NORMALIZED SOLUTIONS FOR A FRACTIONAL SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL GROWTH

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Abstract. In this paper, we study the fractional critical Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda\phi u = \alpha u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

having prescribed mass

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2,$$

where $s,t \in (0,1)$ satisfies $2s+2t > 3, q \in (2, 2_s^*), a > 0$ and $\lambda, \mu > 0$ parameters and $\alpha \in \mathbb{R}$ is an undetermined parameter. Under the L^2-subcritical perturbation $q \in (2, 2+\frac{4s}{3})$, we derive the existence of multiple normalized solutions by means of the truncation technique, concentration-compactness principle and the genus theory. For the L^2-supercritical perturbation $q \in (2+\frac{4s}{3}, 2_s^*)$, by applying the constrain variational methods and the mountain pass theorem, we show the existence of positive normalized ground state solutions.

Contents

1. Introduction

In the last decade, the following time-dependent fractional Schrödinger-Poisson system

$$\begin{cases} i\dfrac{\partial\Psi}{\partial\tau} = (-\Delta)^s\Psi + \lambda\phi\Psi - f(x,|\Psi|), & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = |\Psi|^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

has attracted much attention, where $\Psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, s,t \in (0,1), \lambda \in \mathbb{R}$. It is well-known that, the first equation in (1.1) was used by Laskin (see [17, 18]) to extend the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths. This class of fractional Schrödinger equations with a repulsive nonlocal Coulombic potential can be approximated by the Hartree-Fock equations to describe a quantum mechanical system of many particles; see, for example, [7, 20, 21], and [26, 27] for more applied backgrounds on the fractional Laplacian.

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When we look for standing wave solutions to (1.1), namely to solutions of the form $(\Psi(\tau,x)=e^{-i\alpha\tau}u(x),\phi(x)),\alpha\in\mathbb{R}$, then the function $(u(x),\phi(x))$ solves the equation

$$\begin{cases}(-\Delta)^s u+\lambda\phi u=\alpha u+f(x,u), & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & x\in\mathbb{R}^3.\end{cases}\tag{1.2}$$

Here $(-\Delta)^s$ is a nonlocal operator defined by

$$(-\Delta)^s u(x)=C_s\ \text{P.V.}\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2s}}dy,\quad x\in\mathbb{R}^3,\quad s\in(0,1),$$

and P.V. stands for the Cauchy principal value on the integral, and C_s is a suitable normalization constant.

We note that, when $\alpha\in\mathbb{R}$ is a fixed real number, there was a lot of attention in recent years on the system (1.2) for the existence and multiplicity of ground state solutions, bound state solutions and concentrating solutions, see for examples [34, 36, 37, 39] and references therein. Especially, Zhang, do Ó and Squassina [39] considered the existence and asymptotical behaviors of positive solutions as $\lambda\to 0^+$, for the fractional Schrödinger-Poisson system

$$\begin{cases}(-\Delta)^s u+\lambda\phi u=g(u), & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=\lambda u^2, & x\in\mathbb{R}^3,\end{cases}$$

where $\lambda>0$ and g may be subcritical or critical growth satisfying the Berestycki-Lions conditions. In [31], Teng studied the existence of a nontrivial ground state solution for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent

$$\begin{cases}(-\Delta)^s u+V(x)u+\phi u=\mu|u|^{q-1}u+|u|^{2_s^*-2}u, & x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

where $\mu\in\mathbb{R}^+$ is a parameter, $1<q<2_s^*-1,\ s,t\in(0,1)$ with $2s+2t>3$. The potential V satisfies some suitable hypotheses. By the monotonicity trick, concentration-compactness principe and a global compactness Lemma, the author establishes the existence of ground state solutions. Formally, system (1.1) with $s=t=1$ can be regarded as the following classical Schrödinger-Poisson system

$$\begin{cases}-\Delta u+\lambda\phi u=f(x,u), & \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2, & \text{in } \mathbb{R}^3,\end{cases}$$

which appears in semiconductor theory [26] and also describes the interaction of a charged particle with the electrostatic field in quantum mechanics. The literature on the Schrödinger-Poisson system in presence of a pure power nonlinearity is very rich, we refer to [34, 36, 38] and references therein.

Alternatively, from a physical point of view, it is interesting to find solutions of (1.2) with prescribed L^2-norms, α appearing as Lagrange multiplier. Solutions of this type are often referred to as normalized solutions. The occurrence of the L^2-constraint renders several methods developed to deal with variational problems without constraints useless, and the L^2-constraint induces a new critical exponent, the L^2-critical exponent given by

$$\bar{q}:=2+\frac{4s}{3},$$

and the number $\bar{q}$ can keep the mass invariant by the law of conservation of mass. Precisely for this reason, $2+\frac{4s}{3}$ is called L^2-critical exponent or mass critical exponent, which is the threshold exponent for many dynamical properties such as global existence, blow-up, stability or instability of ground states. In particular, it strongly influences the geometrical structure of the corresponding functional. Meanwhile, the appearance of the L^2-constraint makes some classical methods, used to prove the boundedness of any Palais-Smale sequence for the unconstrained problem, difficult to implement. In [22], Li and Teng proved the existence of normalized solutions to the following fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta)^s u + \phi u = \lambda u + f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.3}$$

where $s \in (0,1), 2s + 2t > 3, \lambda \in \mathbb{R}$ and $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies some general conditions which contain the case $f(u) \sim |u|^{q-2}u$ with $q \in (\frac{4s+2t}{s+t}, 2 + \frac{4s}{3}) \cup (2 + \frac{4s}{3}, 2_s^*)$, i.e., the nonlinearity f is L^2-mass subcritical or L^2-mass supercritical growth, but is Sobolev subcritical growth. In [37], Yang, Zhao, and Zhao showed the existence of infinitely many solutions (u, λ) to (1.3) with subcritical nonlinearity $\mu|u|^{q-2}u$, by using the cohomological index theory.

We note that, when $s = t = 1$, problem (1.3), are related to the the following equation

$$\begin{cases} -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u = a|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^N} |u|^2 dx = c^2, \quad u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.4}$$

Recently, Jeanjean and Trung Le in [15] studied the existence of normalized solutions for (1.4) when $\gamma > 0$ and $a > 0$, both in the Sobolev subcritical case $p \in (10/3, 6)$ and in the Sobolev critical case $p = 6$, they showed that there exists a $c_1 > 0$ such that, for any $c \in (0, c_1)$, (1.4) admits two solutions u_c^+ and u_c^- which can be characterized respectively as a local minima and as a mountain pass critical point of the associated energy functional restricted to the norm constraint. While in the case $\gamma < 0, a > 0$ and $p = 6$ the authors showed that (1.4) does not admit positive solutions. Bellazzini, Jeanjean and Luo [4] proved that for $c > 0$ sufficiently small, there exists a critical point which minimizes with prescribed L^2-norms. In [14], Jeanjean and Luo studied the existence of minimizers for with L^2-norm for (1.4), and they expressed a threshold value of $c > 0$ separating existence and nonexistence of minimizers. In [32], Wang and Qian established the existence of ground state and infinitely many radial solutions to (1.4) with $a|u|^{p-2}u$ replaced by a general subcritical nonlinearity $af(u)$, by constructing a particular bounded Palais-Smale sequence when $\gamma < 0, a > 0$. In [23], Li and Zhang studied the existence of positive normalized ground state solutions for a class of Schrödinger-Popp-Podolsky system. For more results on the existence and no-existence of normalized solutions of Schrödinger-Poisson systems, we refer to [2, 3, 5, 6, 12, 14, 15, 24, 35, 37] and references therein.

After the above bibliography review we have found only two papers [22, 37] considering the normalized solutions for the fractional Schrödinger-Poisson system by the prescribed mass approaches with the nonlinearity $f(u)$, being Sobolev subcritical growth.

A natural question arises: How to obtain solutions to system (1.3) in presence of the nonlinear term $f(u) = \mu|u|^{q-2}u+|u|^{2_s^*-2}u$, combining the Sobolev critical term with a subcritical perturbation?

The main contribution of this paper is to give an affirmative answer to this question and fill this gap. To be specific, in the present paper we aim to study the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda \phi u = \alpha u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

having prescribed L^2-norm

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2, \tag{1.6}$$

where $s,t\in(0,1)$ satisfies $2s+2t>3$, $q\in(2,2_s^*)$ and $\alpha\in\mathbb{R}$ is an undetermined parameter, $\mu,\lambda>0$ are parameters. For this purpose, applying the reduction argument introduced in [39], system (1.5) is equivalent to the following single equation

$$(-\Delta)^s u+\lambda\phi_u^t u=\alpha u+\mu|u|^{q-2}u+|u|^{2_s^*-2}u,\quad x\in\mathbb{R}^3, \tag{1.7}$$

where

$$\phi_u^t(x)=c_t\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|^{3-2t}}dy,\quad \text{and}\quad c_t:=\frac{\Gamma(\frac{3}{2}-2t)}{\pi^3 2^{2t}\Gamma(t)}.$$

We shall look for solutions to (1.5)-(1.6), as a critical points of the action functional

$$I_\mu(u)=\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u^t|u|^2dx-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^q dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx,$$

restricted on the set

$$S_a=\left\{u\in H^s(\mathbb{R}^3):\ \int_{\mathbb{R}^3}|u|^2dx=a^2\right\},$$

with α being the Lagrange multipliers, Clearly, each critical point $u_a\in S_a$ of $I_\mu|_{S_a}$, corresponds a Lagrange multiplier $\alpha\in\mathbb{R}$ such that (u_a,α) solves (1.7). In particular, if $u_a\in S_a$ is a minimizer of problem

$$m(a):=\inf_{u\in S_a}I_\mu(u),$$

then there exists $\alpha\in\mathbb{R}$ as a Lagrange multiplier and then (u_a,α) is a weak solution of (1.7). As far as we know, there is no result about the existence of normalized solutions for Schrödinger-Poisson system with a critical term in the current literature. For this aim, we shall focus our attention on the existence, asymptotic and multiplicity of normalized solutions for problem (1.5)- (1.6).

2. The main results

In this section we formulate the main results. We first deal with the existence of multiple normalized ground state solutions in the L^2-subcritical case: $q\in(2,2+\frac{4s}{3})$. Secondly, we are concerned with the existence and asymptotic behavior of positive normalized ground state solutions of Schrödinger-Poisson system (1.7) in the L^2-supercritical case: $q\in(2+\frac{4s}{3},2_s^*)$.

To state the main results, for $\delta_{q,s}=3(q-2)/2qs$, we introduce the following constants:

$$D_1:=2^{-\frac{q\delta_{q,s}-2}{2_s^*-2}}S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}; \tag{2.1}$$

$$D_2:=D(s,t)^{-1}S^{\frac{3[(2_s^*-2)-q(1-\delta_{q,s})]}{2s(2_s^*-2)}}, \tag{2.2}$$

where

$$D(s,t):=\left(\frac{(3-2t)\lambda\Gamma_t}{2s}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}, \tag{2.3}$$

and Γ_t is given in (3.3).

The first result is concerned with the multiplicity of normalized solutions for the L^2-subcritical perturbation, which can be formulated as

Theorem 2.1. *Let $\mu, \lambda, a > 0$, and $q \in (2, 2+\frac{4s}{3})$. Then, for a given $k \in \mathbb{N}$, there exists $\beta > 0$ independent of k and $\mu_k^* > 0$ large, such that problem (1.5)-(1.6) possesses at least k couples $(u_j, \alpha_j) \in H^s(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $\mu > \mu_k$ and*

$$a \in \left(0, \left(\frac{\beta}{\mu}\right)^{\frac{1}{q(1-\delta_{q,s})}}\right) \tag{2.4}$$

with $\int_{\mathbb{R}^3} |u_j|^2 dx = a^2$, $\alpha_j < 0$ for all $j = 1, \cdots, k$.

The second result of this paper is concerned with the existence and asymptotical behavior of normalized solutions for the L^2-supercritical perturbation when the parameters $\lambda, \mu > 0$ are suitably small.

Theorem 2.2. *Let $q \in (2+\frac{4s}{3}, 2_s^*)$, assume that $\mu, a > 0$ satisfy the following inequality*

$$\mu\delta_{q,s} \max\left\{a^{q(1-\delta_{q,s})}, a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}}\right\} < \min\{D_1, D_2\}, \tag{2.5}$$

where $\delta_{q,s} = 3(q-2)/2qs$. Then, there exists $\Lambda^* > 0$ such that for $0 < \lambda < \Lambda^*$, problem (1.5)-(1.6) possesses a positive normalized ground state solution $u_\alpha \in H^s(\mathbb{R}^3)$ for some $\alpha < 0$.

Finally, we present an existence result of normalized solutions under the L^2-supercritical perturbation, when parameter $\mu > 0$ is large.

Theorem 2.3. *If $2 + \frac{4s}{3} < q < 2_s^*$, there exists $\mu^* = \mu^*(a) > 0$ large, such that as $\mu > \mu^*$, problem (1.5)-(1.6) possesses a couple $(u_a, \alpha) \in H^s(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^3} |u_a|^2 dx = a^2$, $\alpha < 0$.*

Remark 2.1. (i) Theorems 2.1-2.3 improve and complement the main results in [31, 39] in the sense that, we are concerned with the normalized solutions.

(ii) Our studies improve and fill in gaps of the main works of [22, 30, 37], since we consider the existence of normalized solutions to (1.5)-(1.6) with Sobolev critical growth.

2.1. **Remarks on the proofs.** We give some comments on the proof for the main results above. Since the critical terms $|u|^{2_s^*-2}u$ is L^2-supercritical, the functional I_μ is always unbounded from below on S_a, and this makes it difficulty to deal with existence of normalized solutions on the L^2- constraint. One of the main difficulties that one has to face in such context is the analysis of the convergence of constrained Palais-Smale sequences: In fact, the critical growth term in the equation makes the bounded (PS) sequences possibly not convergent; moreover, the Sobolev critical term $|u|^{2_s^*-2}u$ and nonlocal convolution term $\lambda\phi_u^t u$, makes it more complicated to estimate the critical value of mountain pass, and one has to consider how the interaction between the nonlocal term and the nonlinear term, and the energy balance between these competing terms needs to be controlled through moderate adjustments of parameter $\lambda > 0$. Another of difficulty is that sequences of approximated Lagrange multipliers have to be controlled, since α is not prescribed; and moreover, weak limits of Palais-Smale sequences could leave the constraint, since the embeddings $H^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ and also $H^s_{\text{rad}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ are not compact.

To overcome these difficulties, we employ Jeanjean's theory [13] by showing that the mountain pass geometry of $I_\mu|_{S_a}$ allows to construct a Palais-Smale sequence of functions satisfying the Pohozaev identity. This gives boundedness, which is the first step in proving strong H^s-convergence. As naturally expected, the presence of the Sobolev critical term in (1.5) further complicates the study of the convergence of Palais-Smale sequences. To overcome the loss of compactness caused by the critical growth, we shall employ the concentration-compactness principle, mountain pass theorem and energy estimation to obtain the existence of normalized ground states of (1.5), by showing that, suitably combining some of the main ideas from [28, 29], compactness can be restored in the present setting.

Finally, let us sketch the ideas and methods used along this paper to obtain our main results. For the L^2-subcritical perturbation: $q \in (2, 2 + \frac{4s}{3})$, it is difficult to get the boundedness of the (PS) sequence by the idea of [13]. To get over this difficulty, we use the truncation technique; to restore the loss of compactness of the (PS) sequence caused by the critical growth, we apply for the concentration-compactness principle; and to obtain the multiplicity of normalized solutions of (1.5)-(1.6), we employ the genus theory. For the L^2-supercritical perturbation: $q \in (2 + \frac{4s}{3}, 2_s^*)$, we use the Pohozaev manifold and mountain pass theorem to prove the existence of positive ground state solutions for system (1.5)-(1.6) when $\mu > 0$ small. While if the parameter $\mu > 0$ is large, we employ a fiber map and the concentration-compactness principle to prove that the (PS) sequence is strongly convergent, to obtain a normalized solution of (1.5)-(1.6).

2.2. **Paper outline.** This paper is organized as follows.

- Section 2 provides the main results, and Section 3 presents some preliminary results that will be used frequently in the sequel.
- Section 4 presents the multiplicity of normalized ground state solutions for system (1.5)-(1.6) when $q \in (2, 2 + \frac{4s}{3})$, and finish the proof of Theorem 2.1.
- Section 5 proves the existence of normalized positive ground state solutions for problem (1.5)-(1.6) when $q \in (2 + \frac{4s}{3}, 2_s^*)$, and Theorem 2.2 is proved if $\mu, \lambda > 0$ are suitably small.
- In Section 6 we give another existence result for problem (1.5)-(1.6) with $q \in (2 + \frac{4s}{3}, 2_s^*)$, when the parameter $\mu > 0$ is large, and finishes the proof of Theorem 2.3.

Notations. In the sequel of this paper, we denote by $C, C_i > 0$ different positive constants whose values may vary from line to line and are not essential to the problem. We denote by $L^p = L^p(\mathbb{R}^3)$ with $1 < p \leq \infty$ the Lebesgue space with the standard norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{1/p}$.

3. Preliminary stuff

In this section, we first give the functional space setting, and sketch the fractional order Sobolev spaces [27]. We recall that, for any $s \in (0,1)$, the nature functions space associated with $(-\Delta)^s$ is $H := H^s(\mathbb{R}^3)$ which is a Hilbert space equipped with the inner product and norm, respectively given by

$$\langle u, v \rangle := \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv) dx, \ \|u\|_H^2 = \langle u, u \rangle .$$

The homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined by

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy < +\infty \right\},$$

a completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|^2 := \|u\|_{D^{s,2}(\mathbb{R}^3)}^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy,$$

where $2_s^* = 6/(3 - 2s)$ is the critical exponent. From Proposition 3.4 and 3.6 in [27] we have

$$\|u\|^2 = \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy.$$

The best fractional Sobolev constant S is defined as

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^3), u \neq 0} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}}. \tag{3.1}$$

The work space $H^s_{rad}(\mathbb{R}^3)$ is defined by

$$H^s_{rad}(\mathbb{R}^3) := \left\{u \in H^s(\mathbb{R}^3) : \ u \text{ is radially decreasing}\right\}.$$

Let $\mathbb{H} = H \times \mathbb{R}$ with the scalar product $\langle\cdot,\cdot\rangle_H + \langle\cdot,\cdot\rangle_{\mathbb{R}}$, and the corresponding norm $\|(\cdot,\cdot)\|^2_{\mathbb{H}} = \|\cdot,\cdot\|^2_H + |\cdot,\cdot|^2_{\mathbb{R}}$.

The following two inequalities play an important role in the proof of our main results.

Proposition 3.1. *(Hardy-Littlewood-Sobolev inequality [20]) Let $l, r > 1$ and $0 < \mu < N$ be such that $\frac{1}{r} + \frac{1}{l} + \frac{\mu}{N} = 2$, $f \in L^r(\mathbb{R}^N)$ and $h \in L^l(\mathbb{R}^N)$. Then there exists a constant $C(N, \mu, r, l) > 0$ such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)h(y)|x-y|^{-\mu}dxdy\right| \le C(N,\mu,r,l)\|f\|_r\|h\|_l.$$

We recall the fractional Gagliardo-Nirenberg inequality.

Lemma 3.2. *([11]) Let $0 < s < 1$, and $p \in (2, 2^*_s)$. Then there exists a constant $C(p,s) = S^{-\frac{\delta_{p,s}}{2}} > 0$ such that*

$$\|u\|^p_p \le C(p,s)\|(-\Delta)^{\frac{s}{2}}u\|_2^{p\delta_{p,s}}\|u\|_2^{p(1-\delta_{p,s})}, \quad \forall u \in H^s(\mathbb{R}^3), \tag{3.2}$$

where $\delta_{p,s} = 3(p-2)/2ps$.

Lemma 3.3. *(Lemma 5.1 [9]) If $u_n \rightharpoonup u$ in $H^s_{rad}(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx \to \int_{\mathbb{R}^3}\phi_u u^2dx,$$

and

$$\int_{\mathbb{R}^3}\phi^t_{u_n}u_n\varphi dx \to \int_{\mathbb{R}^3}\phi^t_u u\varphi dx, \quad \forall \varphi \in H^s_{rad}(\mathbb{R}^3).$$

From Proposition 3.1, with $l = r = \frac{6}{3+2t}$, then Hardy-Littlewood-Sobolev inequality implies that:

$$\int_{\mathbb{R}^3}\phi^t_u u^2dx = \int_{\mathbb{R}^3}\left(\frac{1}{|x|^{3-2t}} * u^2\right)u^2dx \le \Gamma_t\|u\|^4_{\frac{12}{3+2t}}. \tag{3.3}$$

It is easy to enumerate that

$$q\delta_{q,s}\begin{cases} <2, & \text{if } \ 2 < q < \bar{q};\\ =2, & \text{if } \ q = \bar{q};\\ >2, & \text{if } \ \bar{q} < q < 2^*_s,\end{cases}$$

where $\bar{q} := 2 + \frac{4s}{3}$ is the L^2-critical exponent.

Now, we introduce the Pohozaev mainfold associated to (1.7), which can be derived from [31].

Proposition 3.4. *Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (1.7), then u satisfies the equality*

$$\frac{3-2s}{2}\|u\|^2 + \frac{2t+3}{4}\lambda\int_{\mathbb{R}^3}\phi^t_u u^2dx = \frac{3\alpha}{2}\|u\|^2_2 + \frac{3\mu}{q}\int_{\mathbb{R}^3}|u|^qdx + \frac{3}{2^*_s}\int_{\mathbb{R}^3}|u|^{2^*_s}dx.$$

Lemma 3.5. *Let $u \in H^s(\mathbb{R}^N)$ be a weak solution of (1.7), then we can construct the following Pohozaev manifold*

$$\mathcal{P}_a = \{u \in S_a : P_\mu(u) = 0\},$$

where

$$P_\mu(u) = s\|u\|^2 + \frac{3-2t}{4}\lambda\int_{\mathbb{R}^3}\phi^t_u u^2dx - s\mu\delta_{q,s}\int_{\mathbb{R}^3}|u|^qdx - s\int_{\mathbb{R}^3}|u|^{2^*_s}dx.$$

Proof. From Proposition 3.4, we know that u satisfies the Phohzaev identity as follows

$$\frac{3-2s}{2}\|u\|^2+\frac{2t+3}{4}\lambda\int_{\mathbb{R}^3}\phi_u^t u^2dx=\frac{3\alpha}{2}\|u\|_2^2+\frac{3\mu}{q}\int_{\mathbb{R}^3}|u|^qdx+\frac{3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx. \tag{3.4}$$

Moreover, since u is the weak solution of system (1.7), we have

$$\|u\|^2+\lambda\int_{\mathbb{R}^3}\phi_u^t u^2dx=\alpha\|u\|_2^2+\mu\int_{\mathbb{R}^3}|u|^qdx+\int_{\mathbb{R}^3}|u|^{2_s^*}dx. \tag{3.5}$$

Combining with (3.4) and (3.5), we get

$$s\|u\|^2+\frac{3-2t}{4}\lambda\int_{\mathbb{R}^3}\phi_u^t u^2dx=s\mu\delta_{q,s}\int_{\mathbb{R}^3}|u|^qdx+s\int_{\mathbb{R}^3}|u|^{2_s^*}dx,$$

which finishes the proof. □

Finally, we state the following well-known embedding result.

Lemma 3.6. *([10]). Let $N\geq 2$. The embedding $H^s_{rad}(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is compact for any $2<p<2_s^*$.*

4. Proof of Theorem 2.1

In this section, we aim to show the multiplicity of normalized solutions to (1.5)-(1.6). To begin with, we recall the definition of a genus. Let X be a Banach space and let A be a subset of X. The set A is said to be symmetric if $u\in A$ implies that $-u\in A$. We denote the set

$$\Sigma:=\{A\subset X\setminus\{0\}: A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A\in\Sigma$, define

$$\gamma(A)=\begin{cases}0, & \text{if } A=\emptyset,\\ \inf\{k\in\mathbb{N}:\exists \text{ an odd } \varphi\in C(A,\mathbb{R}^k\setminus\{0\})\},\\ +\infty, \text{if no such odd map},\end{cases}$$

and that $\Sigma_k=\{A\in\Sigma:\gamma(A)\geq k\}$.

In order to overcome the loss of compactness of the (PS) sequences, we need to apply for the following concentration-compactness principle.

Lemma 4.1. *([40]) Let $\{u_n\}$ be a bounded sequence in $D^{s,2}(\mathbb{R}^3)$ converging weakly and a.e. to some $u\in D^{s,2}(\mathbb{R}^3)$. We have that $|(-\Delta)^{\frac{s}{2}}u_n|^2\rightharpoonup\omega$ and $|u_n|^{2_s^*}\rightharpoonup\zeta$ in the sense of measures. Then, there exist some at most a countable set J, a family of points $\{z_j\}_{j\in J}\subset\mathbb{R}^3$, and families of positive numbers $\{\zeta_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that*

$$\omega\geq|(-\Delta)^{\frac{s}{2}}u|^2+\sum_{j\in J}\omega_j\delta_{z_j}, \tag{4.1}$$

$$\zeta=|u|^{2_s^*}+\sum_{j\in J}\zeta_j\delta_{z_j} \tag{4.2}$$

and

$$\omega_j\geq S\zeta_j^{\frac{2}{2_s^*}}, \tag{4.3}$$

where δ_{z_j} is the Dirac-mass of mass 1 concentrated at $z_j\in\mathbb{R}^3$.

Lemma 4.2. *([40]) Let $\{u_n\} \subset D^{s,2}(\mathbb{R}^3)$ be a sequence in Lemma 4.1 and define that*

$$\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx, \quad \zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{2_s^*} dx.$$

Then it follows that

$$\omega_\infty \geq S \zeta_\infty^{\frac{2}{2_s^*}}, \tag{4.4}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^3} d\omega + \omega_\infty \tag{4.5}$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^3} d\zeta + \zeta_\infty. \tag{4.6}$$

For $u \in S_{r,a}$, in view of Lemma 3.2, and the Sobolev inequality, one has that

$$\begin{aligned} I_\mu(u) =& \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ \geq& \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \frac{\mu}{q} a^{q(1-\delta_{q,s})} C_{q,s} \|(-\Delta)^{\frac{s}{2}} u\|_2^{q\delta_{q,s}} - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}} \|(-\Delta)^{\frac{s}{2}} u\|_2^{2_s^*} \\ :=& g(\|(-\Delta)^{\frac{s}{2}} u\|_2), \end{aligned} \tag{4.7}$$

where

$$g(r) = \frac{1}{2} r^2 - \frac{\mu}{q} a^{q(1-\delta_{q,s})} C_{q,s} r^{q\delta_{q,s}} - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}} r^{2_s^*}.$$

Recalling that $2 < q < 2 + \frac{4s}{3}$, we get that $q\delta_{q,s} < 2$, and there exists $\beta > 0$ such that, if $\mu a^{q(1-\delta_{q,s})} \leq \beta$, the function g attains its positive local maximum. More precisely, there exist two constants $0 < R_1 < R_2 < +\infty$, such that

$$g(r) > 0, \ \forall r \in (R_1, R_2); \quad g(r) < 0, \ \forall r \in (0, R_1) \cup (R_2, +\infty).$$

Let $\tau : \mathbb{R}^+ \to [0,1]$ be a nonincreasing and C^∞ function satisfying

$$\tau(r) = \begin{cases} 1, & \text{if } r \in [0, R_1], \\ 0, & \text{if } r \in [R_2, +\infty). \end{cases}$$

In the sequel, let us consider the truncated functional

$$I_{\mu,\tau}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{\tau(\|(-\Delta)^{\frac{s}{2}} u\|_2)}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

For $u \in S_{r,a}$, again by Lemma 3.2, and the Sobolev inequality, it is easy to see that

$$\begin{aligned} I_{\mu,\tau}(u) \geq& \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \frac{\mu}{q} a^{q(1-\delta_{q,s})} C_{q,s} \|(-\Delta)^{\frac{s}{2}} u\|_2^{q\delta_{q,s}} - \frac{\tau(\|(-\Delta)^{\frac{s}{2}} u\|_2)}{2_s^*} S^{-\frac{2_s^*}{2}} \|(-\Delta)^{\frac{s}{2}} u\|_2^{2_s^*} \\ :=& \widetilde{g}(\|(-\Delta)^{\frac{s}{2}} u\|_2), \end{aligned}$$

where

$$\widetilde{g}(r) = \frac{1}{2} r^2 - \frac{\mu}{q} a^{q(1-\delta_{q,s})} C_{q,s} r^{q\delta_{q,s}} - \frac{\tau(r)}{2_s^*} S^{-\frac{2_s^*}{2}} r^{2_s^*}.$$

Then, by the definition of $\tau(\cdot)$, when $a \in (0, (\frac{\beta}{\mu})^{\frac{1}{q(1-\delta_{q,s})}})$, we have

$$\widetilde{g}(r) < 0, \quad \forall r \in (0, R_1); \quad \widetilde{g}(r) > 0, \quad \forall r \in (R_1, +\infty).$$

In what follows, we always assume that $a \in (0, (\frac{\beta}{\mu})^{\frac{1}{q(1-\delta_{q,s})}})$. Without loss of generality, in the sequel, we may assume that

$$\frac{1}{2}r^2 - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}r^{2_s^*} \geq 0, \ \forall \ r \in [0, R_1] \tag{4.8}$$

and

$$R_1 < S^{\frac{3}{4s}}. \tag{4.9}$$

Lemma 4.3. *The functional $I_{\mu,\tau}$ has the following characteristics:*

(i) $I_{\mu,\tau} \in C^1\left(H^s_{rad}(\mathbb{R}^3), \mathbb{R}\right)$;

(ii) $I_{\mu,\tau}$ is coercive and bounded from below on $S_{r,a}$. Moreover, if $I_{\mu,\tau}(u) \leq 0$, then $\|(-\Delta)^{\frac{s}{2}}u\|_2 \leq R_1$ and $I_{\mu,\tau}(u) = I(u)$;

(iii) $I_{\mu,\tau}|_{S_{r,a}}$ satisfies the $(PS)_c$ condition for all $c < 0$, provided that $\mu > \mu_1^* > 0$ large.

Proof. We can obtain conclusions (i) and (ii) by a standard argument. To prove item (iii), let $\{u_n\}$ be a $(PS)_c$ sequence of $I_{\mu,\tau}$ restricted to $S_{r,a}$ with $c < 0$. By (ii), we see that $\|(-\Delta)^{\frac{s}{2}}u_n\|_2 \leq R_1$ for large n, and thus $\{u_n\}$ is a $(PS)_c$ sequence of $I|_{S_{r,a}}$ with $c < 0$; i.e., $I(u_n) \to c < 0$ and $\|I|'_{S_{r,a}}(u_n)\| \to 0$ as $n \to \infty$. Then, $\{u_n\}$ is bounded in $H^s_{rad}(\mathbb{R}^3)$. Therefore, up to a subsequence, there exists $u \in H^s_{rad}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^s_{rad}(\mathbb{R}^3)$ and $u_n \to u$ in $L^p(\mathbb{R}^3)$ for $2 < p < 2_s^*$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. From $2 < q < 2 + \frac{4s}{3} < 2_s^*$ and Lemma 3.3, we infer to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q dx = \int_{\mathbb{R}^3}|u|^q dx, \quad \int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2 dx \to \int_{\mathbb{R}^3}\phi^t_u u^2 dx.$$

Moreover, we have that $u \not\equiv 0$. Indeed, assume by contradiction that, $u \equiv 0$, then $\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q dx = 0$. From (4.8) and the definition of $I_{\mu,\tau}$, we infer that

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} I_{\mu,\tau}(u_n) = \lim_{n\to\infty} I_\mu(u_n) \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2 dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3}|u_n|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right] \\
&\geq \lim_{n\to\infty}\left[\frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{2_s^*} - \frac{\mu}{q}\int_{\mathbb{R}^3}|u_n|^q dx\right] \\
&\geq -\frac{\mu}{q}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q dx = 0,
\end{aligned}$$

which is absurd. On the other hand, setting the function $\Theta(v) : H^s_{rad}(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Theta(v) = \frac{1}{2}\int_{\mathbb{R}^3}|v|^2 dx,$$

it follows that $S_a = \Theta^{-1}(\{\frac{a^2}{2}\})$. Then, by Proposition 5.12 in [33], there exists $\alpha_n \in \mathbb{R}$ such that

$$\|I'_\mu(u_n) - \alpha_n\Theta'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, we have that

$$(-\Delta)^s u_n + \phi^t_{u_n}u_n - \mu|u_n|^{q-2}u_n - |u_n|^{2_s^*-2}u_n = \alpha_n u_n + o_n(1) \ \text{ in } H^{-s}_{rad}(\mathbb{R}^3), \tag{4.10}$$

where $H^{-s}_{rad}(\mathbb{R}^3)$ is the dual space of $H^s_{rad}(\mathbb{R}^3)$. Thus, we have for $\varphi \in H^s_{rad}(\mathbb{R}^3)$, that

$$\begin{aligned}
&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi dx + \int_{\mathbb{R}^3}\phi^t_{u_n}u_n\varphi dx - \mu\int_{\mathbb{R}^3}|u_n|^{q-2}u_n\varphi dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi dx \\
&\qquad = \alpha_n\int_{\mathbb{R}^3}u_n\varphi dx + o_n(1),
\end{aligned} \tag{4.11}$$

and if we choose $\varphi = u_n$, we get

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \mu \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = \alpha_n \int_{\mathbb{R}^3} u_n^2 dx + o_n(1). \tag{4.12}$$

From (4.12), and the boundedness of $\{u_n\}$ in $D^{s,2}(\mathbb{R}^3)$, we can deduce that $\{\alpha_n\}$ is bounded in $\mathbb{R}$. Then we can assume that, up to a subsequence, $\alpha_n \to \alpha$ for some $\alpha \in \mathbb{R}$. Then, by (4.11), we can derive that u solves the following equation

$$(-\Delta)^s u + \phi_u u - \mu|u|^{q-2}u - |u|^{2_s^*-2}u = \alpha u. \tag{4.13}$$

Indeed, for any $\varphi \in H^s_{rad}(\mathbb{R}^3)$, it follows by $u_n \rightharpoonup u$ in $H^s_{rad}(\mathbb{R}^3)$ and $\alpha_n \to \alpha$, that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx \to \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx; \quad \text{and} \quad \alpha_n \int_{\mathbb{R}^3} u_n \varphi dx \to \alpha \int_{\mathbb{R}^3} u\varphi dx.$$

as $n \to \infty$. Since $\{|u_n|^{2_s^*-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, $\{|u_n|^{q-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{q-1}}(\mathbb{R}^3)$, and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$, we obtain that

$$|u_n|^{2_s^*-2}u_n \rightharpoonup |u|^{2_s^*-2}u \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3), \quad \text{and} \quad |u_n|^{q-2}u_n \rightharpoonup |u|^{q-2}u \quad \text{in } L^{\frac{2_s^*}{2_s^*-q+1}}(\mathbb{R}^3),$$

and so,

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*-2} u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^{2_s^*-2} u\varphi dx \quad \text{and} \quad \int_{\mathbb{R}^3} |u_n|^{q-2} u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^{q-2} u\varphi dx,$$

as $n \to \infty$. Recall from Lemma 3.3 that

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u u \varphi dx, \quad \forall \varphi \in H^s_{rad}(\mathbb{R}^3).$$

Thus, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^3} \phi_u^t u \varphi dx - \mu \int_{\mathbb{R}^3} |u|^{q-2} u\varphi dx - \int_{\mathbb{R}^3} |u|^{2_s^*-2} u \varphi dx \\ &\quad = \alpha \int_{\mathbb{R}^3} u\varphi dx. \end{aligned} \tag{4.14}$$

Therefore, u solves equation (4.13).

In the sequel, by the concentration-compactness principle, we can prove that

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to \int_{\mathbb{R}^3} |u|^{2_s^*} dx. \tag{4.15}$$

In fact, since $\|(-\Delta)^{\frac{s}{2}} u_n\|_2 \le R_1$ for n large enough, by Lemma 4.1, there exist two positive measures, $\zeta, \omega \in \mathcal{M}(\mathbb{R}^3)$, such that

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \omega, \quad |u_n|^{2_s^*} \rightharpoonup \zeta \quad \text{in } \mathcal{M}(\mathbb{R}^3) \tag{4.16}$$

as $n \to \infty$. Then, by Lemma 4.1, either $u_n \to u$ in $L^{2_s^*}_{loc}(\mathbb{R}^3)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$ and positive numbers $\{\zeta_j\}_{j\in J}$ such that

$$\zeta = |u|^{2_s^*} + \sum_{j\in J} \zeta_j \delta_{x_j}.$$

Moreover, there exist some at most a countable set $J \subseteq \mathbb{N}$, a corresponding set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and two sets of positive numbers $\{\zeta_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that items (4.1)-(4.3) holds. Now, assume that $J \neq \emptyset$. We split the proof into three steps.

Step 1. We prove that $\omega_j = \zeta_j$, where ω_j, and ζ_j come from Lemma 4.1.

Define $\varphi \in C_0^\infty(\mathbb{R}^3)$ as a cut-off function with $\varphi \in [0,1]$, $\varphi \equiv 1$ in $B_{1/2}(0)$, $\varphi \equiv 0$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $\rho > 0$, define

$$\varphi_\rho(x) := \varphi\left(\frac{x-x_j}{\rho}\right) = \begin{cases} 1, & |x-x_j| \le \frac{1}{2}\rho, \\ 0, & |x-x_j| \ge \rho. \end{cases}$$

By the boundedness of $\{u_n\}$ in $H^s_{rad}(\mathbb{R}^3)$, we have that $\{\varphi_\rho u_n\}$ is also bounded in $H^s_{rad}(\mathbb{R}^3)$. Thus, one has that

$$\begin{aligned} o_n(1) =&\langle I'_\mu(u_n), u_n\varphi_\rho\rangle \\ =&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\varphi_\rho)dx + \lambda\int_{\mathbb{R}^3}\phi^t_{u_n}u_n\varphi_\rho dx - \mu\int_{\mathbb{R}^3}|u_n|^q\varphi_\rho dx \\ &- \int_{\mathbb{R}^3}|u_n|^{2^*_s}\varphi_\rho dx. \end{aligned} \tag{4.17}$$

It is easy to check that

$$\begin{aligned} &\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\varphi_\rho)dx \\ =&\iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)]|u_n(x)-u_n(y)|^2[u_n(x)\varphi_\rho(x)-u_n(y)\varphi_\rho(y)]}{|x-y|^{3+2s}}dxdy \\ =&\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\varphi_\rho(y)}{|x-y|^{3+2s}}dxdy + \iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\varphi_\rho(x)-\varphi_\rho(y)]u_n(x)}{|x-y|^{3+2s}}dxdy \\ :=& T_1 + T_2, \end{aligned} \tag{4.18}$$

where

$$T_1 = \iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\varphi_\rho(y)}{|x-y|^{3+2s}}dxdy$$

and

$$T_2 = \iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\varphi_\rho(x)-\varphi_\rho(y)]u_n(x)}{|x-y|^{3+2s}}dxdy.$$

For T_1, by (4.16), we obtain

$$\begin{aligned} \lim_{\rho\to 0}\lim_{n\to\infty} T_1 =& \lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\varphi_\rho(y)}{|x-y|^{3+2s}}dxdy \\ =& \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\omega = \omega(\{x_j\}) = \omega_j. \end{aligned} \tag{4.19}$$

From Hölder's inequality, we have

$$\begin{aligned} T_2 =& \iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\varphi_\rho(x)-\varphi_\rho(y)]u_n(x)}{|x-y|^{3+2s}}dxdy \\ \le& \left(\iint_{\mathbb{R}^6}\frac{|\varphi_\rho(x)-\varphi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}} \\ \le& C\left(\iint_{\mathbb{R}^6}\frac{|\varphi_\rho(x)-\varphi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}. \end{aligned}$$

Analogously to the proof of Lemma 3.4 in [40], we obtain

$$\lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|\varphi_\rho(x)-\varphi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy = 0,$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\varphi_\rho)dx = \omega(\{x_j\}) = \omega_j.$$

Again by (4.16), we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\varphi_\rho dx=\lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\zeta=\zeta(\{x_j\})=\zeta_j. \tag{4.20}$$

By the definition of φ_ρ, and the absolute continuity of the Lebesgue integral, one has that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q\varphi_\rho dx=\lim_{\rho\to 0}\int_{\mathbb{R}^3}|u|^q\varphi_\rho dx=\lim_{\rho\to 0}\int_{|x-x_j|\le\rho}|u|^q\varphi_\rho dx=0. \tag{4.21}$$

Thus, by Proposition 3.1 and Lemma 3.6, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\varphi_\rho dx&\le C\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{6}}\left(\int_{\mathbb{R}^3}|u_n^2\varphi_\rho|^{\frac{6}{3+2t}}dx\right)^{\frac{3+2t}{6}}\\&\le C\|u_n\|_H^2\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}|\varphi_\rho|^{\frac{6}{3+2t}}dx\right)^{\frac{3+2t}{6}}\\&\le C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}\varphi_\rho dx\right)^{\frac{3+2t}{6}}.\end{aligned} \tag{4.22}$$

Therefore,

$$\begin{aligned}\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\varphi_\rho dx&\le\lim_{\rho\to 0}\lim_{n\to\infty}C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}\varphi_\rho dx\right)^{\frac{3+2t}{6}}\\&=\lim_{\rho\to 0}C_1\left(\int_{\mathbb{R}^3}|u|^{\frac{12}{3+2t}}\varphi_\rho dx\right)^{\frac{3+2t}{6}}\\&=\lim_{\rho\to 0}C_1\left(\int_{|x-x_j|\le\rho}|u|^{\frac{12}{3+2t}}\varphi_\rho dx\right)^{\frac{3+2t}{6}}=0.\end{aligned} \tag{4.23}$$

Summing up, from (4.17)-(4.19) and (4.21), taking the limit as $n\to\infty$, and then the limit as $\rho\to 0$, we arrive at

$$\omega_j=\zeta_j.$$

Step 2. We show that $\omega_\infty=\zeta_\infty$, where ω_∞ and ζ_∞ are given in Lemma 4.2. Let $\psi\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi\in[0,1]$, $\psi\equiv 0$ in $B_{1/2}(0)$, $\psi\equiv 1$ in $\mathbb{R}^3\setminus B_1(0)$. For any $R>0$, define

$$\psi_R(x):=\psi\left(\frac{x}{R}\right)=\begin{cases}0, & |x|\le\frac{1}{2}R,\\ 1, & |x|\ge R.\end{cases}$$

Using again the boundedness of $\{u_n\}$ and $\{u_n\psi_R\}$ in $H^s_{rad}(\mathbb{R}^3)$, we have

$$\begin{aligned}o_n(1)=&\langle I'_\mu(u_n),u_n\psi_R\rangle\\=&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\psi_R)dx+\lambda\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\psi_R dx-\mu\int_{\mathbb{R}^3}|u_n|^q\psi_R dx\\&-\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_R dx.\end{aligned} \tag{4.24}$$

It is easy to derive that

$$\begin{aligned}
&\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\psi_R)dx\\
&=\iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][u_n(x)\psi_R(x)-u_n(y)\psi_R(y)]}{|x-y|^{3+2s}}dxdy\\
&=\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{3+2s}}dxdy+\iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\psi_R(x)-\psi_R(y)]u_n(x)}{|x-y|^{3+2s}}dxdy\\
&:=T_3+T_4,
\end{aligned}$$

where

$$T_3=\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{3+2s}}dxdy$$

and

$$T_4=\iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\psi_R(x)-\psi_R(y)]u_n(x)}{|x-y|^{3+2s}}dxdy.$$

For T_3, by (4.16) and Lemma 4.2, we infer to

$$\lim_{R\to\infty}\lim_{n\to\infty}T_3=\lim_{R\to\infty}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{3+2s}}dxdy=\omega_\infty.$$

By virtue of Hölder's inequality, we get

$$\begin{aligned}
T_4&=\iint_{\mathbb{R}^6}\frac{[u_n(x)-u_n(y)][\psi_R(x)-\psi_R(y)]u_n(x)}{|x-y|^{3+2s}}dxdy\\
&\le\left(\iint_{\mathbb{R}^6}\frac{|\psi_R(x)-\psi_R(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}\\
&\le C\left(\iint_{\mathbb{R}^6}\frac{|\psi_R(x)-\psi_R(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining the above proof, we conclude that

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|\psi_R(x)-\psi_R(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy\\
&=\lim_{R\to\infty}\lim_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|[1-\psi_R(x)]-[1-\psi_R(y)]|^2|u_n(x)|^2}{|x-y|^{3+2s}}dxdy=0.
\end{aligned}$$

Hence,

$$\lim_{R\to\infty}\lim_{n\to\infty}\iint_{\mathbb{R}^6}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n\psi_R)dx=\omega_\infty.$$

By Lemma 4.2, we have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_Rdx=\zeta_\infty. \tag{4.25}$$

Analogous the proof of Lemma 3.3 in [40], we infer to

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^q\psi_Rdx=\lim_{R\to\infty}\int_{\mathbb{R}^3}|u|^q\psi_Rdx=\lim_{R\to\infty}\int_{|x|>\frac{1}{2}R}|u|^q\psi_Rdx=0. \tag{4.26}$$

Moreover, we can obtain

$$
\begin{aligned}
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\psi_R dx &\le \lim_{R\to\infty}\lim_{n\to\infty} C_1\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2t}}\psi_R dx\right)^{\frac{3+2t}{6}}\\
&= \lim_{R\to\infty} C_1\left(\int_{\mathbb{R}^3}|u|^{\frac{12}{3+2t}}\psi_R dx\right)^{\frac{3+2t}{6}}\\
&= \lim_{R\to\infty} C_1\left(\int_{|x|\ge R/2}|u|^{\frac{12}{3+2t}}\psi_R dx\right)^{\frac{3+2t}{6}} = 0.
\end{aligned}
\tag{4.27}
$$

Summing up, from (4.24)-(4.27), taking the limit as $n\to\infty$, and then the limit as $R\to\infty$, we have

$$\omega_\infty = \zeta_\infty.$$

Step 3. We claim that $\zeta_j = 0$ for any $j\in J$ and $\zeta_\infty = 0$.

Suppose by contradiction that, there exists $j_0\in J$ such that $\zeta_{j_0} > 0$ or $\zeta_\infty > 0$. Step 1, Step 2, and Lemmas 4.1, 4.2 imply that

$$\zeta_{j_0} \le (S^{-1}\omega_{j_0})^{\frac{2_s^*}{2}} = (S^{-1}\zeta_{j_0})^{\frac{2_s^*}{2}}, \tag{4.28}$$

and

$$\zeta_\infty = (S^{-1}\omega_\infty)^{\frac{2_s^*}{2}} = (S^{-1}\zeta_\infty)^{\frac{2_s^*}{2}}. \tag{4.29}$$

Consequently, we get $\zeta_{j_0} \ge S^{\frac{3}{2s}}$ or $\zeta_\infty \ge S^{\frac{3}{2s}}$. If the former case occurs, we have

$$
\begin{aligned}
R_1^2 \ge \lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 &\ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}\\
&\ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{2_s^*}\varphi_\rho dx\right)^{\frac{2}{2_s^*}} = S\left(\int_{\mathbb{R}^3}\varphi_\rho d\zeta\right)^{\frac{2}{2_s^*}}.
\end{aligned}
\tag{4.30}
$$

Taking the limit $\rho\to 0$ in the last inequality, we get

$$R_1^2 \ge S(\zeta_{j_0})^{\frac{2}{2_s^*}} \ge S(S^{\frac{3}{2s}})^{\frac{2}{2_s^*}} = S^{\frac{3}{2s}},$$

which contradicts (4.9). If the last case happens, we have

$$
\begin{aligned}
R_1^2 \ge \lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 &\ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}\\
&\ge S\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_R dx\right)^{\frac{2}{2_s^*}}\\
&\ge S\lim_{n\to\infty}\left(\int_{|x|\ge R}|u_n|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}.
\end{aligned}
\tag{4.31}
$$

Taking the limits $n\to\infty$ and $R\to\infty$ in (4.31), we infer to

$$R_1^2 \ge S(\zeta_\infty)^{\frac{2}{2_s^*}} \ge S(S^{\frac{3}{2s}})^{\frac{2}{2_s^*}} = S^{\frac{3}{2s}},$$

which also contradicts (4.9). Therefore, $\zeta_j = 0$ for any $j\in J$ and $\zeta_\infty = 0$. As a result, by Lemma 4.1, we obtain that $u_n\to u$ in $L^{2_s^*}_{loc}(\mathbb{R}^3)$; while by Lemma 4.2, we know that $u_n\to u$ in $L^{2_s^*}(\mathbb{R}^3)$.

Now, we prove there exists $\mu_1^* > 0$ independently on $n\in\mathbb{N}$ such that if $\mu > \mu_1^*$, the Lagrange multiplier $\alpha < 0$ in (4.13). Indeed, note that $\{u_n\}\subset S_{r,s}$ and $\|(-\Delta)^{\frac{s}{2}}u_n\|_2 \le R_1$, as can be seen from the previous proof of this lemma, and (3.2)-(3.3) that, there exists $Q_1 > 0$ independently on n, such that

$$Q_1 \le \int_{\mathbb{R}^3} |u_n|^q dx \le C(q,s)\|(-\Delta)^{\frac{s}{2}} u_n\|_2^{q\delta_{q,s}} \|u_n\|_2^{q(1-\delta_{q,s})} \le C(q,s) R_1^{q\delta_{q,s}} a^{q(1-\delta_{q,s})}, \tag{4.32}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \le \Gamma_t \|u_n\|^4_{\frac{12}{3+2t}} &\le \Gamma_t C\left(12/3+2t, s\right)^{\frac{3+2t}{3}} \|(-\Delta)^{\frac{s}{2}} u_n\|_2^{\frac{3-2t}{s}} \|u_n\|_2^{\frac{2t+4s-3}{s}} \\ &\le \Gamma_t C\left(12/3+2t, s\right)^{\frac{3+2t}{3}} R_1^{\frac{3-2t}{s}} a^{\frac{2t+4s-3}{s}} \\ &:= Q_2,\end{aligned} \tag{4.33}$$

where $Q_2 = Q_2(s,t,R_1,a) > 0$. We define the constant

$$\mu_1^* := \frac{q\lambda(2t+4s-3)Q_2}{2[6-q(3-2s)]Q_1}. \tag{4.34}$$

By (4.32)-(4.34) we have

$$\mu_1^* > \lim_{n\to+\infty} \left\{ \frac{q\lambda(2t+4s-3)\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx}{2[6-q(3-2s)]\int_{\mathbb{R}^3} |u_n|^q dx} \right\} = \frac{q\lambda(2t+4s-3)\int_{\mathbb{R}^3} \phi_u^t u^2 dx}{2[6-q(3-2s)]\int_{\mathbb{R}^3} |u|^q dx} > 0. \tag{4.35}$$

Recall by (4.13) and its Pohozaev identity $P_\mu(u) = 0$, we infer to

$$s\alpha\|u\|_2^2 = \lambda \frac{2t+4s-3}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx + \frac{q(3-2s)-6}{2q} \mu \int_{\mathbb{R}^3} |u|^q dx. \tag{4.36}$$

Now, if $\mu > \mu_1^*$, we conclude from (4.35), that

$$\mu > \frac{q\lambda(2t+4s-3)\int_{\mathbb{R}^3} \phi_u^t u^2 dx}{2[6-q(3-2s)]\int_{\mathbb{R}^3} |u|^q dx}.$$

Thus, from (4.36), we infer to $\lim_{n\to+\infty} \alpha_n = \alpha < 0$. Hence, taking into account (4.12), we derive

$$\begin{aligned}&\lim_{n\to\infty} \left[\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \alpha\|u_n\|_2^2\right] \\ &= \lim_{n\to\infty} \left[\mu\|u_n\|_q^q + \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1)\right] \\ &= \mu\|u\|_q^q + \int_{\mathbb{R}^3} |u|^{2_s^*} dx = \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx - \alpha\|u\|_2^2.\end{aligned} \tag{4.37}$$

Since $\alpha < 0$ for $\mu > \mu_1^*$ large, we obtain by Fatou's Lemma,

$$\begin{aligned}&\lim_{n\to\infty} \left[\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \alpha\|u_n\|_2^2\right] \\ &\ge \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \lambda \int_{\mathbb{R}^3} \phi_u^t u^2 dx + \liminf_{n\to\infty}(-\alpha\|u_n\|_2^2),\end{aligned} \tag{4.38}$$

and from (4.37)-(4.38), one has

$$-\alpha\|u\|_2^2 \ge \liminf_{n\to\infty}(-\alpha\|u_n\|_2^2). \tag{4.39}$$

But by Fatou's Lemma, we see that

$$\liminf_{n\to\infty}(-\alpha\|u_n\|_2^2) \ge -\alpha\|u\|_2^2. \tag{4.40}$$

Combining (4.39) with (4.40) we get

$$\lim_{n\to\infty}\left(-\alpha\|u_n\|_2^2\right) = -\alpha\|u\|_2^2;$$

that is,

$$\lim_{n\to\infty}\|u_n\|_2^2 = \|u\|_2^2.$$

Thus, by (4.37) we have

$$\lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 = \|(-\Delta)^{\frac{s}{2}}u\|_2^2.$$

Theerfore, $u_n \to u$ in $H^s_{rad}(\mathbb{R}^3)$ and $\|u\|_2 = a$. The proof is complete. □

For $\varepsilon > 0$, we introduce the set

$$I_{\mu,\tau}^{-\varepsilon} = \left\{u \in H^s_{rad}(\mathbb{R}^3) \cap S_a : I_{\mu,\tau}(u) \le -\varepsilon\right\} \subset H^s_{rad}(\mathbb{R}^3).$$

By the fact that $I_{\mu,\tau}(u)$ is continuous and even on $H^s_{rad}(\mathbb{R}^3)$, $I_{\mu,\tau}^{-\varepsilon}$ is closed and symmetric.

Lemma 4.4. *For any fixed $k \in \mathbb{N}$, there exists $\varepsilon_k := \varepsilon(k) > 0$ and $\mu_k := \mu(k) > 0$ such that, for $0 < \varepsilon \le \varepsilon_k$ and $\mu \ge \mu_k$, one has that $\gamma(I_{\mu,\tau}^{-\varepsilon}) \ge k$.*

The proof of Lemma 4.4 is similar to Lemma 3.2 in [1], so we omit it here.
In the sequel, we define the set

$$\Sigma_k := \left\{\Omega \subset H^s_{rad}(\mathbb{R}^3) \cap S_a : \Omega \text{ is closed and symmetric}, \gamma(\Omega) \ge k\right\},$$

and by Lemma 4.3-(ii), we know that

$$c_k := \inf_{\Omega\in\Sigma_k}\sup_{u\in\Omega} I_{\mu,\tau}(u) > -\infty$$

for all $k \in \mathbb{N}$. To prove Theorem 2.1, we introduce the critical value, we define

$$K_c := \{u \in H^s_{rad}(\mathbb{R}^3) \cap S_a : I'_{\mu,\tau}(u) = 0, I_{\mu,\tau}(u) = c\}.$$

Then, we can derive the following conclusion:

Lemma 4.5. *If $c = c_k = c_{k+1} = \cdots = c_{k+\ell}$, then one has $\gamma(K_c) \ge \ell + 1$. Especially, $I_{\mu,\tau}(u)$ admits at least $\ell + 1$ nontrivial critical points.*

Proof. For $\varepsilon > 0$, it is easy to check that $I_{\mu,\tau}^{-\varepsilon} \in \Sigma$. For any fixed $k \in \mathbb{N}$, by Lemma 4.4, there exists $\varepsilon_k := \varepsilon(k) > 0$ and $\mu_k := \mu(k) > 0$ such that, if $0 < \varepsilon \le \varepsilon_k$ and $\mu \ge \mu_k$, we have $\gamma(I_{\mu,\tau}^{-\varepsilon_k}) \ge k$. Thus, $I_{\mu,\tau}^{-\varepsilon_k} \in \Sigma_k$, and moreover,

$$c_k \le \sup_{u\in I_{\mu,\tau}^{-\varepsilon_k}} I_{\mu,\tau}(u) = -\varepsilon_k < 0.$$

Assume that $0 > c = c_k = c_{k+1} = \cdots = c_{k+\ell}$. Then, by Lemma 4.3-(iii), $I_{\mu,\tau}(u)$ satisfies the $(PS)_c$-condition at the level $c < 0$. So, K_c is a compact set. By Theorem 2.1 in [1], or Theorem 2.1 in [16], we know that the restricted functional $I_{\mu,\tau}|_{S_a}$ possesses at least $\ell + 1$ nontrivial critical points. □

Proof of Theorem 2.1. Let $\mu \ge \mu_k^* = \max\{\mu_1^*, \mu_k\}$. From Lemma 4.3-(ii), we see that the critical points of $I_{\mu,\tau}(u)$ found in Lemma 4.5 are the critical points of I_μ, which completes the proof. □

5. Proof of Theorem 2.2

From Lemma 3.5, we see that any critical point of $I_\mu|_{S_a}$ belongs to $\mathcal{P}_a$. Consequently, the properties of the manifold $\mathcal{P}_a$ have relation to the mini-max structure of $I_\mu|_{S_a}$. For $u \in S_a$ and $t \in \mathbb{R}$, we introduce the transformation (e.g. [29]):

$$(\theta \star u)(x) := e^{\frac{3\theta}{2}} u(e^{\theta} x), \quad x \in \mathbb{R}^3, \quad \theta \in \mathbb{R}. \tag{5.1}$$

It is easy to check that the dilations preserve the L^2-norm such that $\theta \star u \in S_a$, by direct calculation, one has

$$\begin{aligned} I(u,\theta) = I_\mu((\theta \star u)) =& \frac{e^{2s\theta}}{2}\|u\|^2 + \frac{\lambda e^{(3-2t)\theta}}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q} e^{(\frac{3q}{2}-3)\theta}\int_{\mathbb{R}^3} |u|^q dx \\ & - \frac{1}{2_s^*} e^{3(\frac{2_s^*}{2}-1)\theta}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let $u \in S_a$, then*

(i) $\|(-\Delta)^{\frac{s}{2}}(\theta \star u)\|_2 \to 0$ and $I_\mu((\theta \star u)) \to 0$ as $\theta \to -\infty$;
(ii) $\|(-\Delta)^{\frac{s}{2}}(\theta \star u)\|_2 \to +\infty$ and $I_\mu((\theta \star u)) \to -\infty$ as $\theta \to +\infty$.

Proof. A direct computation shows that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(\theta \star u)|^2 dx = e^{2s\theta}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \tag{5.3}$$

and

$$\|(-\Delta)^{\frac{s}{2}}(\theta \star u)\|_2 \to 0 \quad \text{as} \quad \theta \to -\infty.$$

Notice that

$$\begin{aligned} I_\mu((\theta \star u)) =& \frac{e^{2s\theta}}{2}\|u\|^2 + \frac{\lambda e^{(3-2t)\theta}}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q} e^{(\frac{3q}{2}-3)\theta}\int_{\mathbb{R}^3} |u|^q dx \\ & - \frac{1}{2_s^*} e^{\frac{3(2_s^*-2)}{2}\theta}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned} \tag{5.4}$$

by $q > 2$, we infer to

$$I_\mu((\theta \star u)) \to -\infty, \quad \text{as} \quad \theta \to +\infty.$$

Hence, item (i) follows. Using $2s + 2t > 3$, it is easy to obtain that $\frac{3(2_s^*-2)}{2} > 3 - 2t$, and conclusion (ii) holds. □

Lemma 5.2. *There exist $K = K_a > 0$ and $\widetilde{a} > 0$ such that for all $0 < a < \widetilde{a}$,*

$$0 < \sup_{u \in \mathcal{A}_a} I_\mu(u) < \inf_{u \in \mathcal{B}_a} I_\mu(u), \tag{5.5}$$

where $\mathcal{A}_a := \{u \in S_{r,a} : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \le K_a\}$, $\mathcal{B}_a := \{u \in S_{r,a} : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = 2K_a\}$.

Proof. By Lemma 3.2, we have for any $q \in (2, 2_s^*)$, that

$$\|u\|_q^q \le C(q,s)\|(-\Delta)^{\frac{s}{2}} u\|_2^{q\delta_{q,s}} \|u\|_2^{q(1-\delta_{q,s})}. \tag{5.6}$$

By the Sobolev inequality (3.1), and (5.6), for $u \in S_{r,a}$, we have

$$\begin{aligned} & I_\mu((\theta\star u)) - I_\mu(u) \\ &= \frac{1}{2}\|(\theta\star u)\|^2 - \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi^t_{(\theta\star u)}|(\theta\star u)|^2dx - \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi^t_u u^2 dx \\ &\quad - \frac{\mu}{q}\int_{\mathbb{R}^3}|(\theta\star u)|^q dx + \frac{\mu}{q}\int_{\mathbb{R}^3}|u|^q dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3}|(\theta\star u)|^{2^*_s}dx + \frac{1}{2^*_s}\int_{\mathbb{R}^3}|u|^{2^*_s}dx \\ &\geq \frac{1}{2}\|(\theta\star u)\|^2 - \frac{1}{2}\|u\|^2 - \lambda\Gamma_t K_a^{\frac{3-2t}{2s}}\|u\|_2^{\frac{4s+2t-3}{s}} - \frac{\mu}{q}\int_{\mathbb{R}^3}|(\theta\star u)|^q dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3}|(\theta\star u)|^{2^*_s}dx \\ &\geq \frac{1}{2}\|(\theta\star u)\|^2 - \frac{1}{2}\|u\|^2 - \lambda\Gamma_t K_a^{\frac{3-2t}{2s}} a^{\frac{4s+2t-3}{s}} - \frac{\mu}{q}C(q,s)a^{\frac{6-q(3-2s)}{2s}}\left(\|(\theta\star u)\|^2\right)^{\frac{q\delta_{q,s}}{2}} \\ &\quad - \frac{S^{-\frac{2^*_s}{2}}}{2^*_s}\left(\|(\theta\star u)\|^2\right)^{\frac{2^*_s}{2}}. \end{aligned} \tag{5.7}$$

Let $\|u\|^2 \leq K_a$ and choose $\theta > 0$ such that $\|(\theta\star u)\|^2 = 2K_a$, here K_a will be determined later, set

$$\widetilde{a} = \left(\frac{K_a^{\frac{2t+2s-3}{2s}}}{16\lambda\Gamma_t}\right)^{\frac{s}{4s+2t-3}},$$

then we get

$$\begin{aligned} & I_\mu((\theta\star u)) - I_\mu(u) \\ &\geq \frac{1}{2}K_a - \lambda\Gamma_t K_a^{\frac{3-2t}{2s}}\widetilde{a}^{\frac{4s+2t-3}{s}} - \frac{\mu}{q}2^{\frac{q\delta_{q,s}}{2}}C(q,s)\widetilde{a}^{\frac{6-q(3-2s)}{2s}}K_a^{\frac{3(q-2)}{4s}} - \frac{S^{-\frac{2^*_s}{2}}}{2^*_s}2^{\frac{2^*_s}{2}}K_a^{\frac{2^*_s}{2}} \\ &\geq \frac{1}{2}K_a - \frac{1}{16}K_a - \frac{\mu}{q}2^{\frac{3(q-2)}{4s}}C(q,s)\left(\frac{1}{16\lambda\Gamma_t}\right)^{\frac{6-q(3-2s)}{2(4s+2t-3)}}K_a^{\frac{[6-q(3-2s)][2t+2s-3]}{4s(4s+2t-3)}}K_a^{\frac{3(q-2)}{4s}} \\ &\quad - \frac{S^{-\frac{2^*_s}{2}}}{2^*_s}2^{\frac{2^*_s}{2}}K_a^{\frac{2^*_s}{2}} \\ &= \frac{7}{16}K_a - \frac{\mu 2^{\frac{3(q-2)}{4s}}C(q,s)}{q(16\lambda\Gamma_t)^{\frac{6-q(3-2s)}{2(4s+2t-3)}}}K_a^{\gamma_1}K_a - \frac{2^{\frac{2^*_s}{2}}}{2^*_s S^{\frac{2^*_s}{2}}}K_a^{\frac{2^*_s-2}{2}}K_a \\ &\geq \frac{5}{16}K_a > 0, \end{aligned} \tag{5.8}$$

where $\gamma_1 := \frac{[2t+2s-3][6-q(3-2s)]+[3(q-2)-4s][4s+2t-3]}{4s(4s+2t-3)}$. If we take

$$K_a = \min\left\{\left(\frac{q[16\lambda\Gamma_t]^{\frac{6-q(3-2s)}{2(4s+2t-3)}}}{16\mu 2^{\frac{3(q-2)}{4s}}C(q,s)}\right)^{\gamma_2}, \quad \left(\frac{2^*_s S^{\frac{2^*_s}{2}}}{2^{\frac{2^*_s}{2}}16}\right)^{\frac{2}{2^*_s-2}}\right\}$$

with $\gamma_2 := \frac{4s(4s+2t-3)}{[2t+2s-3][6-q(3-2s)]+[3(q-2)-4s][4s+2t-3]}$, then, we deduce by (5.8) that (5.5) holds. □

By Lemma 5.2, we can deduce the following

Corollary 5.1. *Let $K_a, \widetilde{a}$ be given in Lemma 5.2, and $u \in S_{r,a}$ with $\|u\|^2 \leq K_a$, then $I_\mu(u) > 0$. Furthermore, we have*

$$L_0 := \inf\left\{I_\mu(u): \ u \in S_{r,a}, \|u\|^2 = \frac{1}{2}K_a\right\} > 0.$$

Proof. As in the proof of Lemma 5.2, we have that

$$I_\mu(u) \geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q}C(q,s)a^{\frac{6-q(3-2s)}{2s}}\left(\|u\|^2\right)^{\frac{3(q-2)}{4s}} - \frac{S^{-\frac{2_s^*}{2}}}{2_s^*}\left(\|u\|^2\right)^{\frac{2_s^*}{2}} > 0,$$

if $\|u\|^2 \leq K_a$, and the conclusion follows. □

Next, we study the characterizations of the mountain pass levels for $I(u,\theta)$ and $I_\mu(u)$. Denote the closed set $I_\mu^d := \{u \in S_{r,a} : I_\mu(u) \leq d\}$, and $S_{r,a} := H_r^s(\mathbb{R}^3) \cap S_a$.

Proposition 5.3. *Under assumptions $2 + \frac{4s}{3} < q < 2_s^*$, define*

$$\widetilde{c}_\mu(a) := \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \max_{t\in[0,1]} I(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma}_a = \{\widetilde{\gamma} \in C([0,1], S_{r,a} \times \mathbb{R}) : \ \widetilde{\gamma}(0) \in (\mathcal{A}_a, 0), \widetilde{\gamma}(1) \in (I_\mu^0, 0)\},$$

and

$$c_\mu(a) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)),$$

where

$$\Gamma_a = \{\gamma \in C([0,1], S_{r,a}) : \ \gamma(0) \in \mathcal{A}_a, \gamma(1) \in I_\mu^0\},$$

then we have

$$\widetilde{c}_\mu(a) = c_\mu(a) > 0.$$

Proof. Note that $\Gamma_a \times \{0\} \subset \widetilde{\Gamma}_a$, we see that $\widetilde{c}_\mu(a) \leq c_\mu(a)$. On the other hand, for $\widetilde{\gamma}(t) = (\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)) \in \widetilde{\Gamma}_a$, we denote by $\gamma(t) = \widetilde{\gamma}_1(t) \star \widetilde{\gamma}_2(t)$. Thus, $\gamma(t) \in \Gamma_a$, and so

$$\max_{t\in[0,1]} I(\widetilde{\gamma}(t)) = \max_{t\in[0,1]} I_\mu(\widetilde{\gamma}_1(t) \star \widetilde{\gamma}_2(t)) = \max_{t\in[0,1]} I_\mu(\gamma(t)),$$

which implies that $\widetilde{c}_\mu(a) \geq c_\mu(a) > 0$, using Corollary 5.1. □

Next, we show the existence of the $(PS)_{c_\mu(a)}$-sequence for $I(u,\theta)$ on $S_{r,a} \times \mathbb{R} \subset \mathbb{H}$. It is obtained by a standard argument using Ekeland's variational principle and constructing pseudo-gradient flow, see Proposition 2.2 [13].

Proposition 5.4. *Let $\{h_n\} \subset \widetilde{\Gamma}_a$ satisfying that*

$$\max_{t\in[0,1]} I(h_n(t)) \leq \widetilde{c}_\mu(a) + \frac{1}{n},$$

then there exists a sequence $\{(v_n, \theta_n)\} \subset S_{r,a} \times \mathbb{R}$ such that

(i) $I(v_n, \theta_n) \in [\widetilde{c}_\mu(a) - \frac{1}{n}, \widetilde{c}_\mu(a) + \frac{1}{n}]$,
(ii) $\min_{t\in[0,1]} \|(v_n, \theta_n) - h_n(t)\|_{\mathbb{H}} \leq \frac{1}{\sqrt{n}}$; *and*
(iii) $\|(I|_{S_{r,a}\times\mathbb{R}})'(v_n, \theta_n)\| \leq \frac{2}{\sqrt{n}}$, *that is,*

$$|\langle I'(v_n, \theta_n), z\rangle_{\mathbb{H}^{-1}\times\mathbb{H}}| \leq \frac{2}{\sqrt{n}}\|z\|_{\mathbb{H}},$$

for all

$$z \in \widetilde{T}_{(v_n,\theta_n)} \triangleq \{(z_1, z_2) \in \mathbb{H} : \langle v_n, z_1\rangle_{L^2} = 0\}.$$

It follows from the above proposition, we can obtain a special $(PS)_{c_\mu(a)}$-sequence for $I_\mu(u)$ on $S_{r,a} \subset H^s(\mathbb{R}^3)$.

Proposition 5.5. *Under the assumption $2 + \frac{4s}{3} < q < 2_s^*$, there exists a sequence $\{u_n\} \subset S_{r,a}$ such that*

(1) $I_\mu(u_n) \to c_\mu(a)$ *as* $n \to \infty$;
(2) $P_\mu(u_n) \to 0$ *as* $n \to \infty$;
(3) $(I_\mu|_{S_{r,a}})'(u_n) \to 0$ *as* $n \to \infty$*, i.e.,* $\langle I'_\mu(u_n), z\rangle_{H^{-1}\times H} \to 0$*, uniformly for all* z *satisfying*

$$\|z\|_H \leq 1, \quad \textit{where} \quad z \in T_{u_n} := \{z \in H : \ \langle u_n, z\rangle_{L^2} = 0\}$$

Proof. By Proposition 5.3, $\widetilde{c}_\mu(a) = c_\mu(a)$. Hence, we can take $\{h_n = ((h_n)_1, 0)\} \in \widetilde{\Gamma}_a$ so as to

$$\max_{t\in[0,1]} I(h_n(t)) \leq \widetilde{c}_\mu(a) + \frac{1}{n}.$$

It follows from Proposition 5.4 that, there exists a sequence $\{(v_n, \theta_n)\} \subset S_{r,a} \times \mathbb{R}$ such that as $n \to \infty$, one has

$$I(v_n, \theta_n) \to c_\mu(a), \quad \theta_n \to 0; \tag{5.9}$$

$$(I|_{S_{r,a}\times\mathbb{R}})'(v_n, \theta_n) \to 0. \tag{5.10}$$

Set $u_n = \theta_n \star v_n$. Then, $I_\mu(u_n) = I(v_n, \theta_n)$, and by (5.9), item (1) holds. To prove conclusion (2), we utilize

$$\begin{aligned}
\partial_\theta I(v_n, \theta_n) &= se^{2s\theta_n}\|v_n\|^2 + \frac{(3-2t)\lambda}{4}e^{(3-2t)\theta_n}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2dx - \frac{3\mu(q-2)}{2q}e^{(\frac{3q}{2}-3)\theta_n}\int_{\mathbb{R}^3}|v_n|^q dx \\
&\quad - \frac{3(2_s^*-2)}{22_s^*}e^{\frac{3(2_s^*-2)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx \\
&= s\|(-\Delta)^{\frac{s}{2}}u_n\|^2 + \frac{(3-2t)\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx - \frac{3\mu(q-2)}{2q}\int_{\mathbb{R}^3}|u_n|^q dx \\
&\quad - \frac{3(2_s^*-2)}{22_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \\
&= P_\mu(u_n)
\end{aligned}$$

which implies item (2) by (5.10). To show item (3), we set $z_n \in T_{u_n}$. Then,

$$\begin{aligned}
I'_\mu(u_n)z_n &= \iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y))(z_n(x)-z_n(y))}{|x-y|^{3+2s}}dxdy + \lambda\int_{\mathbb{R}^3}\phi_{u_n}^t u_n z_n dx \\
&\quad - \mu\int_{\mathbb{R}^3}|u_n|^{q-2}u_n z_n dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n z_n dx \\
&= e^{\frac{(4s-3)\theta_n}{2}}\iint_{\mathbb{R}^6}\frac{(v_n(x)-v_n(y))(z_n(e^{-\theta_n}x)-z_n(e^{-\theta_n}y))}{|x-y|^{3+2s}}dxdy \\
&\quad + e^{\frac{3-4t}{2}\theta_n}\int_{\mathbb{R}^3}\phi_{v_n}v_n(x)z_n(e^{-\theta_n}x)dx - \mu e^{\frac{3(q-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{q-2}v_n(x)z_n(e^{-\theta_n}x)dx \\
&\quad - e^{\frac{3(2_s^*-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{2_s^*-2}v_n(x)z_n(e^{-\theta_n}x)dx.
\end{aligned}$$

Denote by $\widetilde{z}_n(x) = e^{-\frac{3s}{2}}z_n(e^{-\theta_n}x)$, then we get

$$\langle I'_\mu(u_n), z_n\rangle_{H^{-1}\times H} = \langle I'(v_n, \theta_n), (\widetilde{z}_n, 0)\rangle_{\mathbb{H}^{-1}\times\mathbb{H}}.$$

It is easy to check that

$$\begin{aligned}\langle v_n, \widetilde{z}_n\rangle_{L^2} &= \int_{\mathbb{R}^3} v_n(x)e^{-\frac{3s}{2}}z_n(e^{-\theta_n}x)dx\\ &= \int_{\mathbb{R}^3} v_n(e^{\theta_n}x)e^{\frac{3s}{2}}z_n(x)dx\\ &= \int_{\mathbb{R}^3} u_n(x)z_n(x)dx = 0\end{aligned}$$

Therefore, we see that $(\widetilde{z}_n, 0) \in \widetilde{T}_{(v_n,\theta_n)}$. On the other hand,

$$\|(\widetilde{z}_n, 0)\|_{\mathbb{H}}^2 = \|\widetilde{z}_n\|_H^2 = \|z_n\|_2^2 + e^{-2s\theta_n}\|z_n\|^2 \le C\|z_n\|^2,$$

where the last inequality follows by $\theta_n \to 0$. Consequently, we conclude item (3). □

Remark 5.1 *From Propositions 5.4,5.5, we know that $u_n := \theta_n \star v_n \subset S_{r,a}$ is a (PS) sequence for I_μ with the level $c_\mu(a)$, that is*

$$I_\mu(u_n) \to c_\mu(a) \quad as \quad n \to +\infty, \tag{5.11}$$

and

$$(I_\mu|_{S_{r,a}})'(u_n) \to 0 \quad as \quad n \to +\infty. \tag{5.12}$$

Lemma 5.6. *The (PS) sequence $\{u_n\}$ mentioned in Remark 5.1 is bounded in $H^s_{rad}(\mathbb{R}^3)$. Moreover, suppose that $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$, and $\lambda < \lambda_1^*$ for some $\lambda_1^* > 0$, then $\lim_{n\to+\infty} \alpha_n = \alpha < 0$.*

Proof. From Remark 5.1 we see that $I_\mu(u_n)$ is bounded. In fact, by $P_\mu(u_n) \to 0$ as $n \to \infty$, we have

$$|(1 + 2t)I_\mu(u_n) + P_\mu(u_n)| \le 3c_\mu(a),$$

which implies that,

$$\begin{aligned}&\frac{1+2s+2t}{2}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \lambda\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx - \mu\left(\frac{1+2t}{2} + s\delta_{q,s}\right)\int_{\mathbb{R}^3}|u_n|^qdx\\ &\quad - \left(\frac{1+2t}{2_s^*} + s\right)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \ge -3c_\mu(a).\end{aligned} \tag{5.13}$$

In view of the boundedness of $I_\mu(u_n)$, we have

$$\|(-\Delta)^{\frac{s}{2}}u_n\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx \le 6c_\mu(a) + \frac{2\mu}{q}\int_{\mathbb{R}^3}|u_n|^qdx + \frac{2}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx. \tag{5.14}$$

By (5.13)-(5.14), we obtain

$$\frac{2s+2t-3}{4}\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx + \mu\frac{(\delta_{q,s}-2)s}{q}\int_{\mathbb{R}^3}|u_n|^qdx + \frac{(2_s^*-2)s}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \le 3c_\mu(a)(2+2s+2t).$$

Note that $2s + 2t > 3, q > 2 + \frac{4s}{3}$, we have that $q\delta_{q,s} - 2 > 0$, and so

$$\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2dx, \quad \int_{\mathbb{R}^3}|u_n|^qdx \quad \text{and} \quad \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx$$

are all bounded. Thus, $\|(-\Delta)^{\frac{s}{2}}u_n\|_2 \le R_2$ for some $R_2 > 0$ independently on $n \in \mathbb{N}$. Since $\{u_n\} \subset S_{r,a}$, we see that $\{u_n\}$ is bounded in $H^s_{rad}(\mathbb{R}^3)$. Thus, passing to a subsequence, and we may assume that $u_n \rightharpoonup u$ for some $u \in H^s_{rad}(\mathbb{R}^3)$, and so $u_n \to u$ in $L^p(\mathbb{R}^3), \forall p \in (2, 2_s^*)$.

Now, we set the functional $\Phi :\ H^s_{rad}(\mathbb{R}^3) \to \mathbb{R}$ as

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3}|u|^2dx,$$

then $S_{r,a} = \Phi^{-1}\left(\frac{a^2}{2}\right)$. As a result, it can be derived from Proposition 5.12 [33] that there is a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that

$$I'_\mu(u_n) - \alpha_n \Phi'(u_n) \to 0 \ \text{ in } H^{-s}_{rad}(\mathbb{R}^3) \ \text{ as } \ n \to \infty.$$

That is, we have

$$(-\Delta)^s u_n + \phi^t_{u_n} u_n - \mu |u_n|^{q-2} u_n - |u_n|^{2^*_s - 2} u_n = \alpha_n u_n + o_n(1) \ \text{ in } H^{-s}_{rad}(\mathbb{R}^3), \tag{5.15}$$

Similar to the proof of Lemma 4.3, we know that u solves the equation

$$(-\Delta)^s u + \phi^t_u u - \mu |u|^{q-2} u - |u|^{2^*_s - 2} u = \alpha u. \tag{5.16}$$

Moreover, $u \not\equiv 0$. In fact, argue by contradiction that $u \equiv 0$. Then $u_n \to 0$ in $L^p(\mathbb{R}^3)$, $\forall\ p \in (2, 2^*_s)$, and by $P_\mu(u_n) = o_n(1)$, (3.3), we have

$$\begin{aligned} o_n(1) &= s\|u_n\|^2 + \lambda \frac{3-2t}{4} \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 dx - \mu s \delta_{q,s} \int_{\mathbb{R}^3} |u_n|^q dx - s \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx \\ &= s\|u_n\|^2 - s \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx + o_n(1). \end{aligned}$$

We may assume that $\lim_{n\to+\infty} \|u_n\|^2 = \lim_{n\to+\infty} \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx = \vartheta \geq 0$. Thus, we have

$$\begin{aligned} c_\mu(a) + o_n(1) &= I_\mu(u_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u_n|^q dx - \frac{1}{2^*_s} \int_{\mathbb{R}^3} |u_n|^{2^*_s} dx \\ &= \frac{1}{2}\vartheta - \frac{1}{2^*_s}\vartheta + o_n(1) = \frac{s}{3}\vartheta + o_n(1). \end{aligned} \tag{5.17}$$

On the other hand, by the Sobolev inequality (3.1), we have $\vartheta \geq S \vartheta^{\frac{2}{2^*_s}}$. Then we have two possible cases: (i) $\vartheta = 0$; (ii) $\vartheta \geq S^{\frac{3}{2s}}$.

If $\vartheta = 0$, then by (5.17) we get $I_\mu(u_n) \to 0$, which contradicts to $I_\mu(u_n) \to c_\mu(a) > 0$. Now if the second case $\vartheta \geq S^{\frac{3}{2s}}$ occurs, then by (5.17) we get $I_\mu(u_n) \to \frac{s}{3}\vartheta \geq \frac{s}{3} S^{\frac{3}{2s}}$, which contradicts to $I_\mu(u_n) \to c_\mu(a) < \frac{s}{3} S^{\frac{3}{2s}}$. Hence, $u \not\equiv 0$. Moreover, by (5.15) and $P_\mu(u_n) = o_n(1)$, we have

$$s\alpha_n \|u_n\|_2^2 = \lambda \frac{2t+4s-3}{4} \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 dx + \frac{q(3-2s)-6}{2q} \mu \int_{\mathbb{R}^3} |u_n|^q dx + o_n(1). \tag{5.18}$$

Since $\{u_n\} \subset S_{r,a}$ is bounded in $H^s_{rad}(\mathbb{R}^3)$, then by Lemma 3.6 and (5.18), we derive that $\{\alpha_n\}$ is bounded and $\lim_{n\to+\infty} \alpha_n = \alpha \in \mathbb{R}$. By a similar argument as in (4.32) and (4.33), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} T_1 \leq \int_{\mathbb{R}^3} |u_n|^q dx &\leq C(q,s) \|(-\Delta)^{\frac{s}{2}} u_n\|_2^{q\delta_{q,s}} \|u_n\|_2^{q(1-\delta_{q,s})} \\ &\leq C(q,s) R_2^{q\delta_{q,s}} a^{q(1-\delta_{q,s})}, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 dx \leq \Gamma_t \|u_n\|^4_{\frac{12}{3+2t}} &\leq \Gamma_t C\left(12/3+2t, s\right)^{\frac{3+2t}{3}} \|(-\Delta)^{\frac{s}{2}} u_n\|_2^{\frac{3-2t}{s}} \|u_n\|_2^{\frac{2t+4s-3}{s}} \\ &\leq \Gamma_t C\left(12/3+2t, s\right)^{\frac{3+2t}{3}} R_2^{\frac{3-2t}{s}} a^{\frac{2t+4s-3}{s}} \\ &:= T_2, \end{aligned} \tag{5.20}$$

where $T_2 = Q_2(s,t,R_2,a) > 0$. We define the positive constant

$$\lambda_1^* := \frac{2[6-q(3-2s)]\mu T_1}{q(2t+4s-3)T_2}. \tag{5.21}$$

Therefore, if $\lambda < \lambda_1^*$, we get

$$\lambda q(2t+4s-3)T_2 < 2[6-q(3-2s)]\mu T_1.$$

Hence, by (5.19),(5.20) we see that

$$\lambda \frac{2t+4s-3}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx < \frac{[6-q(3-2s)]\mu}{2q}\int_{\mathbb{R}^3}|u_n|^q dx. \tag{5.22}$$

Taking the limit in (5.21) as $n\to+\infty$, and applying Lemmas 3.3,3.6, we obtain

$$\lambda \frac{2t+4s-3}{4}\int_{\mathbb{R}^3}\phi_{u}^t u^2 dx < \frac{[6-q(3-2s)]\mu}{2q}\int_{\mathbb{R}^3}|u|^q dx. \tag{5.23}$$

Consequently, passing the limit in (5.18) as $n\to+\infty$, and using (5.23) we deduce that

$$s\alpha a^2 = \lambda \frac{2t+4s-3}{4}\int_{\mathbb{R}^3}\phi_{u}^t u^2 dx + \frac{q(3-2s)-6}{2q}\mu\int_{\mathbb{R}^3}|u|^q dx < 0.$$

Thus, we have that $\alpha<0$, if $\lambda<\lambda_1^*$ small. □

Lemma 5.7. *If* $2+\frac{4s}{3}<q<2_s^*$, *and inequality* (2.5) *holds, then there* $\lambda_2^*>0$, *such that* $c_\mu(a)<\frac{s}{3}S^{\frac{3}{2s}}$ *for* $\lambda<\lambda_2^*$ *small.*

Proof. From [8], we know that S defined in (3.1) is attained in $\mathbb{R}^3$ by functions

$$U_\varepsilon(x) = \frac{C(s)\varepsilon^{3-2s}}{(\varepsilon^2+|x|^2)^{\frac{3-2s}{2}}}$$

for any $\varepsilon>0$ and $C(s)$ being normalized constant such that

$$\|(-\Delta)^{\frac{s}{2}}U_\varepsilon\|_2^2 = \int_{\mathbb{R}^3}|U_\varepsilon|^{2_s^*}dx = S^{\frac{3}{2s}}.$$

We define $u_\varepsilon=\varphi U_\varepsilon$, and

$$v_\varepsilon = a\frac{u_\varepsilon}{\|u_\varepsilon\|_2}\in S_a\cap H^s_{rad}(\mathbb{R}^3),$$

where $\varphi(x)\in C_0^\infty(B_2(0))$ is a radial cutoff function such that $0\le\varphi(x)\le 1$ and $\varphi(x)\equiv 1$ on $B_1(0)$. From Proposition 21 and Proposition 22 in [28], we have

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx = S^{\frac{3}{2s}}+O(\varepsilon^{3-2s}). \tag{5.24}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx = S^{\frac{3}{2s}}+O(\varepsilon^{3}). \tag{5.25}$$

For any $p>1$, by a direct computation [31], we obtain the following estimations:

$$\int_{\mathbb{R}^N}|u_\varepsilon|^p dx = \begin{cases} O(\varepsilon^{\frac{3(2-p)+2sp}{2}}), & \text{if } p>\frac{3}{3-2s};\\ O(\varepsilon^{\frac{3}{2}}|\log\varepsilon|), & \text{if } p=\frac{3}{3-2s};\\ O(\varepsilon^{\frac{(3-2s)p}{2}}), & \text{if } p<\frac{3}{3-2s}, \end{cases} \tag{5.26}$$

and especially,

$$\int_{\mathbb{R}^3}|u_\varepsilon|^2 dx = \begin{cases} C\varepsilon^{2s}, & \text{if } 0<s<\frac{3}{4};\\ C\varepsilon^{2s}|\log\varepsilon|, & \text{if } s=\frac{3}{4};\\ C\varepsilon^{3-2s}, & \text{if } \frac{3}{4}<s<1. \end{cases} \tag{5.27}$$

Define the function

$$\begin{aligned}\Psi^{\mu}_{v_\varepsilon}(\theta) := I_\mu((\theta\star v_\varepsilon)) = \frac{e^{2s\theta}}{2}\|v_\varepsilon\|^2 + \frac{e^{(3-2t)\theta}}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx - \frac{\mu}{q}e^{\frac{3(q-2)}{2}\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx \\ -\frac{1}{2^*_s}e^{2^*_s s\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*_s}dx,\end{aligned}\tag{5.28}$$

then it is easy to see that $\Psi^{\mu}_{v_\varepsilon}(\theta)\to 0^+$ as $\theta\to-\infty$, and $\Psi^{\mu}_{v_\varepsilon}(\theta)\to-\infty$ as $\theta\to+\infty$. Therefore, $\Psi^{\mu}_{v_\varepsilon}$ can obtain its global positive maximum at some $\theta_{\varepsilon,\mu}>0$. A direct computation yields that

$$\begin{aligned}&(\Psi^{\mu}_{v_\varepsilon})'(\theta)\\ &= se^{2s\theta}\|v_\varepsilon\|^2 + \frac{3-2t}{4}e^{(3-2t)\theta}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx \\ &\quad -\frac{3\mu(q-2)}{2q}e^{\frac{3(q-2)}{2}\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx - se^{2^*_s s\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*_s}dx \\ &= s\|\theta\star v_\varepsilon\|^2 + \frac{3-2t}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{\theta\star v_\varepsilon}|\theta\star v_\varepsilon|^2dx - \frac{3\mu(q-2)}{2q}\int_{\mathbb{R}^3}|\theta\star v_\varepsilon|^qdx - s\int_{\mathbb{R}^3}|\theta\star v_\varepsilon|^{2^*_s}dx \\ &= P_\mu(\theta\star v_\varepsilon);\end{aligned}\tag{5.29}$$

and

$$\begin{aligned}(\Psi^{\mu}_{v_\varepsilon})''(\theta) = 2s^2e^{2s\theta}\|v_\varepsilon\|^2 + \frac{(3-2t)^2}{4}e^{(3-2t)\theta}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx \\ -\mu qs^2\delta^2_{q,s}e^{\frac{3(q-2)}{2}\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx - 2^*_s s^2e^{2^*_s s\theta}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*_s}dx.\end{aligned}$$

Let $\theta_{\varepsilon,\mu}$ be the maximum point of $\Psi^{\mu}_{v_\varepsilon}(\theta)$, then $\theta_{\varepsilon,\mu}$ is unique. In fact, combining with $(\Psi^{\mu}_{v_\varepsilon})'(\theta_{\varepsilon,\mu})=0$, and $3-2t-2s<0, 2-q\delta_{q,s}<0, 2-2^*_s<0$, we have

$$\begin{aligned}&(\Psi^{\mu}_{v_\varepsilon})''(\theta_{\varepsilon,\mu})\\ &= 2s^2e^{2s\theta_{\varepsilon,\mu}}\|v_\varepsilon\|^2 + \frac{(3-2t)^2}{4}e^{(3-2t)\theta_{\varepsilon,\mu}}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx \\ &\quad -\mu qs^2\delta^2_{q,s}e^{\frac{3(q-2)}{2}\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx - 2^*_s s^2e^{2^*_s s\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*_s}dx \\ &= 2s^2\|\widetilde{u}_\varepsilon\|^2 + \frac{(3-2t)^2}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{\widetilde{u}_\varepsilon}\widetilde{u}_\varepsilon^2dx - \mu s^2q\delta^2_{q,s}\int_{\mathbb{R}^3}|\widetilde{u}_\varepsilon|^qdx - 2^*_s s^2\int_{\mathbb{R}^3}|\widetilde{u}_\varepsilon|^{2^*_s}dx \\ &= \frac{(3-2t)(3-2t-2s)}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{\widetilde{u}_\varepsilon}\widetilde{u}_\varepsilon^2dx + \mu s^2\delta_{q,s}[2-q\delta_{q,s}]\int_{\mathbb{R}^3}|\widetilde{u}_\varepsilon|^qdx + s^2[2-2^*_s]\int_{\mathbb{R}^3}|\widetilde{u}_\varepsilon|^{2^*_s}dx < 0,\end{aligned}$$

where $\widetilde{u}_\varepsilon = \theta_{\varepsilon,\mu}\star v_\varepsilon$, and the uniqueness of $\theta_{\varepsilon,\mu}$ follows. Using $(\Psi^{\mu}_{v_\varepsilon})'(\theta_{\varepsilon,\mu}) = P_\mu(\theta_{\varepsilon,\mu}\star v_\varepsilon)=0$ again, we have

$$\begin{aligned}se^{2^*_s s\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^{2^*_s}dx &= se^{2s\theta_{\varepsilon,\mu}}\|v_\varepsilon\|^2 + \lambda\frac{3-2t}{4}e^{(3-2t)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx \\ &\quad -\frac{3\mu(q-2)}{2q}e^{\frac{3(q-2)}{2}\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^qdx \\ &\le se^{2s\theta_{\varepsilon,\mu}}\|v_\varepsilon\|^2 + \lambda\frac{3-2t}{4}e^{(3-2t)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx \\ &= e^{2s\theta_{\varepsilon,\mu}}\left(s\|v_\varepsilon\|^2 + \lambda\frac{3-2t}{4}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx\right) \\ &\le e^{2s\theta_{\varepsilon,\mu}}2\max\left\{s\|v_\varepsilon\|^2, \lambda\frac{3-2t}{4}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v_\varepsilon^2dx\right\}.\end{aligned}\tag{5.30}$$

In the sequel, we distinguish the following two possible cases.

Case 1. $s\|v_\varepsilon\|^2 > \lambda\frac{3-2t}{4}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2 dx$.

In this case, we have from (5.30) that

$$se^{2_s^* s\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx < e^{2s\theta_{\varepsilon,\mu}}2s\|v_\varepsilon\|^2 \Longrightarrow e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \le \frac{2\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}, \tag{5.31}$$

and from $(\Psi_{v_\varepsilon}^\mu)'(\theta_{\varepsilon,\mu}) = 0$, we have

$$\begin{aligned}
&e^{(2_s^*-2)s\theta_{\varepsilon,\mu}}\\
&= \frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}} + \lambda\frac{3-2t}{4s}\frac{e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2 dx}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}} - \mu\delta_{q,s}e^{(q\delta_{q,s}-2)s\theta_{\varepsilon,\mu}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&\ge \frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}} - \mu\delta_{q,s}\left(\frac{2\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\right)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&= \frac{\|u_\varepsilon\|_2^{2_s^*-2}\|u_\varepsilon\|^2}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}} - \mu\delta_{q,s}\left(\frac{2\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\frac{\|u_\varepsilon\|^2}{\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\right)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\frac{\|u_\varepsilon\|_2^{2_s^*-q}}{a^{2_s^*-q}}\\
&= \frac{\|u_\varepsilon\|_2^{2_s^*-2}(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\left[(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}} - \frac{\mu\delta_{q,s}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}}a^{q(1-\delta_{q,s})}\|u_\varepsilon\|_q^q}{(\|u_\varepsilon\|_2)^{q(1-\delta_{q,s})}(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}\right].
\end{aligned} \tag{5.32}$$

Notice that, by (5.24)-(5.27), there exist positive constants C_1, C_2 and C_3 depending on s and q such that

$$(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}} \ge C_1, \quad C_2 \le (\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q\delta_{q,s}-2}{2_s^*-2}} \le \frac{1}{C_2}. \tag{5.33}$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} = \begin{cases} C_3\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})} = C_3, & \text{if } 0<s<\frac{3}{4};\\ C_3|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}}, & \text{if } s=\frac{3}{4};\\ C_3\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}}, & \text{if } \frac{3}{4}<s<1; \end{cases} \tag{5.34}$$

Next, we show that

$$e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \ge C\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}, \tag{5.35}$$

under suitable conditions. To this aim, we distinguish the following three subcases.

Subcase (i). $0 < s < \frac{3}{4}$. In this case, it holds that

$$3 - \frac{3-2s}{2}q - sq(1-\delta_{q,s}) = 0, \tag{5.36}$$

and from (5.32)-(5.34) we have

$$e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \ge \frac{C\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\left[C_1 - \mu\delta_{q,s}a^{q(1-\gamma_{q,s})}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\frac{C_3}{C_2}\right],$$

and we see that inequality (5.35) holds only when $\mu\gamma_{q,s}a^{q(1-\delta_{q,s})} < C_1C_2(C_3)^{-1}2^{-\frac{q\delta_{q,s}-2}{2_s^*-2}}$. Thus, we have to give a more precise estimate, let us come back to (5.32) and observe that by well-known interpolation inequality, we have

$$\frac{\|u_\varepsilon\|_q^q}{(\|u_\varepsilon\|_2)^{q(1-\delta_{q,s})}(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q\delta_{q,s}-2}{2_s^*-2}}} \le \frac{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q-2}{2_s^*-2}}(\|u_\varepsilon\|_2^2)^{\frac{2_s^*-q}{2_s^*-2}}}{(\|u_\varepsilon\|_2)^{q(1-\delta_{q,s})}(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q\delta_{q,s}-2}{2_s^*-2}}} = (\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}. \tag{5.37}$$

Therefore, by (5.37) and (5.32) we have

$$e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \ge \frac{\|u_\varepsilon\|_2^{2_s^*-2}(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\left[(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}} - \mu\delta_{q,s}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}}a^{q(1-\delta_{q,s})}(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}\right]. \tag{5.38}$$

From the estimations (5.24),(5.25), we see that the right hand side of (5.38) is positive provided that

$$\begin{aligned}\mu\delta_{q,s}a^{q(1-\gamma_{q,s})}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}} &< \frac{\left(\|u_\varepsilon\|^2\right)^{\frac{2_s^*-q\gamma_{q,s}}{2_s^*-2}}}{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}}\\ &= \frac{\left(S^{\frac{3}{2s}}+O(\varepsilon^{3-2s})\right)^{\frac{2_s^*-q\gamma_{q,s}}{2_s^*-2}}}{\left(S^{\frac{3}{2s}}+O(\varepsilon^{3})\right)^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}} = S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}+O(\varepsilon^{3-2s}).\end{aligned}$$

Therefore, if $0<s<\frac{3}{4}$ and

$$\mu\delta_{q,s}a^{q(1-\delta_{q,s})}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}} < S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}, \tag{5.39}$$

we have

$$e^{(2_s^*-2)s\theta_{\tilde v_\varepsilon}} \ge \frac{C\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}.$$

Subcase (ii). $s=\frac{3}{4}$. In this case, then we have $3<q<4$, and

$$|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = |\ln\varepsilon|^{\frac{q-2_s^*}{4s(3-2s)}} \to 0 \ \text{ as } \ \varepsilon\to 0.$$

Consequently,

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C_3\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = o_\varepsilon(1).$$

Therefore, we get

$$e^{(2_s^*-2)s\theta_{v_\varepsilon}} \ge C\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\left[C_1-\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}2^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\frac{C_3}{C_2}o_\varepsilon(1)\right] \ge \frac{C\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}.$$

Subcase (iii). $\frac{3}{4}<s<1$. By the definition of $\delta_{q,s}$ and a direct computation we infer to

$$\begin{aligned}&3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}\\ &=(3-2s)\left[\frac{3}{3-2s}-q-\frac{3(q-2)}{4s}\right] = \frac{3-4s}{4s}\left[q-\frac{6}{3-2s}\right](3-2s)>0.\end{aligned}$$

Thus, $\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}} \to 0$ as $\varepsilon\to 0$, and so

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}} = o_\varepsilon(1).$$

Therefore, we conclude that,

$$e^{(2_s^*-2)s\theta_{v_\varepsilon}} \geq C\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\left[C_1-\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}o_\varepsilon(1)\right] \geq \frac{C\|u_\varepsilon\|_2^{22_{\alpha,s}^*-2}}{a^{2_s^*-2}}.$$

Case 2. $s\|v_\varepsilon\|^2 \leq \lambda\frac{3-2t}{4}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2dx.$

In this case, we have from (5.30) that $se^{2_s^*s\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}|v_\varepsilon|^{2_s^*}dx < e^{2s\theta_{\varepsilon,\mu}}\frac{3-2t}{2}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\lambda\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2dx$, which implies that

$$e^{(s2_s^*+2t-3)\theta_{\varepsilon,\mu}} \leq \frac{3-2t}{2s}\frac{\lambda\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2dx}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}, \tag{5.40}$$

and from $(\Psi_{v_\varepsilon}^\mu)'(\theta_{\varepsilon,\mu})=0$ and (5.40), together with (3.2)-(3.3) and Hölder inequality, we induce that

$$\begin{aligned}
&e^{(2_s^*-2)s\theta_{\varepsilon,\mu}}\\
&=\frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}+\frac{3-2t}{4s}\frac{e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\lambda\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^t v_\varepsilon^2dx}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}e^{(q\delta_{q,s}-2)s\theta_{\varepsilon,\mu}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&\geq\frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}\left(\frac{3-2t}{2s}\frac{\lambda\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&\geq\frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}\left(\frac{3-2t}{2s}\frac{\lambda\Gamma_t\|v_\varepsilon\|_{\frac{12}{3+2t}}^4}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&\geq\frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}\left(\frac{(3-2t)\lambda\Gamma_t}{2s}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\left(\frac{\|v_\varepsilon\|_2^{4\tau}\|v_\varepsilon\|_{2_s^*}^{4(1-\tau)}}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}\\
&=\frac{\|v_\varepsilon\|^2}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}D(s,t)a^{\frac{4\tau(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\left(\frac{1}{\|v_\varepsilon\|_{2_s^*}^{2_s^*-4(1-\tau)}}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\frac{\|v_\varepsilon\|_q^q}{\|v_\varepsilon\|_{2_s^*}^{2_s^*}} \qquad (5.41)\\
&=\frac{\|u_\varepsilon\|_2^{2_s^*-2}\|u_\varepsilon\|^2}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}-\mu\delta_{q,s}D(s,t)a^{\frac{4\tau(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\\
&\qquad\times\left(\frac{\|u_\varepsilon\|_2^{2_s^*-4(1-\tau)}}{a^{2_s^*-4(1-\tau)}\|u_\varepsilon\|_{2_s^*}^{2_s^*-4(1-\tau)}}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\frac{\|u_\varepsilon\|_2^{2_s^*-q}}{a^{2^*-q}}\\
&=\frac{\|u_\varepsilon\|_2^{2_s^*-2}\|u_\varepsilon\|^2}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}-\frac{\mu\delta_{q,s}D(s,t)a^{\frac{4\tau(q\delta_{q,s}-2)s-(2_s^*-4(1-\tau))(q\delta_{q,s}-2)s}{s2_s^*+2t-3}-2_s^*+q}}{\|u_\varepsilon\|_{2_s^*}^{2_s^*+[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}}\\
&\qquad\times\|u_\varepsilon\|_2^{2_s^*-q+[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\|u_\varepsilon\|_q^q\\
&=\frac{\|u_\varepsilon\|_2^{2_s^*-2}(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}
\end{aligned}$$

$$\times\left[(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}}-\frac{\mu\delta_{q,s}D(s,t)a^{\frac{4\tau(q\delta_{q,s}-2)s-(2_s^*-4(1-\tau))(q\delta_{q,s}-2)s}{s2_s^*+2t-3}+q-2}}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\|u_\varepsilon\|_{2_s^*}^{[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}}\times\|u_\varepsilon\|_2^{2-q+[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}\|u_\varepsilon\|_q^q\right],$$

where $0<\tau=\frac{2t+4s-3}{4s}<1$, and

$$D(s,t)=\left(\frac{(3-2t)\lambda\Gamma_t}{2s}\right)^{\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}}.$$

By a direct computation, we have the following clearer expressions

$$\begin{aligned}[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}&=\left[2_s^*-4\left(1-\frac{2t+4s-3}{4s}\right)\right]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}\\&=\left[2_s^*-\frac{3-2t}{s}\right]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}=q\delta_{q,s}-2;\end{aligned}\tag{5.42}$$

$$2-q+[2_s^*-4(1-\tau)]\frac{(q\delta_{q,s}-2)s}{s2_s^*+2t-3}=2-q+q\delta_{q,s}-2=(\delta_{q,s}-1)q;\tag{5.43}$$

and

$$\begin{aligned}&\frac{4\tau(q\delta_{q,s}-2)s-(2_s^*-4(1-\tau))(q\delta_{q,s}-2)s}{s2_s^*+2t-3}+q-2\\&=\frac{s(q\delta_{q,s}-2)(4-2_s^*)}{s2_s^*+2t-3}+q-2\\&=\frac{1}{s2_s^*+2t-3}\left[(q-2)(s2_s^*+2t-3)-(2_s^*-4)s(q\delta_{q,s}-2)\right]\\&=\frac{1}{s2_s^*+2t-3}\left[(q-2)(s2_s^*+2t-3)-(2_s^*-4)\left(\frac{3(q-2)}{2}-2s\right)\right]\\&=\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}>0,\end{aligned}\tag{5.44}$$

where the last inequality holds true since $q\in(2+\frac{4s}{3},2_s^*), 2s+2t>3$. Consequently, we have

$$\begin{aligned}(q-2)2t+2s(2_s^*-4)&>\frac{4s}{3}2t+2s(2_s^*-4)\\&=2s\left(\frac{4t}{3}+2_s^*-4\right)=2s\frac{24s+12t-18-8st}{3(3-2s)}>0.\end{aligned}$$

Substituting formulas (5.42)-(5.44) into (5.41), we infer to

$$\begin{aligned}e^{(2_s^*-2)s\theta_{\varepsilon,\mu}}\geq&\frac{\|u_\varepsilon\|_2^{2_s^*-2}(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}\\&\times\left[(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}}-\frac{\mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}}}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\|u_\varepsilon\|_{2_s^*}^{q\delta_{q,s}-2}}\times\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\delta_{q,s})}}\right].\end{aligned}\tag{5.45}$$

Notice that, by (5.24)-(5.27), there exist positive constants C_4, C_5 and C_6 depending on s and q such that

$$(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\geq C_4,\quad\frac{1}{C_5}\leq\|u_\varepsilon\|_{2_s^*}^{q\delta_{q,s}-2}\leq C_5.\tag{5.46}$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} = \begin{cases} C_6\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})} = C_6, & \text{if } \ 0 < s < \frac{3}{4}; \\ C_6|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}}, & \text{if } \ s = \frac{3}{4}; \\ C_6\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}}, & \text{if } \ \frac{3}{4} < s < 1; \end{cases} \tag{5.47}$$

Next, we show that

$$e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \geq C\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}, \tag{5.48}$$

for some positive constant $C > 0$. To obtain the estimation (5.48), as in Case 1, we have to consider the three cases: (i) $0 < s < \frac{3}{4}$; (ii) $s = \frac{3}{4}$; and (iii) $\frac{3}{4} < s < 1$.

When $0 < s < \frac{3}{4}$, it holds that

$$3 - \frac{3-2s}{2}q - sq(1-\delta_{q,s}) = 0, \tag{5.49}$$

and from (5.45)-(5.47) we have

$$e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} \geq \frac{C\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\left[C_1 - \mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}}\frac{C_6}{C_4C_5}\right],$$

and we see that inequality (5.48) holds only when $\mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}} < C_1C_4C_5C_6^{-1}$. Thus, we have to give a more precise estimate, let us come back to (5.45) and observe that by well-known interpolation inequality, we have

$$\begin{aligned} \frac{\|u_\varepsilon\|_q^q}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}\|u_\varepsilon\|_{2_s^*}^{q\delta_{q,s}-2}\|u_\varepsilon\|_2^{q(1-\delta_{q,s})}} &\leq \frac{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q-2}{2_s^*-2}}(\|u_\varepsilon\|_2^2)^{\frac{2_s^*-q}{2_s^*-2}}}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}(\|u_\varepsilon\|_2)^{q(1-\delta_{q,s})}(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q\delta_{q,s}-2}{2_s^*-2}}} \\ &= \frac{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}. \end{aligned} \tag{5.50}$$

Therefore, by (5.45) and (5.50) we derive as

$$\begin{aligned} e^{(2_s^*-2)s\theta_{\varepsilon,\mu}} &\geq \frac{\|u_\varepsilon\|_2^{2_s^*-2}(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}{a^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}} \\ &\quad \times \left[(\|u_\varepsilon\|^2)^{\frac{2_s^*-q\delta_{q,s}}{2_s^*-2}} - \mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}}\frac{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}}{(\|u_\varepsilon\|^2)^{\frac{q\delta_{q,s}-2}{2_s^*-2}}}\right]. \end{aligned} \tag{5.51}$$

We observe that the right hand side of (5.51) is positive provided that

$$\begin{aligned} \mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}} &< \frac{\|u_\varepsilon\|^2}{(\|u_\varepsilon\|_{2_s^*}^{2_s^*})^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}} \\ &= \frac{S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(S^{\frac{3}{2s}} + O(\varepsilon^3)\right)^{\frac{q(1-\delta_{q,s})}{2_s^*-2}}} = S^{\frac{3[(2_s^*-2)-q(1-\delta_{q,s})]}{2s(2_s^*-2)}} + O(\varepsilon^{3-2s}). \end{aligned}$$

Therefore, if $0 < s < \frac{3}{4}$ and

$$\mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}} < S^{\frac{3[(2_s^*-2)-q(1-\delta_{q,s})]}{2s(2_s^*-2)}}, \tag{5.52}$$

we see that (5.48) holds for some constant $C > 0$.

For the cases: $s = \frac{3}{4}$, and $\frac{3}{4} < s < 1$, we still have the following estimations as in **Case 1**,

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C_3\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = o_\varepsilon(1);$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}} = o_\varepsilon(1),$$

respectively. Moreover, we derive that

$$e^{(2_s^*-2)s\theta_{\tilde{v}_\varepsilon}} \ge C\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\left[C_1 - \mu\delta_{q,s}D(s,t)a^{\frac{(q-2)2t+2s(2_s^*-4)}{s2_s^*+2t-3}}\frac{C_5}{C_4}o_\varepsilon(1)\right] \ge \frac{C\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}. \tag{5.53}$$

To sum up, condition (2.5) can ensure that (5.39), (5.52) occur, so as to guarantee (5.53) hold.

In what follows we focus on an upper estimate of $\max_{\theta\in\mathbb{R}}\Psi^\mu_{v_\varepsilon}(\theta)$. We split the argument into two steps.

Step 1. We estimate for $\max_{\theta\in\mathbb{R}}\Psi^0_{v_\varepsilon}(\theta)$, where,

$$\Psi^0_{v_\varepsilon}(\theta) := \frac{e^{2s\theta}}{2}\|v_\varepsilon\|^2 - \frac{e^{2_s^*s\theta}}{2_s^*}\int_{\mathbb{R}^N}|v_\varepsilon|^{2_s^*}dx.$$

It is easy to see that for every $v_\varepsilon \in S_{r,a}$ the function $\Psi^0_{v_\varepsilon}(\theta)$ has a unique critical point $\theta_{\varepsilon,0}$, which is a strict maximum point and is given by

$$e^{s\theta_{\varepsilon,0}} = \left(\frac{\|v_\varepsilon\|^2}{\int_{\mathbb{R}^N}|v_\varepsilon|^{2_s^*}dx}\right)^{\frac{1}{2_s^*-2}}. \tag{5.54}$$

Using the fact that

$$\sup_{\theta\ge 0}\left(\frac{\theta^2}{2}a - \frac{\theta^{2_s^*}}{2_s^*}b\right) = \frac{s}{3}\left(\frac{a}{b^{2/2_s^*}}\right)^{\frac{2_s^*}{2_s^*-2}},$$

for any fixed $a, b > 0$. We can deduce by (5.24), (5.25), that

$$\begin{aligned}\Psi^0_{v_\varepsilon}(\theta_{\varepsilon,0}) &= \frac{s}{3}\left(\frac{\|v_\varepsilon\|^2}{(\int_{\mathbb{R}^N}|v_\varepsilon|^{2_s^*}dx)^{\frac{2}{2_s^*}}}\right)^{\frac{2_s^*}{2_s^*-2}} = \frac{s}{3}\left(\frac{\|u_\varepsilon\|^2}{(\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx)^{\frac{2}{2_s^*}}}\right)^{\frac{2_s^*}{2_s^*-2}}\\ &= \frac{s}{3}\left(\frac{S^{\frac{3}{2s}}+O(\varepsilon^{3-2s})}{(S^{\frac{3}{2s}}+O(\varepsilon^3))^{\frac{2}{2_s^*}}}\right)^{\frac{2_s^*}{2_s^*-2}} = \frac{s}{3}S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}).\end{aligned} \tag{5.55}$$

Step 2. We next estimate for $\max_{\theta\in\mathbb{R}}\Psi^{\mu}_{v_\varepsilon}(t)$. Recall (3.3), (5.30) and Hölder inequality, we have

$$
\begin{aligned}
&e^{(2^*_s-2)s\theta_{\varepsilon,\mu}}\\
&\leq \frac{2\max\left\{\|v_\varepsilon\|^2,\lambda^{\frac{3-2t}{4s}}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v^2_\varepsilon dx\right\}}{\|v_\varepsilon\|^{2^*_s}_{2^*_s}}\\
&\leq \frac{2\max\left\{\|v_\varepsilon\|^2,\lambda^{\frac{3-2t}{4s}}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\Gamma_t\|v_\varepsilon\|^{4\tau}_2\|v_\varepsilon\|^{4(1-\tau)}_{2^*_s}\right\}}{\|v_\varepsilon\|^{2^*_s}_{2^*_s}}\\
&= \frac{2\max\left\{a^2\|u_\varepsilon\|^2\|u_\varepsilon\|^{2^*_s-2}_2,\lambda^{\frac{3-2t}{4s}}e^{(3-2t-2s)\theta_{\varepsilon,\mu}}\Gamma_t a^4\|u_\varepsilon\|^{4(1-\tau)}_{2^*_s}\|u_\varepsilon\|^{2^*_s-4(1-\tau)}_2\right\}}{a^{2^*_s}\|u_\varepsilon\|^{2^*_s}_{2^*_s}}.
\end{aligned}\tag{5.56}
$$

From the estimations (5.24)-(5.25) and (5.56), we see that the number $\theta_{\varepsilon,\mu}$ can not go to $+\infty$, and there exists some $\theta^*\in\mathbb{R}$ such that

$$\theta_{\varepsilon,\mu}\leq\theta^*,\quad \text{for all }\ \varepsilon,\mu>0.\tag{5.57}$$

Hence, by virtue of (5.56), (5.57) and (3.3) we derive to

$$
\begin{aligned}
&\max_{\theta\in\mathbb{R}}\Psi^{\mu}_{v_\varepsilon}(\theta)\\
&=\Psi^{\mu}_{v_\varepsilon}(\theta_{\varepsilon,\mu})=\Psi^{0}_{v_\varepsilon}(\theta_{\varepsilon,\mu})+\frac{e^{(3-2t)\theta_{\varepsilon,\mu}}}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v^2_\varepsilon dx-\mu\frac{e^{q\gamma_{q,s}s\theta_{\varepsilon,\mu}}}{q}\int_{\mathbb{R}^N}|v_\varepsilon|^q dx\\
&\leq\sup_{\theta\in\mathbb{R}}\Psi^{0}_{v_\varepsilon}(\theta)+\frac{e^{(3-2t)\theta_{\varepsilon,\mu}}}{4}\lambda\int_{\mathbb{R}^3}\phi^t_{v_\varepsilon}v^2_\varepsilon dx-\mu\frac{e^{q\gamma_{q,s}s\theta_{\varepsilon,\mu}}}{q}\int_{\mathbb{R}^N}|v_\varepsilon|^q dx\\
&\leq\Psi^{0}_{v_\varepsilon}(\theta_{v_{\varepsilon,0}})+C\lambda\left(\int_{\mathbb{R}^3}|v_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}-\frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q dx}{\|u_\varepsilon\|^{q(1-\gamma_{q,s})}_2}\\
&\leq\frac{s}{3}S^{\frac{3}{2s}}+O(\varepsilon^{3-2s})+C\frac{\lambda a^4}{\|u_\varepsilon\|^4_2}\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}-\frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q dx}{\|u_\varepsilon\|^{q(1-\gamma_{q,s})}_2}\\
&\leq\frac{s}{3}S^{\frac{3}{2s}}+C_1\varepsilon^{3-2s}+C_2\lambda\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\|u_\varepsilon\|^4_2}-C_3\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q dx}{\|u_\varepsilon\|^{q(1-\gamma_{q,s})}_2}.
\end{aligned}\tag{5.58}
$$

Next, we separate three cases:

Case 1: $0<s<\frac{3}{4}$. In this case, owing to $2t+8s<9$, we get $p=\frac{12}{3+2t}>\frac{3}{3-2s}$, it following from (5.26)-(5.27) and (5.34) that,

$$
\begin{aligned}
&\frac{s}{3}S^{\frac{3}{2s}}+C_1\varepsilon^{3-2s}+C_2\lambda\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\|u_\varepsilon\|^4_2}-C_3\frac{\int_{\mathbb{R}^3}|u_\varepsilon|^q dx}{\|u_\varepsilon\|^{q(1-\gamma_{q,s})}_2}\\
&=\frac{s}{3}S^{\frac{3}{2s}}+C_1\varepsilon^{3-2s}+C_2\lambda\frac{\varepsilon^{2t+4s-3}}{\varepsilon^{4s}}-C_3\\
&<\frac{s}{3}S^{\frac{3}{2s}},
\end{aligned}\tag{5.59}
$$

if we choose $\lambda=\varepsilon^s$.

Case 2: $s=\frac{3}{4}$. In this case, we still have $2t+8s=2t+6<9$, and also, $p=\frac{12}{3+2t}>\frac{3}{3-2s}$. Moreover, $2+\frac{q(\gamma_{q,s}-1)}{2}=\frac{q(3-2s)}{4s}>0$, hence

$$\varepsilon^{2t+2s-3}\to 0,\quad \varepsilon^{3-2s}(\log\varepsilon)^2\to 0,\quad\text{and}\quad |\ln\varepsilon|^{2+\frac{q(\gamma_{q,s}-1)}{2}}\to+\infty,$$

when $\varepsilon \to 0^+$. Consequently, if we choose $\lambda = \varepsilon^{2s}$, then we have

$$
\begin{aligned}
&\frac{s}{3}S^{\frac{3}{2s}} + C_1\varepsilon^{3-2s} + C_2\lambda\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\|u_\varepsilon\|_2^4} - C_3\frac{\int_{\mathbb{R}^3}|u_\varepsilon|^q dx}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \\
&= \frac{s}{3}S^{\frac{3}{2s}} + C_1\varepsilon^{3-2s} + C_2\lambda\frac{\varepsilon^{2t+4s-3}}{\varepsilon^{4s}|\log\varepsilon|^2} - C_3|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} \\
&= \frac{s}{3}S^{\frac{3}{2s}} + \frac{1}{(\log\varepsilon)^2}\left[C_1\varepsilon^{3-2s}(\log\varepsilon)^2 + C_2\varepsilon^{2t+2s-3} - C_3|\ln\varepsilon|^{2+\frac{q(\gamma_{q,s}-1)}{2}}\right] \\
&< \frac{s}{3}S^{\frac{3}{2s}},
\end{aligned}
\tag{5.60}
$$

when $\varepsilon > 0$ small enough.

Case 3: $\frac{3}{4} < s < 1$. In this case, using the fact that $2t + 2s > 3, q > 2 + \frac{4s}{3}$, we can obtain the inequality by a direct computation,

$$3 - \frac{3-2s}{2}q - \frac{(3-2s)q(1-\gamma_{q,s})}{2} < 3 - 2s.$$

Thus, from (5.26)-(5.27) and (5.34), letting $\lambda = \varepsilon^{6-4s}$ we derive that

$$
\begin{aligned}
&\frac{s}{3}S^{\frac{3}{2s}} + C_1\varepsilon^{3-2s} + C_2\lambda\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\|u_\varepsilon\|_2^4} - C_3\frac{\int_{\mathbb{R}^3}|u_\varepsilon|^q dx}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \\
&= \frac{s}{3}S^{\frac{3}{2s}} + C_1\varepsilon^{3-2s} + C_2\begin{cases} \lambda\dfrac{\varepsilon^{2t+4s-3}}{\varepsilon^{6-4s}}, & \text{if } \dfrac{12}{3+2t} > \dfrac{3}{3-2s}, \\ \lambda\dfrac{\varepsilon^{2t+4s-3}|\ln\varepsilon|^{\frac{3+2t}{3}}}{\varepsilon^{6-4s}}, & \text{if } \dfrac{12}{3+2t} = \dfrac{3}{3-2s}, \\ \lambda\dfrac{\varepsilon^{2(3-2s)}}{\varepsilon^{6-4s}}, & \text{if } \dfrac{12}{3+2t} < \dfrac{3}{3-2s} \end{cases} \\
&\quad - C_3\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}-(3-2s)} \\
&< \frac{s}{3}S^{\frac{3}{2s}}.
\end{aligned}
\tag{5.61}
$$

Since $v_\varepsilon \in S_{r,a}$, from Lemma 5.1 we can take $\theta_1 < 0$ and $\theta_2 > 0$ such that $\theta_1 \star v_\varepsilon \in \mathcal{A}_a$ and $I_\mu(\theta_2 \star v_\varepsilon) < 0$, respectively. Then we can define a path

$$\gamma_{v_\varepsilon} : t \in [0,1] \mapsto ((1-t)\theta_1 + t\theta_2) \star v_\varepsilon \in \Gamma_a.$$

To sum up, by the estimations (5.58)-(5.61), we can derive that

$$c_{r,\mu}(a) \le \max_{t\in[0,1]} I_\mu(\gamma_{v_\varepsilon}(t)) \le \max_{\theta\in\mathbb{R}} \Psi^\mu_{v_\varepsilon}(\theta) < \frac{s}{3}S^{\frac{3}{2s}}, \tag{5.62}$$

for $\varepsilon > 0$ small enough, which is the desired result. □

Lemma 5.8. *Let $\{u_n\}$ be the (PS) sequence in $S_{r,a}$ at level $c_\mu(a)$, with $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$, assume that $u_n \rightharpoonup u$, then, $u \not\equiv 0$.*

Proof. Arguing by contradiction, we suppose that $u \equiv 0$. Noticing that $\{u_n\}$ is bounded in $H^s_{rad}(\mathbb{R}^3)$, going to a subsequence, we may assume that $\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 \to \ell \ge 0$. By Lemma 3.6, $u_n \to 0$ in

$L^p(\mathbb{R}^3), \forall p \in (2, 2_s^*)$. From Proposition 5.5 and Lemmas 3.3,3.6, we have $P_\mu(u_n) \to 0$ such that,

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx &= \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \frac{3-2t}{4s}\lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \mu\delta_{q,s} \int_{\mathbb{R}^3} |u_n|^q dx \\ &= \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + o_n(1) \\ &= \ell + o_n(1),\end{aligned}$$

as $n \to \infty$. Then, using Sobolev's inequality, one has $\ell \geq S\ell^{\frac{2}{2_s^*}}$, and so, either $\ell \geq S^{\frac{3}{2s}}$ or $\ell = 0$. In the case $\ell \geq S^{\frac{3}{2s}}$, from $I_\mu(u_n) \to c_\mu(a), P_\mu(u_n) \to 0$, we know

$$\begin{aligned}&c_\mu(a) + o_n(1) \\ &= I_\mu(u_n) = I_\mu(u_n) - \frac{1}{s2_s^*} P_\mu(u_n) \\ &= \frac{s}{3}\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \lambda \frac{s2_s^* + 2t - 3}{4s2_s^*} \int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 dx - \mu \frac{2_s^* - q}{q2_s^*} \int_{\mathbb{R}^3} |u_n|^q dx + o_n(1) \\ &= \frac{s}{3}\ell + o_n(1)\end{aligned}$$

which means $c_\mu(a) = \frac{s}{3}\ell$, that is $c_\mu(a) \geq \frac{s}{3}S^{\frac{3}{2s}}$, which contradicts the assumption $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$. In the case $\ell = 0$, one has

$$\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 \to 0, \quad \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to 0,$$

and combining with

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \to 0, \quad \int_{\mathbb{R}^3} |u_n|^q dx \to 0,$$

we have, $I_\mu(u_n) \to 0$, which is absurd since $c_\mu(a) > 0$. Therefore, $u \not\equiv 0$. □

Lemma 5.9. *Let $\{u_n\}$ be the (PS) sequence in $S_{r,a}$ at level $c_\mu(a)$, with $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$, assume that $P_\mu(u_n) \to 0$ when $n \to \infty$, and $\lambda < \lambda_1^*$ small. Then one of the following alternatives holds:*

(i) either going to a subsequence $u_n \rightharpoonup u$ weakly in $H_{rad}^s(\mathbb{R}^3)$, but not strongly, where $u \not\equiv 0$ is a solution to

$$(-\Delta)^s u + \lambda\phi_u^t u = \alpha u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3, \tag{5.63}$$

where $\alpha_n \to \alpha < 0$, and

$$I_\mu(u) < c_\mu(a) - \frac{s}{3}S^{\frac{3}{2s}};$$

(ii) or passing to a subsequence $u_n \to u$ strongly in $H_{rad}^s(\mathbb{R}^3), I_\mu(u) = c_\mu(a)$ and u is a solution of (1.5)-(1.6) for some $\alpha < 0$.

Proof. By Lemma 5.6, we have that $\{u_n\} \subset S_{r,a}$ is a bounded (PS) sequence for I_μ in $H_{rad}^s(\mathbb{R}^3)$, and so $u_n \rightharpoonup u$ in $H_{rad}^s(\mathbb{R}^3)$ for some u. By the Lagrange multiplier principle, there exists $\{\alpha_n\} \subset \mathbb{R}$ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx - \alpha_n \int_{\mathbb{R}^3} u_n \varphi dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u\varphi dx - \mu \int_{\mathbb{R}^3} |u_n|^{q-2} u_n \varphi dx \\ &\quad - \int_{\mathbb{R}^3} |u_n|^{2_s^*-2} u_n \varphi dx = o_n(1)\|\varphi\|,\end{aligned} \tag{5.64}$$

for any $\varphi \in H_{rad}^s(\mathbb{R}^3)$. Moreover, one has $\lim_{n\to\infty} \alpha_n = \alpha < 0$. Letting $n \to \infty$ in (5.64), we have

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx + \lambda \int_{\mathbb{R}^3} \phi_u^t u\varphi dx - \mu \int_{\mathbb{R}^3} |u|^{q-2} u\varphi dx - \int_{\mathbb{R}^3} |u|^{2_s^*-2} u\varphi dx - \alpha \int_{\mathbb{R}^3} u\varphi dx = 0,$$

which implies that u solves the equation

$$(-\Delta)^s u + \lambda\phi_u^t u = \alpha u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3, \tag{5.65}$$

and we have the Pohozǎev identity $P_\mu(u) = 0$.

Let $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $H^s_{rad}(\mathbb{R}^3)$. According to Brezis-Lieb lemma [33] and Lemma 3.3, one has

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 = \|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 + o_n(1), \quad \|u_n\|_{2_s^*}^{2_s^*} = \|u\|_{2_s^*}^{2_s^*} + \|v_n\|_{2_s^*}^{2_s^*} + o_n(1), \tag{5.66}$$

and

$$\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3}\phi_u u^2 dx + o_n(1), \quad \|u_n\|_q^q = \|u\|_q^q + \|v_n\|_q^q + o_n(1). \tag{5.67}$$

Then, from $P_\mu(u_n) \to 0, u_n \to u$ in $L^p(\mathbb{R}^3)$, one can derive that

$$\begin{aligned}&\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 + \frac{3-2t}{4s}\lambda\int_{\mathbb{R}^3}\phi_u^t u^2 dx\\ &= \mu\delta_{q,s}\int_{\mathbb{R}^3}|u|^q dx + \int_{\mathbb{R}^3}|u|^{2_s^*}dx + \int_{\mathbb{R}^3}|v_n|^{2_s^*}dx + o_n(1).\end{aligned}$$

By $P_\mu(u) = 0$, we have

$$\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 = \int_{\mathbb{R}^3}|v_n|^{2_s^*}dx + o_n(1). \tag{5.68}$$

Passing to a subsequence, we may assume that

$$\lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx = \ell \geq 0. \tag{5.69}$$

Then, it follows from Sobolev's inequality that $\ell \geq S\ell^{\frac{2}{2_s^*}}$, and so, either $\ell \geq S^{\frac{3}{2s}}$ or $\ell = 0$. In the case $\ell \geq S^{\frac{3}{2s}}$, from $I_\mu(u_n) \to c_\mu(a), P_\mu(u_n) \to 0$, we know

$$\begin{aligned}c_\mu(a) &= \lim_{n\to\infty} I_\mu(u_n) = \lim_{n\to\infty}\left\{I_\mu(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx + o_n(1)\right\}\\ &= I_\mu(u) + \frac{s}{3}\ell \geq I_\mu(u) + \frac{s}{3}S^{\frac{3}{2s}}\end{aligned} \tag{5.70}$$

which means that item (i) holds.

If $\ell = 0$, then $\|u_n - u\| = \|v_n\| \to 0$, one has $u_n \to u$ in $D^{s,2}(\mathbb{R}^3)$, and so $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^3)$. To prove that $u_n \to u$ in $H^s_{rad}(\mathbb{R}^3)$, it remains only to prove that $u_n \to u$ in $L^2(\mathbb{R}^3)$. Fix $\psi = u_n - u$ as a test function in (5.64), and $u_n - u$ as a test function of (5.65), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}(u_n-u)|^2dx - \int_{\mathbb{R}^3}(\alpha_n u_n - \alpha u)(u_n-u)dx + \lambda\int_{\mathbb{R}^3}(\phi_{u_n}^t u_n - \phi_u^t u)(u_n-u)dx\\ &= \mu\int_{\mathbb{R}^3}(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n-u)dx + \int_{\mathbb{R}^3}(|u_n|^{2_s^*-2}u_n - |u|^{2_s^*-2}u)(u_n-u)dx + o_n(1).\end{aligned} \tag{5.71}$$

Passing the limit in (5.71) as $n \to \infty$, we have

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^3}(\alpha_n u_n - \alpha u)(u_n-u)dx = \lim_{n\to\infty}\alpha\int_{\mathbb{R}^3}(u_n-u)^2dx,$$

and then $u_n \to u$ in $L^2(\mathbb{R}^3)$. Therefore, item (ii) holds. □

Now, we are ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\lambda < \Lambda^* := \min\{\lambda_1^*, \lambda_2^*\}$. By virtue of Lemmas 5.1-5.2,5.6-5.7, Propositions 5.3-5.5, there exists a bounded $(PS)_{c_\mu(a)}$-sequence $\{u_n\} \subset S_{r,a}$, with $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$, and $u \in H^s_{rad}(\mathbb{R}^3)$ such that one of the alternatives of Lemma 5.9 holds. We assert that (i) of Lemma 5.9 can not occur. Indeed, suppose by contradiction that, item (i) holds, then u is a nontrivial solution of (5.63), and by Lemma 5.9 and Lemma 5.7, we have

$$I_\mu(u) < c_\mu(a) - \frac{s}{3}S^{\frac{3}{2s}} < 0.$$

On the other hand, we have

$$\begin{aligned} I_\mu(u) &= I_\mu(u) - \frac{1}{2s}P_\mu(u) \\ &= \frac{2s+2t-3}{8}\lambda \int_{\mathbb{R}^3} \phi_u^t u^2 dx + \frac{q\delta_{q,s}-2}{2q}\mu \int_{\mathbb{R}^3} |u|^q dx + \frac{s}{3}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq 0, \end{aligned}$$

which leads to a contradiction. Therefore, $u_n \to u$ strongly in $H^s_{rad}(\mathbb{R}^3)$ with $I_\mu(u) = c_\mu(a)$, and u is a solution of (1.5)-(1.6) for some $\alpha < 0$. Moreover, $u(x) > 0$ in $\mathbb{R}^3$. In fact, we note that all the calculations above can be repeated word by word, replacing I_μ with the functional

$$I_\mu^+(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u^+|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u^+|^{2_s^*} dx. \tag{5.72}$$

Then u is the critical point of I_μ^+ restricted on the set $S_{r,a}$, it solves the equation

$$(-\Delta)^s u + \lambda\phi_u^t u = \alpha u + \mu|u^+|^{q-2}u + |u^+|^{2_s^*-2}u. \quad \text{in } \mathbb{R}^3, \tag{5.73}$$

Using $u^- = \min\{u, 0\}$ as a test function in (5.73), in view of $(a-b)(a^- - b^-) \geq |a^- - b^-|^2, \forall a, b \in \mathbb{R}$, we conclude that

$$\begin{aligned} &\|(-\Delta^{\frac{s}{2}})u^-\|_2^2 = \iint_{\mathbb{R}^6} \frac{|u^-(x) - u(y)|^2}{|x-y|^{3+2s}} dxdy \\ &\leq \|(-\Delta^{\frac{s}{2}})u^-\|_2^2 + \lambda\int_{\mathbb{R}^3} \phi_u^t |u^-|^2 dx - \alpha\int_{\mathbb{R}^3} |u^-|^2 dx \\ &\leq \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))((u^-(x) - u^-(y))}{|x-y|^{3+2s}} dxdy + \lambda\int_{\mathbb{R}^3} \phi_u^t |u^-|^2 dx - \alpha\int_{\mathbb{R}^3} |u^-|^2 dx \\ &= 0. \end{aligned}$$

Thus, $u^- = 0$ and $u \geq 0, \forall x \in \mathbb{R}^3$, is a solution of (5.73). By the regularity result [36] we know that $u \in L^\infty(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$. Suppose $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, then $(-\Delta)^s u(x_0) = 0$ and by the definition of $(-\Delta)^s$, we have [27]:

$$(-\Delta)^s u(x_0) = -\frac{C_s}{2}\int_{\mathbb{R}^3} \frac{u(x_0+y) + u(x_0-y) - 2u(x_0)}{|y|^{3+2s}} dy.$$

Hence, $\int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}} dy = 0$, which implies $u \equiv 0$, a contradiction. Thus, $u(x) > 0, \forall x \in \mathbb{R}^3$. □

6. Proof of Theorem 2.3

In this section, we deal with the L^2-supercritical case $2 + \frac{4s}{3} < q < 2_s^*$, when parameter $\mu > 0$ large. In view of $\frac{3(q-2)}{2s} > 2$, the truncated functional $I_{\mu,\tau}$ defined in Section 4 is still unbounded from below on $S_{r,a}$, and the truncation technique can not be applied to study problem (1.5)-(1.6).

To overcome this difficulty, as in Section 5 we introduce the transformation (e.g. [29]):

$$(\theta \star u)(x) := e^{\frac{3\theta}{2}} u(e^{\theta} x), \quad x \in \mathbb{R}^N, \quad \theta \in \mathbb{R}, \tag{6.1}$$

and the auxiliary functional

$$\begin{aligned} I(u,\theta) = I_\mu((\theta \star u)) =& \frac{e^{2s\theta}}{2}\|u\|^2 + \frac{\lambda e^{(3-2t)\theta}}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\mu}{q} e^{q\delta_{q,s}s\theta} \int_{\mathbb{R}^3} |u|^q dx \\ &- \frac{1}{2_s^*} e^{\frac{3(2_s^*-2)}{2}\theta} \int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned} \tag{6.2}$$

From Lemmas 5.1, 5.2, we have the the mountain pass level value $c_\mu(a)$ by

$$c_\mu(a) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) > 0,$$

where

$$\Gamma_a = \{\gamma \in C([0,1], S_{r,a}) : \ \gamma(0) \in A_a, \gamma(1) \in I_\mu^0\}.$$

In what follows, we set $g(t) = \mu|t|^{q-2}t + |u|^{2_s^*-2}u$, for any $t \in \mathbb{R}$. From Propositions 5.4,5.5, we know that there exist a $(PS)_{c_\mu(a)}$-sequence $\{u_n\} \subset S_{r,a}$ satisfying

$$I_\mu(u_n) \to c_\mu(a), \quad \|I'_\mu|_{S_{r,a}}(u_n)\| \to 0 \ \text{ and } P_\mu(u_n) \to 0, \quad \text{as} \quad n \to \infty,$$

where

$$P_\mu(u_n) = s\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{3-2t}{4}\lambda \int_{\mathbb{R}^3} \phi_u^t u^2 dx + 3\int_{\mathbb{R}^3} G(u_n) dx - \frac{3}{2}\int_{\mathbb{R}^3} g(u_n) u_n dx.$$

Similar to the Section 5, setting the functional $\Psi(v) : H^s_{rad}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\Psi(v) = \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 dx,$$

it follows that $S_{r,a} = \Psi^{-1}(\{\frac{a^2}{2}\})$, and by Proposition 5.12 in [33], there exists $\alpha_n \in \mathbb{R}$ such that

$$\|I'_\mu(u_n) - \alpha_n \Psi'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.$$

That is, we have

$$(-\Delta)^s u_n + \lambda \phi_{u_n}^t u_n - g(u_n) = \alpha_n u_n + o_n(1) \ \text{ in } H^{-s}_{rad}(\mathbb{R}^3). \tag{6.3}$$

Therefore, for any $\varphi \in H^s_{rad}(\mathbb{R}^3)$, one has

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \varphi dx - \int_{\mathbb{R}^3} g(u_n)\varphi dx = \alpha_n \int_{\mathbb{R}^3} u_n \varphi dx + o_n(1). \tag{6.4}$$

In the sequel, we study the asymptotical behavior of the mountain pass level value $c_\mu(a)$ as $\mu \to +\infty$, and the properties of the $(PS)_{c_\mu(a)}$-sequence $\{u_n\} \subset S_{r,a}$ as $n \to +\infty$.

Lemma 6.1. *The limit* $\lim_{\mu \to +\infty} c_\mu(a) = 0$ *holds.*

Proof. Recall Lemmas 5.1, 5.2, we see that for fixed $u_0 \in S_{r,a}$, there exists two constants θ_1, θ_2 satisfying $\theta_1 < 0 < \theta_2$ such that $u_1 := \theta_1 \star u_0 \in A$ and $I_\mu(u_2) < 0$. Then we can define a path

$$\eta_0 : \tau \in [0,1] \to ((1-\tau)\theta_1 + \tau\theta_2) \star u_0 \in \Gamma_a.$$

Thus, we have

$$\begin{aligned} c_\mu(a) &\le \max_{t \in [0,1]} I_\mu(\eta_0(t)) \\ &\le \max_{r \ge 0} \left\{ \frac{r^{2s}}{2}\|u_0\|^2 + \frac{r^{3-2t}}{4}\lambda \int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx - \frac{\mu}{q} r^{\frac{3q-6}{2}} \int_{\mathbb{R}^3} |u_0|^q dx \right\} \\ &:= \max_{r \ge 0} h(r). \end{aligned}$$

Note that $\frac{3q-6}{2} > 2s > 3-2t$, we have that $\lim_{r\to 0^+} h(r) = 0^+, \lim_{r\to+\infty} h(r) = -\infty$, and so, there exists a unique maximum point $r_0 > 0$ such that $\max_{r\geq 0} h(r) = h(r_0) > 0$. Hence, we distinguish two cases: $r_0 \geq 1$ and $0 \leq r_0 < 1$.

If $r_0 \geq 1$, we have by $2s+2t > 3$, that

$$\begin{aligned}
\max_{t\in[0,1]} I_\mu(\eta_0(t)) &\leq h(r_0)\\
&\leq \frac{r_0^{2s}}{2}\|u_0\|^2 + \frac{r_0^{2s}}{4}\lambda \int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx - \frac{\mu}{q} r_0^{\frac{3q-6}{2}} \int_{\mathbb{R}^3} |u_0|^q dx\\
&\leq \max_{r\geq 0}\left\{2\max\left\{\frac{1}{2}\|u_0\|^2, \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx\right\} r^{2s} - \frac{\mu}{q} r^{\frac{3q-6}{2}} \int_{\mathbb{R}^3} |u_0|^q dx\right\}\\
&= 2a(r_{max})^{2s} - \frac{\mu b}{q}(r_{max})^{\frac{3q-6}{2}}\\
&= \frac{2a(3q-6-4s)}{3q-6}\left[\frac{8qsa}{\mu b(3q-6)}\right]^{\frac{4s}{3q-6-4s}},
\end{aligned}$$

where

$$r_{max} = \left[\frac{8qsa}{\mu b(3q-6)}\right]^{\frac{4s}{3q-6-4s}},\ a = \max\left\{\frac{1}{2}\|u_0\|^2, \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx\right\},\ b = \int_{\mathbb{R}^3} |u_0|^q dx.$$

Therefore, for $2 + \frac{4s}{3} < q < 2_s^*$, we have a positive constant $\widetilde{C}$ independent of μ such that

$$\gamma_\mu(a) \leq \widetilde{C}\mu^{-\frac{4s}{3q-6-4s}} \to 0, \quad \text{as } \mu \to +\infty.$$

If $0 \leq r_0 < 1$, we infer to

$$\begin{aligned}
\max_{t\in[0,1]} I_\mu(\eta_0(t)) &\leq \frac{r_0^{2s}}{2}\|u_0\|^2 + \frac{r_0^{3-2t}}{4}\int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx - \frac{\mu}{q} r_0^{\frac{3q-6}{2}} \int_{\mathbb{R}^3} |u_0|^q dx\\
&\leq \max_{r\geq 0}\left\{2\max\left\{\frac{1}{2}\|u_0\|^2, \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_0}^t u_0^2 dx\right\} r^{3-2t} - \frac{\mu}{q} r^{\frac{3q-6}{2}} \int_{\mathbb{R}^3} |u_0|^q dx\right\}\\
&= 2a(\widetilde{r}_{max})^{3-2t} - \frac{\mu b}{q}(\widetilde{r}_{max})^{\frac{3q-6}{2}}\\
&= \frac{2a(3q+4t-12)}{3q-6}\left[\frac{4qa(3-2t)}{\mu b(3q-6)}\right]^{\frac{2(3-2t)}{3q+4t-12}},
\end{aligned}$$

where

$$\widetilde{r}_{max} = \left[\frac{4qa(3-2t)}{\mu b(3q-6)}\right]^{\frac{2}{3q+4t-12}}.$$

Since $2 + \frac{4s}{3} < q < 2_s^*$, and $2s + 2t > 3$, we can deduce that $3q + 4t - 12 > 0$, then there exists a positive constant C_1 independent of μ such that

$$c_\mu(a) \leq C_1\mu^{-\frac{2(3-2t)}{3q+4t-12}} \to 0, \quad \text{as } \mu \to +\infty.$$

This completes the proof. □

Lemma 6.2. *There exists a constant $C = C(q,s) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} G(u_n)dx \leq Cc_\mu(a),$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} g(u_n)u_n dx \leq Cc_\mu(a),$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx \le Cc_\mu(a),\ \limsup_{n\to\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx \le Cc_\mu(a).$$

Proof. Since $I_\mu(u_n)\to c_\mu(a)$ and $P_\mu(u_n)\to 0$ as $n\to\infty$, we have

$$\begin{aligned}
&3c_\mu(a)+o_n(1) = 3I_\mu(u_n)+P_\mu(u_n)\\
&= \frac{3+2s}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx + \lambda\frac{3-t}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx - \frac{3}{2}\int_{\mathbb{R}^3}g(u_n)u_ndx\\
&= \frac{3+2s}{2}\left(2c_\mu(a)-\frac{\lambda}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx + 2\int_{\mathbb{R}^3}G(u_n)dx + o_n(1)\right)\\
&\quad + \lambda\frac{3-t}{2}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx - \frac{3}{2}\int_{\mathbb{R}^3}g(u_n)u_ndx\\
&= (3+2s)\left[c_\mu(a)+\int_{\mathbb{R}^3}G(u_n)dx+o_n(1)\right] - \frac{3}{2}\int_{\mathbb{R}^3}g(u_n)u_ndx - \lambda\frac{2t+2s-3}{4}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx.
\end{aligned} \tag{6.5}$$

Hence,

$$\begin{aligned}
2sc_\mu(a)+o_n(1) &= \lambda\frac{2t+2s-3}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx + \frac{3}{2}\int_{\mathbb{R}^3}g(u_n)u_ndx - (3+2s)\int_{\mathbb{R}^3}G(u_n)dx\\
&\ge \frac{3q}{2}\int_{\mathbb{R}^3}G(u_n)dx - (3+2s)\int_{\mathbb{R}^3}G(u_n)dx\\
&= \frac{3q-2(3+2s)}{2}\int_{\mathbb{R}^3}G(u_n)dx,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}G(u_n)dx \le \frac{4s}{3q-2(3+2s)}c_\mu(a) \le Cc_\mu(a) \tag{6.6}$$

and then

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}g(u_n)u_ndx \le Cc_\mu(a). \tag{6.7}$$

Then, from (6.5)-(6.7), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{\frac{3+2s}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2dx + \lambda\frac{3-t}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\right\}\\
&= \limsup_{n\to\infty}\left\{3c_\mu(a)+\frac{3}{2}\int_{\mathbb{R}^3}g(u_n)u_ndx+o_n(1)\right\} \le Cc_\mu(a).
\end{aligned} \tag{6.8}$$

Consequently, the proof is completed. □

Lemma 6.3. *There exists $\mu_1^* := \mu_1^*(a) > 0$ such that $u \not\equiv 0$ for all $\mu \ge \mu_1^*$.*

Proof. From Lemma 5.6, we know that $\{u_n\}$ is bounded in $H^s_{rad}(\mathbb{R}^3)$, and by Lemma 3.6, up to a subsequence, there exists $u\in H^s_{rad}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^s_{rad}(\mathbb{R}^3)$, $u_n\to u$ strongly in $L^t(\mathbb{R}^3)$, for $t\in(2,2_s^*)$, $u_n\to u$ a.e. on $\mathbb{R}^3$. In view of $2+\frac{4s}{3}<q<2_s^*$, and Lemmas 3.3, 3.6, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^qdx = \int_{\mathbb{R}^3}|u|^qdx,\quad \lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2dx = \int_{\mathbb{R}^3}\phi_u^t u^2dx. \tag{6.9}$$

Suppose by contradiction that, $u \equiv 0$. Then, by (6.9) and $P_\mu(u_n) = o_n(1)$, we deduce as

$$\begin{aligned} o_n(1) &= \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \frac{3-2t}{4s}\lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \mu\delta_{q,s} \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &= \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1). \end{aligned}$$

Without loss of generality, we may assume that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \to \ell, \quad \text{and} \quad \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to \ell,$$

as $n \to \infty$. By Sobolev's inequality we get $\ell \geq S\ell^{\frac{2}{2_s^*}}$, and so, either $\ell \geq S^{\frac{3}{2s}}$ or $\ell = 0$.

If $\ell \geq S^{\frac{3}{2s}}$, then from $I_\mu(u_n) \to c_\mu(a)$, $P_\mu(u_n) \to 0$, we have

$$\begin{aligned} & c_\mu(a) + o_n(1) \\ &= I_\mu(u_n) = I_\mu(u_n) - \frac{1}{s2_s^*} P_\mu(u_n) \\ &= \frac{s}{3}\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 + \lambda \frac{s2_s^* + 2t - 3}{4s2_s^*} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \mu \frac{2_s^* - q\delta_{q,s}}{q2_s^*} \int_{\mathbb{R}^3} |u_n|^q dx + o_n(1) \\ &= \frac{s}{3}\ell + o_n(1), \end{aligned}$$

which implies that $c_\mu(a) = \frac{s}{3}\ell$, and so, $c_\mu(a) \geq \frac{s}{3}S^{\frac{3}{2s}}$, but this is impossible since by Lemma 6.1, there exists some $\mu_1^* := \mu_1^*(a) > 0$ such that $c_\mu(a) < \frac{s}{3}S^{\frac{3}{2s}}$ as $\mu > \mu_1^*$.

If $\ell = 0$, then we have $\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 \to 0$, thus $I_\mu(u_n) \to 0$, which is absurd since $c_\mu(a) > 0$. Therefore, $u \not\equiv 0$. □

Lemma 6.4. *$\{\alpha_n\}$ is bounded in $\mathbb{R}$, and $\limsup_{n\to\infty} |\alpha_n| \leq \frac{C}{a^2} c_\mu(a)$ has the following estimation:*

$$\alpha_n = \frac{1}{a^2}\left[\lambda \frac{2t+4s-3}{4s} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx + \frac{q(3-2s)-6}{2qs}\mu \int_{\mathbb{R}^3} |u_n|^q dx\right] + o_n(1).$$

Moreover, there exists some $\mu_2^* := \mu_2^*(a) > 0$ such that $\lim_{n\to+\infty} \alpha_n = \alpha < 0$, if $\mu > \mu_2^*$ large.

Proof. By (6.3) and the fact that $u_n \in S_{r,a}$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \int_{\mathbb{R}^3} g(u_n) u_n dx &= \alpha_n \int_{\mathbb{R}^3} |u_n|^2 dx + o_n(1) \\ &= \alpha_n a^2 + o_n(1). \end{aligned}$$

It indicates that

$$\alpha_n = \frac{1}{a^2}\left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 dx - \int_{\mathbb{R}^3} g(u_n) u_n dx\right] + o_n(1).$$

By Lemma 5.6 we have the boundedness of $\{u_n\}$ in $H_{rad}^s(\mathbb{R}^3)$, and so, $\{\alpha_n\}$ is bounded in $\mathbb{R}$. By Lemma 6.2 we know that $\limsup_{n\to\infty} |\alpha_n| \leq \frac{C}{a^2} c_\mu(a)$. Moreover, together with $P_\mu(u_n) \to 0$ as $n \to \infty$, we derive as

$$\begin{aligned} \alpha_n =& \frac{1}{a^2}\left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 dx - \int_{\mathbb{R}^3} g(u_n) u_n dx - \frac{1}{s} P_\mu(u_n)\right] + o_n(1) \\ =& \frac{1}{a^2}\left[\lambda \frac{2t+4s-3}{4s} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx + \frac{q(3-2s)-6}{2qs}\mu \int_{\mathbb{R}^3} |u_n|^q dx\right] + o_n(1). \end{aligned}$$

By (6.9) and similar arguments to that of (4.32)-(4.35), we see that there exists $\mu_2^* := \mu_2^*(a) > 0$, such that

$$\begin{aligned}\alpha &= \lim_{n\to\infty} \alpha_n \\ &= \lim_{n\to\infty} \frac{1}{a^2}\left\{\lambda\frac{2t+4s-3}{4s}\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx + \frac{q(3-2s)-6}{2qs}\mu\int_{\mathbb{R}^3}|u_n|^q dx + o_n(1)\right\} \\ &= \frac{1}{a^2}\left[\lambda\frac{2t+4s-3}{4s}\int_{\mathbb{R}^3}\phi_u^t u^2 dx + \frac{q(3-2s)-6}{2qs}\mu\int_{\mathbb{R}^3}|u|^q dx\right] \\ &< 0,\end{aligned} \tag{6.10}$$

for $\mu > \mu_2^*$ large. □

Subsequently, using the concentration-compactness principle, we derive the following lemma, whose proof is similar to that of Lemma 4.3 in Section 5, we omit its details here.

Lemma 6.5. *For $\mu > \mu^* := \max\{\mu_1^*, \mu_2^*\}$, there holds $\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \to \int_{\mathbb{R}^3}|u|^{2_s^*}dx$.*

With the help of the above technical lemmas, we can prove Theorem2.3 as follows.

Proof of Theorem 2.3. Let $\mu > \mu^* := \max\{\mu_1^*, \mu_2^*\}$. From Lemmas 5.1, 5.2, the functional I_μ satisfies the Mountain pass geometry, from Propositions 5.4,5.5, there exist a $(PS)_{c_\mu(a)}$-sequence $\{u_n\} \subset S_{r,a}$ satisfying (6.3), (6.4), which is bounded in $H_{rad}^s(\mathbb{R}^3)$, and there exists $u \in H_{rad}^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H_{rad}^s(\mathbb{R}^3)$, $u_n \to u$ strongly in $L^p(\mathbb{R}^3)$, for $p \in (2, 2_s^*)$. Moreover, by Lemmas 6.1-6.4, we have that $\alpha_n \to \alpha < 0$ as $n \to +\infty$. By the weak convergence of $u_n \rightharpoonup u$ in $H_{rad}^s(\mathbb{R}^3)$, (6.3) and (6.4), we have that u solves the equation

$$(-\Delta)^s u + \phi_u^t u - \mu|u|^{q-2}u - |u|^{2_s^*-2}u = \alpha u. \tag{6.11}$$

Therefore, from (6.9)-(6.11) and Lemma 6.5, it follows that

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \lambda\int_{\mathbb{R}^3}\phi_u^t u^2 dx - \alpha\|u\|_2^2 &= \mu\|u\|_q^q + \int_{\mathbb{R}^3}|u|^{2_s^*}dx \\ &= \lim_{n\to\infty}\left[\mu\|u_n\|_q^q + \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right] \\ &= \lim_{n\to\infty}[\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \lambda\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2 dx - \alpha_n\|u_n\|_2^2] \\ &= \lim_{n\to\infty}[\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 - \alpha_n\|u_n\|_2^2] + \lambda\int_{\mathbb{R}^3}\phi_u^t u^2 dx.\end{aligned}$$

Since $\alpha < 0$, as in the proof of Lemma 4.3, we can derive as

$$\lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 = \|(-\Delta)^{\frac{s}{2}}u\|_2^2 \quad\text{and}\quad \lim_{n\to\infty}\|u_n\|_2^2 = \|u\|_2^2.$$

Therefore, $u_n \to u$ in $H_{rad}^s(\mathbb{R}^3)$ and $\|u\|_2 = a$. This completes the proof. □

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