Geometric rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

Von der Fakultät für Mathematik und Informatik
der Universität Leipzig
angenommene

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

DOCTOR RERUM NATURALIUM
(Dr.rer.nat.)

im Fachgebiet
Mathematik
vorgelegt

von Konstantinos Zemas, M.Sc.
geboren am 18. März 1992 in Agrinio (Griechenland)

Die Annahme der Dissertation wurde empfohlen von:

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Die Verleihung des akademischen Grades erfolgt mit Bestehen der Verteidigung am 11.11.2020 mit dem Gesamtprädikat summa cum laude.

Erklärungen

- Ich erkenne die Promotionsordnung der Fakultät für Mathematik und Informatik der Universität Leipzig vom 3. Mai 2013 in der Fassung der 1. Änderungssatzung vom 16. Oktober 2017 an.

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Bibliographische Daten

Titel: Geometric rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

(Geometrische Starrheitsabschätzungen für isometrische und konformen Abbildungen von $\mathbb{S}^{n-1}$ zu $\mathbb{R}^n$)

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Universität Leipzig, Dissertation, 2020

145 Seiten, 0 Abbildungen, 43 Referenzen

Abstract

In this thesis we study qualitative as well as quantitative stability aspects of isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, when $n \geq 2$ and $n \geq 3$ respectively. Starting from the classical theorem of *Liouville*, according to which the *isometry group* of $\mathbb{S}^{n-1}$ is the group of its *rigid motions* and the *conformal group of* $\mathbb{S}^{n-1}$ is the one of its *Möbius transformations*, we obtain stability results for these classes of mappings among maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ in terms of appropriately defined deficits. Unlike classical *geometric rigidity results* for maps defined on *domains* of $\mathbb{R}^n$ and mapping into $\mathbb{R}^n$, not only an *isometric*$\backslash$*conformal* deficit is necessary in this more *flexible setting*, but also a deficit measuring how much the maps in consideration *distort* $\mathbb{S}^{n-1}$ *in a generalized sense.* The introduction of the latter is motivated by the *classical Euclidean isoperimetric inequality.*

Acknowledgements

First of all I would like to thank my advisor Prof. Dr. Stephan Luckhaus for his guidance, patience and support throughout these years. His scientific expertise and sharp way of thinking, together with his kind personality and sense of humour made all our discussions (be them mathematical or not) an invaluable part of my experience during this period.

I would also like to thank Prof. Dr. Felix Otto and all the members of his research group (of which I was a member in my last year in Leipzig) for creating such a stimulating and interactive environment in the Analysis Group of our Institute.

I would like to thank my family (my mother Maria, my father Giannis and my sister Aspa) for their unconditional love and support since times I cannot even remember, including our dog Rudy who has been a part of our family in the last 7 years.

Many thanks should also be given to all my friends for the beautiful moments we have had together. Starting from my friends since childhood (Vasilis, Dimitris, Thanos, Thodwros, Menios, Sotiris, Stavros, Ksanthi, Pegky, Vicky), the ones since our time as undergraduate students in Athens (Alexandros, Dimitris who was also my first-ever flatmate when we moved in Berlin, Nikos, Andreas, Rodia, Panos, Zervogiannhs, Vaggelhs, Konstantinos, Ilias, Evi, Stathis, Kyriakos, Abel, Zak, Lydia, Iro, Katerina) and also the friends I made in Berlin (Katerina, Apostolos, Thodwra, Mat, Zervas, Drosos).

A separate mention should of course be made for all the friends I made during these 4 years in Leipzig, mostly via the Max Planck Institute but not only. I feel really fortunate that this sometimes quite stressful period was spent together with people that made me feel comfortable since my first day in the city, and with whom we became true friends in a surprisingly short period of time. I would therefore like to thank deeply my former 'group fellows' Gianni, Souhayl, Eris and Daniel as well as my office-mate Giovanni who was unofficially "an honorary member of the group" and my official IT-consultant.

I would like to thank my ex-flatmates Paolo, Renan and Sharwin as well as the official cook of the WG, Niccolo. They for sure have several funny stories to tell from me being their flatmate. Many thanks should also go to all the other friends I made through the MPI and the University of Leipzig, namely Konstantinos, Tim, Marcello, Stefano, Gerardo, Florio, Felix, Bogdan, Pau, Alex, Nono, Maria, Francesco, Caio, Gabriele, Björn, Ivan as well as Vasilis, Anna, Chrisa, Vanessa, Simone, Teresa, Federica, Svenja, Eva and Beatrice, my Krav-Maga Instructor Andreas Lehnert and all the people together with whom we were playing football.

I would especially like to thank Jonas Hirsch for all our mathematical discussions, the nice drawings that he was very often sending to me and also for motivating me to join him in a hobby completely new to both of us, contemporary dancing.

Last but certainly not least, I would like to thank deeply and devote this thesis to my girlfriend Vasiliki for all the love, the support and the sweet moments she has added to my life in the last 4 years, for giving me a very good reason to become a better cook, for being my most beloved flatmate (sorry Dimitris, Paolo, Renan, Sharwin) and also for bearing me and Jonas as students in her contemporary dance classes during the months before Corona came.

Contents

Notations

$n \in \mathbb{N}$	a natural number denoting the dimension of the ambient Euclidean space
$\{e_1, \dots, e_n\}$	the standard orthonormal basis of $\mathbb{R}^n$
$\langle a, b \rangle$	the Euclidean inner product between two vectors $a, b \in \mathbb{R}^n$
$A : B$ or $\langle A, B \rangle$	the Euclidean inner product between two matrices $A, B \in \mathbb{R}^{n\times m}$
$\lvert \cdot \rvert$	the Euclidean norm of vectors or matrices\ linear maps
A^t	the transpose of a matrix $A \in \mathbb{R}^{n\times m}$ (or the adjoint of the corresponding linear map respectively)
$Sym(n)$, $Skew(n)$	the space of $n \times n$ symmetric, antisymmetric matrices respectively
A_{sym}, A_{skew}	the symmetric, antisymmetric part of a matrix A respectively
$\overline{A}$	the mean value of a tensor field A on its domain of definition
$\overline{U}$	the topological closure of an open set $U \subseteq \mathbb{R}^n$
$B^n_\rho(x_0)$	the open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ of radius $\rho > 0$
$\mathbb{S}^{n-1}_\rho(x_0)$	$\partial B^n_\rho(x_0)$
B^n	$B^n_1(0)$
$\mathbb{S}^{n-1} := (\mathbb{S}^{n-1}, g)$	$\mathbb{S}^{n-1}_1(0)$ equipped with the *standard round metric* g
$\mathcal{U}_\delta(\mathbb{S}^{n-1})$	the open δ-tubular neighbourhood of $\mathbb{S}^{n-1}$
$\{\tau_1, \dots, \tau_{n-1}\}$	a positively oriented local orthonormal frame for $T_x\mathbb{S}^{n-1}$ so that for every $x \in \mathbb{S}^{n-1}$, $\{\tau_1(x), \cdots, \tau_{n-1}(x), x\}$ is a positively oriented orthonormal system of n vectors in $\mathbb{R}^n$
dv_g	the standard $(n-1)$-volume form on $\mathbb{S}^{n-1}$
ω_n	the volume of the unit ball in $\mathbb{R}^n$
$\mathcal{H}^k$	the k-dimensional Hausdorff measure
$\mathrm{Vol}(E)$, $\mathrm{Per}(E)$	the volume, the perimeter of an $\mathcal{H}^n$-measurable set $E \subseteq \mathbb{R}^n$
$O(n)$	the orthogonal group of $\mathbb{R}^n$, i.e. the set $\{O \in \mathbb{R}^{n\times n} : O^tO = I_n\}$
$SO(n)$	the special orthogonal group of $\mathbb{R}^n$, i.e. the set $\{R \in O(n) : \det A = 1\}$
$CO_+(n)$	the conformal group of $\mathbb{R}^n$, i.e. the set $\{\lambda R \in \mathbb{R}^{n\times n} : \lambda > 0, R \in SO(n)\}$,

	(actually its "positive cone")
$\mathcal{M}_b(U;\mathbb{R}^d)$	the space of all $\mathbb{R}^d$-valued Radon measures on a bounded domain $U \subseteq \mathbb{R}^n$
$\mathrm{spt}\mu$	the support of a measure $\mu \in \mathcal{M}_b(U;\mathbb{R}^d)$
$\lvert\mu\rvert(U)$	the total variation of a measure $\mu \in \mathcal{M}_b(U;\mathbb{R}^d)$
$\vec{\nu}_r$	the unit normal vector field to $\mathbb{S}^{n-1}_r$, i.e. $\vec{\nu}_r(x) = \frac{x}{r}$ for every $x \in \mathbb{S}^{n-1}_r$
$\mathrm{id}_{\mathbb{S}^{n-1}}$	the standard embedding map of $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$
I_x	the identity transformation on $T_x\mathbb{S}^{n-1}$
$\nabla v, \mathrm{div} v,\ \Delta v$	the Euclidean gradient, divergence and Laplacian of a map $v : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$
$\nabla_T u$	the tangential gradient of $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, represented in the local coordinates $\{\tau_1, \dots, \tau_{n-1}\}$ and the Euclidean coordinates $\{e_1, \dots, e_n\}$ by the $n \times (n-1)$ matrix with entries $(\nabla_T u)_{ij} = \langle \nabla_T u^i, \tau_j \rangle$ for $i = 1, \dots, n,\ j = 1, \dots, n-1$
$\mathrm{div}_{\mathbb{S}^{n-1}} u,\ \Delta_{\mathbb{S}^{n-1}} u$	the tangential divergence, Laplace-Beltrami operator of a map $u : \mathbb{S}^{n-1} \to \mathbb{R}^n$
$P_T(x)$	$\nabla_T \mathrm{id}_{\mathbb{S}^{n-1}}(x) : T_x\mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ for $x \in \mathbb{S}^{n-1}$, i.e. in local coordinates $(P_T)_{ij} = \langle e_i, \tau_j \rangle$ for $i = 1, \dots, n$ and $j = 1, \dots, n-1$
$d_x u$	the *intrinsic gradient* of a map $u : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$, viewed as a linear map $d_x u : T_x\mathbb{S}^{n-1} \mapsto T_{u(x)}\mathbb{S}^{n-1}$ w.r.t. the frame $\{\tau_1, \dots, \tau_{n-1}\}$
$\partial_{\vec{\nu}} f$	the radial derivative of a function $f : \overline{B^n} \mapsto \mathbb{R}$ on $\mathbb{S}^{n-1}$
C^k	the space of k-times continuously differentiable maps, $k \in \mathbb{N}$
$L^p, W^{1,p}$	the standard Lebesque or Sobolev spaces respectively, $1 \leq p < \infty$
$\lVert u\rVert_{L^p(\mathbb{S}^{n-1})}$	the Lebesgue norm of a map $u \in L^p(\mathbb{S}^{n-1};\mathbb{R}^n)$ *with the convention that the integral is taken with respect to the normalized $\mathcal{H}^{n-1}$-measure*, to simplify some dimensional constants appearing later in the content, i.e. $\lVert u\rVert^p_{L^p(\mathbb{S}^{n-1})} := ⨍_{\mathbb{S}^{n-1}} \lvert u\rvert^p \, d\mathcal{H}^{n-1}$
$\lVert u\rVert_{W^{1,p}(\mathbb{S}^{n-1})}$	the Sobolev norm of a map $u \in W^{1,p}(\mathbb{S}^{n-1};\mathbb{R}^n)$ with the same convention, i.e. $\lVert u\rVert^p_{W^{1,p}(\mathbb{S}^{n-1})} := ⨍_{\mathbb{S}^{n-1}} \lvert u\rvert^p \, d\mathcal{H}^{n-1} + ⨍_{\mathbb{S}^{n-1}} \lvert\nabla_T u\rvert^p \, d\mathcal{H}^{n-1}$
$W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n)$	the space of Lipschitz maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ with norm $\lVert u\rVert_{W^{1,\infty}(\mathbb{S}^{n-1})} := \max\left\{\lVert u\rVert_{L^\infty(\mathbb{S}^{n-1})}, \lVert\nabla_T u\rVert_{L^\infty(\mathbb{S}^{n-1})}\right\}$
a_+, a_-	the positive, negative part of $a \in \mathbb{R}$, i.e. $a_+ := \max\{a, 0\},\ a_- := -\min\{a, 0\}$
$\sim_{M_1,M_2,\dots}$	the corresponding equality is valid up to a constant that is allowed to vary from line to line but depends only on the parameters $M_1, M_2, \dots$
$\lesssim_{M_1,M_2,\dots}$	the same meaning as above for inequalities

Chapter 1

Introduction

1.1 An overview of geometric rigidity results for the orthogonal and the conformal group

One of the most classical and well known *rigidity theorems* in differential geometry is *Liouville's theorem* that concerns *isometric* and *conformal maps* defined on domains of the Euclidean space. In modern terms it can be stated as follows.

Theorem 1.1.1. ***(Liouville)***

(i) Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $u \in W^{1,2}(U;\mathbb{R}^n)$ is a generalized orientation-preserving isometric map, that is

$$\nabla u \in SO(n) \quad \text{a.e.} \quad \textit{in } U. \tag{1.1.1}$$

Then u is a rigid motion of U, i.e. there exist $R \in SO(n)$ and $b \in \mathbb{R}^n$ so that

$$u(x) = Rx + b. \tag{1.1.2}$$

(ii) Let $n \geq 3$ and $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $u \in W^{1,n}(U;\mathbb{R}^n)$ is a generalized orientation-preserving conformal map, that is

$$\nabla u \in CO_+(n) \quad \text{a.e.} \quad \textit{in } U. \tag{1.1.3}$$

Then u is the restriction on U of a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$, i.e. there exist $A \in CO_+(n)$, $b \in \mathbb{R}^n$ and *$a \in \mathbb{R}^n \setminus U$ so that*

$$u(x) = Ax + b \quad \text{or} \quad u(x) = AB\frac{x-a}{|x-a|^2} + b, \tag{1.1.4}$$

where $B = \mathrm{diag}(1, ..., 1, -1) \in \mathbb{R}^{n\times n}$.

Another classical fact is that the first part of the theorem fails if the group $SO(n)$ is replaced by the full orthogonal group $O(n)$, unless the maps in consideration are assumed to be more regular, in particular $C^1(U;\mathbb{R}^n)$. The reason is that $O(n)$ has *rank-one connections* and therefore the corresponding *differential inclusion*, even when considered among *Lipschitz mappings*, admits non-trivial solutions (for example the so called *simple laminates*). Regarding the second part of the theorem, it is also well known that *Liouville's theorem for conformal maps does not hold in two dimensions.* Actually, according to the famous *Riemann mapping theorem in complex analysis*, every simply connected domain in $\mathbb{C}$ that is not $\mathbb{C}$ itself is conformally equivalent to the open unit disk. Therefore, the class of conformal maps defined on a fixed open subdomain of the complex plane does not admit a simple characterization as before.

A simple proof of the first statement, as can be found for instance in [FJM02], can be carried out along the following lines. Notice that

$$\nabla u \in SO(n) \;\text{ a.e. in }\; U \implies \operatorname{cof}\nabla u = \nabla u \;\text{ a.e. in }\; U. \tag{1.1.5}$$

By *Piola's identity* we have $\operatorname{div}(\operatorname{cof}\nabla u) = 0$, which in this case implies that $\Delta u = 0$ in U in the sense of distributions. By *Weyl's lemma* for harmonic functions the map u is smooth, and by using *Bochner' formula*

$$\frac{1}{2}\Delta(|\nabla u|^2 - n) = \nabla u \cdot \Delta \nabla u + |\nabla^2 u|^2 = |\nabla^2 u|^2 \quad \text{in} \quad U. \tag{1.1.6}$$

Since $\nabla u \in SO(n)$ a.e. in U, the left hand side vanishes identically, thus $\nabla^2 u \equiv 0$ in U, i.e. u is affine with gradient in $SO(n)$. It is also obvious that the same proof can be carried out if the *integrability exponent* 2 is replaced by any exponent $p \in [1,\infty]$.

Regarding the proof of the second part of the theorem, J. Liouville was the first one to prove it in 1850 for *sufficiently regular* maps, in particular for maps in the class $C^3(U;\mathbb{R}^n)$. The conformality condition on u can be rewritten as a system of PDE, namely

$$\left\{ \begin{array}{ll} \nabla u^t \nabla u = \frac{|\nabla u|^2}{n} I_n & \text{in} \;\; U \\ \det\nabla u > 0 & \text{in} \;\; U \end{array} \right\}. \tag{1.1.7}$$

This system can be solved explicitely and the solutions are precisely given by (1.1.4).

More than a century after the first proof, F. Gehring proved Liouville's theorem for *homeomorphisms in the Sobolev class* $W^{1,n}(U;\mathbb{R}^n)$ in [Geh62] and Y.G. Reshetnyak removed the injectivity assumption in [Res67a], by combining ideas from the original proof and the *regularity theory for the n-harmonic equation.* Later on, T. Iwaniec proved in [Iwa92] that there exists a *critical exponent* $1 < p_n < n$ such that *Liouville's theorem for* *conformal maps* holds in $W^{1,p}(U;\mathbb{R}^n)$ whenever $p \geq p_n$. Below this *integrability threshold* one can construct counterexamples, i.e. for every $p' \in (1, p_n)$ there exists a map $u \in W^{1,p'}(U;\mathbb{R}^n)$ such that $\nabla u \in CO_+(n)$ a.e. in U, but u is not a single Möbius transformation. Actually, T. Iwaniec and G. Martin showed in [IM93] that the sharp threshold is $p_n = \frac{n}{2}$ in case n *is even*, conjecturing that this is also the case when n is odd. Certainly, as it is mentioned in their work, there exist counterexamples to Liouville's theorem for conformal maps below the exponent $\frac{n}{2}$ in all dimensions. However, their beautiful proof regarding the sharpness of the exponent $\frac{n}{2}$ in even dimensions does not seem to adapt in the case of odd dimensions. The reason is that at the core of their proof lies the algebraic expression of the *Jacobian determinant* of a map u in terms of its $\frac{n}{2} \times \frac{n}{2}$ *minors*, a fact that is of course possible only when n is even.

A natural question that can be posed and was subsequently widely explored is whether these *rigidity theorems are stable*, so that one can have *approximate versions* in the following sense.

If for a map u its gradient is close to $SO(n)$ or to $CO_+(n)$ in an average sense, is it true that the map is itself close to a single rigid motion or a Möbius map respectively in an appropriate average sense, both in a qualitative and a quantitative fashion?

In the rest of this introductory Section we give a brief overview and description of some, among several interesting, results related to this question.

1.1.1 On the stability of the orthogonal group when $n \geq 2$

As far as the *stability of solutions* to the *differential inclusion* $\nabla u \in SO(n)$ is concerned, a qualitative result was obtained by Y.G. Reshetnyak in [Res67b].

Theorem 1.1.2. ***(Y.G. Reshetnyak, [Res67b])*** *Let $n \geq 2$, $U \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 \leq p < \infty$. If $(u_j)_{j\in\mathbb{N}} \in W^{1,p}(U;\mathbb{R}^n)$ is a sequence of mappings such that*

$$\lim_{j\to\infty} \|\mathrm{dist}(\nabla u_j, SO(n))\|_{L^p(U)} = 0, \tag{1.1.8}$$

then there exists $R \in SO(n)$ such that up to a non-relabeled subsequence

$$\lim_{j\to\infty} \|\nabla u_j - R\|_{L^p(U)} = 0. \tag{1.1.9}$$

A modern proof of this result that uses the concept of *Young measures* can be found in [Kin87] and a generalization to the setting of approximately orientation-preserving isometries between Riemannean manifolds in [KMS19].

In the quest for quantitative analogues, let us first mention the classical work of F. John (see [Joh61], [Joh72]), in which he considered mappings that are *apriori approximately isometric* in the following sense.

Let $Q := Q(x_0, L)$ be an n-dimensional cube in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$, of side-length $L > 0$ and let $u \in C^1(Q; \mathbb{R}^n)$. The *strain tensor* associated to the *right Cauchy-Green tensor* of u is defined as $e_u := \sqrt{\nabla u^t \nabla u} - I_n$ and the *maximum strain* of u as $\varepsilon_u := \|e_u\|_{L^\infty(Q)}$. Given now $\delta > 0$, a map $u \in C^1(Q; \mathbb{R}^n)$ is called $\delta - quasiisometric$ iff $\varepsilon_u \leq \delta$. With these definitions, John's results can be summarized in the following.

Theorem 1.1.3. ***(F. John, [Joh61],[Joh72])*** *Let $n \geq 2$, $Q := Q(x_0, L)$ be an n-cube in $\mathbb{R}^n$ and $1 < p < \infty$. There exist $\delta := \delta(n, p) > 0$ and $C := C(n, p) > 0$ such that for every $\delta - quasiisometric$ map $u \in C^1(Q; \mathbb{R}^n)$ there exists $O \in O(n)$ so that*

$$\|\nabla u - O\|_{L^p(Q)} \leq C \|e_u\|_{L^p(Q)}. \tag{1.1.10}$$

The theorem implies in particular that for every $\delta - quasiisometric$ map $u \in C^1(Q; \mathbb{R}^n)$, *its gradient has bounded mean oscillation in Q*. By an application of the standard Sobolev inequalities one sees that there also exists a constant $C' := C'(n, p) > 0$ such that for every $\delta - quasiisometric$ map $u \in C^1(Q; \mathbb{R}^n)$ as in the theorem there exists a rigid motion γ of $\mathbb{R}^n$ so that

$$\begin{aligned} &\text{If } 1 < p < n, \quad \|u - \gamma\|_{L^{p^*}(Q)} \leq C' \|e_u\|_{L^p(Q)}, \quad \text{where } p^* := \frac{np}{n-p} > 1, \\ &\text{If } p > n, \qquad \|u - \gamma\|_{L^\infty(Q)} \leq C' L^{1-\frac{n}{p}} \|e_u\|_{L^p(Q)}. \end{aligned}$$

Notice that the previous theorem concerns C^1-*regular maps*, providing thus a stability result for solutions to the differential inclusion $\nabla u \in O(n)$, which is *rigid in this class of mappings*. An improvement of F. John's results was later obtained by R. V. Kohn by proving

Theorem 1.1.4. ***(R. V. Kohn, [Koh82])*** *Let $n \geq 2$, $U \subseteq \mathbb{R}^n$ be a bounded, Lipschitz domain and let $p \geq 1$ with $p \neq n$. There exist $C_1 := C_1(n, U, p) > 0$, $C_2 := C_2(n, U) > 0$ such that for every bi-Lipschitz map $u : U \mapsto \mathbb{R}^n$ there exist a rigid motion γ of $\mathbb{R}^n$ and $O \in O(n)$ so that*

(i) If $1 \leq p < n$, then

$$\|u - \gamma\|_{L^{p^*}(U)} + \|u - \gamma\|_{L^p(\partial U)} \leq C_1 \|\epsilon_u\|_{L^p(U)}, \tag{1.1.11}$$

where $p^* = \frac{np}{n-p} > 1$ *is again the conjugate Sobolev exponent of* p. *The "nonlinear elastic strain"* ϵ_u *is now defined as*

$$\epsilon_u := (\lambda_n - 1)_+ + (\lambda_2 \dots \lambda_n - 1)_+ + |\det(G_u) - 1|, \tag{1.1.12}$$

where $G_u := \sqrt{\nabla u^t \nabla u}$ *and* $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *are the principal stretches of* u, *i.e. the eigenvalues of* G_u.

(ii) If $p > n$, *then*

$$\|u - \gamma\|_{L^\infty(U)} \leq C_1 \|\epsilon_u\|_{L^p(U)}. \tag{1.1.13}$$

(iii) If one sets $\tilde{e}_u := |\nabla u^t \nabla u - I_n|$, *then one also has the estimate*

$$\int_U |\nabla u - O|^2 \, dx \leq C_2 \|\epsilon_u + \tilde{e}_u\|_{L^1(U)}. \tag{1.1.14}$$

It is worth noticing that R. V. Kohn's results include the case $p = 1$ and *do not assume any apriori smallness of the nonlinear elastic strain*, which was required in F. John's framework, but only a *Lipschitz-invertibility assumption*. Since ϵ_u contains terms measuring also *"surface" change* and *"signed-volume" change*, the last theorem also yields stability of the general group of rigid motions in terms of this *nonlinear elastic strain*, even if the maps under consideration are assumed to be less regular than C^1.

A fundamental breakthrough that requires neither a smallness assumption on the elastic energy, nor invertibility assumptions on the maps in consideration, was achieved in the pioneering work of G. Friesecke, R. D. James and S. Müller in [FJM02], where a sharp scaling-invariant quantitative estimate was obtained.

Theorem 1.1.5. ***(G. Friesecke, R. D. James, S. Müller, [FJM02])*** *Let* $n \geq 2$ *and* $U \subseteq \mathbb{R}^n$ *be a bounded, Lipschitz domain. There exists a constant* $C := C(n, U) > 0$ *such that for every* $u \in W^{1,2}(U; \mathbb{R}^n)$ *there exists an associated* $R \in SO(n)$ *so that*

$$\|\nabla u - R\|_{L^2(U)} \leq C \, \|\mathrm{dist}(\nabla u; SO(n))\|_{L^2(U)} \, . \tag{1.1.15}$$

The latter estimate holds true also in $W^{1,p}(U; \mathbb{R}^n)$ for any $p \in (1, \infty)$ as well as in *interpolation spaces* (see [CS06] and [CDM14]). Notice that the exponent with which the norm on the right hand side of the estimate appears, is sharp. Moreover, apart from being translationally and rotationally invariant, the estimate is also *scaling invariant* with respect to the domain, in the sense that if $C := C(n, U) > 0$ stands for *the optimal constant* for which (1.1.15) holds, then $C(n, \lambda R U + b) = C(n, U)$ for every $\lambda > 0$, $R \in SO(n)$ and $b \in \mathbb{R}^n$.

Theorem 1.1.5. has been used widely in the analysis of *variational models for nonlinear elasticity*, for instance in questions related to *dimension reduction*. A nice application appears already in [FJM02], where the authors use their *nonlinear rigidity estimate* together with Γ-*convergence tools*, to rigorously derive *thin-plate theories from a 3-dimensional model of nonlinear elasticity*, as the thickness of the plate goes to zero.

It is also fairly well known but still worth remarking that (1.1.15) is the exact *nonlinear counterpart* of the *classical Korn's inequality*, which is a fundamental tool for problems in the context of *linearized elasticity*.

Theorem 1.1.6. ***(Korn, see for example [Cia88])*** *Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded, Lipschitz domain. There exists a constant $C := C(n,U) > 0$ such that for every $u \in W^{1,2}(U;\mathbb{R}^n)$ there exists $S \in Skew(n)$ (i.e. $S^t = -S$) so that*

$$\|\nabla u - S\|_{L^2(U)} \leq C \, \|(\nabla u)_{\mathrm{sym}}\|_{L^2(U)} . \tag{1.1.16}$$

Korn's inequality also holds in $W^{1,p}(U;\mathbb{R}^n)$ for every $p \in (1,\infty)$ and can also be generalized in a *Riemannean setting* (see [CJ02]). It is clear that Theorem 1.1.5. is the nonlinear analogue of Theorem 1.1.6., since the tangent space of the *finite-dimensional Lie group* $SO(n)$ at I_n is exactly the *Lie-algebra* $\mathfrak{so}(n)$ *of skew-symmetric* $n \times n$ *matrices*. Also, the *linearization of the function* $u \mapsto \mathrm{dist}(\nabla u; SO(n))$ around the identity mapping gives

$$\mathrm{dist}(\nabla u; SO(n)) = \left|(\nabla u)_{\mathrm{sym}} - I_n\right| + \mathcal{O}(|\nabla u - I_n|^2). \tag{1.1.17}$$

Let us now present a very short sketch of the proof of Theorem 1.1.5., and refer the interested reader to the original work [FJM02], Section 3, for the detailed proof.

Sketch of proof of Theorem 1.1.5. The approach of G. Friesecke, R. D. James and S. Müller in Theorem 1.1.5. consists of the following steps. First, the corresponding *interior estimate* is proven (see Proposition 3.4 in [FJM02]), namely

Let Q be an n-dimensional cube in $\mathbb{R}^n$ and Q' be the cube in $\mathbb{R}^n$ having the same center and half the side-length of Q. For every $v \in W^{1,2}(Q;\mathbb{R}^n)$ there exists $R \in SO(n)$ such that

$$\|\nabla v - R\|_{L^2(Q')} \leq C(n) \, \|\mathrm{dist}(\nabla v; SO(n))\|_{L^2(Q)} , \tag{1.1.18}$$

where $C(n) > 0$ is a dimensional constant.

The proof of the *interior estimate*, which is the main part in the proof of Theorem 1.1.5., is itself divided in several steps.

Step 0. By using a suitable *truncation argument* (see Proposition A.1 in [FJM02]), one can *without loss of generality* assume that the map v is *Lipschitz* with an *apriori Lipschitz bound*, i.e. $\|\nabla v\|_{L^\infty(Q)} \leq M$, where $M := M(n) > 0$ is a dimensional constant.

Step 1. Let now $\varepsilon := \|\mathrm{dist}(\nabla v; SO(n))\|_{L^2(Q)}$. Without loss of generality one may assume that $0 < \varepsilon \leq 1$. In the case of the *exact differential inclusion* $v \in W^{1,2}(U;\mathbb{R}^n)$ with $\nabla v \in SO(n)$ a.e. in U, *Piola's identity* implied that actually v was harmonic in U. In this *approximate setting*, if one calls v_h the *harmonic replacement* of v, i.e. the unique solution to the Dirichlet problem

$$\left\{ \begin{array}{ll} -\Delta \tilde{v} = 0 & \text{in } Q \\ \tilde{v} = v & \text{on } \partial Q \end{array} \right\}, \tag{1.1.19}$$

Piola's identity implies that the map $v_h - v$ satisfies the PDE

$$\left\{ \begin{array}{ll} -\Delta(v_h - v) = \mathrm{div}(\nabla u - \mathrm{cof}\nabla u) & \text{in } Q \\ v_h - v = 0 & \text{on } \partial Q \end{array} \right\}. \tag{1.1.20}$$

As a result,

$$\int_Q |\nabla v_h - \nabla v|^2 \; dx \lesssim_n \varepsilon^2 \tag{1.1.21}$$

$$\implies \int_Q \mathrm{dist}^2(\nabla v_h; SO(n)) \; dx \lesssim_n \varepsilon^2. \tag{1.1.22}$$

Step 2. By *Step 1* one can now focus on the harmonic replacement v_h. By testing *Bochner's identity* (1.1.6) with a suitable cut-off function and by using basic properties of harmonic functions, one can arrive at the following L^∞-estimates.

$$\left\|\nabla^2 v_h\right\|_{L^\infty(Q')} \lesssim_n \sqrt{\varepsilon} \implies \|\nabla v_h - R\|_{L^\infty(Q')} \lesssim_n \sqrt{\varepsilon}, \tag{1.1.23}$$

for a constant matrix $R \in \mathbb{R}^{n\times n}$ that can without loss of generality be chosen to lie in $SO(n)$, and even more specifically one can choose $R = I_n$. Notice that (1.1.23) immediately implies that $\|\nabla v_h - R\|_{L^2(Q')} \lesssim_n \sqrt{\varepsilon}$, which is a *suboptimal version of the desired estimate*, with the deficit ε appearing with the *suboptimal exponent* $\frac{1}{2}$ instead of 1.

Step 3. Despite being suboptimal, the L^∞-estimate (1.1.23) allows one to linearize the $\mathrm{dist}(\cdot\,; SO(n))$ around the identity as in (1.1.17) and use *Korn's inequality* (1.1.16) for the displacement map $v_h(x) - x$ in order to *improve the suboptimal exponent* $\frac{1}{2}$ *to the optimal exponent* 1. In particular, one is able to show that

$$\int_{Q'} \left|\nabla v_h - \overline{\nabla v_h}\right|^2 \; dx \lesssim_n \varepsilon^2, \tag{1.1.24}$$

where $\overline{\nabla v_h} := \fint_{Q'} \nabla v_h \, dx$, and since $\operatorname{dist}(\overline{\nabla v_h}; SO(n)) \lesssim_n \varepsilon$, one can replace the average in (1.1.24) with a constant matrix $R \in SO(n)$.

Once the interior estimate in cubes is established, the *global estimate* in an arbitrary Lipschitz domain U is obtained via *covering arguments*. Clearly *Steps* 0 and 1 can be carried out unchanged when the cube Q is replaced by an arbitrary Lipschitz domain U, so that only *Steps 2 and 3* have to be modified. Suitably *covering U with a sequence of cubes*, in each one of which the *interior estimate* applies, and using standard estimates for harmonic functions, the authors in [FJM02] obtain the following *global estimate*.

$$\int_U \operatorname{dist}^2(x; \partial U) |\nabla^2 u_h|^2 \, dx \leq c(n, U) \int_U \operatorname{dist}^2(\nabla u_h; SO(n)) \, dx. \tag{1.1.25}$$

Finally, coupling (1.1.25) with a *weighted version of the Poincare inequality*, namely

$$\int_U |f - \overline{f}|^2 \, dx \leq \tilde{c}(n, U) \int_U \operatorname{dist}^2(x; \partial U) |\nabla f|^2 \, dx, \tag{1.1.26}$$

which holds for all $f \in W^{1,2}(U; \mathbb{R}^m)$, applied to $f := \nabla u_h$, yields (1.1.15) for u_h. $\square$

Let us also remark that generalizations of the previous results to the case of *incompatible fields*, that is fields that do not arise globally as gradients, are important for applications in the analysis of *variational models for crystal plasticity*. For example, generalizations of Theorem 1.1.6. and Theorem 1.1.5. in *two dimensions* are provided by

Theorem 1.1.7. ***(A. Garroni, G. Leoni, M. Ponsiglione, [GLP10])*** *Let $U \subseteq \mathbb{R}^2$ be a bounded, simply-connected, Lipschitz domain. There exists $C := C(U) > 0$ such that for every $A \in L^2(U; \mathbb{R}^{2\times 2})$ with $\mathrm{Curl} A \in \mathcal{M}_b(U; \mathbb{R}^2)$ there exists a skew-symmetric matrix $S \in \mathbb{R}^{2\times 2}$, so that*

$$\|A - S\|_{L^2(U)} \leq C \left(\|A_{\mathrm{sym}}\|_{L^2(U)} + |\mathrm{Curl} A|(U) \right) \tag{1.1.27}$$

and by its nonlinear analogue

Theorem 1.1.8. ***(S. Müller, L. Scardia, C.I. Zeppieri, [MSZ14])*** *Let $U \subseteq \mathbb{R}^2$ be a bounded, simply-connected, Lipschitz domain. There exists $C := C(U) > 0$ such that for every $A \in L^2(U; \mathbb{R}^{2\times 2})$ with $\mathrm{Curl} A \in \mathcal{M}_b(U; \mathbb{R}^2)$, there exists a rotation $R \in SO(2)$ so that*

$$\|A - R\|_{L^2(U)} \leq C \left(\|\operatorname{dist}(A; SO(2))\|_{L^2(U)} + |\mathrm{Curl} A|(U) \right). \tag{1.1.28}$$

Here, for a matrix field $A = (A_{ij})_{i,j=1,2}$, $\mathrm{Curl}A := ((\mathrm{Curl}A)^1, \mathrm{Curl}A)^2)$ denotes its *distributional* Curl, defined componentwise as $(\mathrm{Curl}A)^i := \partial_2 A_{i1} - \partial_1 A_{i2}$ for $i = 1, 2$.

Let us additionally mention that G. Lauteri and S. Luckhaus have obtained *geometric rigidity estimates for incompatible fields in dimensions $n \geq 3$* in [LL17]. In order to describe their results, we recall that a tensor field $A \in L^1(U; \mathbb{R}^{n\times n})$ can be identified with an $\mathbb{R}^n$-valued 1-form, namely with $\omega_A := (\omega_A^i)_{i=1,\dots,n}$, where $\omega_A^i := \sum_{j=1}^n A_{ij}\, dx^j$, and the (distributional) Curl of A can be identified with the $\mathbb{R}^n$-valued 2-form (the space of which we denote by $(\wedge^2(\mathbb{R}^n))^n$) given by the (distributional) *exterior differential of* ω_A, namely $d\omega_A := (d\omega_A^i)_{i=1,\dots,n}$. With the use of an *averaged homotopy operator* similar to the one introduced by T. Iwaniec and A. Lutoborski in [IL93] and techniques from harmonic analysis, the authors of [LL17] have shown the following.

Theorem 1.1.9. ***(G. Lauteri, S. Luckhaus, [LL17])***

(i) *Let $n \geq 2$ and $U \subseteq \mathbb{R}^n$ be a bounded convex domain. There exists a constant $C := C(n,U) > 0$ such that for every $A \in L^{\frac{n}{n-1}}(U; \mathbb{R}^{n\times n})$ with $\mathrm{Curl}A \in \mathcal{M}_b\left(U; (\wedge^2(\mathbb{R}^n))^n\right)$ and $\mathrm{spt}(\mathrm{Curl}A) \Subset U$, there exists $R \in SO(n)$ so that*

$$\|A - R\|_{L^{\frac{n}{n-1},\infty}(U)} \leq C\left(\|\mathrm{dist}(A; SO(n))\|_{L^{\frac{n}{n-1},\infty}(U)} + |\mathrm{Curl}A|(U)\right). \tag{1.1.29}$$

(ii) *Let $n \geq 3$, $p \in \left[\frac{n}{n-1}, 2\right]$ and $M > 0$ be fixed. There exists $C := C(n,p,M) > 0$ such that for every $A \in L^\infty(B^n; \mathbb{R}^{n\times n})$ with $\|A\|_{L^\infty(B^n)} \leq M$, $\mathrm{Curl}A \in \mathcal{M}_b\left(B^n; (\wedge^2(\mathbb{R}^n))^n\right)$ and $\mathrm{spt}(\mathrm{Curl}A) \Subset B^n$, there exists $R \in SO(n)$ so that*

(ii_a) in the supercritical case $\frac{n}{n-1} < p \leq 2$, one has the estimate

$$\int_{B^n} |A - R|^p\, dx \leq C\left(\int_{B^n} \mathrm{dist}^p(A; SO(n))\, dx + (|\mathrm{Curl}A|(B^n))^{\frac{n}{n-1}}\right); \tag{1.1.30}$$

(ii_b) in the critical case $p = \frac{n}{n-1}$, one has the non-scaling invariant estimate

$$\begin{aligned}\int_{B^n} |A - R|^{\frac{n}{n-1}}\, dx \leq \quad & C\int_{B^n} \mathrm{dist}^{\frac{n}{n-1}}(A; SO(n))\, dx \\ & + C\big(|\mathrm{Curl}A|(B^n)\big)^{\frac{n}{n-1}}\big(|\log(|\mathrm{Curl}A|(B^n))| + 1\big).\end{aligned} \tag{1.1.31}$$

1.1.2 On the stability of the conformal group when $n \geq 3$

Turning now our attention to the stability of solutions to the differential inclusion associated with the group $CO_+(n)$ in dimension $n \geq 3$, the reader is referred to the book of Y.G. Reshetnyak [Res13] and the references therein for a detailed collection of the results obtained mostly by the author of the book, and which initiated further research in this direction. Here, we would like to describe some more recent results in the spirit of those presented in the previous Subsection.

Related to the question of the sharp regularity conditions under which Liouville's theorem for conformal mappings holds, a qualitative version appears in the work of B. Yan in [Yan96].

Theorem 1.1.10. ***(B. Yan, [Yan96])*** *Let $n \geq 3$, $U \subseteq \mathbb{R}^n$ be an open bounded Lipschitz domain and $p \geq n$. Let $(u_j)_{j\in\mathbb{N}} \in W^{1,p}(U;\mathbb{R}^n)$ be a weakly convergent sequence such that $u_j \rightharpoonup u$ in $W^{1,p}(U;\mathbb{R}^n)$ and*

$$\lim_{j\to\infty} \|\mathrm{dist}(\nabla u_j; CO_+(n))\|_{L^p(U)} = 0. \tag{1.1.32}$$

Then $\nabla u \in CO_+(n)$ a.e. in U, i.e. u is a Möbius transformation of U and actually $u_j \to u$ strongly in $W^{1,p}(U;\mathbb{R}^n)$.

Similar to the *rigid case*, the above theorem fails for $p < \frac{n}{2}$. A natural question is then whether there exists a critical threshold $p^* \in [\frac{n}{2}, n)$ so that whenever $p \geq p^*$, Theorem 1.1.10. holds for maps in $W^{1,p}(U;\mathbb{R}^n)$ that are *approximately conformal* in the previous sense and fails when $p < p^*$. Another related question is whether $p^* = p_n$, where p_n is the critical threshold for the validity of Liouville's theorem as addressed by T. Iwaniec and G. Martin. The existence of such a p^* that is not too far below n was established in [YZ98] and in [MŠY99] S. Müller, V. Sverak and B. Yan proved that actually $p^* = p_n = \frac{n}{2}$ in the case $n \geq 4$ is even, in complete accordance with the rigid case.

Regarding quantitative estimates, a result in the spirit of the one of G. Friesecke, R. D. James and S. Müller in an L^2-setting was obtained by D. Faraco and X. Zhong in [FZ05]. Due to the *noncompactness of the conformal group and its degeneracy at* $0 \in \mathbb{R}^{n\times n}$, their result concerns *rotationally invariant compact subsets of* $CO_+(n)$ *that are bounded away from* 0 *and infinity*, and can be *represented as a finite union of annuli-type subregions of* $CO_+(n)$. We present for simplicity the result in the case that such a subset has only one connected component, and can therefore be written as

$$CO_+(n; m, M) := \{\lambda R;\ 0 < m \leq \lambda \leq M,\ R \in SO(n)\} \Subset CO_+(n). \tag{1.1.33}$$

Given an open bounded Lipschitz domain $U \subseteq \mathbb{R}^n$ with $n \geq 3$, the *differential inclusion*

$$\phi \in W^{1,2}(U;\mathbb{R}^n) \ \text{ with } \ \nabla\phi \in CO_+(n; m, M) \ \text{ a.e. in } \ U \tag{1.1.34}$$

possesses of course also non-affine solutions. The set of all possible such solutions is comprised by the orientation-preserving Möbius transformations described by (1.1.4), whose gradients are uniformely bounded from below by $\sqrt{n}m$ and from above by $\sqrt{n}M$. We denote the set of all those maps by $\mathbb{M}_n(U;m,M)$ and more generally we denote by $\mathbb{M}_n(U)$ the set of all orientation-preserving Möbius transformations that are finite in U. With these definitions, the main result in [FZ05] is

Theorem 1.1.11. ***(D. Faraco, X. Zhong, [FZ05])*** *Let $n \geq 3$, $U' \Subset U \subseteq \mathbb{R}^n$ be open bounded Lipschitz domains and let $0 < m \leq M < \infty$.*

(i) *There exists $C_1 := C_1(n,m,M,U',U) > 0$ such that for every $u \in W^{1,2}(U;\mathbb{R}^n)$, there exists $\phi \in \mathbb{M}_n(U';m,M)$ so that*

$$\|\nabla u - \nabla\phi\|_{L^2(U')} \leq C_1 \|\mathrm{dist}(\nabla u; CO_+(n;m,M))\|_{L^2(U)}. \tag{1.1.35}$$

(ii) *In the special case $m = M$, there exists $C_2 := C_2(n,m,U) > 0$ such that for every $u \in W^{1,2}(U;\mathbb{R}^n)$, there exists $\phi \in \mathbb{M}_n(U;m,m)$ so that*

$$\|\nabla u - \nabla\phi\|_{L^2(U)} \leq C_2 \|\mathrm{dist}(\nabla u; CO_+(n;m,m))\|_{L^2(U)}. \tag{1.1.36}$$

The second part of the theorem is of course a direct consequence of Theorem 1.1.5.. The interesting issue regarding the first part, where the subannulus of $CO_+(n)$ is generically nontrivial, is that it is an *interior estimate which, unlike the $SO(n)$-case, cannot be extended to a global one.*

As D. Faraco and X. Zhong remark (see Example 3.3 in [FZ05]), given any two parameters $0 < m < M < \infty$, one can construct a sequence of inversions $(\phi_j)_{j\in\mathbb{N}} \in \mathbb{M}_n(B^n)$ with the centers of inversion belonging all to a straight line and converging to a point outside $\overline{B^n}$, with the property that

$$\lim_{j\to\infty} \frac{\inf\limits_{\psi\in\mathbb{M}_n(B^n;m,M)} \int_{B^n} |\nabla\phi_j - \nabla\psi|^2 \, dx}{\int_{B^n} \mathrm{dist}^2(\nabla\phi_j; CO_+(n;m,M)) \, dx} = \infty. \tag{1.1.37}$$

Nevertheless, an interesting open question regarding the *interior estimate* is whether it holds true for the full conformal group $CO_+(n)$ and not only for compact subsets of it of the previous type. It seems that one cannot easily adapt the method of proof in [FZ05] to the *noncompact case*, both because of the *unboundedness* of $CO_+(n)$ and also because of the *degeneracy* of the associated *n-harmonic equation* at the *vertex of the cone.*

Indeed, the proof of Theorem 1.1.11. is performed by a suitable application and modification of the ideas of the proof of Theorem 1.1.5. in the *conformal setting*. First, the estimate is proven when U is a ball B in $\mathbb{R}^n$ and U' is the *concentric ball* of half the radius, the case of general bounded Lipschitz domains $U' \Subset U \subseteq \mathbb{R}^n$ following by standard *covering arguments*. The assumption on the subset of the group $CO_+(n)$ in consideration is bounded from above, allows one to apply essentially the same *truncation argument* as in the $SO(n)$-case, i.e. Proposition A.1 in [FJM02], and assume that u is Lipschitz with an apriori bound on its Lipschitz constant that depends on n and M.

The idea is then again to replace the map u with a map that has the same boundary values and satisfies some suitably associated PDE. The natural choice would be to consider the *n-harmonic replacement* of u, i.e. the solution to the boundary value problem

$$\left\{ \begin{array}{ll} -\mathrm{div}\left(|\nabla v|^{n-2}\nabla v\right) = 0 & \text{in } B \\ v = u & \text{on } \partial B \end{array} \right\}, \tag{1.1.38}$$

which unfortunately leads to the *suboptimal estimate*

$$\int_B |\nabla u - \nabla v|^2 \, dx \lesssim_{n,M} \left(\int_B \mathrm{dist}^2(\nabla u, CO_+(n;m,M)) \, dx \right)^{\frac{1}{n-1}}.$$

Nevertheless, the extra assumption that the subset of $CO_+(n)$ is also bounded from below away from zero, enabled the authors in [FZ05] to consider a suitable *strongly elliptic modification* of (1.1.38) for which, by standard elliptic theory, the analogous to (1.1.21) estimate holds with optimal exponent (see the estimate (4.7) in [FZ05]). Hence, the problem is again reduced to showing (1.1.35) for mappings that satisfy this related *strongly elliptic equation*, the so-called *F-harmonic mappings* in [FZ05]. By similar but somewhat more qualitative arguments than those in [FJM02], the authors are finally again able to reduce to a *linearized setting*, where the following *variant of Korn's inequality for the trace-free part of the symmetrized gradient* is used.

Theorem 1.1.12. ***(Y.G. Reshetnyak, see Theorem 3.3, Chapter 3, [Res13])*** *Let $n \geq 3$ and U be a subdomain of $\mathbb{R}^n$ that is starshaped with respect to a ball. There exists a constant $C := C(n,U) > 0$ such that for every $u \in W^{1,2}(U;\mathbb{R}^n)$,*

$$\|\nabla u - \nabla(\Pi_{\Sigma_n} u)\|_{L^2(U)} \leq C \left\| (\nabla u)_{\mathrm{sym}} - \frac{\mathrm{div} u}{n} I_n \right\|_{L^2(U)}, \tag{1.1.39}$$

where $\Pi_{\Sigma_n} : W^{1,2}(U;\mathbb{R}^n) \mapsto \Sigma_n$ is the $W^{1,2}$-projection on the finite-dimensional kernel of the trace-free symmetrized gradient operator.

The last variant of Korn's inequality has of course an intimate connection with the geometry of $CO_+(n)$, discussed in detail in [Res13], Chapters 2 and 3 and also in [FZ05] and the references therein. If $TCO_+(n)$ stands for the tangent space to the Lie group $CO_+(n)$ at I_n $\left(\text{its dimension being } \frac{(n+1)(n+2)}{2}\right)$, it easy to see that

$$A \in TCO + (n) \iff A_{\mathrm{sym}} = \frac{\mathrm{Tr} A}{n} I_n,$$

so that the function $A \mapsto d(A) := \left|A_{\text{sym}} - \frac{\text{Tr}A}{n} I_n\right|$ is equivalent to the distance of A to $TCO_+(n)$. Therefore, the linear subspace of $W^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$

$$\Sigma_n := \left\{u \in W^{1,2}(\mathbb{R}^n;\mathbb{R}^n) : (\nabla u)_{\text{sym}} = \frac{\text{div}u}{n} I_n\right\} \tag{1.1.40}$$

can be viewed as the *Lie algebra of the Möbius group* of $\overline{\mathbb{R}^n}$, i.e. Σ_n is isomorphic to $\mathfrak{so}(n+2,1)$.

1.2 Description of the main results

Inspired from the results described in the previous Section, in this thesis we study *stability aspects of isometric and conformal maps from* $\mathbb{S}^{n-1}$ *into the ambient Euclidean space* $\mathbb{R}^n$. Except for the complementary Section 4.3 which came as a result of a short private discussion of the author with Dr. Jonas Hirsch, the results presented in this thesis are obtained by the author and S. Luckhaus in a joint (ongoing) work (see [LZ] and especially the upcoming preprint [LZon]).

Now that the starting domain is of codimension 1 in $\mathbb{R}^n$, this case exhibits more *flexibility* than the case of such maps defined on open subsets of $\mathbb{R}^n$ and mapping into $\mathbb{R}^n$.

On the one hand, the *"spherical version"* of *Liouville's theorem* still asserts that the only *isometric diffeomorphisms* of $\mathbb{S}^{n-1}$ are its *rigid motions*, i.e. the restrictions on $\mathbb{S}^{n-1}$ of orthogonal transformations of $\mathbb{R}^n$, and the only *conformal diffeomorphisms* of $\mathbb{S}^{n-1}$ are its *Möbius transformations*, and actually the conclusions hold again under less restrictive regularity and invertibility assumptions.

In **Chapter 2** we revise such versions of *Liouville's theorem* and give some *intrinsic and to the knowledge of the author new proofs of it*, which can also be modified to give *approximate versions* of the theorem.

On the other hand, it is also well known that these are not the only isometric\ conformal maps that one can define on $\mathbb{S}^{n-1}$. In general, there exist such maps from $\mathbb{S}^{n-1}$ onto other closed embedded hypersurfaces. Standard examples come from the theory of *isometric embeddings*, as a consequence of the celebrated *Nash-Kuiper theorem.*

Theorem 1.2.1. ***(J.F. Nash-N.H. Kuiper, [Nas54], [Kui55])*** *Let (M^d,h) be a smooth compact d-dimensional manifold, $m \geq d+1$ and $u : M^d \mapsto \mathbb{R}^m$ be a short embedding, that is an embedding for which $\mathcal{H}^1(u\circ\gamma) \leq \mathcal{H}^1(\gamma)$ for every C^1-curve γ in M^d. Then u can be uniformely approximated by C^1-isometric embeddings.*

While by a classical result in differential geometry the only C^2-isometric embedding of $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ is the standard one modulo rigid motions, the above astonishing theorem implies the following somewhat counterintuitive fact. Given any $\delta \in (0,1)$, in an arbitrarily small C^0-neighbourhood of the *short homothety* $u_\delta : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, $u_\delta(x) := \delta x$, there exist C^1 isometric embeddings which can be visualized as *wrinkling isometrically* $\mathbb{S}^{n-1}$ inside a small ball in a way that produces continuously varying tangent planes. Although beyond the scope of this thesis, let us mention that these type of *flexibility phenomena*, known as the *h-principle*, occur very often in problems in geometry or fluid dynamics, either in smooth solutions of *underdetermined* problems (for example smooth isometric embeddings in *high codimension*), or in relatively *low-regularity* solutions of *determined problems* (for example C^1-isometric embeddings with fixed codimension). The interested reader is referred to [SJ12] for an introduction to this beautiful topic and to the classical treatise by M. Gromov [Gro13].

Other examples of conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ that are not Möbius transformations are also (at least when $n = 3$) used in *cartography*, for instance the inverse of *Jacobi's conformal map projection* that conformally maps $\mathbb{S}^2$ onto the surface of an *ellipsoid*.

Therefore, *Liouville's rigidity theorem* on the one hand, and the above *flexibility phenomena* on the other, naturally motivate the question of *stability of the isometry\ conformal group of* $\mathbb{S}^{n-1}$, described in loose terms by the following.

Question. *Let* $n \geq 2 \backslash n \geq 3$ *and* $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ *be a map which is in an average sense almost isometric\almost conformal. If we have the extra information that* $u(\mathbb{S}^{n-1})$ *is in an average sense close to being a round sphere, can we control the deviation of* u *from a particular rigid motion\Möbius transformation (up to translation and scaling) of* $\mathbb{S}^{n-1}$*?*

This is essentially the guiding question throughout the thesis. Of course, as it is common in many questions regarding the *stability of geometric\functional inequalities* or the *stability of absolute minimizers in geometric variational problems*, the *notions of measurement* are themselves an important feature of the problem. The deficits measuring the deviation of u from being isometric\conformal are analogous to similar ones that have appeared in the literature for these notions for maps defined *in the bulk*, i.e. in open subdomains of $\mathbb{R}^n$. The deficit we choose for the necessary extra information on *how much* u *distorts* $\mathbb{S}^{n-1}$ is in a sense *a generalized isoperimetric deficit for the map* u. Both in the *isometric* and in the *conformal* setting it is motivated by the property of *balls* being the only *isoperimetric sets* in $\mathbb{R}^n$ (modulo sets of measure zero), and also by *their stability among sets of finite perimeter in terms of the isoperimetric deficit* (see for example the beautiful works in [Fus17], [FMP08], [FMP10] and the references therein on the sharp quantitative form of the isoperimetric inequality).

In **Chapter 3** we provide an answer to the previous question in the *isometric case*, when the ambient dimension is $n \geq 2$. The main result of the first two Sections of Chapter 3 is **Theorem 3.2.3.**, according to which

For every $n \geq 2$ and $M > 0$ there exists $C_{n,M} > 0$ such that for every Lipschitz map $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ with $\|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M$, there exists $O \in O(n)$ so that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \, d\mathcal{H}^{n-1} \leq C_{n,M} \left(\delta_u + \varepsilon_u\right).$$

Here, δ_u denotes the *L^2-isometric deficit* of u, defined by

$$\delta_u := \left(\fint_{\mathbb{S}^{n-1}} \left| \sqrt{\nabla_T u^t \nabla_T u} - I_x \right|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}$$

and ε_u its *generalized isoperimetric deficit* (being actually *the positive part of the excess in generalized volume*), defined by

$$\varepsilon_u := \left(1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| \right)_+ .$$

The proof of this result, which is *first presented in* **Section 3.1** *in the particular case that u is an isometric map from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$*, is based on first proving its corresponding *qualitative \compactness analogue* and then reduce to the case of maps that are apriori close to a certain rigid motion of $\mathbb{S}^{n-1}$ in the appropriate topology, namely the *$W^{1,2}$ − topology.*

The assumption on an apriori Lipschitz bound for the maps under consideration can be substantially weakened, as **Proposition 3.2.6.** *in* **Subsection 3.2.2.** suggests. This can be done via the use of the same *Lipschitz truncation argument* as in [FJM02], but in our context the Lipschitz truncation of a Sobolev map $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ should also take care of the possible change in the deficit ε_u. Because of this extra feature, we can show that in the case $n = 2$ or $n = 3$ the assumption can be completely removed. In the case $n \geq 4$ it can be replaced by a much weaker one, namely by the assumption that the maps in consideration enjoy a uniform bound in some homogeneous Sobolev space of order higher than 2 (namely in $\dot{W}^{1,2(n-2)}(\mathbb{S}^{n-1}; \mathbb{R}^n)$).

Note also that the presence of the deficit ε_u which *penalizes deviation from isoperimetry in a generalized sense*, allows us to have a quantitative stability result for the whole group of rigid motions of $\mathbb{S}^{n-1}$, be them orientation-preserving\-reversing. We also exhibit examples showing the optimality of the result in terms of the norm on the left hand side and the exponents of the deficits on the right hand side of the estimate.

Regarding the choice of the deficit ε_u, notice that if u is an *isometric embedding of* $\mathbb{S}^{n-1}$ *into* $\mathbb{R}^n$ and E_u is the open bounded connected set in $\mathbb{R}^n$ with $\partial E_u = u(\mathbb{S}^{n-1})$, then of course $\mathrm{Per}(E_u) = \mathcal{H}^{n-1}(u(\mathbb{S}^{n-1})) = n\omega_n$, while

$$\mathrm{Vol}(E_u) = \frac{1}{n}\left|\int_{\mathbb{S}^{n-1}} \langle u, \nu_u\rangle g_u \; d\mathcal{H}^{n-1}\right| = \frac{1}{n}\left|\int_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \; d\mathcal{H}^{n-1}\right|.$$

By the *classical Euclidean isoperimetric inequality,* $\varepsilon_u = 1 - \left|\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \; d\mathcal{H}^{n-1}\right|$ in this case, and it really represents the isoperimetric deficit of the set E_u. Here we have abused notation and denoted

$$\left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle := \left\langle u, *\left(\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right)\right\rangle,$$

i.e. we have set $\nu_u := *\frac{\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u}{|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u|}$, identifying (by Hodge duality) the *normalized wedge product* of the (linearly independent in this case) vectors $(\partial_{\tau_i} u)_{i=1}^{n-1}$ in $\mathbb{R}^n$ with the *unit normal to the hyperplane they span.* Moreover, $g_u := \left|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right| = \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}$ is the standard *area element induced by u.* From here on, we also use the notation

$$V_n(u) := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \; d\mathcal{H}^{n-1}, \tag{1.2.1}$$

which for a *Lipschitz embedding* $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ gives the *signed enclosed volume* normalized by the volume of the unit ball B^n. For a general map u (for which $|V_n(u)| < \infty$), we may still sometimes refer to it as the *signed-volume term*, although it might not necessarily represent the actual signed enclosed volume. The integral in the definition of V_n is also connected to the notion of *degree* and also to the property of *the Jacobian determinant being a null-Lagrangian.* It is indeed a standard fact that for any *appropriate extension* $U : \overline{B^n} \mapsto \mathbb{R}^n$ of u in the interior of the unit ball,

$$\fint_{B^n} \det\nabla U \; dx = \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \; d\mathcal{H}^{n-1}. \tag{1.2.2}$$

This identity holds true for example for the *harmonic extension* $u_h : \overline{B^n} \mapsto \mathbb{R}^n$ of u, the latter being taken componentwise.

In **Section 3.3** we present a *linear stability result which can be viewed as an analogue of Theorem 1.1.6. for maps from* $\mathbb{S}^{n-1}$ *into* $\mathbb{R}^n$, and discuss some simple consequences of it. According to (3.3.13),

Although the kernel of the quadratic form associated to the generalized full isoperimetric deficit (introduced below) is infinite-dimensional, any positive combination of it with the quadratic form associated to the isometric deficit (even when $n = 2$) has finite dimensional kernel (which is actually isomorphic to $Skew(n)$) and satisfies a corresponding coercivity estimate. In other words, the intersection of the two kernels exactly corresponds to the tangent space to the orthogonal group at the identity matrix.

In **Chapter 4** we study the *conformal* case when the ambient dimension is $n \geq 3$. The corresponding deficit in this case is again in the spirit of the ones appearing in Chapter 3 and is motivated by the following simple observations.

For an arbitrary map $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$, let $0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{n-1}$ be the eigenvalues of the symmetric positive-definite matrix $\sqrt{\nabla_T u^t \nabla_T u}$. In view of the *arithmetic mean-geometric mean inequality* we always have that $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$,

$$\left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} = \left(\frac{\sum_{i=1}^{n-1}\sigma_i^2}{n-1}\right)^{\frac{n-1}{2}} \geq \left(\prod_{i=1}^{n-1}\sigma_i^2\right)^{\frac{1}{2}} = \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}, \tag{1.2.3}$$

and by averaging on $\mathbb{S}^{n-1}$

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1}, \tag{1.2.4}$$

Equality would hold iff $0 \leq \sigma_1(x) = \ldots = \sigma_{n-1}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$, i.e. iff u is a *generalized conformal map from* $\mathbb{S}^{n-1}$ *to* $\mathbb{R}^n$. Therefore, the difference (or alternatively the ratio) between the two sides of (1.2.4) provides an *average measure of deviation from conformality* for maps $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$. This is of course in complete analogy to the case of maps $v \in W^{1,n}(U;\mathbb{R}^n)$, where $U \subseteq \mathbb{R}^n$ is a bounded open Lipschitz domain. In that case, by the same reasoning

$$\int_U \left(\frac{|\nabla v|^2}{n}\right)^{\frac{n}{2}} dx \geq \int_U |\det \nabla v|\, dx \geq \int_U \det \nabla v\, dx,$$

with equalities iff $\nabla v \in CO_+(n)$ for a.e. $x \in U$, or equivalently (as long as $n \geq 3$), iff v is the restriction of a Möbius transformation onto U. The reader is again referred to [Res13], [Yan96], [YZ98] [MŠY99] and the references therein for an exposition of *stability results* for conformal maps in domains of $\mathbb{R}^n$ in terms of such an average measure of non-conformality.

If moreover $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is a *general Lipschitz embedding*, in view of the *isoperimetric inequality* we further have

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} \geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} = \frac{\mathcal{H}^{n-1}(u(\mathbb{S}^{n-1}))}{n\omega_n} \geq |V_n(u)|^{\frac{n-1}{n}}, \tag{1.2.5}$$

and equalities in the above chain of inequalities would hold iff the map u is conformal and $u(\mathbb{S}^{n-1})$ is a round sphere in $\mathbb{R}^n$, i.e. u has to be a conformal embedding of $\mathbb{S}^{n-1}$ onto another round sphere, hence a Möbius transformation up to a translation vector and a scaling factor. The chain of inequalities in (1.2.5) can be rewritten as

$$D_{n-1}(u) \geq P_{n-1}(u) \geq |V_n(u)| \geq V_n(u), \tag{1.2.6}$$

where

$$D_{n-1}(u) := \left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}, \tag{1.2.7}$$

$$P_{n-1}(u) := \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\ d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}, \tag{1.2.8}$$

and $V_n(u)$ is defined in (1.2.1). Actually, the inequality

$$P_{n-1}(u) \geq V_n(u) \tag{1.2.9}$$

is valid for all maps $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$, even if they are not necessarily embeddings. The nonnegative quantity $P_{n-1}(u) - V_n(u)$ is the one that we refer to as the *generalized full isoperimetric deficit*. Having not been able to find a reference for this particular inequality in the literature (without refering to *Almgren's isoperimetric inequality for integral currents*), we include in Appendix A a short proof of it which comes as a simple consequence of another *generalized isoperimetric inequality*, proved by elementary means by S. Müller (see Lemma 1.3 in [Mü90]).

It is also immediate that these three geometric quantities, to which we will refer in the sequel as *the generalized* $(n-1)$*-Dirichlet energy-term, the generalized perimeter-term* and *the generalized signed-volume-term* respectively, enjoy the following *invariance properties.*

(*i*) (*Translational invariance*) For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and every $b \in \mathbb{R}^n$

$$D_{n-1}(u+b) = D_{n-1}(u)\ ,\ P_{n-1}(u+b) = P_{n-1}(u)\ ,\ V_n(u+b) = V_n(u). \tag{1.2.10}$$

(*ii*) (*Rotational invariance*) For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and every rotation map $R \in SO(n)$

$$D_{n-1}(Ru) = D_{n-1}(u)\ ,\ P_{n-1}(Ru) = P_{n-1}(u)\ ,\ V_n(Ru) = V_n(u). \tag{1.2.11}$$

(*iii*) (*Scaling behaviour*) For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and every $\lambda > 0$

$$D_{n-1}(\lambda u) = \lambda^n D_{n-1}(u)\ ,\ P_{n-1}(\lambda u) = \lambda^n P_{n-1}(u)\ ,\ V_n(\lambda u) = \lambda^n V_n(u). \tag{1.2.12}$$

(*iv*) (*Conformal invariance*) For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and every orientation-preserving $\psi \in Conf(\mathbb{S}^{n-1})$

$$D_{n-1}(u\circ\psi) = D_{n-1}(u)\ ,\ P_{n-1}(u\circ\psi) = P_{n-1}(u)\ ,\ V_n(u\circ\psi) = V_n(u). \quad (1.2.13)$$

With these considerations in mind, we say that for a map u its *combined conformal-isoperimetric deficit is ε-small for some* $\varepsilon > 0$ iff

$$D_{n-1}(u) \leq (1+\varepsilon)V_n(u). \quad (1.2.14)$$

Employing this deficit, our main stability results for the conformal case, i.e. Theorem 4.1.2. and Theorem 4.2.1. are of *local nature* and concern actually *maps that are apriori close in a certain topology to a fixed Möbius transformation of* $\mathbb{S}^{n-1}$. Without loss of generality we can take this to be the identity transformation on $\mathbb{S}^{n-1}$, at least as long as we focus on *compact subsets* of the set of its Möbius transformations, with gradient bounded from below and above by fixed positive constants (see part (*ii*) in Remark 4.1.1.).

Without entering into the precise technical assumptions, **Theorem 4.1.2** and **Theorem 4.2.1.** can be described as follows.

For every map $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ which lies in a sufficiently small neighbourhood of the $\mathrm{id}_{\mathbb{S}^{n-1}}$ *in an appropriate topology, there exist an orientation-preserving Möbius transformation of $\mathbb{S}^{n-1}$ which we call ϕ_u, a vector $b_u \in \mathbb{R}^n$ and a positive factor $\lambda_u > 0$ such that*

$$\left\| \left(\frac{u\circ\phi_u - b_u}{\lambda_u}\right) - \mathrm{id}_{\mathbb{S}^{n-1}} \right\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq C\sqrt{\varepsilon}.$$

The constant $C > 0$ depends only on the dimension, as does also the size of the neighbourhood (in the correct topology of course) around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ for the validity of our local estimate. The exponent $\frac{1}{2}$ with which the ε-deficit appears on the right hand side is also optimal, i.e. it cannot generically be improved. This can easily be checked by considering the sequence of affine mappings $(u_\sigma)_{\sigma>0} : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ for $\sigma \to 0^+$, where $u_\sigma(x) := A_\sigma x$, with $A_\sigma := \mathrm{diag}(1,...,1,1+\sigma) \in \mathbb{R}^{n\times n}$.

In **Section 4.1** we present the result and its proof when the dimension of the ambient space is $n = 3$ and in **Section 4.2** in the *higher dimensional* case $n \geq 4$. The argumentation follows the same lines in both cases and some intermediate steps in the proofs are the same. However, *in dimension $n = 3$ some assumptions can be relaxed*, for example the topology in which we require apriori closeness of u to the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is weaker than the one in dimensions $n \geq 4$ (as we will see it will be the $W^{1,2}$-topology instead of $W^{1,\infty}$), and *some of the arguments can be simplified* (compare for instance **Subsection 4.1.2** with **Subsection 4.2.2**). We have therefore chosen to keep these two cases separate and provide in Section 4.2 the necessary details in the arguments that need to be slightly modified.

As in many occasions where local stability of absolute minimizers in variational problems is examined, at the core of the proofs in both cases lies the study of the *coercivity properties* of the second variation\ quadratic form Q_n which appears after a *formal Taylor expansion of the combined conformal-isoperimetric deficit* around the $\mathrm{id}_{\mathbb{S}^{n-1}}$. This is of course performed in a purely $W^{1,2}$-setting, i.e. at this *linearized level of course no closeness to the identity assumption has to be made.* The main ingredient that we will be making use of, is the *fine interplay between the Fourier decomposition of a* $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$*-vector field into spherical harmonics and the invariance properties of the linear first order differential operator associated to the second variation of the signed-volume-term* V_n *around the* $\mathrm{id}_{\mathbb{S}^{n-1}}$.

At a *functional-analytic level*, this gives a decomposition of a $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$-vector field into vector-valued spherical harmonics of a special type, as we prove in **Theorem 4.1.8.**, which might be interesting in its own right. This in turn implies a *coercivity estimate* for the quadratic form Q_n associated to the *combined conformal-isoperimetric deficit*, i.e. **Theorem 4.1.10.** and **Theorem 4.2.7.**, which are the main results in Subsection 4.1.2 and Subsection 4.2.2 respectively. Similarly to Section 3.3, a corollary of these estimates is the following.

Although the kernels of the nonnegative quadratic forms arising as the second variations of the conformal deficit and the generalized full isoperimetric deficit at the $\mathrm{id}_{\mathbb{S}^{n-1}}$ *are both infinite-dimensional, the intersection of the two kernels is finite dimensional and actually isomorphic to the Lie algebra of infinitesimal Möbius transformations of* $\mathbb{S}^{n-1}$.

In this sense, Theorems 4.1.10 and 4.2.7 are the analogues of Theorem 1.1.12. for maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$.

In **Section 4.1.3** we complete the proof of Theorem 4.1.2. and (since the argument there can be carried out essentially unchanged in all dimensions $n \geq 3$) also the proof of Theorem 4.2.1.. This is done by using the *Inverse Function Theorem* and a *topological degree argument* similar to the corresponding ones appearing in [Res70] and [FZ05].

The complementary **Section 4.3** came as a result of a short private discussion with Dr. Jonas Hirsch, whom the author would like to thank. We include it in order to give another application of how some of the ideas that the reader will encounter in Chapters

2 and 3 can be used to *give an alternative and somewhat shorter proof of a recent result due to A. Bernard-Mantel, C.B. Muratov and T. M. Simon in [BMMS]*, where a quantitative stability result for the particular case of degree ± 1 conformal mappings from $\mathbb{S}^2$ onto itself is obtained, the authors there being motivated by the analysis of a variational model from micromagnetics.

In the **Outlook** we list some open questions that originated from this work or that the author finds interesting in general. Finally, in the **Appendices** we include for the convenience of the reader a brief proof of the *generalized isoperimetric inequality* that we have been using, some *basic facts from the theory of spherical harmonics* and also a detailed derivation of the *Taylor expansions of the geometric quantities* that appear throughout the main body of the thesis.

Chapter 2

A new view of Liouville's theorem on $\mathbb{S}^{n-1}$

2.1 The isometry group of $\mathbb{S}^{n-1}$ when $n \geq 2$

Turning now to the main topic of the thesis and before giving some standard definitions, let us first make the following trivial remark for the sake of clarity. As we have mentioned in the table of Notations, we denote by $\mathbb{S}^{n-1} := (\mathbb{S}^{n-1}, g)$ the *standard round sphere* embedded in $\mathbb{R}^n$.

Given a sufficiently regular (say C^1) map $u : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$, at every $x \in \mathbb{S}^{n-1}$ the *gradient* of u can be viewed *extrinsically* as the linear map $\nabla_T u(x) : T_x\mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, as if u was considered a map from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ with $|u| \equiv 1$, and *intrinsically* as the linear map $d_x u : T_x\mathbb{S}^{n-1} \mapsto T_{u(x)}\mathbb{S}^{n-1}$. Choosing the *local orthonormal frame* $\{\tau_1, \dots, \tau_{n-1}\}$ *indicated by the unit normal vector field on* $\mathbb{S}^{n-1}$, the linear maps $(d_x u)^t d_x u : T_x\mathbb{S}^{n-1} \mapsto T_x\mathbb{S}^{n-1}$ and $(\nabla_T u^t \nabla_T u)(x) : T_x\mathbb{S}^{n-1} \mapsto T_x\mathbb{S}^{n-1}$ coincide, so that we can use either of them in the definitions to come without distinction. The same holds true for less regular maps (for example Lipschitz or Sobolev maps) at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ where the gradient is defined.

Definition 2.1.1. Let $n \geq 2$ and $1 \leq p \leq \infty$.

(i) A map $u \in C^1(\mathbb{S}^{n-1}; \mathbb{R}^n)$ is called *isometric* iff at every $x \in \mathbb{S}^{n-1}$

$$(\nabla_T u^t \nabla_T u)(x) = I_x. \tag{2.1.1}$$

(ii) A map $u \in W^{1,p}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ is called *generalized isometric* iff at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$

$$(\nabla_T u^t \nabla_T u)(x) = I_x, \tag{2.1.2}$$

where the gradient is understood in the weak but also in the classical sense $\mathcal{H}^{n-1}$-a.e., since such maps are automatically 1-Lipschitz.

(*iii*) A map $u \in W^{1,p}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ is called a *generalized orientation-preserving\-reversing isometry* of $\mathbb{S}^{n-1}$ iff at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ the *intrinsic gradient* of u is an orientation-preserving\-reversing isometry between $T_x\mathbb{S}^{n-1}$ and $T_{u(x)}\mathbb{S}^{n-1}$.

(*iv*) The group of all isometric diffeomorphisms of $\mathbb{S}^{n-1}$ will be denoted by $Isom(\mathbb{S}^{n-1})$, and the subgroup consisting only of the orientation-preserving ones will be denoted by $Isom_+(\mathbb{S}^{n-1})$.

As we have discussed in the Introduction, isometric and also conformal maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ are more flexible than the ones from n-dimensional subdomains of $\mathbb{R}^n$ into $\mathbb{R}^n$. However, *Liouville's rigidity theorem* still holds for such maps from $\mathbb{S}^{n-1}$ onto itself, a fact that is of course a trivial consequence of the classical version of Liouville's theorem regarding the structure of the isometry and the conformal group of the Euclidean space.

A fairly standard proof of this fact is that for any $n \geq 2$, an isometry $u : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$ can be extended *radially* to $\mathbb{R}^n$ via $U(x) := |x|u\left(\frac{x}{|x|}\right)$ if $x \in \mathbb{R}^n \setminus \{0\}$ and $U(0) := 0$. It is a simple fact of Euclidean geometry that U is an isometry of $\mathbb{R}^n$, hence a *linear orthogonal transformation* and going back, u is the restriction of such a transformation on $\mathbb{S}^{n-1}$.
For the conformal case, for any $n \geq 3$ the *conformal diffeomorphisms* of $\mathbb{S}^{n-1}$ are of course in a *bijective correspondence* to the *conformal diffeomorphisms of the augmented Euclidean space* $\mathbb{R}^{n-1} \cup \{\infty\}$, i.e. the Möbius transformations, via the *stereographic projection* which is itself a conformal map.

However, the main purpose of this and the next Section is to present an *alternative* and more *intrinsic proof of Liouville's theorem on* $\mathbb{S}^{n-1}$, which to the knowledge of the author has not appeared in the literature. This will also provide some motivation for the subsequent *stability analysis*, since the arguments can be *perturbed both in a qualitative and a quantitative way* as we will see later. The proof does not use the corresponding result in the Euclidean space, only the knowledge that orthogonal transformations are isometric and Möbius maps are conformal, and is more *intrinsic* in the sense that it basically relies on the *sharp Poincare inequality on* $\mathbb{S}^{n-1}$ (see the Remark B.0.3. in Appendix B).

Theorem 2.1.2. ***(Liouville's Theorem for the isometry group of*** $\mathbb{S}^{n-1}$***)***
Let $n \geq 2$. Then $u \in Isom(\mathbb{S}^{n-1})$ iff it is the restriction of an orthogonal transformation of $\mathbb{R}^n$ on $\mathbb{S}^{n-1}$, i.e. there exists $O \in O(n)$ so that for every $x \in \mathbb{S}^{n-1}$

$$u(x) = Ox. \tag{2.1.3}$$

Proof. First of all, the restrictions of orthogonal transformations of $\mathbb{R}^n$ on $\mathbb{S}^{n-1}$ are isometries of $\mathbb{S}^{n-1}$. Conversely, if $u \in Isom(\mathbb{S}^{n-1})$ then for every $x \in \mathbb{S}^{n-1}$ one has

$$(\nabla_T u^t \nabla_T u)(x) = (d_x u)^t d_x u = I_x \implies \frac{|\nabla_T u(x)|^2}{n-1} = 1. \tag{2.1.4}$$

Since $u : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$ is an isometric diffeomorphism, $u^\sharp(\omega) = \pm\omega$ for every $(n-1)$-form ω on $\mathbb{S}^{n-1}$. By the *change of variables formula* applied to the vector-valued $(n-1)$-form xdv_g (where dv_g is the standard volume-form on $\mathbb{S}^{n-1}$) and keeping in mind a possible change of sign in case the diffeomorphism u reverses the orientation of $\mathbb{S}^{n-1}$, we obtain

$$0 = \fint_{\mathbb{S}^{n-1}} xdv_g = \pm \fint_{\mathbb{S}^{n-1}} u^\sharp(xdv_g) = \pm \fint_{\mathbb{S}^{n-1}} u(x)dv_g(x), \tag{2.1.5}$$

i.e.

$$\fint_{\mathbb{S}^{n-1}} u \; d\mathcal{H}^{n-1} = 0. \tag{2.1.6}$$

By the *Poincare inequality on* $\mathbb{S}^{n-1}$ for fields with zero mean (see (B.0.6)) and since $|u| \equiv 1$,

$$1 \;=\; \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \; d\mathcal{H}^{n-1} \;\geq\; \fint_{\mathbb{S}^{n-1}} |u|^2 \; d\mathcal{H}^{n-1} \;=\; 1. \tag{2.1.7}$$

The *equality case in the Poincare inequality* implies that *in the Fourier expansion of u in spherical harmonics, no other spherical harmonics except the first order ones should appear.* Since the first order spherical harmonics are the coordinate functions normalized by a suitable constant, we deduce that $u(x) = Ox$ for some $O \in \mathbb{R}^{n\times n}$. Although it is not really important here, as we mention in Remark B.0.1. in the Appendix B, we actually have that $O = \nabla u_h(0)$, where $u_h : \overline{B^n} \mapsto \mathbb{R}^n$ is the *harmonic extension of* u *in* B^n.

This linear map would transform $\mathbb{S}^{n-1}$ into the boundary of an *ellipsoid*, which after possibly an orthogonal change of coordinates is

$$u(\mathbb{S}^{n-1}) := \Big\{ y = (y_1, y_2, ...y_n) \in \mathbb{R}^n : \frac{y_1^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} + ... + \frac{y_n^2}{\sigma_n^2} = 1 \Big\}, \tag{2.1.8}$$

where $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_n$ are the eigenvalues of the symmetric matrix $\sqrt{O^t O}$. By assumption $u(\mathbb{S}^{n-1}) \equiv \mathbb{S}^{n-1}$, and this forces $\sigma_1^2 = \sigma_2^2 = ... = \sigma_n^2 = 1$, i.e. $O \in O(n)$. □

Remark 2.1.3. The previous proof can also be carried out under less restrictive regularity assumptions, as long as the maps in question preserve\- reverse the orientation of $\mathbb{S}^{n-1}$ in the sense of Definition 2.1.1., a condition that should again be imposed because of the *rank-one connectedness of the orthogonal group.*

For instance, if $n \geq 2$, $1 \leq p \leq \infty$ and $u \in W^{1,p}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ is a *generalized orientation-preserving isometry* of $\mathbb{S}^{n-1}$, with the same proof as before we can deduce that there exists $R \in SO(n)$ so that $u(x) = Rx$ for every $x \in \mathbb{S}^{n-1}$. Indeed, any such map is *Lipschitz continuous* and bijective. The surjectivity is a consequence of the more general fact that *any isometry v of a compact metric space (X, d) (in our case $\mathbb{S}^{n-1}$ with the geodesic distance induced by the round metric g) is surjective.* We remind the reader of the elementary proof of this topological fact.

Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Define the sequence $(x_l)_{l\in\mathbb{N}} \in X$ via $x_0 := x$, $x_{l+1} := v(x_l)$. Since X is compact, we can assume (up to passing to a non-relabeled subsequence) that $(x_l)_{l\in\mathbb{N}}$ is a convergent and therefore Cauchy sequence. Thus, for any $l < m$ sufficiently large we have $d(x_l, x_m) < \varepsilon$. By the fact that v is an isometry, we recursively obtain

$$d(x; v(X)) \leq d(x_0, v(x_{m-l-1})) := d(x_0, x_{m-l}) = \cdots = d(x_l, x_m) < \varepsilon, \tag{2.1.9}$$

and since $\varepsilon > 0$ was arbitrary and $v(X)$ is closed, we conclude that $x \in v(X)$.

With the same argument as in the smooth case we obtain that $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0$ and also the rest of the proof can be carried out unchanged.

Remark 2.1.4. An equivalent way to state Theorem 2.1.2. would be to say that *the only isometric maps that transform $\mathbb{S}^{n-1}$ into another round sphere (of radius 1 of course) are the rigid motions.* As we mentioned in Section 1.2, if $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is an *isometric embedding*, $u(\mathbb{S}^{n-1})$ is a *closed Lipschitz hypersurface in* $\mathbb{R}^n$ with the same $(n-1)$-Hausdorff measure as $\mathbb{S}^{n-1}$. If E_u stands for the bounded domain in $\mathbb{R}^n$ with $\partial(E_u) = u(\mathbb{S}^{n-1})$, we know from the classical *Euclidean Isoperimetric Inequality* that

$$\mathrm{Vol}(E_u) \leq \mathrm{Vol}(B^n) = \omega_n, \tag{2.1.10}$$

with equality iff E_u is a ball in $\mathbb{R}^n$, i.e. iff $u(\mathbb{S}^{n-1})$ is a *round sphere of* $\mathbb{R}^n$, i.e. iff *u is a rigid motion of* $\mathbb{S}^{n-1}$ according to Theorem 2.1.2..

With the notations and conventions we introduced in Section 1.2, the last inequality can be rewritten in this case as

$$|V_n(u)| := \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1} \right| \leq 1. \tag{2.1.11}$$

Of course, for a general Lipschitz map $v : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ that is not necessarily an embedding, the last integral may not always represent the actual (signed) enclosed volume. The next theorem can be regarded as a slight generalization of Theorem 2.1.2., since it asserts that for generalized isometric maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ the inequality (2.1.11) can be obtained in a simple way, even without referring immediately to the classical isoperimetric inequality, and the equality case is of course characterized as before.

Theorem 2.1.5. *Let $n \geq 2$ and $1 \leq p \leq \infty$. For every generalized isometric map $u \in W^{1,p}(\mathbb{S}^{n-1};\mathbb{R}^n)$ the inequality (2.1.11) holds, with equality iff u is a rigid motion of $\mathbb{S}^{n-1}$, i.e. iff there exist $O \in O(n)$ and $b \in \mathbb{R}^n$ so that for every $x \in \mathbb{S}^{n-1}$*

$$u(x) = Ox + b. \tag{2.1.12}$$

Proof. Recalling (1.2.10), we have $V_n(u+b) = V_n(u)$ for every constant $b \in \mathbb{R}^n$. Indeed,

$$\begin{aligned} V_n(u+b) &:= \fint_{\mathbb{S}^{n-1}} \left\langle u+b, \bigwedge_{i=1}^{n-1} \partial_{\tau_i}(u+b)\right\rangle \, d\mathcal{H}^{n-1} = \fint_{B^n} \det\nabla(u+b)_h = \fint_{B^n} \det\nabla(u_h+b) \\ &= \fint_{B^n} \det\nabla u_h = \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \, d\mathcal{H}^{n-1} =: V_n(u), \end{aligned}$$

and in particular for $b := -\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1}$,

$$V_n(u) := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left\langle u - \fint_{\mathbb{S}^{n-1}} u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \, d\mathcal{H}^{n-1}. \tag{2.1.13}$$

Since u is assumed to be *generalized isometric*, by taking the trace and the determinant in (2.1.2), we have again that $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$

$$\frac{|\nabla_T u|^2}{n-1} = 1, \qquad \left|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right| = \sqrt{\det(\nabla_T u^t \nabla_T u)} = 1. \tag{2.1.14}$$

Using the *Cauchy-Schwarz inequality* and again the *sharp Poincare inequality on* $\mathbb{S}^{n-1}$,

$$\begin{aligned} |V_n(u)| &= \left| \fint_{\mathbb{S}^{n-1}} \left\langle u - \fint_{\mathbb{S}^{n-1}} u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle \, d\mathcal{H}^{n-1} \right| \\ &\leq \fint_{\mathbb{S}^{n-1}} \left| u - \fint_{\mathbb{S}^{n-1}} u \right| \left| \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right| d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left| u - \fint_{\mathbb{S}^{n-1}} u \right| \, d\mathcal{H}^{n-1} \\ &\leq \left(\fint_{\mathbb{S}^{n-1}} \left| u - \fint_{\mathbb{S}^{n-1}} u \right|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \leq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \\ &= \ 1. \end{aligned}$$

If $|V_n(u)| = 1$, then again equalities must hold in all places in the above chain of inequalities. By the equality case in Poincare's inequality we have again $u(x) = Ax + \fint_{\mathbb{S}^{n-1}} u$ for some $A \in \mathbb{R}^{n\times n}$. It is also fairly easy to check that $A \in O(n)$, even by arguing analytically rather than geometrically this time. On the one hand, by the equality cases above we obtain

$$1 = \fint_{\mathbb{S}^{n-1}} \left| u - \fint_{\mathbb{S}^{n-1}} u \right|^2 \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} |Ax|^2 \, d\mathcal{H}^{n-1} = \frac{|A|^2}{n}, \tag{2.1.15}$$

and also

$$1 = \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| = \left| \fint_{B^n} \det \nabla u_h \right| = \left| \fint_{B^n} \det \nabla (Ax)_h \right| = |\det A| \, . \quad (2.1.16)$$

By a standard argument via the arithmetic mean-geometric mean inequality, if we consider the polar decomposition $A = O\sqrt{A^t A}$, where $O \in O(n)$ and label $0 \leq \sigma_1 \leq \cdots \leq \sigma_n$ the eigenvalues of $\sqrt{A^t A}$, then

$$1 = \frac{|A|^2}{n} = \frac{\sum_{i=1}^n \sigma_i^2}{n} \geq \left(\prod_{i=1}^n \sigma_i^2 \right)^{\frac{1}{n}} = |\det A|^{\frac{2}{n}} = 1. \quad (2.1.17)$$

The equality in this algebraic inequality yields $\sigma_1 = \cdots = \sigma_n = 1$, hence $A^t A = I_n$, i.e. $A = O \in O(n)$, which completes the proof. $\square$

In Chapter 3 we will see how to suitably adapt the proof, in order to obtain the *stability version* of the previous theorem both in a *qualitative and a quantitative manner.*

2.2 The conformal group of $\mathbb{S}^{n-1}$ when $n \geq 3$

Let us now discuss the corresponding result for the conformal case. Similar to Definition 2.1.1., we adopt

Definition 2.2.1. Let $n \geq 3$.

(*i*) A map $u \in C^1(\mathbb{S}^{n-1}; \mathbb{R}^n)$ is called *conformal* iff at every $x \in \mathbb{S}^{n-1}$ its gradient is a nonsingular linear map and u preserves the angle between any two tangent vectors at that point, or equivalently iff at every $x \in \mathbb{S}^{n-1}$

$$\left(\nabla_T u^t \nabla_T u \right)(x) = \left(\frac{|\nabla_T u(x)|^2}{n-1} \right) I_x. \quad (2.2.1)$$

(*ii*) A map $u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ is called *generalized conformal* iff at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$

$$\left(\nabla_T u^t \nabla_T u \right)(x) = \left(\frac{|\nabla_T u(x)|^2}{n-1} \right) I_x, \quad (2.2.2)$$

where the gradient is to be understood here in the weak sense.

(*iii*) A map $u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ is called a *generalized orientation-preserving\-reversing conformal map* of $\mathbb{S}^{n-1}$ iff at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ the *intrinsic gradient* of u is an orientation-preserving\-reversing linear conformal map between $T_x\mathbb{S}^{n-1} \cup \{\infty\}$ and $T_{u(x)}\mathbb{S}^{n-1} \cup \{\infty\}$.

(*iv*) The group of all conformal diffeomorphisms of $\mathbb{S}^{n-1}$ will be denoted by $Conf(\mathbb{S}^{n-1})$, and again the subgroup consisting only of the orientation-preserving ones will be denoted by $Conf_+(\mathbb{S}^{n-1})$.

We can now state Liouville's theorem in this setting and give its *new intrinsic proof.*

Theorem 2.2.2. ***(Liouville's Theorem for the conformal group of*** $\mathbb{S}^{n-1}$***)***
Let $n \geq 3$. Then $u \in Conf(\mathbb{S}^{n-1})$ iff it is a Möbius transformation of $\mathbb{S}^{n-1}$, i.e. iff there exist $O \in O(n)$, $\xi \in \mathbb{S}^{n-1}$ and $\lambda > 0$ so that for every $x \in \mathbb{S}^{n-1}$

$$u(x) = O\phi_{\xi,\lambda}(x). \tag{2.2.3}$$

Here, $\phi_{\xi,\lambda} := \sigma_\xi^{-1} \circ i_\lambda \circ \sigma_\xi$, where σ_ξ is the stereographic projection of $\mathbb{S}^{n-1}$ onto the tangent plane $T_\xi\mathbb{S}^{n-1} \cup \{\infty\}$ and $i_\lambda : T_\xi\mathbb{S}^{n-1} \mapsto T_\xi\mathbb{S}^{n-1}$ is the dilation in $T_\xi\mathbb{S}^{n-1}$ by factor λ. Analytically, $\phi_{\xi,\lambda}$ is given by the formula

$$\phi_{\xi,\lambda}(x) := \frac{-\lambda^2\big(1 - \langle x,\xi\rangle\big)\xi + 2\lambda\big(x - \langle x,\xi\rangle\xi\big) + \big(1 + \langle x,\xi\rangle\big)\xi}{\lambda^2\big(1 - \langle x,\xi\rangle\big) + \big(1 + \langle x,\xi\rangle\big)}. \tag{2.2.4}$$

Proof. The argument is similar to the one in the proof of Theorem 2.1.2.. The maps $(\phi_{\xi,\lambda})_{\xi\in\mathbb{S}^{n-1},\lambda>0}$ are conformal diffeomorphisms of $\mathbb{S}^{n-1}$ and conversely, if $u \in Conf(\mathbb{S}^{n-1})$, by taking the determinant in both sides of (2.2.2) and recalling the very first remark in the beginning of the Chapter, we have that for every $x \in \mathbb{S}^{n-1}$

$$\sqrt{\det\left((\nabla_T u^t \nabla_T u)(x)\right)} = \sqrt{\det\left((d_x u)^t d_x u\right)} = \left(\frac{|d_x u|^2}{n-1}\right)^{\frac{n-1}{2}} = \left(\frac{|\nabla_T u(x)|^2}{n-1}\right)^{\frac{n-1}{2}}.$$

Since u is a conformal diffeomorphism of $\mathbb{S}^{n-1}$, we can use the *area formula*, *Jensen's inequality* and again the *sharp Poincare inequality on* $\mathbb{S}^{n-1}$ to obtain

$$\begin{aligned} 1 = \frac{\mathcal{H}^{n-1}(u(\mathbb{S}^{n-1}))}{n\omega_n} &= \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)}\, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \\ &\geq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1}\right)^{\frac{n-1}{2}} \geq \left(\fint_{\mathbb{S}^{n-1}} \left|u - \fint_{\mathbb{S}^{n-1}} u\right|^2 d\mathcal{H}^{n-1}\right)^{\frac{n-1}{2}}. \end{aligned} \tag{2.2.5}$$

If we assume for the moment that $\fint_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} = 0$, then the last averaged integral is exactly equal to 1 since u takes values on $\mathbb{S}^{n-1}$. In this case, again equalities must hold at each step in the above chain of inequalities, and with the same reasoning as in the proof of Theorem 2.1.2. we would infer that $u(x) = Ox$ for some $O \in O(n)$. Of course, for the above argument to work we need the convexity of the function $t \mapsto t^{\frac{n-1}{2}}$, which is true exactly iff $n \geq 3$ (in any case, for $n = 2$ conformality is a trivial notion for maps from $\mathbb{S}^1$ to itself). We only have to justify why one can always reduce to this case.

If $b_u := \fint_{\mathbb{S}^{n-1}} u \neq 0$, one can show that there always exist $\xi_0 \in \mathbb{S}^{n-1}$ and $\lambda_0 > 0$ so that

$$\fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0} \, d\mathcal{H}^{n-1} = 0. \tag{2.2.6}$$

Indeed, consider the map $F : \mathbb{S}^{n-1} \times [0,1] \mapsto \overline{B^n}$, defined as

$$F(\xi,\lambda) := \fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi,\lambda} \, d\mathcal{H}^{n-1} \quad \text{for } \lambda \in (0,1] \quad \text{and} \quad F(\xi,0) := \lim_{\lambda \downarrow 0^+} F(\xi,\lambda). \tag{2.2.7}$$

The map F is obviously continuous, and

$$F(\xi,0) = u(\xi) \quad \text{for every } \xi \in \mathbb{S}^{n-1} \implies F(\mathbb{S}^{n-1},0) = u(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}, \tag{2.2.8}$$

whereas

$$F(\mathbb{S}^{n-1},1) = \{b_u\}. \tag{2.2.9}$$

In other words, F is a continuous homotopy between $\mathbb{S}^{n-1}$ and the point $b_u \in \overline{B^n} \diagdown \{0\}$, and therefore

$$\exists \; \lambda_0 \in (0,1) \;\; \text{s.t.} \;\; 0 \in F(\mathbb{S}^{n-1},\lambda_0) \iff \exists \; \xi_0 \in \mathbb{S}^{n-1} \;\; \text{s.t.} \;\; F(\xi_0,\lambda_0) = 0. \tag{2.2.10}$$

We can now apply the previous argument to the conformal map $u \circ \phi_{\xi_0,\lambda_0}$ that has zero average on $\mathbb{S}^{n-1}$, to deduce that for a matrix $O \in O(n)$ and for every $x \in \mathbb{S}^{n-1}$

$$(u \circ \phi_{\xi_0,\lambda_0})(x) = Ox \implies u(x) = O\phi^{-1}_{\xi_0,\lambda_0}(x) = O\phi_{\xi,\lambda}(x), \tag{2.2.11}$$

where $\xi := \xi_0 \in \mathbb{S}^{n-1}$ and $\lambda := \frac{1}{\lambda_0} > 0$. $\square$

Remark 2.2.3. The Möbius transformations of $\mathbb{S}^{n-1}$ could of course alternatively be described by performing an *inversion* on $T_\xi\mathbb{S}^{n-1}$ with respect to some center, say the *origin* ξ of the affine hyperplane $T_\xi\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ and some *radius*, say $\sqrt{\lambda} > 0$. These maps however would correspond exactly to the Möbius transformations produced by dilation in $T_\xi\mathbb{S}^{n-1}$ by factor $\frac{1}{\lambda}$, composed finally with a *flip in* $\mathbb{R}^n$, i.e. an orthogonal map that would change back the orientation.

We also gave explicitely the formula (2.2.4), because one can compute directly out of it the representation of the *infinitesimal generators of* $Conf(\mathbb{S}^{n-1})$. The corresponding formula in $\mathbb{R}^n$ has definitely appeared in the literature (see for example Formula (2.1) in [Res13], Chapter 4, Paragraph 2) and probably its version on $\mathbb{S}^{n-1}$ as well. In any case, it is an elementary exercise in differential geometry to obtain it by considering a general C^1-*curve* $\gamma : (-\delta, \delta) \mapsto Conf(\mathbb{S}^{n-1})$ for some $0 < \delta \ll 1$, where

$$\gamma(t) = R(t)\phi_{\xi(t),\lambda(t)} \ ; \quad R : (-\delta,\delta) \mapsto SO(n),\ \xi : (-\delta,\delta) \mapsto \mathbb{S}^{n-1},\ \lambda : (-\delta,\delta) \mapsto (0,\infty)$$

with

$$\gamma(0) = \mathrm{id}_{\mathbb{S}^{n-1}}, \quad \text{i.e. } R(0) = I_n,\ \xi(0) = \xi \in \mathbb{S}^{n-1} \text{ (arbitrary)}, \quad \lambda(0) = 1,$$

and simply compute the derivative $\dot{\gamma}(0)$. This leads to the following characterization, which we will use in Chapter 4.

$$T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf(\mathbb{S}^{n-1}) \equiv \left\{ Sx + \mu\big(\langle x, \xi\rangle x - \xi\big) : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n; S \in Skew(n),\ \xi \in \mathbb{S}^{n-1},\ \mu \in \mathbb{R} \right\}.$$

Remark 2.2.4. The proof of Theorem 2.2.2. can also be carried out in a slightly more general context without considering necessarily conformal diffeomorphisms of $\mathbb{S}^{n-1}$. Indeed, if $n \geq 3$ and $u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ is a *generalized orientation-preserving\-reversing conformal map* of $\mathbb{S}^{n-1}$ of *degree* 1\ -1 respectively, with the same proof as before we can deduce that u is an orientation-preserving\-reversing Möbius transformation of $\mathbb{S}^{n-1}$ of the form (2.2.3). *The only possible subtlety is why one can choose also in this case a Möbius transformation of* $\mathbb{S}^{n-1}$ *to fix the mean value of the map at 0.*

Before justifying this point, let us briefly recall some basic facts regarding the notion of *degree for* $W^{1,n-1}$*-Sobolev maps from* $\mathbb{S}^{n-1}$ *to itself.* We follow [BN95], where the notion and properties of the degree for appropriate classes of Sobolev and BMO maps between smooth closed oriented manifolds of the same dimension are introduced.

In the *regular case*, for a map $u \in C^1(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ there is a classical way to define its *degree* from the point of view of *differential topology.* By *Sard's theorem*, $\mathcal{H}^{n-1}$-a.e. $p \in \mathbb{S}^{n-1}$ is a *regular value* of u, i.e. $u^{-1}(p) = \{x_1, \dots, x_k\}$ for some $k \in \mathbb{N}$ and also for every $j = 1, \dots, k$ the *intrinsic gradient* $d_{x_j}u : T_{x_j}\mathbb{S}^{n-1} \mapsto T_{u(x_j)}\mathbb{S}^{n-1}$ is a *nonsingular linear map.* Therefore, for the *intrinsic Jacobian* $J_{x_j}u$ of $d_{x_j}u$ (computed with respect to the local orthonormal frame $\{\tau_1, \dots, \tau_{n-1}\}$), we have that $\det(J_{x_j}u) \neq 0$. The *degree* of u with respect to the point $p \in \mathbb{S}^{n-1}$ is then defined as

$$\deg(u; p) := \sum_{j=1}^{k} \mathrm{sgn}(\det(J_{x_j}u)). \tag{2.2.12}$$

A basic fact in differential topology is that the degree is independent of the choice of the regular value $p \in \mathbb{S}^{n-1}$ and its unique value can thus be denoted by $\deg u$. Intuitively, it counts *how many times* $\mathbb{S}^{n-1}$ *is covered by* $u(\mathbb{S}^{n-1})$, with the orientation being taken into account each time.

If $u \in C^0(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ the classical way to define its *degree* from the point of view of *algebraic topology* is via the *induced group homomorphism* $u^* : H^{n-1}(\mathbb{S}^{n-1}) \mapsto H^{n-1}(\mathbb{S}^{n-1})$. Here, $H^{n-1}(\mathbb{S}^{n-1})$ is the $(n-1)$-th *homology group* of $\mathbb{S}^{n-1}$, which is known to be *isomorphic* to $\mathbb{Z}$. It follows that there exists $a \in \mathbb{Z}$ so that $u^*(m) := a \cdot m$ for every $m \in \mathbb{Z}$. The integer a is then called the *(algebraic-topological) degree* of u, and these two notions of degree coincide for C^1 maps. Another basic topological fact that we will use, is that a map $u \in C^0(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ of *non-zero degree* must be *surjective*. Indeed, if there exists $p \in \mathbb{S}^{n-1}$ such that the image of u lies entirely in $\mathbb{S}^{n-1} \setminus \{p\}$, which is *contractible*, then u must be *null-homotopic* and therefore must have degree 0.

For a map $u \in C^1(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ there is also an *integral formula* to calculate the degree in terms of *integration of differential forms*. In particular, if ω is a smooth $(n-1)$-form on $\mathbb{S}^{n-1}$, then

$$\fint_{\mathbb{S}^{n-1}} u^\sharp(\omega) \;=\; \deg u \fint_{\mathbb{S}^{n-1}} \omega, \tag{2.2.13}$$

which leads to the analytic expression

$$\deg u := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1}, \tag{2.2.14}$$

i.e. *for a sufficiently regular map from $\mathbb{S}^{n-1}$ to itself, the averaged signed-volume has the extra meaning of being the degree of the map.*

This analytic definition of the degree can be extended to maps $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, that is Sobolev maps $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ such that $|u(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$. A useful *approximation lemma* in this case is

Lemma 2.2.5. ***(see [BN95], Section I.3. Theorem 1, Section I.4. Lemma 7)*** *Let $n \geq 2$. For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, there exists a sequence of smooth maps $(u_j)_{j\in\mathbb{N}} \in C^\infty(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ with the property that*

$$u_j \to u \text{ strongly in } W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}) \;\text{ and }\; \deg u_j = \deg u \quad \forall j \in \mathbb{N}. \tag{2.2.15}$$

Using this lemma, we can prove the following.

Lemma 2.2.6. *Let $n \geq 3$ and $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ be a generalized orientation-preserving\-reversing conformal map of $\mathbb{S}^{n-1}$ of degree $1\backslash -1$. Then,*

(*i*) $\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} = 1.$

(*ii*) *There exist $\xi_0 \in \mathbb{S}^{n-1}$ and $\lambda_0 > 0$ so that $\fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0}\, d\mathcal{H}^{n-1} = 0$.*

Proof. We discuss the case of a generalized orientation-preserving conformal map of degree 1, the other case being essentially the same. Let $(u_j)_{j\in\mathbb{N}} \in C^\infty(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ be the previously mentioned sequence that is strongly approximating u in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ and has the property that $\deg u_j = \deg u = 1$ for every $j \in \mathbb{N}$. Of course, the maps $(u_j)_{j\in\mathbb{N}}$ do not necessarily have to be conformal at the first place. Up to passing to a non-relabeled subsequence, we can without loss of generality also suppose that $u_j \to u$ and $\nabla_T u_j \to \nabla_T u$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. For part (i), we have by all our assumptions that $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$,

$$u^\sharp(dv_g) = \sqrt{\det(\nabla_T u^t \nabla_T u)}\ dv_g = \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} dv_g, \tag{2.2.16}$$

and by approximation, the analytic formula for the degree in terms of integration of $(n-1)$- forms on $\mathbb{S}^{n-1}$ holds true for u as well, i.e.

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} u^\sharp(dv_g) = \deg u \fint_{\mathbb{S}^{n-1}} dv_g = 1. \tag{2.2.17}$$

For part (ii), by the properties of the degree we have that the maps u_j are surjective on $\mathbb{S}^{n-1}$ for every $j \in \mathbb{N}$. Therefore, by the *topological argument in the proof of Theorem 2.2.2., which actually does not rely on whether the maps $(u_j)_{j\in\mathbb{N}}$ are conformal or not,* there exist $(\xi_j)_{j\in\mathbb{N}} \in \mathbb{S}^{n-1}$ and $(\lambda_j)_{j\in\mathbb{N}} \in (0,1]$, so that for every $j \in \mathbb{N}$

$$\fint_{\mathbb{S}^{n-1}} u_j \circ \phi_{\xi_j,\lambda_j}\ d\mathcal{H}^{n-1} = 0. \tag{2.2.18}$$

Up to non-relabeled subsequences we can suppose that $\xi_j \to \xi_0 \in \mathbb{S}^{n-1}$ and $\lambda_j \to \lambda_0 \in [0,1]$, thus $\phi_{\xi_j,\lambda_j} \rightharpoonup \phi_{\xi_0,\lambda_0}$ weakly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ and also pointwise $\mathcal{H}^{n-1}$ - a.e. on $\mathbb{S}^{n-1}$. In fact $\lambda_0 \in (0,1]$, i.e. the Möbius transformations $(\phi_{\xi_j,\lambda_j})_{j\in\mathbb{N}}$ do not converge weakly to the *trivial map* $\phi_{\xi_0,0}(x) = \xi_0$.

Indeed, suppose that this was the case. Since then, $u_j \circ \phi_{\xi_j,\lambda_j} \to u \circ \phi_{\xi_0,0} = u(\xi_0)$ pointwise $\mathcal{H}^{n-1}$-a.e. and $|u_j \circ \phi_{\xi_j,\lambda_j}| = 1$, we could use the *Dominated Convergence Theorem* to infer that

$$u(\xi_0) = \fint_{\mathbb{S}^{n-1}} u(\xi_0)\ d\mathcal{H}^{n-1}(x) = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} u_j \circ \phi_{\xi_j,\lambda_j}\ d\mathcal{H}^{n-1} = 0, \tag{2.2.19}$$

$$|u(\xi_0)| = \fint_{\mathbb{S}^{n-1}} |u(\xi_0)|\ d\mathcal{H}^{n-1}(x) = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} |u_j \circ \phi_{\xi_j,\lambda_j}|\ d\mathcal{H}^{n-1} = 1, \tag{2.2.20}$$

and derive a contradiction. Having justfied that $0 < \lambda \leq 1$, and since $u_j \circ \phi_{\xi_j,\lambda_j} \to u \circ \phi_{\xi_0,\lambda_0}$ pointwise $\mathcal{H}^{n-1}$-a.e. and $|u_j \circ \phi_{\xi_j,\lambda_j}| = 1$, what we actually obtain by the *Dominated Convergence Theorem* is that

$$\fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0}\ d\mathcal{H}^{n-1} = 0. \qquad \square$$

Let us conclude this Section by mentioning that the previous arguments can easily be modified in order to give a *compactness statement for sequences of orientation-preserving\-reversing degree 1\-1 approximately conformal maps* on $\mathbb{S}^{n-1}$. For simplicity, we present again the statement in the case of orientation-preserving degree 1 maps, the other case being completely analogous.

Proposition 2.2.7. *Let $n \geq 3$ and $(u_j)_{j\in\mathbb{N}} \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ be a sequence of $\mathcal{H}^{n-1}-$a.e. orientation-preserving maps of degree 1, which approximate in average the conformal group of $\mathbb{S}^{n-1}$ in the sense that*

$$\lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\left(\frac{|\nabla_T u_j|^2}{n-1} \right)^{\frac{n-1}{2}} - \sqrt{\det\left(\nabla_T u_j^t \nabla_T u_j\right)} \right) \, d\mathcal{H}^{n-1} = 0, \tag{2.2.21}$$

which as a condition is in this case equivalent to

$$\lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u_j|^2}{n-1} \right)^{\frac{n-1}{2}} \, d\mathcal{H}^{n-1} = 1. \tag{2.2.22}$$

Then there exist Möbius transformations $(\phi_j)_{j\in\mathbb{N}} \in Conf_+(\mathbb{S}^{n-1})$ and $R \in SO(n)$ so that up to a non-relabeled subsequence

$$u_j \circ \phi_j \to R\mathrm{id}_{\mathbb{S}^{n-1}} \;\; \text{strongly in} \;\; W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}). \tag{2.2.23}$$

Proof. By the *degree one condition*, we can again find sequences $(\xi_j)_{j\in\mathbb{N}} \in \mathbb{S}^{n-1}$ and $(\lambda_j)_{j\in\mathbb{N}} \in (0,1]$, so that after setting $\phi_j := \phi_{\xi_j,\lambda_j} \in Conf_+(\mathbb{S}^{n-1})$ and $\tilde{u}_j := u_j \circ \phi_j$, we have $\fint_{\mathbb{S}^{n-1}} \tilde{u}_j \, d\mathcal{H}^{n-1} = 0$. Thanks to the *conformal invariance* of the quantities involved, (2.2.22) is equivalent to

$$\lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}_j|^2}{n-1} \right)^{\frac{n-1}{2}} \, d\mathcal{H}^{n-1} = 1. \tag{2.2.24}$$

Since $\fint_{\mathbb{S}^{n-1}} \tilde{u}_j = 0$, the sequence $(\tilde{u}_j)_{j\in\mathbb{N}}$ is also *uniformely bounded* in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, hence up to a non-relabeled subsequence converges weakly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ (and up to a further one also pointwise $\mathcal{H}^{n-1}$-a.e.) to a map $\tilde{u} \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$. Since $\tilde{u}_j \to \tilde{u}$ strongly in $L^{n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, we obtain in particular

$$\fint_{\mathbb{S}^{n-1}} \tilde{u} \, d\mathcal{H}^{n-1} = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \tilde{u}_j \, d\mathcal{H}^{n-1} = 0, \tag{2.2.25}$$

and by lower semicontinuity

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}|^2}{n-1} \right)^{\frac{n-1}{2}} \, d\mathcal{H}^{n-1} \leq \liminf_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}_j|^2}{n-1} \right)^{\frac{n-1}{2}} \, d\mathcal{H}^{n-1} = 1. \tag{2.2.26}$$

We can then apply the same argument as in the proof of Theorem 2.2.2., to end up with the chain of inequalities

$$1 \geq \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} \geq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} \geq \left(\fint_{\mathbb{S}^{n-1}} |\tilde{u}|^2\right)^{\frac{n-1}{2}} = 1, \tag{2.2.27}$$

the last equality holding again because $|\tilde{u}(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$. With the same reasoning as before, $\tilde{u}(x) = Rx$ for some $R \in O(n)$.

Through the previous arguments we actually obtained that

$$\tilde{u}_j \rightharpoonup \tilde{u} \text{ weakly in } W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}), \quad \fint_{\mathbb{S}^{n-1}} \tilde{u} = \fint_{\mathbb{S}^{n-1}} \tilde{u}_j = 0,$$

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \tilde{u}_j|^2}{n-1}\right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} = 1, \tag{2.2.28}$$

so actually $\tilde{u}_j \to \tilde{u}$ *strongly* in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$. Finally, since the degree is *stable* under this notion of convergence,

$$1 = \deg \tilde{u} = \fint_{\mathbb{S}^{n-1}} \left\langle Rx, \bigwedge_{i=1}^{n-1} \partial_{\tau_i}(Rx)\right\rangle d\mathcal{H}^{n-1} = \det R, \quad \text{i.e.} \quad R \in SO(n). \qquad \square$$

Chapter 3

On the stability of $Isom(\mathbb{S}^{n-1})$ among almost isometric maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ when $n \geq 2$

In this Chapter we are concerned with *approximate versions* of Theorem 2.1.2., or actually its slightly more general version, i.e. Theorem 2.1.5.. As we have remarked in Section 1.2, due to the abundance of isometric immersions of $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$ that are less regular than C^2, an extra information about the *generalized isoperimetric deficit* produced by the maps under consideration is necessary to expect stability of rigid motions among (almost) isometric maps defined on $\mathbb{S}^{n-1}$. As the reader will notice, the results of Section 3.1 are actually special cases of the ones in Section 3.2, but since the arguments take a somewhat simpler form, we have chosen to proceed constructively and present the results in two separate Sections.

3.1 The case of isometric maps with small isoperimetric deficit

Recalling the Definition 2.1.1., let us denote by $\mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ the class of all *generalized isometric maps* from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$, i.e. all Lipschitz maps that satisfy (2.1.2) $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. By Theorem 2.1.5., we have that the quantity

$$\varepsilon_u = 1 - |V_n(u)| := 1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u; \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| = 1 - \left| \fint_{B^n} \det\nabla u_h(x) \, dx \right| \quad (3.1.1)$$

is nonnegative for every $u \in \mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$, represents the *generalized isoperimetric deficit* related to u, and vanishes precisely when u is a rigid motion of $\mathbb{S}^{n-1}$. It therefore provides a natural choice for the deficit in terms of which the stability of $Isom(\mathbb{S}^{n-1})$ inside

$\mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ is going to be examined. To begin with, we provide a *compactness statement* related to Theorem 2.1.5., which we are going to use soon.

Lemma 3.1.1. *Let $n \geq 2$, $(u_k)_{k\in\mathbb{N}} \in \mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and suppose that*

$$\lim_{k\to\infty} \varepsilon_{u_k} = 0. \tag{3.1.2}$$

There exists $O \in O(n)$ so that up to a non-relabeled subsequence

$$u_k - \fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} \to O\mathrm{id}_{\mathbb{S}^{n-1}} \text{ strongly in } W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n). \tag{3.1.3}$$

Proof. As the statement suggests, we can translate the maps by their *centers of mass* and suppose without loss of generality that $\fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} = 0$ for all $k \in \mathbb{N}$. Then,

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,2}(\mathbb{S}^{n-1})} \leq \sqrt{\frac{n}{n-1}} \cdot \sup_{k\in\mathbb{N}} \|\nabla_T u_k\|_{L^2(\mathbb{S}^{n-1})} = \sqrt{n} < +\infty,$$

and we can extract a non-relabeled $W^{1,2}$-weakly convergent subsequence $u_k \rightharpoonup u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$. In particular,

$$\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = \lim_{k\to\infty} \fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} = 0, \tag{3.1.4}$$

$$\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u_k|^2}{n-1} = 1, \tag{3.1.5}$$

since the fact that $(u_k)_{k\in\mathbb{N}}$ are isometric implies that $\frac{|\nabla_T u_k|^2}{n-1} = 1$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. For the same reason, and for all $k \in \mathbb{N}$,

$$\fint_{\mathbb{S}^{n-1}} \left| \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right|^2 d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \det(\nabla_T u_k^t \nabla_T u_k) \, d\mathcal{H}^{n-1} = 1, \tag{3.1.6}$$

the integrands being equal to 1 pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. Up to a further subsequence we can also assume that $u_k \to u$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$.

We can now argue as in the proof of Theorem 2.1.5. by estimating

$$\begin{aligned} 1 - \varepsilon_{u_k} &= \left| \fint_{\mathbb{S}^{n-1}} \left\langle u_k, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right\rangle d\mathcal{H}^{n-1} \right| \\ &\leq \left(\fint_{\mathbb{S}^{n-1}} |u_k|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^{n-1}} \left| \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right|^2 d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \\ &= \left(\fint_{\mathbb{S}^{n-1}} |u_k|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.1.7}$$

By letting $k \to \infty$ and since $u_k \to u$ strongly in $L^2(\mathbb{S}^{n-1};\mathbb{R}^n)$, we deduce that

$$\fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \geq 1.$$

By the fact that $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0$ and the Poincare inequality on $\mathbb{S}^{n-1}$ once again,

$$1 \geq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} \geq \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \geq 1. \tag{3.1.8}$$

By the equality case we have again $u(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$ with $|A|^2 = n$. Since

$$1 = \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \leq \lim_{k\to\infty} \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u_k|^2}{n-1} = 1,$$

the weak L^2-convergence of the gradients together with the convergence of their L^2-norms actually imply that $u_k \to u := A\mathrm{id}_{\mathbb{S}^{n-1}}$ strongly $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$. It remains to justify that $|\det A| = 1$, which can be done similarly to the exact case.

Indeed, having deduced that $\nabla_T u_k \to \nabla_T u$ strongly in $L^2(\mathbb{S}^{n-1})$, up to a further non-relabeled subsequence we can also assume that $\nabla_T u_k \longrightarrow \nabla_T u$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$, so that also

$$\begin{aligned}
g_k := \left\langle u_k, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right\rangle &\longrightarrow \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle, && \text{pointwise } \mathcal{H}^{n-1} - \text{a.e.,}\\
|g_k| \leq \left| \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right| |u_k| &= |u_k|, && \text{for every } k \in \mathbb{N},\\
|u_k| \longrightarrow |u| &\qquad\quad , && \text{pointwise } \mathcal{H}^{n-1} - \text{a.e.,}\\
\sup_{k\in\mathbb{N}} \fint_{\mathbb{S}^{n-1}} |u_k| \, d\mathcal{H}^{n-1} \leq \sup_{k\in\mathbb{N}} \|u_k\|_{L^2(\mathbb{S}^{n-1})} &\ll +\infty, && \text{since } \|u_k\|_{L^2(\mathbb{S}^{n-1})} \to 1.
\end{aligned}$$

We recall here once again that we are using the convention that all Lebesgue norms are taken with respect to the normalized $\mathcal{H}^{n-1}$-measure on $\mathbb{S}^{n-1}$. By *Lebesgue's dominated convergence theorem* we obtain

$$\begin{aligned}
0 = \lim_{k\to\infty} \varepsilon_{u_k} &= \lim_{k\to\infty} \left(1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u_k, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right\rangle \, d\mathcal{H}^{n-1} \right| \right) = 1 - \left| \fint_{\mathbb{S}^{n-1}} \lim_{k\to\infty} g_k \, d\mathcal{H}^{n-1} \right|\\
&= 1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| = 1 - \left| \fint_{B^n} \det\nabla u_h \, dx \right| = 1 - |\det A|,
\end{aligned}$$

i.e. $|\det A| = 1$ and together with the fact that $|A|^2 = n$, we conclude that $A \in O(n)$. $\square$

The previous Lemma and its proof can also be made quantitative, as the next Theorem suggests.

Theorem 3.1.2. *Let $n \geq 2$. There exists a dimensional constant $C_n > 0$ so that for every $u \in \mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ there exists $O \in O(n)$ such that*

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \, d\mathcal{H}^{n-1} \leq C_n \varepsilon_u. \tag{3.1.9}$$

Proof. First of all, for every map $u \in \mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and $O \in O(n)$ it is trivial that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \leq 2\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 + \fint_{\mathbb{S}^{n-1}} |OP_T|^2\right) \leq 4(n-1).$$

Therefore, it suffices to prove the theorem in the *regime* where $\varepsilon_u > 0$ is smaller than a sufficiently small dimensional constant that will be chosen later, say $0 < \varepsilon_u \leq \varepsilon_0(n) \ll 1$. Without loss of generality, after possibly translating u by its center of mass if necessary, we can assume as always that $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0$.

Step 1. (Proof of the estimate in the** $W^{1,2}(\mathbb{S}^{n-1})$**-vicinity of the** $\mathrm{id}_{\mathbb{S}^{n-1}}$)***
We first prove a *local version of the estimate* under the extra assumption that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \, d\mathcal{H}^{n-1} \leq \theta^2, \tag{3.1.10}$$

where $\theta := \theta(n) > 0$ is a sufficiently small constant that will also be chosen later. In what follows now, we will always assume that θ is sufficiently small, so that all the subsequent arguments hold.

Since the map u is isometric, as we have already seen, it satisfies the pointwise identity

$$\frac{|\nabla_T u|^2}{n-1} = 1 \quad \mathcal{H}^{n-1} - \text{a.e. on } \ \mathbb{S}^{n-1}, \tag{3.1.11}$$

and averaging this identity on $\mathbb{S}^{n-1}$ results in

$$\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} = 1. \tag{3.1.12}$$

This last equation (3.1.12) enables us to rewrite

$$\begin{aligned} &\fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} = 1 - 1 + \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \\ \iff &\fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} = 1 - \left[\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1}\right]. \end{aligned} \tag{3.1.13}$$

If we now couple (3.1.7) with u in the place of u_k (and raised to the power 2) together with (3.1.13), we obtain

$$\begin{aligned} (1-\varepsilon_u)^2 \leq &\left(\fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1}\right)\left(\fint_{\mathbb{S}^{n-1}} \left|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right|^2 \, d\mathcal{H}^{n-1}\right) = \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \\ 1 - 2\varepsilon_u + \varepsilon_u^2 \leq \ & 1 - \left[\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1}\right], \end{aligned}$$

which implies that

$$\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} |u|^2\, d\mathcal{H}^{n-1} \leq 2\varepsilon_u - \varepsilon_u^2,$$

i.e.

$$\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} |u|^2\, d\mathcal{H}^{n-1} \leq 2\varepsilon_u. \tag{3.1.14}$$

Once again, we encounter a familiar nonnegative quantity on the left hand side, i.e. the *deficit* of u in *the L^2-Poincare inequality for maps with zero average on* $\mathbb{S}^{n-1}$.

For every $k \in \mathbb{N}$ we denote by $H_{n,k}$ the subspace of $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ consisting of vector fields whose components are all *k-th order spherical harmonics* (see also Appendix B), so that $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} H_{n,k}$, the orthogonal sum being taken with respect to the $W^{1,2}$-inner product. Let us also denote by $\Pi_{n,k} : W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) \mapsto H_{n,k}$ the corresponding orthogonal projection. In our case of consideration, $\Pi_{n,0}u = \fint_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} = 0$ and as we mention in Remark B.0.1., one always has that $\Pi_{n,1}u = \nabla u_h(0)x$. The first non-trivial eigenvalue of the *Laplace-Beltrami operator on* $\mathbb{S}^{n-1}$ is $\lambda_{n,1} = n-1$ and the second one is $\lambda_{n,2} = 2n$. By basic properties of the decomposition in spherical harmonics (see the Remark B.0.2.) we have

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} - |u|^2\right) &= \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u - \nabla u_h(0)P_T|^2}{n-1} - |u - \nabla u_h(0)x|^2\right)\\
&\geq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u - \nabla u_h(0)P_T|^2}{n-1} - \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u - \nabla u_h(0)P_T|^2}{2n}\\
&= \left(\frac{1}{n-1} - \frac{1}{2n}\right) \fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0)P_T|^2\\
&= \frac{n+1}{2n(n-1)} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0)P_T|^2,
\end{aligned} \tag{3.1.15}$$

so that by combining (3.1.14) with (3.1.15) we arrive at the estimate,

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0)P_T|^2\, d\mathcal{H}^{n-1} \leq \frac{4n(n-1)}{n+1}\varepsilon_u. \tag{3.1.16}$$

What remains to be justified is why in (3.1.16) *one can replace* $\nabla u_h(0)$ *with a matrix belonging to* $O(n)$*, and in view of assumption* (3.1.10) *with a matrix belonging to* $SO(n)$ *for this first step.* This is actually the point where we are going to make use of this extra assumption, and in the second step we are going to get rid of it by using the *compactness Lemma 3.1.1.*. As said, the parameter $\theta > 0$ will always be considered sufficiently small depending finally only on n, so that all subsequent statements are true.

By the *mean-value property of harmonic functions* and Remark B.0.4.,

$$|\nabla u_h(0) - I_n|^2 = \left|\fint_{B^n} \nabla u_h - I_n\right|^2 \leq \fint_{B^n} |\nabla u_h - I_n|^2 \leq \frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \leq \frac{n\theta^2}{n-1}.$$

In particular, $\det\nabla u_h(0) > 0$ and so $\nabla u_h(0) = R_0\sqrt{\nabla u_h(0)^t \nabla u_h(0)}$ for some $R_0 \in SO(n)$, hence

$$\begin{aligned}\operatorname{dist}^2(\nabla u_h(0), SO(n)) &= |\sqrt{\nabla u_h(0)^t \nabla u_h(0)} - I_n|^2 = |\nabla u_h(0) - R_0|^2 \\ &\leq |\nabla u_h(0) - I_n|^2 \qquad\qquad \leq \frac{n\theta^2}{n-1}.\end{aligned}$$

If we label $0 < \mu_1 \leq \cdots \leq \mu_n$ the eigenvalues of the symmetric positive-definite matrix $\sqrt{\nabla u_h(0)^t \nabla u_h(0)}$, and also for every $i = 1, \ldots, n$ set $\lambda_i := \mu_i - 1$, the previous inequality can be rewritten as

$$\Lambda^2 := \sum_{i=1}^n \lambda_i^2 = \sum_{i=1}^n (\mu_i - 1)^2 = \left|\sqrt{\nabla u_h(0)^t \nabla u_h(0)} - I_n\right|^2 \leq \frac{n\theta^2}{n-1}. \tag{3.1.17}$$

We now claim that there exists a constant $c_{n,\theta} > 0$ so that

$$\left|\det\nabla u_h(0) - 1\right| \leq c_{n,\theta}\varepsilon_u. \tag{3.1.18}$$

Let us suppose for the moment that (3.1.18) *holds, and see how to complete the proof of this first step.*

Since $\det\nabla u_h(0) - 1 = \prod_{i=1}^n \mu_i - 1 = \prod_{i=1}^n(\lambda_i + 1) - 1$, we can expand the polynomial in the eigenvalues to obtain

$$\det\nabla u_h(0) - 1 = \sum_{i=1}^n \lambda_i + \sum_{i\neq j} \lambda_i\lambda_j + \sum_{i\neq j\neq k} \lambda_i\lambda_j\lambda_k + \cdots + \lambda_1\lambda_2\cdots\lambda_n. \tag{3.1.19}$$

By choosing $\theta > 0$ sufficiently small, we can make

$$\Lambda := \left(\sum_{i=1}^n \lambda_i^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{n}{n-1}}\theta \ll 1, \tag{3.1.20}$$

so that for a dimensional constant $c_{1,n} > 0$

$$\sum_{i\neq j\neq k} \lambda_i\lambda_j\lambda_k + \cdots + \lambda_1\lambda_2\cdots\lambda_n = \mathcal{O}(|\Lambda|^3) \leq c_{1,n}|\Lambda|^3 \leq \sqrt{\frac{n}{n-1}}c_{1,n}\theta|\Lambda|^2 \leq \frac{1}{4}\Lambda^2, \tag{3.1.21}$$

if $\theta > 0$ is chosen sufficiently small further. Setting for convenience $\lambda := \sum_{i=1}^n \lambda_i$, and combining (3.1.18), (3.1.19) and (3.1.21), we obtain

$$\begin{aligned}
&\det\nabla u_h(0) - 1 \leq \lambda + \frac{1}{2}\left(\lambda^2 - \Lambda^2\right) + \frac{1}{4}\Lambda^2 \\
\Longrightarrow \qquad & \frac{\Lambda^2}{4} \leq \left(\lambda + \frac{\lambda^2}{2}\right) + |1 - \det\nabla u_h(0)| \\
\Longrightarrow \qquad & \frac{\Lambda^2}{4} \leq \left(\lambda + \frac{\lambda^2}{2}\right) + c_{n,\theta}\varepsilon_u. \qquad (3.1.22)
\end{aligned}$$

In order to estimate the term $\left(\lambda + \frac{\lambda^2}{2}\right)$, we merely observe that

$$\begin{aligned}
&\frac{|\nabla u_h(0)|^2}{n} \leq \fint_{B^n} \frac{|\nabla u_h|^2}{n}\, dx \leq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1} = 1 \\
\Longrightarrow\ & \frac{\sum_{i=1}^n (\lambda_i + 1)^2}{n} = \frac{\sum_{i=1}^n \mu_i^2}{n} = \frac{|\nabla u_h(0)|^2}{n} \leq 1 \\
\Longrightarrow\ & \sum_{i=1}^n \lambda_i^2 + 2\sum_{i=1}^n \lambda_i \leq 0 \\
\Longleftrightarrow\ & \lambda \leq -\frac{\Lambda^2}{2} \leq 0. \qquad (3.1.23)
\end{aligned}$$

In view of (3.1.20) we have $|\lambda| \leq \sqrt{n}\Lambda \leq \frac{n\theta}{\sqrt{n-1}} \ll 1$, and therefore the previously mentioned term in the parenthesis is estimated by

$$\lambda + \frac{\lambda^2}{2} \leq \lambda + \frac{n\theta}{2\sqrt{n-1}}|\lambda| = \left(1 - \frac{n\theta}{2\sqrt{n-1}}\right)\lambda \leq 0,$$

since by choosing $\theta > 0$ even smaller if necessary, we can also achieve $1 - \frac{n\theta}{2\sqrt{n-1}} > 0$. The term $(\lambda + \frac{\lambda^2}{2})$ is therefore nonpositive and (3.1.22) gives

$$\operatorname{dist}^2(\nabla u_h(0); SO(n)) = |\nabla u_h(0) - R_0|^2 = \Lambda^2 \leq 4c_{n,\theta}\varepsilon_u. \qquad (3.1.24)$$

This would complete the proof of the first step, since going back to (3.1.16), we get

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |\nabla_T u - R_0 P_T|^2 &\leq 2\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0)P_T|^2 + \frac{n-1}{n}|\nabla u_h(0) - R_0|^2\right) \\
&\leq c_{2,n}\varepsilon_u, \qquad (3.1.25)
\end{aligned}$$

where for example $c_{2,n} = 8(n-1)\left(\frac{n}{n+1} + \frac{c_{n,\theta}}{n}\right) > 0$, for this $\theta > 0$ that was in the end chosen to be a sufficiently small dimensional constant.

It remains to prove (3.1.18), *which is actually the key estimate for this step.* To do so we use again the assumption (3.1.10), which as we have seen immediately implies that

$\nabla u_h(0)$ is sufficiently close to I_n, in particular $|\nabla u_h(0) - I_n| \leq \sqrt{\frac{n}{n-1}}\theta \ll 1$. This in turn implies that there exists *a sufficiently small constant* $c_{1,n,\theta} > 0$ so that

$$|\det\nabla u_h(0) - 1| \leq c_{1,n,\theta} \quad \text{and} \quad |\nabla u_h(0)^{-1} - I_n| \leq c_{1,n,\theta}. \tag{3.1.26}$$

An explicit value for $c_{1,n,\theta}$ is computable but not extremely important (but obviously can be made small as θ is chosen small depending only on the dimension).

The trick here is to write the *signed-volume-term* which appears in the *isoperimetric deficit* ε_u as

$$\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} = \fint_{B^n} \det\nabla u_h = \det\nabla u_h(0) \cdot \fint_{B^n} \det\big(I_n + \nabla w_h(x)\big), \tag{3.1.27}$$

where

$$w(x) := \nabla u_h(0)^{-1}\big(u(x) - \nabla u_h(0)x\big). \tag{3.1.28}$$

Because of (3.1.26), the map w satisfies

$$\|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} = \|\nabla u_h(0)^{-1}\nabla_T u - P_T\|_{L^\infty} \leq c_{2,n,\theta} := \sqrt{n-1}(\sqrt{n} + c_{1,n,\theta} + 1). \tag{3.1.29}$$

In the right hand side of (3.1.27), we can use the expansion of the *signed-volume-term* around the identity (which we exhibit in Appendix C), to obtain

$$\fint_{B^n} \det(I_n + \nabla w_h) \, dx = 1 + n\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} + Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w) \, d\mathcal{H}^{n-1}.$$

Notice that the linear term in the last expression is vanishing, because

$$\begin{aligned} n\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} &= \fint_{B^n} \mathrm{div} w_h \, d\mathcal{H}^{n-1} = \fint_{B^n} \mathrm{Tr}(\nabla w_h) \, dx = \mathrm{Tr}\left(\fint_{B^n} \nabla w_h \, dx\right) \\ &= \mathrm{Tr}\left[\nabla u_h(0)^{-1}\left(\fint_{B^n} \nabla u_h(x) \, dx - \nabla u_h(0)\right)\right] = 0, \end{aligned} \tag{3.1.30}$$

the last equality following again from the *mean-value property of harmonic functions.*

For the higher order terms including the *quadratic one* in them, the bound on the Lipschitz constant of w implies that also $\|w\|_{L^\infty(\mathbb{S}^{n-1})} \ll_{n,\theta} +\infty$ (note that $\fint_{\mathbb{S}^{n-1}} w = 0$), and therefore there also exists a constant $c_{3,n,\theta} > 0$ (which need not be small of course), such that

$$\left|Q_{V_n}(w)+\fint_{\mathbb{S}^{n-1}} R_{V_n}(w,\nabla_T w)\ d\mathcal{H}^{n-1}\right| \leq c_{3,n,\theta}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1}$$
$$= c_{3,n,\theta}\fint_{\mathbb{S}^{n-1}} |\nabla u_h(0)^{-1}(\nabla_T u - \nabla u_h(0)P_T)|^2$$
$$\overset{(3.1.26)}{\leq} c_{4,n,\theta}\fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0)P_T|^2\ d\mathcal{H}^{n-1}$$
$$\overset{(3.1.16)}{\leq} c_{5,n,\theta}\cdot\varepsilon_u \leq c_{5,n,\theta}\cdot\varepsilon_0(n) \ll 1, \qquad (3.1.31)$$

for some constants $c_{4,n,\theta}>0$ and $c_{5,n,\theta}>0$ that can explicitely be defined in terms of the previous ones. By taking the absolute value in (3.1.27) and by using the expansion of the signed-volumed term on the right hand side, we get

$$\left|\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle\ d\mathcal{H}^{n-1}\right| = \det\nabla u_h(0)\left(1+Q_{V_n}(\tilde{w})+\fint_{\mathbb{S}^{n-1}} R_{V_n}(\tilde{w},\nabla_T\tilde{w})\right), \quad (3.1.32)$$

since (3.1.26) and (3.1.31) imply that both factors of the right hand side are actually positive for $\theta>0$ and $\varepsilon_0>0$ both sufficiently small depending only on n. Rearranging terms, taking the absolute values again and using the *triangle inequality*, we arrive at

$$|1-\det\nabla u_h(0)| \leq \left(1-\left|\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle\ d\mathcal{H}^{n-1}\right|\right)$$
$$+\det\nabla u_h(0)\left|Q_{V_n}(w)+\fint_{\mathbb{S}^{n-1}} R_{V_n}(w,\nabla_T w)\right|$$
$$\overset{(3.1.31)}{\leq} \Big(c_{5,n,\theta}(1+c_{1,n,\theta})+1\Big)\varepsilon_u, \qquad (3.1.33)$$

and the proof of this *first step* is now complete.

Step 2. (Reduction to maps in the $W^{1,2}(\mathbb{S}^{n-1})$***-vicinity of the*** $\mathrm{id}_{\mathbb{S}^{n-1}}$***)***
The reduction to *Step 1* is now a standard compactness argument. What we would like to prove is that there exists a constant $C_n>0$ so that

$$\sup_{u\in\mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)} \frac{\min_{O\in O(n)}\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2\ d\mathcal{H}^{n-1}}{\varepsilon_u} \leq C_n \ll +\infty, \qquad (3.1.34)$$

whenever the denominator is non-zero. We argue by *contradiction* and suppose that the latter claim is false. Then, for every $k\in\mathbb{N}$ there exist $u_k\in\mathcal{I}(\mathbb{S}^{n-1};\mathbb{R}^n)$ with $\varepsilon_{u_k}>0$ and $O_k\in O(n)$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2\ d\mathcal{H}^{n-1} = \min_{O\in O(n)}\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - OP_T|^2\ d\mathcal{H}^{n-1} \qquad (3.1.35)$$

and

$$\frac{\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2 \, d\mathcal{H}^{n-1}}{\varepsilon_{u_k}} \geq k. \tag{3.1.36}$$

In particular,

$$\varepsilon_{u_k} \leq \frac{1}{k} \fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2 \, d\mathcal{H}^{n-1} \leq \frac{4(n-1)}{k},$$

and letting $k \to \infty$ we see that along this sequence, $\lim_{k\to\infty} \varepsilon_{u_k} = 0$.

By Lemma 3.1.1. and up to passing to a subsequence, we can find $O_0 \in O(n)$ so that $u_k - \fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} \to O_0 \mathrm{id}_{\mathbb{S}^{n-1}}$ strongly in $W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$. Without loss of generality (up to considering $O_0^t u_k$ instead of u_k if necessary) we can also suppose that $O_0 = I_n$. Then, for the dimensional constant θ chosen in *Step 1*, we can find $k_0 := k_0(\theta) \in \mathbb{N}$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - P_T|^2 \, d\mathcal{H}^{n-1} \leq \theta^2 \quad \forall k \geq k_0. \tag{3.1.37}$$

In other words, after also translating by the centers of mass if necessary, the subsequence $(u_k)_{k\geq k_0}$ satisfies the condition $\fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} = 0$ and fulfills also the *apriori closeness to the identity assumption* (3.1.10). By *Step 1*, we deduce that there exist $(R_k)_{k\geq k_0} \in SO(n)$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - R_k P_T|^2 \leq c_{2,n} \varepsilon_{u_k} \quad \forall k \geq k_0. \tag{3.1.38}$$

Combining now (3.1.35), (3.1.36) and (3.1.38), we arrive at the desired contradiction. $\square$

A closer inspection of the proofs shows that Lemma 3.1.1. and Theorem 3.1.2. remain true for the more general class of *short maps*, that is maps $u \in W^{1,\infty}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ for which $\nabla_T u^t \nabla_T u \leq I_x$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ *in the sense of quadratic forms*. One only needs to replace the exact equalities by $\frac{|\nabla_T u|^2}{n-1} \leq 1$, $\left|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right| \leq 1$, whenever they are used.

Remark 3.1.3. It is also easy to construct examples showing that the estimate (3.1.9) is *optimal* in the norm appearing on the left hand side and the deficit on the right hand side, i.e. *the exponent* 1 *with which* ε_u *appears cannot be improved.* An example in dimension $n = 2$, which can easily be generalized in higher dimensions is the following.

For $0 < \varepsilon \ll 2\pi$, let $u_\varepsilon : \mathbb{S}^1 \mapsto \mathbb{R}^2$ be defined in polar coordinates via

$$u_\varepsilon(\theta) := \left\{ \begin{array}{l} (\cos\theta, \sin\theta);\ 0 \leq \theta < \frac{3\pi}{2} - \frac{\varepsilon}{2}, \\ \left(\cos\theta, 2\sin\left(\frac{3\pi}{2} - \frac{\varepsilon}{2}\right) - \sin\theta\right);\ \frac{3\pi}{2} - \frac{\varepsilon}{2} \leq \theta < \frac{3\pi}{2} + \frac{\varepsilon}{2} \\ (\cos\theta, \sin\theta);\ \frac{3\pi}{2} + \frac{\varepsilon}{2} \leq \theta < 2\pi \end{array} \right\}.$$

For each $0 < \varepsilon \ll 2\pi$, the map u_ε is an isometric map on $\mathbb{S}^1$, being essentially the identity transformation except for a *small circular arc of angle* ε, where it is a *flip* with respect to the horizontal line at height $y_0 = \sin\left(\frac{3\pi}{2} - \frac{\varepsilon}{2}\right)$. Obviously, $\nabla_T u_\varepsilon \to \nabla_T \mathrm{id}_{\mathbb{S}^1}$ strongly in

$L^2(\mathbb{S}^1;\mathbb{R}^2)$ as $\varepsilon \to 0^+$ and one can readily see that

$$\begin{aligned}
\fint_{\mathbb{S}^1} |\nabla_T u_\varepsilon - \nabla_T \mathrm{id}_{\mathbb{S}^1}|^2 \ d\mathcal{H}^1 &\sim \int_0^{2\pi} |\partial_\theta u_\varepsilon(\theta) - \partial_\theta \mathrm{id}_{\mathbb{S}^1}(\theta)|^2 \ d\theta \\
&= \int_{\frac{3\pi}{2}-\frac{\varepsilon}{2}}^{\frac{3\pi}{2}+\frac{\varepsilon}{2}} |(-\sin\theta, -\cos\theta) - (-\sin\theta, \cos\theta)|^2 \ d\theta \\
&= 4\int_{\frac{3\pi}{2}-\frac{\varepsilon}{2}}^{\frac{3\pi}{2}+\frac{\varepsilon}{2}} \cos^2(\theta) \ d\theta \\
&= 2(\varepsilon - \sin\varepsilon) = \mathcal{O}(\varepsilon^3), \quad \text{for } 0 < \varepsilon \ll 2\pi.
\end{aligned}$$

On the other hand, using elementary plane-geometry formulas for the area of circular triangles, we can compute the area of the "*double arc-region* of the unit disc *missed by* u_ε", so that also

$$\varepsilon_{u_\varepsilon} = \frac{2}{\pi}\left(\pi \cdot \frac{\varepsilon}{2\pi} - \frac{1}{2}\sin\varepsilon\right) = \frac{1}{\pi}(\varepsilon - \sin\varepsilon) = \mathcal{O}(\varepsilon^3), \quad \text{for } 0 < \varepsilon \ll 2\pi.$$

In higher dimensions one can construct similar examples by defining maps that are the identity outside a *small angular neighbourhood of an equator* and in the angular neighbourhood being again *flips* in $\mathbb{R}^n$ with respect to the appropriate affine hyperplanes.

3.2 The case of almost isometric maps with small isoperimetric deficit

3.2.1 Maps with an apriori bound on their Lipschitz constant

We would like now to take a step further and see how the results of the previous Section can be generalized to maps that are not necessarily isometric, but are *almost isometric* in the average sense introduced in Section 1.2. Since most of the arguments in the proofs have appeared already in Section 3.1, we will mainly describe the points that have to be slightly modified, although the experienced reader may easily notice how the proofs should be adapted in this more general setting.

It will be convenient (and is also natural in many contexts) to work with Lipschitz maps that enjoy an *apriori bound in their Lipschitz constant*, something that will be a hypothesis for us in this Subsection, while in the next one we discuss how this hypothesis can be relaxed via the use of a standard *truncation argument*. We first revise the definitions of our two deficits.

Definition 3.2.1. Given a map $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ its *isometric deficit* is defined as

$$\delta_u := \left(\fint_{\mathbb{S}^{n-1}} \left| \sqrt{\nabla_T u^t \nabla_T u} - I_x \right|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \tag{3.2.1}$$

and its *generalized isoperimetric deficit* as

$$\varepsilon_u := \Big(1 - |V_n(u)|\Big)_+ := \left(1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| \right)_+ . \tag{3.2.2}$$

Here, we use the convention that $(-\infty)_+ = 0$. Note that we always have $0 \le \varepsilon_u \le 1$, even when $|V_n(u)| = \infty$ and actually

$$0 \le 1 - \varepsilon_u \le |V_n(u)| \quad \text{for every } u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n). \tag{3.2.3}$$

Recall that by the *generalized isoperimetric inequality*, $|V_n(u)| < \infty$ whenever the map $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$.

In the context of this Subsection, a generalization of Lemma 3.1.1. is provided by

Lemma 3.2.2. *Let $n \ge 2$ and $M > 0$ be given. Consider a sequence of Lipschitz maps $(u_k)_{k\in\mathbb{N}} : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ which for all $k \in \mathbb{N}$ satisfy the Lipschitz bound $\|\nabla_T u_k\|_{L^\infty(\mathbb{S}^{n-1})} \le M$, and suppose that*

$$\lim_{k\to\infty} \delta_{u_k} = \lim_{k\to\infty} \varepsilon_{u_k} = 0. \tag{3.2.4}$$

Then there exists $O \in O(n)$ so that up to a non-relabeled subsequence

$$u_k - \fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} \to O\mathrm{id}_{\mathbb{S}^{n-1}} \quad \text{strongly } W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n). \tag{3.2.5}$$

Proof. Similarly to the proof of Lemma 3.1.1., we can assume without loss of generality that

$$\fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} = 0 \quad \forall\, k \in \mathbb{N}. \tag{3.2.6}$$

Even without the assumption that $(u_k)_{k\in\mathbb{N}}$ have uniformely bounded Lipschitz constants, since $\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k|^2 \, d\mathcal{H}^{n-1} \le 2(\delta_{u_k}^2 + n - 1)$ for all $k \in \mathbb{N}$, by (3.2.4) and (3.2.6) we have again

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,2}(\mathbb{S}^{n-1})} \ll +\infty.$$

Extracting a non-relabeled $W^{1,2}$-weakly convergent subsequence $u_k \rightharpoonup u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ which converges also pointwise $\mathcal{H}^{n-1}$–a.e., we have again

$$\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = \lim_{k\to\infty} \fint_{\mathbb{S}^{n-1}} u_k \, d\mathcal{H}^{n-1} = 0,$$

$$\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} \le \liminf_{k\to\infty} \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u_k|^2}{n-1} \, d\mathcal{H}^{n-1} = 1.$$

The last equality here is justified by the trivial estimate

$$
\begin{aligned}
\left|\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u_k|^2}{n-1}\, d\mathcal{H}^{n-1} - 1\right| &= \frac{1}{n-1}\left|\fint_{\mathbb{S}^{n-1}} \mathrm{Tr}\Big[\Big(\sqrt{\nabla_T u_k^t \nabla_T u_k}\Big)^2 - I_x^2\Big]\, d\mathcal{H}^{n-1}\right| \\
&\leq \frac{1}{\sqrt{n-1}} \fint_{\mathbb{S}^{n-1}} \left|\Big(\sqrt{\nabla_T u_k^t \nabla_T u_k}\Big)^2 - I_x^2\right|\, d\mathcal{H}^{n-1} \\
&\leq \frac{1}{\sqrt{n-1}} \left(\|\nabla_T u_k\|_{L^\infty} + \sqrt{n-1}\right) \fint_{\mathbb{S}^{n-1}} \left|\sqrt{\nabla_T u_k^t \nabla_T u_k} - I_x\right| \\
&\lesssim c_{1,n,M}\delta_{u_k}, \qquad (3.2.7)
\end{aligned}
$$

where $c_{1,n,M} := \frac{M}{\sqrt{n-1}} + 1 > 0$.

Since the determinant is a Lipschitz function and $\sup\limits_{k\in\mathbb{N}} \|\nabla_T u_k\|_{L^\infty(\mathbb{S}^{n-1})} \leq M$, we can replace in this setting the exact identity (3.1.6) by the estimate

$$
\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \det \nabla_T u_k^t \nabla_T u_k\, d\mathcal{H}^{n-1} &\leq 1 + \fint_{\mathbb{S}^{n-1}} \left|\det\big(\nabla_T u_k^t \nabla_T u_k\big) \; - 1\right|\, d\mathcal{H}^{n-1} \\
&\leq 1 + \tilde{c}_{n,M} \fint_{\mathbb{S}^{n-1}} \left|\nabla_T u_k^t \nabla_T u_k - I_x\right|\, d\mathcal{H}^{n-1} \\
&\leq 1 + c_{2,n,M}\delta_{u_k}, \qquad (3.2.8)
\end{aligned}
$$

where $\tilde{c}_{n,M} > 0$ comes from the Lipschitz constant of the determinant in the ball $B_{M^2}(0)$ of $\mathbb{R}^{(n-1)\times(n-1)}$ and $c_{2,n,M} := \tilde{c}_{n,M}(M + \sqrt{n-1}) > 0$.

Since $0 \leq 1 - \varepsilon_u \leq \big|V_n(u)\big|$, we can still use (3.1.7) together with (3.2.8), apply again the Poincare inequality and then (3.2.7), to obtain

$$
\begin{aligned}
(1 - \varepsilon_{u_k})^2 &\leq \left(\fint_{\mathbb{S}^2} |u_k|^2\, d\mathcal{H}^{n-1}\right) \cdot (1 + c_{2,n,M}\delta_{u_k}) \\
&\leq \left(\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u_k|^2}{n-1}\, d\mathcal{H}^{n-1}\right) \cdot (1 + c_{2,n,M}\delta_{u_k}) \qquad (3.2.9) \\
&\leq \big(1 + c_{1,n,M}\delta_{u_k}\big)\big(1 + c_{2,n,M}\delta_{u_k}\big).
\end{aligned}
$$

By the assumptions that $\lim_{k\to\infty} \delta_{u_k} = \lim_{k\to\infty} \varepsilon_{u_k} = 0$ and since $u_k \to u$ strongly in $L^2(\mathbb{S}^{n-1}; \mathbb{R}^n)$, we can let $k \to \infty$ in the last chain of inequalities to obtain again

$$
\fint_{\mathbb{S}^{n-1}} |u|^2\, d\mathcal{H}^{n-1} = \lim_{k\to\infty} \fint_{\mathbb{S}^{n-1}} |u_k|^2\, d\mathcal{H}^{n-1} = 1. \qquad (3.2.10)
$$

Therefore, also in this slightly more general setting the limiting map u is such that $\fint_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} = 0$ and

$$
1 \geq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1}\, d\mathcal{H}^{n-1} \geq \fint_{\mathbb{S}^{n-1}} |u|^2\, d\mathcal{H}^{n-1} = 1,
$$

i.e. again $u(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$ with $|A|^2 = n$, and $u_k - \fint_{\mathbb{S}^{n-1}} u_k \to u := A\mathrm{id}_{\mathbb{S}^{n-1}}$ in the strong $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$-topology.

As in the end of the proof of Lemma 3.1.1. we can justify that $|\det A| = 1$. Indeed, up to a further non-relabeled subsequence we can again assume now that $\nabla_T u_k \to \nabla_T u$ pointwise $\mathcal{H}^{n-1}-$a.e. on $\mathbb{S}^{n-1}$ and by the assumption on the uniform Lipschitz bound,

$$
\begin{aligned}
g_k := \left\langle u_k, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right\rangle &\to \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle, && \text{pointwise } \mathcal{H}^{n-1} - \text{a.e.},\\
|g_k| \leq \left| \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_k \right| |u_k| &\leq \left(\frac{M^2}{n-1} \right)^{\frac{n-1}{2}} |u_k|, && \text{for every } k \in \mathbb{N},\\
|u_k| &\longrightarrow |u|, && \text{pointwise } \mathcal{H}^{n-1} - \text{a.e.},\\
\sup_{k\in\mathbb{N}} \fint_{\mathbb{S}^{n-1}} |u_k| \; d\mathcal{H}^{n-1} &\leq \sup_{k\in\mathbb{N}} \|u_k\|_{L^2(\mathbb{S}^{n-1})} \ll +\infty, && \text{since } \|u_k\|_{L^2(\mathbb{S}^{n-1})} \to 1,
\end{aligned}
$$

and we can again use Lebesgue's Dominated Convergence Theorem in the assumption that $\lim_{k\to\infty} \varepsilon_{u_k} = 0$ to conclude. Indeed, by the continuity of the *positive part function*,

$$
\begin{aligned}
0 = \lim_{k\to\infty} \varepsilon_{u_k} &= \lim_{k\to\infty} \left(1 - \left| \fint_{\mathbb{S}^{n-1}} g_k \; d\mathcal{H}^{n-1} \right| \right)_+ = \left(1 - \left| \fint_{\mathbb{S}^{n-1}} \lim_{k\to\infty} g_k \; d\mathcal{H}^{n-1} \right| \right)_+ \\
&= \Big(1 - |V_n(u)| \Big)_+ = \left(1 - \left| \fint_{B^n} \det(Ax)_h \; dx \right| \right)_+ = \big(1 - |\det A| \big)_+,
\end{aligned}
$$

i.e. $|\det A| \geq 1$. If we now consider the polar decomposition $A = O\sqrt{A^tA}$ where $O \in O(n)$, and label $0 \leq \mu_1 \leq \cdots \leq \mu_n$ the eigenvalues of $\sqrt{A^tA}$, then by the arithmetic mean-geometric mean inequality we obtain again

$$
1 = \left(\frac{|A|^2}{n} \right)^{\frac{n}{2}} \geq |\det A| \geq 1 \quad \Longrightarrow \quad |\det A| = \frac{|A|^2}{n} = 1, \text{ i.e. } A \in O(n). \qquad \square
$$

The proof of the previous Lemma also contains essentially all the modifications that are necessary in order to prove the more general version of Theorem 3.1.2., namely

Theorem 3.2.3. *Let $n \geq 2$ and $M > 0$ be given. There exists a constant $C_{n,M} > 0$ such that for every map $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ which is such that*

$$
u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) \text{ with } \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M, \tag{H_M}
$$

there exists $O \in O(n)$ so that

$$
\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \; d\mathcal{H}^{n-1} \leq C_{n,M} \left(\delta_u + \varepsilon_u \right). \tag{3.2.11}
$$

Before giving the proof of the Theorem, let us make some simple remarks. First of all we can without loss of generality focus on *the regime where the isometric deficit is small*, say $0 \leq \delta_u \leq 1$, because if $\delta_u > 1$ then one trivially has that for every $O \in O(n)$,

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \, d\mathcal{H}^{n-1} &\leq 2\fint_{\mathbb{S}^{n-1}} \left(2\left|\sqrt{\nabla_T u^t \nabla_T u} - I_x\right|^2 + 2|I_x|^2 + |OP_T|^2 \right) \\
&\leq 4\delta_u^2 + 6(n-1) \leq 4\delta_u^2 + 6(n-1)\delta_u^2 \\
&\leq (6n-2)\delta_u^2,
\end{aligned}$$

and the hypothesis (H_M) as well as the presence of the generalized isoperimetric deficit are of course obsolete for this trivial stability result.

In the case $\delta_u = 0$, i.e. the case of isometric maps $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, the hypothesis (H_M) is trivially satisfied with $M = \sqrt{n-1}$ and (3.2.11) reduces to (3.1.9) with a purely dimensional constant.

Another interesting point in the estimate is that under the hypothesis (H_M) (or the relaxed hypotheses that we will mention in the next Subsection), *the contribution of the "isoperimetric deficit" can be absorbed into the isometric one whenever* $|V_n(u)| > 1$.

Proof. First of all, by the above reasoning it suffices to actually prove our Theorem in the *regime* where both deficits are *sufficiently small*, say $0 \leq \delta_u \leq \delta_0 \ll 1$ and $0 \leq \varepsilon_u \leq \varepsilon_0 \ll 1$ for some constants δ_0, ε_0 depending possibly both on n and M and which will again be chosen sufficiently small later. As always, we can also assume that $\fint_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} = 0$. The proof is then divided in the same two steps as before.

Step 1. (Proof for maps in the $W^{1,2}(\mathbb{S}^{n-1})$***-vicinity of the*** $\mathrm{id}_{\mathbb{S}^{n-1}}$***)***
Again, we first assume that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \, d\mathcal{H}^{n-1} \leq \theta^2, \tag{3.2.12}$$

where $\theta > 0$ will be chosen in the end sufficiently small depending possibly both on n and M.

Instead of (3.1.12) which followed by averaging a pointwise identity, we can now use its *approximate version*, i.e. (3.2.7) with u instead of u_k, to obtain

$$\left| \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} - 1 \right| \leq c_{1,n,M}\delta_u \tag{3.2.13}$$

$$\iff -1 - c_{1,n,M}\delta_u \leq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} \leq 1 + c_{1,n,M}\delta_u, \tag{3.2.14}$$

which in turn gives an approximate version of (3.1.13), namely

$$\fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1} \leq 1 + c_{1,n,M}\delta_u - \left[\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} |u|^2 \, d\mathcal{H}^{n-1}\right]. \quad (3.2.15)$$

The first inequality in (3.2.9) for u instead of u_k now gives

$$(1-\varepsilon_u)^2 \leq \left[1 + c_{1,n,M}\delta_u - \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} - |u|^2\right) \, d\mathcal{H}^{n-1}\right] [1 + c_{2,n,M}\delta_u]. \quad (3.2.16)$$

Since $\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} - |u|^2\right) \, d\mathcal{H}^{n-1} \geq 0$, we can rearrange terms, discard the nonpositive terms appearing on the right hand side and use the fact that we have assumed without loss of generality that $0 \leq \delta_u \leq 1$, to arrive with the same arguments as before at the analogue of (3.1.16), i.e.

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla u_h(0) P_T|^2 \, d\mathcal{H}^{n-1} \leq c_{3,n,M}(\delta_u + \varepsilon_u), \quad (3.2.17)$$

for a constant $c_{3,n,M} > 0$ that can be made explicit in terms of $c_{1,n,M}$ and $c_{2,n,M}$.

To justify why $\nabla u_h(0)$ can be replaced by a matrix $R \in SO(n)$, the procedure is the same as in the proof of Theorem 3.1.2. for the corresponding part. Analogously to (3.1.18), *one can use the extra assumption* (3.2.12) *to prove that there exists a constant $c_{n,\theta,M} > 0$ such that*

$$|\det \nabla u_h(0) - 1| \leq c_{n,\theta,M}(\delta_u + \varepsilon_u). \quad (3.2.18)$$

Having established this estimate, the proof of the first step can be completed as before, modulo the following minor difference. Keeping the same notation as in the proof of Theorem 3.1.2., the analogue of (3.1.22) via the use now of (3.2.18) would be

$$\frac{\Lambda^2}{4} \leq \left(\lambda + \frac{\lambda^2}{2}\right) + c_{n,\theta,M}(\delta_u + \varepsilon_u). \quad (3.2.19)$$

The term $\left(\lambda + \frac{\lambda^2}{2}\right)$ is now not necessarily nonpositive, but still

$$\frac{|\nabla u_h(0)|^2}{n} \leq \fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T u|^2}{n-1} \overset{(3.2.13)}{\leq} 1 + c_{1,n,M}\delta_u \implies \lambda \leq -\frac{\Lambda^2}{2} + c_{4,n,M}\delta_u \leq c_{4,n,M}\delta_u,$$

where $c_{4,n,M} := \frac{n c_{1,n,M}}{2} > 0$. In constrast to (3.1.23), λ now does not necessarily have a sign, so we consider two cases:

(*i*) If $\lambda \leq 0$, the argument is identical to the one in Theorem 3.1.2..

(*ii*) If $\lambda > 0$, by (3.2.19) we have

$$\begin{aligned}
\operatorname{dist}^2(\nabla u_h(0); SO(n)) = \Lambda^2 &\leq 4\left(\lambda + \frac{\lambda^2}{2}\right) + 4c_{n,\theta,M}(\delta_u + \varepsilon_u) \\
&\leq 4c_{4,n,M}\delta_u + 2c_{4,n,M}^2\delta_u^2 + 4c_{n,\theta,M}(\delta_u + \varepsilon_u) \\
&\leq c_{5,n,\theta,M}\left(\delta_u + \varepsilon_u\right), \qquad (3.2.20)
\end{aligned}$$

for a constant $c_{5,n,\theta,M} > 0$ that can be defined in terms of $c_{4,n,\theta,M}$.

The proof can then be finished by choosing $\theta > 0$ sufficiently small depending only on n, M.

Returning to the proof of (3.2.18), it is also essentially the same as in the *isometric case*. By writing again

$$V_n(u) = \det\nabla u_h(0)\left(1 + Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\right), \qquad (3.2.21)$$

we have $\det\nabla u_h(0) > 0$ because of (3.2.12), while similarly to (3.1.31), the estimates (3.2.17) and (3.2.12) imply that

$$\left|Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\right| \leq c_{6,n,\theta,M}(\delta_u + \varepsilon_u) \leq c_{6,n,\theta,M}(\delta_0 + \varepsilon_0) \ll 1, \qquad (3.2.22)$$

as long as we choose θ sufficiently small and then the constants $\delta_0 > 0$, $\varepsilon_0 > 0$ small depending on n, M. In particular $V_n(u) > 0$ again, and we can consider the following two cases:

(*i*) If $V_n(u) > 1$, then (3.2.21), (3.2.12), (3.2.22) together with the generalized isoperimetric inequality (A.0.1) imply that

$$\begin{aligned}
|\det\nabla u_h(0) - 1| &\leq (V_n(u) - 1) + \det\nabla u_h(0)\left|Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\right| \\
&\leq \left(\fint_{\mathbb{S}^{n-1}} \det\Big(I_x + \big(\sqrt{\nabla_T u^t \nabla_T u} - I_x\big)\Big)\right)^{\frac{n}{n-1}} - 1 + \tilde{c}_{n,\theta,M}(\delta_u + \varepsilon_u) \\
&\leq \left(\fint_{\mathbb{S}^{n-1}} \Big(1 + \mathcal{O}_{n,M}\big(\big|\sqrt{\nabla_T u^t \nabla_T u} - I_x\big|\big)\Big)\right)^{\frac{n}{n-1}} - 1 + \tilde{c}_{n,\theta,M}(\delta_u + \varepsilon_u) \\
&\leq \big(1 + \mathcal{O}_{n,M}(\delta_u)\big)^{\frac{n}{n-1}} - 1 + \tilde{c}_{n,\theta,M}(\delta_u + \varepsilon_u) \\
&\lesssim_{n,\theta,M} (\delta_u + \varepsilon_u).
\end{aligned}$$

(*ii*) If $0 \leq V_n(u) \leq 1$, then again

$$
\begin{aligned}
|1-\det\nabla u_h(0)| &\leq |1-V_n(u)| + \det\nabla u_h(0)\left|Q_{V_n}(\tilde{w}) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w,\nabla_T w)\right| \\
&\leq (1-V_n(u))_+ + \tilde{c}_{n,\theta,M}(\delta_u+\varepsilon_u) \\
&\lesssim_{n,\theta,M} (\delta_u+\varepsilon_u).
\end{aligned}
$$

Step 2. (Reduction to maps in the $W^{1,2}(\mathbb{S}^{n-1})$***-vicinity of the*** $\mathrm{id}_{\mathbb{S}^{n-1}}$***)***
As in *Step 2* in the proof of Theorem 3.1.2., fixing the centers of mass of the maps to 0, the set of mappings under consideration is

$$
\mathcal{C} := \left\{u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} u \; d\mathcal{H}^{n-1} = 0,\ \|\nabla_T u\|_{L^\infty(\mathbb{S}^{n-1})} \leq M,\ \delta_u + \varepsilon_u > 0\right\},
$$

and what we would like to prove is that there exists a constant $C_{n,M} > 0$ so that

$$
\sup_{u\in\mathcal{C}} \frac{\min_{O\in O(n)} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \; d\mathcal{H}^{n-1}}{\delta_u+\varepsilon_u} \leq C_{n,M} \ll +\infty. \tag{3.2.23}
$$

Arguing again by contradiction, if the latter is false, then for every $k \in \mathbb{N}$ there exists $u_k \in \mathcal{C}$ and $O_k \in O(n)$ such that

$$
\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2 \; d\mathcal{H}^{n-1} = \min_{O\in O(n)} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \; d\mathcal{H}^{n-1} \tag{3.2.24}
$$

and

$$
\frac{\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2 \; d\mathcal{H}^{n-1}}{\delta_{u_k}+\varepsilon_{u_k}} \geq k. \tag{3.2.25}
$$

In particular,

$$
\begin{aligned}
&\delta_{u_k}+\varepsilon_{u_k} \leq \frac{1}{k}\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - O_k P_T|^2 \; d\mathcal{H}^{n-1} \\
\Longrightarrow\; &\delta_{u_k}+\varepsilon_{u_k} \leq \frac{2}{k}\fint_{\mathbb{S}^{n-1}} (|\nabla_T u_k|^2 + |O_k P_T|^2) \; d\mathcal{H}^{n-1} \\
\Longrightarrow\; &\delta_{u_k}+\varepsilon_{u_k} \leq \frac{2(M^2+n-1)}{k},
\end{aligned}
$$

and letting $k \to \infty$ we see that along this sequence $\lim_{k\to\infty}\delta_{u_k} = \lim_{k\to\infty}\varepsilon_{u_k} = 0$. We can now use the *compactness property* provided by Lemma 3.2.2. and derive a contradiction as in the *purely isometric case.* □

Remark 3.2.4. Since the isometric deficit δ_u *does not detect changes in the orientation neither in ambient space nor intrinsically*, (3.2.11) is *optimal* also with respect to the exponent with which δ_u appears on the right hand side . An example to check the optimality (which should be compared to the one at the end of Section 3.1), is now the following.

Identify $\mathbb{S}^1$ with the interval $[0,1]$ by identifying the endpoints. For $0<\varepsilon\ll 1$, consider the maps $f_\varepsilon:[0,1]\mapsto[0,1]$, defined as follows.

$$f_\varepsilon(t):=\left\{\begin{array}{l} t;\quad 0\le t<\varepsilon,\\ 2\varepsilon-t;\quad \varepsilon\le t<2\varepsilon \\ -\frac{2\varepsilon}{1-2\varepsilon}+\frac{1}{1-2\varepsilon}t;\quad 2\varepsilon\le t<1 \end{array}\right\}$$

and let $u_\varepsilon:\mathbb{S}^1\mapsto\mathbb{S}^1$ be the corresponding maps defined on the unit circle. Obviously $\varepsilon(u_\varepsilon)=0$. Geometrically, the maps u_ε *travel back and forth* and produce a triple cover of a small arc on $\mathbb{S}^1$. With similar calculations as in the other example,

$$\fint_{\mathbb{S}^1}\left|\partial_\tau u_\varepsilon-\partial_\tau\mathrm{id}_{\mathbb{S}^1}\right|^2\sim\int_0^1\left|f_\varepsilon'(t)-1\right|^2=\int_\varepsilon^{2\varepsilon}(-2)^2+\int_{2\varepsilon}^1\left(\frac{1}{1-2\varepsilon}-1\right)^2$$
$$\sim 4\varepsilon+\frac{4\varepsilon^2}{1-2\varepsilon}\sim\varepsilon+\mathcal{O}(\varepsilon^2)=\mathcal{O}(\varepsilon)\quad\text{for } 0<\varepsilon\ll 1.$$

Moreover,

$$\delta^2(u_\varepsilon)\sim\int_0^1\left||f_\varepsilon'(t)|-1\right|^2=\int_{2\varepsilon}^1\left(\frac{1}{1-2\varepsilon}-1\right)^2\sim\frac{4\varepsilon^2}{1-2\varepsilon}$$
$$\sim\varepsilon^2\big(1+\mathcal{O}(\varepsilon)\big)=\mathcal{O}(\varepsilon^2)\quad\text{for } 0<\varepsilon\ll 1,$$

which reveals the optimality of the exponent of $\delta(u)$ in the estimate in the generic setting. When $n\ge 3$ one can construct similar examples, for instance by rotating the previous one-dimensional example around a fixed axis.

A closer inspection also reveals that in (3.2.11), δ_u can be replaced by the *slightly sharper deficit* $\|(\sigma_{n-1}-1)_+\|_{L^2(\mathbb{S}^{n-1})}$, where $0\le\sigma_1\le\cdots\le\sigma_{n-1}$ are the eigenvalues of $\sqrt{\nabla_T u^t\nabla_T u}$. The latter vanishes precisely for generalized *short maps* (see the upcoming [LZon] for more details.)

3.2.2 On the hypothesis of Theorem 3.2.3.

We would like to discuss here how the hypothesis (H_M) in Theorem 3.2.3. can be weakened. The standard *Lipschitz truncation arguments* (see Proposition A.1 in [FJM02] and the references therein) are valid in our setting as well, since the constructions rely on *a partition of unity argument*. In particular, what we will make use of, is the following.

Lemma 3.2.5. *Let $n\ge 2$ and $1\le p<\infty$. There exists a constant $C_{n,p}>0$ so that for every $u\in W^{1,p}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and every $\lambda>0$, there exists $u_\lambda\in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n)$ such that*

(*i*) $\|\nabla_T u_\lambda\|_{L^\infty(\mathbb{S}^{n-1})}\le C_{n,p}\lambda$, *(ii)* $\mathcal{H}^{n-1}\Big(\{x \in \mathbb{S}^{n-1} : u(x) \neq u_\lambda(x)\}\Big) \leq \frac{C_{n,p}}{\lambda^p} \int_{\{|\nabla_T u|>\lambda\}} |\nabla_T u|^p \, d\mathcal{H}^{n-1}$,

(iii) $\int_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla_T u_\lambda|^p \, d\mathcal{H}^{n-1} \leq C_{n,p} \int_{\{|\nabla_T u|>\lambda\}} |\nabla_T u|^p \, d\mathcal{H}^{n-1}$.

Let now $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ with $0 \leq \delta_u \leq 1$ be given, so that in particular

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} \leq 2n. \tag{3.2.26}$$

An application of the previous Lemma with $p = 2$ and $\lambda \geq 2\sqrt{n-1}$ gives a Lipschitz map u_λ for which

$$\lambda^2 \mathcal{H}^{n-1}\Big(\{x \in \mathbb{S}^{n-1} : u(x) \neq u_\lambda(x)\}\Big) \leq C_{n,2} \int_{\{|\nabla_T u|>\lambda\}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} \lesssim_n \delta_u^2, \tag{3.2.27}$$

$$\int_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla_T u_\lambda|^2 \, d\mathcal{H}^{n-1} \leq C_{n,2} \int_{\{|\nabla_T u(x)|>\lambda\}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} \lesssim_n \delta_u^2. \tag{3.2.28}$$

Indeed, if $|\nabla_T u| > 2\sqrt{n-1}$, then

$$\left|\sqrt{\nabla_T u^t \nabla_T u} - I_x\right|^2 \geq \left(\left|\sqrt{\nabla_T u^t \nabla_T u}\right| - \sqrt{n-1}\right)^2 \geq \frac{\left|\sqrt{\nabla_T u^t \nabla_T u}\right|^2}{4} = \frac{|\nabla_T u|^2}{4}.$$

As in [FJM02], one can easily check that

$$\delta_{u_\lambda} \lesssim_n \delta_u. \tag{3.2.29}$$

Indeed, for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ we have *by the general polar decomposition*

$$\nabla_T u(x) = U(x)\sqrt{\nabla_T u^t \nabla_T u(x)}, \quad \nabla_T u_\lambda(x) = U_\lambda(x)\sqrt{\nabla_T u_\lambda^t \nabla_T u_\lambda(x)},$$

where $U(x), U_\lambda(x) : T_x\mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ are *partial isometries*, i.e. $|U(x)v| = |U_\lambda(x)v| = |v|$ for every $v \in T_x(\mathbb{S}^{n-1})$. Then, we can estimate pointwise $\mathcal{H}^{n-1}$-a.e.

$$\left|\sqrt{\nabla_T u_\lambda^t \nabla_T u_\lambda} - \sqrt{\nabla_T u^t \nabla_T u}\right| \leq |U_\lambda^t - U^t||\nabla_T u_\lambda| + |U^t|\,|\nabla_T u_\lambda - \nabla_T u| \lesssim_n \lambda + |\nabla_T u_\lambda - \nabla_T u|$$

and then

$$\delta_{u_\lambda}^2 \lesssim_n \int_{\{u_\lambda \neq u\}} \left|\sqrt{\nabla_T u_\lambda{}^t \nabla_T u_\lambda} - \sqrt{\nabla_T u^t \nabla_T u}\right|^2 + \delta_u^2 \lesssim_n \int_{\{u_\lambda \neq u\}} \left(\lambda^2 + |\nabla_T u_\lambda - \nabla_T u|^2\right) + \delta_u^2$$

$$\lesssim_n \left(\lambda^2 \mathcal{H}^{n-1}(\{u_\lambda \neq u\}) + \int_{\mathbb{S}^{n-1}} |\nabla_T u_\lambda - \nabla_T u|^2 \, d\mathcal{H}^{n-1} + \delta_u^2\right) \lesssim_n \delta_u^2,$$

where we used (3.2.27), (3.2.28) and the fact that

$$\{u_\lambda = u\} \overset{\mathcal{H}^{n-1}\text{-a.e.}}{\subseteq} \{\nabla_T u_\lambda = \nabla_T u\} \iff \{\nabla_T u_\lambda \neq \nabla_T u\} \overset{\mathcal{H}^{n-1}\text{-a.e.}}{\subseteq} \{u_\lambda \neq u\}.$$

Of course, *in our setting we also need to control the difference between the generalized isoperimetric deficits.* For this purpose, let $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ be such that $|V_n(u)| < \infty$ and let u_λ be its *Lipschitz truncation* provided by Lemma 3.2.5. for $p = 2$ and some $\lambda \geq 2\sqrt{n-1}$. If $|V_n(u_\lambda)| > 1$ then $\varepsilon_{u_\lambda} = 0 \leq \varepsilon_u$, so we may assume without loss of generality that $|V_n(u_\lambda)| \leq 1$. Then,

$$\varepsilon_{u_\lambda} = 1 - |V_n(u_\lambda)| \leq \varepsilon_u + |V_n(u)| - |V_n(u_\lambda)| \leq \varepsilon_u + \Big|V_n(u) - V_n(u_\lambda)\Big|, \tag{3.2.30}$$

i.e. it suffices to control (the absolute value of) the difference between the corresponding *signed-volume-terms*. In this respect, by denoting here $\overline{v} := \fint_{\mathbb{S}^{n-1}} v$ and using the Poincare inequality and then (3.2.28), we have

$$\begin{aligned}
\Big|V_n(u) - V_n(u_\lambda)\Big| &\leq \left|\fint_{\mathbb{S}^{n-1}} \Big\langle (u_\lambda - u) - \overline{(u_\lambda - u)}, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_\lambda \Big\rangle\right| + \left|\fint_{\mathbb{S}^{n-1}} \Big\langle u - \overline{u}, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \Big\rangle\right| \\
&\lesssim_n \lambda^{n-1} \fint_{\mathbb{S}^{n-1}} \Big|u_\lambda - u - \overline{(u_\lambda - u)}\Big| + \left|\fint_{\mathbb{S}^{n-1}} \Big\langle u - \overline{u}, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \Big\rangle\right| \\
&\lesssim_{n,\lambda} \left(\fint_{\mathbb{S}^{n-1}} \Big|\nabla_T u_\lambda - \nabla_T u\Big|^2\right)^{\frac{1}{2}} + \left|\fint_{\mathbb{S}^{n-1}} \Big\langle u - \overline{u}, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \Big\rangle\right| \\
&\lesssim_{n,\lambda} \delta_u + A_{u,\lambda},
\end{aligned} \tag{3.2.31}$$

where $A_{u,\lambda} := \left|\fint_{\mathbb{S}^{n-1}} \Big\langle u - \overline{u}, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \Big\rangle\right|$. In the same fashion,

$$\begin{aligned}
n\omega_n A_{u,\lambda} &= \left|\int_{\{u \neq u_\lambda\}} \Big\langle u - \overline{u}, (\partial_{\tau_1} u_\lambda - \partial_{\tau_1} u) \wedge \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda + \partial_{\tau_1} u \wedge \left(\bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u\right)\right| \\
&\leq \lambda^{n-2} \left\|u - \overline{u}\right\|_{L^2} \left\|\nabla_T u_\lambda - \nabla_T u\right\|_{L^2} + \int_{\{u \neq u_\lambda\}} \Big|u - \overline{u}\Big| \Big|\partial_{\tau_1} u\Big| \left|\bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u\right| \\
&\overset{(3.2.26)}{\lesssim_{n,\lambda}} \delta_u + \int_{\{u \neq u_\lambda\}} \Big|u - \overline{u}\Big| \Big|\partial_{\tau_1} u\Big| \left|\bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u\right|.
\end{aligned}$$

In this way, we arrive at the estimate

$$\Big|V_n(u) - V_n(u_\lambda)\Big| \lesssim_{n,\lambda} \delta_u + \int_{\{u \neq u_\lambda\}} \Big|u - \overline{u}\Big| \Big|\partial_{\tau_1} u\Big| \left|\bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u\right|. \tag{3.2.32}$$

Actually, in the case $n = 3$, a more precise calculation yields

$$V_3(u) - V_3(u_\lambda) = V_3(u - u_\lambda) + R_1(u, u_\lambda) + R_2(u, u_\lambda) + R_3(u, u_\lambda) + R_4(u, u_\lambda);$$

where

$$\begin{aligned}
\big|R_1(u,u_\lambda)\big| &:= \left|\fint_{\mathbb{S}^2} \Big\langle (u-u_\lambda) - \overline{(u-u_\lambda)}, \partial_{\tau_1} u_\lambda \wedge \partial_{\tau_2} u \Big\rangle\right| \lesssim \lambda \, \|\nabla_T u\|_{L^2} \, \|\nabla_T u - \nabla_T u_\lambda\|_{L^2} \lesssim_\lambda \delta_u, \\
\big|R_2(u,u_\lambda)\big| &:= \left|\fint_{\mathbb{S}^2} \Big\langle (u-u_\lambda) - \overline{(u-u_\lambda)}, \partial_{\tau_1} (u-u_\lambda) \wedge \partial_{\tau_2} u_\lambda \Big\rangle\right| \lesssim \lambda \, \|\nabla_T u - \nabla_T u_\lambda\|_{L^2}^2 \lesssim_\lambda \delta_u^2, \\
\big|R_3(u,u_\lambda)\big| &:= \left|\fint_{\mathbb{S}^2} \Big\langle u_\lambda - \overline{u_\lambda}, \partial_{\tau_1} u_\lambda \wedge \partial_{\tau_2} (u-u_\lambda) \Big\rangle\right| \lesssim \lambda^2 \int_{\{u\neq u_\lambda\}} |\nabla_T u - \nabla_T u_\lambda| \lesssim_\lambda \delta_u^2, \\
\big|R_4(u,u_\lambda)\big| &= \left|\fint_{\mathbb{S}^2} \Big\langle u_\lambda - \overline{u_\lambda}, \partial_{\tau_1} (u-u_\lambda) \wedge \partial_{\tau_2} u \Big\rangle\right| \lesssim \|\nabla_T u_\lambda\|_{L^\infty} \|\nabla_T u\|_{L^2} \|\nabla_T u - \nabla_T u_\lambda\|_{L^2} \lesssim_\lambda \delta_u.
\end{aligned}$$

Moreover, by the generalized isoperimetric inequality and (3.2.28), we can also estimate

$$\big|V_3(u-u_\lambda)\big| \leq \left(\frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T u - \nabla_T u_\lambda|^2\right)^{\frac{3}{2}} \lesssim \delta_u^3.$$

To summarize, *the integrand in $V_n(u)$ has linear growth in u and $(n-1)$-growth in $\nabla_T u$, so that $|V_n(u) - V_n(u_\lambda)|$ cannot generically be controlled only with information on the $W^{1,2}$-Sobolev norm of $u-u_\lambda$, except for the case of low dimensions.* To be more precise, in view of the last estimates, the hypothesis (H_M) of Theorem 3.2.3. can definitely be relaxed as the following Proposition suggests.

Proposition 3.2.6. *(i) Let $n=2,3$. The hypothesis (H_M) can be completely removed, i.e. (3.2.11) holds for all $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ for some constant that is only dimension-dependent.*

(ii) Let $n \geq 4$ and $M > 0$ be given. The hypothesis (H_M) can be replaced by the weaker hypothesis

$$u \in \dot{W}^{1,2(n-2)}(\mathbb{S}^{n-1};\mathbb{R}^n), \text{ with } \|\nabla_T u\|_{L^{2(n-2)}(\mathbb{S}^{n-1})} \leq M. \qquad (H_{n,M})$$

Proof. Let us set

$$B_{u,\lambda} := \int_{\{u\neq u_\lambda\}} \big|u-\overline{u}\big|\big|\partial_{\tau_1} u\big| \left|\bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u\right|.$$

(i) If $n=2$, by the Sobolev embedding $\|u-\overline{u}\|_{L^\infty(\mathbb{S}^1)} \leq \|\partial_\tau u\|_{L^2(\mathbb{S}^1)} \leq 2$, where τ denotes the unit tangent vector field to $\mathbb{S}^1$, and therefore

$$B_{u,\lambda} = \int_{\{u\neq u_\lambda\}} \big|u-\overline{u}\big|\big|\partial_\tau u\big| \leq 2\Big(\mathcal{H}^1(\{u\neq u_\lambda\})\Big)^{\frac{1}{2}} \left(\int_{\mathbb{S}^1} |\partial_\tau u|^2\right)^{\frac{1}{2}} \overset{(3.2.27)}{\lesssim} \frac{1}{\lambda}\delta_u.$$

If $n=3$, by the previous estimates (and since w.l.o.g. $0 \leq \delta_u \leq 1$) we have

$$\big|V_3(u) - V_3(u_\lambda)\big| \leq |V_3(u-u_\lambda)| + \sum_{i=1}^4 |R_i(u,u_\lambda)| \lesssim_\lambda \delta_u^3 + \delta_u^2 + \delta_u \lesssim_\lambda \delta_u.$$

(ii) If $n \geq 4$ and $M > 0$ is given, for every map $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ with $0 \leq \delta_u \leq 1$ that satisfies the hypothesis stated in (ii), we have by *the Sobolev embedding* that $\|u - \overline{u}\|_{L^\infty(\mathbb{S}^{n-1})} \lesssim_n \|u\|_{\dot{W}^{1,2(n-2)}(\mathbb{S}^{n-1})} \lesssim_n M$, and therefore

$$
\begin{aligned}
B_{u,\lambda} &\lesssim_n M \int_{\{u \neq u_\lambda\}} \left|\partial_{\tau_1} u\right| \left| \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda - \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u \right| \\
&\lesssim_n M \left(\int_{\{u \neq u_\lambda\}} |\partial_{\tau_1} u| \left| \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u_\lambda \right| + \int_{\{u \neq u_\lambda\}} |\partial_{\tau_1} u| \left| \bigwedge_{i=2}^{n-1} \partial_{\tau_i} u \right| \right) \\
&\lesssim_{n,\lambda} M \left\|\nabla_T u\right\|_{L^2} \left(\mathcal{H}^{n-1}(\{u \neq u_\lambda\})\right)^{\frac{1}{2}} + M \left\|\nabla_T u\right\|_{L^{2(n-2)}}^{n-2} \left(\int_{\{u \neq u_\lambda\}} |\nabla_T u|^2 \right)^{\frac{1}{2}}. \\
&\lesssim_{n,\lambda,M} \delta_u + \left(\int_{\{u \neq u_\lambda\} \cap \{|\nabla_T u| \leq \lambda\}} |\nabla_T u|^2 + \int_{\{u \neq u_\lambda\} \cap \{|\nabla_T u| > \lambda\}} |\nabla_T u|^2 \right)^{\frac{1}{2}} \\
&\lesssim_{n,\lambda,M} \delta_u + \left(\lambda^2 \mathcal{H}^{n-1}(\{u \neq u_\lambda\}) + \int_{\{|\nabla_T u| > \lambda\}} |\nabla_T u|^2 \right)^{\frac{1}{2}} \\
&\overset{(3.2.27),(3.2.28)}{\lesssim_{n,\lambda,M}} \delta_u.
\end{aligned}
$$

Therefore, the previous estimates altogether give us

(i) If $n = 2$, then $\forall\ u \in W^{1,2}(\mathbb{S}^1;\mathbb{R}^2)$,

$$|V_2(u) - V_2(u_\lambda)| \lesssim_\lambda \delta_u \implies \varepsilon_{u_\lambda} \lesssim_\lambda (\varepsilon_u + \delta_u).$$

If $n = 3$, then $\forall\ u \in W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)$,

$$|V_3(u) - V_3(u_\lambda)| \lesssim_\lambda \delta_u \implies \varepsilon_{u_\lambda} \lesssim_\lambda (\varepsilon_u + \delta_u).$$

(iii) If $n \geq 4$, $M > 0$, then $\forall\ u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ satisfying the hypothesis in (ii),

$$|V_n(u) - V_n(u_\lambda)| \lesssim_{n,\lambda,M} \delta_u \implies \varepsilon_{u_\lambda} \lesssim_{n,\lambda,M} (\varepsilon_u + \delta_u).$$

Therefore, in each one of the above cases (for $M > 0$ fixed when $n \geq 4$, with the new meaning for M), we can start from a map $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ satisfying the corresponding hypothesis (or no hypothesis when $n = 2, 3$), and replace it with the map u_λ for $\lambda = 2\sqrt{n-1}$. Since the estimate in (3.2.11) holds for the map u_λ for some $O \in O(n)$ (with a purely dimensional constant because $\|\nabla_T u_\lambda\|_{L^\infty(\mathbb{S}^{n-1})} \leq 2\sqrt{n-1}C_{n,2}$), we immediately obtain that it also holds as well for u for the same $O \in O(n)$, with the new constant depending however both on n and the prechosen $M > 0$ when $n \geq 4$. $\square$

3.3 A linear stability result for $Isom_+(\mathbb{S}^{n-1})$

In the last Section of this Chapter we would like to give another *quantitative version of Theorem 2.1.5.*, essentially at a *linearized level.* As in the previous Section, for a map $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ we use as its *isometric deficit* the quantity

$$\delta_u := \left(\fint_{\mathbb{S}^{n-1}} \left| \sqrt{\nabla_T u^t \nabla_T u} - I_x \right|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \geq 0. \tag{3.3.1}$$

If u is further assumed to be in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ (which is a sufficient condition to ensure that $|V_n(u)| < \infty$) by the *general version of the isoperimetric inequality* (A.0.1), we have

$$|V_n(u)| := \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \right| \leq \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \right)^{\frac{n}{n-1}} =: P_{n-1}(u). \tag{3.3.2}$$

In particular, the result of Theorem 2.1.5. can alternatively be stated by saying that *if u is such that equalities hold simultaneously in* (3.3.1) *and* (3.3.2) *then* $u \in Isom(\mathbb{S}^{n-1})$ up to a translation vector (more precisely $u - \fint_{\mathbb{S}^{n-1}} u \in Isom(\mathbb{S}^{n-1})$).

From a *variational viewpoint*, we have that the identity map $\mathrm{id}_{\mathbb{S}^{n-1}}$ is an *absolute minimum* both of the *"isometric-deficit" functional*

$$W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) \ni u \mapsto \delta_u \geq 0, \tag{3.3.3}$$

and of the *"full generalized isoperimetric-deficit" functional*

$$W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n) \ni u \mapsto \tilde{\varepsilon}_u := P_{n-1}(u) - V_n(u) \geq 0, \tag{3.3.4}$$

so the *second variation of both functionals* at the $\mathrm{id}_{\mathbb{S}^{n-1}}$ must be *nonnegative quadratic forms in* $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$. These are calculated in detail in Appendix C. Therefore,

$$Q_{n,\mathrm{isom}}(w) := \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2 \, d\mathcal{H}^{n-1} \geq 0 \tag{3.3.5}$$

for all $w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and also

$$Q_{n,\mathrm{isop}}(w) := \frac{n}{n-1} \left[\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2 + (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{2} \right) - Q_{n,isom}(w) \right] - Q_{V_n}(w) \geq 0 \tag{3.3.6}$$

for all $w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$.

The quadratic form Q_{V_n} is the second variation of the *generalized signed-volume-term* at the $\mathrm{id}_{\mathbb{S}^{n-1}}$, given by

$$Q_{V_n}(w) := \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^{n-1}, \tag{3.3.7}$$

(see again the calculations in Appendix C), and is easily seen to be invariant under translations by fixed vectors, i.e. $Q_{V_n}(w) = Q_{V_n}\left(w - \fint_{\mathbb{S}^{n-1}} w\right)$.

While the nonnegativity of the form $Q_{n,\mathrm{isom}}$ is of course obvious, the nonnegativity of the form $Q_{n,\mathrm{isop}}$ may not be so obvious from the first sight. At the level of these quadratic forms one has a *linearized analogue* of Theorem 2.1.5., that we describe below.

First of all, the *kernel* of the nonnegative form $Q_{n,\mathrm{isop}}$ is *infinite dimensional* and we will give an example of *an infinite dimensional subspace of* $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ where the form vanishes at the end of Subsection 4.2.2.. For $Q_{n,\mathrm{isom}}$, in the case that $n=2$ one can easily see that $\dim(\ker Q_{2,\mathrm{isom}}) = \infty$. Indeed, for any $v : \mathbb{S}^1 \mapsto \mathbb{R}^2$ written as

$$v(x) = \phi(x)x + \psi(x)\tau(x), \quad \text{where} \quad \phi, \psi \in C^\infty(\mathbb{S}^1, \mathbb{R})$$

and $\tau(x) := (-x_2, x_1)$ is the unit tangent vector field to $\mathbb{S}^1$, it is an easy calculation to check that actually

$$(P_T^t \nabla_T v)_{\mathrm{sym}} = (P_T^t \nabla_T v)_{11} := \phi - \partial_\tau \psi,$$

i.e. *for every* $\psi \in C^\infty(\mathbb{S}^1;\mathbb{R})$, the map $v_\psi(x) := \psi(x)\tau(x) + \partial_\tau\psi(x)x \in \ker Q_{2,\mathrm{isom}}$. For $n \geq 3$ a classical (but not so immediate to prove) result in differential geometry, referred to as the *infinitesimal rigidity of the sphere*, asserts that actually $\ker Q_{n,isom} \simeq \mathfrak{so}(n)$. What is actually straightforward to prove is the following fact, that would of course be an immediate consequence of the aforementioned infinitesimal rigidity property of $\mathbb{S}^{n-1}$.

Let $w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ be an element in the *common null-space* of these two nonnegative forms, i.e.

$$Q_{n,\mathrm{isom}}(w) = 0 \iff \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} = 0, \tag{3.3.8}$$

and

$$Q_{n,\mathrm{isop}}(w) = 0. \tag{3.3.9}$$

By taking the trace in (3.3.8), we see that $\mathrm{div}_{\mathbb{S}^{n-1}} w \equiv 0$ on $\mathbb{S}^{n-1}$ and then (3.3.9) reduces to

$$\frac{1}{2}\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} - Q_{V_n}\left(w - \fint_{\mathbb{S}^{n-1}} w\right) = 0. \tag{3.3.10}$$

An integration by parts in the right hand side of (3.3.7) yields the alternative formula

$$Q_{V_n}(w) = \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left(2\mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2\right) d\mathcal{H}^{n-1}, \tag{3.3.11}$$

so that (3.3.10) results in

$$\left(\frac{1}{n-1}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 - \fint_{\mathbb{S}^{n-1}} |w - \fint_{\mathbb{S}^{n-1}} w|^2\right) + n\fint_{\mathbb{S}^{n-1}} \left\langle w - \fint_{\mathbb{S}^{n-1}} w, x\right\rangle^2 = 0. \tag{3.3.12}$$

Once again, the quantity in the brackets is nonnegative, being the L^2-Poincare deficit of w and therefore the only solutions to (3.3.12) are maps w for which

$$w(x) - \fint_{\mathbb{S}^{n-1}} w = Ax \ \text{ for } \ A \in \mathbb{R}^{n\times n} \ \text{ and } \ \left\langle w - \fint_{\mathbb{S}^{n-1}} w, x \right\rangle \equiv 0 \text{ on } \mathbb{S}^{n-1}.$$

By the last equation, the matrix $A \in \mathbb{R}^{n\times n}$ should satisfy

$$\langle Ax, x\rangle \equiv 0 \iff \sum_{i,j=1}^{n} A_{ij}x_i x_j = 0 \iff \sum_{1\le i\le j\le n} (A_{ij} + A_{ji})x_i x_j \equiv 0 \text{ on } \mathbb{S}^{n-1},$$

i.e. $A^t = -A$. Reversely, any such map is in the null-space of both quadratic forms. As a result of these simple calculations, without refering to the *infinitesimal rigidity of the sphere* we have verified that

$$\ker Q_{n,\mathrm{isom}} \cap \ker Q_{n,\mathrm{isop}} \simeq Skew(n) \simeq \mathfrak{so}(n). \tag{3.3.13}$$

Arguing quantitatively as in Subsection 4.4.2. (or qualitatively via a standard contradiction\ compactness argument), we obtain that for every *positive combination* of these two quadratic forms the following *coercivity estimate* holds.

Proposition 3.3.1. *For every $\alpha > 0$ $\exists$ $C_{n,\alpha} > 0$ such that $\forall w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$,*

$$\alpha Q_{n,\mathrm{isom}}(w) + Q_{n,\mathrm{isop}}(w) \ge C_{n,\alpha} \fint_{\mathbb{S}^{n-1}} |\nabla_T w - [\nabla w_h(0)]_{\mathrm{skew}} P_T|^2 \, d\mathcal{H}^{n-1}. \tag{3.3.14}$$

The last linear estimate can of course immediately imply some further estimates *in a small neighbourhoud of the* $\mathrm{id}_{\mathbb{S}^{n-1}}$, for example one easily obtains the following.

Corollary 3.3.2. *For every $n \ge 2$ there exists $\delta_0 := \delta_0(n) > 0$ with the following property. For every map $u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ with $\|u - \mathrm{id}|_{\mathbb{S}^{n-1}}\|_{W^{1,\infty}} \le \delta_0$, there exists $R \in SO(n)$ such that*

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - RP_T|^2 \, d\mathcal{H}^{n-1} \le C_n \delta_u^2, \tag{3.3.15}$$

where $C_n > 0$ is another dimensional constant.

Proof. In a small $W^{1,\infty}$-neighbourhood of the $\mathrm{id}_{\mathbb{S}^{n-1}}$ linear and nonlinear estimates are of course essentially equivalent. For $\delta_0 > 0$ that will be chosen sufficiently small depending on n in a bit, and u as in the statement

$$|\nabla u_h(0) - I_n|^2 \le \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \implies |\nabla u_h(0) - I_n| \le \sqrt{\frac{n}{n-1}}\delta_0 \ll 1,$$

and by the polar decomposition, $\nabla u_h(0) = R\sqrt{\nabla u_h(0)^t \nabla u_h(0)}$ for some $R \in SO(n)$. Note also that u has to be a map of degree 1.

If we therefore set $\tilde{u} = R^t u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ and $w := \tilde{u} - \mathrm{id}_{\mathbb{S}^{n-1}}$, $\tilde{u}$ is still an orientation-preserving map of degree 1. Thanks to their rotational invariance, the two deficits are of course left *unchanged*, i.e.

$$\begin{aligned}
\delta_u^2 = \delta_{\tilde{u}}^2 &= Q_{n,\mathrm{isom}}(w) + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3)\ d\mathcal{H}^{n-1}, \\
0 = \tilde{\varepsilon}_u = \tilde{\varepsilon}_{\tilde{u}} &= Q_{n,\mathrm{isop}}(w) + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3)\ d\mathcal{H}^{n-1}.
\end{aligned}$$

Adding these two identities results in

$$\delta_u^2 = Q_{n,\mathrm{isom}}(w) + Q_{n,\mathrm{isop}}(w) + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3)\ d\mathcal{H}^{n-1}. \tag{3.3.16}$$

By the triangle inequality,

$$\begin{aligned}
\|\nabla_T w\|_{L^\infty} &= \|R^t \nabla_T u - P_T\|_{L^\infty} \leq |R - I_n| \|\nabla_T u\|_{L^\infty} + \|\nabla_T u - P_T\|_{L^\infty} \\
&\leq \Big(|\nabla u_h(0) - R| + |\nabla u_h(0) - I_n|\Big)\left(\delta_0 + \sqrt{n-1}\right) + \delta_0 \\
&\leq 2|\nabla u_h(0) - I_n|\left(\delta_0 + \sqrt{n-1}\right) + \delta_0 \\
&\lesssim_n \delta_0 + \delta_0^2 \ \lesssim_n \ \delta_0,
\end{aligned}$$

since $\delta_0 > 0$ will be chosen sufficiently small depending only on n. In this way, the higher order terms are of course absorbed in the *quadratic ones*, i.e.

$$\left|\fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3)\ d\mathcal{H}^{n-1}\right| \leq c_n \delta_0 \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1} \tag{3.3.17}$$

for a certain dimensional constant $c_n > 0$. Proposition 3.3.1. then gives

$$Q_{n,\mathrm{isom}}(w) + Q_{n,\mathrm{isop}}(w) \geq C_{n,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1}, \tag{3.3.18}$$

because $[\nabla w_h(0)]_{\mathrm{skew}} = \left[\sqrt{\nabla u_h(0)^t \nabla u_h(0)} - I_n\right]_{\mathrm{skew}} = 0$. Finally, by (3.3.16) we obtain

$$(C_{n,1} - c_n\delta_0) \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \leq \delta_u^2 \implies \fint_{\mathbb{S}^{n-1}} |\nabla_T u - RP_T|^2\ d\mathcal{H}^{n-1} \leq \frac{2}{C_{n,1}}\delta_u^2 \tag{3.3.19}$$

by choosing $\delta_0 > 0$ even smaller if necessary so that $0 < \delta_0 \leq \frac{C_{n,1}}{2c_n}$. □

Remark 3.3.3. As we had mentioned at the end of Subsection 3.2.1., the exponents in the deficits δ_u, ε_u in the right hand side of (3.2.11) cannot *generically be improved.* Nevertheless, the previous Corollary gives a very simple example of a case in which *the exponent in the isometric deficit* (which is the only one needed when one considers orientation-preserving, degree 1 maps from $\mathbb{S}^{n-1}$ to itself) *can be improved to the expected* *optimal one.*

Actually, for maps $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ a *spherical version of the rigidity estimate* of G. Friesecke, R. D. James and S. Müller can be proven, namely

If $n \geq 2$, there exists $C_n > 0$ such that for every $u \in W^{1,2}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ there exist $R \in SO(n)$ so that

$$\fint_{\mathbb{S}^{n-1}} |d_x(u - R\mathrm{id}_{\mathbb{S}^{n-1}})|^2 \, d\mathcal{H}^{n-1} \leq C_n \fint_{\mathbb{S}^{n-1}} \mathrm{dist}^2(d_x u; SO_x) \, d\mathcal{H}^{n-1}. \tag{3.3.20}$$

For every $x \in \mathbb{S}^{n-1}$, after a choice of particular frames, SO_x can be identified with $SO(n-1)$. The estimate (3.3.20) could be a consequence of the original Theorem 1.1.5., by extending u in a small tubular neighbourhood of $\mathbb{S}^{n-1}$ in a way that the extension is *constant in each radial direction*, applying the *original* Theorem 1.1.5. for the *radial extension*, and then taking the limit as the *thickness* of the tubular neighbourhood tends to zero.

Let us close this Chapter with two simple remarks. Firstly, although for $n \geq 3$ $\mathrm{ker}Q_{n,\mathrm{isom}} \simeq \mathfrak{so}(\mathfrak{n})$, an estimate of the type

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T w - [\nabla w_h(0)]_{\mathrm{skew}} P_T|^2 \lesssim_n Q_{n,\mathrm{isom}}(w)$$

does not hold, the obstacle being (loosely speaking) the derivatives of the normal component of w. For example, if one considers purely normal displacements $w_\phi(x) := \phi(x)x$; $\phi \in W^{1,2}(\mathbb{S}^{n-1})$, then by a straightforward computation one can check that

$$(P_T^t \nabla_T w_\phi)_{\mathrm{sym}} = \phi I_x \implies Q_{n,\mathrm{isom}}(w_\phi) = (n-1) \fint_{\mathbb{S}^{n-1}} |\phi|^2,$$

whereas the full gradient of w_ϕ also has derivatives of ϕ in it. In coordinates,

$$(\nabla_T w)_{ij} = \phi (P_T)_{ij} + x_i \partial_{\tau_j} \phi,$$

so if the estimate above was to be valid, it would resemble some short of reverse-Poincare inequality, which is of course generically false.

Secondly (and finally), since the proof of Corollary 3.3.2. is *intrinsic*, in the sense that it does not rely on the rigidity result in the bulk, it would be interesting to give an *intrinsic proof* of (3.3.20), as an example of a *global quantitative rigidity result for orientation-preserving isometries between Riemannean manifolds* (the orientation-preserving rigid motions of $\mathbb{S}^{n-1}$ in this case).

Chapter 4

On the (local) stability of $Conf_+(\mathbb{S}^{n-1})$ among almost conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ when $n \geq 3$

In this Chapter we discuss the stability of the group of orientation-preserving Möbius transformations of $\mathbb{S}^{n-1}$ when $n \geq 3$, in terms of the *combined conformal-isoperimetric deficit* that was defined in the Introduction. Although the approach is the same in dimension $n = 3$ and in the higher dimensional case $n \geq 4$, due to some *analytic differences*, or better said, *simplifications* that occur in dimension $n = 3$, we have chosen to organize these in two different Sections. Some of the steps in the analysis carry out unchanged in both cases and in order to avoid their repetition in the second Section of the Chapter, we will adopt the following convention. In Section 4.1 we will still denote the ambient dimension 3 by the general letter n in the parts of the analysis which we are going to use also in Section 4.2. The author hopes that no confusion will be caused by the alternate use of n and the number 3 for the ambient dimension in the first Section to come.

4.1 The case $n = 3$

4.1.1 Setup of the local stability estimate

For the convenience of the reader, we first repeat some notation that was introduced in Section 1.2, adjusted in this setting where the domain is $\mathbb{S}^2$. For some $\theta > 0$ that will eventually be chosen sufficiently small to serve our purposes and for $\varepsilon > 0$, recalling (1.2.7) and (1.2.14), the (local) class of mappings inside which the stability of $Conf_+(\mathbb{S}^2)$ will be investigated is defined by

$$\mathcal{A}_{3,2,\theta,\varepsilon} := \left\{ u \in W^{1,2}(\mathbb{S}^2;\mathbb{R}^3) : \begin{array}{l} (i)\ \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} \leq \theta, \\ \\ (ii)\ D_2(u) \leq (1+\varepsilon)V_3(u) \end{array} \right\} \tag{4.1.1}$$

where the subscript 3 in the definition of the set $\mathcal{A}_{3,2,\theta,\varepsilon}$ stands for the dimension of the ambient space, the subscript 2 for the exponent in the Sobolev norm and as we defined in (1.2.7), (1.2.1)

$$D_2(u) := \left(\frac{1}{2} \fint_{\mathbb{S}^2} |\nabla_T u|^2 \, d\mathcal{H}^2 \right)^{\frac{3}{2}}, \quad V_3(u) := \fint_{\mathbb{S}^2} \langle u, \partial_{\tau_1} u \wedge \partial_{\tau_2} u \rangle \, d\mathcal{H}^2. \tag{4.1.2}$$

Remark 4.1.1. (i) Let us mention once again here that the positive parameter ε in the definition of $\mathcal{A}_{3,2,\theta,\varepsilon}$ is the one that is referred to as *the combined conformal-isoperimetric deficit of a map* $u \in \mathcal{A}_{3,2,\theta,\varepsilon}$. For every such map the last defining property in (4.1.1) is *invariant under translations by fixed vectors in* $\mathbb{R}^3$*, dilations by a positive factor, rotations in* $\mathbb{R}^3$ *and compositions (from the right) with orientation-preserving Möbius transformations of* $\mathbb{S}^2$.

(ii) Even though the setup we are presenting here is for a local statement *close to the identity transformation of* $\mathbb{S}^2$, we could of course reach the same conlusions if we were to assume that the map u is apriori close to any other orientation-preserving conformal diffeomorphism of $\mathbb{S}^2$. To be more precise, given any $\psi \in Conf_+(\mathbb{S}^2)$, for our local estimate we could consider as well the set of maps

$$\mathcal{A}_{3,2,\psi,\theta,\varepsilon} := \left\{ u \in W^{1,2}(\mathbb{S}^2;\mathbb{R}^3) : \begin{array}{l} (\tilde{i}) \|\nabla_T u - \nabla_T \psi\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} \leq \theta, \\ \\ (\tilde{ii})\ D_2(u) \leq (1+\varepsilon)V_3(u) \end{array} \right\}.$$

It is then immediate to check that whenever $u \in \mathcal{A}_{3,2,\psi,\theta,\varepsilon}$, the map $u \circ \psi^{-1} \in \mathcal{A}_{3,2,\theta,\varepsilon}$. This follows directly from the conformal invariance of the Dirichlet energy in two dimensions and the invariance of the combined conformal-isoperimetric deficit under precompositions with elements of $Conf_+(\mathbb{S}^2)$. Then, all the arguments that we will present could be applied to the map $u \circ \psi^{-1}$ instead of u. For possible later purposes we also recall here some elementary properties of this precomposed map. First of all, by the chain rule (with the gradients viewed as linear maps between the corresponding tangent spaces), one can easily verify that every conformal map $\psi \in Conf_+(\mathbb{S}^2)$ satisfies

$$\frac{|\nabla_T \psi(\psi^{-1}(x))|^2}{2} \cdot \frac{|\nabla_T \psi^{-1}(x)|^2}{2} = 1 \iff |\nabla_T \psi(\psi^{-1}(x))| \cdot |\nabla_T \psi^{-1}(x)| = 2.$$

For $u \in \mathcal{A}_{3,2,\psi,\theta,\varepsilon}$, the map $u \circ \psi^{-1}$ satisfies pointwise $\mathcal{H}^2$-a.e. on $\mathbb{S}^2$ the inequality

$$\left|\nabla_T (u \circ \psi^{-1})(x)\right| \leq \left|\nabla_T u(\psi^{-1}(x))\right| \left|\nabla_T \psi^{-1}(x))\right| = \frac{2\,|\nabla_T u(\psi^{-1}(x))|}{|\nabla_T \psi(\psi^{-1}(x))|},$$

therefore

$$\left\|\nabla_T (u \circ \psi^{-1})\right\|_{L^\infty(\mathbb{S}^2)} \leq \frac{2\|\nabla_T u\|_{L^\infty}}{\min_{\mathbb{S}^2} |\nabla_T \psi|},$$

even (of course) when the right hand side has to be interpreted as being $+\infty$.

For the $W^{1,2}$-Sobolev norm of $u \circ \psi$ we can estimate separately the two integrals as follows.

$$\fint_{\mathbb{S}^2} |(u \circ \psi^{-1}) - \mathrm{id}_{\mathbb{S}^2}|^2 \; d\mathcal{H}^2 = \fint_{\mathbb{S}^2} |u(y) - \psi(y)|^2 \, g(y) \; d\mathcal{H}^2(y),$$

where the induced *area-element* by the change of variables $x = \psi(y)$ is given by

$$g(y) := \sqrt{\det\left((\nabla_T \psi(y))^t \, \nabla_T \psi(y)\right)} = \frac{|\nabla_T \psi(y)|^2}{2}.$$

Hence,

$$\begin{aligned}
\fint_{\mathbb{S}^2} |(u \circ \psi^{-1}) - \mathrm{id}_{\mathbb{S}^2}|^2 \; d\mathcal{H}^2 &= \frac{1}{2} \fint_{\mathbb{S}^2} |u(y) - \psi(y)|^2 \, |\nabla_T \psi(y)|^2 \; d\mathcal{H}^2 \\
&\leq \frac{\|\nabla_T \psi\|^2_{L^\infty(\mathbb{S}^2)}}{2} \fint_{\mathbb{S}^2} |u - \psi|^2 \; d\mathcal{H}^2,
\end{aligned}$$

while, as we have already mentioned, because of the *conformal invariance* of the *Dirichlet energy on* $\mathbb{S}^2$,

$$\fint_{\mathbb{S}^2} |\nabla_T (u \circ \psi^{-1}) - P_T|^2 \; d\mathcal{H}^2 = \fint_{\mathbb{S}^2} |\nabla_T u - \nabla_T \psi|^2 \; d\mathcal{H}^2.$$

Putting these last estimates together, we have (in case it is necessary later) that

$$\left\|u \circ \psi^{-1} - \mathrm{id}_{\mathbb{S}^2}\right\|_{W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)} \leq \max\left\{1, \frac{\|\nabla_T \psi\|_{L^\infty}}{\sqrt{2}}\right\} \|u - \psi\|_{W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)},$$

completing this easy estimate.

(*iii*) The *apriori closeness to the* $\mathrm{id}_{\mathbb{S}^2}$ *in the* $W^{1,2}$ – *topology* can be thought of as a *non-degeneracy condition.* For example, it prevents the maps in consideration from *concentrating around single points.* Conditions of similar flavour have been posed also in [Res70] and [FZ05] and in further results mentioned therein regarding quantitative stability for compact subsets of the conformal group $CO_+(n)$ in bounded domains of $\mathbb{R}^n$, when $n \geq 3$. As we have discussed in the Introduction, such conditions are imposed there to avoid *degeneracy issues* at the origin and at infinity of the "cone" $CO_+(n)$.

With the notations we adopted, the main result of this Section can be stated as follows.

Theorem 4.1.2. *There exists absolute constants $\theta_0 > 0$ and $C > 0$ with the following property. Given $\varepsilon > 0$ arbitrary, then for every $u \in \mathcal{A}_{3,2,\theta_0,\varepsilon}$ there exist $\phi_u \in Conf_+(\mathbb{S}^2)$, $b_u \in \mathbb{R}^3$ and $\lambda_u > 0$ such that*

$$\left\| \left(\frac{u \circ \phi_u - b_u}{\lambda_u} \right) - \mathrm{id}_{\mathbb{S}^2} \right\|_{W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)} \leq C\sqrt{\varepsilon}. \tag{4.1.3}$$

Hence, loosely speaking, *the map u is $\sqrt{\varepsilon}$-close to $(\lambda_u \phi_u^{-1} + b_u)$ in the $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$-topology.* Actually, *the exponent $\frac{1}{2}$ with which the ε-deficit appears on the right hand side of the estimate is optimal*, as can be checked by considering the sequence of "affine" maps $(u_\sigma)_{\sigma>0} : \mathbb{S}^2 \mapsto \mathbb{R}^3$, where $u_\sigma(x) := A_\sigma x$, $A_\sigma := \mathrm{diag}(1, 1, 1+\sigma) \in \mathbb{R}^{3\times3}$ as $\sigma \to 0^+$.

The proof of Theorem 4.1.2. will be given in several steps. We first start with an easy Lemma that allows us to *fix the center of mass and the scale of the map u* and will be of use later.

Lemma 4.1.3. *Given $\theta > 0$ (sufficiently small) and $\varepsilon > 0$, there exists $\tilde{\theta} > 0$ that depends only on θ, so that after possibly replacing θ with $\tilde{\theta}$, we can assume that every $u \in \mathcal{A}_{3,2,\tilde{\theta},\varepsilon}$ has the following additional properties:*

(i) $\fint_{\mathbb{S}^2} u \; d\mathcal{H}^2 = 0$,

(ii) $\fint_{\mathbb{S}^2} \langle u, x\rangle \; d\mathcal{H}^2 = 1$.

Proof. The first property is trivially obtained by considering $u - \fint_{\mathbb{S}^2} u \; d\mathcal{H}^2$ instead of u if necessary. Regarding the second one, by the mean value property of harmonic functions,

$$\fint_{\mathbb{S}^2} \langle u, x\rangle \; d\mathcal{H}^2 = \frac{1}{3}\fint_{B^3} \mathrm{div} u_h \; dx \; = \; \frac{1}{3}\mathrm{Tr}\left(\fint_{B^3} \nabla u_h \; dx\right) \; = \frac{\mathrm{Tr}\nabla u_h(0)}{3}, \tag{4.1.4}$$

and by a simple inequality that we also used in Chapter 3,

$$\left|\frac{\mathrm{Tr}\nabla u_h(0)}{3} - 1\right|^2 \leq \frac{|\nabla u_h(0) - I_3|^2}{3} \; \leq \fint_{B^3} \frac{|\nabla u_h - I_3|^2}{3} \; \leq \fint_{\mathbb{S}^2} \frac{|\nabla_T u - P_T|^2}{2} \leq \frac{\theta^2}{2}.$$

If we thus choose $\theta > 0$ sufficiently small, we have

$$0 < 1 - \frac{\theta}{\sqrt{2}} < \frac{\mathrm{Tr}\nabla u_h(0)}{3} < 1 + \frac{\theta}{\sqrt{2}}. \tag{4.1.5}$$

In order to achieve both desired properties simultaneously, we can set

$$\lambda_u := \frac{\mathrm{Tr}\nabla u_h(0)}{3}$$

and replace u with the map

$$u_1 := \frac{u - \fint_{\mathbb{S}^2} u \; d\mathcal{H}^2}{\lambda_u} \tag{4.1.6}$$

if necessary.

The defining properties of the set $\mathcal{A}_{3,2,\theta,\varepsilon}$ are of course only slightly affected by this simple transformation. Regarding the first one,

$$\begin{aligned}
\|\nabla_T u_1 - P_T\|_{L^2(\mathbb{S}^2)} &= \left\| \nabla_T \left(\frac{u - \fint_{\mathbb{S}^{n-1}} u}{\lambda_u} \right) - P_T \right\|_{L^2(\mathbb{S}^2)} = \left\| \frac{1}{\lambda_u} \nabla_T u - P_T \right\|_{L^2(\mathbb{S}^2)} \\
&= \left\| \frac{1}{\lambda_u} (\nabla_T u - P_T) + \left(\frac{1}{\lambda_u} - 1 \right) P_T \right\|_{L^2(\mathbb{S}^2)} \\
&\leq \frac{1}{\lambda_u} \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2)} + \left| \frac{1}{\lambda_u} - 1 \right| \|P_T\|_{L^2(\mathbb{S}^2)} \\
&= \frac{1}{\lambda_u} \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2)} + \left| \frac{\lambda_u - 1}{\lambda_u} \right| \|P_T\|_{L^2(\mathbb{S}^2)} \\
&\overset{(4.1.5)}{\leq} \frac{1}{1 - \frac{\theta}{\sqrt{2}}} \theta + \frac{1}{1 - \frac{\theta}{\sqrt{2}}} \frac{\theta}{\sqrt{2}} \cdot \sqrt{2} \\
&\leq \left(\frac{2}{1 - \frac{\theta}{\sqrt{2}}} \right) \theta.
\end{aligned}$$

Therefore,

$$\|\nabla_T u_1 - P_T\|_{L^2(\mathbb{S}^2)} \leq \tilde{\theta} := \left(\frac{2}{1 - \frac{\theta}{\sqrt{2}}} \right) \theta. \tag{4.1.7}$$

The second defining property of $\mathcal{A}_{3,2,\theta,\varepsilon}$ is of course left unchanged due to the translational and scaling invariance of the deficit, i.e. it holds for the map u_1 as well, with the same ε. Altogether, we have proven that

$$u \in \mathcal{A}_{3,2,\theta,\varepsilon} \implies u_1 := \frac{u - \fint_{\mathbb{S}^2} u \; d\mathcal{H}^2}{\lambda_u} \in \mathcal{A}_{3,2,\tilde{\theta},\varepsilon}, \tag{4.1.8}$$

with the constant $\tilde{\theta}$ being given explicitely above. Although the precise value of the new constant $\tilde{\theta}$ is not of major importance, what is more important is that $\lim_{\theta\to 0^+}\tilde{\theta}=0$, so that when we will finally choose $\theta>0$ sufficiently small, $\tilde{\theta}>0$ will be sufficiently small as well. $\square$

According to the previous Lemma, after possibly replacing the parameter θ with $\tilde{\theta}$ (which we will not relabel in the sequel), we can focus on the set of maps

$$\tilde{\mathcal{A}}_{3,2,\theta,\varepsilon} := \left\{ u \in W^{1,2}(\mathbb{S}^2;\mathbb{R}^3) : \begin{array}{l} (i)\ \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} \leq \theta \\ (ii)\ \fint_{\mathbb{S}^2} u \ d\mathcal{H}^2 = 0 \\ (iii)\ \fint_{\mathbb{S}^2} \langle u, x\rangle \ d\mathcal{H}^2 = 1 \\ (iv)\ D_2(u) \leq (1+\varepsilon)V_3(u) \end{array} \right\}. \tag{4.1.9}$$

Let us now consider $u \in \tilde{\mathcal{A}}_{3,2,\theta,\varepsilon}$ and set $w := u - \mathrm{id}|_{\mathbb{S}^2}$, as if at first place the *optimal candidate* for being the closest Möbius transformation to u in terms of the *combined conformal-isoperimetric deficit* is really the identity map on $\mathbb{S}^2$. We can then perform a formal *Taylor expansion of the deficit around the identity* and calculate its *second variation*, i.e. the *quadratic term* appearing in the expansion, as well as the *growth behaviour of the higher order terms.* This is a standard computation that we exhibit here in our particular case $n=3$, and in Appendix C in the higher dimensional case $n\geq 4$. Using the properties of the map $u \in \tilde{\mathcal{A}}_{3,2,\theta,\varepsilon}$, we can calculate

$$D_2(u) = \left(\frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T u|^2 \ d\mathcal{H}^2\right)^{\frac{3}{2}} = \left(1 + \fint_{\mathbb{S}^2} \mathrm{div}_{\mathbb{S}^2} w \ d\mathcal{H}^2 + \frac{1}{2}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \ d\mathcal{H}^2\right)^{\frac{3}{2}}.$$

We can rewrite *property (iii)* of u in terms of w, as

$$\fint_{\mathbb{S}^2} \langle w, x\rangle \ d\mathcal{H}^2 = \fint_{\mathbb{S}^2} \langle u, x\rangle \ d\mathcal{H}^2 - 1 = 0, \tag{4.1.10}$$

or equivalently,

$$\fint_{\mathbb{S}^2} \mathrm{div}_{\mathbb{S}^2} w \ d\mathcal{H}^2 = 2\fint_{\mathbb{S}^2} \langle w, x\rangle \ d\mathcal{H}^2 = 0. \tag{4.1.11}$$

Hence,

$$D_2(u) = \left(1 + \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T w|^2\right)^{\frac{3}{2}} = 1 + \frac{3}{4}\fint_{\mathbb{S}^2} |\nabla_T w|^2 + \mathcal{O}\left(\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\right)^2\right). \tag{4.1.12}$$

Since $\frac{d^2}{dt^2}\Big|_{t=0}(1+t)^{\frac{3}{2}} = \frac{3}{4}$, we can take $\theta>0$ small enough so that by property (i) of (4.1.9), the higher-order terms in the expansion of D_2 are estimated as

$$\left|\mathcal{O}\left(\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2\right)^2\right)\right| \le \frac{1}{2}\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2\right)^2 \le \frac{\theta^2}{2}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2. \tag{4.1.13}$$

For the *signed-volume-term* $V_3(u)$ we can write its expression as a *bulk integral*, i.e.

$$\begin{aligned} V_3(u) &= \fint_{\mathbb{S}^2} \langle u, \partial_{\tau_1} u \wedge \partial_{\tau_2} u\rangle \;\, d\mathcal{H}^2 \\ &= \fint_{B^3} \det\nabla u_h(x)\, dx = \fint_{B^3} \det(I_3 + \nabla w_h(x))\, dx \\ &= 1 + \fint_{B^3} \mathrm{div} w_h dx + \frac{1}{2}\fint_{B^3} \big((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2\big)\, dx + \fint_{B^3} \det\nabla w_h \, dx. \end{aligned} \tag{4.1.14}$$

All these terms are well-known *null-Lagrangians*, and can be written back as *boundary integrals* in the following way.

$$\fint_{B^3} \mathrm{div} w_h \, dx = 3\fint_{\mathbb{S}^2} \langle w, x\rangle \, d\mathcal{H}^2 = 0, \tag{4.1.15}$$

$$\begin{aligned} \int_{B^3} (\mathrm{div} w_h)^2 &= \int_{B^3} \mathrm{div}\big((\mathrm{div} w_h) w_h\big) - \int_{B^3} \langle w_h, \nabla \mathrm{div} w_h\rangle \\ &= \int_{\mathbb{S}^2} \langle w, (\mathrm{div} w_h) x\rangle - \int_{B^3} \langle w_h, \nabla \mathrm{div} w_h\rangle \\ &= \int_{\mathbb{S}^2} \langle w, (\mathrm{div}_{\mathbb{S}^2} w) x\rangle + \int_{\mathbb{S}^2} \langle w, x\rangle\langle(\nabla w_h)x, x\rangle - \int_{B^3} \langle w_h, \nabla \mathrm{div} w_h\rangle, \end{aligned}$$

and

$$\begin{aligned} \int_{B^3} \mathrm{Tr}(\nabla w_h)^2 &= \sum_{i,j=1}^{3} \int_{B^3} \partial_j w_h^i \partial_i w_h^j = \sum_{i,j=1}^{3} \int_{B^3} (\partial_j (w_h^i \partial_i w_h^j) - w_h^i \partial_i \partial_j w_h^j) \\ &= \sum_{j=1}^{3} \int_{B^3} \partial_j \big(\langle w_h, \nabla w_h^j\rangle\big) - \sum_{i=1}^{3} \int_{B^3} w_h^i \partial_i (\mathrm{div} w_h) \\ &= \sum_{j=1}^{3} \int_{\mathbb{S}^2} \langle w, x_j \nabla w_h^j\rangle - \int_{B^3} \langle w_h, \nabla \mathrm{div} w_h\rangle \\ &= \int_{\mathbb{S}^2} \Big\langle w, \sum_{j=1}^{3} x_j \nabla_T w^j \Big\rangle + \int_{\mathbb{S}^2} \langle w, x\rangle\langle(\nabla w_h)x, x\rangle - \int_{B^3} \langle w_h, \nabla \mathrm{div} w_h\rangle, \end{aligned}$$

and subtracting these two identities we arrive at

$$\frac{1}{2}\fint_{B^3} \Big((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2\Big)\, dx = \frac{3}{2}\fint_{\mathbb{S}^2} \Big\langle w, (\mathrm{div}_{\mathbb{S}^2} w) x - \sum_{j=1}^{3} x_j \nabla_T w^j \Big\rangle \, d\mathcal{H}^2. \tag{4.1.16}$$

Of course this calculation can be carried out in every dimension. A more intrinsic calculation for this quadratic form without passing through the expression as a null-Lagrangian, is included in Appendix C. The *higher order term* in the expansion in this case is of course

$$\fint_{B^3} \det\nabla w_h \, dx = \fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \, d\mathcal{H}^2. \tag{4.1.17}$$

Going back to (4.1.14), we obtain

$$V_3(u) = 1 + Q_{V_3}(w) + \fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \, d\mathcal{H}^2, \tag{4.1.18}$$

where the *quadratic term in the expansion of* V_3 is

$$Q_{V_3}(w) := \frac{3}{2} \fint_{\mathbb{S}^2} \left\langle w, (\mathrm{div}_{\mathbb{S}^2} w)x - \sum_{j=1}^{3} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^2. \tag{4.1.19}$$

As far as the remainder term is concerned, if we had assumed a uniform Lipschitz bound on u, we could estimate its *growth* via the *Sobolev inequality on* $\mathbb{S}^{n-1}$. Since we will need this observation in Subsection 4.2.1., we first recall here the inequality in every dimension $n \geq 3$ and refer the reader to [Bec93], [DEKL14] for its proof and more details.

Theorem 4.1.4. ***(Sobolev inequality on*** $\mathbb{S}^{n-1}$***)*** *Let* $n \geq 3$, $m \geq 1$. *For every* $v \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^m)$ *the following interpolation inequality holds.*

$$\left(\fint_{\mathbb{S}^{n-1}} \left| v - \fint_{\mathbb{S}^{n-1}} v \right|^p \, d\mathcal{H}^{n-1} \right)^{\frac{2}{p}} \leq c_{n,p} \fint_{\mathbb{S}^{n-1}} |\nabla_T v|^2 \, d\mathcal{H}^{n-1} \tag{4.1.20}$$

for every $p \in [2, \infty)$ *in the case* $n = 3$ *and for every* $p \in [2, 2^*]$, *where* $2^* := \frac{2n-2}{n-3}$, *in the case* $n \geq 4$. *The constant* $c_{n,p} > 0$ *depends only on* n *and* p *and its sharp value is also known.*

Having reduced in our case of interest to $\fint_{\mathbb{S}^2} w \, d\mathcal{H}^2 = 0$, we have that for every $p \geq 2$ there exists a constant $c_p > 0$ such that

$$\left(\fint_{\mathbb{S}^2} |w|^p \, d\mathcal{H}^2 \right)^{\frac{2}{p}} \leq c_p \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2. \tag{4.1.21}$$

However, in this case that $n = 3$, we can merely use the isoperimetric inequality to prove

Lemma 4.1.5. *For every* $u \in \tilde{\mathcal{A}}_{3,2,\theta,\varepsilon}$ *the map* $w := u - \mathrm{id}_{\mathbb{S}^{n-1}}$ *satisfies the estimate*

$$\left| \fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \, d\mathcal{H}^2 \right| \leq \left(\frac{1}{2} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2 \right)^{\frac{3}{2}}, \tag{4.1.22}$$

Therefore, one further has

$$\left| \fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w \rangle \, d\mathcal{H}^2 \right| \leq \frac{\theta}{2\sqrt{2}} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2. \tag{4.1.23}$$

Proof. The first inequality follows from (1.2.6) for $n = 3$. The second then follows immediately using the first defining property of the set $\tilde{\mathcal{A}}_{3,2,\theta,\varepsilon}$, since

$$\|\nabla_T w\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} \leq \|\nabla_T u - P_T\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} \leq \theta. \qquad \square$$

The *quadratic term* $Q_{V_3}(w)$ can also be easily estimated by the *Dirichlet energy*, as

$$\begin{aligned}
|Q_{V_3}(w)| &\leq \frac{3}{2} \fint_{\mathbb{S}^2} |w| \left| (\mathrm{div}_{\mathbb{S}^2} w) x + \sum_{j=1}^{3} x_j \nabla_T w^j \right| d\mathcal{H}^2 \\
&\leq \frac{3}{2} \left(\fint_{\mathbb{S}^2} |w|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^2} |(\mathrm{div}_{\mathbb{S}^2} w) x|^2 + \left| \sum_{j=1}^{3} x_j \nabla_T w^j \right|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \\
&\leq \frac{3}{2} \left(\fint_{\mathbb{S}^2} \frac{|\nabla_T w|^2}{2} \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^2} |\nabla_T w : P_T|^2 + \left(\sum_{j=1}^{3} x_j^2 \right) \left(\sum_{j=1}^{3} |\nabla_T w^j|^2 \right) \, d\mathcal{H}^2 \right)^{\frac{1}{2}} \\
&\leq \frac{3\sqrt{3}}{2\sqrt{2}} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2. \qquad (4.1.24)
\end{aligned}$$

Notice that it suffices to prove Theorem 4.1.2. in the *small conformal-isoperimetric deficit regime*, i.e. we can assume without loss of generality that actually $0 \leq \varepsilon \leq \theta_0^2 \ll 1$, where $\theta_0 > 0$ is the constant that will be finally chosen in the statement of the Theorem, since if $\varepsilon \geq \theta_0^2 > 0$ (4.1.3) holds trivially for $\phi_u = \mathrm{id}_{\mathbb{S}^2}$, $b_u = \fint_{\mathbb{S}^2} u$ and $\lambda_u = 1$. Indeed, since $\fint_{\mathbb{S}^2} \mathrm{id}_{\mathbb{S}^2} = 0$, by the Poincare inequality we trivially obtain

$$\|(u - \overline{u}) - \mathrm{id}_{\mathbb{S}^2}\|_{W^{1,2}} = \|(u - \mathrm{id}_{\mathbb{S}^2}) - \overline{(u - \mathrm{id}_{\mathbb{S}^2})}\|_{W^{1,2}} \lesssim \|\nabla_T u - P_T\|_{L^2} \leq \theta_0 \leq \sqrt{\varepsilon}.$$

Assuming in the next that $0 < \varepsilon \leq \theta_0^2 \ll 1$, we can now use the expansions (4.1.12) and (4.1.18) in the *combined conformal-isoperimetric deficit* $\varepsilon > 0$ to prove

Lemma 4.1.6. *Let $\theta_0 > 0$ sufficiently small and $0 < \varepsilon \leq \theta_0^2$ be fixed. For every $u \in \tilde{\mathcal{A}}_{3,2,\theta_0,\varepsilon}$, the map $w := u - \mathrm{id}_{\mathbb{S}^2}$ satifies the estimate*

$$Q_3(w) \leq \varepsilon + c_{\theta_0} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2, \qquad (4.1.25)$$

where Q_3 is the quadratic form defined as

$$Q_3(w) := \frac{3}{4} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2 - Q_{V_3}(w), \qquad (4.1.26)$$

and $c_{\theta_0} := \frac{3\sqrt{3}}{2\sqrt{2}}\theta_0^2 + \frac{\theta_0}{2\sqrt{2}}(1+\theta_0^2) + \frac{\theta_0^2}{2} > 0$ *is a constant that becomes arbitrarily small as* θ_0 *becomes arbitrarily small.*

Proof. As said, by using (4.1.12) and (4.1.18) and rearranging terms we have

$$
\begin{aligned}
&D_2(u) \leq (1+\varepsilon)V_3(u) \\
\iff\ & 1 + \frac{3}{4}\fint_{\mathbb{S}^2} |\nabla_T w|^2 + \mathcal{O}\left(\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2\right)^2\right) \leq (1+\varepsilon)\Big(1 + Q_{V_3}(w) + \fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w\rangle\Big) \\
\iff\ & \frac{3}{4}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2 - Q_{V_3}(w) \leq \varepsilon + \varepsilon Q_{V_3}(w) + (1+\varepsilon)\fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w\rangle \ d\mathcal{H}^2 \\
&\qquad - \mathcal{O}\left(\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2\right)^2\right) \\
&\qquad \leq \varepsilon + \varepsilon |Q_{V_3}(w)| + (1+\varepsilon)\left|\fint_{\mathbb{S}^2} \langle w, \partial_{\tau_1} w \wedge \partial_{\tau_2} w\rangle \ d\mathcal{H}^2\right| \\
&\qquad + \left|\mathcal{O}\left(\left(\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2\right)^2\right)\right|
\end{aligned}
$$

We finally use (4.1.13), (4.1.23), (4.1.24) for the remainder terms and the fact that we have assumed without loss of generality that $0 < \varepsilon \leq \theta_0^2$ to obtain (4.1.25) with the value of the constant c_{θ_0} as exhibited in the Lemma. □

If we thus choose the parameter $\theta_0 > 0$ *sufficiently small*, we see that for every $0 < \varepsilon \leq \theta_0^2$ the last term on the right hand side of (4.1.25) can be set to be an *arbitrarily small multiple of the Dirichlet energy of* w. We can therefore move our focus of attention on the *coercivity properties of the quadratic form* Q_3, which can be thought of as the *linearization of the nonlinear combined conformal-isoperimetric deficit*. This will be the content of the next Subsection. The underlying geometric idea behind the arguments is the *invariance of the deficit under the action of the rotation group of the sphere*, so that actually *the self-adjoint operator associated to* Q_3 *can be diagonalized simultaneously with the Laplace-Beltrami operator*.

4.1.2 On the coercivity of the quadratic form Q_3

For the most part of this Subsection the results hold true in every dimension $n \geq 3$ and since we will use them also in Section 4.2, we also denote here the ambient dimension 3 with the general letter n to avoid the repetition of the arguments in the next Section. Our goal is to examine the *coercivity properties* (in a *purely* $W^{1,2}$*-setting*) of the quadratic form Q_n. By the reductions we have performed, this can be considered in the space

$$H_n := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w \; d\mathcal{H}^{n-1} = 0 \;,\; \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \; d\mathcal{H}^{n-1} = 0 \right\}. \quad (4.1.27)$$

For every $k \geq 1$ we define now $H_{n,k}$ to be the linear subspace of H_n consisting of those maps in H_n, all the components of which are *k-th order spherical harmonics*. As we mention in the Appendix B, it is well-known that every element of $H_{n,k}$ is the restriction on $\mathbb{S}^{n-1}$ of an ($\mathbb{R}^n$-valued) *homogeneous harmonic polynomial of degree* k, the subspaces $(H_{n,k})_{k=1}^{\infty}$ are pairwise orthogonal with respect to the L^2-inner product, each one is finite dimensional and H_n admits the L^2-orthogonal decomposition $H_n = \bigoplus_{k=1}^{\infty} H_{n,k}$. We also define the subspaces

$$\tilde{H}_{n,k} := \left\{ w_h : \overline{B^n} \mapsto \mathbb{R}^n : \begin{array}{rcl} \Delta w_h & = & 0 \text{ in } B^n \\ w_h|_{\mathbb{S}^{n-1}} \in H_{n,k} & & \end{array} \right\}, \quad (4.1.28)$$

so that $\bigoplus_{k=1}^{\infty} \tilde{H}_{n,k}$ is an L^2-orthogonal decomposition of the vector space of harmonic mappings $w_h : \overline{B^n} \mapsto \mathbb{R}^n$ for which $w_h(0) = 0$ and $\mathrm{Tr}\nabla w_h(0) = 0$.

Actually, for every $k \geq 1$ we can further consider the L^2-*orthogonal Hodge decomposition*

$$\tilde{H}_{n,k} = \tilde{H}_{n,k,\mathrm{sol}} \bigoplus \tilde{H}^{\perp}_{n,k,\mathrm{sol}}, \quad (4.1.29)$$

where

$$\tilde{H}_{n,k,\mathrm{sol}} := \left\{ w_h \in \tilde{H}_{n,k} : \mathrm{div} w_h \;\equiv\; 0 \text{ in } B^n \right\}, \quad (4.1.30)$$

and $\tilde{H}^{\perp}_{n,k,\mathrm{sol}}$ is its *orthogonal complement in* L^2. In view of the k-homogeneity of the maps in $\tilde{H}_{n,k}$, we can write the latter L^2-decomposition also on $\mathbb{S}^{n-1}$, namely

$$H_{n,k} = H_{n,k,\mathrm{sol}} \bigoplus H^{\perp}_{n,k,\mathrm{sol}}, \quad (4.1.31)$$

where

$$H_{n,k,\mathrm{sol}} := \left\{ w \in H_{n,k} : w_h \in \tilde{H}_{n,k,\mathrm{sol}} \right\}, \quad (4.1.32)$$

and $H^{\perp}_{n,k,\mathrm{sol}}$ is its $L^2(\mathbb{S}^{n-1};\mathbb{R}^n)$-orthogonal complement. Let $N(n,k) := \mathrm{dim} H_{n,k} < \infty$, $N_1(n,k) := \mathrm{dim} H_{n,k,\mathrm{sol}}$, $N_2(n,k) := \mathrm{dim} H^{\perp}_{n,k,\mathrm{sol}}$, so that $N(n,k) = N_1(n,k) + N_2(n,k)$.

According to (4.1.19), the bilinear form that corresponds to the *signed-volume-term* V_n is

$$Q_{V_n}(v,w) := \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \langle v, A(w) \rangle \; d\mathcal{H}^{n-1} \quad \text{for } v, w \in H_n,$$

where the associated *linear first order differential operator* A is defined as

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j \quad \text{for } w \in H_n. \tag{4.1.33}$$

The main feature that we are going to use here is the *fine interplay between the operator* A and the above defined spaces, as it is properly described in the following.

Lemma 4.1.7. *The linear operator A is self-adjoint with respect to the L^2-inner product and leaves each one of the subspaces $(H_{n,k})_{k\geq 1}$ invariant. Even more specifically, for every $k \geq 1$ A is a linear self-adjoint isomorphism of the spaces $H_{n,k,\mathrm{sol}}$ and $H^{\perp}_{n,k,\mathrm{sol}}$ with respect to the L^2-inner product.*

Proof. The fact that A is self-adjoint with respect to the L^2-inner product is immediate, since it arises from the second variation of V_n at the $\mathrm{id}_{\mathbb{S}^{n-1}}$, but it is also easy to verify directly after integrating by parts that for any $w, v \in H_n$

$$\fint_{\mathbb{S}^{n-1}} \langle w, A(v)\rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \langle A(w), v\rangle \, d\mathcal{H}^{n-1}.$$

Let now $k \geq 1$ be fixed. Since for every w in the finite dimensional space $H_{n,k}$, its harmonic extension w_h in $\overline{B^n}$ is a (vector-valued) homogeneous harmonic polynomial of degree k, all its derivatives will be polynomials again and hence analytic up to the boundary. In particular, for every $j = 1, 2, \ldots, n$ we have

$$\nabla w_h^j \;=\; \nabla_T w^j + (\partial_{\vec{\nu}} w_h^j)x = \; \nabla_T w^j + k w^j x \;\text{ on }\; \mathbb{S}^{n-1}$$

and

$$\mathrm{div} w_h \;=\; \mathrm{div}_{\mathbb{S}^{n-1}} w + \langle \partial_{\vec{\nu}} w_h, x\rangle \;= \mathrm{div}_{\mathbb{S}^{n-1}} w + k\langle w, x\rangle \quad \text{on } \;\mathbb{S}^{n-1}.$$

Therefore,

$$\begin{aligned} A(w) &= (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j = (\mathrm{div} w_h - k\langle w, x\rangle)x - \sum_{j=1}^{n} x_j(\nabla w_h^j - k w^j x) \\ &= (\mathrm{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j \;\text{ on }\; \mathbb{S}^{n-1}. \end{aligned}$$

Writing the operator A in terms of the full gradient and divergence operators, we easily see that

$$\begin{aligned} \fint_{\mathbb{S}^{n-1}} A(w)\, d\mathcal{H}^{n-1} &= \fint_{\mathbb{S}^{n-1}} (\mathrm{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j \, d\mathcal{H}^{n-1} = \frac{1}{n}\fint_{B^n} \left(\nabla \mathrm{div} w_h - \sum_{j=1}^{n} \partial_j \nabla w_h^j\right) \\ &= \fint_{B^n} (\nabla \mathrm{div} w_h - \nabla \mathrm{div} w_h) \; dx = 0 \end{aligned}$$

and also

$$\fint_{\mathbb{S}^{n-1}} \langle A(w), x\rangle \; d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left\langle (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j, x \right\rangle \; d\mathcal{H}^{n-1}$$
$$= \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \; d\mathcal{H}^{n-1} = (n-1) \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \; d\mathcal{H}^{n-1} = 0.$$

It is straightforward to verify that $[A(w)]_h = (\mathrm{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j$ in B^n, since first of all it matches the boundary condition and also

$$\begin{aligned}
\Delta\left[(\mathrm{div} w_h)x - \sum_{j=1}^{n} x_j \nabla w_h^j\right] &= [\Delta(\mathrm{div} w_h)]x + 2\nabla(\mathrm{div} w_h) - \sum_{j=1}^{n} \left(x_j \Delta(\nabla w_h^j) + 2\partial_j \nabla w_h^j\right) \\
&= [\mathrm{div}(\Delta w_h)]x + 2\nabla(\mathrm{div} w_h) - \sum_{j=1}^{n} x_j \nabla\Delta w_h^j - 2\nabla(\mathrm{div} w_h) \\
&= 0.
\end{aligned}$$

It is also clear that since w_h is an $\mathbb{R}^n$-valued k-homogeneous harmonic polynomial, $[A(w)]_h$ is also an $\mathbb{R}^n$-valued k-homogeneous harmonic polynomial and therefore its restriction on $\mathbb{S}^{n-1}$ is an $\mathbb{R}^n$-valued k-th order spherical harmonic. This finishes the verification of the implication $w \in H_{n,k} \implies A(w) \in H_{n,k}$.

In order to check that $\ker A = \{0\}$ in $H_{n,k,\mathrm{sol}}$, let $w \in H_{n,k,\mathrm{sol}}$ be such that

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j = 0 \quad \text{on} \quad \mathbb{S}^{n-1}.$$

Since $w \in H_{n,k,\mathrm{sol}}$ is smooth, $A(w)$ is smooth as well, so that the last equation can also be understood in the strong sense. Thus, both the *normal* and the *tangential* part of $A(w)$ have to vanish identically, i.e.

$$\mathrm{div}_{\mathbb{S}^{n-1}} w = 0 \quad \text{and} \quad \sum_{j=1}^{n} x_j \nabla_T w_j = 0 \text{ on } \mathbb{S}^{n-1}.$$

By the first of these last two equations and the definition of $H_{n,k,\mathrm{sol}}$ (in which $\mathrm{div} w_h \equiv 0$), we deduce that for such a w,

$$\langle w, x\rangle = \frac{1}{k}(\mathrm{div} w_h - \mathrm{div}_{\mathbb{S}^{n-1}} w) = 0 \quad \text{on} \quad \mathbb{S}^{n-1}.$$

Testing now the second one of the previous equations with the vector field w itself and integrating by parts on $\mathbb{S}^{n-1}$, we obtain

$$
\begin{aligned}
0 \;&=\; -\fint_{\mathbb{S}^{n-1}} \Big\langle \sum_{j=1}^{n} x_j \nabla_T w^j, w \Big\rangle \, d\mathcal{H}^{n-1} = -\sum_{j=1}^{n} \fint_{\mathbb{S}^{n-1}} \Big\langle \nabla_T w^j, x_j w^T \Big\rangle \, d\mathcal{H}^{n-1} \\
&= \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \big(\mathrm{div}_{\mathbb{S}^{n-1}} w - \mathrm{div}_{\mathbb{S}^{n-1}}(\langle w, x\rangle x)\big) + \sum_{j=1}^{n} \fint_{\mathbb{S}^{n-1}} w^j \langle e_j - x_j x, w\rangle \\
&= -\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \Big((n-1)\langle w, x\rangle + \langle \nabla_T \langle w, x\rangle, x\rangle\Big) + \fint_{\mathbb{S}^{n-1}} |w|^2 - \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle^2 \\
&= \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} - n \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle^2 d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1},
\end{aligned}
$$

i.e. $w \equiv 0$ on $\mathbb{S}^{n-1}$.

We finally check that A leaves $H_{n,k,\mathrm{sol}}$ invariant. Indeed, if $w \in H_{n,k,\mathrm{sol}}$ then

$$
\begin{aligned}
\mathrm{div}[A(w)]_h = \mathrm{div}\left((\mathrm{div} w_h) x - \sum_{j=1}^{n} x_j \nabla w_h^j\right) &= -\mathrm{div}\left(\sum_{j=1}^{n} x_j \nabla w_h^j\right) \\
&= -\mathrm{div} w_h - \langle x, \Delta w_h\rangle = 0 \quad \text{in} \quad B^n,
\end{aligned}
$$

i.e. $A(w) \in H_{n,k,\mathrm{sol}}$ as well. This concludes the proof that A is a self-adjoint linear isomorphism of $H_{n,k,\mathrm{sol}}$. Thus A leaves also $H^{\perp}_{n,k,\mathrm{sol}}$ invariant and is actually also an isomorphism of it as we will show next. $\square$

As a consequence of the previous Lemma each one of the finite-dimensional subspaces $(H_{n,k,\mathrm{sol}})_{k\geq 1}$ and $(H^{\perp}_{n,k,\mathrm{sol}})_{k\geq 1}$ admit *an eigenvalue decomposition with respect to A.*

Theorem 4.1.8. *The following statements are true.*

(i) For every $k \geq 1$, the subspace $H_{n,k,\mathrm{sol}}$ has an eigenvalue decomposition with respect to A as

$$
H_{n,k,\mathrm{sol}} = H_{n,k,1} \bigoplus H_{n,k,2},
$$

where $H_{n,k,1}$ is the eigenspace of A corresponding to the eigenvalue $\sigma_{n,k,1} := -k$ and $H_{n,k,2}$ is the one corresponding to the eigenvalue $\sigma_{n,k,2} := 1$.

(ii) For every $k \geq 1$ the subspace $H_{n,k,3} := H^{\perp}_{n,k,\mathrm{sol}}$ is an eigenspace with respect to A corresponding to the eigenvalue $\sigma_{n,k,3} := k + n - 2$.

Proof. As we just have remarked, for every $k \geq 1$ there exists an L^2-orthonormal basis of eigenfunctions $w_{n,k,1}, ..., w_{n,k,N_1(n,k)}$ of $H_{n,k,\mathrm{sol}}$ and $w_{n,k,N_1(n,k)+1}, \ldots, w_{n,k,N(n,k)}$

of $H^{\perp}_{n,k,\mathrm{sol}}$, i.e. for every $i = 1, \ldots, N(n,k)$ the $\mathbb{R}^n$-valued map $w_{n,k,i}$ satisfies the eigenvalue equation

$$A(w_{n,k,i}) := (\mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,i})x - \sum_{j=1}^{n} x_j \nabla_T w^j_{n,k,i} = \sigma_{n,k,i} w_{n,k,i} \quad \text{on} \quad \mathbb{S}^{n-1}. \tag{4.1.34}$$

For each such eigenvalue $\sigma_{n,k,i}$ we denote its corresponding eigenspace by $H_{n,k,i}$. If in the previous *eigenvalue equation* we take the inner product with the unit normal vector field on $\mathbb{S}^{n-1}$, we obtain further that each eigenfunction $w_{n,k,i}$ satisfies the equation

$$\mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,i} = \sigma_{n,k,i} \langle w_{n,k,i}, x\rangle \quad \text{on} \quad \mathbb{S}^{n-1} \tag{4.1.35}$$

and in terms of the full-divergence

$$\mathrm{div}(w_{n,k,i})_h \;=\; \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,i} + \langle \partial_{\vec{\nu}} (w_{n,k,i})_h, x\rangle \;=\; (\sigma_{n,k,i} + k)\, \langle w_{n,k,i}, x\rangle \quad \text{on} \quad \mathbb{S}^{n-1}. \tag{4.1.36}$$

We now fix the index $k \geq 1$ and consider all the different possible cases that will allow us to find the eigenvalues of A in the invariant subspaces $H_{n,k,\mathrm{sol}}$ and $H^{\perp}_{n,k,\mathrm{sol}}$ respectively.

(a_1) Let w be a non-trivial eigenfunction of A in $H_{n,k,\mathrm{sol}}$, so $\mathrm{div} w_h \equiv 0$ in $\overline{B^n}$. By the $(k-1)$-homogeneity of the function $\mathrm{div} w_h$, this is equivalent to $\mathrm{div} w_h \equiv 0$ on $\mathbb{S}^{n-1}$. By (4.1.36) we see that one possibility for the last equation to hold is for $\sigma = -k$. We thus set $\sigma_{n,k,1} := -k$ and label $H_{n,k,1} := \mathrm{span}\{w_{n,k,1}, ..., w_{n,k,p_{n,k}}\}$ its corresponding eigenspace, where $p_{n,k} := \dim H_{n,k,1}$.

(a_2) Let now w be a non-trivial eigenfunction of A in $H_{n,k,\mathrm{sol}}$, with $w \in H^{\perp}_{n,k,1}$. The only possibility for (4.1.36) to hold then is iff

$$\langle w, x\rangle \equiv 0 \quad \text{on} \quad \mathbb{S}^{n-1}.$$

In this case w is a tangential vector field and by (4.1.35) we have $\mathrm{div}_{\mathbb{S}^{n-1}} w \equiv 0$ on $\mathbb{S}^{n-1}$ as well. The eigenvalue equation (4.1.34) reduces then to

$$\sigma w = -\sum_{j=1}^{n} x_j \nabla_T w^j \quad \text{on} \quad \mathbb{S}^{n-1}.$$

With the very same calculation that we performed in the proof of the previous Lemma, we test this last equation with w and integrate by parts to obtain

$$\begin{aligned} \sigma \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} &= -\fint_{\mathbb{S}^{n-1}} \left\langle \sum_{j=1}^{n} x_j \nabla_T w^j, w \right\rangle \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1} - n \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle^2 \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} |w|^2 \, d\mathcal{H}^{n-1}. \end{aligned}$$

We label this eigenvalue $\sigma_{n,k,2} := 1$ and $H_{n,k,2} := \text{span}\{w_{n,k,p_{n,k}+1}, ..., w_{n,k,N_1(n,k)}\}$ will be its corresponding eigenspace. We have thus obtained the full eigenvalue decomposition $H_{n,k,\text{sol}} = H_{n,k,1} \bigoplus H_{n,k,2}$.

(*b*) Let us now look at eigenfunctions of A in the subspace $H^{\perp}_{n,k,\text{sol}}$, in which the divergence of $w_h \in \tilde{H}_{n,k}$ does not vanish identically in $\overline{B^n}$. Since w_h is a vector-valued k-homogeneous harmonic polynomial, we have that $\text{div}w_h$ is a scalar $(k-1)$-homogeneous harmonic polynomial and therefore its restriction on $\mathbb{S}^{n-1}$ is a scalar $(k-1)$-spherical harmonic. We can then apply the Laplace-Beltrami operator on both sides of (4.1.36), to obtain

$$\begin{aligned}(k-1)(k+n-3)\text{div}w_h &= -\Delta_{\mathbb{S}^{n-1}}(\text{div}w_h) = -(\sigma+k)\Delta_{\mathbb{S}^{n-1}}\big(\langle w, x\rangle\big) \\ &= (\sigma+k)\Big(\langle -\Delta_{\mathbb{S}^{n-1}}w, x\rangle - 2\nabla_T w : P_T + \langle w, -\Delta_{\mathbb{S}^{n-1}}x\rangle\Big) \\ &= \Big(k(k+n-2) - 2\sigma + (n-1)\Big)(\sigma+k)\langle w, x\rangle \\ &= \Big(k(k+n-2) - 2\sigma + (n-1)\Big)\text{div}w_h \quad \text{on} \quad \mathbb{S}^{n-1}.\end{aligned}$$

Since in this case $\text{div}w_h$ does not vanish identically, we conclude that

$$k(k+n-2) - 2\sigma + (n-1) = (k-1)(k+n-3) \implies \sigma = k+n-2.$$

We label this eigenvalue as $\sigma_{n,k,3} := k+n-2$ and its corresponding eigenspace as $H_{n,k,3}$. In particular we have found that $H^{\perp}_{n,k,\text{sol}} = H_{n,k,3}$. □

We have obtained in total the L^2-orthogonal decomposition of our space of interest into eigenspaces of A as

$$H_n := \bigoplus_{k=1}^{\infty}\Big(H_{n,k,1} \bigoplus H_{n,k,2} \bigoplus H_{n,k,3}\Big). \tag{4.1.37}$$

It is easy to construct examples showing that except for $H_{n,1,3}$, none of these eigenspaces are apriori trivial. The triviality of $H_{n,1,3}$ is a consequence of the fact that we have already *scaled properly* our initial maps u, so that the corresponding maps w satisfy the condition $⨍_{\mathbb{S}^{n-1}}\langle w, x\rangle \, d\mathcal{H}^{n-1} = 0$. Indeed, let $w(x) := Bx \in H_{n,1,3}$ for some $B \in \mathbb{R}^{n\times n}$. By assumption,

$$0 = ⨍_{\mathbb{S}^{n-1}}\langle w, x\rangle \, d\mathcal{H}^{n-1} = ⨍_{\mathbb{S}^{n-1}}\langle Bx, x\rangle \, d\mathcal{H}^{n-1} = \frac{1}{n}\text{Tr}B$$

Therefore, $\text{div}w_h \equiv \text{Tr}B \equiv 0$ in $\overline{B^n}$, i.e. $w \in H_{n,1,\text{sol}} = H^{\perp}_{n,1,3}$, which forces $w \equiv 0$.

This eigenvalue decomposition of the space H_n into eigenspaces of A is valid in every dimension $n \geq 3$. In dimension $n = 3$ it immediately gives the desired coercivity estimate for the quadratic form Q_3 of Lemma 4.1.6. *with optimal constant.* In the higher-dimensional case, the quadratic form associated with the combined conformal-isoperimetric deficit has an extra term and the study of its coercivity properties via the previous eigenvalue decomposition is slightly more complicated. Since this is going to be the content of Subsection 4.2.2, *for the rest of this Subsection we switch back to denoting the ambient dimension by the number* 3. As a consequence of Theorem 4.1.8., we have

Lemma 4.1.9. *(i) The quadratic forms Q_{V_3} and Q_3 diagonalize on each one of the subspaces $(H_{3,k,i})_{k\geq 1,\ i=1,2,3}$, i.e. there exist constants $(c_{3,k,i})_{i=1,2,3}$ and $(C_{3,k,i})_{i=1,2,3}$, so that for every $w \in H_{3,k,i}$*

$$Q_{V_3}(w) \;=\; c_{3,k,i}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2 \quad \text{and} \quad Q_3(w) = C_{3,k,i}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2. \tag{4.1.38}$$

(ii) For every $k, l \geq 1$ and $i, j = 1, 2, 3$ with $(k,i) \neq (l,j)$, the subspaces $H_{3,k,i}$ and $H_{3,l,j}$ are also Q_{V_3}- and Q_3-orthogonal, i.e. for every $w_{k,i} \in H_{3,k,i}$ and $w_{l,j} \in H_{3,l,j}$

$$Q_{V_3}(w_{k,i}, w_{l,j}) = 0 \quad \text{and} \quad Q_3(w_{k,i}, w_{l,j}) = 0. \tag{4.1.39}$$

Proof. The proof is immediate. For part (i) of the Lemma, by (B.0.5) we know that

$$\fint_{\mathbb{S}^2} |w|^2 \, d\mathcal{H}^2 = \frac{1}{\lambda_{3,k}} \fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2 \quad \text{for all } w \in H_{3,k}, \text{ with } \lambda_{3,k} := k(k+1).$$

For every $i = 1, 2, 3$, if $w \in H_{3,k,i}$ we have

$$Q_{V_3}(w) = \frac{3}{2}\fint_{\mathbb{S}^2} \langle w, A(w) \rangle \, d\mathcal{H}^2 = \frac{3\sigma_{3,k,i}}{2}\fint_{\mathbb{S}^2} |w|^2 \, d\mathcal{H}^2 = \frac{3\sigma_{3,k,i}}{2\lambda_{3,k}}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2,$$

which is precisely (4.1.38) for $c_{3,k,i} := \frac{3\sigma_{3,k,i}}{2\lambda_{3,k}}$ and then

$$Q_3(w) = C_{3,k,i}\fint_{\mathbb{S}^2} |\nabla_T w|^2 \, d\mathcal{H}^2,$$

where $C_{3,k,i} := \frac{3}{4} - c_{3,k,i}$. We list below the precise values of the constants, which are important in this case since we will need to *sum up* the identities in order to obtain an estimate on the full space H_3.

$$c_{3,k,1} = \frac{-3}{2(k+1)}, \quad c_{3,k,2} = \frac{3}{2k(k+1)}, \quad c_{3,k,3} = \frac{3}{2k} \tag{4.1.40}$$

and

$$C_{3,k,1} = \frac{3(k+3)}{4(k+1)}, \quad C_{3,k,2} = \frac{3(k-1)(k+2)}{4k(k+1)}, \quad C_{3,k,3} = \frac{3(k-2)}{4k}. \tag{4.1.41}$$

For part (ii) of the Lemma, the proof is a trivial calculation. Using that the subspaces $(H_{3,k,i})_{k\geq 1, i=1,2,3}$ are mutually orthogonal in $L^2(\mathbb{S}^2;\mathbb{R}^3)$, for any $(k,i),(l,j)\in\mathbb{N}^*\times\{1,2,3\}$ with $(k,i)\neq(l,j)$ and any $w_{k,i}\in H_{3,k,i}$, $w_{l,j}\in H_{3,l,j}$ we immediately have

$$Q_{V_3}(w_{k,i},w_{l,j})=\frac{3\sigma_{3,l,j}}{2}\fint_{\mathbb{S}^2}\langle w_{k,i},w_{l,j}\rangle\;d\mathcal{H}^2=\frac{3\sigma_{3,l,j}}{2}\delta^{kl}\delta^{ij}=0$$

and of course also

$$\fint_{\mathbb{S}^2}\langle\nabla_T w_{k,i},\nabla_T w_{l,j}\rangle\;d\mathcal{H}^2=\lambda_{3,k}\fint_{\mathbb{S}^2}\langle w_{k,i},w_{l,j}\rangle\;d\mathcal{H}^2=\lambda_{3,k}\delta^{kl}\delta^{ij}=0.\quad\square$$

Since $H_{3,1,3}=\{0\}$ and having the precise values of the constants $(C_{3,k,i})_{k\geq 1,i=1,2,3}$, we see that $C_{3,1,2}=C_{3,2,3}=0$, but otherwise it is easy to verify that

$$\tilde{C}:=\min_{\substack{k\geq 1,\ i\in\{1,2,3\}\\ (k,i)\neq(1,2),(1,3),(2,3)}} C_{3,k,i}=C_{3,3,3}=\frac{1}{4}.\tag{4.1.42}$$

Lemma 4.1.9. gives now the desired *coercivity estimate for the quadratic form* Q_3 on the space H_3 with *the sharp constant*. Indeed, for any $w\in H_3$ we write it as a *Fourier series* in terms of the previous *eigenspace decomposition* as

$$w=\sum_{k=1}^{\infty}\sum_{i=1,2,3}w_{3,k,i},$$

where $w_{3,k,i}\in H_{3,k,i}$ for every $k\geq 1$, $i=1,2,3$ (and as we have justified $w_{3,1,3}=0$). Expanding the quadratic form, we obtain

$$\begin{aligned}
Q_3(w)&=\sum_{(k,i),(l,j)\ \in\ \mathbb{N}^*\times\{1,2,3\}}Q_3(w_{3,k,i},w_{3,l,j})\\
&=\sum_{(k,i)\ \in\ \mathbb{N}^*\times\{1,2,3\}}Q_3(w_{3,k,i})+\sum_{(k,i)\neq(l,j)\ \in\ \mathbb{N}^*\times\{1,2,3\}}Q_3(w_{3,k,i},w_{3,l,j})\\
&=\sum_{(k,i)\ \neq\ (1,2),(1,3),(2,3)}C_{3,k,i}\fint_{\mathbb{S}^2}|\nabla_T w_{3,k,i}|^2\;d\mathcal{H}^2\\
&\geq\frac{1}{4}\sum_{(k,i)\ \neq\ (1,2),(1,3),(2,3)}\fint_{\mathbb{S}^2}|\nabla_T w_{3,k,i}|^2\;d\mathcal{H}^2\\
&=\frac{1}{4}\fint_{\mathbb{S}^2}\left|\nabla_T w-\nabla_T(w_{3,1,2}+w_{3,2,3})\right|^2\;d\mathcal{H}^2
\end{aligned}$$

To summarize, if for every $(k,i)\in\mathbb{N}^*\times\{1,2,3\}$ we define $\Pi_{H_{n,k,i}}:H_n\mapsto H_{n,k,i}$ to be the L^2-orthogonal projection of H_n on the subspace $H_{n,k,i}$, we have finally proven the following.

Theorem 4.1.10. *For every $w \in H_3$ the following coercivity estimate holds.*

$$Q_3(w) \geq \frac{1}{4} \fint_{\mathbb{S}^2} \left|\nabla_T w - \nabla_T(\Pi_{3,0} w)\right|^2 \, d\mathcal{H}^2, \tag{4.1.43}$$

where $H_{3,0} := H_{3,1,2} \bigoplus H_{3,2,3}$ is the kernel of Q_3 in H_3 and $\Pi_{3,0} : H_3 \mapsto H_{3,0}$ is the $W^{1,2}$-orthogonal projection of H_3 onto $H_{3,0}$. The constant $\frac{1}{4}$ in the previous estimate is sharp.

4.1.3 Completion of proof of Theorem 4.1.2.

The presence of the *degenerate space* $H_{3,0}$ is a small but natural obstacle to overcome for the proof of Theorem 4.1.2. to be completed. At an *infinitesimal level*, it basically means that although the map u is apriori supposed to be θ_0-*close* to the $\mathrm{id}_{\mathbb{S}^2}$ in the $W^{1,2}$-topology, there might be another Möbius transformation of $\mathbb{S}^2$ that is also θ_0-*close* to the $\mathrm{id}_{\mathbb{S}^2}$ and is a better candidate for the *nearest Möbius map* to u in terms of its *combined conformal-isoperimetric deficit*. Similarly to [Res70] and [FZ05], a *topological argument* will allow us to identify this more suitable candidate. Before doing this, let us present a useful fact about the structure of the subspace $H_{n,0}$. *The characterizations given in the next Lemma hold true in every dimension $n \geq 3$ and this is why we denote the ambient dimension again by the general letter n in it.*

Lemma 4.1.11. *The following statements are true.*

(i) The subspace $H_{n,1,2}$ admits the following characterization

$$H_{n,1,2} = \{w \in H_n : w(x) = Ax; \text{ where } A \in Skew(n)\}, \tag{4.1.44}$$

and its dimension is $\mathrm{dim} H_{n,1,2} = \frac{n(n-1)}{2}$. The projection on this subspace is therefore characterized by

$$\Pi_{H_{n,1,2}} w = 0 \iff \nabla w_h(0) = \nabla w_h(0)^t. \tag{4.1.45}$$

(ii) The subspace $H_{n,2,\mathrm{sol}}$ admits the following characterization

$$H_{n,2,\mathrm{sol}} = \left\{ w \in H_n : \begin{array}{l} \forall\ k = 1, \ldots, n : \quad w^k(x) = \langle A^k x, x \rangle, \\ A^k \in Sym(n) : \mathrm{Tr} A^k = 0, \ \sum_{l=1}^n A^l_{lk} = 0 \end{array} \right\} \tag{4.1.46}$$

and therefore

$$\mathrm{dim} H_{n,2,3} = \mathrm{dim} H_{n,2} - \mathrm{dim} H_{n,2,\mathrm{sol}} = n.$$

The projection on the subspace $H_{n,2,3}$ is characterized by

$$\Pi_{H_{n,2,3}} w = 0 \iff \fint_{\mathbb{S}^{n-1}} (\mathrm{div} w_h(x)) x \, d\mathcal{H}^{n-1}(x) = 0. \tag{4.1.47}$$

Proof. For part (i) of the Lemma, if $w \in H_{n,1,2}$, we can write it as $w(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$. In this space,

$$\langle w, x\rangle \equiv 0 \ \text{ on } \ \mathbb{S}^{n-1} \iff \sum_{1\le i\le j\le n} (A_{ij} + A_{ji})x_i x_j \equiv 0 \ \text{ on } \ \mathbb{S}^{n-1} \iff A^t = -A.$$

The characterization of the projection $\Pi_{H_{n,1,2}}$ is then immediate. Regarding part (ii), let $w \in H_{n,2,\mathrm{sol}}$. Its harmonic extension is a homogeneous *solenoidal* harmonic polynomial of degree 2, so for each $k = 1, \dots, n$, there exist $A^k \in Sym(n)$ such that

$$w_h^k(x) = \langle A^k x, x\rangle = \sum_{i,j=1}^n A^k_{ij} x_i x_j \ \in \overline{B^n}.$$

In particular, for each $k, l = 1, \dots, n$,

$$\partial_l w_h^k(x) = 2\sum_{i=1}^n A^k_{li} x_i \implies \begin{cases} 0 = \frac{1}{2}\Delta w_h^k = \mathrm{Tr} A^k \\ 0 = \frac{1}{2}\mathrm{div} w_h = \sum_{k=1}^n \left(\sum_{l=1}^n A^l_{lk}\right) x_k \iff \sum_{l=1}^n A^l_{lk} = 0. \end{cases}$$

For the last characterization,

$$\Pi_{H_{n,2,3}} w = 0 \iff \Pi_{H_{n,2}} w \in H_{n,2,\mathrm{sol}} \iff \sum_{l=1}^n \left(\nabla^2 w_h^l(0)\right)_{lk} = 0 \text{ for every } k = 1, \dots, n.$$

By the mean value property of harmonic functions again,

$$0 = \sum_{l=1}^n \fint_{B^n} \partial_{lk} w_h^l(x)\, dx = \fint_{B^n} \partial_k(\mathrm{div} w_h(x))\, dx = n\fint_{\mathbb{S}^{n-1}} (\mathrm{div} w_h(x)) x_k \, d\mathcal{H}^{n-1}(x),$$

which completes the proof. $\square$

It is worth noticing here that simply by counting dimensions,

$$\mathrm{dim} H_{n,0} = \mathrm{dim} H_{n,1,2} + \mathrm{dim} H_{n,2,3} = \frac{n(n+1)}{2},$$

which is also the dimension of $Conf(\mathbb{S}^{n-1})$, the latter seen as a *finite-dimensional Lie group*. The next Lemma in which we still denote the ambient dimension 3 by the general letter n, follows from a suitable application of the *Inverse Function Theorem* and is the final ingredient for the completion of the proof of Theorem 4.1.2.. Note that in the statement we still denote $H_{n,0} := H_{n,1,2} \bigoplus H_{n,2,3}$ the linear subspace of $W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ that enjoys the characterizing properties described in Lemma 4.1.11., and in these subspaces the defining properties of H_n, i.e. $\fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0$ and $\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} = 0$ still hold. In any case, these two properties will in the end be fixed by making use of Lemma 4.1.3..

Lemma 4.1.12. *Let $\theta > 0$ sufficiently small and $\varepsilon > 0$ be given. There exist $\tilde{\theta} > 0$ depending on θ with the following property. For every $u \in \tilde{\mathcal{A}}_{n,2,\theta,\varepsilon}$ there exists $\phi_u \in Conf_+(\mathbb{S}^{n-1})$ such that*

$$u \circ \phi_u \in \mathcal{A}_{n,2,\tilde{\theta},\varepsilon} \quad \text{and} \quad \Pi_{H_{n,0}}(u \circ \phi_u) = 0. \tag{4.1.48}$$

Proof. Given $u \in \tilde{\mathcal{A}}_{n,2,\theta,\varepsilon}$, we define the map $\Psi_u : Conf_+(\mathbb{S}^{n-1}) \mapsto \mathbb{R}^{\frac{n(n+1)}{2}}$ as follows. For every $\phi \in Conf_+(\mathbb{S}^{n-1})$ let,

$$\begin{aligned}
\Psi_u(\phi) : &= \left(\fint_{\mathbb{S}^{n-1}} \big(\mathrm{div}(u\circ\phi)_h(x)\big)x, \frac{1}{n}\Big(\partial_j(u\circ\phi)^i_h(0) - \partial_i(u\circ\phi)^j_h(0)\Big)_{1\le i<j\le n}\right) \\
&= \left(\fint_{\mathbb{S}^{n-1}} \Big(\mathrm{div}(u\circ\phi)_h(x)\Big)x, \left(\fint_{\mathbb{S}^{n-1}} \big((u\circ\phi)^i x_j - (u\circ\phi)^j x_i\big)\right)_{1\le i<j\le n}\right).
\end{aligned}$$

Our goal is to show that $0 \in \mathrm{Im}(\Psi_u)$. To simplify notation, let us also set $\Psi := \Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}$.

Clearly $\Psi(\mathrm{id}_{\mathbb{S}^{n-1}}) = 0$. In order to apply the *Inverse Function Theorem*, we look at the differential $d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}} : T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf_+(\mathbb{S}^{n-1}) \mapsto \mathbb{R}^{\frac{n(n+1)}{2}}$ and we prove that *it is a non-degenerate linear map.* Indeed, as we have seen in Remark 2.2.3. of Section 2.2

$$T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf_+(\mathbb{S}^{n-1}) \equiv \left\{ Sx + \mu\big(\langle x,\xi\rangle x - \xi\big) : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n; S \in Skew(n),\ \xi \in \mathbb{S}^{n-1},\ \mu \in \mathbb{R} \right\}.$$

The differential of Ψ at the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is also easy to compute. Indeed, by the *linearity of all the operations* involved, for every $Y \in T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf_+(\mathbb{S}^{n-1})$, defined as before via

$$Y(x) = Sx + \mu\big(\langle x,\xi\rangle x - \xi\big) : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n; \quad S^t = -S,\ \xi \in \mathbb{S}^{n-1},\ \mu \in \mathbb{R},$$

we can calculate (with a slight abuse of notation in the domain of definition of Ψ)

$$\begin{aligned}
d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}(Y) : &= \frac{d}{dt}\Big|_{t=0} \Psi\left(\exp_{\mathrm{id}_{\mathbb{S}^{n-1}}}(tY)\right) = \Psi\left(\frac{d}{dt}\Big|_{t=0} \exp_{\mathrm{id}_{\mathbb{S}^{n-1}}}(tY)\right) = \Psi(Y) \\
&= \left(\fint_{\mathbb{S}^{n-1}} \mathrm{div} Y_h(x) x \, d\mathcal{H}^{n-1}, \left(\fint_{\mathbb{S}^{n-1}} \big(Y^i(x)x_j - Y^j(x)x_i\big)\, d\mathcal{H}^{n-1}\right)_{1\le i<j\le n}\right).
\end{aligned}$$

The harmonic extension of Y in B^n is given by the vector field

$$Y_h(x) = Sx + \mu\left(\langle x,\xi\rangle x - \left(\frac{|x|^2+n-1}{n}\right)\xi\right),$$

the divergence of which is $\mathrm{div} Y_h(x) = \frac{(n+2)(n-1)}{n}\mu\langle x,\xi\rangle$. In particular,

$$\fint_{\mathbb{S}^{n-1}} \mathrm{div} Y_h(x) x \, d\mathcal{H}^{n-1} = \frac{(n+2)(n-1)}{n^2}\mu\xi,$$

while for every $1 \le i < j \le n$,

$$\fint_{\mathbb{S}^{n-1}} \left(Y^i(x)x_j - Y^j(x)x_i\right) \, d\mathcal{H}^{n-1} = \frac{2}{n} S_{ij}.$$

These short calculations show that indeed $\ker(d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}) = \{0\}$, i.e. $d\Psi|_{\mathrm{id}_{\mathbb{S}^{n-1}}}$ is a *linear isomorphism* between $T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf_+(\mathbb{S}^{n-1})$ and $\mathbb{R}^{\frac{n(n+1)}{2}}$.

Since the *exponential mapping* $\exp_{\mathrm{id}_{\mathbb{S}^{n-1}}}(\cdot)$ is a *local diffeomorphism* between a neighbourhood of 0 in $T_{\mathrm{id}_{\mathbb{S}^{n-1}}} Conf_+(\mathbb{S}^{n-1})$ and a neighbourhood of $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $Conf_+(\mathbb{S}^{n-1})$, we can use the *Inverse Function Theorem* to find a *sufficiently small open neighbourhood* $\mathcal{U}_0$ of $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $Conf_+(\mathbb{S}^{n-1})$ inside which the map

$$\Psi|_{\mathcal{U}_0} : \mathcal{U}_0 \subseteq Conf_+(\mathbb{S}^{n-1}) \mapsto \Psi(\mathcal{U}_0) \subseteq \mathbb{R}^{\frac{n(n+1)}{2}} \quad \text{is a} \quad C^1 - \text{diffeomorphism}.$$

In particular, $\deg(\Psi; 0; \mathcal{U}_0) = 1$.

As a next step, we justify that Ψ *is homotopic to* Ψ_u *in* $\mathcal{U}_0$. Indeed, for every $\phi \in \mathcal{U}_0$, we can estimate

$$\begin{aligned}
\left|\Psi_u(\phi) - \Psi(\phi)\right|^2 &= \sum_{k=1}^{n} \left(\fint_{\mathbb{S}^{n-1}} \operatorname{div}\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]_h x_k \right)^2 \\
&\quad + \sum_{1 \le i < j \le n} \left(\fint_{\mathbb{S}^{n-1}} \left(\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]^i x_j - \left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]^j x_i \right) \right)^2 \\
&\le \fint_{\mathbb{S}^{n-1}} \left(\operatorname{div}\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]_h\right)^2 + \sum_{i \ne j} \fint_{\mathbb{S}^{n-1}} \left(\left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]^i \right)^2 x_j^2 \\
&\le 2n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T \left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]\right|^2 + \fint_{\mathbb{S}^{n-1}} \left|(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right|^2.
\end{aligned}$$

In the last step we used the estimate

$$\fint_{\mathbb{S}^{n-1}} (\operatorname{div} v_h)^2 \, d\mathcal{H}^{n-1} \le n \fint_{\mathbb{S}^{n-1}} |\nabla v_h|^2 \, d\mathcal{H}^{n-1} \le 2n \fint_{\mathbb{S}^{n-1}} |\nabla_T v|^2 \, d\mathcal{H}^{n-1},$$

which follows from Cauchy-Schwartz and Remark B.0.4.. With similar estimates as the ones we have exhibited in Remark 4.1.1., part (ii), we can also estimate separately

$$\fint_{\mathbb{S}^{n-1}} \left|\nabla_T \left[(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi\right]\right|^2 d\mathcal{H}^{n-1} \le C_1(\mathcal{U}_o) \fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^2 \le C_1(\mathcal{U}_o)\theta^2, \quad (4.1.49)$$

where

$$C_1(\mathcal{U}_0) \sim_n \sup_{\phi \in \mathcal{U}_0} \inf_{x \in \mathbb{S}^{n-1}} \left|\nabla_T \phi\right|^{3-n} > 0.$$

Of course, in the case of this Subsection, i.e. for $n = 3$, the presence of this constant is obsolete since the Dirichlet energy is conformally invariant, so the first one of the last two inequalities is actually an exact equality (without the presence of a constant). Similarly,

$$\fint_{\mathbb{S}^{n-1}} |(u-\mathrm{id}_{\mathbb{S}^{n-1}})\circ\phi|^2 \ d\mathcal{H}^{n-1} \leq C_2(\mathcal{U}_0) \fint_{\mathbb{S}^{n-1}} |u-\mathrm{id}_{\mathbb{S}^{n-1}}|^2 \ d\mathcal{H}^{n-1} \leq C_2(\mathcal{U}_0)\theta^2, \quad (4.1.50)$$

where we have used the fact that $\fint_{\mathbb{S}^{n-1}}(u-\mathrm{id}_{\mathbb{S}^{n-1}})\, d\mathcal{H}^{n-1}=0$ for $u\in\tilde{\mathcal{A}}_{n,2,\theta,\varepsilon}$, the Poincare inequality and property (i) of the class $\tilde{\mathcal{A}}_{n,2,\theta,\varepsilon}$ in the last inequality above. Here,

$$C_2(\mathcal{U}_0) \sim_n \sup_{\phi\in\mathcal{U}_0} \inf_{x\in\mathbb{S}^{n-1}} \left|\nabla_T\phi\right|^{1-n} > 0.$$

The strict positivity of the constants $C_1(\mathcal{U}_0), C_2(\mathcal{U}_0)$ is ensured by the fact that we can take the neighbourhood $\mathcal{U}_0$ to be sufficiently small around the $\mathrm{id}_{\mathbb{S}^{n-1}}$. Hence,

$$\|\Psi_u-\Psi\|_{L^\infty(\mathcal{U}_0)} \leq C(\mathcal{U}_0)\theta, \quad (4.1.51)$$

where $C(\mathcal{U}_0) := \max\left\{\sqrt{2nC_1(\mathcal{U}_0)}, \sqrt{C_2(\mathcal{U}_0)}\right\} > 0$.

We can now continue as in Proposition 4.7 of [FZ05]. For the sake of making the proof self-contained we present the argument here, adapted to our setting.

Let $(\Gamma_s)_{s\in[0,1]}$ be a *foliation of* $\mathcal{U}_0$ (which can be taken for example to be a small geodesic ball around the $\mathrm{id}_{\mathbb{S}^{n-1}}$) *by closed hypersurfaces in* $Conf_+(\mathbb{S}^{n-1})$, so that $\Gamma_0=\{\mathrm{id}_{\mathbb{S}^{n-1}}\}$ and Γ_1 is the topological boundary of $\mathcal{U}_0$. Let us define

$$m(s) := \min\nolimits_{\phi\in\Gamma_s}|\Psi(\phi)| \ \ \text{for every } s\in[0,1]. \quad (4.1.52)$$

This is obviously a continuous function of s. Since $\Psi|_{\Gamma_0}\equiv 0$ and $\Psi|_{\mathcal{U}_0}$ is a homeomorphism onto its image, we infer that

$$m(s)>0 \ \ \text{for all } s\in(0,1] \ \ \text{and} \ \lim_{s\to 0^+} m(s)=0. \quad (4.1.53)$$

Notice that $m(1)>0$ depends only on the size of $\mathcal{U}_0$. We can then choose $\theta>0$ sufficiently small so that $\big(C(\mathcal{U}_0)+1\big)\theta\leq\frac{m(1)}{2}$ and then define

$$s_\theta := \inf\left\{s\in[0,1] : m(s)\geq \big(C(\mathcal{U}_0)+1\big)\theta\right\}. \quad (4.1.54)$$

Clearly, $\lim_{\theta\to 0^+} s_\theta = 0$.

Let us now consider the *linear homotopy* between Ψ and Ψ_u. For every $t\in[0,1]$ and $\phi\in\Gamma_{s_\theta}\subseteq\mathcal{U}_0\subseteq Conf_+(\mathbb{S}^{n-1})$,

$$\begin{aligned}\left|\big((1-t)\Psi+t\Psi_u\big)(\phi)\right| &\geq |\Psi(\phi)| - t|(\Psi_u-\Psi)(\phi)| \\ &\geq \ \min\nolimits_{\phi\in\Gamma_{s_\theta}}|\Psi(\phi)| - \|\Psi_u-\Psi\|_{L^\infty(\mathcal{U}_0)} \\ &\geq m_{s_\theta} - C(\mathcal{U}_0)\theta \ \geq \ \theta > 0.\end{aligned}$$

In particular,

$$\big((1-t)\Psi + t\Psi_u\big)(\phi) \neq 0 \;\text{ for every } t \in [0,1] \text{ and } \phi \in \Gamma_{s_\theta}. \tag{4.1.55}$$

Since *the degree around* 0 *remains constant through this linear homotopy*, if $\mathcal{U}_{s_\theta}$ is the open neighbourhood around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ in $Conf_+(\mathbb{S}^{n-1})$ such that $\partial\mathcal{U}_{s_\theta} = \Gamma_{s_\theta}$, then

$$\deg(\Psi_u, 0;\, \mathcal{U}_{s_\theta}) = \deg(\Psi, 0;\, \mathcal{U}_{s_\theta}) = 1. \tag{4.1.56}$$

Therefore, there exists $\phi_u \in \mathcal{U}_{s_\theta} \subseteq Conf_+(\mathbb{S}^{n-1})$ so that

$$\Psi_u(\phi_u) = 0 \iff \Pi_{H_{n,0}}(u \circ \phi_u) = 0. \tag{4.1.57}$$

In the same fashion as have estimated before,

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |u \circ \phi_u - \mathrm{id}_{\mathbb{S}^{n-1}}|^2 \; d\mathcal{H}^{n-1} &\leq \; 2\left(\fint_{\mathbb{S}^{n-1}} |(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi_u|^2 + \fint_{\mathbb{S}^{n-1}} |\phi_u - \mathrm{id}_{\mathbb{S}^{n-1}}|^2\right) \\
&\leq 2\left(C_2(\mathcal{U}_0)\theta^2 + \fint_{\mathbb{S}^{n-1}} |\phi_u - \mathrm{id}_{\mathbb{S}^{n-1}}|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |\nabla_T(u \circ \phi_u) - P_T|^2 \; d\mathcal{H}^{n-1} &\leq \; 2\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T(u - \mathrm{id}_{\mathbb{S}^{n-1}}) \circ \phi_u|^2 + \fint_{\mathbb{S}^{n-1}} |\nabla_T\phi_u - P_T|^2\right) \\
&\leq 2\Big(C_1(\mathcal{U}_0)\theta^2 + \fint_{\mathbb{S}^{n-1}} |\nabla_T\phi_u - P_T|^2\Big),
\end{aligned}$$

so that

$$\begin{aligned}
\|\nabla_T(u \circ \phi_u) - P_T\|_{L^2(\mathbb{S}^{n-1})} &\leq \sqrt{2}\Big(\sqrt{C_1(\mathcal{U}_0)}\theta + \|\phi_u - \mathrm{id}_{\mathbb{S}^{n-1}}\|_{W^{1,2}(\mathbb{S}^{n-1})}\Big) \\
&\leq \sqrt{2}\Big(\sqrt{C_1(\mathcal{U}_0)}\theta + C(s_\theta)\Big),
\end{aligned} \tag{4.1.58}$$

where

$$C(s_\theta) := \max_{\phi \in \overline{\mathcal{U}_{s_\theta}}} \|\phi_u - \mathrm{id}_{\mathbb{S}^{n-1}}\|_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{4.1.59}$$

Of course, *all topologies in the finite dimensional manifold $Conf_+(\mathbb{S}^{n-1})$ are equivalent* and since $\lim_{\theta\to 0^+} s_\theta = 0$, we also have that $\lim_{\theta\to 0^+} C(s_\theta) = 0$. Hence, we can take the neighbourhood $\mathcal{U}_0$ small enough and then $\theta > 0$ sufficiently small so that $u \circ \phi_u \in \mathcal{A}_{n,2,\tilde{\theta},\varepsilon}$, where $\tilde{\theta} := \sqrt{2}\Big(\sqrt{C_1(\mathcal{U}_0)}\theta + C(s_\theta)\Big) > 0$ is again sufficiently small (depending only on θ and the dimension). $\square$

We can now combine all the previous steps to complete the proof of our main Theorem.

Proof of Theorem 4.1.2. For $\theta_0 > 0$ that will be chosen sufficiently small in the end (and as we have assumed without loss of generality $0 < \varepsilon \le \theta_0^2$), let us consider a map $u \in \mathcal{A}_{3,2,\theta_0,\varepsilon}$. Using Lemma 4.1.12., we can find a Möbius transformation $\phi_u \in Conf_+(\mathbb{S}^2)$ such that

$$\left(u - \fint_{\mathbb{S}^2} u\right) \circ \phi_u \in \mathcal{A}_{3,2,\theta_0,\varepsilon} \text{ and } \Pi_{H_{3,0}}\left(\left(u - \fint_{\mathbb{S}^2} u\right) \circ \phi_u\right) = 0,$$

where we have abused notation by not replacing θ_0 with $\tilde{\theta}_0$, something that we can do without loss of generality as we have justified just above. Applying Lemma 4.1.3. to the map $\left(u - \fint_{\mathbb{S}^2} u\right) \circ \phi_u$, we can further find and $b_u \in \mathbb{R}^3$ and $\lambda_u > 0$, so that by abusing notation once more,

$$\tilde{u} := \frac{u \circ \phi_u - b_u}{\lambda_u} \in \tilde{\mathcal{A}}_{3,2,\theta_0,\varepsilon},$$

with $b_u := \fint_{\mathbb{S}^2} u \circ \phi_u \, d\mathcal{H}^2$, $\lambda_u := \frac{\mathrm{Tr}[\nabla(u \circ \phi_u)_h(0)]}{3}$. Setting $\tilde{w} := \tilde{u} - \mathrm{id}_{\mathbb{S}^2}$, we have as a consequence of these two lemmata that

$$\fint_{\mathbb{S}^2} \tilde{w} \, d\mathcal{H}^2 = 0, \quad \fint_{\mathbb{S}^2} \langle \tilde{w}, x\rangle \, d\mathcal{H}^2 = 0 \iff \Pi_{H_{3,1,3}}\tilde{w} = 0, \quad \text{and also } \Pi_{H_{3,0}}\tilde{w} = 0.$$

Thanks to the invariances of the combined conformal-isoperimetric deficit, the map $\tilde{u}$ still satisfies the inequality

$$D_2(\tilde{u}) \le (1 + \varepsilon)V_3(\tilde{u}).$$

Expanding the deficit around the identity, we arrive again at (4.1.25) and since we have assumed without loss of generality that $0 < \varepsilon \le \theta_0^2$,

$$Q_3(\tilde{w}) \le \varepsilon + c_{\theta_0}\fint_{\mathbb{S}^2} |\nabla_T \tilde{w}|^2 \, d\mathcal{H}^2. \tag{4.1.60}$$

Since $\tilde{w} \in H_n$ and $\Pi_{H_{3,0}}\tilde{w} = 0$, the inequality (4.1.43) gives us.

$$Q_3(\tilde{w}) \ge \frac{1}{4}\fint_{\mathbb{S}^2} |\nabla_T \tilde{w}|^2 \, d\mathcal{H}^2. \tag{4.1.61}$$

By choosing now θ_0 small enough so that

$$c_{\theta_0} \le \frac{1}{8},$$

we can combine the last two estimates to infer that

$$\fint_{\mathbb{S}^2} \left|\nabla_T \left(\frac{u \circ \phi_u - b_u}{\lambda_u}\right) - P_T\right|^2 \, d\mathcal{H}^2 = \fint_{\mathbb{S}^2} |\nabla_T \tilde{w}|^2 \, d\mathcal{H}^2 \le 8\varepsilon.$$

Finally, by the *Poincare inequality on* $\mathbb{S}^{n-1}$,

$$\left\|\left(\frac{u \circ \phi_u - b_u}{\lambda_u}\right) - \mathrm{id}_{\mathbb{S}^2}\right\|_{W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)} \le C\sqrt{\varepsilon}; \tag{4.1.62}$$

for an absolute constant $C > 0$, for example $C = 2\sqrt{3}$. $\square$

4.2 The higher dimensional case, $n \geq 4$

In this Section we would like to discuss how the results regarding the local stability of $Conf_+(\mathbb{S}^2)$ among maps from $\mathbb{S}^2$ to $\mathbb{R}^3$ that are *almost conformal* and produce *small generalized isoperimetric deficit* can be generalized in higher dimensions. As we have remarked, we are going to follow the same lines as before and therefore we will focus more on the parts that exhibit a slight difference, adjusting the setting and the remaining parts of the analysis without repeating the proofs.

4.2.1 The setup of the estimate, revisited

We first revise shortly the setup in which the local stability of $Conf_+(\mathbb{S}^{n-1})$ will be investigated. This differs from the one of Subsection 4.1.1 only in the choice of the topology in which the maps in consideration are assumed to be *apriori close to the* $\mathrm{id}_{\mathbb{S}^{n-1}}$, the reason for this being a difference in the growth behaviour of the higher than quadratic order terms in the expansion of the deficit around the $\mathrm{id}_{\mathbb{S}^{n-1}}$. For some $\theta > 0$ that will again be chosen sufficiently small eventually and $\varepsilon > 0$, the relevant class of mappings is now

$$\mathcal{A}_{n,\infty,\theta,\varepsilon} := \left\{ u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \begin{array}{l} (i)\ \|\nabla_T u - P_T\|_{L^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)} \leq \theta, \\ (ii)\ D_{n-1}(u) \leq (1+\varepsilon)V_n(u) \end{array} \right\}, \tag{4.2.1}$$

where now

$$D_{n-1}(u) := \left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}, V_n(u) := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1}. \tag{4.2.2}$$

In this higher dimensional setting, *the local stability result is similar to Theorem 4.1.2.*, namely

Theorem 4.2.1. *Let $n \geq 4$. There exists a constant $\theta_0 := \theta_0(n) > 0$ and $C := C(n) > 0$ with the following property. Given $\varepsilon > 0$, then for every $u \in \mathcal{A}_{n,\infty,\theta_0,\varepsilon}$ there exist $\phi_u \in Conf_+(\mathbb{S}^{n-1})$, $b_u \in \mathbb{R}^n$ and $\lambda_u > 0$ such that*

$$\left\| \left(\frac{u \circ \phi_u - b_u}{\lambda_u} \right) - \mathrm{id}_{\mathbb{S}^{n-1}} \right\|_{W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)} \leq C\sqrt{\varepsilon}. \tag{4.2.3}$$

The exponent $\frac{1}{2}$ in the ε-deficit is again optimal, for the same reason as in the case $n = 3$ and the proof of the theorem will follow the same steps.

First of all, since $\mathcal{A}_{n,\infty,\theta,\varepsilon} \subset \mathcal{A}_{n,2,\theta,\varepsilon}$ Lemma 4.1.3. applies also here with the same proof also for maps $u \in \mathcal{A}_{n,\infty,\theta,\varepsilon}$, just replacing the dimension 3 with the general ambient dimension n, so that again without loss of generality we can move our attention to the set of maps

$$\tilde{\mathcal{A}}_{n,\infty,\theta,\varepsilon} := \left\{ u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n) : \begin{array}{l} (i)\ \|\nabla_T u - P_T\|_{L^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)} \leq \theta \\ (ii)\ \fint_{\mathbb{S}^{n-1}} u\ d\mathcal{H}^{n-1} = 0 \\ (iii)\ \fint_{\mathbb{S}^{n-1}} \langle u, x\rangle\ d\mathcal{H}^{n-1} = 1 \\ (iv)\ D_{n-1}(u) \leq (1+\varepsilon)V_n(u) \end{array} \right\}. \tag{4.2.4}$$

We can now consider $u \in \tilde{\mathcal{A}}_{n,\infty,\theta,\varepsilon}$, set again $w := u - \mathrm{id}|_{\mathbb{S}^{n-1}}$ and then perform a *Taylor expansion of the deficit around the identity.* The computations are once again standard (we include them for convenience in Appendix C), but the outcome is slightly different than before. Actually, in this case that $n \geq 4$, we have

Lemma 4.2.2. *Let $n \geq 4$ and $\theta > 0$. There exist constants $C_1, C_2 > 0$ depending only on n and θ such that for every $u \in \tilde{\mathcal{A}}_{n,\infty,\theta,\varepsilon}$ and after setting $w := u - \mathrm{id}|_{\mathbb{S}^{n-1}}$,*

(a) The $(n-1)$-Dirichlet-energy-term $D_{n-1}(u)$ has the formal Taylor expansion

$$D_{n-1}(u) \ = \ 1 + Q_{D_{n-1}}(w) + \fint_{\mathbb{S}^{n-1}} R_{D_{n-1}}(\nabla_T w)\ d\mathcal{H}^{n-1}, \tag{4.2.5}$$

where $Q_{D_{n-1}}(w)$ is the quadratic form defined via

$$Q_{D_{n-1}}(w) := \frac{1}{2}\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right)\ d\mathcal{H}^{n-1} \tag{4.2.6}$$

and the remainder term is of growth

$$\begin{aligned} \left| \fint_{\mathbb{S}^{n-1}} R_{D_{n-1}}(\nabla_T w)\ d\mathcal{H}^{n-1} \right| &= \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3) + \mathcal{O}\left(\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \right)^2 \right) \\ \Longrightarrow \left| \fint_{\mathbb{S}^{n-1}} R_{D_{n-1}}(\nabla_T w)\ d\mathcal{H}^{n-1} \right| &\leq C_1 \|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1}. \end{aligned} \tag{4.2.7}$$

(b) The signed-volume-term $V_n(u)$ has the formal Taylor expansion

$$V_n(u) \ = \ 1 + Q_{V_n}(w) + \fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\ d\mathcal{H}^{n-1}, \tag{4.2.8}$$

where $Q_{V_n}(w)$ is the quadratic form defined via the following equivalent formulas:

$$Q_{V_n}(w) := \begin{cases} \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^n x_j \nabla_T w^j \right\rangle\ d\mathcal{H}^{n-1} \\ \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left(2\ \mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2 \right)\ d\mathcal{H}^{n-1} \\ \frac{1}{2}\fint_{B^n} \left((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2 \right) dx. \end{cases} \tag{4.2.9}$$

The remainder term has the algebraic structure

$$\fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\ d\mathcal{H}^{n-1} = \sum_{k=3}^{n} \fint_{\mathbb{S}^{n-1}} R_{V_n,k}(w, \nabla_T w)\ d\mathcal{H}^{n-1},$$

where for every $k = 3, \ldots, n$, the k-th summand is of the form

$$\fint_{\mathbb{S}^{n-1}} R_{V_n,k}(w, \nabla_T w)\ d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \Big\langle w, A_{n,k}(w) \Big\rangle\ d\mathcal{H}^{n-1},$$

$A_{n,k}$ being a nonlinear first order differential operator that is a "homogeneous polynomial" of order $k-1$ in the first derivatives of w. Regarding its growth behaviour, one always has

$$\fint_{\mathbb{S}^{n-1}} |R_{V_n}(w, \nabla_T w)|\ d\mathcal{H}^{n-1} \leq C_2 \left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\ d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}. \tag{4.2.10}$$

Remark 4.2.3. As the last Lemma suggests, there are two basic differences in the expansion of the *combined conformal-isoperimetric deficit* around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ compared to the case $n = 3$, both coming from the expansion of the $(n-1)-$Dirichlet-energy-term. The first one is already seen in $Q_{D_{n-1}}(w)$, where a *divergence term* appears with a certain dimensional coefficient, which for $n = 3$ was vanishing. The increase in the dimension and the presence of this term will be the two new features in the study of the coercivity of Q_n via the *eigenspace decomposition* described at the beginning of Subsection 4.1.2.

The second difference is seen at the *growth behaviour* of the remainder term in the expansion of $D_{n-1}(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$. In the case $n = 3$, we saw that this term was growing *quadratically in the L^2-Dirichlet energy of w*. Imposing the condition that the map u is apriori close to the identity in $W^{1,2}$ was therefore perfectly enough to absorb this term in the final (*local in nature*) estimate. Its behaviour in the case $n \geq 4$ is different, since it grows like *the cubic power of the L^3-norm of the gradient of w*. This is the reason why in this setting we impose apriori-closeness of u to the $\mathrm{id}_{\mathbb{S}^{n-1}}$ in a stronger topology, namely in the $W^{1,\infty}$-topology, to finally absorb this term in the local estimate. It seems that a slightly weaker condition for our estimates to be valid, would be to impose that $\nabla_T u$ is close to P_T in BMO (see [SS19] for details), but the discussion of this anyway small refinement goes beyond the scope of the thesis.

Regarding the Taylor expansion of the term $V_n(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ and the structure as well as the growth behaviour of the remainder terms in it, one readily sees that it is the exact higher dimensional analogue of the one in the case $n = 3$, so that this part of the deficit does not exhibit any differences in its treatment. The precise algebraic expression of $\fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\ d\mathcal{H}^{n-1}$ is of course more complicated than in (4.1.18), but its structure and the assumption that $\|\nabla_T u - P_T\|_{L^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)} \leq \theta$ still imply that

$$|R_{V_n}(w, \nabla_T w)| \lesssim_{n,\mathcal{O}(1)+\theta} |w||\nabla_T w|^2 \quad \mathcal{H}^{n-1} - \text{a.e. on } \mathbb{S}^{n-1}, \text{which implies}$$
$$\fint_{\mathbb{S}^{n-1}} |R_{V_n}(w, \nabla_T w)| \lesssim_{n,\theta} \|w\|_{L^{\frac{2(n-1)}{n-3}}} \||\nabla_T w|^2\|_{L^{\frac{2(n-1)}{n+1}}} \lesssim_{n,\theta} \theta^{\frac{2(n-3)}{n+1}} \|w\|_{L^{\frac{2(n-1)}{n-3}}} \|\nabla_T w\|_{L^2}^{\frac{n+1}{n-1}}.$$

For the last estimates only the information that $\|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})}$ is apriori bounded is necessary and not that it is actually bounded by a small constant. We can then apply the Sobolev inequality with the optimal exponent $p_n = \frac{2n-2}{n-3}$ in this case, to obtain the analogue of (4.1.22), i.e. (4.2.10) with the optimal exponent $\frac{n}{n-1}$ and with $C_2(n, \theta) = o(\theta)$.

As in Lemma 4.1.6. we use the expansions (4.2.5) and (4.2.8) in the *mixed conformal-isoperimetric deficit* $\varepsilon > 0$, rearrange terms and use the defining conditions of the set $\tilde{\mathcal{A}}_{n,\infty,\theta,\varepsilon}$ to estimate the higher order terms by

$$\left|\fint_{\mathbb{S}^{n-1}} R_{D_{n-1}}(\nabla_T w)\, d\mathcal{H}^{n-1}\right| \leq C_1 \|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \leq C_1 \theta \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\, d\mathcal{H}^{n-1},$$

and

$$\left|\fint_{\mathbb{S}^{n-1}} R_{V_n}(w, \nabla_T w)\, d\mathcal{H}^{n-1}\right| \leq C_2 \left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\right)^{\frac{n}{n-1}} \leq C_2 \theta^{\frac{2}{n-1}} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\, d\mathcal{H}^{n-1},$$

and finally arrive (again assuming without loss of generality that $0 < \varepsilon \leq \theta_0^2$) at

Lemma 4.2.4. *Let $n \geq 4$, $\theta_0 > 0$ sufficiently small and $0 < \varepsilon \leq \theta_0^2$ arbitrary but fixed. For every $u \in \tilde{\mathcal{A}}_{n,\infty,\theta,\varepsilon}$, the map $w := u - \mathrm{id}_{\mathbb{S}^{n-1}}$ satifies the estimate*

$$Q_n(w) \leq \varepsilon + c_{n,\theta_0} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\, d\mathcal{H}^{n-1}, \tag{4.2.11}$$

where Q_n is the quadratic form defined as

$$Q_n(w) := Q_{D_{n-1}}(w) - Q_{V_n}(w) \tag{4.2.12}$$

and $c_{n,\theta_0} := \frac{n\sqrt{n}}{2\sqrt{n-1}} \cdot \theta_0^2 + C_2(n, \theta)(1+\theta_0^2) \cdot \theta^{\frac{2}{n-1}} + C_1(n, \theta_0) \cdot \theta_0 > 0$, where $C_1 := C_1(n, \theta_0) > 0$ and $C_2 := C_2(n, \theta_0) > 0$ are the constants of Lemma 4.2.2..

For $\theta_0 > 0$ small enough, $\|\nabla_T w\|_{L^\infty(\mathbb{S}^{n-1})} \leq \theta_0$ and as we have seen both constants are actually such that $C_1(n, \theta_0), C_2(n, \theta_0) \sim_n o(\theta_0)$. Once again, choosing $\theta_0 > 0$ sufficiently small depending only on n, the constant c_{n,θ_0} can be set to be arbitrarily small and then the important feature for the *local stability estimate* is again the behaviour of the quadratic term Q_n. This will be studied using again the results of Subsection 4.1.2, addressing the differences in this higher dimensional setting.

4.2.2 On the coercivity of the quadratic form Q_n

We would like to examine the quadratic form

$$Q_n(w) := \frac{n}{2}\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2}{n-1} + \frac{(n-3)(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{(n-1)^2} - \langle w, A(w) \rangle \right) \, d\mathcal{H}^{n-1}, \qquad (4.2.13)$$

where

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j, \qquad (4.2.14)$$

again in the space

$$H_n := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w \, d\mathcal{H}^{n-1} = 0 \ , \ \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} = 0 \right\}, \qquad (4.2.15)$$

where now $n \geq 4$. As we have already mentioned, Lemma 4.1.7. and Theorem 4.1.8. hold true in every dimension $n \geq 3$. In the case $n = 3$, where the *divergence-term* was dropping out, we had a *precise splitting* of the quadratic form Q_3 in the eigenspaces $(H_{n,k,i})_{(k,i) \in \mathbb{N}^* \times \{1,2,3\} \setminus (1,3)}$, i.e. Lemma 4.1.9. and as a consequence of it we obtained the desired coercivity estimate (4.1.43) with sharp constant. In this higher dimensional case, Lemma 4.1.9. holds *partially*, in the sense that *the quadratic form Q_{V_n} is of course splitting among these eigenspaces, but the full form Q_n is not due to the presence of the divergence term.* Nevertheless, *the form Q_n is still proportional to the Dirichlet energy* in each one of the eigenspaces separately with constants that can be computed explicitely.

Lemma 4.2.5. *Let $k \geq 1$ and $i = 1, 2, 3$. For every $w \in H_{n,k,i}$ one has*

$$Q_{V_n}(w) \ = \ c_{n,k,i}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}; \qquad \text{where} \qquad c_{n,k,i} := \frac{n\sigma_{n,k,i}}{2\lambda_{n,k}} \qquad (4.2.16)$$

$$\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 = \alpha_{n,k,i}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}; \qquad \alpha_{n,k,i} := \frac{\sigma^2_{n,k,i}(2\lambda_{n,k}c_{n,k,i} - n)}{n\lambda_{n,k}(2\sigma_{n,k,i} - n)} \qquad (4.2.17)$$

$$Q_n(w) = C_{n,k,i}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1}; C_{n,k,i} := \frac{n}{2(n-1)} + \frac{n(n-3)}{2(n-1)^2}\alpha_{n,k,i} - c_{n,k,i}. \qquad (4.2.18)$$

Proof. We just need to justify the second identity. To do so, recall that for every $v \in H_n$ we can alternatively write $Q_{V_n}(v)$ in the form

$$Q_{V_n}(v) = \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \left(2\mathrm{div}_{\mathbb{S}^{n-1}} v \langle v, x \rangle - n\langle v, x \rangle^2 + |v|^2 \right) \, d\mathcal{H}^{n-1}.$$

For $w \in H_{n,k,i}$ we have $Q_{V_n}(w) = c_{n,k,i}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2$, $\fint_{\mathbb{S}^{n-1}} |w|^2 = \frac{1}{\lambda_{n,k}}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2$ and recalling (4.1.35), $\mathrm{div}_{\mathbb{S}^{n-1}} w = \sigma_{n,k,i}\langle w, x \rangle$. Substituting these identities above yields the desired identity for $\fint_{\mathbb{S}^{n-1}}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2$ and the one for Q_n follows then immediately. □

Let us list the values of the previously mentioned constants.

$$\text{For } k \geq 1; \left\{ \begin{aligned} c_{n,k,1} &= \tfrac{-n}{2(k+n-2)} \\ \alpha_{n,k,1} &= \tfrac{k(k+1)}{(k+n-2)(2k+n)} \\ C_{n,k,1} &= \tfrac{n}{2}\Big(\tfrac{1}{n-1} + \tfrac{1}{k+n-2} + \tfrac{(n-3)k(k+1)}{(n-1)^2(k+n-2)(2k+n)}\Big) \end{aligned} \right\},$$

$$\text{For } k \geq 1; \left\{ \begin{aligned} c_{n,k,2} &= \tfrac{n}{2k(k+n-2)} \\ \alpha_{n,k,2} &= 0 \\ C_{n,k,2} &= \tfrac{n}{2}\tfrac{(k-1)(k+n-1)}{(n-1)k(k+n-2)} \end{aligned} \right\}, \tag{4.2.19}$$

$$\text{For } k \geq 2; \left\{ \begin{aligned} c_{n,k,3} &= \tfrac{n}{2k} \\ \alpha_{n,k,3} &= \tfrac{(k+n-2)(k+n-3)}{k(2k+n-4)} \\ C_{n,k,3} &= \tfrac{n(k-2)\big((3n-5)k+(n^2-6n+7)\big)}{2(n-1)^2k(2k+n-4)} \end{aligned} \right\},$$

the last set of constants being considered for $k \geq 2$, because in any case $\Pi_{H_{n,1,3}} w = 0$ for every $w \in H_n$. By taking a closer look at the values of the constants, it is seen that one cannot merely *neglect* the *divergence term*. Indeed, *although the quadratic form*

$$\frac{1}{2}\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} - Q_{V_n}(w)$$

is splitting among the eigenspaces $(H_{n,k,i})_{(k,i)\in\mathbb{N}^*\times\{1,2,3\}\setminus(1,3)}$, *for* $n \geq 4$ *it does not have a sign.*

The last quadratic form is actually *negative* in $H_{n,k,3}$ for every $k = 2, \ldots, n-2$, *zero* in $H_{n,1,2}, H_{n,n-1,3}$ and strictly positive in each one of the other eigenspaces. Therefore the *contribution of the nonnegative term* $\frac{1}{2}\frac{n(n-3)}{(n-1)^2}\fint_{\mathbb{S}^{n-1}}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2$ has to be taken into account, the presence of which however *produces mixed terms* in the expression of Q_n. The interesting feature is that these *mixed terms* are of a particular form as the next Lemma reveals.

Lemma 4.2.6. *Let* $w \in H_n$ *be written in Fourier series as*

$$w = \sum_{\substack{(k,i)\in\mathbb{N}^*\times\{1,2,3\} \\ (k,i)\neq(1,3)}} w_{n,k,i}, \quad \text{where} \quad w_{n,k,i} \in H_{n,k,i}.$$

Then,

$$\begin{aligned} Q_n(w) = \sum_{\substack{(k,i)\in\mathbb{N}^*\times\{1,2,3\} \\ (k,i)\neq(1,3)}} Q_n(w_{n,k,i}) &+ \frac{1}{2}\frac{n(n-3)}{(n-1)^2}\sum_{k\geq 1}\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,1} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,3} \\ &+ \frac{1}{2}\frac{n(n-3)}{(n-1)^2}\sum_{k\geq 3}\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,3} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,1}. \end{aligned}$$

Proof. Since the form $\frac{1}{2}\frac{n}{n-1}\fint_{\mathbb{S}^{n-1}}|\nabla_T w|^2\,d\mathcal{H}^{n-1}-Q_{V_n}(w)$ *splits completely* in the eigenspaces $(H_{n,k,i})_{(k,i)\in\mathbb{N}^*\times\{1,2,3\}\setminus(1,3)}$, what needs to be checked is that

$$\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,k,i}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,l,j}\;d\mathcal{H}^{n-1}=0 \tag{4.2.20}$$

for all pairs $(k,i),(l,j)\in\mathbb{N}^*\times\{1,2,3\}\setminus(1,3)$ with $(k,i)\neq(l,j)$, *except for the pairs of the form* $(k,1),(k+2,3)$ *and* $(k,3),(k+2,1)$.

This can be checked again using the different formulas for the quadratic form Q_{V_n}. Indeed, let $w_{n,k,i}\in H_{n,k,i}, w_{n,l,j}\in H_{n,l,j}$ with $(k,i)\neq(l,j)$. Since $\mathrm{div}_{\mathbb{S}^{n-1}}w\equiv 0$ whenever $w\in H_{n,k,2}$, we may suppose without loss of generality that $i,j\in\{1,3\}$. Then,

$$\begin{aligned}Q_{V_n}(w_{n,k,i},w_{n,l,j})&=\frac{n}{2}\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,k,i}\langle w_{n,l,j},x\rangle+\frac{n}{2}\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,l,j}\langle w_{n,k,i},x\rangle\\&-\frac{n^2}{2}\fint_{\mathbb{S}^{n-1}}\langle w_{n,k,i},x\rangle\langle w_{n,l,j},x\rangle+\frac{n}{2}\fint_{\mathbb{S}^{n-1}}\langle w_{n,k,i},w_{n,l,j}\rangle,\end{aligned}$$

and by the fact that actually

$$Q_{V_n}(w_{n,k,i},w_{n,l,j})=\frac{n\sigma_{n,l,j}}{2}\fint_{\mathbb{S}^{n-1}}\langle w_{n,k,i},w_{n,l,j}\rangle=\frac{n\sigma_{n,l,j}}{2}\delta^{kl}\delta^{ij}=0$$

and (4.1.35), we obtain

$$\begin{aligned}0&=\frac{n}{2}\left(\frac{1}{\sigma_{n,l,j}}+\frac{1}{\sigma_{n,k,i}}-\frac{n}{\sigma_{n,k,i}\sigma_{n,l,j}}\right)\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,k,i}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,l,j}\;d\mathcal{H}^{n-1}\\ \iff 0&=(\sigma_{n,k,i}+\sigma_{n,l,j}-n)\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,k,i}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,l,j}\;d\mathcal{H}^{n-1},\end{aligned}$$

i.e. (4.2.20) holds, unless the pairs $(k,i)\neq(l,j)$ are such that $\sigma_{n,k,i}+\sigma_{n,l,j}=n$. In this respect,

(i) If $i=j=1$, $\sigma_{n,k,i}+\sigma_{n,l,j}=-k-l<0<n$,

(ii) If $i=j=3$, $\sigma_{n,k,i}+\sigma_{n,l,j}=k+l+2n-4\geq 2n-2>n$,

(iii) If $i=1,j=3$, $\sigma_{n,k,i}+\sigma_{n,l,j}=n\iff -k+l+n-2=n\iff l=k+2$,

(iv) If $i=3,j=1$, $\sigma_{n,k,i}+\sigma_{n,l,j}=n\iff k+n-2-l=n\iff k=l+2$,

which proves the desired claim and the formula for Q_n follows by the bilinearity of the expression. *Another interesting point in the formula is that the summation in the last term of the expression starts from* $k=3$. The reason for this is that in any case $w_{n,1,3}\equiv 0$ whenever $w\in H_n$, but also

$$\fint_{\mathbb{S}^{n-1}}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,2,3}\mathrm{div}_{\mathbb{S}^{n-1}}w_{n,4,1}\;d\mathcal{H}^{n-1}=0. \tag{4.2.21}$$

To prove this last identity, recall that $\mathrm{div}(w_{n,4,1})_h = 0$ in $\overline{B^n}$ and therefore

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,2,3} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,4,1} \; d\mathcal{H}^{n-1} &= -4n \fint_{\mathbb{S}^{n-1}} \langle w_{n,2,3}, x\rangle \langle w_{n,4,1}, x\rangle \; d\mathcal{H}^{n-1} \\
&= -4n \fint_{\mathbb{S}^{n-1}} \big\langle \langle (w_{n,2,3})_h, x\rangle (w_{n,4,1})_h, x \big\rangle \; d\mathcal{H}^{n-1} \\
&= -4 \fint_{B^n} \mathrm{div}\big(\langle (w_{n,2,3})_h, x\rangle (w_{n,4,1})_h\big) \; dx \\
&= -4 \fint_{B^n} \langle \nabla \langle (w_{n,2,3})_h, x\rangle, (w_{n,4,1})_h \rangle \; dx.
\end{aligned}$$

To justify that the last integral is zero, observe that

$$\begin{aligned}
\fint_{B^n} \langle \nabla \langle (w_{n,2,3})_h, x\rangle, (w_{n,4,1})_h \rangle &= \fint_{B^n} \sum_{i=1}^{n} (w_{n,4,1})_h^i \langle \partial_i (w_{n,2,3})_h, x\rangle + \langle (w_{n,4,1})_h, (w_{n,2,3})_h \rangle \; dx \\
&= \sum_{i=1}^{n} \fint_{B^n} (w_{n,4,1})_h^i \langle \partial_i (w_{n,2,3})_h, x\rangle \; dx,
\end{aligned}$$

because $\fint_{B^n} \langle (w_{n,4,1})_h, (w_{n,2,3})_h \rangle = 0$, the reason being simply that the vector-valued homogeneous harmonic polynomials $(w_{n,4,1})_h, (w_{n,2,3})_h$ are of different degree. Moreover, we observe that for every $i = 1, \ldots, n$,

$$\Delta\big(\langle \partial_i (w_{n,2,3})_h, x\rangle\big) = 2\partial_i \mathrm{div}(w_{n,2,3})_h \quad \text{in} \quad B^n. \tag{4.2.22}$$

Since $(w_{n,2,3})_h$ is an $\mathbb{R}^n-$ valued 2nd order homogeneous harmonic polynomial, $\partial_i \mathrm{div}(w_{n,2,3})_h$ is simply a constant. The function $\langle \partial_i (w_{n,2,3})_h, x\rangle - \frac{\partial_i \mathrm{div}(w_{n,2,3})_h |x|^2}{n}$ is therefore a homogeneous harmonic polynomial of degree 2, hence also L^2-orthogonal to $(w_{n,4,1})_h$. Thus,

$$\begin{aligned}
&\fint_{B^n} (w_{n,4,1})_h^i \langle \partial_i (w_{n,2,3})_h, x\rangle \; dx = \frac{\partial_i \mathrm{div}(w_{n,2,3})_h}{n} \fint_{B^n} (w_{n,4,1})_h^i |x|^2 \; dx \\
\Longleftrightarrow &\fint_{B^n} (w_{n,4,1})_h^i \langle \partial_i (w_{n,2,3})_h, x\rangle \; dx = \frac{\partial_i \mathrm{div}(w_{n,2,3})_h}{n+6} \fint_{\mathbb{S}^{n-1}} (w_{n,4,1})^i \; d\mathcal{H}^{n-1} = 0,
\end{aligned}$$

where we used the fact that the function $(w_{n,4,1})_h^i |x|^2$ is 6-homogeneous, so that we can write its integral over B^n as an integral on $\mathbb{S}^{n-1}$, up to a multiplicative constant. The last integral is of course zero for every nontrivial spherical harmonic. Note that the previous argument relies on the fact that $\partial_i \mathrm{div}(w_{n,2,3})_h$ is constant and of course cannot be implemented for the *mixed terms of higher order*. □

Looking again at the values of the constants in (4.2.19), we see that the quadratic form Q_n vanishes again in the desired space $H_{n,0} := H_{n,1,2} \bigoplus H_{n,2,3}$, an issue that can be handled applying again Lemma 4.1.12.. Therefore, *what we need to check is that the presence of the mixed divergence-terms is harmless, i.e. it does not produce any further zeros in Q_n.* A quantitative way to see this using some elementary estimates is the following.

For every $k \geq 1$ we can apply a *weighted Cauchy-Schwartz inequality, with weight* $\varepsilon_{n,k} = \frac{k+n}{k} > 0$ to estimate

$$
\begin{aligned}
\left| \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,1} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,3} \right| &\leq \left(\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k,1})^2 \right)^{\frac{1}{2}} \left(\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,3})^2 \right)^{\frac{1}{2}} \\
&= \left(\alpha_{n,k,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,1}|^2 \right)^{\frac{1}{2}} \left(\alpha_{n,k+2,3} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\alpha_{n,k,1}\varepsilon_{n,k}}{2} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,1}|^2 + \frac{\alpha_{n,k+2,3}}{2\varepsilon_{n,k}} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \\
&= \frac{(k+1)(k+n)}{2(k+n-2)(2k+n)} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,1}|^2 \\
&\quad + \frac{k(k+n-1)}{2(k+2)(2k+n)} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2.
\end{aligned}
$$

For the last summand in the expression obtained in Lemma 4.2.6., after shifting the summation index we can rewrite it as

$$
\sum_{k \geq 1} \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,3} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+4,1} \; d\mathcal{H}^{n-1}
$$

and for every $k \geq 1$ we can estimate as before,

$$
\begin{aligned}
\left| \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+2,3} \mathrm{div}_{\mathbb{S}^{n-1}} w_{n,k+4,1} \right| &\leq \left(\alpha_{n,k+2,3} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \right)^{\frac{1}{2}} \left(\alpha_{n,k+4,1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+4,1}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\alpha_{n,k+2,3}}{2\varepsilon_{n,k}} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2 + \frac{\alpha_{n,k+4,1}\varepsilon_{n,k}}{2} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+4,1}|^2 \\
&= \frac{k(k+n-1)}{2(k+2)(2k+n)} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+2,3}|^2 \\
&\quad + \frac{(k+4)(k+5)(k+n)}{2k(k+n+2)(2k+n+8)} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k+4,1}|^2.
\end{aligned}
$$

The choice of the weights was such that some of the terms match. The series appearing are all *absolutely summable*, $Q_n(w_{n,1,2}) = Q_n(w_{n,2,3}) = 0$ and we can therefore estimate the form Q_n from *below* by

$$Q_n(w) \geq \sum_{\substack{(k,i)\in\mathbb{N}^*\times\{1,2,3\}\\(k,i)\neq(1,2),(1,3),(2,3)}} Q_n(w_{n,k,i}) - \frac{1}{4}\frac{n(n-3)}{(n-1)^2}\sum_{k\geq 1}\frac{(k+1)(k+n)}{(k+n-2)(2k+n)}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k,1}|^2$$

$$-\frac{1}{4}\frac{n(n-3)}{(n-1)^2}\sum_{k\geq 1}\frac{(k+4)(k+5)(k+n)}{k(k+n+2)(2k+n+8)}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k+4,1}|^2$$

$$-\frac{1}{2}\frac{n(n-3)}{(n-1)^2}\sum_{k\geq 1}\frac{k(k+n-1)}{(k+2)(2k+n)}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k+2,3}|^2.$$

After rearranging terms we arrive at the estimate,

$$Q_n(w) \geq \sum_{k\geq 1}\tilde{C}_{n,k,1}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k,1}|^2 + \sum_{k\geq 2}\tilde{C}_{n,k,2}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k,2}|^2 + \sum_{k\geq 3}\tilde{C}_{n,k,3}\fint_{\mathbb{S}^{n-1}}|\nabla_T w_{n,k,3}|^2$$

where the new constants are defined as

$$\tilde{C}_{n,k,1} := \left[C_{n,k,1} - \frac{n(n-3)(k+1)}{4(n-1)^2(k+n-2)(2k+n)}\left(\frac{k(k+n-4)\chi_{m\geq 5}(k)}{k-4} + (k+n)\right)\right],$$

$$\tilde{C}_{n,k,2} = C_{n,k,2}, \quad k \geq 2$$

and

$$\tilde{C}_{n,k,3} := \left[C_{n,k,3} - \frac{1}{2}\frac{n(n-3)}{(n-1)^2}\frac{(k-2)(k+n-3)}{k(2k+n-4)}\right], \quad k \geq 3.$$

By elementary algebraic calculations that we omit here one can verify that

$$\tilde{C}_{n,l,1} = \frac{n}{2}\left[\frac{1}{n-1} + \frac{1}{l+n-2} - \frac{(n-3)(n-l)(l+1)}{2(n-1)^2(l+n-2)(2l+n)}\right], \quad \text{for } l = 1,2,3,4,$$

$$\tilde{C}_{n,k,1} = \frac{n}{2}\left[\frac{1}{n-1} + \frac{1}{k+n-2} - \frac{n(n-3)(k-2)(k+1)}{(n-1)^2(k-4)(k+n-2)(2k+n)}\right], \quad \text{for } k \geq 5,$$

$$\tilde{C}_{n,k,2} = \frac{n}{2}\frac{(k-1)(k+n-1)}{(n-1)k(k+n-2)}, \quad \text{for } k \geq 1,$$

$$\tilde{C}_{n,k,3} = \frac{n(k-2)\left((n-1)k-1\right)}{(n-1)^2 k(2k+n-4)}, \quad \text{for } k \geq 2,$$

and in particular,

$$\min_{k\geq 1}\tilde{C}_{n,k,1} =: C_{n,1} > 0, \quad \min_{k\geq 2}\tilde{C}_{n,k,2} =: C_{n,2} > 0, \quad \min_{k\geq 3}\tilde{C}_{n,k,3} =: C_{n,3} > 0. \tag{4.2.23}$$

Labelling $C_n := \min\{C_{n,1}, C_{n,2}, C_{n,3}\} > 0$, we can estimate again from below,

$$Q_n(w) \geq C_n \left(\sum_{k\geq 1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,1}|^2 + \sum_{k\geq 2} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,2}|^2 + \sum_{k\geq 3} \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,k,3}|^2 \right)$$
$$= C_n \left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 - \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,1,2}|^2 - \fint_{\mathbb{S}^{n-1}} |\nabla_T w_{n,2,3}|^2 \right).$$

In this way we arrive again at the desired estimate, namely

Theorem 4.2.7. *There exists a dimensional constant $C_n > 0$ such that for every $w \in H_n$ the following coercivity estimate holds.*

$$Q_n(w) \geq C_n \fint_{\mathbb{S}^{n-1}} \left| \nabla_T w - \nabla_T (\Pi_{n,0} w) \right|^2 \, d\mathcal{H}^{n-1}, \tag{4.2.24}$$

where $H_{n,0} := H_{n,1,2} \bigoplus H_{n,2,3}$ is the kernel of Q_n in H_n, and $\Pi_{n,0} : H_n \mapsto H_{n,0}$ is the $W^{1,2}$-orthogonal projection of H_n onto $H_{n,0}$.

The proof of the *local stability* Theorem 4.2.1. is now essentially the same as the one of Theorem 4.1.2. for the case $n = 3$. The *degenerate space* $H_{n,0}$ is again characterized by Lemma 4.1.11. and then Lemma 4.1.12. applies again, so that the proof carries out unchanged, except of course for replacing the initial set of maps $\mathcal{A}_{3,2,M,\theta,\varepsilon}$ with $\mathcal{A}_{n,\infty,\theta,\varepsilon}$ and using (4.2.11) instead of (4.1.25).

Remark 4.2.8. A more abstract way to argue about the coercivity of the quadratic form Q_n that would be similar to the argument in Section 3.3, would be to notice that for every $w \in H_n$,

$$Q_n(w) = Q_{n,\mathrm{conf}}(w) + Q_{n,\mathrm{isop}}(w),$$

where

$$Q_{n,\mathrm{conf}}(w) := Q_{D_{n-1}}(w) - Q_{P_{n-1}}(w) = \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} \left| (P_T^t \nabla_T w)_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} w}{n-1} I_x \right|^2 \, d\mathcal{H}^{n-1} \geq 0,$$

and

$$Q_{n,\mathrm{isop}}(w) := \frac{n}{n-1} \left[\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2 + (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{2} \right) - Q_{n,isom}(w) \right] - Q_{V_n}(w) \geq 0,$$

where by recalling (3.3.5),

$$Q_{n,\mathrm{isom}}(w) := \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2 \, d\mathcal{H}^{n-1} \geq 0.$$

Again, *the quadratic forms* $Q_{n,\mathrm{conf}}$ *and* $Q_{n,\mathrm{isop}}$ *are both nonnegative and the kernel of each one of them is actually infinite-dimensional.* Indeed, for $Q_{n,\mathrm{conf}}(w)$ we observe that for every $\phi \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R})$ with $\fint_{\mathbb{S}^{n-1}} \phi \, d\mathcal{H}^{n-1} = 0$ and $\fint_{\mathbb{S}^{n-1}} \phi(x)x \, d\mathcal{H}^{n-1} = 0$ the map $w_\phi(x) := \phi(x)x \in H_n$ and by an easy short calculation

$$P_T^t \nabla_T w_\phi = \phi I_x \implies (P_T^t \nabla_T w_\phi)_{\mathrm{sym}} - \frac{\mathrm{div}_{\mathbb{S}^{n-1}} w_\phi}{n-1} I_x = 0.$$

Since the space of such ϕ's is obviously infinite dimensional we have in particular that also $\dim(\ker Q_{n,\mathrm{conf}}) = \infty$.

Regarding the quadratic form $Q_{n,isop}$, we will prove the following.

Claim. $H_{n,2} := \bigoplus_{k=1}^{\infty} H_{n,k,2} \subseteq \ker Q_{n,isop} \implies \dim(\ker Q_{n,\mathrm{isop}}) = \infty.$

To verify the claim we first use the following identity, which is referred to as *Korn's identity*, that is interesting in its own right and whose derivation is a simple computation which is also included at the end of Appendix C.

For every $w \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ *the following identity holds*

$$Q_{n,\mathrm{isom}}(w) = \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 - \left| \sum_{j=1}^{n} x_j \nabla_T w^j \right|^2 + (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right) - \frac{n-2}{n} Q_{V_n}(w). \quad (4.2.25)$$

The interesting point of this identity is that when $n \geq 3$ the quadratic form Q_{V_n} of the expansion of the *signed-volume-term* appears in the right hand side as some short of *curvature contribution*, and it is really a *surface identity* in the sense that *the corresponding identity in the bulk* is

$$\int_U |(\nabla w)_{\mathrm{sym}}|^2 \; dx = \frac{1}{2}\int_U \left(|\nabla w|^2 + (\mathrm{div} w)^2\right) \; dx - \frac{1}{2}\int_U \left((\mathrm{div} w)^2 - \mathrm{Tr}(\nabla w)^2\right) \; dx,$$

but the last term on the right hand side is a null-Lagrangian and should be interpreted as a *boundary-term contribution.*

By using *Korn's identity*, the quadratic form $Q_{n,\mathrm{isop}}$ can be rewritten in a simpler form as

$$Q_{n,\mathrm{isop}}(w) = \frac{n}{2(n-1)}\fint_{\mathbb{S}^{n-1}} \left(\left| \sum_{j=1}^{n} x_j \nabla_T w^j \right|^2 - \langle w, A(w) \rangle \right) d\mathcal{H}^{n-1}.$$

But if $w \in H_{n,2}$, then in this *infinite-dimensional space*, $A(w) = w$ and $\mathrm{div}_{\mathbb{S}^{n-1}} w = 0$ on $\mathbb{S}^{n-1}$, i.e. $-\sum_{j=1}^{n} x_j \nabla_T w^j = w$, and therefore $Q_{n,\mathrm{isop}}(w) = 0$ which proves the claim.

If now $w \in \ker Q_n \iff w \in \ker Q_{n,\mathrm{conf}} \cap \ker Q_{n,\mathrm{isop}}$, then again the following two equations must be satisfied simultaneously.

$$\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} = \frac{\operatorname{div}_{\mathbb{S}^{n-1}} w}{n-1} I_x \quad \text{and} \quad Q_{n,\text{isop}}(w) = 0. \tag{4.2.26}$$

Because of the first one, the second results in the equation

$$\begin{aligned} &\frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} \left[\frac{|\nabla_T w|^2}{2} + \frac{(\operatorname{div}_{\mathbb{S}^{n-1}} w)^2}{2} - \frac{(\operatorname{div}_{\mathbb{S}^{n-1}} w)^2}{n-1} \right] = Q_{V_n}(w) \\ \iff &\frac{1}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 + \frac{n-3}{(n-1)^2} \fint_{\mathbb{S}^{n-1}} (\operatorname{div}_{\mathbb{S}^{n-1}} w)^2 - \fint_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle = 0, \end{aligned} \tag{4.2.27}$$

i.e. we ended up back to the original equation $Q_n(w) = 0$. Arguing directly with the eigenvalue decomposition with respect to A also has the extra benefit of showing explicitely how the form Q_n behaves in each one of the eigenspaces separately, as well as giving a lower bound for the value of the optimal constant C_n in the coercivity estimate.

4.3 On a stability result for degree ± 1 Möbius transformations of $\mathbb{S}^2$

This last Section is of complementary character and its purpose is to show how the proof of Theorem 2.2.2. can be *perturbed in a quantitative manner* to give an alternative and somewhat shorter proof of a recent result due to A.B.-Mantel, C.B. Muratov and T.M. Simon (see [BMMS]) regarding the *rigidity of degree* ± 1 *harmonic maps from* $\mathbb{S}^2$ *onto itself.*

As it is well-known in the theory of harmonic maps, a map between *two-dimensional Riemannean manifolds* is *harmonic*, i.e. a *critical point of the Dirichlet-energy functional* iff it is generalized conformal and in particular, according to Liouville's theorem, the class of orientation-preserving/-reversing, degree ± 1 harmonic maps from $\mathbb{S}^2$ onto itself is precisely the group $Conf(\mathbb{S}^2)$.

We discuss here the case of maps of degree 1, the case of maps of degree -1 being completely analogous, or it can be derived from the first case by a *single flip* in the *ambient space* $\mathbb{R}^3$. Following the notation we had in the previous Chapters that differs only slightly from the ones in [BMMS], let us define

$$\mathcal{A}_{\mathbb{S}^2} := \left\{ u \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2) : \ \deg u := \fint_{\mathbb{S}^2} \langle u, \partial_{\tau_1} u \wedge \partial_{\tau_2} u \rangle \, d\mathcal{H}^2 = 1 \right\}. \tag{4.3.1}$$

As a particular case of (1.2.6) for $n = 3$ and since $V_3(u) = 1$ for every $u \in \mathcal{A}_{\mathbb{S}^2}$ (see also Lemma A.3 in [BMMS] and the references therein), in this class of mappings we know that

$$\frac{1}{2} \fint_{\mathbb{S}^2} |\nabla_T u|^2 \geq 1 \quad \text{for every } u \in \mathcal{A}_{\mathbb{S}^2},$$

with equality iff $u \in Conf_+(\mathbb{S}^2)$. Therefore, for every $u \in \mathcal{A}_{\mathbb{S}^2}$ the quantity

$$D(u) := \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T u|^2 - 1 \tag{4.3.2}$$

is nonnegative, invariant under both the left and the right action of $Conf_+(\mathbb{S}^2)$, and vanishes precisely when $u \in Conf_+(\mathbb{S}^2)$ providing thus an appropriate notion of *conformal deficit* for maps in the above defined class.

A direct application of Proposition 2.2.7 (which for our argument we use instead of *Step 1* in the proof of *Theorem 2.4* in [BMMS]), for $n = 3$ in this case, gives

Lemma 4.3.1. *Let $(u_j)_{j\in\mathbb{N}} \in \mathcal{A}_{\mathbb{S}^2}$ be a sequence of mappings such that*

$$\lim_{j\to\infty} D(u_j) = 0. \tag{4.3.3}$$

Then, up to a non-relabeled subsequence there exist $(\psi_j)_{j\in\mathbb{N}} \in Conf_+(\mathbb{S}^2)$ and $R \in SO(3)$ such that

$$\fint_{\mathbb{S}^2} u_j \circ \psi_j \, d\mathcal{H}^2 = 0 \tag{4.3.4}$$

and

$$u_j \circ \psi_j \longrightarrow R\mathrm{id}_{\mathbb{S}^2} \quad \text{strongly in } W^{1,2}(\mathbb{S}^2;\mathbb{S}^2). \tag{4.3.5}$$

With these notations, *we present now our alternative proof of the following.*

Theorem 4.3.2. ***(A.B.-Mantel, C.B. Muratov, T.M. Simon, [BMMS], Theorem 2.4.)*** *There exists a constant $c > 0$ such that for every $u \in \mathcal{A}_{S_2}$ there exists $\phi \in Conf_+(\mathbb{S}^2)$ so that*

$$\fint_{\mathbb{S}^2} |\nabla_T u - \nabla_T \phi|^2 \, d\mathcal{H}^2 \leq cD(u). \tag{4.3.6}$$

Proof. As we have already seen in the proof of Lemma 2.2.6., given $u \in \mathcal{A}_{\mathbb{S}^2}$ one can always find a Möbius transformation $\psi \in Conf_+(\mathbb{S}^2)$ so that $\fint_{\mathbb{S}^2} u \circ \psi \, d\mathcal{H}^2 = 0$. Hence, if we set $\tilde{u} := u \circ \psi$, thanks to the *invariance of the Dirichlet-energy under conformal reparametrizations in two dimensions* and the *invariance of the degree under orientation-preserving conformal reparametrizations*, we have

$$\tilde{u} \in \mathcal{A}_{\mathbb{S}^2} \;\text{ with }\; D(\tilde{u}) = D(u), \;\; \mathrm{deg}\tilde{u} = \mathrm{deg}u = 1 \;\text{ and }\; \fint_{\mathbb{S}^2} \tilde{u} \, d\mathcal{H}^2 = 0. \tag{4.3.7}$$

The proof is again divided in two steps, where in the first one we prove a *local version* of the statement under the assumption that our map $\tilde{u}$ is *apriori close* to the $\mathrm{id}_{\mathbb{S}^2}$ in the $W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$-topology and in the second step where we use the *compactness* Lemma 4.3.1. to conclude.

Step 1. Let us first prove (4.3.6) under the extra assumption that

$$\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - P_T|^2 \, d\mathcal{H}^2 \leq \theta^2, \tag{4.3.8}$$

where θ is a sufficiently small positive constant that will be chosen later. This assumption also implies a trivial upper bound for the *conformal deficit*, since

$$D(\tilde{u}) = \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u}|^2 \, d\mathcal{H}^2 - 1 \leq \fint_{\mathbb{S}^2} (|\nabla_T \tilde{u} - P_T|^2 + |P_T|^2) \, d\mathcal{H}^2 - 1 \leq 1 + \theta^2. \tag{4.3.9}$$

Since $u \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ and $\psi \in Conf_+(\mathbb{S}^2)$ their composition $\tilde{u} := u \circ \psi$ also satisfies the pointwise identity

$$|\tilde{u}| = 1, \ \mathcal{H}^2 - \text{a.e. on } \mathbb{S}^2. \tag{4.3.10}$$

With a trick similar to one that we have used earlier, we can alternatively write the conformal deficit as

$$\begin{aligned} D(u) &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T u|^2 \, d\mathcal{H}^2 - 1 \\ &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T (u \circ \psi)|^2 \, d\mathcal{H}^2 - 1 \\ &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u}|^2 \, d\mathcal{H}^2 - 1 \\ &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u}|^2 \, d\mathcal{H}^2 - \fint_{\mathbb{S}^2} |\tilde{u}|^2 \, d\mathcal{H}^2, \end{aligned} \tag{4.3.11}$$

where we used that (4.3.10) in particular implies that

$$\fint_{\mathbb{S}^{n-1}} |\tilde{u}|^2 \, d\mathcal{H}^2 = 1.$$

In other words, a feature similar to one that appeared in Chapter 3 appears in this *two dimensional conformal setting* as well, i.e. *the conformal deficit of* u (or equivalently of $\tilde{u}$) transforms into the *deficit in the* L^2-*Poincare inequality* for the *zero-average map* $\tilde{u}$. Once again by expanding in spherical harmonics and by using the sharp Poincare inequality (with constant $\frac{1}{6}$) for the map $\tilde{u} - \nabla \tilde{u}_h(0)\mathrm{id}_{\mathbb{S}^2}$ which has vanishing mean and vanishing linear part,

$$\begin{aligned} D(u) &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u}|^2 \, d\mathcal{H}^2 - \fint_{\mathbb{S}^2} |\tilde{u}|^2 \, d\mathcal{H}^2 \\ &= \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - \nabla \tilde{u}_h(0) P_T|^2 \, d\mathcal{H}^2 - \fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0) x|^2 \, d\mathcal{H}^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{6}\right)\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - \nabla \tilde{u}_h(0) P_T|^2 \, d\mathcal{H}^2, \end{aligned}$$

and the last estimate can be rewritten as

$$\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - \nabla \tilde{u}_h(0) P_T|^2 \, d\mathcal{H}^2 \leq 3D(u). \tag{4.3.12}$$

As the reader can notice, although we are now in a different setting, (4.3.12) is similar to (3.1.16) and (3.2.17) and once again *we only have to justify why we can replace* $\nabla \tilde{u}_h(0)$ *with a matrix* $R \in SO(3)$ *in the last estimate.* The proof of this fact follows the lines of the analogous proofs in Theorem 3.1.2. and Theorem 3.2.3., by using the *degree one condition* for $\tilde{u}$, the extra assumption (4.3.8) and the extra information that $\tilde{u}$ takes values $\mathcal{H}^2$-a.e. on $\mathbb{S}^2$. As it was argued in Chapter 3, (4.3.8) implies that

$$|\nabla \tilde{u}_h(0) - I_3|^2 \leq \frac{3\theta^2}{2} \tag{4.3.13}$$

and by choosing $\theta > 0$ sufficiently small, we can take $\nabla \tilde{u}_h(0)$ to be invertible and such that

$$|\nabla \tilde{u}_h(0)|^2, |[\nabla \tilde{u}_h(0)]^{-1}|^2 \in [2, 4] \ \text{ and } \ \det \nabla \tilde{u}_h(0) \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{4.3.14}$$

By writing again $\nabla \tilde{u}_h(0) = R_0 \sqrt{\nabla \tilde{u}_h(0)^t \nabla \tilde{u}_h(0)}$ with $R_0 \in SO(3)$, label the eigenvalues of $\sqrt{\nabla \tilde{u}_h(0)^t \nabla \tilde{u}_h(0)} \in Sym_+(3)$ as $0 < \mu_1 \leq \mu_2 \leq \mu_3$ and set $\lambda_i := \mu_i - 1$, $\lambda := \lambda_1 + \lambda_2 + \lambda_3$ and $\Lambda^2 := \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, we have as in (3.1.17),

$$\Lambda^2 = \operatorname{dist}^2(\nabla \tilde{u}_h(0); SO(3)) \leq |\nabla \tilde{u}_h(0) - I_3|^2 \leq \frac{3\theta^2}{2}. \tag{4.3.15}$$

Setting again,

$$\tilde{w}(x) := \nabla \tilde{u}_h(0)^{-1} (\tilde{u}(x) - \nabla \tilde{u}_h(0) x), \tag{4.3.16}$$

we can use the fact that $\deg(\tilde{u}) = 1$ and the computations we performed to arrive at (3.1.32), to obtain

$$1 = \det \nabla \tilde{u}_h(0) \left(1 + Q_{V_3}(\tilde{w}) + \fint_{\mathbb{S}^2} \langle \tilde{w}, \partial_{\tau_1} \tilde{w} \wedge \partial_{\tau_2} \tilde{w} \rangle \, d\mathcal{H}^2 \right). \tag{4.3.17}$$

By writing again $\det \nabla \tilde{u}_h(0)$ as a polynomial in the eigenvalues as

$$\det \nabla \tilde{u}_h(0) = 1 + \lambda + \frac{1}{2}(\lambda^2 - \Lambda^2) + \lambda_1 \lambda_2 \lambda_3, \tag{4.3.18}$$

instead of (3.1.22) or (3.2.19) we now have the exact indentity

$$\frac{\Lambda^2}{2} = \left(\lambda + \frac{\lambda^2}{2} \right) + \lambda_1 \lambda_2 \lambda_3 + \det \nabla \tilde{u}_h(0) \left(Q_{V_3}(\tilde{w}) + \fint_{\mathbb{S}^2} \langle \tilde{w}, \partial_{\tau_1} \tilde{w} \wedge \partial_{\tau_2} \tilde{w} \rangle \, d\mathcal{H}^2 \right). \tag{4.3.19}$$

In the way that we have already encountered, the last identity leads to the desired estimate, i.e.

$$\operatorname{dist}^2(\nabla \tilde{u}_h(0); SO(3)) = \Lambda^2 \leq c_1 D(u), \tag{4.3.20}$$

for an absolute constant $c_1 > 0$.

Indeed, the summand $\left(\lambda + \frac{\lambda^2}{2}\right)$ can be handled as in the proofs we saw in Chapter 3 by observing that now

$$\frac{|\nabla \tilde{u}_h(0)|^2}{3} \leq \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u}|^2 \, d\mathcal{H}^2 = 1 + D(u) \implies \lambda \leq \frac{3}{2}D(u) - \frac{\Lambda^2}{2} \leq \frac{3}{2}D(u)$$

and distinguish again between the case $\lambda \geq 0$, where

$$\lambda + \frac{\lambda^2}{2} \leq \frac{3}{2}D(u) + \frac{9}{8}[D(u)]^2 \overset{(4.3.9)}{\leq} \left(\frac{3}{2} + \frac{9}{8}(1+\theta^2)\right) D(u),$$

and $\lambda < 0$, where due to (4.3.8),

$$\lambda + \frac{\lambda^2}{2} \leq \left(1 - \frac{3}{2\sqrt{2}}\theta\right) \lambda < 0.$$

Alternatively, by decomposing the $\mathbb{S}^2$-valued map $\tilde{u}$ into its *linear part* given by the map $x \mapsto \nabla \tilde{u}_h(0)x$ and the part consisting of the *higher order spherical harmonics*, one can obtain

$$1 = \fint_{\mathbb{S}^2} |\tilde{u}|^2 = \fint_{\mathbb{S}^2} |\nabla \tilde{u}_h(0)x|^2 + \fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2 = \frac{1}{3}|\nabla \tilde{u}_h(0)|^2 + \fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2$$

$$\implies 1 - \fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2 = \frac{1}{3}(\mu_1^2 + \mu_2^2 + \mu_3^2) = \frac{1}{3}[(\lambda_1+1)^2 + (\lambda_2+1)^2 + (\lambda_3+1)^2]$$

$$\implies \lambda = -\frac{1}{2}\Lambda^2 - \frac{3}{2}\fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2, \tag{4.3.21}$$

which is an exact identity relating λ and Λ. If we use this identity (4.3.19) results in

$$\begin{aligned} \Lambda^2 = \; & \lambda_1\lambda_2\lambda_3 + \left[\frac{1}{8}\Lambda^4 + \frac{3}{4}\Lambda^2 \fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2 + \frac{9}{8}\left(\fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2\right)^2\right] \\ & - \frac{3}{2}\fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2 + \det \nabla \tilde{u}_h(0)\left(Q_{V_3}(\tilde{w}) + \fint_{\mathbb{S}^2} \langle \tilde{w}, \partial_{\tau_1}\tilde{w} \wedge \partial_{\tau_2}\tilde{w}\rangle\right). \end{aligned} \tag{4.3.22}$$

Of course,

$$\lambda_1\lambda_2\lambda_3 \overset{AM-GM}{\leq} \left(\frac{\Lambda^2}{3}\right)^{\frac{3}{2}} \overset{(4.3.15)}{\leq} \frac{\theta}{3\sqrt{2}}\Lambda^2 \quad \text{and} \quad \frac{1}{8}\Lambda^4 \overset{(4.3.15)}{\leq} \frac{3\theta^2}{16}\Lambda^2. \tag{4.3.23}$$

By the *Poincare inequality* for the map $\tilde{u} - \nabla \tilde{u}_h(0)x$ whose linear part is vanishing, we get

$$\fint_{\mathbb{S}^2} |\tilde{u} - \nabla \tilde{u}_h(0)x|^2 \, d\mathcal{H}^2 \leq \frac{1}{6}\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - \nabla \tilde{u}_h(0)P_T|^2 \, d\mathcal{H}^2 \overset{(4.3.12)}{\leq} \frac{D(u)}{2},$$

so that

$$\frac{3}{4}\Lambda^2 \fint_{\mathbb{S}^2} |\tilde{u}-\nabla\tilde{u}_h(0)x|^2\, d\mathcal{H}^2 \leq \frac{3D(u)}{8}\Lambda^2 \quad\text{and}\quad \frac{9}{8}\left(\fint_{\mathbb{S}^2} |\tilde{u}-\nabla\tilde{u}_h(0)x|^2\, d\mathcal{H}^2\right)^2 \leq \frac{9}{32}[D(u)]^2.$$

Regarding the last summands, the first term in the second line of (4.3.22) is non-positive, while the *quadratic term in the expansion of the degree* can be easily estimated as in (4.1.24) (with a slightly better constant this time because the linear part of $\tilde{w}$ is vanishing), so that

$$\begin{aligned} |Q_{V_3}(\tilde{w})| &\leq \frac{3}{2\sqrt{2}}\fint_{\mathbb{S}^2} |\nabla_T\tilde{w}|^2\, d\mathcal{H}^2 \leq \frac{3}{2\sqrt{2}}\fint_{\mathbb{S}^2} \left|\nabla\tilde{u}_h(0)^{-1}(\nabla_T\tilde{u}-\nabla\tilde{u}_h(0)P_T)\right|^2\, d\mathcal{H}^2 \\ &\leq \frac{3|\nabla\tilde{u}_h(0)^{-1}|^2}{2\sqrt{2}}\fint_{\mathbb{S}^2} \left|\nabla_T\tilde{u}-\nabla\tilde{u}_h(0)P_T\right|^2\, d\mathcal{H}^2 \\ &\leq \frac{18}{\sqrt{2}}D(u), \end{aligned} \tag{4.3.24}$$

the last inequality following from (4.3.12) and (4.3.14).

The last term can be estimated by using again the functional form of the conformal-isoperimetric inequality (1.2.6) for $n=3$ and $\tilde{w}$, i.e.

$$\begin{aligned} \left|\fint_{\mathbb{S}^2} \langle\tilde{w}, \partial_{\tau_1}\tilde{w}\wedge\partial_{\tau_2}\tilde{w}\rangle\, d\mathcal{H}^2\right| &\leq \left(\frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T\tilde{w}|^2\, d\mathcal{H}^2\right)^{\frac{3}{2}} \\ &\leq \frac{\left|[\nabla\tilde{u}_h(0)]^{-1}\right|^3}{2\sqrt{2}}\left(\fint_{\mathbb{S}^2} \left|\nabla_T\tilde{u}-\nabla\tilde{u}_h(0)P_T\right|^2\right)^{\frac{3}{2}} \\ &\leq \frac{4\sqrt{27}}{\sqrt{2}}\left[D(u)\right]^{\frac{3}{2}} \\ &\leq \frac{4\sqrt{27(1+\theta^2)}}{\sqrt{2}}D(u). \end{aligned} \tag{4.3.25}$$

where we used again (4.3.12) and (4.3.14).

By plugging in all the estimates (4.3.23)-(4.3.25) into the identity (4.3.22) and keeping in mind (4.3.9), we obtain

$$\Lambda^2 \leq \left(\frac{\theta}{3\sqrt{2}}+\frac{3\theta^2}{16}+\frac{3D(u)}{8}\right)\Lambda^2 + \frac{3}{2}\left(\frac{18}{\sqrt{2}}+\frac{4\sqrt{27(1+\theta^2)}}{\sqrt{2}}\right)D(u)+\frac{9}{32}[D(u)]^2$$

$$\Longrightarrow \left(\frac{5}{8}-\frac{\theta}{3\sqrt{2}}-\frac{9\theta^2}{16}\right)\Lambda^2 \leq \left(\frac{27}{\sqrt{2}}+\frac{6\sqrt{27(1+\theta^2)}}{\sqrt{2}}+\frac{9}{32}(1+\theta^2)\right)D(u). \tag{4.3.26}$$

This final estimate is precisely (4.3.20), after choosing $\theta > 0$ sufficiently small to additionally satisfy for example

$$\frac{5}{8} - \frac{\theta}{3\sqrt{2}} - \frac{9\theta^2}{16} \geq \frac{1}{2} \quad \text{and then} \quad c_1 := 27\sqrt{2} + 6\sqrt{54(1+\theta^2)} + \frac{9}{16}(1+\theta^2).$$

Therefore, by combining (4.3.12) and (4.3.20) we conclude that

$$\begin{aligned}
&\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - R_0 P_T|^2 \leq 2\fint_{\mathbb{S}^2} |\nabla_T \tilde{u} - \nabla \tilde{u}_h(0) P_T|^2 + \frac{2}{3}\Lambda^2 \\
\Longrightarrow &\fint_{\mathbb{S}^2} |\nabla_T \big[(u - R_0\psi^{-1}) \circ \psi\big]|^2 \leq \left(6 + \frac{2}{3}c_1\right) D(u).
\end{aligned}$$

By the conformal invariance of the Dirichlet-energy in two dimensions we can rewrite the last estimate as desired, i.e.

$$\fint_{\mathbb{S}^2} |\nabla_T u - \nabla_T \phi|^2 \, d\mathcal{H}^2 \leq c D(u), \tag{4.3.27}$$

where $\phi := R_0\psi^{-1} \in Conf_+(\mathbb{S}^2)$, and $c := 6 + \frac{2}{3}c_1 > 0$.

Step 2. Arguing again by *contradiction*, suppose that the statement of the theorem is false. Then, for every $k \in \mathbb{N}$ there exists a map $u_k \in \mathcal{A}_{\mathbb{S}^2}$ with $D(u_k) > 0$ such that

$$\fint_{\mathbb{S}^2} |\nabla_T u_k - \nabla_T \phi|^2 \, d\mathcal{H}^2 \geq k D(u_k) \quad \text{for all } \phi \in Conf_+(\mathbb{S}^2). \tag{4.3.28}$$

In particular, for every $\phi \in Conf_+(\mathbb{S}^2)$ which we can fix for the following computation, and for every $k \geq 5$,

$$\begin{aligned}
kD(u_k) &\leq \fint_{\mathbb{S}^2} |\nabla_T u_k - \nabla_T \phi|^2 \, d\mathcal{H}^2 = \fint_{\mathbb{S}^2} \left|\nabla_T (u_k \circ \phi^{-1}) - P_T\right|^2 \, d\mathcal{H}^2 \\
&\leq 2\fint_{\mathbb{S}^2} \left(|\nabla_T (u_k \circ \phi^{-1})|^2 + |P_T|^2\right) \, d\mathcal{H}^2 \\
&= 2\fint_{\mathbb{S}^2} |\nabla_T u_k|^2 \, d\mathcal{H}^2 + 4 = 4D(u_k) + 8 \\
\Longrightarrow D(u_k) &\leq \frac{8}{k-4}, \qquad\qquad \text{for every } k \geq 5.
\end{aligned}$$

By letting $k \to \infty$ we obtain $\lim_{k\to\infty} D(u_k) = 0$. We can then use the *compactness* Lemma 4.3.1. and *Step 1* to obtain a contradicition as in the end of the proof of Theorem 3.1.2. or Theorem 3.2.3 and conclude. □

Outlook

Inspired from rigidity and stability results for isometric and conformal maps from open bounded subdomains of $\mathbb{R}^n$ into $\mathbb{R}^n$, in this thesis we studied similar features for maps defined on $\mathbb{S}^{n-1}$ and mapping into $\mathbb{R}^n$. We would also like to collect here some open questions that we mentioned throughout the thesis, that either originate from our study, or that the author finds interesting problems to be explored in general.

(1) Prove or disprove the *conjecture* of T. Iwaniec and G. Martin regarding the sharpness of the integrability exponent $\frac{n}{2}$ for the validity of *Liouville's theorem for conformal maps* in $W^{1,p}(U;\mathbb{R}^n)$ whenever $p \geq \frac{n}{2}$, also in *odd dimensions*. One can then explore the *approximate version* of the previous question (see Theorem 1.1.10. and the subsequent comments), which for the case of *even* $n \geq 4$ has been settled by S. Müller, V. Sverak and B. Yan in [MŠY99].

(2) Prove more general versions of the *local (with respect to the domains)* estimate (1.1.35) of D. Faraco and X. Zhong where the *compact, rotationally invariant, annuli-type subsets of* $CO_+(n)$ are replaced by more general subsets of $CO_+(n)$, or even $CO_+(n)$ itself if possible. A new PDE approach to this question would be very interesting.

(3) It would be interesting to find alternative proofs of the results of Section 3.2 that do not rely somehow on the stability of the L^2-Poincare inequality and can give also the L^p-version (with respect to the definition of the isometric deficit) of Theorem 3.2.3. for $1 \leq p < \infty$. By using the *standard truncation argument* that we described in Subsection 3.2.2, one should expect that the analogue of Theorem 3.2.3. for $p \geq n-1$ should be free of any hypothesis regarding apriori boundedness in some Sobolev norm, even when $n \geq 4$.

(4) Find intrinsic proofs of quantitative rigidity estimates for maps from $\mathbb{S}^{n-1}$ to itself, or to other closed embedded hypersurfaces in $\mathbb{R}^n$ with optimal exponent in the isometric deficit (see the comments in Remark 3.3.3.). Although not completely concrete, some questions in this direction could be the following.

(4_a) Give an intrinsic proof of (3.3.20) without the assumption made in Corollary 3.3.2., i.e. the *apriori closeness to the identity assumption.*

(4_b) Obtain appropriate generalizations for maps $u : \mathbb{S}^{n-1} \mapsto N^{n-1}$, where N^{n-1} is a closed embedded hypersurface in $\mathbb{R}^n$ that has *small isoperimetric deficit*, or satisfies some *appropriate curvature condition*.

(4_c) Try to obtain general quantitative estimates for orientation-preserving isometries between orientable Riemannean manifolds of the same dimension (see [KMS19] in this respect).

(5) Prove more *global in nature results* with respect to the *combined conformal-isoperimetric deficit*, starting from the local stability results for conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$ ($n \geq 3$) that were presented in Chapter 4. Explore the analogues of the questions (4_b), (4_c) in the conformal setting.

Appendix A

A generalized isoperimetric inequality for maps on $\mathbb{S}^{n-1}$

We would like to give here a proof of *a generalized version of the isoperimetric inequality in functional form*, that we mentioned and used in the main body of the thesis, namely

Lemma A.0.1. *Let $n \geq 2$, $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$. Then the following inequality holds,*

$$\left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \right| \leq \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}. \tag{A.0.1}$$

As we have mentioned, it is exactly because of this inequality that the integral on the left hand side is finite for maps in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$. The interested reader is refered to [NB11], [DP12] and the references therein for related results and details concerning the regularity assumptions under which the *signed-volume-functional* is finite for maps defined on domains of $\mathbb{R}^{n-1}$ and mapping into $\mathbb{R}^n$. Of course, if the map $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is an *embedding*, then (A.0.1) is the *classical Euclidean isoperimetric inequality* for the open bounded set E_u in $\mathbb{R}^n$ with $\partial E_u = u(\mathbb{S}^{n-1})$. Here, we simply want to mention how (without refering to *Almgren's general isoperimetric inequality for integral currents*) Lemma A.0.1. is a simple consequence of the following *generalized isoperimetric inequality* due to S. Müller.

Lemma A.0.2 (S. Müller, [Mü90], Lemma 1.3). *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, Lipschitz domain, let $v \in W^{1,n}(\Omega;\mathbb{R}^n)$, let $x \in \Omega$ and let $R < \mathrm{dist}(x, \partial\Omega)$. Then, for a.e. $r \in (0,R)$*

$$\left| \int_{B_r(x)} \det Dv \, dy \right|^{\frac{n-1}{n}} \leq C \int_{\partial B_r(x)} |\mathrm{adj} Dv| \, d\mathcal{H}^{n-1}, \tag{A.0.2}$$

where the constant C depends only on n.

By a careful look at the proof, which is given in Section 3 in [Mü90] and relies on a *degree argument* and the classical *Sobolev embedding in* $BV(\mathbb{R}^n)$, one can verify that the inequality holds for the same constant as the classical isoperimetric inequality, i.e. for $C = n^{-1}\omega_n^{-\frac{1}{n}}$, although this might not be the optimal one for (A.0.2).

Let us see how we can obtain (A.0.1). Let $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$. By *a standard density argument* and *Fatou's lemma* we can assume without loss of generality that $u \in C^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)$. Indeed, if $(u_k)_{k\in\mathbb{N}} \in C^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)$ is such that $u_k \to u$ strongly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ and up to subsequences $u_k \to u$, $\nabla_T u_k \to \nabla_T u$ pointwise $\mathcal{H}^{n-1}$-a.e., then $|V_n(u)| \le \liminf_{k\to\infty} |V_n(u_k)|$, $P_{n-1}(u) = \lim_{k\to\infty} P_{n-1}(u_k)$. We can then extend u in B^n in a small (one sided) annular neighbourhood around $\mathbb{S}^{n-1}$ as follows.

Given $0 \le \delta \ll 1$, let $\phi_\delta \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ be a smooth cut-off such that $0 \le \phi_\delta \le 1$, $\phi_\delta \equiv 1$ in $\mathcal{U}_\delta(\mathbb{S}^{n-1})$, $\mathrm{spt}\phi_\delta \subset\subset \mathcal{U}_{2\delta}(\mathbb{S}^{n-1})$ and then extend u to $U_\delta : \overline{B^n} \mapsto \mathbb{R}^n$ by setting

$$U_\delta(y) := \left\{ \begin{array}{ll} \phi_\delta(y)u\big(\frac{y}{|y|}\big); & y \neq 0, \\ 0; & y = 0 \end{array} \right\}.$$

In particular, $U_\delta \in C^\infty(\overline{B^n};\mathbb{R}^n)$ and in polar coordinates $U_\delta(r,\theta) = u(\theta) \ \ \forall\, r \in [1-\delta, 1]$ and $\theta \in \mathbb{S}^{n-1}$, $U_\delta(r,\theta) = 0 \ \ \forall\, r \in [0, 1-2\delta]$ and $\theta \in \mathbb{S}^{n-1}$. Therefore (A.0.2) applies to U_δ in B^n with the constant $C = n^{-1}\omega_n^{-\frac{1}{n}}$, and can be rewritten as

$$\left| \fint_{B^n} \mathrm{det}\nabla U_\delta \, dx \right| \le \left(\fint_{\mathbb{S}^{n-1}} |\mathrm{adj}\nabla U_\delta| \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}. \tag{A.0.3}$$

The last inequality is precisely (A.0.1). Indeed, recalling (1.2.2), i.e. the property of the *Jacobian determinant being a null-Lagrangian*, for the left hand side we have that

$$\fint_{B^n} \mathrm{det}\nabla U_\delta \, dx = \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle d\mathcal{H}^{n-1} \quad \text{since } U_\delta \equiv u \ \text{ on } \ \mathbb{S}^{n-1},$$

while for the right hand side the two integrands agree pointwise.

Indeed, since the extension U_δ has *no radial derivative* on $\mathbb{S}^{n-1}$, for every $i, j = 1, \dots, n$ we have pointwise on $\mathbb{S}^{n-1}$,

$$(\nabla U_\delta)_{ij} = \sum_{m=1}^{n-1} \langle \nabla U_\delta^i, \tau_m \rangle \langle e_j, \tau_m \rangle + \partial_{\vec{\nu}} U_\delta^i \langle e_j, x \rangle = \sum_{m=1}^{n-1} \langle \nabla_T u^i, \tau_m \rangle \langle e_j, \tau_m \rangle = \big(\nabla_T u P_T^t\big)_{ij},$$

i.e.

$$\nabla U_\delta = \nabla_T u P_T^t = \Big(\nabla_T u \Big| 0\Big) \cdot \begin{pmatrix} P_T^t \\ 0 \end{pmatrix} \quad \text{on } \ \mathbb{S}^{n-1},$$

where (with respect to the local orthonormal coordinates $\{\tau_1, \dots, \tau_{n-1}\}$) we have *augmented* the $n \times (n-1)$- matrix $\nabla_T u$ to an $n \times n$-matrix with an extra column of zeros, and the $(n-1)\times n$-matrix P_T^t to an $n\times n$-matrix with an extra row of zeros. By a simple fact from linear algebra,

$$\mathrm{adj}\nabla U_\delta = \mathrm{adj}\left(\frac{P_T^t}{0}\right)\cdot \mathrm{adj}\left(\nabla_T u\Big|0\right) = \mathrm{cof}\left(P_T|0\right)\cdot\mathrm{cof}\left(\frac{\nabla_T u^t}{0}\right) \quad \text{on} \quad \mathbb{S}^{n-1}.$$

It is easy to check that for every $i,j=1,\dots,n$

$$\left[\mathrm{cof}\left(P_T|0\right)\right]_{ij} = (-1)^{i+n}\delta^{nj}\det(\langle e_{\bar{i}},\tau_l\rangle), \quad \left[\mathrm{cof}\left(\frac{\nabla_T u^t}{0}\right)\right]_{ij} = (-1)^{n+j}\delta^{ni}\det(\langle\nabla_T u^{\bar{j}},\tau_m\rangle),$$

where $(\langle e_{\bar{i}},\tau_l\rangle)$ is the $(n-1)\times(n-1)$ minor of P_T with the i-th row ommited and similarly for $(\langle\nabla_T u^{\bar{j}},\tau_m\rangle)$. Multiplying these two matrices, we get

$$(\mathrm{adj}\nabla U_\delta)_{ij} = (-1)^{i+j}\left[\det(\langle e_{\bar{i}},\tau_l\rangle)\right]\left[\det(\langle\nabla_T u^{\bar{j}},\tau_m\rangle)\right].$$

Finally,

$$\begin{aligned}
|\mathrm{adj}\nabla U_\delta| &= \left(\sum_{i,j=1}^n(\mathrm{adj}\nabla U_\delta)_{ij}^2\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^n\left[\det(\langle e_{\bar{i}},\tau_l\rangle)\right]^2\left[\det(\langle\nabla_T u^{\bar{j}},\tau_m\rangle)\right]^2\right)^{\frac{1}{2}}\\
&= \left(\sum_{i=1}^n\left[\det(\langle e_{\bar{i}},\tau_l\rangle)\right]^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^n\left[\det(\langle\nabla_T u^{\bar{j}},\tau_m\rangle)\right]^2\right)^{\frac{1}{2}}\\
&= \sqrt{\det(P_T^tP_T)}\cdot\sqrt{\det(\nabla_T u^t\nabla_T u)} = \sqrt{\det(I_x)}\sqrt{\det(\nabla_T u^t\nabla_T u)}\\
&= \sqrt{\det(\nabla_T u^t\nabla_T u)}
\end{aligned}$$

and (A.0.1) follows. In the passage from the second to the third line in the previous chain of equalities we have applied the *Cauchy-Binet formula*, according to which for any $n\times d$-matrix A,

$$\det(A^tA) = \sum_{\tilde{A}}(\det\tilde{A})^2, \tag{A.0.4}$$

the sum being taken over all $d\times d$ *minors* $\tilde{A}$ of A.

Appendix B

Spherical Harmonics

It is well known that the Hilbert space $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ *admits an orthonormal basis consisting of eigenfunctions of the Laplace-Beltrami operator*. In particular, for every $k \in \mathbb{N}$ there exists a finite number (denoted by $G(n,k)$) of linearly independent maps $(\psi_{n,k,j})_{j=1,2,...,G(n,k)}$, which are called the *k-th order spherical harmonics*, are restrictions on $\mathbb{S}^{n-1}$ of ($\mathbb{R}^n$-valued) homogeneous harmonic polynomials in $\mathbb{R}^n$ of degree k respectively and enjoy the following properties.

(*i*) For every $k, k' \in \mathbb{N},\ j = 1,2,...,G(n,k),\ j' = 1,2,...,G(n,k')$,

$$\int_{\mathbb{S}^{n-1}} \langle \psi_{n,k,j}, \psi_{n,k',j'} \rangle \ d\mathcal{H}^{n-1} = \delta^{kk'}\delta^{jj'} \tag{B.0.1}$$

(*ii*) For every $k \in \mathbb{N}$, $j = 1,2,...,G(n,k)$

$$-\Delta_{\mathbb{S}^{n-1}}\psi_{n,k,j} = \lambda_{n,k}\psi_{n,k,j}, \text{ where } \lambda_{n,k} := k(k+n-2), \tag{B.0.2}$$

or, in distributional formulation, for every $\phi \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$

$$\int_{\mathbb{S}^{n-1}} \langle \nabla_T\psi_{n,k,j}, \nabla_T\phi \rangle \ d\mathcal{H}^{n-1} = \lambda_{n,k}\int_{\mathbb{S}^{n-1}} \langle \psi_{n,k,j}, \phi \rangle \ d\mathcal{H}^{n-1}. \tag{B.0.3}$$

The dimension of each eigenspace in the scalar case is $G(n,0) = 1,\ G(n,1) = n$, and for $k \geq 2$ it is $G(n,k) = \binom{n+k-1}{k}$-$\binom{n+k-3}{k-2}$. The reader can refer to [Gro96] for more information on spherical harmonics.

Remark B.0.1. For every vector field $u := (u^1,\dots,u^n) \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$ we have a formal expansion of each one of its components into a Fourier series as

$$u^i = \sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)} a^i_{n,k,j}\psi_{n,k,j}, \quad \text{where } a^i_{n,k,j} := \int_{\mathbb{S}^{n-1}} u^i\psi_{n,k,j} \ d\mathcal{H}^{n-1},$$

where now by an abuse of notation, $(\psi_{n,k,j})_{k\geq 0, j\in G(n,k)}$ are the scalar spherical harmonics. Let $P_{n,k,j}$ be the k-th homogeneous harmonic polynomial in $\mathbb{R}^n$ whose restriction on $\mathbb{S}^{n-1}$ is $\psi_{n,k,j}$. In polar coordinates $(r,\theta)\in[0,\infty)\times\mathbb{S}^{n-1}$, we can write these polynomials as $P_{n,k,j}(r,\theta)=r^k\psi_{n,k,j}(\theta)$.

For each $i=1,\dots,n$, the harmonic extension u_h^i has the same power series expansion in the interior of the unit ball, namely

$$u_h^i=\sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)}a_{n,k,j}^iP_{n,k,j}\ \text{ in } B^n.$$

If the vector field u has zero average on $\mathbb{S}^{n-1}$, then

$$u_h^i(0)=\fint_{\mathbb{S}^{n-1}}u^i\ d\mathcal{H}^{n-1}=0\ \text{ for every } i=1,2,\dots,n.$$

In view of the homogeneity of $P_{n,k,j}$ this is equivalent to $a_{n,0}^i=0$ for all $i=1,2,\dots,n$.

Another immediate but as we saw useful observation, which is based on the fact that *the first order spherical harmonics are the coordinate functions* $\psi_{n,1,j}(\theta)=\frac{\theta_j}{\sqrt{\omega_n}}$, is that the *linear part of* u is given by the linear map $x\mapsto\nabla u_h(0)x$.

Remark B.0.2. The following *Parseval identities on* $\mathbb{S}^{n-1}$ hold true: If $\phi\in W^{1,2}(\mathbb{S}^{n-1})$ with its Fourier expansion in spherical harmonics being $\phi=\sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)}a_{n,k,j}\psi_{n,k,j}$, then

$$\int_{\mathbb{S}^{n-1}}|\phi|^2=\sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)}(a_{n,k,j})^2\quad\text{and}\quad\int_{\mathbb{S}^{n-1}}|\nabla_T\phi|^2=\sum_{k=1}^{\infty}\sum_{j=1}^{G(n,k)}\lambda_{n,k}(a_{n,k,j})^2. \tag{B.0.4}$$

In particular, for every $k\in\mathbb{N}$ and every $j=1,2,...,G(n,k)$, we have the identity

$$\int_{\mathbb{S}^{n-1}}|\nabla_T\psi_{n,k,j}|^2\ d\mathcal{H}^{n-1}=\lambda_{n,k}\int_{\mathbb{S}^{n-1}}|\psi_{n,k,j}|^2\ d\mathcal{H}^{n-1}. \tag{B.0.5}$$

Remark B.0.3. The *sharp Poincare inequality* for functions $f\in W^{1,2}(\mathbb{S}^{n-1})$ is then easily deduced. Let $f=\sum_{k=0}^{\infty}\sum_{j=1}^{G(n,k)}f_{n,k,j}\psi_{n,k,j}$. Since $\lambda_{n,k}\geq n-1$ for every $k\geq 1$, we obtain

$$\int_{\mathbb{S}^{n-1}}|\nabla_T f|^2\geq(n-1)\sum_{k=1}^{\infty}\sum_{j=1}^{G(n,k)}(f_{n,k,j})^2=(n-1)\int_{\mathbb{S}^{n-1}}\left|f-\fint_{\mathbb{S}^{n-1}}f\right|^2. \tag{B.0.6}$$

Of course, depending on the number of vanishing first *Fourier modes* in the expansion of f the constant in the above inequality can be improved in an obvious way. Obviously, the *Poincare inequality* holds then true also for vector-valued maps $u\in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^m)$.

By expanding a function in spherical harmonics one can often obtain useful estimates. In the next remark we mention two of them that we have used earlier.

Remark B.0.4. If $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^m)$ and $u_h : \overline{B} \mapsto \mathbb{R}^m$ is as usual its harmonic extension, the following estimates hold true:

(i) $⨍_{B^n} |\nabla u_h|^2 \, dx \leq \frac{n}{n-1} ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1}$,

(ii) $\frac{n}{n-1} ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} \leq ⨍_{\mathbb{S}^{n-1}} |\nabla u_h|^2 \, d\mathcal{H}^{n-1} \leq 2 ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1}$.

Let us give the proof of these two simple estimates in the case that u is scalar-valued, the case of vector-valued u being an immediate consequence. We write again

$$u = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} a_{n,k,j} \psi_{n,k,j},$$

and its harmonic extension in polar coordinates as

$$u_h(r, \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} r^k a_{n,k,j} \psi_{n,k,j}(\theta).$$

For the first estimate, we have

$$\begin{aligned}
⨍_{B^n} |\nabla u_h|^2 \, dx &= ⨍_{B^n} \operatorname{div}(u_h \nabla u_h) \, dx = n ⨍_{\mathbb{S}^{n-1}} u \, \partial_{\vec{\nu}} u_h \, d\mathcal{H}^{n-1} \\
&= \sum_{k=0}^{\infty} \sum_{j=1}^{G(n,k)} nk (a_{n,k,j})^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \frac{n \lambda_{n,k}}{k+n-2} (a_{n,k,j})^2 \\
&\leq \frac{n}{n-1} \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \lambda_{n,k} (a_{n,k,j})^2 = \frac{n}{n-1} ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1},
\end{aligned}$$

while for the second one, we have

$$\begin{aligned}
⨍_{\mathbb{S}^{n-1}} |\nabla u_h|^2 \, d\mathcal{H}^{n-1} &= ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + ⨍_{\mathbb{S}^{n-1}} |\partial_{\vec{\nu}} u_h|^2 \, d\mathcal{H}^{n-1} \\
&= ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} k^2 (a_{n,k,j})^2 \\
&= ⨍_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \, d\mathcal{H}^{n-1} + \sum_{k=1}^{\infty} \sum_{j=1}^{G(n,k)} \frac{k}{k+n-2} \lambda_{n,k} (a_{n,k,j})^2,
\end{aligned}$$

and since $\frac{1}{n-1} \leq \frac{k}{k+n-2} \leq 1$ for every $k \geq 1$, the desired estimate follows immediately.

Appendix C

Taylor expansions of the deficits and proof of Korn's identity

In this Appendix we calculate in detail the *Taylor expansions* up to second order of the geometric quantities that we used in the main body of the thesis. The computations presented here are *formal* and we assume without further clarification that the maps in question are always regular enough so that we can perform the expansions. For such a map $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ we set as always $w := u - \mathrm{id}_{\mathbb{S}^{n-1}}$.

The *isometric deficit* of u can be expanded as

$$\delta_u^2 := \fint_{\mathbb{S}^{n-1}} \left| \sqrt{\nabla_T u^t \nabla_T u} - I_x \right|^2 \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \left| \sqrt{(\nabla_T w^t + P_T^t)(\nabla_T w + P_T)} - I_x \right|^2 \, d\mathcal{H}^{n-1}$$

$$= \fint_{\mathbb{S}^{n-1}} \left| \sqrt{I_x + P_T^t \nabla_T w + (P_T^t \nabla_T w)^t + \nabla_T w^t \nabla_T w} - I_x \right|^2 \, d\mathcal{H}^{n-1}$$

Formally,

$$\sqrt{I_x + P_T^t \nabla_T w + (P_T^t \nabla_T w)^t + \nabla_T w^t \nabla_T w} = I_x + \frac{1}{2} \left(P_T^t \nabla_T w + (P_T^t \nabla_T w)^t \right) + \mathcal{O}(|\nabla_T w|^2),$$

so that

$$\begin{aligned} \delta_u^2 &= \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} + \mathcal{O}(|\nabla_T w|^2) \right|^2 \, d\mathcal{H}^{n-1} \\ &= \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2 \, d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3) \, d\mathcal{H}^{n-1}. \end{aligned}$$

Therefore, the quadratic term appearing in the expansion of the *isometric deficit* δ_u around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is

$$Q_{n,\mathrm{isom}}(w) := \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2 \, d\mathcal{H}^{n-1}.$$

For the expansion of *the generalized* $(n-1)$*-Dirichlet energy-term* of u around the $\mathrm{id}_{\mathbb{S}^{n-1}}$, we had seen in detail the computation in the case $n=3$ in Subsection 4.1.1. For $n \geq 4$,

$$
\begin{aligned}
D_{n-1}(u) &:= \left(\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}\right)^{\frac{n}{n-1}} = \left(\fint_{\mathbb{S}^{n-1}} \left(1 + \frac{2}{n-1}\mathrm{div}_{\mathbb{S}^{n-1}} w + \frac{|\nabla_T w|^2}{n-1}\right)^{\frac{n-1}{2}}\right)^{\frac{n}{n-1}} \\
&= \left(\fint_{\mathbb{S}^{n-1}} \left(1 + \mathrm{div}_{\mathbb{S}^{n-1}} w + \frac{1}{2}|\nabla_T w|^2 + \frac{1}{2}\frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 + \mathcal{O}(|\nabla_T w|^3)\right)\right)^{\frac{n}{n-1}} \\
&= \left[1 + \fint_{\mathbb{S}^{n-1}} \left((n-1)\langle w, x\rangle + \frac{1}{2}|\nabla_T w|^2 + \frac{1}{2}\frac{n-3}{n-1}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\right) + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3))\right]^{\frac{n}{n-1}} \\
&= 1 + n\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle \, d\mathcal{H}^{n-1} + \frac{n}{2(n-1)}\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \, d\mathcal{H}^{n-1} \\
&\quad + \frac{n(n-3)}{2(n-1)^2}\fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \, d\mathcal{H}^{n-1} + \frac{n}{2}\left(\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle\right)^2 \\
&\quad + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3) + \mathcal{O}\left(\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\right)^2 + \left|\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle\right| \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2\right).
\end{aligned}
$$

Therefore, the quadratic term appearing in the expansion of $D_{n-1}(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is

$$
Q_{D_{n-1}}(w) := \frac{n}{2}\left[\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2}{n-1} + \frac{n-3}{(n-1)^2}(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\right) + \left(\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle\right)^2\right].
$$

Notice that in Subsection 4.2.1 we had already translated and scaled the initial map u properly, so that the map w was satisfying $\fint_{\mathbb{S}^{n-1}} w = 0$, $\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle = 0$. Thus, the last term in $Q_{D_{n-1}}(w)$ was dropping out and also the structure of the higher order terms was simplifying.

For the expansion of *the generalized perimeter-term* around the $\mathrm{id}_{\mathbb{S}^{n-1}}$, we have

$$
\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \sqrt{\det(I_x + A)} \, d\mathcal{H}^{n-1},
$$

where $A := P_T^t \nabla_T w + (P_T^t \nabla_T w)^t + \nabla_T w^t \nabla_T w$. The Taylor expansion of the determinant around the identity matrix gives,

$$
\det(I + A) = 1 + \mathrm{Tr}A + \frac{1}{2}\Big((\mathrm{Tr}A)^2 - \mathrm{Tr}(A^2)\Big) + \mathcal{O}(|A|^3)
$$

and since in our case,

(a) $\mathrm{Tr}A = 2\mathrm{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2$

(b) $(\mathrm{Tr}A)^2 = 4(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 + \mathcal{O}(|\nabla_T w|^3)$

(c) $\mathrm{Tr}(A^2) = |P_T^t \nabla_T w + (P_T^t \nabla_T w)^t|^2 + \mathcal{O}(|\nabla_T w|^3)$,

we obtain the formal expansion

$$\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\; d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \sqrt{1 + \Theta(w) + \mathcal{O}(|\nabla_T w|^3)}\; d\mathcal{H}^{n-1},$$

where

$$\Theta(w) := 2\mathrm{div}_{\mathbb{S}^{n-1}} w + |\nabla_T w|^2 + 2(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 - 2\left|\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2}\right|^2.$$

Since $(\Theta(w))^2 = 4(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 + \mathcal{O}(|\nabla_T w|^3)$, we can perform a Taylor expansion of the square root inside the integral to get

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\; d\mathcal{H}^{n-1} &= \fint_{\mathbb{S}^{n-1}} \left(1 + \frac{1}{2}\Theta(w) - \frac{1}{8}(\Theta(w))^2 + \mathcal{O}\left(|\nabla_T w|^3\right)\right) \\
&= 1 + (n-1)\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle + \frac{1}{2}\fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2\right) \\
&\quad - \fint_{\mathbb{S}^{n-1}} \left|\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2}\right|^2 + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right).
\end{aligned}$$

A final Taylor expansion of the function $t \mapsto t^{\frac{n}{n-1}}$ gives,

$$\begin{aligned}
P_{n-1}(u) :&= \left(\fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)}\; d\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}} \\
&= 1 + n\fint_{\mathbb{S}^{n-1}} \langle w, x\rangle\; d\mathcal{H}^{n-1} + \frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2}{2} + \frac{(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{2}\right) d\mathcal{H}^{n-1} \\
&\quad - \frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left|\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2}\right|^2 d\mathcal{H}^{n-1} + \fint_{\mathbb{S}^{n-1}} \mathcal{O}\left(|\nabla_T w|^3\right)\; d\mathcal{H}^{n-1}.
\end{aligned}$$

The quadratic term appearing in the expansion of $P_{n-1}(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is therefore

$$Q_{P_{n-1}}(w) := \frac{n}{n-1}\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2}{2} + \frac{(\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{2} - \left|\frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2}\right|^2\right) d\mathcal{H}^{n-1}.$$

For the expansion of *the generalized signed-volume-term* around the $\mathrm{id}_{\mathbb{S}^{n-1}}$, we can argue as in Subsection 4.1.1. An *intrinsic way* to perform the calculation is the following.

$$
\begin{aligned}
V_n(u) := & \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \, d\mathcal{H}^{n-1} \; = \; \fint_{\mathbb{S}^{n-1}} \left\langle w + x, \bigwedge_{i=1}^{n-1} \left(\partial_{\tau_i} w + \partial_{\tau_i} x \right) \right\rangle \, d\mathcal{H}^{n-1} \\
= & \fint_{\mathbb{S}^{n-1}} \sum_{k=0}^{n-1} \sum_{|\alpha|=k} \sigma(\alpha, \bar{\alpha}) \left\langle w + x, \left(\bigwedge_{\alpha} \partial_{\tau_\alpha} w \right) \wedge \left(\bigwedge_{\bar{\alpha}} \partial_{\tau_{\bar{\alpha}}} x \right) \right\rangle \, d\mathcal{H}^{n-1} \\
= & \, I_0(w) + I_1(w) + I_2(w) + I_3(w).
\end{aligned}
$$

Here we have used standard multiindex notation. For every $k \in \{0, 1, ..., n-1\}$ and for every multiindex $\alpha := (\alpha_1, ..., \alpha_k)$, where $(a_i)_{i=1}^{k} \in \mathbb{N}$ with $1 \leq \alpha_1 < ... < \alpha_k \leq n-1$, we denote $\bar{\alpha}$ its complementary multiindex (with its entries also in increasing order), $\sigma(\alpha, \bar{\alpha})$ the sign of the permutation that maps $(\alpha, \bar{\alpha})$ to the standard ordering $(1, ..., n)$ and $\partial_{\tau_\alpha} w := \partial_{\tau_{\alpha_1}} w \wedge ... \wedge \partial_{\tau_{\alpha_k}} w$. We have also denoted by $(I_i(w))_{i=0,1,2}$ the zeroth, first and second order terms with respect to w and $\nabla_T w$ in the expansion of $V_n(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ respectively and by $I_3(w)$ the remaining term which is a *polynomial* of order at least 3 and at most n in w and its first derivatives. Keeping in mind that $\partial_{\tau_i} x = \tau_i$ for $i = 1, ..., n-1$ and that by an abuse of notation $\tau_1 \wedge \tau_2 \wedge ... \wedge \tau_{n-1} \equiv x$, we can compute each term separately.

$$
I_0(w) : \; = \; \fint_{\mathbb{S}^{n-1}} \langle x, \partial_{\tau_1} x \wedge ... \wedge \partial_{\tau_{n-1}} x \rangle \, d\mathcal{H}^{n-1} \; = \; \fint_{\mathbb{S}^{n-1}} |x|^2 \, d\mathcal{H}^{n-1} \; = \; 1.
$$

$$
\begin{aligned}
I_1(w) : \; = \; & \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle + \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle x, \left(\bigwedge_{l=1}^{i-1} \tau_l \right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{m=i+1}^{n-1} \tau_m \right) \right\rangle \\
= & \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \; + \fint_{\mathbb{S}^{n-1}} \sum_{i=1}^{n} \langle \partial_{\tau_i} w, \tau_i \rangle = \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle + \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \\
= & \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} + (n-1) \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1} \\
= & \, n \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle \, d\mathcal{H}^{n-1}.
\end{aligned}
$$

For the quadratic term, we observe that we can write it as $I_2(w) := I_{2,1}(w) + I_{2,2}(w)$, where

$$I_{2,1}(w) := \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle w, \left(\bigwedge_{l=1}^{i-1} \partial_{\tau_l} x\right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{m=i+1}^{n-1} \partial_{\tau_m} x\right)\right\rangle \, d\mathcal{H}^{n-1}$$

$$= \sum_{i=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \left\langle w, \left(\bigwedge_{l=1}^{i-1} \tau_l\right) \wedge \left(\sum_{j=1}^{n-1} \langle \partial_{\tau_i} w, \tau_j\rangle \tau_j + \langle \partial_{\tau_i} w, x\rangle x\right) \wedge \left(\bigwedge_{m=i+1}^{n-1} \tau_m\right)\right\rangle$$

$$= \fint_{\mathbb{S}^{n-1}} \mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle \, d\mathcal{H}^{n-1} - \fint_{\mathbb{S}^{n-1}} \sum_{i=1}^{n-1} \langle w, \tau_i\rangle \langle \partial_{\tau_i} w, x\rangle \, d\mathcal{H}^{n-1}$$

$$= \fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^{n-1}.$$

The change of sign in the one before the last equality is due to orientation reasons, since we have taken the local orthonormal basis $\{\tau_1, ..., \tau_{n-1}\}$ of $T_x \mathbb{S}^{n-1}$ in such a way that at every $x \in \mathbb{S}^{n-1}$ the set of vectors $\{\tau_1(x), ..., \tau_{n-1}(x), x\}$ is a positively oriented frame of $\mathbb{R}^n$. Moreover,

$$I_{2,2}(w) := \fint_{\mathbb{S}^{n-1}} \sum_{1 \le i < j \le n-1} \left\langle x, \left(\bigwedge_{k=1}^{i-1} \partial_{\tau_k} x\right) \wedge \partial_{\tau_i} w \wedge \left(\bigwedge_{l=i+1}^{j-1} \partial_{\tau_l} x\right) \wedge \partial_{\tau_j} w \wedge \left(\bigwedge_{m=j+1}^{n-1} \partial_{\tau_m} x\right)\right\rangle$$

$$= \frac{1}{2} \fint_{\mathbb{S}^{n-1}} \sum_{1 \le i, j \le n-1} \big(\langle \partial_{\tau_i} w, \tau_i\rangle \langle \partial_{\tau_j} w, \tau_j\rangle - \langle \partial_{\tau_i} w, \tau_j\rangle \langle \partial_{\tau_j} w, \tau_i\rangle\big) \, d\mathcal{H}^{n-1}.$$

After integrating by parts it is easy to see that the first term is

$$\fint_{\mathbb{S}^{n-1}} \sum_{1 \le i, j \le n-1} \big\langle \partial_{\tau_i} w, \tau_i\big\rangle \big\langle \partial_{\tau_j} w, \tau_j\big\rangle \, d\mathcal{H}^{n-1} = \fint_{\mathbb{S}^{n-1}} \big\langle w, (n-1)(\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \nabla_T \mathrm{div}_{\mathbb{S}^{n-1}} w\big\rangle \, d\mathcal{H}^{n-1},$$

while

$$\fint_{\mathbb{S}^{n-1}} \sum_{1 \le i, j \le n-1} \big\langle \partial_{\tau_i} w, \tau_j\big\rangle \big\langle \partial_{\tau_j} w, \tau_i\big\rangle = \fint_{\mathbb{S}^{n-1}} \big\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \nabla_T \mathrm{div}_{\mathbb{S}^{n-1}} w + (n-2) \sum_{j=1}^{n} x_j \nabla_T w^j\big\rangle.$$

Subtracting these two identities, we arrive at

$$I_{2,2}(w) := \Big(\frac{n}{2} - 1\Big) \fint_{\mathbb{S}^{n-1}} \left\langle w, \ (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^{n-1}.$$

Therefore, the quadratic term appearing in the expansion of $V_n(u)$ around the $\mathrm{id}_{\mathbb{S}^{n-1}}$ is

$$
\begin{aligned}
Q_{V_n}(w) := I_2(w) &= \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \left\langle w, (\mathrm{div}_{\mathbb{S}^{n-1}} w)x - \sum_{j=1}^{n} x_j \nabla_T w^j \right\rangle \, d\mathcal{H}^{n-1} \\
&= \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \left(2 \,\mathrm{div}_{\mathbb{S}^{n-1}} w \langle w, x\rangle - n\langle w, x\rangle^2 + |w|^2 \right) \, d\mathcal{H}^{n-1} \\
&= \frac{1}{2} \fint_{B^n} \left((\mathrm{div} w_h)^2 - \mathrm{Tr}(\nabla w_h)^2 \right) dx.
\end{aligned}
$$

The identity between the first and the second line above follows from a simple integration by parts, and the one between the third and the first line was justified in Subsection 4.1.1. By the same procedure that we followed to calculate $I_2(w)$ one can also obtain the algebraic structure of *higher order terms* in the expansion which was described in Lemma 4.2.2..

Let us conclude by giving a proof of *Korn's identity on* $\mathbb{S}^{n-1}$ which was mentioned in Remark 4.2.8..

Proof of Korn's identity. With the notation we introduced before, we have

$$
\begin{aligned}
\fint_{\mathbb{S}^{n-1}} \mathrm{Tr}\left((P_T^t \nabla_T w)^2\right) &= \sum_{i,j=1,\ k,l=1}^{n-1,n} \fint_{\mathbb{S}^{n-1}} \langle e_k, \tau_i\rangle \langle e_l, \tau_j\rangle \langle \nabla_T w^k, \tau_j\rangle \langle \nabla_T w^l, \tau_i\rangle \\
&= \sum_{i,j=1}^{n-1} \fint_{\mathbb{S}^{n-1}} \langle \partial_{\tau_i} w, \tau_j\rangle \langle \partial_{\tau_j} w, \tau_i\rangle \ d\mathcal{H}^{n-1} \\
&= \fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \ d\mathcal{H}^{n-1} - 2 I_{2,2}(w) \\
&= \fint_{\mathbb{S}^{n-1}} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \ d\mathcal{H}^{n-1} - \frac{2(n-2)}{n} Q_{V_n}(w).
\end{aligned}
$$

Therefore,

$$
\fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2 = \frac{1}{2} \fint_{\mathbb{S}^{n-1}} |P_T^t \nabla_T w|^2 + \frac{1}{2} \fint_{\mathbb{S}^{n-1}} \mathrm{Tr}\left((P_T^t \nabla_T w)^2\right)
$$

and the identity follows since $\fint_{\mathbb{S}^{n-1}} |P_T^t \nabla_T w|^2 = \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 - \fint_{\mathbb{S}^{n-1}} \left| \sum_{j=1}^n x_j \nabla_T w^j \right|^2$. $\square$

Bibliography

[Bec93] William Beckner. *Sharp Sobolev Inequalities on the Sphere and the Moser-Trudinger Inequality. Annals of Mathematics*, pages 213–242, 1993.

[BMMS] Anne Bernand-Mantel, Cyrill B. Muratov, and Theresa M. Simon. *A quantitative description of skyrmions in ultrathin ferromagnetic films and rigidity of degree±1 harmonic maps from* $\mathbb{R}^2$ *to* $\mathbb{S}^2$. *arXiv preprint, arXiv:1912.09854.*

[BN95] Haım Brezis and Louis Nirenberg. *Degree theory and BMO; part I: Compact manifolds without boundaries. Selecta Mathematica New Series*, 1(2):197–264, 1995.

[CDM14] Sergio Conti, Georg Dolzmann, and Stefan Müller. *Korn's second inequality and geometric rigidity with mixed growth conditions. Calculus of Variations and Partial Differential Equations*, 50(1-2):437–454, 2014.

[Cia88] Philippe G. Ciarlet. *Three-dimensional elasticity*, volume 20. Elsevier, 1988.

[CJ02] Wenyi Chen and Jürgen Jost. *A Riemannian version of Korn's inequality. Calculus of Variations and Partial Differential Equations*, 14(4):517–530, 2002.

[CS06] Sergio Conti and Ben Schweizer. *Rigidity and Gamma convergence for solid-solid phase transitions with* $SO(2)$ *invariance. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 59(6):830–868, 2006.

[DEKL14] Jean Dolbeault, Maria J. Esteban, Michael Kowalczyk, and Michael Loss. *Sharp interpolation inequalities on the sphere: new methods and consequences.* In *Partial Differential Equations: Theory, Control and Approximation*, pages 225–242. Springer, 2014.

[DP12] Guido De Philippis. *Weak notions of Jacobian determinant and relaxation. ESAIM: Control, Optimisation and Calculus of Variations*, 18(1):181–207, 2012.

[FJM02] Gero Friesecke, Richard D. James, and Stefan Müller. *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 55(11):1461–1506, 2002.

[FMP08] Nicola Fusco, Francesco Maggi, and Aldo Pratelli. *The sharp quantitative isoperimetric inequality. Annals of mathematics*, pages 941–980, 2008.

[FMP10] Alessio Figalli, Francesco Maggi, and Aldo Pratelli. *A mass transportation approach to quantitative isoperimetric inequalities. Inventiones mathematicae*, 182(1):167–211, 2010.

[Fus17] Nicola Fusco. *The stability of the Isoperimetric inequality.* In *Vector-Valued Partial Differential Equations and Applications*, pages 73–123. Springer, 2017.

[FZ05] Daniel Faraco and Xiao Zhong. *Geometric rigidity of conformal matrices. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 4(4):557–585, 2005.

[Geh62] Frederick W. Gehring. *Rings and quasiconformal mappings in space. Transactions of the American Mathematical Society*, 103(3):353–393, 1962.

[GLP10] Adriana Garroni, Giovanni Leoni, and Marcello Ponsiglione. *Gradient theory for plasticity via homogenization of discrete dislocations. JEMS*, 12(5):1231–1266, 2010.

[Gro96] Helmut Groemer. *Geometric applications of Fourier series and spherical harmonics*, volume 61. Cambridge University Press, 1996.

[Gro13] Misha Gromov. *Partial differential relations*, volume 9. Springer Science & Business Media, 2013.

[IL93] Tadeusz Iwaniec and Adam Lutoborski. *Integral estimates for null Lagrangians. Archive for Rational Mechanics and Analysis*, 125(1):25–79, 1993.

[IM93] Tadeusz Iwaniec and Gaven Martin. *Quasiregular mappings in even dimensions. Acta Mathematica*, 170(1):29–81, 1993.

[Iwa92] Tadeusz Iwaniec. *p-Harmonic tensors and quasiregular mappings. Annals of mathematics*, 136(3):589–624, 1992.

[Joh61] Fritz John. *Rotation and strain. Communications on Pure and Applied Mathematics*, 14(3):391–413, 1961.

[Joh72] Fritz John. *Bounds for deformations in terms of average strains. Inequalities III*, pages 129–144, 1972.

[Kin87] David Kinderlehrer. *Remarks about equilibrium configurations of crystals.* 1987.

[KMS19] Raz Kupferman, Cy Maor, and Asaf Shachar. *Reshetnyak rigidity for Riemannian manifolds. Archive for Rational Mechanics and Analysis*, 231(1):367–408, 2019.

[Koh82] Robert V. Kohn. *New integral estimates for deformations in terms of their nonlinear strains. Archive for Rational Mechanics and Analysis*, 78(2):131–172, 1982.

[Kui55] Nicolaas H. Kuiper. *On C^1-isometric imbeddings. I*. In *Indagationes Mathematicae (Proceedings)*, volume 58, pages 545–556. Elsevier, 1955.

[LL17] Gianluca Lauteri and Stephan Luckhaus. *Geometric Rigidity Estimates for Incompatible Fields in Dimension $n \geq 3$. arXiv preprint arXiv:1703.03288*, 2017.

[LZ] Stephan Luckhaus and Konstantinos Zemas. *Stability estimates for the conformal group of $\mathbb{S}^{n-1}$ in dimension $n \geq 3$. arXiv preprint, arXiv:1910.01862, 2019.*

[LZon] Stephan Luckhaus and Konstantinos Zemas. *Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$*. In preparation.

[Mü90] Stefan Müller. *Higher integrability of determinants and weak convergence in L1. Journal für die reine und angewandte Mathematik*, 412:20–34, 1990.

[MŠY99] Stefan Müller, Vladimir Šverák, and Baisheng Yan. *Sharp stability results for almost conformal maps in even dimensions. The Journal of Geometric Analysis*, 9(4):671, 1999.

[MSZ14] Stefan Müller, Lucia Scardia, and Caterina Ida Zeppieri. *Geometric rigidity for incompatible fields, and an application to strain-gradient plasticity. Indiana University Mathematics Journal*, pages 1365–1396, 2014.

[Nas54] John Nash. *C^1 isometric embeddings. Annals of mathematics*, pages 383–396, 1954.

[NB11] Hoài-Minh Nguyên and H. Brezis. *Jacobian determinant revisited. Inventiones mathematicae*, 185(ARTICLE):17–54, 2011.

[Res67a] Yu G. Reshetnyak. *Liouville's theorem on conformal mappings for minimal regularity assumptions. Siberian Mathematical Journal*, 8(4):631–634, 1967.

[Res67b] Yu G. Reshetnyak. *On the stability of conformal mappings in multidimensional spaces. Siberian Mathematical Journal*, 8(1):69–85, 1967.

[Res70] Yu G. Reshetnyak. *On stability bounds in the Liouville theorem on conformal mappings of multidimensional spaces. Siberian Mathematical Journal*, 11(5):833–846, 1970.

[Res13] Yu G. Reshetnyak. *Stability theorems in geometry and analysis*, volume 304. Springer Science & Business Media, 2013.

[SJ12] László Székelyhidi Jr. *From isometric embeddings to turbulence. HCDTE lecture notes. Part II. Nonlinear hyperbolic PDEs, dispersive and transport equations*, 7:63, 2012.

[SS19] Daniel E. Spector and Scott J. Spector. *Uniqueness of Equilibrium with Sufficiently Small Strains in Finite Elasticity. Archive for Rational Mechanics and Analysis*, 233(1):409–449, 2019.

[Yan96] Baisheng Yan. *Remarks on $W^{1,p}$-stability of the conformal set in higher dimensions.* In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 13, pages 691–705. Elsevier, 1996.

[YZ98] Baisheng Yan and Zhengfang Zhou. *Stability of weakly almost conformal mappings. Proceedings of the American Mathematical Society*, 126(2):481–489, 1998.

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