Coarsening phenomena in the network flow

Alessandra Pluda [*] Yoshihiro Tonegawa [†]

February 1, 2024

Abstract

In this short note we summarize recent results on the asymptotic behaviour of the network flow and we give indications of an expected coarsening–type behaviour for the network flow past singularities. The paper is complemented with a discussion on critical points and local minimizers of the length functional.

Keywords: motion by curvature, network, coarsening, flow past singularities, basin of attraction of critical points.

1 Introduction

A network $\mathcal{N}$ is a 1–dimensional connected and planar set composed of a finite number of smooth, regular and embedded curves $\{\gamma^i\}_{i=1}^N$ that meet only at their end-points in junctions. We are interested in the so–called *network flow*, a geometric flow that can be understood as the gradient flow of the length functional

$$L(\mathcal{N}) := \sum_i \int_0^1 |\partial_x \gamma^i(x)|\,\mathrm{d}x = \sum_i \int_{\gamma^i} 1\,\mathrm{d}s\,.$$

Formally, we derive the motion equations computing the first variation of L. Each curve moves with normal velocity equal to its curvature

$$v^\perp(t,x) = \vec{k}(x)\,,$$

or, equivalently $\langle \partial_t\gamma(t,x), \nu(t,x)\rangle = \kappa(t,x)$. Moreover, to interpret the curvature as the gradient of the length, we shall put to zero the contribution of the boundary, obtaining that during the evolution the junctions are "balanced", in the sense that at the junctions the unit tangent vectors of the concurring curves sum up to zero.
The network flow, the 1–dimensional case of the *multi-phase mean curvature flow*, brings the study of the *mean curvature flow* a step further, allowing the evolution of a specific class of singular objects (regular networks) instead of immersions of a single smooth manifold.

[*]alessandra.pluda@unipi.it, Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy.

[†]tonegawa@math.titech.ac.jp, Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan.

Recently the research on this topic has been particularly flourishing and numerous results have been obtained both for weak [10, 11, 21, 5, 4] and strong solutions [9, 3, 14, 15, 19].
Even though the flow has become fashionable among researchers in geometric analysis, the origin of this evolution is definitely more applied: the flow has been indeed proposed as a model of the growth of polycrystals in metals [16].
One of the motivations of the study of this flow is the tentative formalisation of a "coarsening–type behaviour" of the flow, that ultimately would indicate how good as a model of grain growth this flow is. Generically a network flow with a highly complicated initial datum (with hundreds of loops, for instance) is expected to converge, as time goes to infinity, to a critical point of the length with a much simpler structure than the initial network. This hypothetical behaviour is evident from numerical simulations (see for instance experiment posted on the webpage of Selim Esedoglu: dept.math.lsa.umich.edu/esedoglu and of Ken Brakke: kenbrakke.com).

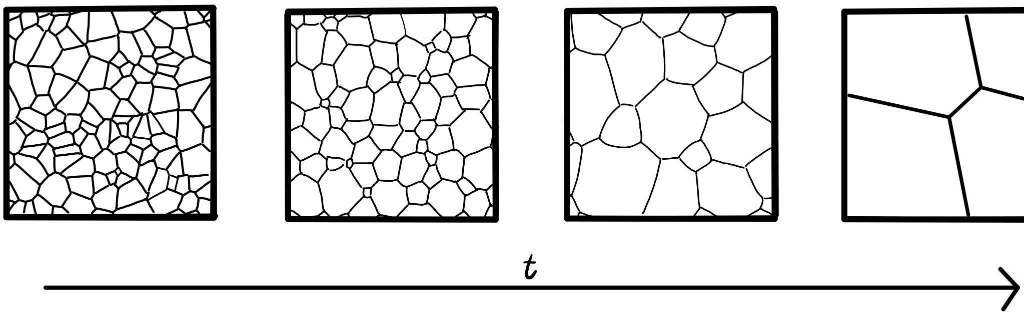


Figure 1: Expected evolution of a complicated network

In this note we will describe the arguments supporting this expectation and summarize some of the tools developed till now to get a very accurate description of the evolution.
We will adopt a classical PDE approach, and all the results will be presented in a informal and very accessible way.

Acknowledgments. *This note has been developed during a visits of the second author and Tom Ilmanen to Pisa. We warmly thank Tom Ilmanen for various conversation on the subject. A.P. is partially supported by projects PRA2022 "GEODOM" and BIHO2023. Y.T. is partially supported by JSPS Kakenhi 23H00085, 19H00639.*

2 Network flow

Consider a smooth planar curve $\gamma : [0,1] \to \mathbb{R}^2$. We say that γ is *regular* if for every $x \in [0,1]$ we have $\partial_x \gamma(x) \neq 0$. For a regular curve γ, define

$$\tau := \frac{\partial_x \gamma}{|\partial_x \gamma|}\,, \qquad \nu := \mathrm{R}(\tau)\,,$$

the tangent and the normal vector, respectively, where R denotes the anticlockwise rotation centred in the origin of $\mathbb{R}^2$ of angle $\frac{\pi}{2}$. As usual we define $\mathrm{d}s := |\partial_x \gamma|\,\mathrm{d}x$ the arclength element and $\partial_s := |\partial_x \gamma|^{-1}\partial_x$ the arclength derivative. The curvature of γ is

$$\boldsymbol{k} := \partial_s^2 \gamma = \frac{\partial_x^2 \gamma(x)}{|\partial_x \gamma(x)|^2} - \frac{\partial_x \gamma(x)\,\big\langle \partial_x^2 \gamma(x), \partial_x \gamma(x) \big\rangle}{|\partial_x(x)\gamma|^4} = \kappa(x)\nu(x)\,,$$

where κ is the oriented curvature.

Definition 2.1. A network $\mathcal{N}$ is a connected set in the Euclidean plane, composed of finitely many regular, embedded smooth curves that meet only at their end–points in junctions.

We distinguish between interior and exterior vertices of the network: at the firsts, more than one curve concur, and the latter are the termini of the network.
We say that a network is a *tree* if it does not contain loops. We call *grain* a bounded region enclosed by one or more curve of the network.
We denote by L^i the length of the i–th curve of a network, namely

$$L^i := \int_0^1 |\partial_x \gamma^i(x)| \,\mathrm{d}x = \int_{\gamma^i} 1 \,\mathrm{d}s \,,$$

and the length of the network is nothing but the sum of the length of all its curves.

Definition 2.2. A network whose interior vertices are only triple junctions, where the unit tangent vectors form angles of 120 degrees, is called *regular*.
A network is said to be *minimal* if it is regular and it is composed of straight segments.

We define now the network flow, namely the formal geometric gradient flow of the length functional. In this paper the flow will be described as solution of a system of partial differential equation. We require that each curve of the network moves by curvature

$$(\partial_t \gamma^i)^\perp = \vec{\kappa}^i \,. \tag{2.1}$$

Moreover, apart from the initial time, the evolving network will be regular: for all times $t > 0$ we impose the following balancing condition at each triple junction

$$\begin{aligned} &\gamma^{i_1} = \gamma^{i_2} = \gamma^{i_3} \,, \quad \text{and} \\ &\tau^{i_1} + \tau^{i_2} + \tau^{i_3} = 0, \end{aligned} \tag{2.2}$$

Definition 2.3. A time dependent family of networks $\mathcal{N}_t$, with $0 \leq t < T$, is a solution of the motion by curvature of regular networks if $\mathcal{N}_t$ converges to the initial network $\mathcal{N}_0$ as $t \searrow 0$, $\mathcal{N}_t$ is regular for all $t > 0$ and satisfies (2.1), (2.2) for all $0 < t < T$. We say that the solution is maximal if it does not exists another solution defined on $[0, \widetilde{T})$ with $\widetilde{T} > T$ that coincide with $\mathcal{N}_t$ on $[0, T)$.

Remark 2.4. To maintain the presentation as simple as possible, we have not specified the type of convergence towards the initial datum. One can think that the set $\mathcal{N}_t$ converges in Hausdorff distance or that the collection of maps $(\gamma_t^1, \dots, \gamma_t^N)$ describing the network converges uniformly to the collection of maps $(\gamma_0^1, \dots, \gamma_0^N)$ that describes $\mathcal{N}_0$ (some of the γ_0^i could be the constant map).

If we suppose that the initial datum is a regular network, with linearization and a fixed point argument, one can prove that there exists a unique (up to reparametrization) maximal solution to the network flow with initial datum $\mathcal{N}_0$ in the maximal time interval $[0, T_{\max})$ (see [1, 7, 14]).
With definitely much more effort one can still prove a short-time existence result with irregular initial data [8, 13].

3 Singularities

In this paper we focus our attention on the asymptotic behaviour of the flow. Ideally, one would like to prove that either the maximal time of existence T is finite and everything vanishes at T (as in the case of closed curves) or $T = \infty$ and the evolving family of networks convergence to a critical point of the length functional. Unfortunately complications arises during the evolution in the form of "singularities". With the current Definition 2.3 of the flow, one can describe the long–time behaviour as follows [14]:

Theorem 3.1. *Let $T > 0$ and let $(\mathcal{N}_t)$, with $0 \leq t < T$, be a maximal solution of the motion by curvature of regular networks in the maximal time interval $[0, T)$. If $T = +\infty$ the family of evolving networks converges (up to subsequences) to a network composed of straight segments and balanced junctions (the sum of the unit tangent vector at the junctions equals zero). If T is finite, as $t \to T$ at least one of the following happens:*

- *i) the inferior limit of the length of at least one curve of the network is zero;*
- *ii) the superior limit of the L^2-norm of the curvature is $+\infty$;*

and the two possibilities are not mutually exclusive.

If $T = \infty$, in certain cases, the result can be strengthened, as we will discuss in Section 7.1. Let us instead elaborate a bit more on the case $T < \infty$.

Consider a grain of the network enclosed in a loop ℓ composed of m curves $(\gamma^1, \ldots, \gamma^m)$ and let $A(t)$ be the area of a grain. Using Gauss–Bonnet theorem we get that the time-derivative of the area is given by

$$A'(t) = -\sum_{i=1}^{m} \int_{\gamma^i} \langle \partial_t \gamma^i, \nu^i \rangle = -\sum_{i=1}^{m} \int_{\gamma^i} \kappa^i = -\left(2 - \frac{m}{3}\right)\pi\,. \tag{3.1}$$

Thus $A(t)$ increases linearly in time if $m > 6$, remains constant if $m = 6$, and decreases linearly in time if $m < 6$. In this last case, in particular, the area is zero at

$$\hat{T} = \frac{A_0}{(2 - {}^{m}\!/_{3})\,\pi}$$

where A_0 is the initial area enclosed by the loop. If we consider symmetric networks with a loop of $m < 6$ curves both the area and the length of ℓ will go to zero as $t \to \hat{T}$, producing an example of singularity in which an entire region enclosed by several curves vanishes in finite time [2, 18, 14]. When the length of a loop goes to zero, the L^2–norm of the curvature blows up: indeed by Hölder inequality we get

$$\left(2 - \frac{m}{3}\right)\pi \leq \int_\ell |\vec{\kappa}| \leq \left(\int_\ell \kappa^2\right)^{1/2} \sqrt{\mathrm{L}(\ell)}\,,$$

that is

$$\int_\ell \kappa^2 \,\mathrm{d}s \geq \frac{C}{\mathrm{L}(\ell)} \quad \text{with } C \neq 0 \text{ for non–hexagonal cells}\,. \tag{3.2}$$

We thus have examples of singularities in which both *i)* and *ii)* simultaneously happens. There are also explicit examples of evolution in which as $t \to T$ a single curve disappear [14, 19]. Whenever two triple junctions coalesce without the disappearance of a region, the curvature of the networks remains bounded [15]. It is instead widely believed that there are no singularities where the length of each curve is uniformly bounded away from zero and the curvature is unbounded.

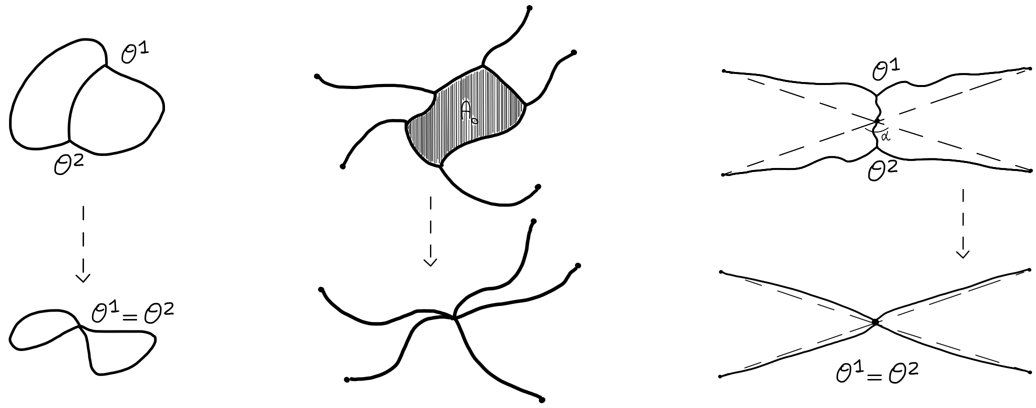


Figure 2: Examples of singularity.

The take-home message of this section is that singularities actually arise. To describe the asymptotic behaviour of the evolution as $t \to \infty$ we need to introduce a notion of flow past singularities.

4 Flow past singularities

We now give a different definition of the network flow, it can still be describe by smooth solutions of a system of partial differential equation and it is still a motion by curvature of regular networks apart from a finite set of singular times $\{a_1, \dots, a_\ell\}$.

Definition 4.1. A time dependent family of networks $\mathcal{N}_t$, $0 \leq t < T$, is a solution of the network flow if $\mathcal{N}_t$ converges to the initial network $\mathcal{N}_0$ as $t \searrow 0$, and if $[0, T)$ decomposes as a finite union of subintervals $[0, t_1) \cup [t_1, t_2) \cup \dots, [t_\ell, T)$ so that $\mathcal{N}_t$ is regular for all t except possibly $t_1, \dots, t_\ell$. On each open interval (t_j, t_{j+1}) the family $\mathcal{N}_t$ satisfies (2.1), (2.2) and it is continuous across each t_j. The times t_j are the singular times. The solution is assumed to be maximal.

For every $j \in \{1, \ell\}$, as $t \nearrow t_j$ the length of some arcs of $\mathcal{N}_t$ tends to zero, while for every $t_j < t < t_{j+1}$ the flow $\mathcal{N}_t$ has a collection of new arcs emanating from all vertices in $\mathcal{N}_{t_j}$ with order greater than 3.

At singular times t_j irregular networks appears. In many situations, we are able to "restart" the flow after such singularities [8, 13].
A key feature of the solution past singularity is that a single irregular junction gives birth to a cluster of triple junctions. The irregular junction is somehow locally replaced by a regular network whose combinatorics/topology is the same as one of the expanding solitons of the flow. Expanding solitons are solutions that self-similarly dilates during the evolution, each evolving curve has the form $\gamma(t, x) = \lambda(t)\eta(x)$ where the expanding factor $\lambda(t)$ equals $\sqrt{2t}$.

Note that not any soliton generates a solution, but it has to be compatible with the irregular junctions. Let the irregular junction coincides with the origin, and let $\gamma^{i_1}, \dots \gamma^{i_k}$ be the concurring curves, with unit tangent vectors $\tau^{i_1}, \dots, \tau^{i_k}$. Consider k halflines from the origin, whose direction coincides with $\tau^{i_1}, \dots, \tau^{i_k}$ and a small disk centred at O. Replace the part of the network inside the disk with a miniatures of a expanding soliton. The expanding soliton must have k non-compact branches whose directions at infinity coincide with the k halflines. Connect the soliton with the remaining part of the network nicely and let it flow. We get our evolution past singularity. We stress the fact that the number of solutions past singularities coincides with the number of expanding solutions compatible with the irregular junction.

Now, to discuss the asymptotic behaviour of the flow, we should know that the singular times are finite. Unfortunately at the moment we are still not able to get such a result: not only the singular times could in principle be infinite, but may even "accumulate".
The following figures shows an example of these (maybe) possible situations.

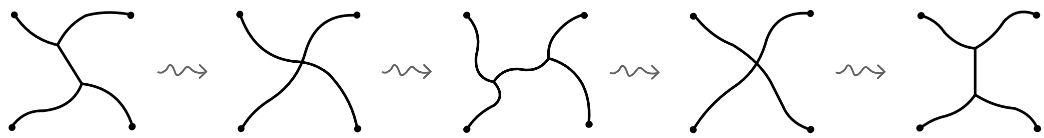


Figure 3: The family of evolving networks switches between two different topologies.

Excluding these unfortunate possibilities goes far beyond the purpose of this note, for a partial result in this direction see [17]. From now on, we simply suppose to be always able to restart the flow after a singularity and we exclude any "pathological" behaviour.

5 Simplification of the topology through singularities

Suppose that to restart the flow after a singularity we consider only solitons without loops. Then, we can easily show also that the number of grains, of curves and of triple junctions is non–increasing during the evolution.
To be precise, when a singularity occurs with no vanishing of regions, the number of grains, of curves and of triple junctions is preserved. On the other hand, when a bounded region disappears and we desingularize the irregular junctions by gluing–in a tree like soliton, we do not add grains to the network, the number of grains is non–increasing, the total number of curves decreases at least by three and the total number of triple junctions decreases at least by two.

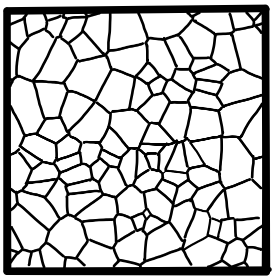
One may wonder if it is too restrictive to consider only tree like solitons. If an expanding soliton contains a grain, then by (3.1) it is bounded by a loop composed of at least seven curves. Thus to desingularise a junctions where at most five curves concur we can use only trees. Our choice is supported by the following:

Conjecture 5.1 (T. Ilmanen)**.** Let $\mathcal{N}_t$ be a solution of the network flow in $[0, T)$, let $\tilde{T} > 0$ be a singular time of the evolution and let $\mathcal{O}$ be an irregular junction of $\mathcal{N}_{\tilde{T}}$. Then, at most 5 curves concur at $\mathcal{O}$.

If the statement were true, then extra grains can appear only in the desingularisation of the initial datum.

6 Average growth of the area of the grains

For simplicity we set the evolution in the flat torus $\mathbb{T}^2$. We suppose that the initial network $\mathcal{N}_0$ is composed by a large number of curves and triple junctions, let's say that it has N^2 grains. Then the average diameter of grains is of order $1/N$, the average area of order $1/N^2$ and the global length of the network is of order N. We have shown that grains bounded by less than six curve should disappear during the evolution. We argue that the average area of the (surviving) grains grows linearly. By formula (3.2) we have that along each loop ℓ there holds



$$\int_\ell k^2\,\mathrm{d}s \geq \frac{C}{L(\ell)} \gtrsim N\,,$$

with $C \neq 0$ for non-hexagonal grains.

Till the percentage of non–hexagonal grains is sufficiently high, we can pass from an estimate on a single loop to an integral estimate on the whole network:

$$\int_{\mathcal{N}} k^2\,\mathrm{d}s \gtrsim N\sharp\,(\text{non–hexagonal grains}) = N^3\,. \tag{6.1}$$

Computing the evolution of the total length of $\mathcal{N}$, from the gradient flow structure of the problem we get

$$\frac{d}{dt}L(\mathcal{N}) = -\int_{\mathcal{N}} k^2\,\mathrm{d}s\,. \tag{6.2}$$

Putting together (6.1) and (6.2) we derive the following differential inequality:

$$\frac{d}{dt}N(t) \leq -N^3(t)\,, \tag{6.3}$$

from which we get

$$\frac{1}{N(t)^2} \geq 2t + c_0\,, \quad \text{that is} \quad N(t) \leq \frac{1}{\sqrt{2t+c_0}}\,, \tag{6.4}$$

with c_0 a constant encoding the number of initial grains. From this computation we obtain that the average area of the grains grows at least linearly in time: $\frac{1}{N(t)^2} \geq 2Ct$. However, there are two main limitations: we basically supposed that all grains are very similar to each other. Moreover, the inequality (6.3) remains valid till the number of non–hexagonal grains is of order N^2.

7 Stability

We analyse the flow as $t \to +\infty$. A "soft" statement (part of Theorem 3.1) reads as follow: if $\mathcal{N}_t$ is a solution to the network flow in $[0,+\infty)$, then as $t \to \infty$, the evolving networks $\mathcal{N}_t$ converge, up to subsequences, in $C^{1,\alpha} \cap W^{2,2}$, for every $\alpha \in (0, 1/2)$, to a critical point of the length functional. We stress the fact that the limit is not necessary a regular network, but it is merely composed of straight segments and balanced junctions (sum of the unit tangent vectors equals zero) and it is not necessary a global minimizer of the length functional [19]. At this point three questions naturally arise:

- does the full sequence $\mathcal{N}_t$ converges to a limit network as $t \to +\infty$?
- Under which hypothesis on the initial datum are we able to ensure global existence?
- When is it possible to prove that there exists a time $\widetilde{T}$ such that $\mathcal{N}_t$ with $t \in [\widetilde{T}, \infty)$ and the limit network $\mathcal{N}_\infty$ have the same topology?

It turns out that these three questions are intimately related, as we proved in [19]:

Theorem 7.1. *Let $\mathcal{N}_*$ be a minimal network. Then, there exists $\varepsilon = \varepsilon(\mathcal{N}_*) > 0$ such that the network flow starting from any regular network $\mathcal{N}_0$ with*

$$\|\mathcal{N}_* - \mathcal{N}_0\|_{H^2} < \varepsilon$$

exists for all times and converges to a network $\mathcal{N}_\infty$ with the same topology and same length of $\mathcal{N}_*$.

One may criticise Theorem 7.1 for two main reasons:

- In general, the result does not ensure $\mathcal{N}_\infty$ to coincide with $\mathcal{N}_*$. It is easy to prove that if $\mathcal{N}_*$ is a tree, then $\mathcal{N}_\infty = \mathcal{N}_*$.
- Being H^2–close is a very strong condition.

  There are examples of global solutions with initial data not H^2–close to minimal networks. Consider for instance the case of triods: a *triod* is the simplest example of regular network, composed of three curves that meet at one junctions, and the other end-points of the curves P^1, P^2, P^3 are fixed in the plane. Suppose that P^1, P^2, P^3 are the vertices of a triangle with all angles smaller than 120 degrees. Then the network with minimal length connecting them is still a triod. If the network flow starts from an initial datum contained in the triangle, then no lengths go to zero during the evolution, the maximal time of existence is infinite and the *full* sequence of triods converge to the unique minimal network connecting P^1, P^2, P^3 [12]. Hence the basin of attraction of the minimal network is at least the entire convex envelop of P^1, P^2, P^3. Note that the example presents again a bound *from below* on the thickness of the basin of attraction of the critical point.

Apparently Theorem 7.1 goes against the supposed coarsening–type behaviour of the evolution, under the hypothesis the evolving network does not simplify, its topology is preserved. However, we expect the cases in which one has stability to be scarce. The length functional lacks deeply convexity in the class of networks with fixed termini, and the number of critical points is bigger and bigger as the number of termini increases. We thus expect the basin of attraction of a critical point to become smaller and smaller.
We would like to have a bound from above on the thickness of the basin of attraction of critical points. As a first step in the estimate of the basin of attraction of minimal networks for the flow, we now present a result on local minimizers of the length functional. We give a quantitative bound from below on their local minimality. One should hope that the order of the bound from above would coincide with the order of the bound from below.

7.1 Critical points of the length functional

To consider the grains of the network as a partition of an open subset of $\mathbb{R}^2$ could be particularly convenient. We briefly summarise the relevant jargon relative to Cacciopoli partitions.

Let $\Omega \subset \mathbb{R}^2$ be open. A partition $\mathbf{E} = (E_1, \ldots, E_n)$ of Ω is a collection of finite perimeter sets $E_i \subset \Omega$ such that $|E_i \cap E_j| = 0$ for $i \neq j$ and $|\Omega \setminus \cup_{i=1}^n E_i| = 0$. We define the perimeter of the partition of Ω as

$$P(\mathbf{E}, \Omega) = \frac{1}{2} \sum_{i=1}^{N} P(E_i, \Omega)$$

where $P(E_i, \Omega) := |D\chi_{E_i}|(\Omega)$ is the (relative) perimeter of E_i in Ω.
We denote by $\Sigma_{ij}^{\mathbf{E}} := \partial^* E_i \cap \partial^* E_j$ and by $\nu_{ij} = \nu_i = -\nu_j$ the unit normal to Σ_{ij}, where ν_i is the generalized outer unit normal to the set E_i. In particular we can think of ν_{ij} as a normal pointing from E_i into E_j. We informally refer to $\bigcup_{i=1}^N (\partial^* E_i \cap \Omega) = \bigcup_{i<j=1}^N \left(\Sigma_{ij}^{\mathbf{E}} \cap \Omega\right)$ as the *boundary* of the partition $\mathbf{E}$.

Given a network $\mathcal{N}$ we denote by d be the minimum between the minimal distance of any two external vertices of the network and the length of the shortest edge of $\mathcal{N}$.

We say that a compact and connected set $\mathcal{N}$ disconnects two points x, y of $\partial\Omega$ if any continuous path σ from x to y intersect $\mathcal{N}$.
Let Ω be a open subset of the plane. Suppose that a network $\mathcal{N}^* \subset \overline{\Omega}$ has all its termini on $\partial\Omega$. We call $\mathcal{A}(\mathcal{N}^*, \Omega)$ the class of all networks with the following property: if $\mathcal{N}^*$ disconnects two points x, y of $\partial\Omega$ then also $\mathcal{N}$ disconnects x and y.

Theorem 7.2. *Let $\mathcal{N}^*$ be a minimal network. Then, there exists a δ–neighbourhood Ω of $\mathcal{N}^*$ with $0 < \delta \leq \frac{\sqrt{3}}{8} d$ such that $\mathcal{N}^*$ is a minimizer of the length among all networks $\mathcal{N}$ in $\mathcal{A}(\mathcal{N}^*, \Omega)$.*

Remark 7.3. For the sake of clarity in the statement we refer to Ω as a δ–neighbourhood of $\mathcal{N}^*$. To be precise, Ω is truncated as in Figure 4

Idea of the proof
The result is a direct consequence of [20, Theorem 3.9]. Instead of explaining only how our current statement fits in the framework of Theorem 3.9, we prefer, in addition, to summarise its proof here.
Construct Ω as in Figure 4 Since locally $\mathcal{N}^*$ is an hexagonal lattice, one shows that $\mathcal{N}^*$ can be interpreted as the boundary of a suitable partition $\mathbf{E} = (E_1, E_2, E_3)$ of Ω.

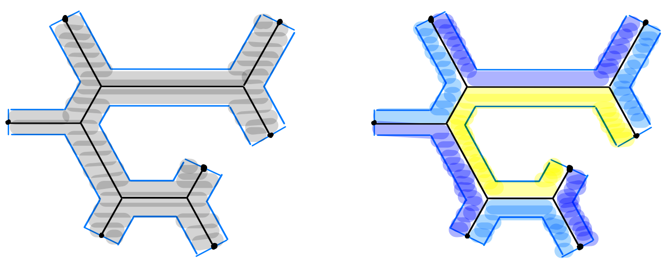


Figure 4: Left: A minimal network and the truncated neighborhood Ω. Right: the associated partition of three phases $\mathbf{E} = (E_1, E_2, E_3)$.

Now to prove that $\mathbf{E}$ is perimeter minimizing among all partitions $\mathbf{F}$ of Ω with the same trace on the boundary of Ω of $\mathbf{E}$ we construct an explicit *calibration* of $\mathbf{E}$ in Ω.
A calibration for $\mathbf{E}$ is a collection of three (sufficiently regular) vector fields (Φ_1, Φ_2, Φ_3), $\Phi_i : \overline{\Omega} \to \mathbb{R}^2$ with (distributional) divergence equal to zero fulfilling the following properties:

$$\begin{aligned} |\Phi_i - \Phi_j| \leq 1 \quad & \mathcal{H}^1 - \text{a.e. in } \Omega \,, \text{for } i,j = 1,2,3\,,\,, i \neq j\,, \\ (\Phi_i - \Phi_j) \cdot \nu_{ij} = 1 \quad & \mathcal{H}^1 - \text{a.e. in } \Sigma^{\mathbf{E}}_{ij}\,, \text{for } i,j = 1,2,3\,, i \neq j\,. \end{aligned}$$

We then have

$$\begin{aligned} \mathcal{P}(\mathbf{E}) &= \int_{\Sigma^{\mathbf{E}}_{12}\cap\Omega} (\Phi_1 - \Phi_2)\cdot \nu_{12}\, \mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{E}}_{23}\cap\Omega} (\Phi_2 - \Phi_3)\cdot \nu_{23}\, \mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{E}}_{31}\cap\Omega} (\Phi_3 - \Phi_1)\cdot \nu_{31}\, \mathrm{d}\mathcal{H}^1 \\ &= \sum_{i=1}^{3} \int_\Omega \Phi_i \cdot D\chi_{E_i} = \sum_{i=1}^{3} \int_\Omega \Phi_i \cdot D\chi_{F_i} \\ &\leq \int_{\Sigma^{\mathbf{F}}_{12}\cap\Omega} |\Phi_1 - \Phi_2|\, \mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{F}}_{23}\cap\Omega} |\Phi_2 - \Phi_3|\, \mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{F}}_{31}\cap\Omega} |\Phi_3 - \Phi_1|\, \mathrm{d}\mathcal{H}^1 = \mathcal{P}(\mathbf{F})\,. \end{aligned}$$

for every partition $\mathbf{F}$ that have the same trace of the boundary of Ω of $\mathbf{E}$.

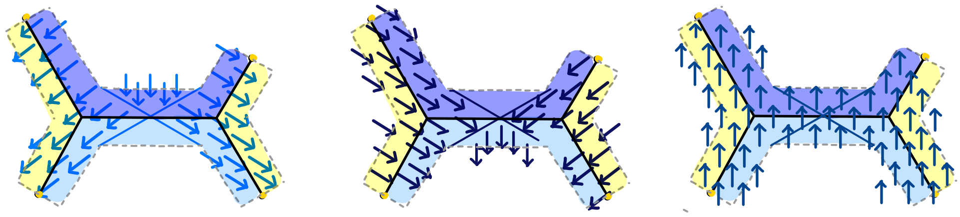


Figure 5: A calibration of a minimal network

Note that the differences $\Phi_i - \Phi_j$ play a crucial role, we can actually focus directly on the differences. Indeed, any time we are able to find three divergence free vector fields $\Psi_{12}, \Psi_{23}, \Psi_{31} : \overline{\Omega} \to \mathbb{R}^2$ such that

- $|\Psi_{12}|, |\Psi_{23}|, |\Psi_{31}| \leq 1$ $\mathcal{H}^1$-a.e. in Ω,
- $\Psi_{ij} \cdot \nu_{ij} = 1$ $\mathcal{H}^1$-a.e. in Σ_{ij}, for $i,j = 1,2,3$ such that Ψ_{ij} is defined,
- $\Psi_{12} + \Psi_{23} + \Psi_{31} = 0$ $\mathcal{H}^1$-a.e. in Ω.

We are then able to exhibit a calibration (Φ_1, Φ_2, Φ_3). Indeed, we can fix for example, $\Phi_1(x,y) := (0,0)|_{\overline{\Omega}}$ and set $\Phi_2 := \Phi_1 - \Psi_{12}$ and $\Phi_3 := \Psi_{31} + \Phi_1$. The three vector fields $\Psi_{12}, \Psi_{23}, \Psi_{31}$ that do the job are the unitary vector fields depicted in Figure 5.
To conclude it remains to prove that to any network $\mathcal{N}$ satisfying the hypothesis of the theorem we can associate a partition (F_1, F_2, F_3) with $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$ whose boundary is contained in $\mathcal{N}$.
We associate to $\mathcal{N}$ a partition $(F_1, \ldots, F_n)$. Then each connected component of $\Omega \setminus \mathcal{N}$ corresponds to one of the F_i and there exists a unique $j \in \{1,2,3\}$ such that $F_i \cap \partial\Omega$ coincides with $E_j \cap \partial\Omega$. It is enough to rename F_i as F_j with $j \in \{1,2,3\}$. $\square$

By checking the proof, one realise that the theorem can be immediately improved to the following stronger but less direct/transparent statement:

Corollary 7.4. *Let $\mathcal{N}^*$ be a minimal network. Then there exists Ω as in Theorem 7.2 and there exists a partition $\mathbf{E} = (E_1, E_2, E_3)$ of Ω whose boundary coincides with $\mathcal{N}^*$ such that $\mathcal{N}^*$ is a minimizer of the length functional in Ω among all networks $\mathcal{N}$ inducing a partition $\mathbf{F} = (F_1, F_2, F_3)$ of Ω with* $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$.

As one can see in the following picture, there are networks that are competitors in the sense of the corollary but not in the sense of the theorem.

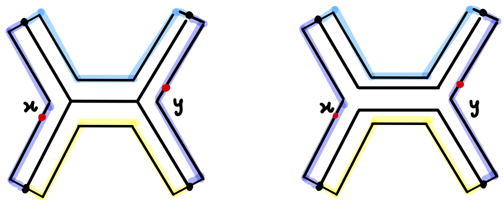


Figure 6: Left: a minimal network. Right: a competor.

We have repeatedly use the fact that a network in Ω induces a partition of Ω. On the contrary, the boundary of a partition can be understood as a network in the plane. Based on this observation it is not hard to imagine that Theorem 7.2 can be stated directly in the language of partitions.

Corollary 7.5. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $\mathbf{E} = (E_1, \dots, E_n)$ be a partition of Ω whose boundary is a minimal network and let d be the minimum of the distance between any two end-points and the shortest edge of $\mathcal{N}$. Then there exists a δ–neighbourhood D of $\mathcal{N}$ with $0 < \delta \leq \frac{\sqrt{3}}{8} d$ such that $\mathbf{E}$ is a minimizer in D for the perimeter among all partitions $\mathbf{F} = (F_1, \dots, F_n)$ with* $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$.

Proof. Let $\mathbf{E}$ be a partition as in the statement and $\mathbf{F}$ a competitor. To apply [20, Theorem 3.9] it is enough to canonically associate in D a partition of three sets $\widetilde{\mathbf{E}}$ to $\mathbf{E}$ and $\widetilde{\mathbf{F}}$ to $\mathbf{F}$, so that $P(\widetilde{\mathbf{E}}) = P(\mathbf{E})$ and $P(\widetilde{\mathbf{F}}) \leq P(\mathbf{F})$. In a δ–neighbourhood of $\mathcal{N}$ it is always possible to associate a partition of three sets $\widetilde{\mathbf{E}} = (\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3)$ to $\mathbf{E} = (E_1, \dots, E_n)$ (see [20, Theorem 3.9] for details). Consider now a competitor $\mathbf{F}$. Since $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$, we can associate to each F_i such that $F_i \cap \partial D \neq \emptyset$ one of the three $\widetilde{F}_i$ in such a way that $\mathrm{tr}_D \chi_{\widetilde{F}_i} = \mathrm{tr}_D \chi_{\widetilde{E}_i}$. To conclude we associate to all the remaining F_i the set $\widetilde{F}_1$. □

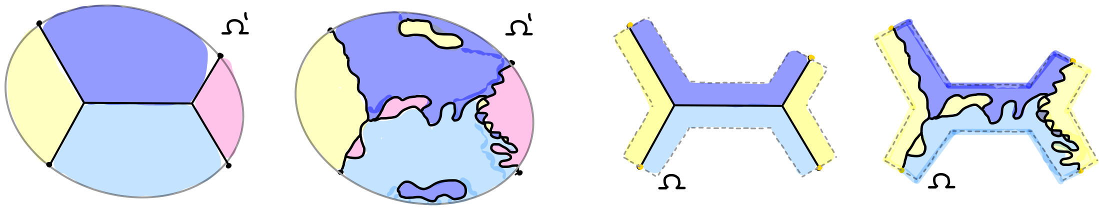


Figure 7: Left: two partition of Ω', the boundary of the first is a minimal network. Right: construction of Ω and relabelling of the two partitions inside of Ω.

The corollary establishes the local minimality of the partition among all partitions that are close to the minimal candidate in a L^∞ sense. With extra work, one can obtain a statement of the same flavour but with the L^1 distance in place of the L^∞ distance [6].

8 Conclusions

In the paper we listed the properties of the flow that indicate that the structure/topology of the networks should simplify during the evolution.

- When at most five curves concur at an irregular junction, locally all flowouts are without loops. Hence, the number of grains, of curves and of triple junctions is non–increasing during the evolution. In particular, when a region enclosed by a loop vanishes, the total number of curves decreases at least by three and the total number of triple junctions decreases at least by two.
- If we suppose that all grains are very similar to each other and that percentage of non–hexagonal grains is sufficiently high during the evolution, then we proved that grains bound by less than six curve should disappear during the evolution and the average area of the (surviving) grains grows linearly.
- It is well-known that regular networks with straight segments are critical points of the length functional, thus steady states of the network flow. However, we expect the volume of the basin of attraction of all the many critical point of the length functional to be small in the space of networks. A first step in this direction is the quantitative estimate of the size of the basin of local minimality of regular networks with straight segments obtained by local calibrations.

Because of the stability result, it is clearly impossible to get a deterministic theorem for all initial data. We conclude the paper with the following conjecture:

Conjecture 8.1. Consider a solution to the network flow in the torus whose initial datum is the Voronoi partition associated with n points randomly chosen. Then, there exists $\varphi(n)$ negligible with respect to n such that the probability that the limit network (as $t \to \infty$) has more than $\varphi(n)$ cells goes to zero as $n \to \infty$.

References

[1] Lia Bronsard and Fernando Reitich, *On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation*, Arch. Rational Mech. Anal. **124** (1993), no. 4, 355–379. MR 1240580

[2] Xinfu Chen and Jong-Shenq Guo, *Self-similar solutions of a 2-D multiple-phase curvature flow*, Phys. D **229** (2007), no. 1, 22–34. MR 2340176

[3] Y. Epshteyn, C. Liu, and M. Mizuno, *Motion of grain boundaries with dynamic lattice misorientations and with triple junctions drag*, SIAM J. Math. Anal. **53** (2021), no. 3, 3072–3097. MR 4263432

[4] S. Esedoḡlu and F. Otto, *Threshold dynamics for networks with arbitrary surface tensions*, Comm. Pure Appl. Math. **68** (2015), no. 5, 808–864. MR 3333842

[5] J. Fischer, S. Hensel, T. Laux, and T. Simon, *The local structure of the energy landscape in multiphase mean curvature flow: weak–strong uniqueness and stability of evolutions*, 2020, arXiv:2003.05478.

[6] Julian Fischer, Sebastian Hensel, Tim Laux, and Theresa Simon, *Local minimizers of the interface length functional based on a concept of local paired calibrations*, preprint 2023, arXiv:2212.11840.

[7] Michael Gößwein, Julia Menzel, and Alessandra Pluda, *Existence and uniqueness of the motion by curvature of regular networks*, Interfaces Free Bound. **25** (2023), no. 1, 109–154. MR 4578340

[8] Tom Ilmanen, André Neves, and Felix Schulze, *On short time existence for the planar network flow*, J. Differential Geom. **111** (2019), no. 1, 39–89. MR 3909904

[9] T. Kagaya, M. Mizuno, and K. Takasao, *Long time behavior for a curvature flow of networks related to grain bundary motion with the effect of lattice misoriantations*, 2021, arXiv:2112.11069.

[10] Lami Kim and Yoshihiro Tonegawa, *On the mean curvature flow of grain boundaries*, Ann. Inst. Fourier (Grenoble) **67** (2017), no. 1, 43–142. MR 3612327

[11] ———, *Existence and regularity theorems of one-dimensional Brakke flows*, Interfaces Free Bound. **22** (2020), no. 4, 505–550. MR 4184584

[12] David Kinderlehrer and Chun Liu, *Evolution of grain boundaries*, Math. Models Methods Appl. Sci. **11** (2001), no. 4, 713–729. MR 1833000

[13] Jorge Lira, Rafe Mazzeo, Alessandra Pluda, and Mariel Sez, *Short-time existence for the network flow*, Communications on Pure and Applied Mathematics (2023).

[14] C. Mantegazza, M. Novaga, A. Pluda, and F. Schulze, *Evolution of networks with multiple junctions*, to appear: Asterisque, ArXiv:1611.08254.

[15] Carlo Mantegazza, Matteo Novaga, and Alessandra Pluda, *Type-0 singularities in the network flow - evolution of trees*, Journal fur die Reine und Angewandte Mathematik **2022** (2022), no. 792, 189 221.

[16] W. M. Mullins, *Two–dimensional motion of idealized grain boundaries*, J. Appl. Phys. **27** (1956), 900–904.

[17] M. Novaga and L. Sciaraffia, *Singularities of the network flow with symmetric initial data*, ArXiv Preprint Server – http://arxiv.org, 2023.

[18] Alessandra Pluda, *Evolution of spoon-shaped networks*, Netw. Heterog. Media **11** (2016), no. 3, 509–526. MR 3541532

[19] Alessandra Pluda and Marco Pozzetta, *Łojasiewiczsimon inequalities for minimal networks: stability and convergence*, Mathematische Annalen (2023).

[20] ______, *Minimizing properties of networks via global and local calibrations*, Bulletin of the London Mathematical Society **55** (2023), no. 6, 3029 3052.

[21] S. Stuvard and Y. Tonegawa, *On the Existence of Canonical Multi–phase Brakke Flows*, 2021. To appear: Adv. Calc. Var. arXiv:2109.14415.