential**:

Definition 5.16 ([15], Def. 3.5.11). *The space $D(\delta_k) \subset L^2(\Lambda^k T^*M)$ is the space of k -forms ω for which there exists a form $\delta\omega \in L^2(\Lambda^{k-1} T^*M)$ called the codifferential of ω , such that*

$$\int \langle \delta\omega, \eta \rangle d\mu = \int \langle \omega, d\eta \rangle d\mu$$

for all $\eta \in \text{TestForm}_{k-1}(M)$. In the case $k = 0$, we set $D(\delta_0) = L^2(\mu)$ and define δ to be identically zero.

For 1-forms, note that $\omega \in D(\delta_1)$ if and only if $\omega^\sharp \in D(\text{div})$ and, in this case, $\delta\omega = -\text{div}(\omega^\sharp)$. In [15], it is shown that for each k , $\text{TestForm}_k(M) \subset D(\delta_k)$. Hence, the following definition makes sense:

Definition 5.17 ([15], Def. 3.5.13). *The space $W_H^{1,2}(\Lambda^k T^*M)$ is defined as $W_d^{1,2}(\Lambda^k T^*M) \cap D(\delta_k)$ endowed with the norm $\|\cdot\|_{W_H^{1,2}(\Lambda^k T^*M)}$ defined by*

$$\|\omega\|_{W_H^{1,2}(\Lambda^k T^*M)}^2 := \|\omega\|_{W_d^{1,2}(\Lambda^k T^*M)}^2 + \|\delta\omega\|_{L^2(\Lambda^{k-1} T^*M)}^2.$$

The space $H_H^{1,2}(\Lambda^k T^*M)$ is defined as the $W_H^{1,2}$ -closure of $\text{TestForm}_k(M)$.

It is not hard to check that $\text{TestForm}_k(M) \subset W_H^{1,2}(\Lambda^k T^*M)$.

Definition 5.18 ([15], Def. 3.6.3). *The space $H_H^{1,2}(TM) \subset L^2(TM, \mu)$ is the space of vector fields $X \in L^2(TM)$ such that $X^\flat \in H_H^{1,2}(T^*M)$ equipped with the norm $\|X\|_{H_H^{1,2}(TM)} := \|X^\flat\|_{H_H^{1,2}(T^*M)}$.*

Definition 5.19 ([15], Def. 3.5.14). *Given $k \in \mathbb{N}$ the domain $D(\Delta_{H,k}) \subset H_H^{1,2}(\Lambda^k T^*M)$ of the Hodge Laplacian is defined as the set of $\omega \in H_H^{1,2}(\Lambda^k T^*M)$ for which there exists an $\alpha \in L^2(\Lambda^k T^*M)$ such that*

$$\int \langle \alpha, \eta \rangle d\mu = \int \langle d\omega, d\eta \rangle d\mu + \int \langle \delta\omega, \delta\eta \rangle d\mu \quad \forall \eta \in H_H^{1,2}(\Lambda^k T^*M).$$

In this case α is unique and we denote it by $\Delta_{H,k}\omega$.

Note that

$$\int \langle \Delta_{H,k}\omega, \omega \rangle d\mu = \int (\langle d\omega, d\omega \rangle + \langle \delta\omega, \delta\omega \rangle) d\mu.$$

In [15, Prop. 3.6.1], it is shown that $\text{TestForm}_1(M) \subset D(\Delta_{H,1})$. Moreover, $\Delta f = -\Delta_{H,0}f$ for all $f \in \text{TestF}(M)$. Finally, we recall the generalised **Ricci curvature tensor**:

Theorem 5.20 ([15], Thm. 3.6.7). *There is a unique continuous map $\mathbf{Ric} : [H_H^{1,2}(TM)]^2 \rightarrow \text{Meas}(M)$ such that, for every $X, Y \in \text{TestV}(M)$, it holds:*

$$\mathbf{Ric}(X, Y) = \Delta \frac{\langle X, Y \rangle}{2} + \left(\frac{1}{2} \langle X, (\Delta_H Y^\flat)^\sharp \rangle + \frac{1}{2} \langle Y, (\Delta_H X^\flat)^\sharp \rangle - \nabla X : \nabla Y \right) \mu.$$

This map is bilinear, symmetric and satisfies:

$$\mathbf{Ric}(X, X) \geq Kg(X, X)\mu.$$

Moreover, setting $X = Y$, we get that

$$\mathbf{Ric}(X, X) + (|\nabla X|_{HS}^2 - \langle X, (\Delta_H X^\flat)^\sharp \rangle) \mu = \Delta \frac{|X|^2}{2}.$$

We will now connect the measure valued Ricci tensor in the sense of Theorem 5.20 and the distributional Ricci tensor in the sense of Subsection 3.3.

Proposition 5.21. *Let M be a smooth manifold with a C^0 -Riemannian metric g with L^2_{loc} Christoffel symbols. Consider $d\mu = h^2 \text{dvol}_g$ for a positive function $h \in C^0 \cap W^{1,2}_{\text{loc}}$. Assume that (M, \mathbf{d}_g, μ) is a $\text{CD}(K, \infty)$ -space. For $X \in \text{TestV}(M)$ we have that $\mathbf{Ric}(X, X) = \text{Ric}_{\mu, \infty}(X, X)$ in the sense of (3.12).*

Proof. By the previous observations we have that for $\varphi \in C_c \cap W^{1,2}_{\text{loc}}(M)$ (which we assume to be supported in one coordinate patch),

$$\begin{aligned} \int_M \varphi d\mathbf{Ric}(X, X) &= \int_M \varphi d\Delta \frac{|X|^2}{2} + \int_M (\langle X, (\Delta_H X^\flat)^\sharp \rangle - |\nabla X|_{HS}^2) \varphi d\mu \\ &= -\frac{1}{2} \int_M \langle \nabla |X|^2, \nabla \varphi \rangle d\mu + \int_M (\langle X, (\Delta_{\mu, H} X^\flat)^\sharp \rangle - |\nabla X|_{HS}^2) \varphi d\mu \\ &= -\frac{1}{2} \int_M \langle \nabla |X|^2, \nabla \varphi \rangle d\mu + \int_M (|dX^\flat|^2 + |\delta_\mu X^\flat|^2 - |\nabla X|_{HS}^2) \varphi d\mu. \end{aligned}$$

Now the last line equals the right hand side of (3.12) as it appears in Lemma 4.16 so we get that locally

$$\begin{aligned} &\int_M \varphi d\mathbf{Ric}(X, X) \\ &= \int X^j X^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \varphi h^2 \sqrt{|g|} dx^1 \dots dx^n - \int \Gamma_{jk}^p \partial_p (X^j X^k \varphi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\ &\quad + \int \Gamma_{pk}^p \partial_j (X^j X^k \varphi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\ &\quad + 2 \int X^j X^k (\partial_j h \partial_k h + h \partial_s h \Gamma_{kj}^s) \varphi \sqrt{|g|} dx^1 \dots dx^n + 2 \int \partial_j h \partial_k (X^j X^k h \varphi \sqrt{|g|}) dx^1 \dots dx^n. \end{aligned}$$

□

As in [15], we have that for all $f \in \text{TestF}(M)$ it holds

$$\mathbf{Ric}(\nabla f, \nabla f) + |\nabla^2 f|_{HS}^2 \mu = \Delta \frac{|\nabla f|^2}{2} - \langle \nabla f, \nabla \Delta f \rangle \mu = \Gamma_2(f, f). \quad (5.1)$$

Proposition 5.22. *Let M be a smooth manifold and g a C^0 -Riemannian metric with L^2_{loc} -Christoffel symbols and $h \in C^0(M, (0, \infty)) \cap W^{1,2}_{\text{loc}}$ such that (M, \mathbf{d}_g, μ) is a $\text{CD}(K, \infty)$ -space, where $d\mu = h^2 \text{dvol}_g$. Then, for all $f \in H^{2,2}(M)$, it holds that*

$$\int_M |\nabla^2 f|^2 h^2 \text{dvol}_g \leq \int_M |\Delta f|^2 h^2 \text{dvol}_g - K \int_M |\nabla f|^2 h^2 \text{dvol}_g.$$

If $f \in D(\Delta)$, we have that $H_t f \rightarrow f$ strongly in $H^{2,2}$.

Proof. By [15, Cor. 3.3.9], we have that

$$\int_M |\nabla^2 u|_{HS}^2 d\mu \leq \int_M ((\Delta u)^2 - K |\nabla u|^2) d\mu$$

for all $u \in D(\Delta)$. If $f \in H^{2,2} \cap D(\Delta)$, we have that $\Delta f \in L^2$ and hence $H_t \Delta f \rightarrow \Delta f$ strongly in L^2 as $t \rightarrow 0$. As the semi-group H_t is generated by Δ , we have that $\Delta H_t f = H_t \Delta f$ for all $t \geq 0$. It follows that $\Delta H_t f \rightarrow \Delta f$ strongly in L^2 as $t \rightarrow 0$. Setting $u = f - H_t f$, we infer

$$\begin{aligned} \|\nabla^2(f - H_t f)\|_{L^2((T^*)^{\otimes 2} M)}^2 &\leq -K \int_M |\nabla(f - H_t f)|^2 h^2 \text{dvol}_g + \int_M |\Delta(f - H_t f)|^2 h^2 \text{dvol}_g \\ &\leq (|K| + 1) (\|f - H_t f\|_{H^2_1(M, \mu)} + \|\Delta(f - H_t f)\|_{L^2(M, \mu)}) \rightarrow 0 \end{aligned}$$

as $t \rightarrow 0$.

□

6 RCD implies distributional Ricci curvature lower bounds

6.1 Some useful approximation results

We start the section with some local considerations, so we work in \mathbb{R}^n for simplicity of presentation. Recall the following consequence of the Poincaré inequality:

Lemma 6.1. *Let $R > 0$, $1 \leq p < \infty$ and $u \in W^{1,p}(B_R(0))$. Let $m := \frac{1}{|B_R(0)|} \int_{B_R(0)} u \, d\mathcal{L}^n$. Then there exists a constant $C = C(p) > 0$ independent of R such that*

$$\|u - m\|_{L^p(B_R(0))} \leq CR \|\nabla u\|_{L^p(B_R(0))}.$$

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^n$ be open and ν be a σ -finite measure on Ω such that $C_c^\infty(\Omega)$ is dense in $L^p(\nu)$ for each $p \in [1, \infty)$. Let $f, (f_k)_{k \geq 1}$ be measurable functions on Ω such that $\|f_k\|_{L^\infty}$ and $\|f\|_{L^\infty}$ are bounded and such that $f_k \rightharpoonup f$ weakly in $L^p(\nu)$ for some $p \in [1, \infty)$. Then $f_k \xrightarrow{*} f$ in the weak* topology of $(L^1)^* = L^\infty$.*

Proof. As ν is σ -finite, we have that indeed $(L^1)^* = L^\infty$. By the Banach-Alaoglu theorem, it follows that there is a weakly* convergent subsequence of f_k , which we will still denote by f_k . Hence, f_k converges weakly* to some $\tilde{f} \in L^\infty$. Suppose $\tilde{f} \neq f$. Then there exists an L^1 -function ϕ such that

$$\int_{\Omega} f \phi \, d\nu \neq \int_{\Omega} \tilde{f} \phi \, d\nu.$$

As $C_c^\infty(\Omega)$ is dense in L^1 , we can assume that $\phi \in C_c^\infty \subset L^q$, where $q = \frac{p}{p-1}$. This contradicts the weak convergence in L^p . \square

Lemma 6.3. *Let $K \subset B_1(0) \subset \mathbb{R}^n$ be compact and denote $d = \text{dist}(K, \partial B_1(0)) > 0$. Then there exists a $\delta_0 > 0$ and a constant $C = C(n)$ such that for all $\delta \in (0, \delta_0)$, there exist an integer $m \leq C\delta^{-n}$ and points $y_1, \dots, y_m \in B_1(0)$ such that $B_{3\delta}(y_i) \subset B_{2\delta + \frac{d}{12}}(y_i) \subset B_1(0)$ for all $1 \leq i \leq m$, $K \subset \bigcup_{i=1}^m B_\delta(y_i)$ and for all $x \in B_1(0)$, $|\{i : x \in B_{2\delta}(y_i)\}| \leq C$.*

Proof. Let $\delta_0 = \frac{d}{12}$ and fix $\delta \in (0, \delta_0)$. Now let $\{y_1, \dots, y_m\} := \frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_{1-\frac{3d}{4}}(0)$. It follows that $K \subset B_{1-\frac{3d}{4}-\frac{\delta}{2}}(0) \subset \bigcup_{i=1}^m B_\delta(y_i) \subset \bigcup_{i=1}^m B_{2\delta + \frac{d}{12}}(y_i) \subset B_{1-\frac{d}{4}}(0)$. Moreover, there exists a constant $C_1(n)$ such that for each $x \in \mathbb{R}^n$, $|\frac{1}{2\sqrt{n}}\mathbb{Z}^n \cap B_2(x)| \leq C_1(n)$ so for each $x \in B_1(0)$, there are at most $C_1(n)$ points in $\frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_{2\delta}(x)$. Finally, there exists a constant $C_2(n)$ such that for each $0 < \lambda \leq 1$, $|\frac{\lambda}{2\sqrt{n}}\mathbb{Z}^n \cap B_1(0)| \leq C_2(n)\lambda^{-n}$. Taking $C(n) = \max(C_1(n), C_2(n))$ finishes the proof. \square

Lemma 6.4. *Let $R > 0$, $\varphi \in C_c^\infty(B_R(0))$. Then there exists a constant $C = C(n, R) > 0$ and $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$, there exists a set $\{\chi_1, \dots, \chi_m\} \subset C_c^\infty(B_R(0), [0, 1])$ and a family $\{y_1, \dots, y_m\} \subset B_{R-3\delta}(0)$ such that the following holds:*

- (i) $m \leq C\delta^{-n}$,
- (ii) $\sum_{i=1}^m \chi_i(x) \leq 1$ for all $x \in B_R(0)$ and $\sum_{i=1}^m \chi_i(x) = 1$ for all $x \in \text{supp } \varphi$,
- (iii) For each $1 \leq i \leq m$, $\|\nabla \chi_i\|_{L^\infty} \leq C\delta^{-1}$.
- (iv) For all i , it holds $\chi_i \in C_c^\infty(B_{2\delta}(y_i))$.
- (v) For all $x \in B_R(0)$, it holds $|\{i : x \in B_{2\delta}(y_i)\}| \leq C$.

Proof. By rescaling, we can assume $R = 1$. Then take $K = \text{supp } \varphi$ and apply Lemma 6.3 to get δ_0 . Choose $0 < \delta \leq \delta_0$ and take $m, \{y_1, \dots, y_m\}$ as in Lemma 6.3. It then follows that for all $i = 1, \dots, m$, $B_{3\delta}(y_i) \subset B_R(0)$, hence $y_i \in B_{R-3\delta}(0)$. Now, for each $i \in \{1, \dots, m\}$ take a function $\eta_i \in C_c^\infty(B_{2\delta}(y_i), [0, 1])$ such that $\eta_i(x) = 1$ for all $x \in B_\delta(y_i)$ and $|\nabla \eta_i| \leq C\delta^{-1}$. It follows that $\sum_{i=1}^m \eta_i \in C_c^\infty(B_1(0))$ and $\sum_{i=1}^m \eta_i \geq 1$

on $\text{supp } \varphi$. Now take a function $h \in C^\infty(\mathbb{R})$ such that $|h'| \leq 1$, $h(x) \geq \max(|x|, \frac{1}{4})$ and $h(x) = x$ for $x \geq 1$. Define

$$\chi_j(x) = \frac{\eta_j(x)}{h(\sum_{i=1}^m \eta_i(x))}.$$

It follows that $\sum_{i=1}^m \chi_i(x) \leq 1$ for all $x \in B_1(0)$ and $\sum_{i=1}^m \chi_i(x) = 1$ for all $x \in \text{supp } \varphi$. To see (iii), note that for all $x \in B_1(0)$

$$|\nabla \chi_j(x)| \leq \frac{|\nabla \eta_j|(x) + |h'(\sum_{i=1}^m \eta_i(x))| \sum_{i=1}^m |\nabla \eta_i|(x)}{h(\sum_{i=1}^m \eta_i(x))^2} \leq 16C \left(\delta^{-1} + \sum_{i: x \in \text{supp } \eta_i}^m \delta^{-1} \right) \leq 16(1 + C(n))\delta^{-1}.$$

□

Definition 6.5. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $\text{CompV}(\Omega)$ the linear span of functions of the form $\rho \nabla h$, where $\rho, h \in C_c^\infty(\Omega)$. For a smooth manifold M with a C^0 -Riemannian metric g , we define $\text{CompV}(M)$ as the linear span of functions of the form $\rho \nabla h$, where $\rho, h \in C_c^\infty(M)$.

Lemma 6.6. Let $R > 0$, $B_R(0) \subset \mathbb{R}^n$. Let $X \in C_c^\infty(B_R(0); \mathbb{R}^n)$. Then there exists a constant $C = C(n, R) > 0$ and a constant $C_2 = C_2(n) > 0$ such that for all $\varepsilon > 0$ there is a $\tilde{X} = \sum_{p=1}^q h_p \nabla f_p \in \text{CompV}(B_R(0))$ satisfying the following

- (i) $\|X - \tilde{X}\|_{W^{1,1}(B_R(0))} \leq C\varepsilon$ and $\|X - \tilde{X}\|_{L^\infty(B_R(0))} \leq C\varepsilon$,
- (ii) $\|\tilde{X}\|_{W^{1,\infty}(B_R(0))} \leq C(1 + \|X\|_{W^{2,\infty}(B_R(0))})$,
- (iii) $q \leq C_2$,
- (iv) for all p , we have that $\|h_p\|_{L^\infty(B_R(0))} \leq C$ and $\|f_p\|_{L^\infty(B_R(0))} \leq C\|X\|_{C^1(B_R(0))}\varepsilon$,
- (v) for all p , we have that $\|\nabla h_p\|_{L^\infty(B_R(0))} \leq C(1 + \|X\|_{C^2(B_R(0))})\varepsilon^{-1}$ and $\|\nabla f_p\|_{L^\infty(B_R(0))} \leq C(1 + \|X\|_{C^1(B_R(0))})$.

Proof of (i). Assume (by potentially readjusting) that $\varepsilon \leq \min(\frac{1}{2}, \delta_0)$, where $\delta_0 > 0$ is as in Lemma 6.4. We will denote by DX the Jacobi matrix of X . We choose

$$\delta = \frac{\varepsilon}{8n^2(1 + \|DX\|_{L^\infty} + \|D^2X\|_{L^\infty})}, \quad (6.1)$$

so by the mean value theorem we get that for all $y, z \in B_R(0)$ with $|y - z| \leq 4\delta$, it holds $|X(y) - X(z)| < \varepsilon$ and $|DX(y) - DX(z)| < \varepsilon$. Note that we also have $\delta < \delta_0$. For this chosen δ , take $m, \{\chi_1, \dots, \chi_m\}$ and $\{y_1, \dots, y_m\}$ as in Lemma 6.4. Let $\psi \in C_c^\infty(B_{3\delta}(0))$ such that $\psi(B_{2\delta}(0)) = \{1\}$ and $|\nabla \psi| \leq 2\delta^{-1}$. Fix an $i \in \{1, \dots, m\}$. Let

$$A_i = \frac{1}{|B_{2\delta}(y_i)|} \int_{B_{2\delta}(y_i)} X d\mathcal{L}^n \in \mathbb{R}^n,$$

and

$$\alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \psi(x - y_i) \langle A_i, (x - y_i) \rangle.$$

Then $\nabla \alpha_i = A_i$ and $DA_i = 0$ on $B_{2\delta}(y_i)$. For $1 \leq l, k \leq n$, let $B_i^{lk} = \partial_k X^l(y_i) \in \mathbb{R}$. Moreover, define

$$\begin{aligned} \rho_i^k(x) &:= \psi(x - y_i)(x_k - (y_i)_k) \in C_c^\infty(B_R(0)), \\ \eta_i^{lk}(x) &:= \psi(x - y_i) B_i^{lk}(x_l - (y_i)_l) \in C_c^\infty(B_R(0)) \text{ and} \\ \beta_i^{lk}(x) &:= (\rho_i^k \nabla \eta_i^{lk})(x) = \psi(x - y_i)(x_k - (y_i)_k) \cdot \nabla(\psi(x - y_i) B_i^{lk}(x_l - (y_i)_l)) \in \text{CompV}(B_R(0)). \end{aligned}$$

Then, for all $x \in B_{2\delta}(y_i)$, we have that

$$\begin{aligned}\beta_i^{lk}(x) &= (x_k - (y_i)_k)B_i^{lk}e_l \in \mathbb{R}^n, \quad \text{and} \\ D\beta_i^{lk}(x) &= (\delta_m^l \delta_p^k B_i^{lk})_{m,p=1,\dots,n} \in \mathbb{R}^{n \times n}.\end{aligned}$$

Notice that

$$\int_{B_{2\delta}(y_i)} \beta_i^{lk} d\mathcal{L}^n = 0 \in \mathbb{R}^n,$$

as β_i^{lk} is linear on $B_{2\delta}(y_i)$. Now define $\tilde{X} := \sum_{i=1}^m \chi_i \cdot (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}) \in \text{CompV}(B_R(0))$. As $X, \tilde{X} \in W_0^{1,1}(B_R(0))$, the Poincaré inequality implies that

$$\|X - \tilde{X}\|_{W^{1,1}(B_R(0))} \leq C \cdot (1 + R) \|DX - D\tilde{X}\|_{L^1(B_R(0))},$$

for some $C = C(n) > 0$. Now, as $X = \sum_{i=1}^m \chi_i X$, we get that

$$\begin{aligned}\|DX - D\tilde{X}\|_{L^1(B_R(0))} &= \int_{B_R(0)} \left| \sum_{i=1}^m D(\chi_i X) - D(\chi_i (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk})) \right| d\mathcal{L}^n \\ &\leq \int_{B_R(0)} \sum_{i=1}^m |\nabla\chi_i| \left| X - (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}) \right| + \left| \sum_{i=1}^m \chi_i (DX - D \sum_{k,l=1}^n \beta_i^{kl}) \right| d\mathcal{L}^n.\end{aligned}$$

For each i , we get that

$$\left\| DX - D \left(\sum_{k,l=1}^n \beta_i^{kl} \right) \right\|_{L^\infty(B_{2\delta}(y_i))} = \|DX - DX(y_i)\|_{L^\infty(B_{2\delta}(y_i))} \leq n^2 \varepsilon. \quad (6.2)$$

Thus, using that $\chi_i \geq 0$, we get that

$$\int_{B_R(0)} \left| \sum_{i=1}^m \chi_i (DX - D \sum_{k,l=1}^n \beta_i^{kl}) \right| d\mathcal{L}^n \leq \int_{B_R(0)} \sum_{i=1}^m \chi_i \left| (DX - D \sum_{k,l=1}^n \beta_i^{kl}) \right| d\mathcal{L}^n \leq n^2 |B_R(0)| \varepsilon.$$

For each i , we have $|\nabla\chi_i| \leq C(n, R)\delta^{-1}$ and

$$\begin{aligned}\int_{B_R(0)} |\nabla\chi_i| \left| X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl} \right| d\mathcal{L}^n &= \int_{B_{2\delta}(y_i)} |\nabla\chi_i| \left| X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl} \right| d\mathcal{L}^n \\ &\leq C(n, R)\delta^{-1} \left\| X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl} \right\|_{L^1(B_{2\delta}(y_i))}.\end{aligned}$$

By Lemma 6.1 and (6.2), we have that

$$\begin{aligned}\left\| X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl} \right\|_{L^1(B_{2\delta}(y_i))} &\leq 2C\delta \left\| DX - D \left(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl} \right) \right\|_{L^1(B_{2\delta}(y_i))} \\ &= 2C\delta \left\| DX - D \left(\sum_{k,l=1}^n \beta_i^{kl} \right) \right\|_{L^1(B_{2\delta}(y_i))} \\ &\leq C\delta^{n+1} \varepsilon.\end{aligned}$$

Thus,

$$\int_{B_R(0)} |\nabla \chi_i| |X - \nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{kl}| d\mathcal{L}^n \leq C(n, R) \varepsilon \delta^n.$$

Now, as $m \leq C(n, R) \delta^{-n}$, we get that

$$\int_{B_R(0)} \sum_{i=1}^m |\nabla \chi_i| |X - \left(\nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{lk} \right)| \leq C(n, R)^2 \varepsilon.$$

All together this gives

$$\|X - \tilde{X}\|_{W^{1,1}(B_R(0))} \leq C(1+R) \|DX - D\tilde{X}\|_{L^1(B_R(0))} \leq C(n, R) \varepsilon.$$

Moreover, note that for each $x \in B_R(0)$,

$$|X - \tilde{X}|(x) \leq \left| \sum_{i=1}^m \chi_i (X - \nabla \alpha_i) \right|(x) + \sum_{i=1}^m \sum_{l,k} |\chi_i \beta_i^{lk}|(x) \leq C(n) \varepsilon.$$

This concludes the proof of (i).

Proof of (ii). Next, we want to investigate the L^∞ -norm of $D\tilde{X}$. We get that for a point $y \in B_R(0)$

$$\begin{aligned} |DX - D\tilde{X}|(y) &= \left| \sum_{i=1}^m D(\chi_i X) - D\left(\chi_i (\nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{lk})\right) \right|(y) \\ &\leq \left(\sum_{i=1}^m |\nabla \chi_i| \left| X - \left(\nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{lk} \right) \right| \right)(y) + \left| \sum_{i=1}^m \chi_i \left(DX - D \sum_{k,l=1}^n \beta_i^{lk} \right) \right|(y). \end{aligned}$$

We will investigate the two sums separately. For a fixed i , we know that $|\nabla \chi_i|(y) \leq C(n, R) \delta^{-1}$ and $|\chi_i|(y) \leq 1$. It suffices to only consider $y \in B_{2\delta}(y_i)$, in which case we have that $B_{2\delta}(y_i) \subset B_{4\delta}(y)$. By our choice of δ , we have that for all $z \in B_{2\delta}(y_i)$ it holds $|X(z) - X(y)| \leq \varepsilon$. Hence,

$$\begin{aligned} &|X(y) - \nabla \alpha_i| \\ &= \left| X(y) - \frac{1}{|B_{2\delta}(y_i)|} \int_{B_{2\delta}(y_i)} X(z) d\mathcal{L}^n(z) \right| \\ &\leq \frac{1}{|B_{2\delta}(y_i)|} \int_{B_{2\delta}(y_i)} |X(z) - X(y)| d\mathcal{L}^n(z) \leq \varepsilon. \end{aligned}$$

Moreover,

$$\left| \sum_{k,l=1}^n \beta_i^{lk} \right|(y) \leq n^2 \delta (1 + |DX|_{L^\infty}).$$

This gives that

$$\begin{aligned} |\nabla \chi_i| \left| X - \left(\nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{lk} \right) \right|(y) &\leq C(n) \delta^{-1} \left(|X(y) - \nabla \alpha_i| + \left| \sum_{k,l=1}^n \beta_i^{lk} \right|(y) \right) \\ &\leq C(n) \delta^{-1} (\varepsilon + \delta |DX|_{L^\infty}) \\ &\leq C(n) (1 + |DX|_{L^\infty} + |D^2 X|_{L^\infty}), \end{aligned}$$

where in the last estimate we used the precise dependence of ε in terms of δ , (6.1). The choice of δ also gives

$$\left|DX - D \sum_{k,l=1}^n \beta_i^{kl}\right|(y) \leq n^2 \varepsilon.$$

Noting that $|i : y \in \text{supp} \chi_i| \leq C(n)$, we get that

$$\begin{aligned} |DX - D\tilde{X}|(y) &\leq \sum_{i=1}^m |\nabla \chi_i| \left| X - \left(\nabla \alpha_i + \sum_{k,l=1}^n \beta_i^{lk} \right) \right|(y) + \left| \sum_{i=1}^m \chi_i \left(DX - D \sum_{k,l=1}^n \beta_i^{kl} \right) \right|(y) \\ &\leq C(n) \left((1 + \|DX\|_{L^\infty} + \|D^2 X\|_{L^\infty}) + n^2 \varepsilon \right). \end{aligned}$$

It follows that there exists a constant $C(n, R)$ such that

$$\|D\tilde{X}\|_{L^\infty} \leq \|DX - D\tilde{X}\|_{L^\infty} + \|DX\|_{L^\infty} \leq C(n, R)(1 + \|X\|_{C^2(B_R(0))}).$$

As $\tilde{X} \in C_c^\infty(B_R(0))$, we get that also $\|\tilde{X}\|_{L^\infty} \leq CR(1 + \|X\|_{C^2(B_R(0))})$, which proves (ii).

Proof of (iii) For the last three statements, let us explicitly spell out how we can build \tilde{X} out of finitely many (bounded by $C_2 = C_2(n)$ independent of δ) C_c^∞ -functions and their gradients. Recall that by the proof of Lemma 6.3, we have chosen the y_i to be in $\frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_R(0)$. Note that $\left| \left(\mathbb{Z}/12n\mathbb{Z} \right)^n \right| = (12n)^n$. For $\xi \in \left(\mathbb{Z}/12n\mathbb{Z} \right)^n$, let

$$Z_\xi := \{ \zeta \in \mathbb{Z}^n : \zeta_s \equiv \xi_s \pmod{12n}, \forall 1 \leq s \leq n \}.$$

Define $Y_\xi := \{ i : y_i \in \{y_1, \dots, y_m\} \cap \frac{\delta}{2\sqrt{n}} Z_\xi \}$. If $Y_\xi = \emptyset$, define $\chi_\xi = \alpha_\xi = \rho_\xi^k = \eta_\xi^{lk} = 0 \in C_c^\infty(B_R(0))$, where $k, l \in \{1, \dots, n\}$. Otherwise, define

$$\begin{aligned} \chi_\xi &:= \sum_{i \in Y_\xi} \chi_i \in C_c^\infty(B_R(0)), \\ \alpha_\xi &:= \sum_{i \in Y_\xi} \alpha_i \in C_c^\infty(B_R(0)), \\ \rho_\xi^k &:= \sum_{i \in Y_\xi} \rho_i^k \in C_c^\infty(B_R(0)), \text{ for } 1 \leq k \leq n \text{ and} \\ \eta_\xi^{lk} &:= \sum_{i \in Y_\xi} \eta_i^{lk} \in C_c^\infty(B_R(0)), \text{ for } 1 \leq k, l \leq n. \end{aligned}$$

We note that

$$B_{3\delta}(y_i) \cap B_{3\delta}(y_{i'}) = \emptyset, \text{ if } i, i' \in Y_\xi \text{ and } i \neq i', \quad (6.3)$$

as then $|y_i - y_{i'}| \geq 6\sqrt{n}\delta$. Now, for all i , we have that

$$\text{supp } \alpha_i \cup \text{supp } \chi_i \cup \bigcup_{k=1}^n \text{supp } \rho_i^k \cup \bigcup_{k,l=1}^n \text{supp } \eta_i^{lk} \subset B_{3\delta}(y_i),$$

so

$$\begin{aligned} \chi_\xi \nabla \alpha_\xi + \sum_{k,l=1}^n \chi_\xi \rho_\xi^k \nabla \eta_\xi^{lk} &= \left(\sum_{i \in Y_\xi} \chi_i \right) \left(\sum_{i \in Y_\xi} \nabla \alpha_i \right) + \sum_{k,l=1}^n \left(\sum_{i \in Y_\xi} \chi_i \right) \left(\sum_{i \in Y_\xi} \rho_i^k \right) \left(\sum_{i \in Y_\xi} \nabla \eta_i^{lk} \right) \\ &= \sum_{i \in Y_\xi} \chi_i \nabla \alpha_i + \sum_{i \in Y_\xi} \chi_i \sum_{k,l=1}^n \rho_i^k \nabla \eta_i^{lk}. \end{aligned}$$

Hence,

$$\tilde{X} = \sum_{\xi \in \left(\frac{\mathbb{Z}}{12n\mathbb{Z}}\right)^n} \left(\chi_\xi \nabla \alpha_\xi + \sum_{k,l=1}^n \chi_\xi \rho_\xi^k \nabla \eta_\xi^{lk} \right).$$

These are at most $2 \cdot (12n)^n (n^2 + 1) =: C_2(n)$ functions, proving (iii).

Proof of (iv) and (v). To prove the last two statements, we may again fix a $\xi \in \left(\frac{\mathbb{Z}}{12n\mathbb{Z}}\right)^n$ as above. Take an $x \in B_R(0)$. If $x \notin B_{3\delta}(y_i)$ for some $i \in Y_\xi$, then all of the above functions and their gradients are zero when evaluated at x . Otherwise, by (6.3), there exists exactly one $i \in Y_\xi$ such that $x \in B_{3\delta}(y_i)$. Then we have,

$$\begin{aligned} |\chi_\xi(x)| &= |\chi_i(x)| \leq 1, \\ |\nabla \chi_\xi(x)| &= |\nabla \chi_i(x)| \leq C\delta^{-1} \leq C(1 + \|X\|_{C^2})\varepsilon^{-1}, \\ |\alpha_\xi(x)| &= |\alpha_i(x)| \leq C\|X\|_{C^0}\delta \leq C(1 + \|DX\|_{C^1})\delta \leq C\varepsilon, \end{aligned}$$

and

$$\begin{aligned} |\nabla \alpha_\xi(x)| &= |\nabla \alpha_i(x)| \leq |\nabla \psi(x - y_i) \langle A_i, (x - y_i) \rangle|(x) + |\psi(x - y_i) A_i|(x) \leq \delta^{-1} \cdot C\|X\|_{C^0}\delta + C\|X\|_{C^0} \\ &\leq C(1 + \|X\|_{C^0}). \end{aligned}$$

Moreover,

$$|\chi_\xi \rho_\xi^k(x)| = |\chi_i \rho_i^k(x)| \leq C\delta \leq C\varepsilon$$

and

$$\begin{aligned} |\nabla(\chi_\xi \rho_\xi^k)|(x) &= |\nabla(\chi_i \rho_i^k)|(x) \leq |\nabla \chi_i|(x) |\rho_i^k|(x) + |\chi_i|(x) (|\nabla \psi(x - y_i)| |(x_k - (y_i)_k)| + |\psi(x - y_i)|) \\ &\leq C(\delta^{-1} \cdot \delta) + C(\delta^{-1}\delta + 1) \leq C. \end{aligned}$$

Finally,

$$|\eta_\xi^{lk}(x)| = |\eta_i^{lk}(x)| = |\psi(x - y_i)| |B_i^{lk}| |(x_l - (y_i)_l)| \leq C\|X\|_{C^1}\delta$$

and

$$\begin{aligned} |\nabla \eta_\xi^{lk}(x)| &= |\nabla \eta_i^{lk}(x)| \leq |\nabla \psi(x - y_i)| |B_i^{lk}| |x_l - (y_i)_l| + |\psi(x - y_i)| |B_i^{lk}| \\ &\leq C\delta^{-1}\|X\|_{C^1}\delta + \|X\|_{C^1} \leq C\|X\|_{C^1}. \end{aligned}$$

As x was arbitrary in $B_R(0)$, this holds for all x , and as ξ was arbitrary, this holds for all ξ , which proves the lemma. \square

We now deduce a corollary about Sobolev spaces in \mathbb{R}^n . This is not related to our further study but it is worth mentioning.

Corollary 6.7. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. Then $\text{CompV}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ for all $p \in [1, \infty)$.*

We are now going to apply the previous results to manifolds. The next lemma follows from partitioning the manifold in balls and patching the approximations from Lemma 6.6 together.

Lemma 6.8. *Let M be a smooth manifold with a C^0 -Riemannian metric g that admits L_{loc}^2 -Christoffel symbols and a measure μ defined via $d\mu = h^2 \text{dvol}_g$, where $h \in C^0(M, (0, \infty))$. Let X be a compactly supported smooth vector field on M and let $\varepsilon > 0$. Then there exist a constant $C = C(M, g, h, X)$ and a vector field $\tilde{X} = \sum_{p=1}^q h_p \nabla f_p \in \text{CompV}(M)$ such that*

- (i) $\|X - \tilde{X}\|_{H^1_1(TM, \mu)} \leq C\varepsilon,$
- (ii) $\sup_M(|\tilde{X}|_g + |\nabla \tilde{X}|_g) \leq C,$
- (iii) $q \leq C,$
- (vi) for all p , we have that $\|h_p\|_{L^\infty} \leq C$ and $\|f_p\|_{L^\infty} \leq C\varepsilon,$
- (v) for all p , we have that $\|\nabla h_p\|_{L^\infty} \leq C\varepsilon^{-1}$ and $\|\nabla f_p\|_{L^\infty} \leq C.$

In the next proposition we invoke the parabolic De Giorgi-Nash-Moser theory [21] to obtain $C^{0,\alpha}$ convergence of the heat flow to the initial datum. The non-triviality of the statement relies on the fact that the Riemannian metric is merely C^0 with L^2_{loc} Christoffel symbols, and the weight on the measure is merely $C^0 \cap W^{1,2}_{\text{loc}}(M)$.

Proposition 6.9. *Let M be a smooth manifold and g a C^0 -Riemannian metric with L^2_{loc} Christoffel symbols. Let moreover $h \in C^0(M, (0, \infty)) \cap W^{1,2}_{\text{loc}}(M)$. Define the measure μ on M via $d\mu = h^2 d\text{vol}_g$. Let $K \in \mathbb{R}$ and assume that (M, \mathbf{d}_g, μ) is an $\text{RCD}(K, \infty)$ -space. Denote by H_t the heat flow on M . Let $\Omega \subset M$ be open such that Ω is contained in one coordinate patch (U, ψ) and $\bar{\Omega}$ is compact. Then there exists an $\alpha \in (0, 1)$ such that for each $\rho \in C^\infty_c(M)$, $T > 0$, and for each open $\tilde{\Omega} \subset \subset \Omega$, it holds $(\text{id}_{[0,1]} \times \psi)_* H_t \rho|_{\tilde{\Omega}} \in C^{0,\alpha}([0, T], \tilde{\Omega})$.*

Proof. Throughout this proof we write ρ_t for $\psi_* H_t \rho|_{\Omega}$. As $\bar{\Omega}$ is compact, we get that there exists a constant $\lambda > 0$ such that $\frac{1}{\lambda} \leq h^2 \sqrt{|g|} \leq \lambda$ and $\frac{1}{\lambda} \text{Id}_n \leq g^{-1} \leq \lambda \text{Id}_n$ on Ω . We note that ρ_t weakly solves

$$\frac{d}{dt} \rho_t = \Delta_\mu \rho_t$$

on Ω . Observing that for any two functions $f, \phi \in C^\infty_c(\Omega)$ it holds

$$\int_M \Delta_\mu f \cdot \phi \, d\mu = - \int_M \langle \nabla_g f, \nabla_g \phi \rangle_g \, d\mu = - \int_M h^2 \sqrt{|g|} g^{ij} \partial_i f \partial_j \phi \, d\mathcal{L}^n,$$

we get that ρ_t weakly solves

$$\partial_t \rho_t - \text{div}(A D \rho_t) = 0, \tag{6.4}$$

where

$$A_{ij} = h^2 \sqrt{|g|} g^{ij}. \tag{6.5}$$

By the previous observations, A is uniformly elliptic on Ω . Recalling that $\rho \in C^\infty_c(M)$, the result follows from the parabolic DeGiorgi-Nash-Moser theory, see for instance Theorem 1.1 in Chapter V of [21]. \square

We are now ready to prove the main approximation result of the section that will be key in the proof of the main theorem of the paper.

Lemma 6.10. *Let M be a smooth manifold endowed with a C^0 -Riemannian metric g that admits L^2_{loc} -Christoffel symbols and a continuous positive weight h . Let μ be the measure defined via $d\mu = h^2 d\text{vol}_g$. Assume that the metric measure space (M, \mathbf{d}_g, μ) satisfies the $\text{CD}(K, \infty)$ -condition. Then for each relatively compact, open $U \subset M$ and vector field $X \in C^\infty_c(TM)$ there exists a sequence $(W_j)_j \subset \text{TestV}(M)$ such that $W_j \rightarrow X$ in $H^1_1(TU, \mu)$ and $(W_j)_j$ is bounded in $L^\infty(U, \mu)$.*

Proof. Let $\varepsilon > 0$. Applying Lemma 6.8, we can find a constant $C = C(M, g, h, X)$ and functions $f_p, h_p \in C^\infty_c(M)$ where $p \in \{1, \dots, q\}$ such that, denoting $\tilde{X} := \sum_{p=1}^q h_p \nabla f_p$,

- $\|X - \tilde{X}\|_{H^1_1(TM, \mu)} \leq C\varepsilon,$
- $\sup_M(|\tilde{X}|_g + |\nabla \tilde{X}|_g) \leq C,$

- $q \leq C$,
- for all p , we have that $\|h_p\|_{L^\infty} \leq C$ and $\|f_p\|_{L^\infty} \leq C\varepsilon$,
- for all p , we have that $\|\nabla h_p|g\|_{L^\infty} \leq C\varepsilon^{-1}$ and $\|\nabla f_p|g\|_{L^\infty} \leq C$.

We first note that by interpolation, $\|X - \tilde{X}\|_{H_1^2(TM,\mu)} \leq C\sqrt{\varepsilon}$. We have that $C_c^\infty \subset D(\Delta_\mu) \cap W^{2,2}(M,\mu) \cap L^\infty(M)$. Thanks to Proposition 5.22, we get that for each p , there exists a $t \in (0, \frac{1}{|K|}]$, such that

$$\max(\|h_p - H_t h_p\|_{H_2^2(M,\mu)}, \|f_p - H_t f_p\|_{H_2^2(M,\mu)}) < \frac{\varepsilon^2}{\max_p (\|f_p\|_{H_2^2(M,\mu)} + \|h_p\|_{H_2^2(M,\mu)})}.$$

Fix a relatively compact, open set $U \subset M$. Then by using Proposition 6.9 on a finite, relatively compact cover, we can potentially decrease t , to get

$$\max(\|h_p - H_t h_p\|_{L^\infty(U,\mu)}, \|f_p - H_t f_p\|_{L^\infty(U,\mu)}) < \frac{\varepsilon^2}{\max_p (\|f_p\|_{H_2^2(M,\mu)} + \|h_p\|_{H_2^2(M,\mu)})},$$

for all $p = 1, \dots, q$. Define $\tilde{f}_p := H_t f_p$ and $\tilde{h}_p := H_t h_p$ for all p . Moreover, recall that for each $t \in (0, \frac{1}{|K|}]$ and each $f \in W^{2,2} \cap L^\infty$, we have that

$$\begin{aligned} \|H_t f\|_{L^\infty(M,\mu)} &\leq \|f\|_{L^\infty(M,\mu)} \text{ and} \\ \|\nabla H_t f|g\|_{L^\infty(M,\mu)} &\leq e^{-Kt} \|\nabla f|g\|_{L^\infty(M,\mu)}, \end{aligned}$$

by Theorem 4.10 and the maximum principle of the heat flow. Hence,

$$\begin{aligned} \|\tilde{f}_p\|_{L^\infty} &\leq \|f_p\|_{L^\infty} \leq C\varepsilon, \\ \|\tilde{h}_p\|_{L^\infty} &\leq \|h_p\|_{L^\infty} \leq C, \\ \|\nabla \tilde{f}_p|g\|_{L^\infty} &\leq C, \text{ and} \\ \|\nabla \tilde{h}_p|g\|_{L^\infty} &\leq C\varepsilon^{-1}. \end{aligned}$$

Define $W := \sum_{p=1}^q \tilde{h}_p \nabla \tilde{f}_p$. We directly get that

$$\|W|g\|_{L^\infty(M,\mu)} = \left\| \sum_{p=1}^q \tilde{h}_p \nabla \tilde{f}_p \right\|_{L^\infty(M,\mu)} \leq \sum_{p=1}^q \|\tilde{h}_p \nabla \tilde{f}_p|g\|_{L^\infty(M,\mu)} \leq qC^2.$$

To conclude, we estimate that

$$\begin{aligned} \|W - \tilde{X}\|_{L^2(TM,\mu)} &\leq \sum_{p=1}^q \|\tilde{h}_p \nabla \tilde{f}_p - h_p \nabla f_p\|_{L^2(TM,\mu)} \\ &\leq \sum_{p=1}^q \|\tilde{h}_p \nabla \tilde{f}_p - \tilde{h}_p \nabla f_p\|_{L^2(TM,\mu)} + \|\tilde{h}_p \nabla f_p - h_p \nabla f_p\|_{L^2(TM,\mu)} \\ &\leq \sum_{p=1}^q \|\tilde{h}_p\|_{L^\infty(M,\mu)} \|\nabla \tilde{f}_p - \nabla f_p\|_{L^2(TM,\mu)} + \|\tilde{h}_p - h_p\|_{L^2(M,\mu)} \|\nabla f_p\|_{L^\infty(TM,\mu)} \\ &\leq C(M, g, h, X)\varepsilon. \end{aligned}$$

For the covariant derivative, we note that

$$\nabla W = \left(\nabla_c \sum_{p=1}^q \tilde{h}_p \nabla \tilde{f}_p \right)^\# = \left(\sum_{p=1}^q \tilde{h}_p \text{Hess} \tilde{f}_p + d\tilde{h}_p \otimes \nabla \tilde{f}_p \right)^\# = \sum_{p=1}^q \tilde{h}_p (\text{Hess} \tilde{f}_p)^\# + \nabla \tilde{h}_p \otimes \nabla \tilde{f}_p.$$

Then, by similar arguments as the previous ones, we get that

$$\begin{aligned} \left\| \nabla W - \nabla \tilde{X} \right\|_{L^2(T^{\otimes 2}U, \mu)} &\leq \sum_{p=1}^q \left\| \nabla(\tilde{h}_p \nabla \tilde{f}_p) - \nabla(h_p \nabla f_p) \right\|_{L^2(T^{\otimes 2}U, \mu)} \\ &\leq \sum_{p=1}^q \left\| \tilde{h}_p \right\|_{L^\infty(U, \mu)} \left\| \text{Hess}(\tilde{f}_p - f_p)^\# \right\|_{L^2(T^{\otimes 2}U, \mu)} + \left\| \tilde{h}_p - h_p \right\|_{L^\infty(U, \mu)} \left\| (\text{Hess} \tilde{f}_p)^\# \right\|_{L^2(T^{\otimes 2}U, \mu)} \\ &\quad + \sum_{p=1}^q \left\| \nabla \tilde{h}_p \right\|_{L^\infty(TU)} \left\| \nabla \tilde{f}_p - \nabla f_p \right\|_{L^2(TU, \mu)} + \left\| \nabla(\tilde{h}_p - h_p) \right\|_{L^2(TU, \mu)} \left\| \nabla \tilde{f}_p \right\|_{L^\infty(TU, \mu)} \\ &\leq C(M, U, g, h, X) \varepsilon. \end{aligned}$$

Finally, we estimate that

$$\|X - W\|_{H_1^2(TU, \mu)} \leq \|X - \tilde{X}\|_{H_1^2(TU, \mu)} + \|\tilde{X} - W\|_{H_1^2(TU, \mu)} \leq C(M, U, g, h, X) \sqrt{\varepsilon}.$$

□

6.2 The case $N = \infty$

We are now able to prove the first main result:

Theorem 6.11. *Let M be a smooth manifold and $g \in C^0(M)$ be a metric with L_{loc}^2 Christoffel symbols and let $h \in C^0(M, (0, \infty)) \cap W_{\text{loc}}^{1,2}(M)$ be such that (M, d_g, μ) , with $d\mu = h^2 \text{dvol}_g$ is a $\text{CD}(K, \infty)$ -space. Then*

$$\text{Ric}_{\mu, \infty} \geq Kg$$

in the distributional sense.

Proof. From Corollary 4.27, we get that (M, d_g, μ) is indeed an $\text{RCD}(K, \infty)$ -space, so all the observations from the previous chapter apply. Our aim is to show that $\text{Ric}_g \geq Kg$ distributionally, so we fix a smooth vector field $X \in \mathcal{T}_0^1$ and a test volume $\omega = \phi h^2 \text{vol}_g$, where $\phi \in C_c \cap W_{\text{loc}}^{1,2}(M)$. Using a smooth cut-off function, we can assume that X is compactly supported. We can (using a partition of unity) again assume that ϕ is supported in one coordinate patch. We will therefore directly work in an open, relatively compact

set $U \subset \mathbb{R}^n$. From (3.8), we know that

$$\begin{aligned}
\int_U \text{Ric}_{\mu,\infty}(X, X)\omega &= \int_U X^j X^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \phi h^2 \sqrt{|g|} dx^1 \dots dx^n \\
&\quad - \int_U \Gamma_{jk}^p \partial_p (X^j X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n + \int_U \Gamma_{pk}^p \partial_j (X^j X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
&\quad + 2 \int_U X^j X^k (\partial_j h \partial_k h + h \partial_s h \Gamma_{kj}^s) \phi \sqrt{|g|} dx^1 \dots dx^n + 2 \int_U \partial_j h \partial_k (X^j X^k h \phi \sqrt{|g|}) dx^1 \dots dx^n \\
&= \int_U X^j X^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \phi h^2 \sqrt{|g|} dx^1 \dots dx^n - \int_U \Gamma_{jk}^p \partial_p (X^j) X^k \phi h^2 \sqrt{|g|} dx^1 \dots dx^n \\
&\quad - \int_U \Gamma_{jk}^p X^j \partial_p (X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n + \int_U \Gamma_{pk}^p \partial_j (X^j) X^k \phi h^2 \sqrt{|g|} dx^1 \dots dx^n \\
&\quad + \int_U \Gamma_{pk}^p X^j \partial_j (X^k \phi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
&\quad + 2 \int_U X^j X^k (\partial_j h \partial_k h + h \partial_s h \Gamma_{kj}^s) \phi \sqrt{|g|} dx^1 \dots dx^n + 2 \int_U \partial_j h \partial_k X^j X^k h \phi \sqrt{|g|} dx^1 \dots dx^n \\
&\quad + 2 \int_U \partial_j h X^j \partial_k (X^k h \phi \sqrt{|g|}) dx^1 \dots dx^n.
\end{aligned}$$

By the assumptions on g and h and Lemma 3.8, there exist functions $\Psi_{i,j,k} \in L^2(U, \mathcal{L}^n)$ and $\Phi_{j,k} \in L^1(U, \mathcal{L}^n)$ all compactly supported in U such that

$$\int_U \text{Ric}_{\mu,\infty}(X, X)\omega = \int_U X^j X^k \Phi_{j,k} d\mathcal{L}^n + \int_U \partial_i X^j X^k \Psi_{i,j,k} d\mathcal{L}^n.$$

By Lemma 6.10, there exists a sequence $V_\delta \in \text{TestV}(M)$ such that $\|V_\delta|_g\|_{L^\infty(U)} \leq C$ for some constant $C > 0$ that does not depend on δ and $\|V_\delta - X\|_{H_1^2(TU, \mu)} \rightarrow 0$ as $\delta \rightarrow 0$. Fix an $\varepsilon > 0$. Note that $V_\delta \otimes V_\delta$ is bounded in L^∞ and converges to $X \otimes X$ in L^1 . By Lemma 6.2, $V_\delta \otimes V_\delta \xrightarrow{*} X \otimes X$ in $L^\infty = (L^1)^*$ and $V_\delta \xrightarrow{*} X$ in $L^\infty = (L^1)^*$. Hence, we can find a $\delta_0 > 0$ such that $\|V_\delta - X\|_{H_1^2(TU, \mu)} < \varepsilon$ and $\|V_\delta \otimes V_\delta - X \otimes X\|_{L^1(T \otimes^2 U, \mu)} < \varepsilon$ for all $\delta \in (0, \delta_0)$. As g and $h^2 \sqrt{|g|}$ are uniformly bounded from above and from below on U , we get that $\|V_\delta - X\|_{W^{1,2}(\mathbb{R}^n, \mathcal{L}^n)} < C\varepsilon$ and $\|V_\delta \otimes V_\delta - X \otimes X\|_{L^1(\mathbb{R}^{n \times n}, \mathcal{L}^{n \times n})} < C\varepsilon$ for all $\delta \in (0, \delta_0)$ and a $C = C(M, g, h, U) > 0$. Now we can use the weak* convergence, to choose a $\delta_1 \in (0, \delta_0)$ such that

$$\begin{aligned}
\left| \int_U X^j X^k \Phi_{j,k} d\mathcal{L}^n - \int_U V_{\delta_1}^j V_{\delta_1}^k \Phi_{j,k} d\mathcal{L}^n \right| &\leq \varepsilon \text{ and} \\
\left| \int_U \partial_i X^j X^k \Psi_{i,j,k} d\mathcal{L}^n - \int_U \partial_i V_{\delta_1}^j V_{\delta_1}^k \Psi_{i,j,k} d\mathcal{L}^n \right| &\leq \varepsilon.
\end{aligned}$$

From now on we will denote $V_{\delta_1} = V$ for simplicity. We get that

$$\left| \int_U \partial_i X^j V^k \Psi_{i,j,k} d\mathcal{L}^n - \int_U \partial_i V^j V^k \Psi_{i,j,k} d\mathcal{L}^n \right| \leq C(M, g, h, U, \phi, X)\varepsilon.$$

Now note that

$$\left| \int_M \langle V, V \rangle_g \phi h^2 d\text{vol}_g - \int_M \langle X, X \rangle_g \phi h^2 d\text{vol}_g \right| \leq C \|V \otimes V - X \otimes X\|_{L^1(\mathbb{R}^{n \times n})} \leq C\varepsilon,$$

where again $C = C(M, g, h, U, \phi, X)$. Finally, we use that

$$\int_M \phi h^2 d\mathbf{Ric}(V, V) = \int_U \text{Ric}_{\mu,\infty}(V, V)\omega = \int_U V^j V^k \Phi_{j,k} d\mathcal{L}^n + \int_U \partial_i V^j V^k \Psi_{i,j,k} d\mathcal{L}^n. \quad (6.6)$$

All together, we get that

$$\begin{aligned}
\int_M \text{Ric}(X, X)\omega &\geq \int_M \phi h^2 d\mathbf{Ric}(V, V) - C\varepsilon \\
&\geq K \int_M \langle V, V \rangle_g \phi h^2 d\text{vol}_g - C\varepsilon \\
&\geq K \int_M \langle X, X \rangle_g \phi h^2 d\text{vol}_g - (2 + |K|)C\varepsilon \\
&= K \int_M g(X, X)\omega - (2 + |K|)C\varepsilon.
\end{aligned}$$

Sending $\varepsilon \rightarrow 0$ yields the result. \square

Lemma 6.12. *Let $\Omega \subset \mathbb{R}^n$ be open and $M \in C_c^\infty(\Omega; \mathbb{R}_{sym}^{n \times n})$ be pointwise positive semidefinite. Let $\varepsilon > 0$. Then there exist a function $\varphi \in C_c^\infty(\Omega, [0, 1])$ and vector fields $b_k \in C^\infty(\Omega, \mathbb{R}^n)$ for $k = 1, \dots, n$, such that for $M_\varepsilon := \sum_k \varphi b_k \otimes b_k$, it holds*

$$|M - M_\varepsilon|_{C^1(\mathbb{R}^n)} \leq C\varepsilon,$$

where $C = C(\Omega, \text{supp } M, n)$ does not depend on ε .

Proof. First we recall a basic fact about matrices. Let $P \in \mathbb{R}^{n \times n}$ be positive definite. Then there exists a unique positive definite matrix $A \in \mathbb{R}^{n \times n}$ such that $A^T A = P$. Writing $A = (A_{ij})_{ij}$, we get that $P_{ij} = (A^T A)_{ij} = \sum_{k=1}^n A_{ki} A_{kj} = \sum_k (A_k \otimes A_k)_{ij}$, where A_k denotes the k -th row vector of A and \otimes the dyadic product.

Now define $\tilde{M} := M + \varepsilon \text{Id}_n \in C^\infty(\Omega; \mathbb{R}_{sym}^{n \times n})$ and note that \tilde{M} is constant outside $\text{supp } M$. By construction, we have that \tilde{M} is positive definite everywhere in Ω and its smallest eigenvalue is bounded below by ε . Similarly, its largest eigenvalue is bounded above by $\|M\|_{HS|_{C^0(\Omega)}} + \varepsilon =: m$. Set

$$U := \left\{ a + ib : a \in \left(\frac{1}{2}\varepsilon, m + 2 \right), b \in (-1, 1) \right\}$$

and $K := [\varepsilon, m + 1] \subset \mathbb{R}_+ \subset \mathbb{C}$. For a matrix $B \in \mathbb{C}^{n \times n}$, we denote by $\sigma_{\mathbb{C}^{n \times n}}(B)$ the set of its complex eigenvalues. Note that for each $p \in \Omega$, $\sigma_{\mathbb{C}^{n \times n}}(\tilde{M}(p)) \subset K$. Let γ be a smooth cycle in $U \setminus K$ such that $n(\gamma, w) = 1$ for all $w \in K$, where $n(\gamma, w)$ denotes the winding number of γ around w . We denote by $s : U \rightarrow \mathbb{C}$ the unique holomorphic function which is defined by $s^2(z) = z$ and $s(w) \geq 0$ for $w \in \mathbb{R} \cap U$. Define

$$B(p) := \frac{1}{2\pi i} \int_\gamma s(z)(z \text{Id}_n - \tilde{M}(p))^{-1} dz.$$

Using holomorphic functional calculus we infer that $B(p)$ is the unique positive square root of $\tilde{M}(p)$ for all $p \in \Omega$. As γ is a fixed curve with positive distance to K , all functions in the integral are smooth and bounded on the domain of integration. As $\tilde{M}(p)$ is a smooth function of p , so is $B(p)$ and each of its rows, which we will denote by b_k for $k = 1, \dots, n$. Let $\varphi \in C_c^\infty(\Omega, [0, 1])$ be a non-negative cut-off function such that $\varphi \equiv 1$ on $\text{supp } M$. We can choose φ such that $|\nabla \varphi|$ is bounded by $C(n) \cdot \text{dist}(\partial\Omega, \text{supp } M)$. Now define

$$M_\varepsilon := \varphi \sum_k b_k \otimes b_k.$$

By definition, we have that

$$|M_\varepsilon - \tilde{M}|(p) = \begin{cases} 0 & \text{if } p \in \text{supp } M, \\ \varphi|\varepsilon \text{Id}_n| \leq C(n)\varepsilon & \text{if } p \notin \text{supp } M. \end{cases}$$

Similarly,

$$|\nabla(M_\varepsilon - \tilde{M})|(p) = \begin{cases} 0 & \text{if } p \in \text{supp } M, \\ \nabla\varphi|\varepsilon\text{Id}_n| \leq C(\Omega, \text{supp } M, n)\varepsilon & \text{if } p \notin \text{supp } M. \end{cases}$$

By definition, we know that $|M - \tilde{M}|_{C^1} \leq C(n)\varepsilon$, so combining the estimates above, we get that

$$|M - M_\varepsilon|_{C^1} \leq |M - \tilde{M}|_{C^1} + |M_\varepsilon - \tilde{M}|_{C^1} \leq C\varepsilon.$$

□

Theorem 6.13. *Let M be a smooth manifold endowed with a C^0 -Riemannian metric g that admits L^2_{loc} -Christoffel symbols and a positive function $h \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$. Define the measure μ via $d\mu = h^2 \text{dvol}_g$. Suppose the distributional Bakry-Émery ∞ -Ricci curvature as defined in (3.8) is bounded below by K for some $K \in \mathbb{R}$. For each coordinate patch U , there exist regular Radon measures ν_{ij} for $i, j = 1, \dots, n$ such that for a test volume ω with $\text{supp } \omega \subset U$ (locally written as $\omega = \varphi h^2 \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$) and smooth vector fields X, Y , we have that in local coordinates it holds*

$$\text{Ric}_{\mu, \infty}(X, Y)\omega = \sum_{i,j} \int X^i Y^j \varphi h^2 \sqrt{|g|} d\nu_{ij}(x_1, \dots, x_n). \quad (6.7)$$

Proof. Using the local trivialisation $\psi : U \rightarrow \Omega \subset \mathbb{R}^n$, we may assume that we are working on the open set Ω and will express everything in local coordinates. Define

$$D_{ij} := \partial_p \Gamma_{ij}^p - \partial_i \Gamma_{pj}^p + \Gamma_{ij}^s \Gamma_{ps}^p - \Gamma_{jp}^s \Gamma_{is}^p - \frac{2}{h^2} (h \partial_{ij} h - \partial_i h \partial_j h - \Gamma_{ij}^s h \partial_s h) - K g_{ij} \in \mathcal{D}'(\Omega).$$

By definition, D_{ij} is a distribution of order at most 1 and for any smooth vector field X and any test volume $\omega = \varphi h^2 \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$, for a $\varphi \in C_c \cap W^{1,2}_{\text{loc}}(M)$ we get that

$$(\text{Ric}(X, X)_{\mu, \infty} - K g(X, X))\omega = \sum_{i,j} \langle D_{ij}, X^i X^j \varphi h^2 \sqrt{|g|} \rangle_{\mathcal{D}', \mathcal{D}}.$$

Now define $D \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times n}_{\text{sym}})$ via $\langle D, M \rangle_{\mathcal{D}', \mathcal{D}} := \sum_{i,j} \langle D_{ij}, M_{ij} \rangle_{\mathcal{D}', \mathcal{D}}$. Denote $B(X, \varphi) := \varphi h^2 \sqrt{|g|} (X \otimes X)$. Then

$$(\text{Ric}(X, X)_{\mu, \infty} - K g(X, X))\omega = \sum_{ij} \langle D_{ij}, B(X, \varphi)_{ij} \rangle_{\mathcal{D}', \mathcal{D}} = \langle D, B(X, \varphi) \rangle_{\mathcal{D}', \mathcal{D}}.$$

Now let $B \in C_c^\infty(\Omega; \mathbb{R}^{n \times n}_{\text{sym}})$ be pointwise positive semidefinite and $\varepsilon > 0$. Let B_ε be as in Lemma 6.12. As $\text{Ric}_{\mu, \infty} - K g \geq 0$, we get that

$$\langle D, B_\varepsilon \rangle_{\mathcal{D}', \mathcal{D}} \geq 0.$$

Moreover,

$$|\langle D, (B_\varepsilon - B) \rangle_{\mathcal{D}', \mathcal{D}}| \leq C(\Omega, \text{supp } B, g, h) |B - B_\varepsilon|_{C^1(\Omega)} \leq C(\Omega, B, n, g, h)\varepsilon.$$

Hence,

$$\langle D, B \rangle_{\mathcal{D}', \mathcal{D}} \geq -C(\Omega, B, n, g, h)\varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we get that for any such B ,

$$\langle D, B \rangle_{\mathcal{D}', \mathcal{D}} \geq 0. \quad (6.8)$$

Now fix $S \subset \Omega$ compact and let $A \in C_c^\infty(\Omega; \mathbb{R}_{sym}^{n \times n})$ (not necessarily positive semidefinite) such that $\text{supp } A \subset S$ and $\sup_S |A|_{HS} \leq 1$. Then A only has eigenvalues in $[-1, 1]$ at each point in Ω . Let $\rho \in C_c^\infty(\Omega, [0, 1])$ such that $\rho \equiv 1$ on S . It then follows that $\rho \text{Id}_n - A$ is positive semidefinite in Ω . Then by (6.8),

$$\langle D, A \rangle_{\mathcal{D}', \mathcal{D}} \leq \langle D, \rho \text{Id}_n \rangle_{\mathcal{D}', \mathcal{D}}.$$

In other words:

$$\sup_{A \in C_c^\infty(\Omega; \mathbb{R}_{sym}^{n \times n}), \text{supp } A \subset S, \|A\|_{HS} \leq 1} \langle D, A \rangle_{\mathcal{D}', \mathcal{D}} \leq \langle D, \rho \text{Id}_n \rangle_{\mathcal{D}', \mathcal{D}} < \infty.$$

As S is arbitrary, we have proven that $D \in C_c(\Omega; \mathbb{R}_{sym}^{n \times n})' \cong (C_c(\Omega, \mathbb{R})^{\frac{n(n+1)}{2}})' \cong (C_c(\Omega, \mathbb{R})')^{\frac{n(n+1)}{2}}$. By Riesz' Theorem, $C_c(\Omega, \mathbb{R})'$ equals the space of regular Radon measures on Ω , so it follows that there exist regular Radon measures $\tilde{\nu}_{ij}$ on Ω for $i, j = 1, \dots, n$ such that

$$\langle D, M \rangle = \sum_{i,j} \int_{\Omega} M_{ij} d\tilde{\nu}_{ij}, \quad \text{for all } M \in C_c(\Omega; \mathbb{R}_{sym}^{n \times n}).$$

Define the regular Radon measure

$$\nu_{ij} := \tilde{\nu}_{ij} + K g_{ij} \mathcal{L}^n.$$

Now for any vector field X and any compactly supported test volume $\varphi h^2 \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$, we have that

$$\int_{\Omega} X^i X^j \varphi h^2 \sqrt{|g|} g_{ij} K d\mathcal{L}^n = \langle K g(X, X), \omega \rangle_{\mathcal{D}', \mathcal{D}}.$$

By the definition of $B(X, \varphi)$ and D , we get that

$$\langle \text{Ric}(X, X)_{\mu, \infty}, \omega \rangle_{\mathcal{D}', \mathcal{D}} = \sum_{i,j} \int X^i X^j \varphi h^2 \sqrt{|g|} d\nu_{ij}(x_1, \dots, x_n).$$

By using that $\text{Ric}_{\mu, \infty}(\cdot, \cdot)$ is a symmetric bilinear form and considering $\text{Ric}_{\mu, \infty}(X + Y, X + Y)$, we conclude that (6.7) holds. \square

Remark 6.14. *From the proof of Theorem 6.13 it also follows that $\sum_{i,j} |\nu_{ij}|(S) < \infty$ for each compact subset $S \subset \Omega$. Moreover, the components $(\nu_{ij})_{ij}$ change tensorially under coordinate transformations.*

6.3 The case $N \in [n, \infty)$

Finally, we want to examine the case when (M, \mathbf{d}_g, μ) is a $\text{CD}^*(K, N)$ space for some $N \geq 1$. By Corollary 4.27, we know that (M, \mathbf{d}_g, μ) is infinitesimally Hilbertian, hence we know that it is $\text{RCD}^*(K, N)$. Then, by Theorem 5.4, we get that (M, \mathbf{d}_g, μ) satisfies the $\text{BE}(K, N)$ -condition

$$\mathbf{\Gamma}_2(f, f) = \frac{1}{2} \Delta \langle \nabla f, \nabla f \rangle - \langle \nabla \Delta f, \nabla f \rangle \mu \geq (K |\nabla f|^2 + \frac{1}{N} (\Delta f)^2) \mu, \quad \text{for all } f \in \text{TestF}(M).$$

Testing with a non-negative function $\varphi \in C_c \cap W_{\text{loc}}^{1,2}(M)$, that we may assume to be supported in one coordinate patch, we obtain:

$$\begin{aligned} & -\frac{1}{2} \int_M \langle \nabla \langle \nabla f, \nabla f \rangle, \nabla \varphi \rangle h^2 \sqrt{|g|} dx^1 \dots dx^n - \int_M \varphi \langle \nabla \Delta f, \nabla f \rangle h^2 \sqrt{|g|} dx^1 \dots dx^n \\ & \geq \int_M (K |\nabla f|^2 + \frac{1}{N} (\Delta f)^2) \varphi h^2 \sqrt{|g|} dx^1 \dots dx^n. \end{aligned}$$

Using 3.12 and Proposition 5.21 and (5.1), we get that

$$\begin{aligned}
& \int (\nabla f)^j (\nabla f)^k (\Gamma_{kj}^s \Gamma_{ps}^p - \Gamma_{kp}^s \Gamma_{js}^p) \varphi h^2 \sqrt{|g|} dx^1 \dots dx^n - \int \Gamma_{jk}^p \partial_p ((\nabla f)^j (\nabla f)^k \varphi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
& + \int \Gamma_{pk}^p \partial_j ((\nabla f)^j (\nabla f)^k \varphi h^2 \sqrt{|g|}) dx^1 \dots dx^n \\
& + \int \frac{2}{h^2} (\nabla f)^j (\nabla f)^k (\partial_k h \partial_j h + h \Gamma_{jk}^s \partial_s h) \varphi h^2 \sqrt{|g|} dx^1 \dots dx^n \\
& + 2 \int \partial_k h \partial_j (h (\nabla f)^j (\nabla f)^k \varphi \sqrt{|g|}) dx^1 \dots dx^n + \int_M \varphi |\nabla^2 f|_{HS}^2 h^2 \sqrt{|g|} dx^1 \dots dx^n \\
& \geq \int_M (K |\nabla f|^2 + \frac{1}{N} (\Delta_\mu f)^2) \varphi h^2 \sqrt{|g|} dx^1 \dots dx^n. \tag{6.9}
\end{aligned}$$

Now take a function $\tilde{f} \in C_c^2(M, \mu)$. It follows that $\tilde{f} \in H^{2,2}(M, \mu) \cap D(\Delta_\mu)$, so by Proposition 5.22 the heat flow $H_t \tilde{f} =: \tilde{f}_t$ converges strongly to \tilde{f} in $H^{2,2}(M) \subset H_2^2(TM, \mu)$. Then $\tilde{f}_t \rightarrow \tilde{f}$ in $H^{2,2} \subset H_2^2$ so we can conclude that (6.9) holds for any $\tilde{f} \in C_c^2(M)$ and any $\varphi \in C_c^1(M)$. With Theorem 6.13, this gives:

Proposition 6.15. *Let M be a smooth manifold endowed with a C^0 -Riemannian metric g that admits L_{loc}^2 -Christoffel symbols and a measure μ defined via $d\mu = h^2 \text{dvol}_g$, for a positive function $h \in C^0 \cap W_{\text{loc}}^{1,2}(M)(M)$. If (M, d_g, μ) is a $\text{CD}^*(K, N)$ -space, then for all $f \in C_c^2(M)$, it holds*

$$\sum_{ij} (\nabla f)_i (\nabla f)_j \nu_{ij} + (|\nabla^2 f|_{HS}^2 - K |\nabla f|^2 - \frac{1}{N} (\Delta_\mu f)^2) \mu \geq 0 \tag{6.10}$$

in fixed local coordinates.

An application of Theorem 4.7 in [12] shows:

Lemma 6.16. *Let M be a smooth manifold, g a C^0 -Riemannian metric on M and $U \subset V \subset M$ both open such that $\bar{U} \subset V$ and \bar{V} is compact. Moreover denote by $g_\varepsilon = g * \rho_\varepsilon$. Denote $G = \|g\|_{C^0(\bar{V})} + \|g^{-1}\|_{C^0(\bar{V})}$. Then there exists an $\varepsilon_0 > 0$ such that for all $x \in U$ and $0 < \varepsilon < \varepsilon_0$, it holds*

$$i_\varepsilon(x) \geq C(\varepsilon, G, U, V, \rho) > 0,$$

where $i_\varepsilon(x)$ denotes the injectivity radius with respect to the metric g_ε .

Lemma 6.17. *Let M be a smooth manifold, g a C^0 -Riemannian metric that admits L_{loc}^2 -Christoffel symbols on M and $U \subset V \subset M$ both open such that $\bar{U} \subset V$, and \bar{V} is compact and contained in one coordinate patch. Moreover let $g_\varepsilon := g * \rho_\varepsilon$. Denote $G = \|g\|_{C^0(\bar{V})} + \|g^{-1}\|_{C^0(\bar{V})}$. For all $p \in U$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C = C(U, V, \varepsilon, G, p, \rho) > 0$ with the following property: for all vectors $\tilde{v} \in T_p M$ with $\|\tilde{v}\|_{g_\varepsilon} \leq \min(1, C)$, it holds*

$$|D \exp_p^{g_\varepsilon} |_{\tilde{v}}|_{euc} \leq 3, \quad \text{and} \quad |D^2 \exp_p^{g_\varepsilon} |_{\tilde{v}}|_{euc} \leq 3. \tag{6.11}$$

Proof. By covering and rescaling, we can assume that $U = B_R^{euc}(0) \subset \mathbb{R}^n$ and $V = B_{R+2}^{euc}(0) \subset \mathbb{R}^n$. Let $\varepsilon_0 > 0$ small enough for Lemma 6.16 to hold, and such that $\varepsilon_0 \leq \frac{1}{2}$ and for all $\varepsilon \in (0, \varepsilon_0)$, g_ε is a Riemannian metric on $B_R(0)$ satisfying

$$\|g_\varepsilon\|_{C^0(\overline{B_R^{euc}(0)})} + \|g_\varepsilon^{-1}\|_{C^0(\overline{B_R^{euc}(0)})} \leq 2G.$$

Choose $\varepsilon \in (0, \varepsilon_0)$ and denote by i_ε the injectivity radius with respect to g_ε . Fix a point $p \in U$ and $v \in T_p U$ such that $\|v\|_{euc} \leq 1$. Note that there exists $\lambda = \lambda(G) \geq 1$ such that g, g_ε , and $|\cdot|_{euc}$ are pairwise λ -equivalent on V and their induced metrics are pairwise λ -equivalent on $B_{1/2}^{euc}(p)$. For a curve $c : [0, 1] \rightarrow M$ and W a vector field along the c , we denote by \dot{W} the covariant time derivative and by W' the derivative in local

coordinates. Denote $\gamma := \gamma_v^\varepsilon$ the geodesic with respect to g_ε , originating in p and tangent to v . Note that $|\dot{\gamma}(t)|_{g_\varepsilon}$ is constant in the existence domain of γ , hence we have that for all such t ,

$$|\dot{\gamma}(t)|_{euc} \leq \lambda^2 |\dot{\gamma}(0)|_{euc} = \lambda^2 |v|_{euc}. \quad (6.12)$$

For any vector $w \in T_p U$ define $J_{v,w}^\varepsilon$ to be the Jacobi field in metric g_ε along γ with $J_{v,w}^\varepsilon(0) = \nabla_{\dot{\gamma}(0)} J_{v,w}^\varepsilon(0) = w$. Recall the Jacobi equation for the smooth metric g_ε :

$$\ddot{J}_{v,w}^\varepsilon(t) = \nabla_{\dot{\gamma}(t)} (\nabla_{\dot{\gamma}(t)} J_{v,w}^\varepsilon)(t) = R(J_{v,w}^\varepsilon(t), \dot{\gamma}(t)) \dot{\gamma}(t).$$

In order to keep notation short, we write J_w for $J_{v,w}^\varepsilon$. In local coordinates, we get:

$$\begin{aligned} (\dot{J}_w(t))^k &= (\nabla_{\dot{\gamma}(t)} J_w(t))^k = \frac{d}{dt} J_w^k(t) + J^j(t) \dot{\gamma}^i(t) \Gamma_{\varepsilon ij}^k(\gamma(t)), \text{ and} \\ (\ddot{J}_w(t))^l &= \frac{d}{dt} \left(\frac{d}{dt} J_w^l(t) + J^j(t) \dot{\gamma}^i(t) \Gamma_{\varepsilon ij}^l(\gamma(t)) \right) + \Gamma_{\varepsilon pk}^l(\gamma(t)) \dot{\gamma}^p(t) \left(\frac{d}{dt} J_w^k(t) + J^j(t) \dot{\gamma}^i(t) \Gamma_{\varepsilon ij}^k(\gamma(t)) \right). \end{aligned}$$

Hence, the Jacobi equation in local coordinates gives

$$\begin{aligned} (R_\varepsilon)_{ijk}^l J_w^i(t) \dot{\gamma}^j(t) \dot{\gamma}^k(t) &= \frac{d^2}{dt^2} J_w^l(t) + \frac{d}{dt} J^j(t) \dot{\gamma}^i(t) \Gamma_{\varepsilon ij}^l(\gamma(t)) + J^j(t) \dot{\gamma}^{i''}(t) \Gamma_{\varepsilon ij}^l(\gamma(t)) \\ &\quad + J^j(t) \dot{\gamma}^i(t) d\Gamma_{\varepsilon ij}^l(\gamma(t))(\dot{\gamma}(t)) + \Gamma_{\varepsilon pk}^l(\gamma(t)) \dot{\gamma}^p(t) \left(\frac{d}{dt} J_w^k(t) \right. \\ &\quad \left. + J^j(t) \dot{\gamma}^i(t) \Gamma_{\varepsilon ij}^k(\gamma(t)) \right). \end{aligned}$$

We have that

$$|D^2 g_\varepsilon| \leq C(\rho, G) \varepsilon^{-2}, \text{ and } |Dg_\varepsilon^{-1}| \leq C(\rho, G) \varepsilon^{-1}, \quad (6.13)$$

where we may assume that $C > 1$. Moreover, recall that we can use the geodesic equation and (6.12) to bound $|\dot{\gamma}''(t)|_{euc}$ in terms of $|v|_{euc}$, $G\varepsilon^{-1}$ and λ . Hence, all coefficients of the above linear system of ODEs are bounded by $C\varepsilon^{-2}$. Denote $Y_w^T(t) := (J^T(t), (J')^T(t))$. We get that

$$\begin{aligned} Y_w'(t) &= P(t)Y(t), \\ Y_w(0)^T &= (0^T, w^T), \end{aligned} \quad (6.14)$$

where P is a matrix with smooth entries whose L^∞ -norm is bounded above by $C\varepsilon^{-2}$. Gronwall's inequality gives that for $u \in T_p M$

$$|Y_w(t) - Y_{w+su}(t)|_{euc} \leq |s||u|_{euc} e^{Ct\varepsilon^{-2}}.$$

Hence,

$$|J_w'(t) - J_{w+su}'(t)|_{euc} \leq |s||u|_{euc} e^{Ct\varepsilon^{-2}}.$$

Thus,

$$|J_w(t) - J_{w+su}(t)|_{euc} \leq |s||u|_{euc} \frac{\varepsilon^2}{C} (e^{Ct\varepsilon^{-2}} - 1).$$

It is known that $J_w(t) = D \exp_p^{g_\varepsilon}|_{tv}(tw)$. Hence, denoting $a := C\varepsilon^{-2}$, we get that for $0 < t \leq \frac{1}{a}$,

$$|D \exp_p^{g_\varepsilon}|_{tv}(w) - D \exp_p^{g_\varepsilon}|_{tv}(w + su)|_{euc} \leq |s||u|_{euc} \frac{1}{at} (e^{at} - 1) \leq 3|s||u|_{euc},$$

which shows local Lipschitz continuity of the first derivative at the point $\gamma(t)$. The second derivative of the exponential map exists as g_ε is smooth and by the Lipschitz continuity, it is bounded. Setting $su = -w$, and recalling that $J_0(t) = 0$ for all t , we get that

$$|D \exp_p^{g_\varepsilon}|_{tv}(w)|_{euc} \leq 3|w|_{euc}.$$

Hence the first differential is bounded at the point tv . Recall that v is arbitrary, given that the geodesic tangent to v and its variation are well defined for $t \in [0, \frac{1}{a(\varepsilon)}]$. Then the above holds for each $\tilde{v} \in (0, \frac{1}{a(\varepsilon)}]v$ and (6.11) holds for each \tilde{v} such that

$$0 < \|\tilde{v}\|_{g_\varepsilon} \leq \frac{a}{2\lambda} \min(i_\varepsilon(p), 1).$$

By continuity of the differential and the second differential this also holds for $\|\tilde{v}\|_{g_\varepsilon} = 0$. Since p was arbitrary, taking Lemma 6.16 into account, we get the result. \square

Lemma 6.18. *Let $K, K' \subset \mathbb{R}^n$ be compact sets such that $K \subset K'$. Let furthermore \mathcal{F} be a family of closed balls in \mathbb{R}^n such that for every point $x \in K$, there exists a radius $R_x > 0$ such that for all positive $R \leq R_x$, it holds $\overline{B_R(x)} \in \mathcal{F}$. Let $\varphi \in C_c(K', [0, \infty))$ and $\varepsilon > 0$. Then there exists a finite family of functions $\{\chi_p\}_{p=1}^q$ such that the following holds:*

- (i) *For every $p = 1, \dots, q$, there exists an $x \in K$ and a positive $R \leq R_x$ such that $\chi_p \in C_c(B_R(x), [0, \infty))$ and χ_p is rotationally symmetric centered at x .*
- (ii) $\sum_{p=1}^q \chi_p \leq \varphi$ and $\left\| \sum_{p=1}^q \chi_p - \varphi \right\|_{C^0(K)} \leq \varepsilon$.

Proof. We first recall that, by Besicovitch's covering theorem, there exists a constant $c_n \geq 1$ such that for any family \mathcal{G} of closed balls in \mathbb{R}^n , such that the set A of their centers is bounded, there exist c_n countable subfamilies $(\mathcal{G}_h)_{h=1}^{c_n}$ such that for each $h = 1, \dots, c_n$, any two different balls in \mathcal{G}_h are disjoint and $A \subset \bigcup_{h=1}^{c_n} \bigcup_{B \in \mathcal{G}_h} B$.

Claim. For every function $\varphi \in C_c(K', [0, \infty))$, there exists a finite family of functions $\{\psi_k\}_{k=1}^m$ such that for every $k = 1, \dots, m$ the following holds:

1. There exists $x \in K$ and a positive $R \leq R_x$ such that $\psi_k \in C_c(B_R(x), [0, \infty))$.
2. ψ_k is rotationally symmetric centered at x .
3. The following estimate holds:

$$\sum_{k=1}^m \psi_k \leq \varphi, \quad \text{and} \quad \left\| \sum_{k=1}^m \psi_k - \varphi \right\|_{C^0(K)} \leq \left(1 - \frac{1}{2c_n}\right) \|\varphi\|_{C^0(K)}. \quad (6.15)$$

Proof of the claim. If $\|\varphi\|_{C^0(K)} = 0$, we do not need any function to approximate φ , so we set $m = 0$. Otherwise, using that φ is uniformly continuous, we can find $\delta > 0$ such that for all $x, y \in K'$ with $|x - y| \leq 4\delta$, we have that $|\varphi(x) - \varphi(y)| \leq \|\varphi\|_{C^0(K)} \cdot \frac{1}{2c_n}$. For each $x \in K$ we define $\rho_x = \min(R_x, \delta)$. Now $\bigcup_{x \in K} B_{\frac{\rho_x}{2}}(x)$ is an open cover of K , so by compactness of K , there exists a finite subcover $\bigcup_{k=1}^m B_{\frac{\rho_k}{2}}(x_k)$ of K , where we write $\rho_k = \rho_{x_k}$ to keep notation short. Now we apply Besicovitch's covering theorem to find c_n finite families $\mathcal{F}_h \subset \{\overline{B_{\frac{\rho_k}{2}}(x_k)}\}_k$ such that $K \subset \bigcup_{h=1}^{c_n} \bigcup_{B \in \mathcal{F}_h} B$ and the balls in each family are pairwise disjoint. For each h denote by I_h the set of indices k such that $\overline{B_{\frac{\rho_k}{2}}(x_k)} \in \mathcal{F}_h$. Fix such an h . Note that I_h is finite, hence for each $k \in I_h$ we can find a radius $\frac{\rho_k}{2} < r_k \leq \rho_k$ such that for $k, k' \in I_h$, $k \neq k'$, we have

that $B_{r_{k'}}(x_{k'}) \cap B_{r_k}(x_k) = \emptyset$. For each $k = 1, \dots, m$, define a function $\eta_k \in C_c(B_{r_k}(x_k), [0, 1])$ such that η_k is rotationally symmetric (with center x_k) and $\eta_k = 1$ on $\overline{B_{\frac{\rho_k}{2}}(x_k)}$. Define

$$\theta_k := \min_{y \in B_{r_k}(x_k)} \varphi(y) \in [0, \infty).$$

Now define $\psi_k \in C_c(B_{r_k}(x_k), [0, \infty))$ via

$$\psi_k(x) := \frac{\theta_k}{c_n} \eta_k(x).$$

We now claim that (6.15) holds. Pick a point $x \in K'$. Let $L_x := \{l : x \in B_{r_l}(x_l)\}$. For each $h = 1, \dots, c_n$, we have that $|I_h \cap L_x| \leq 1$, hence $|L_x| \leq c_n$. By definition, we have that $\theta_l \leq \varphi(x)$ for each $l \in L_x$, hence

$$\sum_{k=1}^m \psi_k(x) = \sum_{l \in L_x} \psi_l(x) = \sum_{l \in L_x} \frac{\theta_l}{c_n} \eta_l(x) \leq \varphi(x).$$

Now let $x \in K$. We have that for each $l \in L_x$ and $y \in \overline{B_{r_l}(x_l)}$ it holds $|x - y| \leq 2r_l \leq 2\rho_l \leq 2\delta$. Hence, for each $l \in L_x$, we have that $|\varphi(x) - \theta_l| \leq \|\varphi\|_{C^0(K)} \cdot \frac{1}{2c_n}$. By the construction of our cover, we have that there exists at least one $\lambda \in L_x$ such that $x \in B_{\frac{\rho_\lambda}{2}}(x_\lambda)$. Hence

$$\sum_{k=1}^m \psi_k(x) \geq \frac{\theta_\lambda}{c_n} \eta_\lambda(x) = \frac{\theta_\lambda}{c_n}.$$

Thus,

$$\begin{aligned} 0 \leq \varphi(x) - \sum_{k=1}^m \psi_k(x) &\leq \varphi(x) - \frac{\theta_\lambda}{c_n} \leq \varphi(x) - \frac{\varphi(x)}{c_n} + \left| \frac{\varphi(x)}{c_n} - \frac{\theta_\lambda}{c_n} \right| = \left(1 - \frac{1}{c_n}\right) \varphi(x) + \frac{1}{c_n} |\varphi(x) - \theta_\lambda| \\ &\leq \left(1 - \frac{1}{c_n}\right) \|\varphi\|_{C^0(K)} + \frac{1}{2c_n^2} \|\varphi\|_{C^0(K)} \leq \left(1 - \frac{1}{2c_n}\right) \|\varphi\|_{C^0(K)}. \end{aligned}$$

This proves the claim.

Now given a function $\varphi \in C_c(K', [0, \infty))$, we define a sequence of functions $(\varphi_j)_{j=0}^\infty \subset C_c(K', [0, \infty))$ as follows: $\varphi_0 := \varphi$. Given φ_j , we apply the claim to find $m_j \geq 0$ and functions $\psi_k^{(j)}$, for $k = 1, \dots, m_j$ such that

- $\psi_k^{(j)} \in C_c(B_{R_{x_j, k}}(x_{j, k}), [0, \infty))$ for some $x_{j, k} \in K$.
- $\psi_k^{(j)}$ is rotationally symmetric centered at $x_{j, k}$.
- The following estimate holds:

$$\sum_{k=1}^{m_j} \psi_k^{(j)} \leq \varphi_j, \quad \text{and} \quad \left\| \sum_{k=1}^{m_j} \psi_k^{(j)} - \varphi_j \right\|_{C^0(K)} \leq \left(1 - \frac{1}{2c_n}\right) \|\varphi_j\|_{C^0(K)}.$$

Then set $\varphi_{j+1} := \varphi_j - \sum_{k=1}^{m_j} \psi_k^{(j)} \geq 0$. Inductively, we get that

$$\|\varphi_j\|_{C^0(K)} \leq \left(1 - \frac{1}{2c_n}\right)^j \|\varphi\|_{C^0(K)}. \quad (6.16)$$

Now we can choose $J \geq \frac{\ln \varepsilon - \ln \|\varphi\|_{C^0(K)}}{\ln(1 - \frac{1}{2c_n})}$. We choose the family of functions $\{\psi_k^{(j)} : k = 1, \dots, m_j, j = 0, \dots, J\}$

Then

$$\varphi - \sum_{j=0}^J \sum_{k=1}^{m_j} \psi_k^{(j)} = \varphi_0 - \sum_{j=0}^J (\varphi_j - \varphi_{j+1}) = \varphi_{J+1} \geq 0.$$

By our choice of J and (6.16), we get that

$$\left\| \varphi - \sum_{j=0}^J \sum_{k=1}^{m_j} \psi_k^{(j)} \right\|_{C^0(K)} = \|\varphi_{J+1}\|_{C^0(K)} \leq \varepsilon.$$

This finishes the proof. \square

Lemma 6.19. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $h \in C^0(\mathbb{R}^n)$. Then for each $p \in \mathbb{R}^n$, we have that if p is a Lebesgue point of f , then p is also a Lebesgue point of $f \cdot h$.*

We are now able to state and prove the second main result of the paper, namely that the $\text{CD}^*(K, N)$ condition implies that the distributional N -Bakry-Émery Ricci tensor is bounded below by K , on a smooth manifold endowed with a continuous Riemannian metric with L^2_{loc} -Christoffel symbols and a $C^0 \cap W^{1,2}_{\text{loc}}$ -weight on the volume measure.

Theorem 6.20. *Let M be a smooth manifold, $g \in C^0$ a Riemannian metric that admits L^2_{loc} -Christoffel symbols and $h \in C^0 \cap W^{1,2}_{\text{loc}}(M)$ a positive function, $V := -2 \log h \in C^0 \cap W^{1,2}_{\text{loc}}(M)$. Define the measure μ via $\mu := e^{-V} \text{dvol}_g$. Let $K \in \mathbb{R}$ and $N \in [n, \infty)$. If (M, \mathbf{d}_g, μ) is a $\text{CD}^*(K, N)$ -space, then*

$$\text{Ric}_{\mu, N} = \text{Ric}_{\mu, \infty} - \frac{1}{N-n} \nabla V \otimes \nabla V \geq Kg \quad \text{in } \mathcal{D}'\mathcal{T}_2^0.$$

Proof. We first notice that by Lemma 4.9 and Corollary 4.27, (M, \mathbf{d}_g, μ) is an $\text{RCD}(K, \infty)$ -space, so by Theorem 6.13, we have that $\text{Ric}_{\mu, \infty} \in \mathcal{D}'\mathcal{T}_2^0$ can be expressed via Radon measures $(\nu_{ij})_{i,j}$ as in (6.7). Let $U \subset W \subset M$ be open such that \overline{W} is compact, $\overline{U} \subset W$ and W lies in one coordinate patch. It is enough to prove the statement for $M = U$, as we can cover the manifold with sets like U . Let $\rho \in C_c^\infty(B_1^{\text{euc}}(0), [0, \infty))$ be a rotationally symmetric standard mollifier and $\rho_\delta(x) := \frac{1}{\delta^n} \rho(\frac{x}{\delta})$, $\delta > 0$. For $0 < \delta \leq \delta_0 < \text{dist}(\partial W, U)$, we define $g_\delta = \rho_\delta * g$. We assume that for all these δ , g_δ is a Riemannian metric on U and that $\max(\|g_\delta\|_{C^0(U)}, \|g_\delta^{-1}\|_{C^0(U)}) < 2 \max(\|g\|_{C^0(U)}, \|g^{-1}\|_{C^0(U)})$, as this is the case for $\delta_0 > 0$ small enough. Then $g_\delta \rightarrow g$ everywhere as $\delta \rightarrow 0$. Then, by the proof of Proposition 4.19 there exists a $\zeta > 0$ and a $\lambda \geq 1$ such that g, g_δ and $|\cdot|_{\text{euc}}$ are pairwise λ -equivalent on U and $\mathbf{d}_g, \mathbf{d}_{g_\delta}$, and $|\cdot|_{\text{euc}}$ are pairwise λ -equivalent on $B_\zeta^{\text{euc}}(p)$ for all $p \in U$. Fix a point $p \in U$ such that

$$p \text{ is a Lebesgue point of } Dg, \text{ and of } Dg \cdot Dg. \quad (6.17)$$

Then, as ρ is rotationally symmetric, we get that

$$Dg_\delta(p) \rightarrow Dg(p). \quad (6.18)$$

Fix $\varepsilon > 0$. We can assume $p = 0$. In order to compute local coordinates with a vanishing Levi-Cevita connection with respect to g at the point p , we define the map $\psi : U \rightarrow \tilde{U} \subset \mathbb{R}^n, x \mapsto y$ via

$$y^k = \psi^k(x) := 0 + \alpha_i^k x^i + \beta_{ij}^k x^i x^j,$$

where $\alpha_i^k, \beta_{ij}^k = \beta_{ji}^k \in \mathbb{R}$ are constants. Then

$$\begin{aligned} (D\psi)_{ki}(x) &= \partial_{x^i} y^k = \alpha_i^k + 2\beta_{ij}^k x^j, \\ \partial_{x^t} (D\psi)_{ki}(x) &= 2\beta_{it}^k, \quad \text{and} \\ (\psi_* g)_{ij} &= ((D\psi)^{-T} g (D\psi)^{-1})_{ij}. \end{aligned}$$

Moreover, it holds

$$\begin{aligned} (\partial_{x^t} (D\psi)^{-1})_{ij}(0) &= -((D\psi)^{-1} \partial_{x^t} D\psi (D\psi)^{-1})_{ij}(0) = -(D\psi)_{it}^{-1} \partial_{x^t} D\psi_{ts} (D\psi)_{sj}^{-1}(0) \\ &= -2(D\psi)_{it}^{-1} \beta_{st}^t (D\psi)_{sj}^{-1}(0). \end{aligned}$$

We assume α to be invertible and define $\alpha^{-1} =: \sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{n \times n}$. In order to be “normal coordinates” at $p = 0$, we need that at the point $\psi(0) = 0 \in \tilde{U}$, it holds

$$(\psi_*g)_{ij}(0) = (\alpha^{-T}g\alpha^{-1})_{ij}(0) = \text{Id}_n. \quad (6.19)$$

We want the first derivative of the metric with respect to the y -coordinates to vanish at $p = 0$, hence we compute:

$$\begin{aligned} 0 &= \partial_{y^k}(\psi_*g)_{ij}(0) = \partial_{x^l}((D\psi^{-1})_{ri}(D\psi^{-1})_{qj}g_{rq}) \frac{\partial x^l}{\partial y^k}(0) \\ &= (\partial_{x^l}(D\psi^{-1})_{ri}(D\psi^{-1})_{qj}g_{rq} + (D\psi^{-1})_{ri}\partial_{x^l}(D\psi^{-1})_{qj}g_{rq} + (D\psi^{-1})_{ri}(D\psi^{-1})_{qj}\partial_{x^l}g_{rq}) \frac{\partial x^l}{\partial y^k}(0) \\ &= (-2\sigma_{rt}\beta_{sl}^t\sigma_{si}\sigma_{qj}g_{rq} - 2\sigma_{ri}\sigma_{qt}\beta_{sl}^t\sigma_{sj}g_{rq} + \sigma_{ri}\sigma_{qj}\partial_{x^l}g_{rq})\sigma_{lk}. \end{aligned} \quad (6.20)$$

This gives $\frac{1}{2}n^2(n+1)$ equations and $\frac{1}{2}n^2(n+1)$ variables β_{ij}^k . To keep notation short, we rewrite this to

$$\zeta_{ijk}^m(g, \sigma)\beta_{ij}^k = \xi^m(Dg, \sigma), \quad m = 1, \dots, \frac{1}{2}n^2(n+1), \quad (6.21)$$

for some polynomials $\zeta_{ijk}^m : (\mathbb{R}^{n \times n})^2 \rightarrow \mathbb{R}$ and $\xi^m : \mathbb{R}^{n \times n \times n} \times \mathbb{R}^{n \times n}$. In order to find a solution of this system of equations, consider $g_\delta, \delta \in (0, \delta_0)$. By using the exponential map, we can find a map ψ_δ defined as

$$\psi_\delta = (\exp_p^{g_\delta})^{-1} : \Omega_\delta \rightarrow B_{r_\delta}(0) \subset T_pU \cong \mathbb{R}^n, x \mapsto y_\delta,$$

where $\Omega_\delta \subset U$ is an open neighbourhood of 0 and $r_\delta > 0$. Define the distance function $d_\delta^p : \Omega_\delta \rightarrow [0, \infty), y \mapsto d_{g_\delta}(p, z) = (\sum_{i=1}^n (y_\delta^i)^2)^{\frac{1}{2}}$. It is known that $(\psi_\delta)_*(g_\delta)_{ij} = \delta_{ij} + O_\delta((d_\delta^p)^2)$. More precisely, this means that there exists a constant $C_{g_\delta} > 0$ such that for all $x \in \Omega_\delta$,

$$\|(\psi_\delta)_*g_\delta(x) - \text{Id}_n\|_{\mathbb{R}^{n \times n}} \leq C_{g_\delta}(d_\delta^p)^2(x). \quad (6.22)$$

Note that $|Dg_\delta| \leq C\lambda\delta^{-1}$ for some $C > 0$. Take $\tilde{\Omega}_\delta := \Omega_\delta \cap \{x : |x|_{euc} = |x - p|_{euc} \leq \frac{\delta}{2nC}, (d_\delta^p(x))^2 < \frac{1}{2nC_{g_\delta}} \frac{1}{4\lambda^2}\}$. Then in $\tilde{\Omega}_\delta$, we get that

$$\frac{1}{2\lambda}\text{Id}_n \leq g_\delta \leq 2\lambda\text{Id}_n, \quad \text{and} \quad \frac{1}{2}\text{Id}_n \leq (\psi_\delta)_*g_\delta \leq 2\text{Id}_n. \quad (6.23)$$

Using that $\|v\|_{g_\delta} = \|(\psi_\delta)_*v\|_{(\psi_\delta)_*g_\delta}$, we get that the transition map $\Psi_\delta := D\psi_\delta : T\tilde{\Omega}_\delta \rightarrow T\psi_\delta(\tilde{\Omega}_\delta)$ is bounded from above and from below in the Euclidean norm, meaning that for all $0 \neq v \in T_q\tilde{\Omega}_\delta$, we have that

$$\frac{1}{4\lambda} = \frac{1}{2\lambda} \leq \frac{\|\Psi_\delta(v)\|_{euc}}{\|v\|_{euc}} \leq \frac{2}{\frac{1}{2\lambda}} = 4\lambda. \quad (6.24)$$

By the inverse function theorem, we know that on $\tilde{\Omega}_\delta$

$$D\psi_\delta = (D(\exp_p^{g_\delta}))^{-1}, \quad \text{hence} \quad \partial_i D\psi_\delta = -D\psi_\delta \partial_i (D(\exp_p^{g_\delta})) D\psi_\delta.$$

Hence, using (6.24) and (6.11), we get that

$$|(D\psi_\delta)^{-1}|_{euc}(0) \leq 4\lambda \quad \text{and} \quad |D^2\psi_\delta|_{euc}(0) \leq 12\lambda^2. \quad (6.25)$$

Now for all $\delta > 0$, we have that

$$\zeta_{ijk}^m(g_\delta, (D\psi_\delta)^{-1}) \frac{1}{2} \partial_j (D\psi_\delta)_{ik} = \xi^m(Dg_\delta, (D\psi_\delta)^{-1}), \quad m = 1, \dots, \frac{1}{2}n^2(n+1).$$

With (6.24) and (6.25), we get that there exists a convergent subsequence $\delta' \rightarrow 0$ such that $(D\psi_{\delta'}(0), D^2\psi_{\delta'}(0)) \rightarrow (D\psi(0), D^2\psi(0))$ and

$$|D\psi|_{euc}(0) \leq 4\lambda, \quad |D\psi^{-1}|_{euc}(0) \leq 4\lambda, \quad \text{and} \quad |D^2\psi|_{euc}(0) \leq 12\lambda^2. \quad (6.26)$$

By (6.17), (6.18), and the continuity of g , it follows that $(D\psi(0), D^2\psi(0))$ satisfies (6.19) and (6.20). Now we have found coefficients for our map $\psi : U \rightarrow \tilde{U}$ and we notice that for now the map is defined everywhere on U . By (6.19), $D\psi(0)$ must be invertible, so by the inverse function theorem we can shrink U to a neighbourhood U' of p such that ψ is a diffeomorphism on U' . Note that $D^2\psi$ is constant, so

$$|D^2\psi|_{euc}(x) \leq 12\lambda^2, \quad (6.27)$$

for all $x \in U'$. As $\psi_*g = \text{Id}_n$, we get that the transition map $\Psi|_0 := D\psi|_0 : T_0U' \rightarrow T_0\psi(U')$ is bounded, meaning that for all $0 \neq v \in T_0\psi(U')$, we have that

$$\frac{1}{\lambda} \leq \frac{\|\Psi(v)\|_{euc}}{\|v\|_{euc}} \leq \lambda. \quad (6.28)$$

Using (6.27), we can shrink the domain U' and find an open $\Omega \subset U'$ such that $p \in \Omega$ and the transition map $\Psi := D\psi : T\Omega \rightarrow T\psi(\Omega)$ satisfies

$$\frac{1}{2\lambda} \leq \frac{\|\Psi(v)\|_{euc}}{\|v\|_{euc}} \leq 2\lambda, \quad (6.29)$$

for each $v \in T_q\Omega$, $q \in \Omega$. As p satisfies (6.17), we can use Lemma 6.19, to get that for each $f \in C_c^2(M)$,

$$p \text{ is a Lebesgue point of } x \mapsto |\nabla_g^2 f(x)|_{HS}^2, \text{ and } x \mapsto (\Delta_g f(x))^2. \quad (6.30)$$

Here Δ_g denotes the Laplacian with respect to the volume measure induced by g . As, by the above construction, $y = \psi(x)$ defines normal coordinates at p with respect to g , we have that

$$\begin{aligned} ((\psi)_*(\nabla_g^2 f)(0))_{ij} &= \partial_{y^i, y^j}^2 f(0), \quad \text{and} \\ \Delta_g f &= \sum_i \partial_{y^i, y^i}^2 f(0) = \sum_i ((\psi)_*(\nabla_g^2 f)(p))_{ii}. \end{aligned}$$

Now fix a smooth vector field X on U and denote $B := \psi_*(X(p)) \in \mathbb{R}^n$. Let $f : \Omega \rightarrow \mathbb{R}$ be defined by $\psi_*f := \sum_i B_i y^i + \frac{a}{2\sqrt{n}}(y^i)^2$ for some $a \in \mathbb{R}$. Then

$$\psi_*\nabla_g f(0) = B, \quad \psi_*\nabla_g^2 f(0) = \frac{a}{\sqrt{n}}\text{Id}_n, \quad \text{and} \quad \psi_*\Delta_g f(0) = \sqrt{n}a.$$

Note that f is smooth because ψ is smooth. Writing $h^2 = e^{-V}$ for a $C^0 \cap W_{loc}^{1,2}$ -function V , we get:

$$\Delta_{g,\mu} f = \Delta_g f + \langle \nabla_g V, \nabla_g f \rangle_g.$$

Hence,

$$|\nabla_g^2 f|_{HS,g}^2(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p) = \frac{N-n}{N}a^2 - \frac{2\sqrt{n}}{N}a\langle X, \nabla_g V \rangle_g(p) - \frac{1}{N}\langle X, \nabla_g V \rangle_g^2(p).$$

The choice $a = \frac{\sqrt{n}\langle X, \nabla_g V \rangle_g(p)}{N-n}$ minimises the right hand side and yields

$$|\nabla_g^2 f|_{HS,g}^2(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p) = -\frac{1}{N-n}\langle X, \nabla_g V \rangle_g^2(p). \quad (6.31)$$

Now there is a $\tau_0 > 0$, $\tau \leq \min(\varepsilon, 1)$ such that $B_{\tau_0}(p) := B_{\tau_0}^{euc}(p) \subset \Omega$ and for all $z \in B_{\tau_0}(p)$, we have

$$|\nabla_g f(z) - X(p)|_{euc} \leq \varepsilon \quad \text{and} \quad |X(z) - X(p)|_{euc} \leq \varepsilon. \quad (6.32)$$

Finally note $f = \psi^* \psi_* f$, so we can bound $\|f\|_{W^{2,\infty}}$ in a neighbourhood of p in terms of $|\psi|, |D\psi|, |D\psi^{-1}|, |D^2\psi|, a$, and B . Hence, by (6.27) and (6.29), we get that

$$\|f\|_{W^{2,\infty}(B_{\tau_0}(p))} \leq C(X, V, \lambda, n, N). \quad (6.33)$$

Now, by (6.10), we have

$$\sum_{i,j} (\nabla_g f)_i (\nabla_g f)_j \nu_{ij} + (|\nabla_g^2 f|_{HS,g}^2 - K|\nabla_g f|_g^2 - \frac{1}{N}(\Delta_{g,\mu} f)^2) \mu \geq 0.$$

Recall that $\mu = e^{-V} \sqrt{|g|} \mathcal{L}^n$, where $V, g \in C^0(U)$. Then by the triangle inequality, (6.30) together with Lemma 6.19, and by (6.32), we get that there exists a $\tau \in (0, \tau_0)$ such that for all $\tau' \in (0, \tau)$, it holds

$$\begin{aligned} & \left| \left(\sum_{i,j} (\nabla_g f)_i(p) (\nabla_g f)_j(p) \nu_{ij} + (|\nabla_g^2 f|_{HS,g}^2(p) - K|\nabla_g f|_g^2(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p)) \mu \right) \right. \\ & \quad \left. - \left(\sum_{i,j} (\nabla_g f)_i (\nabla_g f)_j \nu_{ij} + (|\nabla_g^2 f|_{HS,g}^2 - K|\nabla_g f|_g^2 - \frac{1}{N}(\Delta_{g,\mu} f)^2) \mu \right) \right|_{TV} (B_{\tau'}(p)) \\ & \leq C(n, U, g, V, X, N, K) \left(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n \right) (B_{\tau'}(p)) \cdot \varepsilon. \end{aligned}$$

Similarly, using again (6.32), we get that

$$\begin{aligned} & \left| \sum_{i,j} (\nabla_g f)_i(p) (\nabla_g f)_j(p) \nu_{ij} - (K|\nabla_g f|_g^2(p) + \frac{1}{N-n}(\langle \nabla_g f, \nabla_g V \rangle_g^2)(p)) \mu \right. \\ & \quad \left. - \sum_{i,j} X^i X^j \nu_{ij} + (K|X|_g^2 + \frac{1}{N-n}(\langle X, \nabla_g V \rangle_g^2)) \mu \right|_{TV} (B_{\tau'}(p)) \\ & \leq C(n, U, g, V, X, N, K) \varepsilon \left(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n \right) (B_{\tau'}(p)). \end{aligned}$$

Together with (6.31), this yields

$$\begin{aligned} & \left(\sum_{i,j} X^i X^j \nu_{ij} - (K|X|_g^2 + \frac{1}{N-n}(\langle X, \nabla_g V \rangle_g^2)) \mu \right) (B_{\tau'}(p)) \\ & \geq -C(n, U, g, V, X, N, K) \varepsilon \left(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n \right) (B_{\tau'}(p)). \end{aligned}$$

We infer that for $\chi \in C_c(B_{\tau}(p), [0, \infty))$ with χ rotationally symmetric around p , it holds

$$\begin{aligned} & \int_{B_{\tau}(p)} \chi d \left(\sum_{i,j} X^i X^j \nu_{ij} - (K|X|_g^2 + \frac{1}{N-n}(\langle X, \nabla_g V \rangle_g^2)) \mu \right) \\ & \geq -C(n, U, g, V, X, N, K) \varepsilon \int_{B_{\tau}(p)} \chi d \left(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n \right). \end{aligned} \quad (6.34)$$

As p was chosen arbitrarily among all points satisfying (6.17), we have that for all $p \in U$, satisfying (6.17), and each $\varepsilon > 0$, there exists a $\tau_{p,\varepsilon} > 0$ such that (6.34) holds for $\chi \in C_c(B_{\tau_{p,\varepsilon}}^{euc}(p), [0, \infty))$ such that χ is rotationally symmetric around p . Now suppose there exists a function $\phi \in C_c(\bar{U}, [0, \infty))$ such that

$$\langle \text{Ric}_{\mu,\infty}(X, X) - K|X|_g^2 \mu - \frac{1}{N-n}(\nabla V \otimes \nabla V)(X, X)\mu, \phi \, dx^1 \dots dx^n \rangle_{\mathcal{D}', \mathcal{D}} = \kappa < 0.$$

We can assume that $\|\phi\|_{sup} = 1$. Choose

$$0 < \varepsilon \leq \frac{-\kappa}{8C \cdot (\mathcal{L}^n(U) + \sum_{i,j} |\nu_{ij}|_{TV}(U))}, \quad (6.35)$$

where $C \geq 1$ is as in (6.34). Write

$$\text{Ric}_{\mu,\infty}(X, X) := \mathfrak{r}_X^s + \mathfrak{r}_X^a,$$

where \mathfrak{r}_X^s denotes the singular part and \mathfrak{r}_X^a denotes the absolutely continuous part with respect to the Lebesgue measure. As $\text{Ric}_{\mu,\infty}(X, X) - K|X|_g^2 \mu \geq 0$, and $K|X|_g^2 \mu$ is absolutely continuous with respect to the Lebesgue measure, we know that

$$\mathfrak{r}_X^s \geq 0. \quad (6.36)$$

Moreover, there exist two Borel sets $U_s, U_a \subset U$ such that $U_s \cap U_a = \emptyset$, $U_s \cup U_a = U$, $\mathfrak{r}_X^s(U_a) = 0$, and $\mathcal{L}^n(U_s) = 0$. Define

$$\mathcal{N} := \{p \in U : p \text{ does not satisfy (6.17)}\}.$$

Then $\mathcal{L}^n(\mathcal{N}) = 0$ and hence $\mathcal{L}^n(\mathcal{N} \cup U_s) = 0$. By the outer regularity of Borel measures, there exists an open set $O \subset U$ such that

$$\begin{aligned} (\mathcal{N} \cup U_s) \subset O, \text{ and} \\ |\mathfrak{r}_X^a|_{TV}(O) + (|K| |X|_g^2 + \frac{1}{N-n}(\nabla V \otimes \nabla V)(X, X))\mu(O) \leq \frac{\varepsilon}{4}. \end{aligned} \quad (6.37)$$

Denote $\text{supp } \phi = K'$, $K := K' \setminus O$ and note that both K and K' are compact. Moreover, denote $\mathcal{F} := \{B_\tau(p) : p \in K, 0 < \tau \leq \frac{1}{2}\tau_{p,\varepsilon}\}$. We can now apply Lemma 6.18 to find an $m \in \mathbb{N}$ and functions $\chi_k \in C_c(B_{\tau_k}(p_k), [0, \infty))$ for $k = 1, \dots, m$, $p_k \in K$, χ_k is rotationally symmetric centered at p_k , $\tau_k \leq \frac{\tau_{p_k,\varepsilon}}{2}$, such that $\sum_{k=1}^m \chi_k \leq \phi$ and $\|\sum_{k=1}^m \chi_k - \phi\|_{C^0(K)} \leq \varepsilon$. Denote $\psi := \sum_{k=1}^m \chi_k$. Define the signed Radon measure \mathfrak{s} on U as

$$\mathfrak{s} = -K|X|_g^2 \mu - \frac{1}{N-n}(\nabla V \otimes \nabla V)(X, X)\mu.$$

By (6.35), (6.37), and the bound $\|\phi - \psi\|_{C^0(K)} \leq \varepsilon$, we have that

$$\begin{aligned} \left| \int_U (\phi - \psi) d(\mathfrak{s} + \mathfrak{r}_X^a) \right| &= \left| \int_{K'} (\phi - \psi) d(\mathfrak{s} + \mathfrak{r}_X^a) \right| \\ &\leq \left| \int_K (\phi - \psi) d(\mathfrak{s} + \mathfrak{r}_X^a) \right| + \left| \int_O (\phi - \psi) d(\mathfrak{s} + \mathfrak{r}_X^a) \right| \\ &\leq \varepsilon \cdot (|\mathfrak{s}|_{TV} + |\mathfrak{r}_X^a|_{TV})(U) + \|\phi\|_{sup} (|\mathfrak{r}_X^a|_{TV} + |\mathfrak{s}|_{TV})(O) \\ &\leq -\frac{\kappa}{8} + \frac{\varepsilon}{4} \leq -\frac{\kappa}{2}. \end{aligned}$$

Then, using (6.36) and that $\phi \geq \psi$, we get that

$$\begin{aligned} \int_U \phi d(\mathfrak{s} + \mathfrak{r}_X^s + \mathfrak{r}_X^a) &= \int_U \psi d(\mathfrak{s} + \mathfrak{r}_X^s + \mathfrak{r}_X^a) + \int_U (\phi - \psi) d(\mathfrak{s} + \mathfrak{r}_X^a) + \int_U (\phi - \psi) d(\mathfrak{r}_X^s) \\ &\geq \int_U \psi d(\mathfrak{s} + \mathfrak{r}_X^s + \mathfrak{r}_X^a) + \frac{\kappa}{2}. \end{aligned}$$

Finally, using (6.34), and $\|\psi\|_{sup} \leq \|\phi\|_{sup} \leq 1$, we get that

$$\begin{aligned} &\int_U \psi d(\mathfrak{s} + \mathfrak{r}_X^s + \mathfrak{r}_X^a) \\ &= \sum_{k=1}^m \left(\sum_{i,j} \int_{B_{\tau_k}(p_k)} X^i X^j \chi_k d\nu_{ij} - \int_{B_{\tau_k}(p_k)} \left(K|X|_g^2 + \frac{1}{N-n} \langle (X, \nabla_g V)_g \rangle \right) \chi_k d\mu \right) \\ &\geq -C(n, U, g, V, X, N, K) \varepsilon \sum_{k=1}^m \left(\int_{B_{\tau_k}(p_k)} \sum_{i,j} \chi_k d|\nu_{ij}|_{TV} + \int_{B_{\tau_k}(p_k)} \chi_k d\mathcal{L}^n \right) \\ &\geq \frac{\kappa}{8(\mathcal{L}^n(U) + \sum_{i,j} |\nu_{ij}|_{TV}(U))} \left(\int_U \sum_{i,j} \psi d|\nu_{ij}|_{TV} + \int_U \psi d\mathcal{L}^n \right) \\ &\geq \frac{\kappa}{8}. \end{aligned}$$

But then

$$\begin{aligned} \kappa &= \left\langle \text{Ric}_{\mu, \infty}(X, X) - K|X|_g^2 \mu - \frac{1}{N-n} (\nabla V \otimes \nabla V)(X, X) \mu, \phi dx^1 \dots dx^n \right\rangle_{\mathcal{D}', \mathcal{D}} \\ &= \int_U \phi d(\mathfrak{s} + \mathfrak{r}_X^s + \mathfrak{r}_X^a) \geq \frac{5\kappa}{8}. \end{aligned}$$

Since $\kappa < 0$, this is a contradiction. As X was arbitrary, the proof is complete. \square

7 Distributional Ricci curvature lower bounds imply RCD

In this section, we prove the reverse implication, namely from distributional to synthetic Ricci lower bounds, under the assumption of the volume growth condition (4.1). Note that such a condition is necessary, as all $\text{CD}(K, \infty)$ spaces satisfy it (see Remark 4.11). At the end of the section, we establish such volume growth condition for weighted manifolds with C^1 -metrics and distributional Bakry-Émery N -Ricci curvature bounded below by K for finite N .

Theorem 7.1. *Let M be a smooth manifold, g a continuous Riemannian metric with L_{loc}^2 -Christoffel symbols and let $h \in C^0(M) \cap W_{\text{loc}}^{1,2}(M)$ be a positive function. Denote $V := -2 \log h \in C^0 \cap W_{\text{loc}}^{1,2}(M)$. Define the measure μ via $d\mu := e^{-V} d\text{vol}_g$. Let $N \in [n, \infty]$ and $K \in \mathbb{R}$. Assume that the metric measure space (M, d_g, μ) satisfies (4.1).*

If the distributional Bakry-Émery N -Ricci curvature tensor is bounded below by K , i.e.

$$\text{Ric}_{\mu, N} \geq Kg \text{ in } \mathcal{D}'\mathcal{T}_2^0,$$

then (M, d_g, μ) satisfies the $\text{RCD}^(K, N)$ -condition if $N \in [n, \infty)$, or the $\text{RCD}(K, \infty)$ -condition if $N = \infty$.*

Proof. By Theorem 5.4, it suffices to prove that (M, \mathbf{d}_g, μ) satisfies the $\text{BE}(K, N)$ -condition. For any smooth function $f \in C^\infty(M)$ and any non-negative, compactly supported test volume ω , we have that

$$\begin{aligned} & \langle \text{Ric}_{\mu, N}(\nabla f, \nabla f) - Kg(\nabla f, \nabla f), \omega \rangle_{\mathcal{D}', \mathcal{D}} \\ &= \left\langle \text{Ric}_{\mu, \infty}(\nabla f, \nabla f) - \frac{1}{N-n} \langle \nabla V, \nabla f \rangle_g^2 - Kg(\nabla f, \nabla f), \omega \right\rangle_{\mathcal{D}', \mathcal{D}} \geq 0. \end{aligned} \quad (7.1)$$

Let $n \geq 1$ be the dimension of M and denote by Δ_g the Laplacian induced by the volume measure dvol_g .

Claim. The following inequality holds at μ -almost every point $p \in M$:

$$|\nabla^2 f|_{HS}^2 - \frac{1}{N} (\Delta_g f + \langle \nabla V, \nabla f \rangle)^2 \geq -\frac{1}{N-n} \langle \nabla V, \nabla f \rangle^2. \quad (7.2)$$

Proof of the claim. Fix $p \in M$ such that p is a Lebesgue point of Dg and $(Dg)^2$. Denote $B := |\langle \nabla V, \nabla f \rangle|(p)$. For $(e_i)_{i=1}^n$, a g -orthonormal basis at $T_p M$, we have that

$$|\nabla^2 f|_{HS}^2(p) = \sum_{i,j=1}^n \nabla^2 f(e_i, e_j)^2 \quad \text{and} \quad \Delta_g f(p) = \sum_{i=1}^n \nabla^2 f(e_i, e_i).$$

From the Cauchy-Schwartz inequality, we get that

$$n|\nabla^2 f|_{HS}^2(p) \geq (\Delta_g f(p))^2.$$

Denote $a = |\nabla^2 f|_{HS}$. Then

$$\begin{aligned} & |\nabla^2 f|_{HS}^2 - \frac{1}{N} (\Delta_g f + \langle \nabla V, \nabla f \rangle)^2 + \frac{1}{N-n} \langle \nabla V, \nabla f \rangle^2 \\ & \geq |\nabla^2 f|_{HS}^2 - \frac{1}{N} (|\Delta_g f| + |\langle \nabla V, \nabla f \rangle|)^2 + \frac{1}{N-n} \langle \nabla V, \nabla f \rangle^2 \\ & \geq a^2 - \frac{1}{N} (\sqrt{n}a + B)^2 + \frac{1}{N-n} B^2 \\ & = \frac{1}{N} (\sqrt{N-n}a + \frac{\sqrt{n}}{\sqrt{N-n}} B)^2 \geq 0. \end{aligned}$$

This proves the claim.

We can now plug (7.2) into (7.1) and infer that

$$\left\langle \text{Ric}_{\mu, \infty}(\nabla f, \nabla f) + |\nabla^2 f|_{HS}^2 - \frac{1}{N} (\Delta_\mu f)^2 - Kg(\nabla f, \nabla f), \omega \right\rangle_{\mathcal{D}', \mathcal{D}} \geq 0.$$

Using a partition of unity, we may assume that ω is supported in one coordinate patch and we can locally assume that $\omega = \phi h^2 \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$ for some $\phi \in C_c^0(M, [0, \infty)) \cap W_{\text{loc}}^{1,2}(M)$. In local coordinates, we get that (6.9) holds for any $f \in C^2(M)$. Using Lemma 4.17 and Proposition 5.11, we get that (6.9) holds for each $f \in H^{2,2}(M) \subset H_2^2(M, \mu)$. This means exactly that the Bakry-Émery condition $\text{BE}(K, N)$ is satisfied. \square

The combination of Theorem 6.11, Theorem 6.20, and Theorem 7.1 yields:

Theorem 7.2. *Let M be a smooth manifold, g a continuous Riemannian metric with L_{loc}^2 -Christoffel symbols and let $h \in C^0 \cap W_{\text{loc}}^{1,2}(M)$ be a positive function. Denote $V := -2 \log h \in C^0 \cap W_{\text{loc}}^{1,2}(M)$ and define the measure μ via $d\mu := e^{-V} \text{dvol}_g$. Let $N \in [n, \infty]$ and $K \in \mathbb{R}$. The following are equivalent:*

- (i) (M, \mathbf{d}_g, μ) is a $\text{CD}^*(K, N)$ -space.
- (ii) The distributional Bakry-Émery N -Ricci curvature tensor is bounded below by K and (M, \mathbf{d}_g, μ) satisfies (4.1).

7.1 Remarks on the volume growth condition

Proposition 7.3. *Let M be a smooth manifold and g be a C^1 -Riemannian metric on M . Let furthermore $h \in C^1(M, (0, \infty))$. Let $N \in [n, \infty)$ and $K \in \mathbb{R}$. Suppose that the distributional Bakry-Émery- N -Ricci curvature tensor is bounded below by K . Then $(M, d_g, h^2 d\text{vol}_g)$ satisfies the volume growth condition (4.1).*

Proof. Fix a point $p \in M$ and a sequence $R_j \in (0, \infty)$ such that $\lim_{j \rightarrow \infty} R_j = \infty$. For each j , the set $B_j := B_{R_j}^{d_g}(p)$ is compact. For $\varepsilon \in (0, 1)$, define $g_\varepsilon = \rho_\varepsilon * g$ and $h_\varepsilon = \rho_\varepsilon * h$. Noting that [17, Lemma 4.6] holds similarly for $K < 0$, we get that, for each $j \in \mathbb{N}$, there exists an $\varepsilon_j \in (0, 1)$ such that

- for each Borel set $\Omega \subset B_j$, it holds $\frac{1}{2}h^2 \text{vol}_g(\Omega) \leq h_{\varepsilon_j}^2 \text{vol}_{g_{\varepsilon_j}}(\Omega) \leq 2h^2 \text{vol}_g(\Omega)$;
- for each $x \in B_j$ we have that g and g_{ε_j} are 2-equivalent, i.e. $\frac{1}{2}g \leq g_{\varepsilon_j} \leq 2g$;
- for each $X \in TB_j$, we have that $\text{Ric}_{\mu, \infty}[g_{\varepsilon_j}](X, X) \geq (K - 1)g_{\varepsilon_j}(X, X)$.

Now fix a $j \in \mathbb{N}$. We write g_j for g_{ε_j} and h_j for h_{ε_j} to keep notation short. For each $R < R_j$, we can apply the weighted Bishop-Gromov theorem for $(B_j, d_{g_j}, h_j^2 d\text{vol}_{g_j})$ together with the fact that $H := \sup_j \sup_{B_1} |h_j| < \infty$ to get constants $C, D > 0$, depending only on K, N, H , and $\text{vol}_g(B_1(p))$, such that

$$h_j^2 \text{vol}_{g_j}(B_R^{d_{g_j}}(p)) \leq C e^{DR^2}.$$

From the choice of ε_j , it follows that for all $R \leq \frac{1}{2}R_j$, it holds

$$h^2 \text{vol}_g(B_R^d(p)) \leq 2C e^{4DR^2}.$$

As j was arbitrary, the proof is finished. □

Corollary 7.4. *Let M be a smooth manifold, g a C^1 -Riemannian metric and let $h \in C^1$ be a positive function. Denote $V := -2 \log h \in C^1$ and define the measure μ via $d\mu := e^{-V} d\text{vol}_g$. Let $N \in [n, \infty)$ and $K \in \mathbb{R}$. The following are equivalent:*

- (i) (M, d_g, μ) is a $\text{CD}^*(K, N)$ -space.
- (ii) The distributional Bakry-Émery N -Ricci curvature tensor is bounded below by K .

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