A SHARP ISOPERIMETRIC-TYPE INEQUALITY FOR LORENTZIAN SPACES SATISFYING TIMELIKE RICCI LOWER BOUNDS

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Abstract. The paper establishes a sharp and rigid isoperimetric-type inequality in Lorentzian signature under the assumption of Ricci curvature bounded below in the timelike directions. The inequality is proved in the high generality of Lorentzian pre-length spaces satisfying timelike Ricci lower bounds in a synthetic sense via optimal transport, the so-called $\mathsf{TCD}^e_p(K,N)$ spaces. The results are new already for smooth Lorentzian manifolds. Applications include an upper bound on the area of Cauchy hypersurfaces inside the interior of a black hole (original already in Schwarzschild) and an upper bound on the area of Cauchy hypersurfaces in cosmological space-times.

Contents

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1. Introduction

Brief account on the isoperimetric problem in Riemannian signature. The isoperimetric problem is one of the most classical problems in Mathematics, having its roots in the Greek legend of Dido, queen of Cartaghe. In *Riemannian* signature, it amounts to answer the following question:

"What is the maximal volume that can be enclosed by a given area?"

Equivalently, it can be stated as the problem of finding the *maximal* function $I_{(M,g)}(\cdot) : [0, \infty) \to [0, \infty)$ such that for every subset $E \subset M$ with smooth boundary ∂E in the $(n+1)$-dimensional Riemannian manifold (M^{n+1}, g), it holds

$$\mathrm{Vol}_g^n(\partial E) \geq I_{(M,g)}(\mathrm{Vol}_g^{n+1}(E)), \tag{1.1}$$

where $\mathrm{Vol}_g^{n+1}(E)$ (resp. $\mathrm{Vol}_g^n(\partial E)$) denotes the $(n+1)$-dimensional measure of E with respect to g (resp. the n-dimensional measure of ∂E with respect to the restriction of g).

The literature on the isoperimetric problem in Riemannian signature is highly extensive (see, for instance, [48, 50]). Even in Euclidean spaces, the complete solution is relatively recent and required several significant breakthroughs. In the broader context of sets with finite perimeter, a complete proof was established by De Giorgi [19] (refer to [20] for the English translation), following a long series of noteworthy intermediate results. It is worth mentioning Steiner [53], who introduced a now-classical symmetrization technique that now bears his name, with this objective in mind.

In the framework of Riemannian manifolds with Ricci curvature bounded below, an isoperimetric inequality in the form (1.1) (with the function $I_{(M,g)}(\cdot)$ here depending only on the dimension and on the Ricci lower bound) was proved by Gromov [29] in case of positive Ricci lower bound, following a previous work by Lévy [39]. After several contributions, the case of bounded diameter and a general Ricci lower bound was established in the sharp form by E. Milman [44] to which we refer for a complete account on the bibliography.

Isoperimetric bounds of the type (1.1) have proven to be extremely influential in mathematical general relativity. Noteworthy developments include the concept of isoperimetric mass introduced by Huisken [31, 32]; the establishment that initial data sets (with non-negative scalar curvature and positive mass) are foliated at infinity by isoperimetric spheres, providing a canonical center of coordinates, as demonstrated by Chodosh-Eichmair-Shi-Yu [17] following prior work by Eichmair-Metzger [22], as well as the pioneering paper by Huisken-Yau [34]; the examination of the isoperimetric problem in the doubled Riemannian Schwarzschild metric by Brendle-Eichmair [10] following Bray [7]; and lastly the work by Brendle [9], generalizing the Alexandrov theorem to a class of Riemannian warped products, which encompasses Riemannian deSitter-Schwarzschild and Riemannian Reissner-Nordstrom metrics.

An alike area bound, playing a pivotal role in mathematical general relativity, is the celebrated *Penrose inequality*, relating the area of a black hole horizon with the total mass of a space-time. More precisely, in 1973, Penrose [49] put forth an argument suggesting that the total mass of a space-time containing black holes with event horizons totaling an area A should be bounded below by $\sqrt{A/16\pi}$. This statement has a significant mathematical implication in Riemannian geometry, known as the Riemannian Penrose inequality. This inequality was initially established by Huisken-Ilmanen [33] for a single black hole, and later by Bray [8] for any number of black holes. The two approaches employ distinct geometric flow techniques.

Brief account on the isoperimetric problem in Lorentzian signature. If (M^{n+1}, g) is a *Lorentzian* manifold, the *maximal* function $I_{(M,g)}(\cdot) : [0,\infty) \to [0,\infty)$ satisfying (1.1) is identically 0 - at least for small volumes and, in several examples (including Minkowski space-time), for all volumes. The reason is that the causal diamonds have positive $(n+1)$-volume, but their boundary is a null hypersurface (with singularities of negligible measure) with zero n-volume.

Indeed, due to the different signature, a geometric *minimization* problem in *Riemannian* signature is turned into a *maximization* problem in *Lorentzian* signature. The landmark example is given by geodesics, which locally *minimize* length in *Riemannian* signature, and instead locally *maximize* time separation (i.e. Lorentzian length) in *Lorentzian* signature. The same phenomenon appears for the isoperimetric problem which, in Lorentzian signature, reads as:

"What is the maximal area that can be used to enclose a given volume?"

Equivalently, it can be stated as the problem to find the *minimal* function $J_{(M,g)}(\cdot) : [0,\infty) \to [0,\infty)$ such that for every subset $E \subset M$ with smooth boundary ∂E in the $(n+1)$-dimensional Lorentzian manifold (M^{n+1}, g), it holds

$$\mathrm{Vol}_g^n(\partial E) \leq J_{(M,g)}(\mathrm{Vol}_g^{n+1}(E)), \tag{1.2}$$

where $\mathrm{Vol}_g^{n+1}(E)$ (resp. $\mathrm{Vol}_g^n(\partial E)$) denotes the $(n+1)$-dimensional measure of E with respect to $|g|$ (resp. the n-dimensional measure of ∂E with respect to the restriction of $|g|$).

In sharp contrast to the Riemannian signature, where the literature on the isoperimetric problem is extensive, the literature on the isoperimetric problem in Lorentzian signature is rather limited. To the best of our knowledge, the following are the only results in the literature at the moment:

- Bahn-Ehrlich [5] in 1999 provided an upper bound on the area of a compact space-like achronal hypersurface S contained in the future of a point O in the $(n+1)$-dimensional Minkowski space-time, in terms of the volume (raised to the appropriate power to obtain scale-invariance) of a suitably constructed past cone $C(S)$ with base point O:

$$\mathrm{Vol}_g^n(S) \le C(n)\ (\mathrm{Vol}_g^{n+1}(C(S)))^{n/(n+1)}. \tag{1.3}$$

- Bahn [4] in 1999 obtained a similar inequality to [5] for two-dimensional Lorentzian surfaces with Gaussian curvature bounded from above.
- Abedin-Corvino-Kapita-Wu [1] in 2009 generalized the isoperimetric inequality of [5] to spacetimes $I \times \mathbb{H}^n$ with a warped metric $g = -(dt)^2 + a(t)^2 g_{\mathbb{H}^n}$ satisfying $a'' \le 0$. This corresponds to a subclass of Friedman-Robertson-Walker space-times satisfying the strong energy condition $\mathrm{Ric}(v,v) \ge 0$ for timelike vectors v.
- Lambert-Scheuer [38] in 2021 extended [1] to spacetimes $N = (a,b)\times S_0$ with metric $g = -dr^2 + \theta(r)^2\hat{g}$ satisfying the null convergence condition and with $(S_0, \hat{g})$ compact. The relation between the area and the volumes is as follows: if $\Sigma \subset N$ is a spacelike, compact, achronal, and connected hypersurface, then
$$\mathrm{Vol}_g^n(\Sigma) \le \varphi(\mathrm{Vol}_g^{n+1}(C(\Sigma))),$$
where $C(\Sigma)$ is the region between Σ and $a \times S$ and φ is the function which gives equality on the coordinate slices.

We also mention the work of Tsai-Wang [56] in 2020 which established an isoperimetric-type inequality for maximal, spacelike submanifolds in Minkowski space. With a slightly different perspective, Graf-Sormani in [28] have recently improved on Truede-Grant [55] by establishing upper bounds on the Lorentzian area and volumes of slices of the time-separation function from a Cauchy hypersurface in terms of its mean curvature.

Arguably, one of the main motivations for such a short bibliography compared with the Riemannian case is the lack of a regularity theory for critical points of the area functional which may fail to be elliptic due to the Lorentzian signature of the ambient metric. We overcome this issue by adopting an optimal transport approach which bypasses the regularity problems.

Main results of the paper. The results will be proved under very low regularity assumptions both on the space-time and on the subsets, namely in the framework of Loretzian pre-length spaces satisfying timelike Ricci curvature lower bounds in a synthetic sense via optimal transport, the so-called $\mathsf{TCD}_p^e(K,N)$ spaces.
The setting of Loretzian pre-length spaces was introduced by Kunzinger-Sämann [37] building on the notion of causal spaces pioneered by Kronheimer-Penrose [36]. The framework comprises Lorentzian manifolds with metrics of low regularity; namely, locally Lipschitz Lorentzian metrics and, more generally, continuous causally plain Lorentzian metrics, studied by Chruściel-Grant [18]. An optimal transport characterization of Ricci curvature lower bounds in the timelike directions for smooth Loretzian manifolds

was obtained by McCann [43] and by Mondino-Suhr [46]. The theory of $\mathsf{TCD}^e_p(K,N)$ spaces has been developed by the authors of the present paper in [14], see also the related work by Braun [6] and the survey [15].

For the sake of the introduction, the statements will be presented in a simplified setting (both on the ambient space-time and on the subsets in consideration), referring to the body of the paper for the more general case. Let us stress that the results seem to be original already in the simplified form below.

Let (M^{n+1},g) be a smooth, globally hyperbolic, Lorentzian manifold. Denote by $\ll$ the chronological relation: for $x,y\in M$, we say that $x\ll y$ if there exists a Lipschitz timelike curve from x to y. Let $\tau: M\times M\to[0,\infty]$ be the time-separation function on M defined by

$$\tau(x,y):=\begin{cases} \sup\{L_g(\gamma)\mid \gamma: I\to M \ \text{ timelike Lipschitz curve}\}, & \text{if } x\ll y,\\ 0 \qquad \text{otherwise}, \end{cases}$$

where

$$L_g(\gamma):=\int_I \sqrt{|g(\dot\gamma_t,\dot\gamma_t)|}\,dt$$

is the length of the timelike Lipschitz curve $\gamma: I\to M$.

The time-separation function satisfies the reverse triangle inequality (on timelike triples) and should be thought of as a Lorentzian counterpart of the distance function in Riemannian geometry. Given a subset $V\subset M$, we denote its chronological future by $I^+(V)$:

$$I^+(V):=\{y\in M\mid \exists x\in V \text{ such that } x\ll y\}.$$

The time-separation function from V is defined by

$$\tau_V: I^+(V)\to(0,\infty],\quad \tau_V(x):=\sup_{y\in V}\tau(y,x),$$

and it should be thought as the Lorentzian distance from V.

Let $S,V\subset M$ be Cauchy hypersurfaces in M, with S contained in the chronological future of V, i.e. $S\subset I^+(V)$. Define the "distance" from V to S by

$$\mathrm{dist}(V,S):=\inf_{x\in S}\tau_V(x)$$

and consider the geodesically conical region from V to S defined by

$$C(V,S):=\{\gamma_t\mid t\in[0,1],\ \text{such that } \gamma_0\in V,\,\gamma_1\in S,\,L_g(\gamma)=\tau_V(\gamma_1)\},$$

i.e. $C(V,S)$ is the region spanned by timelike geodesics from V to S, realizing τ_V. We can now state the main result of the paper (in a simplified form, for the sake of the introduction).

Theorem 1.1 (A sharp isoperimetric-type inequality)**.** *Let (M^{n+1},g) be a globally hyperbolic Lorentzian manifold satisfying Hawking-Penrose's strong* *energy condition (i.e.* $\mathrm{Ric} \geq 0$ *on timelike vectors). Let* $V, S \subset M$ *be Cauchy hypersurfaces with* $S \subset I^+(V)$*. Then*

$$\mathrm{Vol}_g^n(S)\,\mathrm{dist}(V,S) \leq (n+1)\,\mathrm{Vol}_g^{n+1}(C(V,S)).$$

Remark 1.2.
- All the results in the literature about isoperimetric-type inequalities in Lorentzian manifolds (or in Riemannian space-like slices) assume the metric g to be a warped product. Let us stress that there is no symmetry assumption on the space-time in Theorem 1.1, but merely a lower bound on the Ricci curvature in the timelike directions.
- Theorem 1.1 is stated for non-negative Ricci curvature just for the sake of simplicity. A completely analogous statement holds for Ricci curvature bounded below by $K \in \mathbb{R}$ in the timelike directions, see Theorem 4.11.
- The isoperimetric-type inequality in Theorem 1.1 is sharp (see Proposition 4.13) and rigid (see Proposition 4.14): the equality is attained if and only if the space-time is conical.
- The isoperimetric-type inequality in Theorem 1.1 will be proved in the higher generality of Loretzian pre-length spaces satisfying timelike Ricci curvature lower bounds in a synthetic sense via optimal transport, the so-called $\mathsf{TCD}_p^e(K,N)$ spaces (see Definition 2.17). Also the assumption on V can be relaxed considerably: it is enough to assume that V is a Borel, achronal, timelike complete subset. For the general statement, refer to Theorem 4.11.

As applications we will establish:

- An upper bound on the area of Cauchy hypersurfaces inside the interior of a black hole, see Remark 4.15. The bound seems to be new already in the interior of the Schwarzschild black hole, see Example 4.16.
- An upper bound on the area of Cauchy hypersurfaces in cosmological space-times. The novelty with respect to previous results (see for instance [1, 26]) is that no symmetry is assumed; this higher generality seems to have advantages also for applications (see for instance [24]). We refer to Remark 4.17 for more details.

Let us also mention the next result, establishing a monotonicity formula for the area of the level sets of the distance function from a Cauchy hypersurface.

Theorem 1.3 (Area Monotonicity)**.** *Let* (M^{n+1}, g) *be a globally hyperbolic Lorentzian manifold satisfying Hawking-Penrose's strong energy condition (i.e.* $\mathrm{Ric} \geq 0$ *on timelike vectors). Let* $V \subset M$ *be a Cauchy hypersurface and let* $V_t := \{\tau_V = t\}$ *be the achronal slice at distance* $t > 0$ *from* V*. Then the map*

$$(0,\infty) \ni t \longmapsto \frac{\mathrm{Vol}_g^n(V_t)}{t^{N-1}} \tag{1.4}$$

is monotonically non-increasing.

Remark 1.4.
- In the setting of CMC Einstein flows, a pointwise monotonicity formula similar to (1.4) goes back to [25] and [2]. There the spacelike hypersurfaces Σ_t considered are constant mean curvature compact surfaces parametrized by the Hubble time $t = -n/H$. Such monotonicity has then been used to study the convergence as $t \to \infty$ of the metric; we refer to [40] for more details (see also [41] for similar result when $t \to 0$). Theorem 1.3 is stated for non-negative Ricci curvature just for the sake of simplicity. A completely analogous statement holds for Ricci curvature bounded below by $K \in \mathbb{R}$ in the timelike directions, see Theorem 4.9.
- The monotonicity formula for the area (1.4) is sharp (see Remark 4.10): the equality is attained if and only if the space-time is conical.
- The monotonicity of the area Theorem 1.1 will be proved in the higher generality of $\mathsf{TCD}_p^e(K,N)$ Loretzian pre-length spaces satisfying time-like Ricci curvature lower bounds in a synthetic sense via optimal transport. Also the assumption on V can be relaxed considerably: it is enough to assume that V is a Borel, achronal, timelike complete subset. For the general statement, refer to Theorem 4.9.

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Structure of the paper. Section 2 covers the basics of the theory: first we review the definition and some properties of Lorentzian length spaces (Section 2.1), then we recall some concepts on Optimal Transport in the Lorentzian setting (Section 2.2), and finally Section 2.3 recalls the definition of synthetic timelike Ricci curvature lower bounds. In particular, Section 2.2.2 and in Section 2.3.1 review the constructions related to the time-separation function and the localization of TMCP. Section 3 contains the localization of the TCD condition and in Section 4 we obtain the main result on the isoperimetric-type inequality. Finally, Section 5 is devoted to the proof of the sharp timelike Brunn-Minkowski inequality.

2. Preliminaries

2.1. **Lorentzian (pre-)length spaces.** Following the work of Kunzinger-Sämann [37], in this section we briefly recall some basic notions from the theory of Lorentzian (pre-)length spaces. We refer to [37] and to [14] for further details and for the proofs.

Definition 2.1 (Causal space, Kronheimer-Penrose [36])**.** A *causal space* $(X, \ll, \leq)$ is a set X endowed with a preorder $\leq$ and a transitive relation $\ll$ contained in $\leq$.

We write $x < y$ when $x \leq y, x \neq y$. We say that x and y are timelike (resp. causally) related if $x \ll y$ (resp. $x \leq y$). Let $A \subset X$ be an arbitrary subset of X. We define the chronological (resp. causal) future of A the set

$$
\begin{aligned}
I^+(A) &:= \{y \in X \,:\, \exists x \in A \text{ such that } x \ll y\}, \\
J^+(A) &:= \{y \in X \,:\, \exists x \in A \text{ such that } x \leq y\},
\end{aligned}
$$

respectively. Analogously, we define the chronological (resp. causal) past of A. In case $A = \{x\}$ is a singleton, with a slight abuse of notation, we will write $I^\pm(x)$ (resp. $J^\pm(x)$) instead of $I^\pm(\{x\})$ (resp. $J^\pm(\{x\})$). Moreover we also introduce the following notations

$$
X^2_{\leq} := \{(x,y) \in X \times X \,:\, x \leq y\}, \qquad X^2_{\ll} := \{(x,y) \in X \times X \,:\, x \ll y\}.
$$

Definition 2.2 (Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$)**.** A *Lorentzian pre-length space* $(X, \mathsf{d}, \ll, \leq, \tau)$ is a causal space $(X, \ll, \leq)$ additionally equipped with a proper metric d (i.e. closed and bounded subsets are compact) and a lower semicontinuous function $\tau : X \times X \to [0, \infty]$, called *time-separation function*, satisfying

$$
\begin{aligned}
&\tau(x,y) + \tau(y,z) \leq \tau(x,z) \quad \forall x \leq y \leq z \quad \text{reverse triangle inequality} \\
&\qquad \tau(x,y) = 0, \text{ if } x \not\leq y, \quad \tau(x,y) > 0 \Leftrightarrow x \ll y.
\end{aligned} \tag{2.1}
$$

The lower semicontinuity of τ implies that $I^\pm(x)$ is open, for any $x \in X$. The set X is endowed with the metric topology induced by d. All the topological concepts on X will be formulated in terms of such metric topology. For instance, we will denote by $\overline{C}$ the topological closure (with respect to d) of a subset $C \subset X$.

Definition 2.3 (Causal/timelike curves)**.** If $I \subset \mathbb{R}$ is an interval, a non-constant curve $\gamma : I \to X$ is called (future-directed) *timelike* (resp. *causal*) if γ is locally Lipschitz continuous (with respect to d) and if for all $t_1, t_2 \in I$, with $t_1 < t_2$, it holds $\gamma_{t_1} \ll \gamma_{t_2}$ (resp. $\gamma_{t_1} \leq \gamma_{t_2}$). We say that γ is a *null* curve if, in addition to being causal, no two points on $\gamma(I)$ are related with respect to $\ll$.

The length of a causal curve is defined via the time separation function, in analogy to the theory of length metric spaces: for $\gamma : [a,b] \to X$ future-directed causal we set

$$
L_\tau(\gamma) := \inf \left\{ \sum_{i=0}^{N-1} \tau(\gamma_{t_i}, \gamma_{t_{i+1}}) \,:\, a = t_0 < t_1 < \ldots < t_N = b,\ N \in \mathbb{N} \right\}.
$$

In case the interval is half-open, say $I = [a, b)$, then the infimum is taken over all partitions with $a = t_0 < t_1 < \ldots < t_N < b$ (and analogously for the other cases).

Under fairly general assumptions, these definitions coincide with the classical ones in the smooth setting, see [37, Prop. 2.32, Prop. 5.9].

A future-directed causal curve $\gamma : [a, b] \to X$ is *maximal* if it realises the time separation, i.e. if $L_\tau(\gamma) = \tau(\gamma_a, \gamma_b)$.
In case the time separation function is continuous with $\tau(x, x) = 0$ for every $x \in X$, then any maximal timelike curve γ with finite τ-length has a (continuous, monotonically strictly increasing) reparametrization λ by τ-arc-length, i.e. $\tau(\gamma_{\lambda(s_1)}, \gamma_{\lambda(s_2)}) = s_2 - s_1$ for all $s_2 \leq s_1$ in the corresponding interval (see [37, Cor. 3.35]).

We therefore adopt the following convention: a curve γ will be called *(causal) geodesic* if it is maximal and continuous when parametrized by τ-arc-lenght. In other words, the set of (causal) geodesics is

$$\mathrm{Geo}(X) := \{\gamma \in C([0,1], X) \,:\, \tau(\gamma_s, \gamma_t) = (t - s)\,\tau(\gamma_0, \gamma_1)\,\forall s < t\}. \tag{2.2}$$

The set of *timelike geodesic* is defined as follows:

$$\mathrm{TGeo}(X) := \{\gamma \in \mathrm{Geo}(X) :\, \tau(\gamma_0, \gamma_1) > 0\}. \tag{2.3}$$

Given $x \leq y \in X$ we set

$$\mathrm{Geo}(x, y) := \{\gamma \in \mathrm{Geo}(X) \,:\, \gamma_0 = x,\, \gamma_1 = y\} \tag{2.4}$$

$$\mathfrak{I}(x, y, t) := \{\gamma_t \,:\, \gamma \in \mathrm{Geo}(x, y)\} \tag{2.5}$$

respectively the space of geodesics, and the set of t-intermediate points from x to y.
If $x \ll y \in X$, we call

$$\mathrm{TGeo}(x, y) := \{\gamma \in \mathrm{TGeo}(X) \,:\, \gamma_0 = x,\, \gamma_1 = y\}.$$

Given two subsets $A, B \subset X$, we denote

$$\mathfrak{I}(A, B, t) := \bigcup_{x \in A, y \in B} \mathfrak{I}(x, y, t) \tag{2.6}$$

the subset of t-intermediate points of geodesics from points in A to points in B.

Definition 2.4 (Timelike non-branching)**.** A Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$ is said to be *forward timelike non-branching* if and only if for any $\gamma^1, \gamma^2 \in \mathrm{TGeo}(X)$, it holds:

$$\exists\, \bar{t} \in (0,1) \text{ such that } \;\forall t \in [0, \bar{t}\,] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

It is said to be *backward timelike non-branching* if the reversed causal structure is forward timelike non-branching. In case it is both forward and backward timelike non-branching it is said *timelike non-branching*.

By Cauchy Theorem, it is clear that if (M, g) is a space-time whose Christoffel symbols are locally-Lipschitz (e.g. in case $g \in C^{1,1}$) then the associated synthetic structure is timelike non-branching. For spacetimes with a metric of lower regularity (e.g. $g \in C^1$ or $g \in C^0$) timelike branching may occur.

2.1.1. *Causal Ladder.* Concerning the causal ladder, we follow [45]. In order to streamline the presentation we will only consider *Lorentzian geodesic spaces,* i.e. Lorentzian pre-length spaces $(X, \mathsf{d}, \ll, \leq, \tau)$ that additionally are:

- d-*Compatible:* every $x \in X$ admits a neighbourhood U and a constant C such that $L_{\mathsf{d}}(\gamma) \leq C$ for every causal curve γ contained in U;
- *Geodesic:* for all $x, y \in X$ with $x < y$ there is a future-directed causal curve γ from x to y with $\tau(x, y) = L_\tau(\gamma)$.

A Lorentzian geodesic space is in particular a Lorentzian length space, see [37, Def. 3.22].

Hence from [45, Cor. 3.8] we can consider the following version of global hyperbolicity that fits with the previous literature. A Lorentzian geodesic space $(X, \mathsf{d}, \ll, \leq, \tau)$ is called

- *Causal*: if $\leq$ is also antisymmetric, i.e. $\leq$ is an order;
- *Globally hyperbolic*: if it is causal and for every $x, y \in X$ the causal diamond $J^+(x) \cap J^-(y)$ is compact in X.

From [45, Thm. 3.7] this definition of global hyperbolicity is equivalent with the one adopted in [37] (that we omit). Also global hyperbolicity implies that the relation $\leq$ is a closed subset of $X \times X$. It was proved in [37, Thm. 3.28] that for a globally hyperbolic Lorentzian geodesic space $(X, \mathsf{d}, \ll, \leq, \tau)$, the time-separation function τ is finite and continuous: in particular the previous remark on the existence of constant τ-speed parametrizations for maximal causal curves applies, thus any two distinct causally related points are joined by a causal geodesic.
From [45] it also follows that if X is globally hyperbolic and $K_1, K_2 \Subset X$ are compact subsets then

$$\mathfrak{I}(K_1, K_2, t) \Subset \bigcup_{t \in [0,1]} \mathfrak{I}(K_1, K_2, t) \Subset X, \quad \forall t \in [0, 1].$$

2.2. **Optimal transport in Lorentzian geodesic spaces.** We briefly review some basics on optimal transport in the Lorentzian synthetic setting. For simplicity of presentation, we will assume that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic Lorentzian geodesic space; we refer to [14] for more general results.

Given $\mu, \nu \in \mathcal{P}(X)$, the space of probability measures, the set of *transport plans* is

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(X \times X) \, : \, (P_1)_\sharp \pi = \mu, \, (P_2)_\sharp \pi = \nu\}.$$

The set of *causal* and *timelike transport plans* are defined by

$$\Pi_{\leq}(\mu,\nu) := \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_{\leq}) = 1\},$$
$$\Pi_{\ll}(\mu,\nu) := \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_{\ll}) = 1\}.$$

As $X^2_{\leq} \subset X^2$ is a closed subset, $\pi \in \Pi_{\leq}(\mu,\nu)$ if and only if $\operatorname{supp}\pi \subset X^2_{\leq}$. For $p \in (0,1]$, given $\mu,\nu \in \mathcal{P}(X)$, the p-Lorentz-Wasserstein distance is defined by

$$\ell_p(\mu,\nu) := \sup_{\pi \in \Pi_{\leq}(\mu,\nu)} \left(\int_{X\times X} \tau(x,y)^p \, \pi(dxdy) \right)^{1/p}. \tag{2.7}$$

If $\Pi_{\leq}(\mu,\nu) = \emptyset$ we set $\ell_p(\mu,\nu) := -\infty$. A plan $\pi \in \Pi_{\leq}(\mu,\nu)$ maximising in (2.7) is said ℓ_p-*optimal*. The set of ℓ_p-*optimal* plans from μ to ν is denoted by $\Pi^{p\text{-opt}}_{\leq}(\mu,\nu)$.

An alternative formulation of (2.7) can be obtained by using the following function:

$$\ell(x,y)^p := \begin{cases} \tau(x,y)^p & \text{if } x \leq y \\ -\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

Clearly, if $\pi \in \Pi_{\leq}(\mu,\nu)$, then π-a.e. one has $\tau(x,y) = \ell(x,y)$. Moreover, using the convention that $\infty - \infty = -\infty$, if $\pi \in \Pi(\mu,\nu)$ satisfies $\int_{X\times X} \ell(x,y)^p \, \pi(dxdy) > -\infty$ then $\pi \in \Pi_{\leq}(\mu,\nu)$. Thus the maximization problem (2.7) is equivalent (i.e. the sup and the set of maximisers coincide) to the maximisation problem

$$\sup_{\pi \in \Pi(\mu,\nu)} \left(\int_{X\times X} \ell(x,y)^p \, \pi(dxdy) \right)^{1/p}. \tag{2.9}$$

The advantage of the formulation (2.9) is that ℓ^p is upper semi-continuous on $X \times X$. One can therefore invoke standard optimal transport techniques (e.g. [57]) to ensure the existence of a solution of the Monge-Kantorovich problem (2.9).

Proposition 2.5. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $\mu,\nu \in \mathcal{P}(X)$. If $\Pi_{\leq}(\mu,\nu) \neq \emptyset$ and if there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu\otimes\nu)$ such that $\ell^p \leq a \oplus b$ on* $\operatorname{supp}\mu \times \operatorname{supp}\nu$ *(e.g. when μ and ν are compactly supported) then the* sup *in* (2.7) *is attained and finite.*

In Proposition 2.5 we used the following standard notation: given $\mu,\nu \in \mathcal{P}(X)$, $\mu\otimes\nu \in \mathcal{P}(X^2)$ is the product measure; given $u, v : X \to \mathbb{R} \cup \{+\infty\}$, $u \oplus v : X^2 \to \mathbb{R} \cup \{+\infty\}$ is defined by $u \oplus v(x,y) := u(x) + v(y)$.

The Lorentzian-Wasserstein ditance ℓ_p satisfies the reverse triangle inequality:

$$\ell_p(\mu_0,\mu_1) + \ell_p(\mu_1,\mu_2) \leq \ell_p(\mu_0,\mu_2), \quad \forall \mu_0,\mu_1,\mu_2 \in \mathcal{P}(X), \tag{2.10}$$

where we adopt the convention that $\infty - \infty = -\infty$ to interpret the left hand side of (2.10).

We also recall two relevant notions of cyclical monotonicity.

Definition 2.6 (τ^p-cyclical monotonicity and ℓ^p-cyclical monotonicity)**.** A subset $\Gamma \subset X^2_{\leq}$ is said to be τ^p-cyclically monotone (resp. ℓ^p-cyclically monotone) if, for any $N \in \mathbb{N}$ and any family $(x_1, y_1), \dots, (x_N, y_N)$ of points in Γ, the next inequality holds:

$$\sum_{i=1}^{N} \tau(x_i, y_i)^p \geq \sum_{i=1}^{N} \tau(x_{i+1}, y_i)^p, \tag{2.11}$$

(resp. $\sum_{i=1}^{N} \ell^p(x_i, y_i) \geq \sum_{i=1}^{N} \ell^p(x_{i+1}, y_i)$) with the convention $x_{N+1} = x_1$.

Accordingly, a transport plan π is said to be τ^p-cyclically monotone (resp. ℓ^p-cyclically monotone) if there exists a τ^p-cyclically monotone set (resp. ℓ^p-cyclically monotone set) Γ such that $\pi(\Gamma) = 1$. It is straightforward to check that τ^p-cyclical monotonicity implies ℓ^p-cyclical monotonicity; moreover, if $P_1(\Gamma) \times P_2(\Gamma) \subset X^2_{\leq}$ then ℓ^p-cyclical monotonicity is equivalent to τ^p-cyclical monotonicity.

Under fairly general assumptions, cyclical monotonicity and optimality are equivalent. Indeed:

Proposition 2.7 (Prop. 2.8 and Thm 2.26 in [14])**.** *If $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic Lorentzian geodesic space, $p \in (0, 1]$, $\mu, \nu \in \mathcal{P}(X)$ with $\ell_p(\mu, \nu) \in (0, \infty)$, then for any $\pi \in \Pi_{\leq}(\mu, \nu)$ the following holds:*

(1) If π is ℓ_p-optimal then π is ℓ^p-cyclically monotone.
(2) If $\pi(X^2_{\ll}) = 1$ and π is ℓ^p-cyclically monotone then π is ℓ_p-optimal.
(3) If π is τ^p-cyclically monotone then π is ℓ_p-optimal.

Finally, we recall from [14] the definition of (strongly) timelike p-dualisable probability measures.

Definition 2.8 ((Strongly) Timelike p-dualisable measures)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0, 1]$. We say that $(\mu, \nu) \in \mathcal{P}(X)^2$ is *timelike p-dualisable (by $\pi \in \Pi_{\ll}(\mu, \nu)$)* if

(1) $\ell_p(\mu, \nu) \in (0, \infty)$;
(2) $\pi \in \Pi^{p\text{-opt}}_{\leq}(\mu, \nu)$ and $\pi(X^2_{\ll}) = 1$;
(3) there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\ell^p \leq a \oplus b$ on $\operatorname{supp} \mu \times \operatorname{supp} \nu$.

We say that $(\mu, \nu) \in \mathcal{P}(X)^2$ is *strongly timelike p-dualisable* if, in addition, it satisfies:

4. there exists a measurable ℓ^p-cyclically monotone set $\Gamma \subset X^2_{\ll} \cap (\operatorname{supp} \mu \times \operatorname{supp} \nu)$ such that a coupling $\pi \in \Pi_{\leq}(\mu, \nu)$ is ℓ_p-optimal if and only if π is concentrated on Γ, i.e. $\pi(\Gamma) = 1$.

The above notions are tighten with the validity of Kantorovich duality (see [14, Sec. 2.4]). The notion of strongly timelike p-dualisability is non-empty:

Lemma 2.9 (Cor. 2.29 in [14]). *Fix $p \in (0,1]$. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and assume that $\mu, \nu \in \mathcal{P}(X)$ satisfy:*

(1) there exist measurable functions $a, b : X \to \mathbb{R}$ with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\tau^p \leq a \oplus b$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$;

(2) $\operatorname{supp}\mu \times \operatorname{supp}\nu \subset X^2_{\ll}$.

Then (μ, ν) is strongly timelike p-dualisable.

2.2.1. *Geodesics of probability measures in the Lorentz-Wasserstein space.* Let us start by introducing some classical notation. The evaluation map is defined by

$$\mathrm{e}_t : C([0,1], X) \to X, \quad \gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t, \quad \forall t \in [0,1]. \tag{2.12}$$

The stretching/restriction operator $\mathrm{restr}_{s_1}^{s_2} : C([0,1], X) \to C([0,1], X)$ is defined by

$$(\mathrm{restr}_{s_1}^{s_2}\gamma)_t := \gamma_{(1-t)s_1 + ts_2}, \quad \forall s_1, s_2 \in [0,1], s_1 < s_2, \, \forall t \in [0,1]. \tag{2.13}$$

Definition 2.10 (ℓ_p-optimal dynamical plans and ℓ_p-geodesics)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. We say that $\eta \in \mathcal{P}(\mathrm{Geo}(X))$ is an *ℓ_p-optimal dynamical plan* from $\mu_0 \in \mathcal{P}(X)$ to $\mu_1 \in \mathcal{P}(X)$ if $(\mathrm{e}_0)_\sharp\eta = \mu_0, (\mathrm{e}_1)_\sharp\eta = \mu_1$ and

$$(\mathrm{e}_0, \mathrm{e}_1)_\sharp\eta \quad \text{belongs to } \Pi_{\leq}^{p\text{-opt}}((\mathrm{e}_0)_\sharp\eta, (\mathrm{e}_1)_\sharp\eta). \tag{2.14}$$

The set of ℓ_p-optimal dynamical plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$. We say that a curve curve $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}(X)$ is an ℓ_p-geodesic if there exists an ℓ_p-optimal dynamical plan η from μ_0 to μ_1 such that $\mu_t = (\mathrm{e}_t)_\sharp\eta$, for all $t \in [0,1]$.

Notice that if $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$, then the ℓ_p-geodesic

$$\mu_t := (\mathrm{e}_t)_\sharp\eta, \quad \forall t \in [0,1],$$

is continuous in narrow topology and satisfies $\ell_p(\mu_s, \mu_t) = (t-s)\ell_p(\mu_0, \mu_1)$, for all $s, t \in [0,1]$.

Let us recall that if $\mu_0, \mu_1 \in \mathcal{P}(X)$ have compact support, then there always exists an ℓ_p-optimal dynamical plan $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ (and thus an ℓ_p-geodesic) from μ_0 to μ_1, see [14, Prop. 2.33] for the proof and for other properties of ℓ_p-optimal dynamical plans.

2.2.2. *Time-separation functions from sets and their transport relations.* A subset $V \subset X$ is called *achronal* if $x \not\ll y$ for every $x, y \in V$. In particular, if V is achronal, then $I^+(V) \cap I^-(V) = \emptyset$, so we can define the *signed time-separation* to V, $\tau_V : X \to [-\infty, +\infty]$, by

$$\tau_V(x) := \begin{cases} \sup_{y\in V} \tau(y,x), & \text{for } x \in I^+(V) \\ -\sup_{y\in V} \tau(x,y), & \text{for } x \in I^-(V) \\ 0 & \text{otherwise} \end{cases}. \tag{2.15}$$

Note that τ_V is lower semi-continuous on $I^+(V)$ as supremum of continuous functions, and is upper semi-continuous on $I^-(V)$.
In order for these suprema to be attained, global hyperbolicity and geodesic property of X alone are not sufficient. One should rather demand additional compactness properties of the set V. The following notion, introduced by Galloway [27] in the smooth setting, is well suited to this aim.

Definition 2.11 (Future timelike complete (FTC) subsets)**.** A subset $V \subset X$ is *future timelike complete* (FTC), if for each point $x \in I^+(V)$, the intersection $J^-(x) \cap V \subset V$ has compact closure (w.r.t. d) in V. Analogously, one defines *past timelike completeness* (PTC). A subset that is both FTC and PTC is called *timelike complete*.

Lemma 2.12 (Lemma 4.1, [14])**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal FTC (resp. PTC) subset. Then for each $x \in I^+(V)$ (resp. $x \in I^-(V)$) there exists a point $y_x \in V$ with $\tau_V(y_x) = \tau(y_x, x) > 0$ (resp. $\tau_V(y_x) = -\tau(x, y_x) < 0$).*

Moreover for all $x, z \in I^+(V) \cup V$,

$$\tau_V(z) - \tau_V(x) \geq \tau(y_x, z) - \tau(y_x, x) \geq \tau(x, z), \tag{2.16}$$

provided $(x, z) \in X^2_{\leq}$. Analogous statement is valid for $x \in I^-(V)$.

By considering a non-smooth analogue of the gradient flow lines of τ_V, one can obtain a partition into timelike geodesics of the future of V (up to a set of measure zero). We briefly review this construction and refer to [14, Sec. 4.1] for more details.
First, notice that (2.16) can be extended to the whole X^2 by replacing τ with ℓ, defined in (2.8):

$$\tau_V(z) - \tau_V(x) \geq \ell(x, z), \qquad \forall x, z \in (I^+(V) \cup I^-(V) \cup V)^2. \tag{2.17}$$

For ease of writing, we will use the following notation

$$I^{\pm}(V) := (I^+(V) \cup I^-(V) \cup V).$$

We associate to V the following set:

$$\begin{aligned} \Gamma_V := \{(x, z) \in I^{\pm}(V)^2 \cap X^2_{\leq} :\ \tau_V(z) - \tau_V(x) = \tau(x, z) > 0\} \\ \cup \{(x, x) \,:\, x \in I^{\pm}(V)\}. \end{aligned} \tag{2.18}$$

From the inequality (2.17), it follows straightforwardly that the set Γ_V is ℓ-cyclically monotone. This implies the well-known alignment along geodesics of the pairs belonging to Γ_V: for instance if $(x,z) \in \Gamma_V$ with $x \neq z$ and $x \in I^+_V$, there exist $y \in V, \gamma \in \mathrm{TGeo}(y,z)$ and $t \in (0,1)$ such that

$$x = \gamma_t, \qquad \tau(y,\gamma_s) = \tau_V(\gamma_s) \quad \forall s \in [0,1], \qquad (\gamma_s,\gamma_t) \in \Gamma_V \quad \forall s \in [0,t].$$

An analogous property holds true if $z \in I^-(V)$. Again for all the details we refer to [14]. Next we set $\Gamma_V^{-1} := \{(x,y) : (y,x) \in \Gamma_V\}$ and we consider the *transport relation* R_V and the *transport set with endpoints* $\mathcal{T}_V^{end}$

$$R_V := \Gamma_V \cup \Gamma_V^{-1}, \qquad \mathcal{T}_V^{end} := P_1(R_V \setminus \{x = y\}), \tag{2.19}$$

where P_1 denotes the projection on the first coordinate. The transport relation will be an equivalence relation on a suitable subset of $\mathcal{T}_V^{end}$, constructed below. Define the following subsets:

$$\begin{aligned} \mathfrak{a}(\mathcal{T}_V^{end}) &:= \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t.\ (y,x) \in \Gamma_V, y \neq x\} \\ \mathfrak{b}(\mathcal{T}_V^{end}) &:= \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t.\ (x,y) \in \Gamma_V, y \neq x\}, \end{aligned} \tag{2.20}$$

called the set of *initial* and *final points*, respectively. Define the *transport set without endpoints*

$$\mathcal{T}_V := \mathcal{T}_V^{end} \setminus (\mathfrak{a}(\mathcal{T}_V^{end}) \cup \mathfrak{b}(\mathcal{T}_V^{end})). \tag{2.21}$$

Lemma 2.13 (Lemma 4.4, [14])**.** *If $V \subset X$ is a Borel achronal timelike complete subset, then the following identity holds true:*

$$I^+(V) \cup I^-(V) = \mathcal{T}_V^{end} \setminus V.$$

If additionally X is assumed to be timelike (backward and forward) non-branching, then the transport relation R_V is an equivalence relation over $\mathcal{T}_V$. The next lemma gives a clear description of the equivalences classes.

Lemma 2.14. *For each equivalence class $[x]$ of $(\mathcal{T}_V, R_V)$ there exists a convex set $I \subset \mathbb{R}$ of the real line and a bijective map $F : I \to [x]$ satisfying:*

$$\tau(F(t_1), F(t_2)) = t_2 - t_1, \quad \forall\, t_1 \leq t_2 \in I. \tag{2.22}$$

Moreover, calling $\overline{\{z \in [x]\}}$ the topological closure of $\{z \in [x]\} \subset X$, it holds

$$\overline{\{z \in [x]\}} \setminus \{z \in [x]\} = \overline{\{z \in [x]\}} \setminus \mathcal{T}_V \subset \mathfrak{a}(\mathcal{T}_V^e) \cup \mathfrak{b}(\mathcal{T}_V^e). \tag{2.23}$$

The equivalence classes of R_V inside $\mathcal{T}_V$ will be called *rays.*

Concerning the measurability properties of the sets we have considered so far, the set $I^+(x) = \{y \in X \colon \tau(x,y) > 0\}$ is open by continuity of τ (ensured by global hyperbolicity) and the same is valid for $I^-(x)$. Accordingly, $I^+(V) = \bigcup_{x\in V} I^+(x)$ is an open subset of X, and the same is valid for $I^-(V)$. Since τ_V is sup of continuous functions, it is lower semi-continuous. It follows that the set Γ_V is Borel measurable (see (2.18)). It follows that also R_V is Borel measurable, yielding that $\mathcal{T}_V^{end}$ defined in (2.19) is an analytic set (recall that analytic sets are precisely projections of Borel subsets of complete and separable metric spaces and the σ-algebra they generate is denoted by $\mathcal{A}$, we refer to [52] for more details). The transport set $\mathcal{T}_V$ defined in (2.21) can be proved to be an analytic set as well [14, Lem. 4.7].

Finally, the existence an $\mathcal{A}$-measurable quotient map $\mathfrak{Q}$ of the equivalence relation R_V over $\mathcal{T}_V$ can be obtained by a careful use of selection theorems:

Lemma 2.15 (Prop. 4.9, [14]). *There exists an $\mathcal{A}$-measurable quotient map $\mathfrak{Q} : \mathcal{T}_V \to X$ of the equivalence relation R_V over $\mathcal{T}_V$, i.e.*

$$\mathfrak{Q} : \mathcal{T}_V \to \mathcal{T}_V, \quad (x, \mathfrak{Q}(x)) \in R_V, \quad (x,y) \in R_V \Rightarrow \mathfrak{Q}(x) = \mathfrak{Q}(y). \tag{2.24}$$

We will denote by $Q := \mathfrak{Q}(\mathcal{T}_V) \subset X$ the quotient set (which is $\mathcal{A}$-measurable), and by X_α, with $\alpha \in Q$, the rays. Recall that each X_α is isometric to a real, possibly unbounded, open interval.

We refer to [14, Sect. 4.1, 4.2] for the missing details.

2.3. **Synthetic Timelike Ricci curvature lower bounds.** We briefly recall the synthetic formulation of timelike Ricci lower bounds for a globally hyperbolic measured Lorentzian geodesic space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ as given in [14] (after [43] and [46]), see also [15] for a survey.

Definition 2.16 (Measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$). A *measured Lorentzian pre-length space* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is a Lorentzian pre-length space endowed with a Radon non-negative measure $\mathfrak{m}$. We say that $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is globally hyperbolic (resp. geodesic) if $(X, \mathsf{d}, \ll, \leq, \tau)$ is so.

We denote $\mathcal{P}_{ac}(X)$ (resp. $\mathcal{P}_c(X)$) the space of probability measures absolutely continuous with respect to $\mathfrak{m}$ (resp. the space of probability measures with compact support). Given $\mu \in \mathcal{P}(X)$ its relative entropy w.r.t. $\mathfrak{m}$ is given by

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int_M \rho \log \rho \, \mathfrak{m},$$

if $\mu = \rho\, \mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $(\rho \log \rho)_+$ is $\mathfrak{m}$-integrable. Otherwise we set $\mathrm{Ent}(\mu|\mathfrak{m}) = +\infty$. A simple application of Jensen inequality using the convexity of $(0,\infty) \ni t \mapsto t \log t$ gives

$$\mathrm{Ent}(\mu|\mathfrak{m}) \geq -\log \mathfrak{m}(\operatorname{supp} \mu) > -\infty, \quad \forall \mu \in \mathcal{P}_c(X). \tag{2.25}$$

We set $\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) : \mathrm{Ent}(\mu|\mathfrak{m}) \in \mathbb{R}\}$ to be the finiteness domain of the entropy. An important property of the relative entropy is the lower-semicontinuity under narrow convergence.

The following is the definition of the synthetic timelike Ricci curvature lower bounds.

Definition 2.17 ($\mathsf{TCD}^e_p(K,N)$ and $\mathsf{wTCD}^e_p(K,N)$ conditions). Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. We say that a measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TCD}^e_p(K,N)$ (resp. $\mathsf{wTCD}^e_p(K,N)$) if the following holds. For any pair $(\mu_0, \mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})))^2$ which is timelike p-dualisable (resp. $(\mu_0,\mu_1) \in [\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_c(X)]^2$ which is strongly timelike p-dualisable) by some $\pi \in \Pi^{p\text{-opt}}_{\ll}(\mu_0, \mu_1)$, there exists an ℓ_p-geodesic

$(\mu_t)_{t\in[0,1]}$ such that the function $[0,1] \ni t \mapsto e(t) := \mathrm{Ent}(\mu_t|\mathrm{Vol}_g)$ is semi-convex (and thus in particular it is locally Lipschitz in $(0,1)$) and it satisfies

$$e''(t) - \frac{1}{N} e'(t)^2 \geq K \int_{X\times X} \tau(x,y)^2 \, \pi(dxdy), \tag{2.26}$$

in the distributional sense on $[0,1]$.

Definition 2.17 corresponds to a differential/infinitesimal formulation of the $\mathsf{TCD}^e_p(K,N)$ condition. In order to have also an integral/global formulation it is convenient to introduce the following entropy (cf. [23])

$$U_N(\mu|\mathfrak{m}) := \exp\left(-\frac{\mathrm{Ent}(\mu|\mathfrak{m})}{N}\right). \tag{2.27}$$

It is straightforward to check that $[0,1] \ni t \mapsto e(t)$ is semi-convex and satisfies (2.26) if and only if $[0,1] \ni t \mapsto u_N(t) := \exp(-e(t)/N)$ is semi-concave and satisfies

$$u_N'' \leq -\frac{K}{N} \|\tau\|^2_{L^2(\pi)} \, u_N. \tag{2.28}$$

Set
(2.29)

$$\mathfrak{s}_\kappa(\vartheta) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\vartheta), & \kappa > 0 \\ \vartheta, & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\vartheta), & \kappa < 0 \end{cases}, \qquad \mathfrak{c}_\kappa(\vartheta) := \begin{cases} \cos(\sqrt{\kappa}\vartheta), & \kappa \geq 0 \\ \cosh(\sqrt{-\kappa}\vartheta), & \kappa < 0 \end{cases},$$

and

$$\sigma^{(t)}_\kappa(\vartheta) := \begin{cases} \frac{\mathfrak{s}_\kappa(t\vartheta)}{\mathfrak{s}_\kappa(\vartheta)}, & \kappa\vartheta^2 \neq 0 \text{ and } \kappa\vartheta^2 < \pi^2 \\ t, & \kappa\vartheta^2 = 0 \\ +\infty, & \kappa\vartheta^2 \geq \pi^2 \end{cases}. \tag{2.30}$$

Note that the function $\kappa \mapsto \sigma^{(t)}_\kappa(\vartheta)$ is non-decreasing for every fixed ϑ, t. With the above notation, the differential inequality (2.28) is equivalent to the integrated version (cf. [23, Lemma 2.2]):

$$u_N(t) \geq \sigma^{(1-t)}_{K/N}\left(\|\tau\|_{L^2(\pi)}\right) u_N(0) + \sigma^{(t)}_{K/N}\left(\|\tau\|_{L^2(\pi)}\right) u_N(1). \tag{2.31}$$

We thus have that both the $\mathsf{TCD}^e_p(K,N)$ and the $\mathsf{wTCD}^e_p(K,N)$ can be formulated as in terms of (2.31). It has recently been shown in [6, Thm. 3.35] that under the timelike non-branching assumption TCD^e_p and wTCD^e_p are equivalent conditions for any choice of the parameters K and N.

By considering (K,N)-convexity properties only of those ℓ_p-geodesics $(\mu_t)_{t\in[0,1]}$ where μ_1 is a Dirac delta one obtains the following weaker condition [14] (see also Sturm [54] and Ohta [47]) independent on p.

Definition 2.18. Fix $K \in \mathbb{R}$, $N \in (0,\infty)$. The measured globally hyperbolic Lorentz geodesic space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TMCP}^e(K,N)$ if and only if for any $\mu_0 \in \mathcal{P}_c(X) \cap \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$ and for any $x_1 \in X$ such that

$x \ll x_1$ for μ_0-a.e. $x \in X$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to $\mu_1 = \delta_{x_1}$ such that

$$U_N(\mu_t|\mathfrak{m}) \geq \sigma^{(1-t)}_{K/N}\left(\|\tau(\cdot, x_1)\|_{L^2(\mu_0)}\right) U_N(\mu_0|\mathfrak{m}), \quad \forall t \in [0,1). \tag{2.32}$$

As expected, the $\mathsf{wTCD}^e_p(K,N)$ condition implies the $\mathsf{TMCP}^e(K,N)$, see [14, Prop. 3.12]. We next recall some useful results concerning the existence and uniqueness of optimal plans in timelike non-branching $\mathsf{TMCP}^e(K,N)$ spaces.

Theorem 2.19 (Prop. 3.19, Thm. 3.20 and 3.21 in [14])**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic Lorentzian geodesic space satisfying $\mathsf{TMCP}^e(K,N)$.*
Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$, with $\mu_0 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$. Assume that there exists $\pi \in \Pi^{p\text{-}opt}_{\leq}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$.

Then there exists a unique optimal coupling $\pi \in \Pi^{p\text{-}opt}_{\leq}(\mu_0, \mu_1)$ such that $\pi(\{\tau > 0\}) = 1$ and it is induced by a map T, i.e. $\pi = (\mathrm{Id}, T)_\sharp \mu_0$ and

$$\ell_p(\mu_0, \mu_1)^p = \int_X \tau(x, T(x))^p \, \mu_0(dx).$$

Moreover there exists a unique $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ with $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta(\{\tau > 0\}) = 1$ and such η is induced by a map, i.e. there exists $\mathfrak{T} : X \to \mathrm{TGeo}(X)$ such that $\eta = \mathfrak{T}_\sharp \mu_0$; in particular, $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta = \pi$. Finally, the ℓ_p-geodesic $\mu_t = (\mathrm{e}_t)_\sharp \eta$ satisfies $\mu_t = \rho_t \mathfrak{m} \ll \mathfrak{m}$.

2.3.1. *Disintegration of $\mathfrak{m}$ and regularity of conditional measures.* The partition in rays recalled in Section 2.2.2 has a natural interplay with the synthetic curvature conditions: via Disintegration Theorem (after Lemma 2.15) one can associate to the partition of the transport set a decomposition in conditional measures of the reference measure $\mathfrak{m}$ that inherits the synthetic curvature-dimension properties.

Below, we briefly summarise the results from [14, Sect. 4]. We will denote by $\mathcal{M}_+(X)$ the space of non-negative Radon measures over (X, d).

Theorem 2.20. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a globally hyperbolic timelike non-branching Lorentzian geodesic space satisfying $\mathsf{TMCP}^e(K,N)$, assume that the causally-reversed structure satisfies the same conditions and let $V \subset X$ be a Borel achronal timelike complete subset.*

Considering $\mathcal{T}^{end}_V, \mathfrak{a}(\mathcal{T}^{end}_V), \mathfrak{b}(\mathcal{T}^{end}_V)$ and $\mathcal{T}_V$ defined in (2.19), (2.20), (2.21), then $\mathfrak{m}(\mathfrak{a}(\mathcal{T}^{end}_V)) = \mathfrak{m}(\mathfrak{b}(\mathcal{T}^{end}_V) = 0$ and the following disintegration formula is valid:

$$\mathfrak{m}\llcorner_{\mathcal{T}^{end}_V} = \mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) \tag{2.33}$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ satisfies the following properties:

(1) for any $\mathfrak{m}$-measurable set B, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;

(2) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ (strong consistency);
(3) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha \ll \mathcal{L}^1 \llcorner_{X_\alpha}$;
(4) writing $\mathfrak{m}_\alpha = h(\alpha,\cdot)\mathcal{L}^1 \llcorner_{X_\alpha}$, then for $\mathfrak{q}$-a.e. $\alpha \in Q$ it holds that $h(\alpha,\cdot) \in L^1_{loc}(X_\alpha, \mathcal{L}^1 \llcorner_{X_\alpha})$; moreover $h(\alpha,\cdot)$ has an almost everywhere representative that is continuous on $\overline{X_\alpha}$, and locally Lipschitz and positive in the interior of X_α.

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique in the following sense: if any other map $Q \ni \alpha \mapsto \bar{\mathfrak{m}}_\alpha \in \mathcal{P}(X)$ satisfies points (1)-(2), then $\bar{\mathfrak{m}}_\alpha = \mathfrak{m}_\alpha$ for $\mathfrak{q}$-a.e. $\alpha \in Q$.

To localise curvature bounds, a larger family of Lorentz-Wasserstein geodesics was needed: we recall a second way to construct ℓ^p-cyclically monotone sets (introduced in [11] for the metric setting and adapted to the Lorentzian framework in [14]).

Proposition 2.21. *Let $\Delta \subset \Gamma_V$ be such that, for all $(x_0,y_0),(x_1,y_1) \in \Delta$:*

$$(\tau_V(x_0) - \tau_V(x_1))(\tau_V(y_0) - \tau_V(y_1)) \geq 0. \tag{2.34}$$

Then Δ is ℓ^p-cyclically monotone for each $p \in (0,1)$.

3. Localization of Timelike Ricci curvature bounds

We will improve on the results of [14] concerning the regularity properties of the marginal measures associated to the decomposition induced by τ_V.

To obtain the estimates for the one-dimensional densities, it is more convenient to use an equivalent form of the TCD^e_p condition. This equivalent form of TCD^e_p, whose Riemannian counterpart is the well known $\mathsf{CD}^*(K,N)$ condition of Bacher and Sturm [3], has recently been presented also in the Lorentzian setting in [6] and is denoted by TCD^*_p.

We will not use the full equivalence between TCD^e_p and TCD^*_p proven in [6] (see also [23] for the earlier equivalence between CD^e and CD^*), but merely that TCD^e_p implies TCD^*_p. For readers' convenience we now include a self-contained proof of this implication.

Proposition 3.1. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a globally hyperbolic, timelike non-branching Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(K,N)$ for some $p \in (0,1), K \in \mathbb{R}, N \in [1,\infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_c(X)$ with $\mu_0, \mu_1 \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$ and assume that there exists $\pi \in \Pi^{p\text{-}opt}_{\leq}(\mu_0,\mu_1)$ such that $\pi(\{\tau > 0\}) = 1$.*

Then π is the unique element of $\Pi^{p\text{-}opt}_{\leq}(\mu_0,\mu_1)$ concentrated on $\{\tau > 0\}$. Accordingly, there exists a unique optimal dynamical plan η such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta = \pi$. Moreover, the ℓ_p-geodesic $\mu_t = (\mathrm{e}_t)_\sharp \eta$ satisfies $\mu_t = \rho_t \mathfrak{m} \ll \mathfrak{m}$ for every $t \in [0,1]$, and for η-a.e. $\gamma \in \mathrm{TGeo}(X)$ it holds

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma^{(1-t)}_{K/N}(\tau(\gamma_0,\gamma_1))\,\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma^{(t)}_{K/N}(\tau(\gamma_0,\gamma_1))\,\rho_1(\gamma_1)^{-\frac{1}{N}}, \tag{3.1}$$

for all $t \in [0,1]$.

Proof. The first part of the claim is simply Theorem 2.19. We are left to prove (3.1) for the ℓ_p-geodesic induced by the unique optimal dynamical plan η. Fix the map T such that $(Id, T)_\sharp \mu_0 = \pi$.
Since $\pi(\{\tau > 0\}) = 1$, there exists a countable collection of Borel sets A_n such that

$$A_n \times T(A_n) \subset \{\tau > 0\}, \qquad \mu_0 \left(\cup_{n\in\mathbb{N}} A_n\right) = 1.$$

Without loss of generality, we can assume that the sets $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint and $\mu_0(A_n) > 0$ for each $n \in \mathbb{N}$. Consider $\pi_n := (Id, T)_\sharp \mu_0 \llcorner_{A_n} / \mu_0(A_n)$. Such π_n is the unique ℓ_p-optimal plan between its marginal measures (that we denote by $\mu_{0,n}$ and $\mu_{1,n}$) and, accordingly, $\eta\llcorner_{\mathrm{e}_0^{-1}(A_n)}/\mu_0(A_n)$ the unique optimal dynamical plan. Hence from Definition 2.17 and (2.31) it follows that

$$U_N(\mu_{t,n}|\mathfrak{m}) \geq \sigma^{(1-t)}_{K/N}\left(\|\tau\|_{L^2(\pi_n)}\right) U_N(\mu_{0,n}|\mathfrak{m}) + \sigma^{(t)}_{K/N}\left(\|\tau\|_{L^2(\pi_n)}\right) U_N(\mu_{1,n}|\mathfrak{m}),$$

for all $t \in [0, 1]$ and all $n \in \mathbb{N}$. From here we can repeat verbatim a classical argument already present in the literature (see [23, Thm. 3.12]) that permits, by restricting to finer subsets of timelike geodesics via η, to obtain the inequality (3.1) and therefore the claim. □

3.1. **Localization of timelike Ricci lower bounds to τ_V-transport rays.** The goal of this section is to localize the timelike Ricci curvature lower bounds $\mathsf{TCD}^e_p(K,N)$ along the transport set of the signed Lorentzian-distance function from V, namely τ_V.

Theorem 3.2. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(K,N)$ and assume that the causally-reversed structure satisfies the same conditions.*

Let $V \subset X$ be a Borel achronal timelike complete subset and consider the disintegration formula given by Theorem 2.20.

Then, for $\mathfrak{q}$-a.e. α, the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the classical $\mathsf{CD}(K,N)$; namely, writing $\mathfrak{m}_\alpha = h(\alpha,\cdot)\mathcal{L}^1\llcorner_{X_\alpha}$, in holds that $h(\alpha,\cdot)$ is semi-concave (and thus twice differentiable $\mathcal{L}^1$-a.e. on X_α) and it satisfies the differential inequality

$$\frac{\partial^2}{\partial x^2}\log h(\alpha,x) + \frac{1}{N-1}\left(\frac{\partial}{\partial x}\log h(\alpha,x)\right)^2 \leq -K, \tag{3.2}$$

at any point x in the interior of X_α where $h(\alpha,\cdot)$ is twice differentiable.

Proof. For $x \in \mathcal{T}_V$ we will write $R(x)$ to denote its equivalence class in $(\mathcal{T}_V, R_V)$, i.e. the "ray passing through x" (recall Lemma 2.14). For a subset $B \subset \mathcal{T}_V$, we denote $R(B) := \bigcup_{x\in B} R(x)$.
Let $\bar{Q} \subset Q$ be an arbitrary compact subset of positive $\mathfrak{q}$-measure for which

there exist $\varepsilon > 0$ and $a_0 < a_1$ such that

$$\sup_{x,y\in X_\alpha} \tau(x,y) > \varepsilon, \quad X_\alpha \cap \{\tau_V = a_0\} \neq \emptyset, \quad X_\alpha \cap \{\tau_V = a_1\} \neq \emptyset \quad \forall \alpha \in \bar{Q},$$
$$R(\bar{Q}) \cap \tau_V^{-1}([a_0, a_1]) \Subset X,$$
$$\{(x,y) \in \Gamma_V \,:\, x, y \in R(\bar{Q}),\, \tau_V(x) = a_0,\, \tau_V(y) = a_1\} \Subset \{\tau > 0\}.$$

For any $A_0, A_1 \in (a_0, a_1)$ with $A_0 < A_1$, and $L_0, L_1 > 0$ satisfying $A_0 + L_0 < A_1 + L_1 < a_1$, consider the probability measures

$$\mu_0 = \int_{\bar{Q}} \frac{\mathcal{L}^1 \llcorner_{X_\alpha \cap [A_0, A_0+L_0]}}{L_0}\, \mathfrak{q}(d\alpha), \qquad \mu_1 = \int_{\bar{Q}} \frac{\mathcal{L}^1 \llcorner_{X_\alpha \cap [A_1, A_1+L_1]}}{L_1}\, \mathfrak{q}(d\alpha).$$

Proposition 2.21 ensures that the transport plan π defined as the monotone rearrangement along each ray X_α of the normalized Lebesgue measure $\mathcal{L}^1 \llcorner_{X_\alpha \cap [A_0, A_0+L_0]}/L_0$ to $\mathcal{L}^1 \llcorner_{X_\alpha \cap [A_1, A_1+L_1]}/L_1$ is is ℓ^p-cyclically monotone. Since $\pi(X^2_{\ll}) = 1$, we infer that π is ℓ_p-optimal thanks to Proposition 2.7.

From Theorem 2.20 it is immediate to observe that $\mu_0, \mu_1 \ll \mathfrak{m}$, notice indeed that the density $h(\alpha, \cdot)$ is strictly positive in the interior of the transport ray X_α, for $\mathfrak{q}$-a.e. α. Hence we can invoke Proposition 3.1 to deduce that π is the unique element in $\Pi^{p\text{-opt}}_{\ll}(\mu_0, \mu_1)$. Moreover, there exists a unique optimal dynamical plan η such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta = \pi$, and the ℓ_p-geodesic $\mu_t = (\mathrm{e}_t)_\sharp \eta$ satisfies $\mu_t = \rho_t \mathfrak{m} \ll \mathfrak{m}$ for every $t \in [0,1]$, and for η-a.e. $\gamma \in \mathrm{TGeo}(X)$, the concavity estimate (3.1) holds.

The ℓ_p-geodesic μ_t can be written explicitly. Indeed, consider

$$\bar{\mu}_t := \int_{\bar{Q}} \frac{\mathcal{L}^1_{X_\alpha \cap [A_t, A_t+L_t]}}{L_t}\, \mathfrak{q}(d\alpha), \quad A_t = A_0(1-t) + A_1 t,\ L_t = L_0(1-t) + L_1 t.$$

Such $(\bar{\mu}_t)_{t\in[0,1]}$ can be lifted to an optimal dynamical plan $\bar{\eta}$ such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \bar{\eta} = \pi$. By the uniqueness discussed above, we infer that $\bar{\eta} = \eta$ and thus $\mu_t = \bar{\mu}_t$ for all $t \in [0,1]$. Since

$$\mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) = \int_Q h(\alpha, \cdot)\, \mathcal{L}^1 \llcorner_{X_\alpha}\, \mathfrak{q}(d\alpha),$$

one has that $\mu_s = \rho_s \mathfrak{m}$, with $\rho_s(\alpha, t) = \frac{1}{L_s h(\alpha,t)}$, for all $t \in [A_s, A_s + L_s]$. Hence, the concavity estimate (3.1) on $\rho_s(\alpha, \gamma_s)$ writes as:

$$\begin{aligned}(L_s)^{\frac{1}{N}} h(\alpha, (1-s)t_0 + st_1)^{\frac{1}{N}} \geq & \sigma^{(1-s)}_{K/N}(t_1 - t_0)(L_0)^{\frac{1}{N}} h(\alpha, t_0)^{\frac{1}{N}} \\ & + \sigma^{(s)}_{K/N}(t_1 - t_0)(L_1)^{\frac{1}{N}} h(\alpha, t_1)^{\frac{1}{N}},\end{aligned}$$

for every $s \in [0,1]$, for $\mathcal{L}^1$-a.e. $t_0 \in [A_0, A_0 + L_0]$ and t_1 obtained as the image of t_0 through the monotone rearrangement of $[A_0, A_0 + L_0]$ to $[A_1, A_1 + L_1]$. Specializing the previous inequality for $s = 1/2$ and noticing that $t_0 =$

$A_0 + \tau L_0$ gives $t_1 = A_1 + \tau L_1$, we obtain:

$$
\begin{aligned}
&(L_0 + L_1)^{\frac{1}{N}} h(\alpha, A_{1/2} + \tau L_{1/2})^{\frac{1}{N}} \\
&\quad \geq 2^{\frac{1}{N}} \sigma_{K/N}^{(1/2)}(A_1 - A_0 + \tau|L_1 - L_0|)\big\{(L_0)^{\frac{1}{N}} h(\alpha, A_0 + \tau L_0)^{\frac{1}{N}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (L_1)^{\frac{1}{N}} h(\alpha, A_1 + \tau L_1)^{\frac{1}{N}}\big\},
\end{aligned}
$$

for $\mathcal{L}^1$-a.e. $\tau \in [0,1]$, where we used the notation $A_{1/2} := \frac{A_0+A_1}{2}, L_{1/2} := \frac{L_0+L_1}{2}$. Recalling from Theorem 2.20 that the map $s \mapsto h(\alpha, s)$ is continuous, we infer that the previous inequality also holds for $\tau = 0$:

$$
\begin{aligned}
&(L_0 + L_1)^{\frac{1}{N}} h(\alpha, A_{1/2})^{\frac{1}{N}} \\
&\quad \geq 2^{\frac{1}{N}} \sigma_{K/N}^{(1/2)}(A_1 - A_0)\left\{(L_0)^{\frac{1}{N}} h(\alpha, A_0)^{\frac{1}{N}} + (L_1)^{\frac{1}{N}} h(\alpha, A_1)^{\frac{1}{N}}\right\},
\end{aligned}
\tag{3.3}
$$

for all $A_0 < A_1$ with $A_0, A_1 \in (a_0, a_1)$, all sufficiently small L_0, L_1 and $\mathfrak{q}$-a.e. $\alpha \in Q$, with exceptional set depending on A_0, A_1, L_0 and L_1.

Noticing that (3.3) depends in a continuous way on A_0, A_1, L_0 and L_1, it follows that there exists a common exceptional set $N \subset Q$ with the following properties: $\mathfrak{q}(N) = 0$ and for each $\alpha \in Q \setminus N$ the inequality (3.3) holds true for all A_0, A_1, L_0 and L_1. Then one can make the following (optimal) choice

$$
L_0 := L\frac{h(\alpha, A_0)^{\frac{1}{N-1}}}{h(\alpha, A_0)^{\frac{1}{N-1}} + h(\alpha, A_1)^{\frac{1}{N-1}}}, \qquad L_1 := L\frac{h(\alpha, A_1)^{\frac{1}{N-1}}}{h(\alpha, A_0)^{\frac{1}{N-1}} + h(\alpha, A_1)^{\frac{1}{N-1}}},
$$

for any $L > 0$ sufficiently small, and obtain that

$$
h(\alpha, A_{1/2})^{\frac{1}{N-1}} \geq 2^{\frac{1}{N-1}} \sigma_{K/N}^{(1/2)}(A_1 - A_0)^{\frac{N}{N-1}} \left\{h(\alpha, A_0)^{\frac{1}{N-1}} + h(\alpha, A_1)^{\frac{1}{N-1}}\right\}.
\tag{3.4}
$$

By [3, Prop. 5.5] we now that for any $K' < \tilde{K} < K$ there exists $\Theta^* > 0$ such that for all $0 \leq \Theta \leq \Theta^*$ it holds $\sigma_{\tilde{K}/N}^{(t)}(\theta) \geq \sigma_{K'/(N-1)}^{(t)}(\theta)^{\frac{N-1}{N}} t^{\frac{1}{N}}$, for all $t \in [0,1]$. Hence

$$
2^{\frac{1}{N-1}} \sigma_{\tilde{K}/N}^{(1/2)}(\theta)^{\frac{N}{N-1}} \geq \sigma_{K'/(N-1)}^{(1/2)}(\theta).
$$

Plugging the last inequality into (3.4) gives

$$
h(\alpha, A_{1/2})^{\frac{1}{N-1}} \geq \sigma_{K'/N-1}^{(1/2)}(A_1 - A_0) \left\{h(\alpha, A_0)^{\frac{1}{N-1}} + h(\alpha, A_1)^{\frac{1}{N-1}}\right\},
$$

for all A_0, A_1 sufficiently close. In particular this shows that $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ verifies the $\mathsf{CD}_{loc}(K', N)$ condition that is easily seen in dimension one to be equivalent to the full $\mathsf{CD}(K', N)$ condition, as they are both equivalent to the differential inequality (3.2), with K replaced by K'. To prove the claim is then enough to let K' converge to K and invoke the stability of the CD condition [42, 54] to obtain that $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ verifies $\mathsf{CD}(K, N)$. $\square$

To conclude this part we recall a straightforward consequence of the $\mathsf{CD}(K,N)$ condition along the transport rays: for all $x_0, x_1 \in X_\alpha$,

(3.5)
$$\left(\frac{\mathfrak{s}_{K/(N-1)}(b-\tau_V(x_1))}{\mathfrak{s}_{K/(N-1)}(b-\tau_V(x_0))}\right)^{N-1} \leq \frac{h(\alpha,x_1)}{h(\alpha,x_0)} \leq \left(\frac{\mathfrak{s}_{K/(N-1)}(\tau_V(x_1)-a)}{\mathfrak{s}_{K/(N-1)}(\tau_V(x_0)-a)}\right)^{N-1},$$

with $a < \tau_V(x_0) < \tau_V(x_1) < b$ and $b-a \leq \pi\sqrt{(N-1)/(K\vee 0)}$. In other words, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ also satisfies the weaker $\mathsf{MCP}(K,N)$.

4. A Lorentzian isoperimetric type inequality

The goal of the next definition is to define a notion of "area" for an achronal set $A \subset X$. The rough idea is to use the signed time-separation function τ_A from A to define a "future ϵ-tubular neighbourhood" of A, and then define the "area of A" as the first variation of the volume of such future-tubular neighbourhoods. This can be seen as a Lorentzian counterpart of the outer Minkowski content in metric measure spaces.

Definition 4.1 (Timelike Minkowski content)**.** Let $A \subset X$ be a Borel achronal set and consider the signed time-separation function τ_A from A, see (2.15). We define the *future Minkowski content* of A by

(4.1)
$$\mathfrak{m}^+(A) := \inf_{U\in\mathcal{U}} \limsup_{\varepsilon\to 0} \frac{\mathfrak{m}(\tau_A^{-1}((0,\varepsilon))\cap U)}{\varepsilon}, \quad \mathcal{U} := \{U \subset X \colon U \text{ open}, \ A \subset U\}.$$

We define the *past Minkowski content* of A by

$$\mathfrak{m}^-(A) := \inf_{U\in\mathcal{U}} \limsup_{\varepsilon\to 0} \frac{\mathfrak{m}(\tau_A^{-1}((-\varepsilon,0))\cap U)}{\varepsilon}. \tag{4.2}$$

The presence of the infimum over the collection $\mathcal{U}$ of open sets containing A is necessary to avoid infinite volume of the future ε-enlargement of A with respect to τ; indeed, tipically (e.g. in Minkowski spacetime), $\tau_A^{-1}((0,\varepsilon))$ has infinite volume for every $\varepsilon > 0$.

Remark 4.2 (Timelike Minkowski content equals area in the smooth framework)**.** In the smooth framework, the timelike Minkowski content can be related to the classical area of A, as illustrated below. If (M,g) is a globally hyperbolic spacetime and $A \subset M$ a smooth, spacelike, acausal and future causally complete hypersurface, then the signed time-separation function τ_A is smooth on $I^+(A)$ – outside of a set of measure zero. As τ_A has timelike gradient $\nabla\tau_A$ with $g(\nabla\tau_A, \nabla\tau_A) = -1$, the level sets $\tau_A^{-1}(t)$ are spacelike hypersurfaces of $I^+(A)$, for almost every $t > 0$. Denoting by $A_t = \tau_A^{-1}(t)$, coarea formula (see for instance [55, Prop. 3]) implies that

$$\mathrm{Vol}_{g \llcorner I^+(A)} = \int_{(0,+\infty)} \mathrm{Vol}_{g_t}\, dt,$$

where Vol_{g_t} is the volume measure induced by g_t, the Riemannian metric induced by g over A_t. Hence, in the smooth setting above, the timelike Minkowski content coincides with the area induced by the ambient Lorentzian metric g.

Concerning the case when A is a smooth null hypersurface, then $\mathfrak{m}^+(A)$ has to be zero like the induced volume. Let us briefly sketch the argument. If $x \in I^+(A)$, then the supremum defining $\tau_A(x)$ cannot be realized as a maximum. Otherwise, by the classical first variation argument, the optimal path from x to A has to be a geodesic normal to A; but since A is null, the normal directions are contained in the tangent space of A. This forces the geodesic to never leave A and yields a contradiction. It follows that all the optimal directions should leave A from its "boundary" (of higher codimension). Because of the scaling limit in the definition, such a lower dimensional contribution is not detected by the timelike Minkowski content which thus has to vanish.

We will use the localization associated to a timelike complete achronal set V to bound from above the future and past Minkowski content of an achronal set A. To exclude lightlike variations, we introduce a stronger condition for achronal sets.

Definition 4.3 (Empty future V-boundary, $\partial_V^+ A = \emptyset$)**.** Let $V \subset X$ be a timelike complete achronal set. We say that an achronal set $A \subset I^+(V)$ has *empty future V-boundary*, and we write $\partial_V^+ A = \emptyset$, if the following property is satisfied: for every $x \in I^+(A)$ and every geodesic $\gamma : [0,1] \to X$ with $\gamma_0 \in V$, $\gamma_1 = x$, such that $\tau_V(x) = \tau(\gamma_0, x)$, it holds that $\gamma_{[0,1]} \cap A \neq \emptyset$.

If A satisfies the reversed condition in the past we say that A has *empty past V-boundary* and we write $\partial_V^- A = \emptyset$. In case A has both past and future V-boundaries empty, then we say that A has empty V-boundary and we write $\partial_V A = \emptyset$.

It is natural to compare Definition 4.3 with the fundamental notion of Cauchy hypersurface. Recall that a Cauchy hypersurface is a closed achronal set intersected exactly once by any inextendible causal curve [58, Sect. 8.3]. In case X is a smooth manifold with a continuous Lorentzian metric, then a Cauchy hypersurface is a closed achronal topological hypersurface [51, Prop. 5.2].

Lemma 4.4. *If A is a Cauchy hypersurface, then A has empty V-boundary.*

Proof. Let γ be a geodesic as in Definition 4.3 and let $\bar\gamma$ to be any maximal causal extension of γ. Then, by definition, $\bar\gamma$ has to meet A and by the reverse triangle inequality γ has to meet A. □

Remark 4.5. An elementary example (take for instance a Y shaped spacetime) shows that the reverse implication of Lemma 4.4 fails to be valid.

For the dimensional reduction we need also to specify the following notation. Let $(I, |\cdot|, \nu)$ be a one-dimensional metric measure space with I a closed interval and ν a non-negative Radon measure.

For a Borel set $A \subset I$, we denote

$$\nu^+(A) := \limsup_{\varepsilon \to 0} \frac{\nu(\cup_{x\in A}(x, x+\varepsilon))}{\varepsilon}, \qquad \nu^-(A) := \limsup_{\varepsilon \to 0} \frac{\nu(\cup_{x\in A}(x-\varepsilon, x))}{\varepsilon}. \tag{4.3}$$

We will say that $\nu^+(A)$ (resp. $\nu^-(A)$) is the future (resp. past) Minkowski content of A. Note that if A is bounded and $\nu = f(x)\,dx$ with f continuous, then the lim sup in (4.3) is actually a limit.

4.1. **Properties of timelike Minkowski contents.**

Proposition 4.6. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(K,N)$ and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel achronal timelike complete subset and consider the disintegration given by Theorem 2.20. Then the following hold:*

- *For any Borel achronal set $A \subset I^+(V)$ with $\partial^+_V A = \emptyset$ and $\inf_{x\in A} \tau_V(x) > 0$, it holds:*
$$\mathfrak{m}^+(A) \leq \int_Q \mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha), \tag{4.4}$$
where we adopt the notation (4.3) for $\mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})$.
If $A \subset I^-(V)$ is a Borel, achronal set with $\partial^-_V A = \emptyset$ and $\inf_{x\in A} -\tau_V(x) > 0$, then (4.4) holds replacing $\mathfrak{m}^+(A)$ by $\mathfrak{m}^-(A)$ and $\mathfrak{m}^+_\alpha$ by $\mathfrak{m}^-_\alpha$.
- *The following inequality holds true*
$$\mathfrak{m}^+(V) \geq \int_Q \mathfrak{m}^+_\alpha(V \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha). \tag{4.5}$$
The inequality (4.5) remains valid if we replace $\mathfrak{m}^+(V)$ by $\mathfrak{m}^-(V)$ and $\mathfrak{m}^+_\alpha$ by $\mathfrak{m}^-_\alpha$ (recall (4.3)).

Proof. We start by proving the first claim for $A \subset I^+(V)$. The one for $A \subset I^-(V)$ follows by reversing the causal structure.

From $A \subset I^+(V)$, it follows that $I^+(A) \subset I^+(V)$ implying the inclusion $\tau_A^{-1}((0,\varepsilon)) \subset I^+(V)$. Therefore applying Lemma 2.13 and Theorem 2.20 to V, we get

$$\mathfrak{m}(\tau_A^{-1}((0,\varepsilon))) = \int_Q \mathfrak{m}_\alpha(\tau_A^{-1}((0,\varepsilon)) \cap X_\alpha)\, \mathfrak{q}(d\alpha).$$

Consider X_α such that $\tau_A^{-1}((0,\varepsilon)) \cap X_\alpha \neq \emptyset$. By definition, it holds

$$\tau_A^{-1}((0,\varepsilon)) \cap X_\alpha = \{y \in X_\alpha \cap I^+(A) \mid \tau(x,y) < \varepsilon, \forall x \in A\}.$$

Since $\partial^+_V A = \emptyset$, necessarily $A \cap \overline{X_\alpha} \neq \emptyset$. As A is achronal there cannot be two distinct points in $A \cap \overline{X_\alpha}$. Therefore $A \cap \overline{X_\alpha} = \{a_\alpha\}$ and it holds

$$\begin{aligned}\tau_A^{-1}((0,\varepsilon)) \cap X_\alpha &\subset \{y \in X_\alpha \cap I^+(A) \mid \tau(x,y) < \varepsilon, \forall x \in A \cap \overline{X_\alpha}\} \\ &= (A \cap \overline{X_\alpha})^\varepsilon \cap X_\alpha,\end{aligned}$$

where with $(A \cap \overline{X_\alpha})^\varepsilon \cap X_\alpha$ we denote the right ε-enlargement of the set $A \cap \overline{X_\alpha}$ in the metric measure space $(\overline{X_\alpha}, |\cdot|, \mathfrak{m}_\alpha)$, i.e. in the sense of (4.3). Hence,

$$\frac{\mathfrak{m}(\tau_A^{-1}((0,\varepsilon)))}{\varepsilon} \leq \int_Q \frac{\mathfrak{m}_\alpha((A \cap \overline{X_\alpha})^\varepsilon)}{\varepsilon}\, \mathfrak{q}(d\alpha), \quad \forall \varepsilon > 0. \tag{4.6}$$

Since $A \cap \overline{X_\alpha} = \{a_\alpha\}$, then $\mathfrak{m}_\alpha(A) = \mathfrak{m}_\alpha(A \cap \overline{X_\alpha}) = 0$. To conclude the argument, we wish to use Fatou's Lemma in order to pass to the limit in the right hand side of (4.6). To this aim, we look for a function $g \in L^1(Q, \mathfrak{q})$ such that

$$\frac{\mathfrak{m}_\alpha((A \cap \overline{X_\alpha})^\varepsilon) - \mathfrak{m}_\alpha(A)}{\varepsilon} \leq \; g(\alpha), \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

By applying (3.5) (taking as initial point 0 that can be identified with $X_\alpha \cap V$) we obtain for all $\varepsilon \in (0, \varepsilon_0(K,N))$:

$$\begin{aligned}
\frac{\mathfrak{m}_\alpha((A \cap \overline{X_\alpha})^\varepsilon) - \mathfrak{m}_\alpha(A)}{\varepsilon} &= \frac{1}{\varepsilon} \int_{(a_\alpha, a_\alpha+\varepsilon)} h(\alpha, s)\, ds \\
&\overset{(3.5)}{\leq} \frac{h(\alpha, a_\alpha)}{\varepsilon} \int_{(a_\alpha, a_\alpha+\varepsilon)} \left(\frac{\mathfrak{s}_{K/(N-1)}(s)}{\mathfrak{s}_{K/(N-1)}(a_\alpha)} \right)^{N-1} ds \\
&\leq C(K, N, \inf_A \tau_V)\; h(\alpha, a_\alpha) \\
&= C(K, N, \inf_A \tau_V)\; \mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha}).
\end{aligned}$$

If $Q \ni \alpha \mapsto \mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})$ is $\mathfrak{q}$-integrable, then we can choose $g(\alpha) := C(K, N, \inf_A \tau_V)\; \mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})$ as majorant and use Fatou's Lemma to pass to the limit in (4.6) and obtain

$$\mathfrak{m}^+(A) \leq \int_Q \mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha).$$

If $\mathfrak{m}^+_\alpha(A \cap \overline{X_\alpha})$ is not $\mathfrak{q}$-integrable then the claim holds trivially.

We now turn to the second claim. Fix any ray X_α from the disintegration associated to V. Since V is achronal, there exists a unique x_α such that $V \cap \overline{X_\alpha} = \{x_\alpha\}$ and therefore for any $y \in X_\alpha$ it holds $\tau(x_\alpha, y) = \tau_V(y) = \sup_{x \in V} \tau(x, y)$. Hence:

$$\begin{aligned}
\tau_V^{-1}((0,\varepsilon)) \cap \overline{X_\alpha} &= \{y \in \overline{X_\alpha} \mid 0 < \tau_V(y) < \varepsilon\} \\
&= \{y \in \overline{X_\alpha} \mid 0 < \tau(x_\alpha, y) < \varepsilon\} \\
&= (V \cap \overline{X_\alpha})^\varepsilon \cap X_\alpha,
\end{aligned}$$

where by $(V \cap \overline{X_\alpha})^\varepsilon \cap X_\alpha$ we denote the right ε-enlargement of the set $V \cap \overline{X_\alpha}$ in X_α, see (4.3). If U is any open set containing V, then $\tau_V^{-1}((0,\varepsilon)) \cap \overline{X_\alpha} \cap U = (V \cap \overline{X_\alpha})^\varepsilon \cap X_\alpha \cap U$.

Hence (recall that $\mathfrak{m}(V) = 0$):

$$\frac{\mathfrak{m}(\tau_V^{-1}((0,\varepsilon)) \cap U)}{\varepsilon} = \int_Q \frac{\mathfrak{m}_\alpha((V \cap \overline{X_\alpha})^\varepsilon \cap U)}{\varepsilon}\, \mathfrak{q}(d\alpha).$$

Using Fatou's Lemma we deduce that

$$\liminf_{\varepsilon \to 0} \frac{\mathfrak{m}(\tau_V^{-1}((0,\varepsilon) \cap U)}{\varepsilon} \geq \int_Q \mathfrak{m}_\alpha^+(V \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha),$$

notice indeed that the dependance on U on the right hand side disappear after the liminf. Finally, taking the infimum over all open sets U containing V, we obtain the claim. □

Remark 4.7.
- Notice that in the proof of the first claim of Proposition 4.6 we have also shown that, for achronal sets with empty future V-boundary, the restriction to open sets present in the definition of future Minkowski content (4.1) is not necessary to obtain a finite quantity.
- It is clear from the proof that, in the first claim in Proposition 4.6, instead of $\inf_A \tau_V > 0$ (resp. $\inf_A -\tau_V > 0$) it is sufficient to assume that

  $$\mathfrak{q}\text{-ess}\inf\{\tau_V(X_\alpha \cap A) \colon \alpha \in Q \text{ s.t. } X_\alpha \cap A \neq \emptyset\} > 0,$$

  or, respectively, $\mathfrak{q}\text{-ess}\inf\{-\tau_V(X_\alpha \cap A) \colon \alpha \in Q \text{ s.t. } X_\alpha \cap A \neq \emptyset\} > 0.$

In the next section we will obtain a monotonicity formula for the rescaled area of the spacelike hypersurface $V_t := \{\tau_V = t\}$. As not all the integral lines of τ_V will be longer than t, it will be enough to consider the following subset of Q:

$$Q_t := \{\alpha \in Q \colon \sup_{x \in X_\alpha} \tau_V(x) > t\}, \quad t > 0, \tag{4.7}$$

$$Q_t := \{\alpha \in Q \colon \inf_{x \in X_\alpha} \tau_V(x) < t\}, \quad t < 0. \tag{4.8}$$

Notice that the equidistant set $\{\tau_V = t\}$ can be obtained as the translation at (signed) distance t along the rays X_α, for $\alpha \in Q_t$.

Proposition 4.8. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ and $V \subset X$ satisfy the same assumption of Proposition 4.6. Then*

$$\mathfrak{m}^+(V_t) = \int_{Q_t} \mathfrak{m}_\alpha^+(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha), \qquad \mathfrak{m}^-(V_t) = \int_{Q_t} \mathfrak{m}_\alpha^-(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha), \tag{4.9}$$

for $t > 0$ and $t < 0$, respectively.

Proof. By symmetry, we will only deal with the case $t > 0$.

First, we verify that $\partial_V^+ V_t = \emptyset$. If $z \in I^+(V_t)$ then $\tau_V(z) > t$. By continuity of τ_V, any maximizing geodesic realizing τ_V going from z to V has to meet V_t.

Next, we claim that

$$\int_Q \mathfrak{m}^+_\alpha(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha) = \int_{Q_t} \mathfrak{m}^+_\alpha(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha). \tag{4.10}$$

Indeed, thanks to the hypothesis $t \neq 0$, the only rays contributing in the integral in the left hand side of (4.10) are those X_α for which $\overline{X_\alpha} \cap V_t \neq \emptyset$. Since the conditional measures $\mathfrak{m}_\alpha$ are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ on X_α (see Theorem 2.20), it is enough to take the integral in the left hand side over Q_t, i.e. on those rays strictly longer than t (otherwise the right Minkowski content on the ray would be 0). This proves (4.10). Then the inequality $\leq$ in the first identity of (4.9) follows from (4.4) combined with (4.10).

We next show the reverse inequality $\geq$ in the first identity of (4.9). First, we claim that τ_{V_t}, the time-separation function from V_t, satisfies:

$$\tau_{V_t}(w) = \tau_V(w) - t, \quad \forall w \in I^+(V_t). \tag{4.11}$$

Let $w \in I^+(V_t) \subset I^+(V)$. By Lemma 2.13 there exist $z_t \in V_t$ and $\alpha \in Q$ such that $z_t \in X_\alpha$ and $w \in \overline{X_\alpha}$; denoting by z the unique element of the set $\overline{X_\alpha} \cap V$, we obtain

$$\tau_{V_t}(w) \geq \tau(z_t, w) = \tau(z, w) - \tau(z, z_t) = \tau(z, w) - t = \tau_V(w) - t.$$

To show that $\tau_{V_t}(w) \leq \tau_V(w) - t$ we argue as follows: for each $\varepsilon > 0$ there exists $\zeta_t \in V_t$ such that $\tau_{V_t}(w) \leq \tau(\zeta_t, w) + \varepsilon$. Since V is timelike complete, by Lemma 2.12 there exists $\zeta \in V$ such that $t = \tau_V(\zeta_t) = \tau(\zeta, \zeta_t)$. The reverse triangle inequality $\tau(\zeta, w) \geq \tau(\zeta, \zeta_t) + \tau(\zeta_t, w)$ implies that

$$\tau_{V_t}(w) \leq \tau(\zeta_t, w) + \varepsilon \leq \tau(\zeta, w) - \tau(\zeta, \zeta_t) + \varepsilon = \tau(\zeta, w) - t + \varepsilon \leq \tau_V(w) - t + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\tau_{V_t}(w) \leq \tau_V(w) - t$. This completes the proof of (4.11).

From (4.11) and the disintegration (2.33), it follows that for every open set $U \supset V_t$ it holds

$$\begin{aligned} \mathfrak{m}\left(\tau_{V_t}^{-1}((0,\varepsilon)) \cap U\right) &= \mathfrak{m}\left(\tau_V^{-1}((t, t+\varepsilon)) \cap U\right) \\ &= \int_{Q_t} \mathfrak{m}_\alpha(\tau_V^{-1}((t, t+\varepsilon)) \cap X_\alpha \cap U)\, \mathfrak{q}(d\alpha). \end{aligned} \tag{4.12}$$

Reasoning as in the second part of the proof of Proposition 4.6 using Fatou's Lemma, we obtain

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{Q_t} \mathfrak{m}_\alpha(\tau_V^{-1}((t, t+\varepsilon)) \cap X_\alpha \cap U)\, \mathfrak{q}(d\alpha) \geq \int_{Q_t} \mathfrak{m}^+_\alpha(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(d\alpha). \tag{4.13}$$

To obtain the inequality $\geq$ in the first of (4.9) it is then enough to take the $\limsup$ as $\varepsilon \to 0^+$ in (4.12) divided by $\varepsilon > 0$, use (4.13) and finally take the infimum over all open sets $U \supset V_t$.

$\square$

4.2. **A sharp monotonicity formula for the area of τ_V-level sets.** Combining the results from the previous sections, we prove the following monotonicity formula for the area of the level sets of τ_V.

Theorem 4.9 (Monotonicity formula for the area)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(K,N)$ and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel achronal timelike complete subset.*

Denote by V_t the achronal slice at τ_V-distance t from V, i.e. $V_t := \{\tau_V = t\}$. Then

$$(0,\infty) \ni t \longmapsto \frac{\mathfrak{m}^+(V_t)}{(\mathfrak{s}_{K/(N-1)}(t))^{N-1}} \quad \textit{is monotonically non-increasing, and}$$

$$(-\infty,0) \ni t \longmapsto \frac{\mathfrak{m}^-(V_t)}{(\mathfrak{s}_{K/(N-1)}(-t))^{N-1}} \quad \textit{is monotonically non-decreasing.}$$

In the case $K = 0$, i.e. non-negative timelike Ricci (aka Hawking-Penrose strong energy condition), the monotonicity formula takes the following neat expression:

$$(0,\infty) \ni t \longmapsto \frac{\mathfrak{m}^+(V_t)}{t^{N-1}} \quad \text{is monotonically non-increasing, and}$$

$$(-\infty,0) \ni t \longmapsto \frac{\mathfrak{m}^-(V_t)}{(-t)^{N-1}} \quad \text{is monotonically non-decreasing.}$$

Proof. From Proposition 4.8 by the continuity of the densities $h(\alpha,\cdot)$ (see (4) in Theorem 2.20) we deduce that:

$$\mathfrak{m}^+(V_t) = \int_{Q_t} h(\alpha,t)\,\mathfrak{q}(d\alpha).$$

By the second inequality of (3.5) we have that

$$h(\alpha,t) \geq \left(\frac{\mathfrak{s}_{K/(N-1)}(t)}{\mathfrak{s}_{K/(N-1)}(T)}\right)^{N-1} h(\alpha,T), \quad \forall \alpha \in Q_T,\ \forall T > t > 0,$$

and thus, for all $T > t > 0$:

$$\begin{aligned} \frac{\mathfrak{m}^+(V_T)}{(\mathfrak{s}_{K/(N-1)}(T))^{N-1}} &= \int_{Q_T} \frac{h(\alpha,T)}{(\mathfrak{s}_{K/(N-1)}(T))^{N-1}}\,\mathfrak{q}(d\alpha) \\ &\leq \int_{Q_T} \frac{h(\alpha,t)}{(\mathfrak{s}_{K/(N-1)}(t))^{N-1}}\,\mathfrak{q}(d\alpha). \end{aligned}$$

Since by the very definition, for $T > t > 0$, it holds that $Q_t \supset Q_T$, we infer that, for all $T > t > 0$:

$$\frac{\mathfrak{m}^+(V_T)}{(\mathfrak{s}_{K/(N-1)}(T))^{N-1}} \leq \int_{Q_t} \frac{h(\alpha,t)}{(\mathfrak{s}_{K/(N-1)}(t))^{N-1}}\,\mathfrak{q}(d\alpha) = \frac{\mathfrak{m}^+(V_t)}{(\mathfrak{s}_{K/(N-1)}(t))^{N-1}}.$$

In the symmetric situation of $0 > t > T$, we can use the first inequality of (3.5) to obtain

$$h(\alpha,-t) \geq \left(\frac{\mathfrak{s}_{K/(N-1)}(-t)}{\mathfrak{s}_{K/(N-1)}(-T)}\right)^{N-1} h(\alpha,-T)$$

and thus

$$\begin{aligned}\frac{\mathfrak{m}^-(V_T)}{(\mathfrak{s}_{K/(N-1)}(-T))^{N-1}} &= \int_{Q_T} \frac{h(\alpha,T)}{(\mathfrak{s}_{K/(N-1)}(-T))^{N-1}}\,\mathfrak{q}(d\alpha) \\ &\leq \int_{Q_t} \frac{h(\alpha,t)}{(\mathfrak{s}_{K/(N-1)}(-t))^{N-1}}\,\mathfrak{q}(d\alpha) = \frac{\mathfrak{m}^-(V_t)}{(\mathfrak{s}_{K/(N-1)}(-t))^{N-1}}.\end{aligned}$$

□

Remark 4.10 (Sharpness of the monotonicity formula in Theorem 4.9)**.** The area mononoticity in Theorem 4.9 is sharp, as equality is achieved in conical regions in the model spaces.

More precisely, for $K = 0$ and $N \in \mathbb{N} \geq 2$, consider the N-dimensional Minkowski space with coordinates $(x_1, \dots, x_N)$ and Lorentzian metric $dx_1^2 + \dots + dx_{N-1}^2 - dx_N^2$. Let X be the conical region

$$X := \{0\} \cup \{(x_1, \dots, x_N) \colon x_N^2 \geq a\,(x_1^2 + \dots + x_{N-1}^2)\}, \quad \text{for some } a > 1.$$

Note that X, endowed with the standard metric and Lorentzian structure, is a timelike non-branching Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(0,N)$. Observe that $X \setminus \{0\}$ is a subset of the open cone of timelike vectors and that $V = \{0\}$ is a Borel achronal timelike complete subset of X. A direct computation (see the proof of Proposition 4.13) shows that there exists $c = c(a,N)$ such that

$$\mathfrak{m}^+(V_t) = ct^{N-1}, \text{ for all } t > 0.$$

In particular, equality is achieved in the monotonocity formula. For $K > 0$ (resp. $K < 0$), one can construct an analogous example replacing the N-dimensional Minkowski space by the N-dimensional de Sitter space of constant sectional curvature $K/(N-1)$ (resp. the N-dimensional anti-de Sitter space of of constant sectional curvature $K/(N-1)$).

4.3. **A sharp and rigid isoperimetric-type inequality.** We next deduce from Theorem 3.2 and Proposition 4.6 an isoperimetric type inequality.

For $V \subset X$, Borel achronal timelike complete subset, and $S \subset I^+(V)$ Borel achronal set with $\partial^+_V S = \emptyset$, we will consider the conically shaped region $C(V,S)$ spanned by the set of τ_V-maximizing geodesics from V to S, i.e.:

(4.14)
$$C(V,S) := \{\gamma_t \colon \gamma \in \mathrm{Geo}(X),\ t \in [0,1],\ \gamma_0 \in V, \gamma_1 \in S,\ L_\tau(\gamma) = \tau_V(\gamma_1)\}.$$

Define also

$$\mathfrak{D}_{K,N}(t) := \frac{1}{\mathfrak{s}_{K/(N-1)}(t)^{N-1}} \int_0^t \mathfrak{s}_{K/(N-1)}(s)^{N-1}\,ds, \quad t \in (0, T_{K,N}) \tag{4.15}$$

where $\mathfrak{s}_{K/(N-1)}(t)$ was defined in (2.29) and

$$T_{K,N} := \sup\{t > 0 : \mathfrak{s}_{K/(N-1)}(t) > 0\} \in (0, \infty].$$

Note that, for $K = 0$, one obtains simply

$$\mathfrak{D}_{0,N}(t) = \frac{t}{N}.$$

The function $\mathfrak{D}_{K,N}$ admits equivalent expression: since

$$\sigma^{(s/t)}_{K/(N-1)}(t) = \frac{\mathfrak{s}_{K/(N-1)}(s)}{\mathfrak{s}_{K/(N-1)}(t)},$$

it follows by a change of variable that

$$\mathfrak{D}_{K,N}(t) = \int_0^t \sigma^{(s/t)}_{K/(N-1)}(t)^{N-1}\, ds = t \int_0^1 \sigma^{(r)}_{K/(N-1)}(t)^{N-1}\, dr.$$

Then the following isoperimetric type inequality holds true.

Theorem 4.11 (An isoperimetric-type inequality)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(K, N)$, and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel achronal timelike complete subset and $S \subset I^+(V)$ a Borel achronal set with $\partial^+_V S = \emptyset$ (in particular this holds if $S \subset I^+(V)$ is a Cauchy hypersurface). Then*

$$\mathfrak{m}^+(S)\, \mathfrak{D}_{K,N}(\mathrm{dist}(V, S)) \leq \mathfrak{m}(C(V, S)), \tag{4.16}$$

where

$$\mathrm{dist}(V, S) := \inf\{\tau_V(x) \colon x \in S\}. \tag{4.17}$$

If $K = 0$, the bound (4.16) *reads as*

$$\mathfrak{m}^+(S)\, \mathrm{dist}(V, S) \leq N \mathfrak{m}(C(V, S)). \tag{4.18}$$

Proof. Consider the disintegration formula associated to τ_V. Since $S \subset I^+(V)$, then $C(V, S) \subset I^+(V)$ and therefore

$$\mathfrak{m}(C(V, S)) = \int_Q \mathfrak{m}_\alpha(X_\alpha \cap C(V, S))\, \mathfrak{q}(d\alpha).$$

Since both V and S are achronal and $\partial^+_V S = \emptyset$, then $X_\alpha \cap C(V, S)$ can be identified via τ_V to a real interval $[0, b_\alpha]$ for some $b_\alpha > 0$. Then

$$\mathfrak{m}(C(V, S)) = \int_Q \int_{[0, b_\alpha]} h(\alpha, s)\, ds\, \mathfrak{q}(d\alpha). \tag{4.19}$$

By (3.5), we have that for $\mathfrak{q}$-a.e. $\alpha \in Q$ it holds that $b_\alpha \in (0, T_{K,N})$; moreover, if $0 < s < b_\alpha$, then

$$h(\alpha, s) \geq h(\alpha, b_\alpha)\, \frac{\mathfrak{s}_{K/(N-1)}(s)^{N-1}}{\mathfrak{s}_{K/(N-1)}(b_\alpha)^{N-1}}. \tag{4.20}$$

Hence,

$$\int_0^{b_\alpha} h(\alpha,s)\,ds \geq \frac{h(\alpha,b_\alpha)}{\mathfrak{s}_{K/(N-1)}(b_\alpha)^{N-1}} \int_0^{b_\alpha} \mathfrak{s}_{K/(N-1)}(s)^{N-1} ds = h(\alpha,b_\alpha)\,\mathfrak{D}_{K,N}(b_\alpha).$$

Notice that the function $(0, T_{K,N}) \ni t \mapsto \mathfrak{D}_{K,N}(t)$ is increasing. Moreover, if we denote by $\{z_\alpha\} = X_\alpha \cap S$, then $b_\alpha = \tau_V(z_\alpha)$, yielding $b_\alpha \geq \mathrm{dist}(V,S)$. We infer that

$$\int_0^{b_\alpha} h(\alpha,s)\,ds \geq h(\alpha,b_\alpha)\,\mathfrak{D}_{K,N}(\mathrm{dist}(V,S)), \quad \mathfrak{q}\text{-a.e. } \alpha \in Q. \tag{4.21}$$

The combination of (4.19) and (4.21) gives

$$\begin{aligned}\mathfrak{m}(C(V,S)) &\geq \mathfrak{D}_{K,N}(\mathrm{dist}(V,S)) \int_Q h(\alpha,b_\alpha)\,\mathfrak{q}(d\alpha) \\ &= \mathfrak{D}_{K,N}(\mathrm{dist}(V,S)) \int_Q \mathfrak{m}^+_\alpha(S\cap \overline{X_\alpha})\,\mathfrak{q}(d\alpha),\end{aligned}$$

which, together with (4.4), concludes the proof of the claim. □

Remark 4.12 (Related literature in Riemannian signature)**.** At a formal level, the proof of Theorem 4.11 is performed following the integral lines of the gradient flow of τ_V, the Lorentzian distance from V. This should be compared with the celebrated Heintze-Karcher inequality in the Riemannian setting [30], where one obtains a volume bound of a smooth Riemannian manifold M in terms of the co-dimensional one volume of a smooth hypersurface V, the maximal value of the mean curvature of V and the maximal distance from V in M. The proof of the Heintze-Karcher inequality is also performed following the integral lines of the gradient flow of the distance from V, however the volume bound on each integral line depends on the mean curvature of V. The main advantage of the proof of Theorem 4.11 is that, in addition to considerably relaxing the regularity assumed on the space, it does not assume any bound on the mean curvature of V.

We now show that Theorem 4.11 is sharp.

Proposition 4.13 (Sharpness of Theorem 4.11)**.** *The inequality* (4.18) *is sharp, in the following sense. For $N \in \mathbb{N}$, $N \geq 2$:*

- *for $K = 0$, the equality in* (4.18) *is achieved for a conical region in N-dimensional Minkowski space-time.*
- *for $K > 0$, the equality in* (4.16) *is achieved for a conical region in N-dimensional de Sitter space-time with constant sectional curvature $K/(N-1)$;*
- *for $K < 0$, the equality in* (4.16) *is achieved for a conical region in N-dimensional anti-de Sitter space-time with constant sectional curvature $K/(N-1)$;*

Proof. We will first consider the two dimensional Minkowski space-time $\mathbb{M}^2$ with metric $-dy^2 + dx^2$ and reference measure the volume measure, i.e. the

2 dimensional Lebesgue measure. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 \colon -y^2 + x^2 = -1\}$$

and we restrict the space to

$$X := \left\{ (x, y) \colon -a \le x \le a,\ y \ge |x| \frac{\sqrt{1 + a^2}}{a} \right\}.$$

Taking $V = \{(0, 0)\}$, then

$$C(V, S) = \{(x, y) \in X \colon -y^2 + x^2 \le 1\}, \qquad S = \{(x, \sqrt{1 + x^2}) \colon x \in (-a, a)\}.$$

Then

$$\begin{aligned} \mathcal{L}^2(C(V, S)) &= 2 \left(\int_{(0,a)} \sqrt{1 + x^2}\, dx - \frac{a\sqrt{1 + a^2}}{2} \right) \\ &= (x\sqrt{1 + x^2} + \sinh^{-1}(x)) \Big|_0^a - a\sqrt{1 + a^2}. \end{aligned}$$

The length ℓ of S is instead given by

$$\ell = 2 \int_0^a \sqrt{1 - y'(x)^2}\, dx = 2 \int_0^a \frac{1}{\sqrt{1 + x^2}}\, dx = 2 \sinh^{-1}(x) \big|_0^a.$$

Since by construction $\mathrm{dist}(V, S) = 1$, for this example (4.18) is an identity for $N = 2$.

In higher dimension $N = n + 1 \ge 3$, we consider the $n + 1$-dimensional Minkowski space $\mathbb{M}^{n+1}$ with metric $g = -dt^2 + dx_1^2 + \ldots + dx_n^2$. Consider the cone

$$X = \{(x, t) \colon \|x\| \le a, t \ge \|x\|\sqrt{1 + a^2}/a\}, \quad \text{for any } a > 0,$$

and the surface

$$S = \{(x, t) \in X \colon t^2 - \|x\|^2 = 1\}. \tag{4.22}$$

The achronal set V will be the origin O.

The volume of $C(V, S)$ will be the difference between the volume of the cone

$$W := X \cap \{0 \le t \le \sqrt{1 + a^2}\}$$

and the volume of E, the epigraph in W of the function $t = \sqrt{1 + \|x\|^2}$. Then

$$\begin{aligned} \mathcal{L}^{n+1}(W) &= \int_0^{\sqrt{1+a^2}} \omega_n \left(\frac{a}{\sqrt{1 + a^2}} \right)^n r^n\, dr = \frac{\omega_n}{n + 1} \left(\frac{a}{\sqrt{1 + a^2}} \right)^n \sqrt{1 + a^2}^{n+1} \\ &= \sqrt{1 + a^2} a^n \frac{\omega_n}{n + 1}, \\ \mathcal{L}^{n+1}(E) &= \omega_n \int_1^{\sqrt{1+a^2}} (\sqrt{r^2 - 1})^n dr. \end{aligned}$$

Thus

$$\mathcal{L}^{n+1}(C(V,S)) = \omega_n \left(\frac{a^n}{n+1}\sqrt{1+a^2} - \int_1^{\sqrt{1+a^2}} (\sqrt{r^2-1})^n dr \right).$$

For computing the area of S, we parametrize S via the graph of the function $\sqrt{1+r^2}$ over the polar coordinates $(r,\Theta) \in [0,\infty)\times\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The tangent space of S is spanned by ∂_t and $\partial_{\Theta_i}, i = 1,\dots,n-1$. Restricting the Minkowski metric to S, the area form is given by $r^{n-1}/\sqrt{1+r^2}\, dr d\Theta$. Then

$$\begin{aligned}\mathrm{Area}(S) &= \int_{\mathbb{S}^{n-1}} \int_0^a r^{n-1}/\sqrt{1+r^2}\, dr\, d\Theta = n\omega_n \int_0^a r^{n-1}/\sqrt{1+r^2}\, dr \\ &= n\omega_n \int_1^{\sqrt{1+a^2}} (\sqrt{x^2-1})^{n-2} dx,\end{aligned}$$

where, in the last identity, we performed the change of variables $x = \sqrt{1+r^2}$. It is possible to check that, for all $a > 0$, $n \geq 2$:

$$\int_1^{\sqrt{1+a^2}} n(\sqrt{x^2-1})^{n-2} + (n+1)(\sqrt{x^2-1})^n dx = a^n\sqrt{1+a^2}, \tag{4.23}$$

yielding

$$\mathrm{Area}(S) = (n+1)\mathcal{L}^{n+1}(C(V,S)), \quad \text{for all } a > 0,\, n \geq 2.$$

Since, by construction, all the points in S are at distance 1 from the origin O, we just showed that S defined in (4.22) achieves the equality in (4.18) for $V = \{O\}$.
This shows sharpness for $K = 0$, $N \in \mathbb{N}, N \geq 2$.

For $K \neq 0$, $N \in \mathbb{N}, N \geq 2$, up to scaling we can assume that $K = N-1$ (if $K > 0$) or $K = -(N-1)$ (if $K < 0$). One can check that equality in (4.16) is achieved by the following choices. In the arguments above, replace the Minkowski space by the de Sitter space (in case $K = N-1$) or by the anti-de Sitter space (in case $K = -(N-1)$), the cone X by the exponential of $\exp_p(X)$, the surface S by $\exp_p(S)$, the domain W by $\exp_p(W)$ and set $V = \{p\}$. $\square$

We next show that Theorem 4.11 is also rigid.

Proposition 4.14 (Rigidity of Theorem 4.11)**.** *The inequality* (4.16) *is rigid, in the following sense. In addition to the assumptions of Theorem 4.11, assume that*

(i) *S is a smooth spacelike hypersurface;*
(ii) *$C(V,S)\setminus V$ is isometric to a smooth Lorentzian manifold (M^{n+1}, g), non-complete and with boundary;*
(iii) *Equality is achieved in* (4.16)*, namely*

$$\mathrm{Vol}_g^n(S)\, \mathfrak{D}_{K,N}(\mathrm{dist}(V,S)) = \mathrm{Vol}_g^{n+1}(C(V,S)), \tag{4.24}$$

where Vol_g^{n+1} *(resp.* Vol_g^n*) denotes the* $n+1$*-dimensional volume measure associated to* g *(resp. the* n*-dimensional volume measure associated to the restriction of* g*).*

Then

(a) $V = \{\bar{x}\}$ *is a singleton; denote by* $g_S := \lambda^{-2} g \llcorner TS$ *the normalised restriction of* g *to* TS*, where* $\lambda := \mathrm{dist}(V,S)^{-2}$ *is a normalization constant;*

(b) Let

$$\mathrm{C}(S) := [0, \mathrm{dist}(V,S)] \times S/\sim, \text{ where } (0,x) \sim (0,y) \text{ for all } x,y \in S, \tag{4.25}$$

be a (truncated) cone over S *and endow it with the Lorentzian metric (defined outside the tip* $\{r=0\}$*)*

$$g_{\mathrm{C}(S)} := -dr^2 + \mathfrak{s}_{K/n}(r)^2 g_S, \tag{4.26}$$

where we use the notation $(r,x) \in [0, \mathrm{dist}(V,S)] \times S/\sim$*, and* $\mathfrak{s}_{(\cdot)}(\cdot)$ *is defined in* (2.29).

Then there exists an isometry $\Psi : \mathrm{C}(S) \to C(V,S)$ *such that* $\Psi(\{r=0\}) = V$ *is the tip of the cone and* $\Psi(\{r = \mathrm{dist}(V,S)\}) = S$*.*

If (ii) *is replaced by the stronger*

(ii') $C(V,S)$ *is contained in a smooth Lorentzian manifold* (M^{n+1}, g) *complete and without boundary,*

then (b) *can be improved into*

(b') (S, g_S) *is isometric to a subset of the* n*-dimensional sphere of constant curvature* 1*,* $(\mathbb{S}^n, g_{\mathbb{S}^n})$*, and* $C(V,S)$ *is isometric to a cone in the model space with metric* $-dr^2 + \mathfrak{s}_{K/n}(r)^2 g_{\mathbb{S}^n}$ *(note that* $K=0$ *gives Minkowski,* $K=n$ *de Sitter, and* $K=-n$ *anti-de Sitter) and with tip at* $V = \{\bar{x}\}$*.*

Proof. For the sake of brevity we only sketch the proof.

Following the proof of Theorem 4.11, it is clear that equality in (4.16) forces equality in (4.20) for $\mathfrak{q}$-a.e. α, namely:

$$\frac{h(\alpha,s)}{h(\alpha,b_\alpha)} = \frac{\mathfrak{s}_{K/(N-1)}(s)^{N-1}}{\mathfrak{s}_{K/(N-1)}(b_\alpha)^{N-1}}, \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha, \text{ for all } s \in [0, b_\alpha]. \tag{4.27}$$

Moreover, the fact that $(0, T_{K,N}) \ni t \mapsto \mathfrak{D}_{K,N}(t)$ is increasing forces

$$b_\alpha = \mathrm{dist}(V,S), \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha. \tag{4.28}$$

This means that (up to a set of $\mathfrak{q}$-measure zero) one can identify S with the quotient set Q and parametrize $C(V,S)$, up to a set of $\mathfrak{m}$-measure zero, by the ray map

$$\Psi : [0, \mathrm{dist}(V,S)] \times S \to C(V,S), \quad \Psi(s,\alpha) := X_\alpha(s), \tag{4.29}$$

so that Ψ is a Borel bijection (up to a set of $\mathfrak{m}$-measure zero). Notice that, by construction, $\Psi(\{0\} \times S) = V$ and $\Psi(\{\mathrm{dist}(V,S)\} \times S) = S$. Notice also that (4.27) yields that the co-dimension one volume of the s-section

$\Psi(\{s\} \times S)$ tends to 0 as $s \to 0$.
Now, using the smoothness assumption on $C(V,S)$ and standard Jacobi fields computations, one can mimic the proof of the rigidity in Bishop-Gromov inequality (see for instance [21, 16]) in order to infer that Ψ defined in (4.29) passes to the quotient (4.25) and defines an isometry between $(C(V,S), g)$ and $(\mathrm{C}(S), g_{\mathrm{C}(S)})$. This shows (a) and (b). In order to prove (b'), it suffices to observe that the only cone metrics (4.26) which extend to a smooth Lorentzian manifold (complete and without boundary) are if the form in (b'). □

In the next remark, we discuss a direct application of the isoperimetric inequality (4.18).

Remark 4.15 (An upper bound on the area of Cauchy hypersurfaces in a black hole interior)**.** Let (M^{n+1}, g) be a globally hyperbolic space-time of finite volume and satisfying the Strong Energy Condition (SEC for short, i.e. $\mathrm{Ric}(v,v) \geq 0$ for all $v \in TM$ timelike). M shall be thought as a finite slab in the interior of a black hole. Of course any black hole metric satisfying the vacuum Einstein equations $\mathrm{Ric} \equiv 0$ (such as Schwarzschild or Kerr) also satisfies the SEC.

Assume there exists a subset $\Sigma \subset M$ achronal and past complete. It is natural to expect that the "singular set at the center of the black hole" satisfies such properties (when M is the black hole interior), at least for a generic black hole.

If $S \subset I^-(\Sigma)$ is a Cauchy hypersurface for M, so in particular S is achronal and $\partial_\Sigma S = \emptyset$ (see Lemma 4.4). The quantity $\mathrm{dist}(S, \Sigma)$ shall be thought as the Lorentzian distance from the Cauchy surface S to the singular set Σ.

Applying Theorem 4.11 to the causally reversed structure (i.e. backward in time), we obtain

$$\mathrm{Vol}_g^3(S)\,\mathrm{dist}(S,\Sigma) \leq (n+1)\,\mathrm{Vol}_g^4(M), \tag{4.30}$$

giving an upper bound on the area of the Cauchy surface S (with respect to the 3-dimensional volume measure Vol_g^3 associated to the restriction of g to S) in terms of its time-distance from the singular set Σ and the volume $\mathrm{Vol}_g^4(M)$ (of the slab) of the black hole interior M (with respect to the 4-dimensional volume measure Vol_g^4 associated to g).

Example 4.16 (An upper bound on the area of Cauchy hypersurfaces in the Schwarzschild black hole interior)**.** To fix the ideas by an explicit example, consider a finite slab $\{t \in [a,b]\}$ in the interior of the Schwarzschild black hole

$$M := \{t \in [a,b]\} \cap \{r \leq 2m\} \tag{4.31}$$

with metric

$$g := -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{4.32}$$

and with time orientation so that $-\frac{\partial}{\partial r}$ is future oriented. Note that in the black hole interior, r is a timelike variable while t is a spacelike variable. It is clear that the singulat set $\Sigma := \{r = 0, t \in [a, b]\}$ is achronal and $I^-(\Sigma) = M$.
Let $S \subset M$ be any Cauchy hypersurface for the Schwarzschild slab (4.31). The bound (4.30) reads as

$$\mathrm{Vol}_g^3(S) \inf_S \tau_\Sigma(r) \leq 4\mathrm{Vol}_g^4(\{r \leq 2m, t \in [a, b]\}) = \frac{128}{3}\pi m^3(b-a), \tag{4.33}$$

where τ_Σ depends only on the r-coordinate and is given by the expression

$$\tau_\Sigma(r) = \pi m - \sqrt{2mr - r^2} - 2m \arctan\left(\sqrt{\frac{2m-r}{r}}\right). \tag{4.34}$$

Notice that $\tau_\Sigma(r_0)$ is the maximal proper time that may lapse for a massive observer initially at $r = r_0 \in (0, 2m]$ before hitting the singularity Σ.

While well-known to experts, let us briefly sketch the proof of the expression (4.34) for completeness of presentation. Let $\gamma_\tau = (t(\tau), r(\tau), \theta(\tau), \varphi(\tau))$, $\tau \in [0, T]$, be a future directed timelike curve parametrized by proper time τ:

$$1 = \left(\frac{2m}{r} - 1\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \left(\frac{2m}{r} - 1\right)\left(\frac{dt}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2.$$

From the fact that γ is future directed, we infer that $\tau \mapsto r(\tau)$ is strictly decreasing and

$$\frac{dr}{d\tau} \leq -\sqrt{\frac{2m}{r} - 1}, \tag{4.35}$$

with equality if and only if

$$\left(\frac{2m}{r} - 1\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 = 0. \tag{4.36}$$

Recall that the aim here is, given $\gamma_0 = (t_0, r_0, \theta_0, \varphi_0) \in M$, find (if it exists) the future timelike curve $(\gamma_\tau)_{\tau \in [0,T]}$ with $\gamma_T \in \Sigma$ (i.e. $r(\gamma_T) = 0$) parametrized by proper time, and having the maximal T. This amounts to find the future directed timelike curve having maximal $\frac{dr}{d\tau}$. From (4.35) and (4.36), it is clear that such a curve has to be radial, i.e. $\gamma_\tau = (t_0, r(\tau), \theta_0, \varphi_0)$, and such that

$$T = \int_0^{r_0}\left(\frac{2m}{r} - 1\right)^{-1/2} = \pi m - \sqrt{2mr_0 - r_0^2} - 2m\arctan\left(\sqrt{\frac{2m - r_0}{r_0}}\right).$$

This completes the proof of (4.34).

One can deduce analogous bounds for de-Sitter Schwarzschild and anti de Sitter-Schwarzschild black holes, by using the more general (4.16).

Remark 4.17 (An upper bound on the area of Cauchy hypersurfaces in cosmological space-times)**.** Another situation where Theorem 4.11 seems to give some new geometric information, is for cosmological spacetimes.

In this case, the manifold is homeomorphic to a Lorentzian cone $C(\Sigma)$, with coordinates $(t,x), t \geq 0, x \in \Sigma$ (and diffeomorphic on the open subset $\{t>0\}$), with $\Sigma \times \{t\}$ spacelike slices for any $t>0$ and with $\frac{\partial}{\partial t}$ timelike. In such a model, the point $\{t=0\}$ corresponds to the origin of the universe, i.e. the "big-bang".

In Theorem 4.11, we can choose $V=\{t=0\}$ and S be any Cauchy hypersurface. The quantity $\mathrm{dist}(V,S)$ could be loosely interpreted as a kind of "age" of S, while the lower bound K on the timelike Ricci curvature is related to the cosmologiocal constant and the energy momentum tensor via the Einstein equations.
In previous literature [1, Thm. 2 and Prop. 3], area bounds on Cauchy hypersurfaces were proved in Friedman-LeMaître-Robertson-Walker spacetimes with non-negative timelike Ricci curvature (see also [26] for related results). These are cosmological spacetimes where the time-slices $\Sigma \times \{t\}$ have constant sectional curvature for all $t>0$, i.e. are homogenous and isotropic. Although such symmetries are satisfied at a very good level of approximation at the scale of the universe, recent observations detected some anomalies in the cosmic microwave background that are challenging such a model (see for instance [24]). Since Theorem 4.11 does not assume any symmetry and allows any $K \in \mathbb{R}$, it gives an area bound on any Cauchy hypersurface S also in the case when the timelike Ricci curvature is bounded below by a negative constant K (thus allowing more freedom to the cosmological constant and to the energy-momentum tensor), and the time slices $\Sigma \times \{t\}$, $t>0$, are not necessarily homogenous and isotropic.

5. Further Localization results

For completeness, we include a brief discussion on a generalisation of Theorem 3.2. The results of Section 2.2.2 and Section 2.3.1 are indeed valid for a wider class of functions then time separation functions from achronal timelike complete sets, namely for solutions of the dual Kantorovich problem for $p=1$. In analogy with the metric theory, this class coincides with the class of timelike reverse 1-Lipschitz functions defined as follows:

$$u : X \to \mathbb{R}, \qquad u(x)-u(y) \geq \ell(x,y), \quad \forall\, x,y \in X.$$

For a fixed timelike reverse 1-Lipschitz function u, define the transport relation as

$$\Gamma_u := \{(x,y) \in X_{\leq} \colon u(x)-u(y) = \tau(x,y)\}.$$

One can check that Γ_u is ℓ-cyclically monotone: for any $n \in \mathbb{N}$ and any family $(x_1, y_1), \dots, (x_n, y_n)$ of points in Γ_u:

$$\begin{aligned}\sum_{i=1}^{n} \tau(x_i, y_i) &= \sum_{i=1}^{n} u(x_i) - u(y_i) \\ &= \sum_{i=1}^{n} u(x_{i+1}) - u(y_i) \geq \sum_{i=1}^{N} \ell(x_{i+1}, y_i).\end{aligned}$$

It is then natural to define $R_u, \mathcal{T}_u^{end}, \mathfrak{a}(\mathcal{T}_u^{end}), \mathfrak{b}(\mathcal{T}_u^{end})$ and $\mathcal{T}_u$ as in (2.19), (2.20), (2.21) with the replacement of Γ_V by Γ_u. By the timelike non-branching property, $\mathcal{T}_u$ is partitioned by transport rays induced by u, precisely as for τ_V. Then Proposition 2.21 can be applied to Γ_u to obtain, repeating verbatim the calculations done for τ_V, the following disintegration result for $\mathfrak{m}$.

Theorem 5.1. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}_p^e(K,N)$ and assume that the causally-reversed structure satisfies the same conditions.*
Let $u : X \to \mathbb{R}$ be a timelike reverse 1-Lipschitz function.
Then $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_u^{end})) = \mathfrak{m}(\mathfrak{b}(\mathcal{T}_u^{end}) = 0$ and the following disintegration formula holds true:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^{end}} = \mathfrak{m}\llcorner_{\mathcal{T}_u} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) = \int_Q h(\alpha, \cdot) \, \mathcal{L}^1\llcorner_{X_\alpha} \, \mathfrak{q}(d\alpha), \tag{5.1}$$

where

- *$\mathfrak{q}$ is a probability measure over the Borel quotient set $Q \subset \mathcal{T}_u$;*
- *$h(\alpha, \cdot) \in L^1_{loc}(X_\alpha, \mathcal{L}^1\llcorner_{X_\alpha})$ for $\mathfrak{q}$-a.e. $\alpha \in Q$;*
- *the map $\alpha \mapsto \mathfrak{m}_\alpha(A) = h(\alpha, \cdot)\mathcal{L}^1\llcorner_{X_\alpha}(A)$ is $\mathfrak{q}$-measurable for every Borel set $A \subset \mathcal{T}_V$.*
- *For $\mathfrak{q}$-a.e. α the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the classical $\mathsf{CD}(K,N)$ condition, i.e. (3.2) holds.*

Finally let us note notice that if one replaces the $\mathsf{TCD}_p^e(K,N)$ assumption by the weaker $\mathsf{TMCP}^e(K,N)$, then all the claims of Theorem 5.1 remain valid except from the last point that has to be replaced by "$(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the classical $\mathsf{MCP}(K,N)$, i.e. (3.5) holds."

5.1. **Sharp Brunn-Minkowski inequality via localization.** The metric version of Theorem 5.1 goes back to [12] and, previously, the Riemannian version to [35]. There, many geometric inequalities (in their sharp form) where obtained by applying the theorem to a special function u associated to an optimal transport problem induced by a Borel function $f : X \to \mathbb{R}$ having zero mean $\int_X f \, \mathfrak{m} = 0$: the function u was a Kantorovich potential for the W_1 optimal transport problem between the measures $\mu_0 := f^+ \, \mathfrak{m}$ and $\mu_1 := f^- \, \mathfrak{m}$, where $f^\pm$ are the positive and the negative part of f. In this cases, the disintegration induced by u localises also the zero mean property of f: for $\mathfrak{q}$-a.e. $\alpha \in Q$

$$\int_{X_\alpha} f \, \mathfrak{m}_\alpha = 0.$$

It would be therefore desirable to included to the list of properties of $\mathfrak{m}_\alpha$ of Theorem 5.1, in the case u is the Kantorovich potential associated to f, also the localization of the zero mean condition.

It will be obtained by directly adapting the argument of [12] with the additional difficulty that we cannot rely on the existence of a solution to the dual Kantorovich potential.

For ease of notation, we will drop the normalisation factor by directly assuming that μ_0 and μ_1 are probability measures.

Theorem 5.2. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}_p^e(K,N)$ and assume that the causally-reversed structure satisfies the same conditions.*

Moreover assume that $\int f \, \mathfrak{m} = 0$ for some real valued Borel function f and that the pair of probability measure $(\mu_0 := f^+ \, \mathfrak{m}, \mu_1 := f^- \, \mathfrak{m})$ is timelike 1-dualisable. Then there exists an ℓ-cyclically monotone set Γ_f inducing the following decomposition of the space: $X = Z \cup \mathcal{T}$, with $\mathfrak{m}(Z \setminus \{f = 0\}) = 0$ and $\mathcal{T}$ obtained as the disjoint union of a family of timelike geodesics $\{X_\alpha\}_{\alpha \in Q}$ inducing the following disintegration formula:

$$\mathfrak{m} \llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) = \int_Q h(\alpha, \cdot) \, \mathcal{L}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha), \qquad \mathfrak{q} \in \mathcal{P}(Q), \quad Q \subset \mathcal{T}.$$

Moreover, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the following hold:

- $h(\alpha, \cdot) \in L^1_{loc}(X_\alpha, \mathcal{L}^1 \llcorner_{X_\alpha})$;
- $\int_{X_\alpha} f \, \mathfrak{m}_\alpha = 0$;
- *the one-dimensional m.m.s. $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the $\mathsf{CD}(K,N)$ condition, i.e.* (3.2) *holds.*

Proof. By assumption, there exists an ℓ-cyclically monotone set $\Gamma_f \subset X^2_{\ll}$ and $\pi \in \Pi_{\leq}(\mu_0, \mu_1)$ that is ℓ-optimal and $\pi(\Gamma_f) = 1$.

We now enlarge Γ_f by filling its possible holes so to restore the regularity we would have if Γ_f was included in the subdifferential of a Kantorovich potential. To this aim, define

$$\Gamma := \{(x, y) \in X^2_{\ll} : \tau(z, w) = \tau(z, x) + \tau(x, y) + \tau(y, w), (z, w) \in \Gamma_f\}.$$

We claim that Γ is ℓ-cyclically monotone; indeed, for $(x_i, y_i) \in \Gamma$

$$\begin{aligned}
\sum_i \tau(x_i, y_i) = &\sum_i \tau(z_i, w_i) - \tau(z_i, x_i) - \tau(y_i, w_i) \\
\geq &\sum_i \ell(z_{i+1}, w_i) - \tau(z_i, x_i) - \tau(y_i, w_i) \\
\geq &\sum_i \ell(z_{i+1}, x_{i+1}) + \ell(x_{i+1}, w_i) - \tau(z_i, x_i) - \tau(y_i, w_i) \\
\geq &\sum_i \ell(z_{i+1}, x_{i+1}) + \ell(x_{i+1}, y_i) + \ell(y_i, w_i) - \tau(z_i, x_i) - \tau(y_i, w_i) \\
= &\sum_i \ell(x_{i+1}, y_i).
\end{aligned}$$

This implies that the pairs belonging to Γ are aligned along geodesics: if $(x, y) \in \Gamma$ and $\gamma \in \mathrm{TGeo}(x, y)$ then $(\gamma_s, \gamma_t) \in \Gamma$ for all $0 \leq s \leq t \leq 1$. Then we can proceed by defining $R, \mathcal{T}^{end}, \mathfrak{a}(\mathcal{T}^{end}), \mathfrak{b}(\mathcal{T}^{end})$ and $\mathcal{T}$ like few lines before Theorem 5.1. Thanks to Theorem 5.1 we obtain

$$\mathfrak{m}\llcorner_{\mathcal{T}^{end}} = \mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) = \int_Q h(\alpha, \cdot) \, \mathcal{L}^1\llcorner_{X_\alpha} \, \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a probability measure over the Borel quotient set $Q \subset \mathcal{T}$, and for $\mathfrak{q}$-a.e. α the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the $\mathsf{CD}(K, N)$ condition. In particular $\mathfrak{m}_\alpha = h(\alpha, \cdot)\mathcal{L}^1\llcorner_{X_\alpha}$. We are only left to prove that if $Z = X \setminus \mathcal{T}$, then $f = 0$ $\mathfrak{m}$-a.e. over Z, and that $\int_Q f \, \mathfrak{m}_\alpha = 0$, for $\mathfrak{q}$-a.e. α.

Since $\mu_0 \perp \mu_1$, then $\pi(\{(x, x) \colon x \in X\}) = 0$. Denoting by $\Delta := \{(x, x) \colon x \in X\}$, since $\pi(\Gamma) = 1$, we have

$$\mu_0(\mathcal{T}) = \mu_0(P_1(\Gamma \setminus \Delta)) \geq \pi(\Gamma \setminus \Delta) = 1;$$

in the same way $\mu_1(\mathcal{T}) = 1$. This implis that $\mu_0(Z) = \mu_1(Z) = 0$. Since $\mu_0 = f^+\mathfrak{m}$ and $\mu_1 = f^-\mathfrak{m}$, then Z is a subset of $\{f = 0\}$, up to a set of $\mathfrak{m}$-measure zero.

We are left to show the balance condition $\int_Q f \, \mathfrak{m}_\alpha = 0$, for $\mathfrak{q}$-a.e. α. For any Borel subset $A \subset Q$, consider the saturated set of A, $R(A) := P_2(A \times X \cap R)$ and compute:

$$\mu_0(R(A)) = \pi((R(A) \times X) \cap \Gamma) = \pi((X \times R(A)) \cap \Gamma) = \mu_1(R(A)).$$

Hence for any Borel subset $A \subset Q$

$$\int_A \int_{X_\alpha} f^+ \mathfrak{m}_\alpha \mathfrak{q}(d\alpha) = \int_A \int_{X_\alpha} f^- \mathfrak{m}_\alpha \mathfrak{q}(d\alpha)$$

implying that for any Borel subset $A \subset Q$, $\int_A \int_{X_\alpha} f \, \mathfrak{m}_\alpha \mathfrak{q}(d\alpha) = 0$. The claim is therefore proved. $\square$

Using Theorem 5.2 is routine to derive several geometric inequalities in their sharp form (following for instance [13]). Here we simply report the Brunn-Minkowski inequality.

Proposition 5.3 (Sharp timelike Brunn-Minkowski inequality)**.** *Let* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ *be a timelike non-branching measured Lorentzian pre-length space satisfying* $\mathsf{TCD}^e_p(K,N)$*, for some* $K \in \mathbb{R}, N \in [1,\infty), p \in (0,1)$*.*
Let $A_0, A_1 \subset X$ *be measurable subsets with* $\mathfrak{m}(A_0), \mathfrak{m}(A_1) \in (0,\infty)$*. Calling* $\mu_i := 1/\mathfrak{m}(A_i)\, \mathfrak{m}\llcorner_{A_i}$*,* $i = 1,2$*, assume that* (μ_0, μ_1) *is timelike* 1*-dualisable.*
Then

$$\mathfrak{m}(A_t)^{1/N} \geq \tau^{(1-t)}_{K,N}(\theta)\, \mathfrak{m}(A_0)^{1/N} + \tau^{(t)}_{K,N}(\theta)\, \mathfrak{m}(A_1)^{1/N}$$

where $A_t := \mathcal{I}(A_0, A_1, t)$ *defined in* (2.6) *is the set of* t*-intermediate points of geodesics from* A_0 *to* A_1*, and* Θ *is the maximal/minimal time-separation between points in* A_0 *and* A_1*, i.e.:*

$$\theta := \begin{cases} \sup\{\tau(x_0,x_1) : x_0 \in A_0, x_1 \in A_1\} & \text{if } K < 0, \\ \inf\{\tau(x_0,x_1) : x_0 \in A_0, x_1 \in A_1\} & \text{if } K \geq 0. \end{cases}$$

Finally $\tau^{(t)}_{K/N}(\theta) := t^{\frac{1}{N}} \sigma^{(t)}_{K/N}(\theta)^{\frac{N-1}{N}}$.

Once Theorem 5.2 is at disposal, Proposition 5.3 can be proved following verbatim the proof of [13, Thm. 3.1].

References

[1] Farhan Abedin, Justin Corvino, Shelvean Kapita, and Haotian Wu. On isoperimetric surfaces in general relativity. II. *J. Geom. Phys.*, 59(11):1453–1460, 2009.

[2] Michael T. Anderson. On long-time evolution in general relativity and geometrization of 3-manifolds. *Comm. Math. Phys.*, 222(3):533–567, 2001.

[3] Kathrin Bacher and Karl-Theodor Sturm. Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. *J. Funct. Anal.*, 259(1):28–56, 2010.

[4] Hyoungsick Bahn. Isoperimetric inequalities and conjugate points on Lorentzian surfaces. *J. Geom.*, 65(1-2):31–49, 1999.

[5] Hyoungsick Bahn and Paul Ehrlich. A Brunn-Minkowski type theorem on the Minkowski spacetime. *Canad. J. Math.*, 51(3):449–469, 1999.

[6] Mathias Braun. Rényi's entropy on Lorentzian spaces. Timelike curvature-dimension conditions. *J. Math. Pures Appl. (9)*, 177:46–128, 2023.

[7] Hubert L. Bray. *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature.* ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–Stanford University.

[8] Hubert L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geom.*, 59(2):177–267, 2001.

[9] Simon Brendle. Constant mean curvature surfaces in warped product manifolds. *Publ. Math. Inst. Hautes Études Sci.*, 117:247–269, 2013.

[10] Simon Brendle and Michael Eichmair. Isoperimetric and Weingarten surfaces in the Schwarzschild manifold. *J. Differential Geom.*, 94(3):387–407, 2013.

[11] Fabio Cavalletti. Decomposition of geodesics in the Wasserstein space and the globalization problem. *Geom. Funct. Anal.*, 24(2):493–551, 2014.

[12] Fabio Cavalletti and Andrea Mondino. Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *Invent. Math.*, 208(3):803–849, 2017.

[13] Fabio Cavalletti and Andrea Mondino. Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. *Geom. Topol.*, 21(1):603–645, 2017.

[14] Fabio Cavalletti and Andrea Mondino. Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. *Cambridge J. Math. (in press); Preprint arXiv:2004.08934*, 2020.

[15] Fabio Cavalletti and Andrea Mondino. A review of Lorentzian synthetic theory of timelike Ricci curvature bounds. *Gen. Relativity Gravitation*, 54(11):Paper No. 137, 39, 2022.

[16] Isaac Chavel. *Riemannian geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2006. A modern introduction.

[17] Otis Chodosh, Michael Eichmair, Yuguang Shi, and Haobin Yu. Isoperimetry, scalar curvature, and mass in asymptotically flat Riemannian 3-manifolds. *Comm. Pure Appl. Math.*, 74(4):865–905, 2021.

[18] Piotr T. Chruściel and James D. E. Grant. On Lorentzian causality with continuous metrics. *Classical Quantum Gravity*, 29(14):145001, 32, 2012.

[19] Ennio De Giorgi. Sulla proprietá isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat., Sez. I,*, 8:33–44, 1958.

[20] Ennio De Giorgi. Selected papers, (L. Ambrosio, G. Dal Maso, M. Forti and S. Spagnolo eds.,). *Springe-Verlag, New York*, 2005.

[21] Paul Ehrlich and Miguel Sánchez. Some semi-Riemannian volume comparison theorems. *Tohoku Math. J.*, 52:331–348, 2000.

[22] Michael Eichmair and Jan Metzger. Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions. *Invent. Math.*, 194(3):591–630, 2013.

[23] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. *Invent. Math.*, 201(3):993–1071, 2015.

[24] European Space Agency, Science and Tecnology. Simple but challenging: the universe according to Planck. *https://sci.esa.int/web/planck/-/51551-simple-but-challenging-the-universe-according-to-planck*, 2023.

[25] Arthur E. Fischer and Vincent Moncrief. Hamiltonian reduction and perturbations of continuously self-similar $(n+1)$-dimensional Einstein vacuum spacetimes. *Classical Quantum Gravity*, 19(21):5557–5589, 2002.

[26] Marco Flaim. An isoperimetric-type inequality in Lorentzian space-times using geometric flows. *Master Thesis at Universität Tübingen and Universitá di Trento*, 2022.

[27] Gregory J. Galloway. Curvature, causality and completeness in space-times with causally complete spacelike slices. *Math. Proc. Cambridge Philos. Soc.*, 99(2):367–375, 1986.

[28] Melanie Graf and Christina Sormani. Lorentzian area and volume estimates for integral mean curvature bounds. In *Developments in Lorentzian Geometry*, pages 105–128, Cham, 2022. Springer International Publishing.

[29] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, english edition, 2007. Based on the 1981 French original.

[30] Ernst Heintze and Hermann Karcher. A general comparison theorem with applications to volume estimates for submanifolds. *Ann. Sci. École Norm. Sup.*, 11(4):451–470, 1978.

[31] Gerhard Huisken. An isoperimetric concept for mass and quasilocal mass. *Report 2/2006 – Mathematical aspects of general relativity, Oberwolfach Rep. 3, European Mathematical Society, Zürich*, pages 87–88, 2006.

[32] Gerhard Huisken. An isoperimetric concept for the mass in general relativity. *https://video.ias.edu/node/234*, 2009.

[33] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.

[34] Gerhard Huisken and Shing-Tung Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Invent. Math.*, 124(1-3):281–311, 1996.

[35] Bo'az Klartag. Needle decompositions in Riemannian geometry. *Mem. Amer. Math. Soc.*, 249(1180):v+77, 2017.

[36] E. H. Kronheimer and R. Penrose. On the structure of causal spaces. *Proc. Cambridge Philos. Soc.*, 63:481–501, 1967.

[37] Michael Kunzinger and Clemens Sämann. Lorentzian length spaces. *Ann. Global Anal. Geom.*, 54(3):399–447, 2018.

[38] Ben Lambert and Julian Scheuer. Isoperimetric problems for spacelike domains in generalized Robertson-Walker spaces. *J. Evol. Equ.*, 21(1):377–389, 2021.

[39] Paul Lévy. *Problèms concretes d'analyse fonctionnelle.* Paris, 1951.

[40] John Lott. Collapsing in the Einstein flow. *Ann. Henri Poincaré*, 19(8):2245–2296, 2018.

[41] John Lott. On the initial geometry of a vacuum cosmological spacetime. *Classical Quantum Gravity*, 37(8):085017, 29, 2020.

[42] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[43] Robert J. McCann. Displacement convexity of Boltzmann's entropy characterizes the strong energy condition from general relativity. *Camb. J. Math.*, 8(3):609–681, 2020.

[44] Emanuel Milman. Sharp isoperimetric inequalities and model spaces for curvature-dimension diameter condition. *J. Eur. Math. Soc.*, 17(5):1041–1078, 2015.

[45] E. Minguzzi. Further observations on the definition of global hyperbolicity under low regularity. *Classical Quantum Gravity*, 40(18):Paper No. 185001, 9, 2023.

[46] Andrea Mondino and Stefan Suhr. An optimal transport formulation of the Einstein equations of general relativity. *J. Eur. Math. Soc. (JEMS)*, 25(3):933–994, 2023.

[47] Shin-ichi Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.

[48] Robert Osserman. The isoperimetric inequality. *Bull. Am. Math. Soc.*, 84(6):1182–1238, 1978.

[49] Roger Penrose. Gravitational collapse: the role of general relativity. *Gen. Relativity Gravitation*, 34(7):1141–1165, 2002. Reprinted from Rivista del Nuovo Cimento **1969**, Numero Speziale I, 252–276.

[50] Antonio Ros. The isoperimetric problem. *Lecture series at the Clay Mathematics Institute, Summer School on the Global Theory of Minimal Surfaces, MSRI, Berkeley, California*, 2001.

[51] Clemens Sämann. Global hyperbolicity for spacetimes with continuous metrics. *Ann. Henri Poincaré*, 17(6):1429–1455, 2016.

[52] S. M. Srivastava. *A course on Borel sets*, volume 180 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[53] Jacob Steiner. Einfacher beweis der isoperimetrischen hauptsätze. *J. reine angew Math.*, 18:281–296, 1838.

[54] Karl-Theodor Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.

[55] Jan-Hendrik Treude and James D. E. Grant. Volume comparison for hypersurfaces in Lorentzian manifolds and singularity theorems. *Ann. Global Anal. Geom.*, 43(3):233–251, 2013.

[56] Chung-Jun Tsai and Kai-Hsiang Wang. An isoperimetric-type inequality for spacelike submanifold in the Minkowski space. *Int. Math. Res. Not. IMRN*, (1):128–139, 2022.

[57] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[58] Robert M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.

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