SIZE EFFECTS ON QUASISTATIC GROWTH OF FRACTURES

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Abstract. We perform an analysis of the size effect for quasistatic growth of fractures in linearly isotropic elastic bodies under antiplanar shear. In the framework of the variational model proposed by G.A. Francfort and J.-J. Marigo in [14], we prove that if the size of the body tends to infinity, and even if the surface energy is of cohesive form, under suitable boundary displacements the fracture propagates following the Griffith's functional.

Keywords : variational models, energy minimization, free discontinuity problems, crack propagation, quasistatic evolution, brittle fracture.

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Contents

1. Introduction

A well known fact in fracture mechanics is that *ductility* is also influenced by the size of the structure, and in particular the structure tends to become brittle if its size increases (see for example [8], and references therein). The aim of this paper is to capture this fact for the problem of quasistatic growth of fractures in linearly elastic bodies in the framework of the variational theory of crack propagation formulated by Francfort and Marigo in [14].

The model proposed in [14] is inspired to classical Griffith's criterion and determines the evolution of the fracture through a competition between volume and surface energies. Let us illustrate it and the variant we investigate in the case of *generalized antiplanar shear*.

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded, and with Lipschitz boundary. A fracture $\Gamma \subseteq \overline{\Omega}$ is any rectifiable set, and a displacement u is any function defined almost everywhere in Ω whose set of discontinuities $S(u)$ is contained in Γ (we will make precise the functional setting later). The total energy of the configuration (u, Γ) is given by

$$\int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx + \mathcal{H}^{N-1}(\Gamma). \tag{1.1}$$

The first term in (1.1) implies that we assume to apply linearized elasticity in the unbroken part of Ω. The second term can be considered as the work done to create Γ.

As suggested in [14], more general fracture energies can be considered in (1.1), especially those of Barenblatt's type [5], and here we consider energies of the form

$$\int_\Gamma \varphi(|[u]|(x))\, d\mathcal{H}^{N-1}(x), \tag{1.2}$$

where $[u](x) := u^+(x) - u^-(x)$ is the difference of the traces of u on both sides of Γ, and $\varphi : [0,+\infty[\to [0,+\infty[$ (which depends on the material) is such that $\varphi(0) = 0$. In order to get a physical interpretation of (1.2), let us set $\sigma := \varphi'$: we interpret $\sigma(|[u]|(x))$ as density of force in x that act between the two lips of the crack Γ whose displacement are $u^+(x)$ and $u^-(x)$ respectively. Typically σ is decreasing, and $\sigma(s) = 0$ for $s \geq \bar{s}$: this means that the interaction between the two lips of the fracture decreases as the opening increases, and disappear when the opening is greater than a critical length $\bar{s}$. As a consequence, φ is increasing and concave, and $\varphi(s)$ is constant for $s \geq \bar{s}$. We will then consider φ increasing, concave, with $\varphi(0) = 0$, $c = \varphi'(0) < +\infty$, and $\lim_{s\to+\infty} \varphi(s) = 1$. We can interpret

$$\int_\Gamma \varphi(|[u]|(x))\, d\mathcal{H}^{N-1}(x)$$

as the work made to create Γ with an opening given by $[u]$. Assuming linearized elasticity to hold in $\Omega \setminus \Gamma$, we consider a total energy of the form

$$\|\nabla u\|^2 + \int_\Gamma \varphi(|[u]|)\, d\mathcal{H}^{N-1}, \tag{1.3}$$

where $\|\cdot\|$ denotes the L^2 norm. The problem of irreversible quasistatic growth of fractures in the cohesive case can be addressed through a *time discretization process* in analogy to what proposed in [14] for the energy (1.1).

Let $g(t)$ be a time dependent boundary displacement defined on $\partial_D\Omega \subseteq \partial\Omega$ with $t \in [0,T]$. Let $\delta > 0$ and let $I_\delta := \{0 = t_0^\delta < t_1^\delta < \cdots < t_{N_\delta}^\delta = T\}$ be a subdivision of $[0,T]$ with $\max(t_{i+1}^\delta - t_i^\delta) < \delta$, and let $g_i^\delta := g(t_i^\delta)$. Let us consider a preexisting crack configuration $(\bar{\Gamma}, \bar{\psi})$, with $\bar{\psi}$ a positive function on $\bar{\Gamma}$: this means that $\int_{\bar{\Gamma}} \varphi(\bar{\psi})\, d\mathcal{H}^{N-1}$ is the work done on the fracture $\bar{\Gamma}$ before we apply the boundary displacement $g(t)$. At time $t = 0$ we consider u_0^δ as a minimum of

$$\|\nabla u\|^2 + \int_{S^{g(0)}(u)\cup\bar{\Gamma}} \varphi(|[u]| \vee \bar{\psi})\, d\mathcal{H}^{N-1}. \tag{1.4}$$

Here $S^{g(0)}(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(0)(x)\}$, and for all $x \in \partial_D\Omega$ we consider $[u](x) := g(x) - \tilde{u}(x)$, where $\tilde{u}$ is the trace of u on $\partial\Omega$. Moreover we intend that $\bar{\psi} = 0$ outside $\bar{\Gamma}$. We define the fracture Γ_0^δ at time $t = 0$ as $S^{g(0)}(u_0^\delta) \cup \bar{\Gamma}$. We also set $\psi_0^\delta := \bar{\psi} \vee |[u_0^\delta]|$ on Γ_0^δ. The presence of $S^{g(0)}(u)$ in (1.4) indicates that the points at which the boundary displacement is not attained are considered as a part of the fracture. Notice moreover that problem (1.4) takes into account an *irreversibility condition* assumption in the growth of the fracture. Indeed, while on $S^{g(0)}(u) \setminus \bar{\Gamma}$ the surface energy which comes in minimization of (1.4) is exactly as in (1.2), on $S^{g(0)}(u) \cap \bar{\Gamma}$ the surface energy involved takes into account the previous work made on $\bar{\Gamma}$. The surface energy is of the form of (1.2) only if $|[u]| > \bar{\psi}$, that is only if the opening is increased. If $|[u]| \leq \bar{\psi}$ no energy is gained, that is displacements of this form along the fracture are in a sense surface energy free. Notice finally that the irreversibility condition involves only the modulus of $[u]$: this is an assumption which is reasonable since we are considering only antiplanar displacement. Clearly more complex irreversibility conditions can be formulated, involving for example a partial release of energy: the one we study is the first straightforward extension of the irreversibility condition given in [14] for the energy (1.1).

Supposing to have constructed Γ_i^δ and ψ_i^δ at time t_i^δ, we consider a minimum u_{i+1}^δ of the problem

$$\|\nabla u\|^2 + \int_{S^{g(t_{i+1}^\delta)}(u)\cup\Gamma_i^\delta} \varphi(|[u]| \vee \bar{\psi}_i^\delta)\, d\mathcal{H}^{N-1} \tag{1.5}$$

and define $\Gamma^\delta_{i+1} := \Gamma^\delta_i \cup S^{g(t^\delta_{i+1})}(u^\delta_{i+1})$ and $\psi^\delta_{i+1} := \psi^\delta_i \vee |[u^\delta_{i+1}]|$ on Γ^δ_{i+1}. As noted previously, problem (1.5) involves an irreversibility condition: the surface energy density on Γ^δ_i increases only where $|[u]| > \psi^\delta_i$.

The *discrete in time evolution* of the fracture relative to the boundary datum $g(t)$, the subdivision I_δ, the initial crack configuration $(\bar\Gamma, \bar\psi)$ is given by $\{(u^\delta_i, \Gamma^\delta_i, \psi^\delta_i) : i = 0, \dots, N_\delta\}$.

The *irreversible quasistatic evolution* of fracture relative to the boundary datum $g(t)$ and the initial crack configuration $(\bar\Gamma, \bar\psi)$ is obtained as a limit for $\delta \to 0$ of $(u^\delta(t), \Gamma^\delta(t), \psi^\delta(t))$, where $u^\delta(t) := u^\delta_i$, $\Gamma^\delta(t) := \Gamma^\delta_i$ and $\psi^\delta(t) := \psi^\delta_i$ for $t^\delta_i \le t < t^\delta_{i+1}$.

This program has been studied in detail in several papers in the case $\varphi \equiv 1$, that is for energy of the form (1.1). A first mathematical formulation has been given in [12], where the authors consider the case of dimension $N = 2$ and fractures which are compact and with a uniform bound on the number of connected components. This analysis has been extended to the case of plane elasticity in [9]. In [13] the authors consider the general dimension N, and remove the bound on the number of the connected components of the fractures: the key point is to introduce a weak formulation of the problem considering displacements in the space SBV (see Section 2). Finally in [11] the authors treat the case of finite elasticity not restricted to antiplanar shear, with volume energy depending on the full gradient under suitable growth condition, and in presence of volume forces and traction forces: the appropriate functional space for the displacements is now $GSBV$ (see for example [4] for a precise definition).

In all these papers ([12],[9],[13],[11]), the analysis of the limit reveals three basic properties (irreversibility, minimality and nondissipativity, see Theorem 2.2) which are taken as definition of irreversible quasistatic growth of brittle fractures: the time discretization procedure is considered as a privileged way to get an existence result.

In the case of energy (1.3), several difficulties arise in the analysis of the discrete in time evolution, and in the analysis as $\delta \to 0$. In Section 3, we prove that the functional space we need for the step by step minimization is the space of functions with bounded variation BV (see Section 2): moreover we prove that a relaxed version of (1.3) has to be employed, namely

$$\int_\Omega f(\nabla u)\,dx + \int_\Gamma \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(\Omega), \tag{1.6}$$

where $a = \varphi'(0)$, f is defined in (3.5), and $D^c u$ indicates the Cantorian part of the derivative of u. An existence result for *discrete in time evolution* in this context of BV space is given in Proposition 3.1.

The analysis for $\delta \to 0$ presents several difficulties, the main one being the stability of the minimality property of the discrete in time evolutions. The main purpose of this paper is to prove that these difficulties disappear as the size of the reference configuration increases, thank to the fact that the body response tends to become more and more brittle in spite of the presence of cohesive forces on the fractures given by the function φ. More precisely we consider a crack configuration $(\bar\Gamma, \bar\psi)$ in Ω with $\mathcal{H}^{N-1}(\bar\Gamma) < +\infty$, and prove this fact for the discrete evolutions in $\Omega_h := h\Omega$ with preexisting crack configuration $(\bar\Gamma_h, \bar\psi_h)$ of the form $\bar\Gamma_h := h\bar\Gamma$ and $\bar\psi_h(x) := \bar\psi(\frac{x}{h})$, under suitable boundary displacements. The idea is to rescale displacements and fractures to the fixed configuration Ω, and take advantage from the form of the problem in this new setting. The boundary displacements on $\partial_D\Omega_h := h\partial_D\Omega$ will be taken of the form

$$g_h(t,x) := h^\alpha g\left(t, \frac{x}{h}\right), \qquad g \in AC([0,T]; H^1(\Omega)), \qquad \|g(t)\|_\infty \le C, \qquad t \in [0,T],\ x \in \Omega_h,$$

where $\alpha > 0$ and $C > 0$. We indicate by $(u^{\delta,h}(t), \Gamma^{\delta,h}(t), \psi^{\delta,h}(t))$ the piecewise constant interpolation of the discrete in time evolution of fracture in Ω_h relative to the boundary displacement g_h and the preexisting crack configuration $(\bar\Gamma_h, \bar\psi_h)$. Let us moreover set for every $t \in [0,T]$

$$\mathcal{E}^{\delta,h}(t) := \int_{h\Omega} f(\nabla u^{\delta,h}(t))\,dx + \int_{\Gamma^{\delta,h}(t)} \varphi(\psi^{\delta,h}(t))\,d\mathcal{H}^{N-1} + a|D^c u^{\delta,h}|(\Omega_h).$$

In the case $\alpha = \frac{1}{2}$, we make the following rescaling

$$v^{\delta,h}(t,x) := \frac{1}{\sqrt{h}} u^{\delta,h}(t,hx), \qquad K^{\delta,h}(t) := \frac{1}{h}\Gamma^{\delta,h}(t), \qquad x \in \Omega.$$

The main result of the paper is the following (see Theorem 4.1 for a more precise statement).

Theorem 1.1. *If $\delta \to 0$ and $h \to +\infty$, there exists a quasistatic evolution of brittle fractures $\{t \to (v(t), K(t))\}$ in Ω relative to the preexisting crack $\bar{\Gamma}$ and boundary displacement g in the sense of* [13] *(see Theorem 2.2) such that for all $t \in [0,T]$ we have*

$$\nabla v^{\delta,h}(t) \rightharpoonup \nabla v(t) \qquad \textit{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{1.7}$$

Moreover for all $t \in [0,T]$ we have

$$\frac{1}{h^{N-1}} \mathcal{E}^{\delta,h}(t) \to \|\nabla v(t)\|^2 + \mathcal{H}^{N-1}(K(t)); \tag{1.8}$$

in particular $h^{-N+1}|D^c u^{\delta,h}(t)|(\Omega_h) \to 0$,

$$\frac{1}{h^{N-1}} \int_{\Omega_h} f(\nabla u^{\delta,h}(t))\, dx \to \|\nabla v(t)\|^2, \tag{1.9}$$

and

$$\frac{1}{h^{N-1}} \int_{\Gamma^{\delta,h}(t)} \varphi(\psi^{\delta,h}(t))\, d\mathcal{H}^{N-1} \to \mathcal{H}^{N-1}(K(t)). \tag{1.10}$$

Theorem 1.1 proves that as the size of the reference configuration increases, the response of the body in the problem of quasistatic growth of fractures tends to become brittle, so that energy (1.1) can be considered. Moreover we have convergence results for the volume and surface energies involved.

The particular value $\alpha = \frac{1}{2}$ comes out because a problem of quasistatic evolution has been considered. In fact if we consider an infinite plane with a crack-segment of length l and subject to a uniform stress σ at infinity, following Griffith's theory the crack propagates quasistatically if $\sigma = \frac{K_{IC}}{\sqrt{\pi l}}$, where K_{IC} is the critical *stress intensity factor*. So if the crack has length hl, the stress rescale as $\frac{1}{\sqrt{h}}$. This is precisely what we are prescribing in the case $\alpha = \frac{1}{2}$: in fact the stress that intuitively we prescribe at the boundary can be reconstructed from ∇u_h and rescales precisely as $\frac{1}{\sqrt{h}}$.

For the proof of Theorem 1.1, the first step is to recognize that $(v^{\delta,h}(t), K^{\delta,h}(t), \psi^{\delta,h}(t))$ is a discrete in time evolution relative to the boundary displacement g and the preexisting crack configuration $(\bar{\Gamma}, \bar{\psi})$ for a total energy of the form

$$\int_\Omega f_h(\nabla u)\, dx + \int_\Gamma \varphi_h(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u|(\Omega),$$

where $\varphi_h(s) \nearrow 1$ for all $s \in [0,+\infty[$, and $f_h(\xi) \nearrow |\xi|^2$ for all $\xi \in \mathbb{R}^N$. From the fact that $\varphi_h \nearrow 1$ we recognize that the structure tends to become brittle. Bound on total energy for the discrete in time evolution is available, so that compactness in the space BV can be applied: it turns out that the limits of the displacements are of class SBV with gradient in $L^2(\Omega;\mathbb{R}^N)$. Limits for the fractures are constructed through a Γ-convergence procedure (see Lemma 5.4). Now the main point is to recover the minimality property (see point (c) of Theorem 2.2)

$$\|\nabla v(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v) \setminus K(t)), \qquad v \in SBV(\Omega)$$

from the minimality property of $(v^{\delta,h}(t), K^{\delta,h}(t), \psi^{\delta,h}(t))$. This is done in Lemma 5.5 by means of a refined version of the Transfer of Jump Lemma of [13]: the main difference here is that we have to deal with BV functions and we have to transfer the jump on the part of $K^{\delta,h}(t)$ where $\psi^{\delta,h}(t)$ is greater than a given small constant.

We also consider the cases $\alpha \in]0, \frac{1}{2}[$ and $\alpha > \frac{1}{2}$. It turns out that in the case $\alpha \in]0, \frac{1}{2}[$, the body is not solicited enough to make the preexisting crack $\bar{\Gamma}_h$ propagate, and Ω_h tends to behave elastically in the complement of $\bar\Gamma_h$: more precisely we prove that (Theorem 4.2) in the case $\bar\psi > \varepsilon > 0$, setting

$$v^{\delta,h}(t,x) := \frac{1}{h^\alpha} u^{\delta,h}(t,hx), \tag{1.11}$$

for all $t \in [0,T]$ we have that $v^{\delta,h}(t)$ converges to the displacement of the elastic problem in $\Omega \setminus \bar\Gamma$ under boundary displacement given by $g(t)$.

In the case $\alpha > \frac{1}{2}$ we have that the preexisting fracture $\bar\Gamma_h$ tends to propagate brutally toward *rupture*: in fact in Theorem 4.3 we prove that $v^{\delta,h}(0)$ given by (1.11) converges to a piecewise constant function v in Ω, so that $\bar\Gamma \cup S^{g(0)}(v)$ disconnects Ω. This phenomenon is a consequence of the variational approach based on the search for global minimizers: as the size of Ω_h increases, fractures carry an energy of order h^{N-1}, while non rigid displacements carry an energy of greater order: in this way fracture is preferred to deformation.

The paper is organized as follows: in Section 2 we recall some basic definitions and introduce the functional setting for the problem. In Section 3 we deal with the problem of discrete in time evolutions for fractures in the cohesive case. The main theorems are listed in Section 4, and Sections 5, 6 and 7 are devoted to their proofs. In Section 8 we prove a relaxation result which is used in the problem of discrete in time evolution of fractures, while in Section 9 we prove some auxiliary results employed in the study of the asymptotic behavior of the evolutions.

2. Preliminaries

In this section we state the notation and prove some preliminary results employed in the rest of the paper.

Basic notation. We will employ the following basic notation:

- Ω is an open and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary;
- $\partial_D\Omega$ is a subset of $\partial\Omega$ open in the relative topology;
- $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure;
- we say that $A \tilde\subseteq B$ if $A \subseteq B$ up to a set of $\mathcal{H}^{N-1}$-measure zero;
- $\Gamma \subseteq \Omega$ is rectifiable if there exists a sequence of C^1 manifolds $(M_i)_{i\in\mathbb{N}}$ such that $\Gamma \tilde\subseteq \cup_i M_i$;
- for all $A \subseteq \mathbb{R}^N$, $|A|$ denotes the Lebesgue measure of A;
- for all $A \subseteq \mathbb{R}^N$, $\mathbb{1}_A$ denotes the characteristic function of A;
- if μ is a measure on $\mathbb{R}^N$ and A is a Borel subset of $\mathbb{R}^N$, $\mu \llcorner A$ denotes the restriction of μ to A, i.e. $(\mu\llcorner A)(B) := \mu(B\cap A)$ for all Borel sets $B \subseteq \mathbb{R}^N$;
- $\|u\|_\infty$ and $\|u\|$ denote the sup-norm and the L^2 norm of u respectively;
- if $u, g \in BV(\Omega;\mathbb{R}^m)$, $S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}$;
- if $a, b \in \mathbb{R}$, $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$.

Functions of bounded variation. For the general theory of functions of bounded variation, we refer to [4]; here we recall some basic definitions and theorems we need in the sequel. We say that $u \in BV(A)$ if $u \in L^1(A)$, and its distributional derivative Du is a bounded vector-valued Radon measure on A. In this case it turns out that the set $S(u)$ of points $x \in A$ which are not Lebesgue points of u is rectifiable, that is there exists a sequence of C^1 manifolds $(M_i)_{i\in\mathbb{N}}$ such that $S(u) \subseteq \cup_i M_i$ up to a set of $\mathcal{H}^{N-1}$-measure zero. As a consequence $S(u)$ admits a normal $\nu_u(x)$ at $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$. Moreover for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$, there exist $u^+(x), u^-(x) \in \mathbb{R}$ such that

$$\lim_{r\to 0} \frac{1}{|B_r^\pm(x)|} \int_{B_r^\pm(x)} |u(y) - u^\pm(x)|\, dy = 0,$$

where $B_r^\pm(x) := \{y \in B_r(x) \,:\, (y-x)\cdot\nu_u(x) \gtrless 0\}$, and $B_r(x)$ is the ball with center x and radius r. It turns out that Du can be represented as

$$Du(A) = \int_A \nabla u(x)\, dx + \int_{A\cap S(u)} (u^+(x) - u^-(x))\nu_u(x)\, d\mathcal{H}^{N-1}(x) + D^c u(A),$$

where ∇u denotes the approximate gradient of u and $D^c u$ is the Cantor part of Du. $BV(A)$ is a Banach space with respect to the norm $\|u\|_{BV(A)} := \|u\|_{L^1(A)} + |Du|(A)$.

We will often use the following result: if A is bounded and Lipschitz, and if $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $BV(A)$, then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and $u \in BV(A)$ such that

$$\begin{aligned} u_{k_h} &\to u \qquad \text{strongly in } L^1(A), \\ Du_{k_h} &\overset{*}{\rightharpoonup} Du \qquad \text{weakly}^* \text{ in the sense of measures.} \end{aligned} \tag{2.1}$$

We say that $u_k \overset{*}{\rightharpoonup} u$ weakly* in $BV(A)$ if (2.1) holds.

We say that $u \in SBV(A)$ if $u \in BV(A)$ and $D^c u = 0$. The space $SBV(A)$ is called the space of *special functions of bounded variation.* Note that if $u \in SBV(A)$, then the singular part of Du is concentrated on $S(u)$.

The space SBV is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to L.Ambrosio ([1], [3]).

Theorem 2.1. *Let A be an open and bounded subset of $\mathbb{R}^N$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $SBV(A)$. Assume that there exists $q > 1$ and $c \in [0; +\infty[$ such that*

$$\int_A |\nabla u_k|^q \, dx + \mathcal{H}^{N-1}(S(u_k)) + ||u_k||_\infty \le c$$

for every $k \in \mathbb{N}$. Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and a function $u \in SBV(A)$ such that

$$\begin{gathered} u_{k_h} \to u \quad \textit{strongly in } L^1(A), \\ \nabla u_{k_h} \rightharpoonup \nabla u \quad \textit{weakly in } L^1(A;\mathbb{R}^N), \\ \mathcal{H}^{N-1}(S(u)) \le \liminf_h \mathcal{H}^{N-1}(S(u_{k_h})). \end{gathered} \tag{2.2}$$

In the rest of the paper, we will say that $u_k \rightharpoonup u$ weakly in $SBV(A)$ if u_k and u satisfy (2.2). It will also be useful the following fact which can be derived from Ambrosio's Theorem: if $u_k \rightharpoonup u$ weakly in $SBV(A)$ and if $\mathcal{H}^{N-1} \llcorner S(u_k) \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures, then $\mathcal{H}^{N-1} \llcorner S(u) \le \mu$ as measures.

Finally in the context of fracture problems we will use the following notation: if A is Lipschitz, and if $\partial_D A \subseteq \partial A$, then for all $u, g \in BV(A)$ we set

$$S^g(u) := S(u) \cup \{x \in \partial_D A \,:\, u(x) \neq g(x)\}, \tag{2.3}$$

where the inequality on $\partial_D A$ is intended in the sense of traces. Moreover, we set for all $x \in S(u)$

$$[u](x) := u^+(x) - u^-(x),$$

and for all $x \in \partial_D A$ we set $[u](x) := u(x) - g(x)$, where the traces of u and g on ∂A are used.

Quasi-static evolution of brittle fractures. Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and let $\partial_D\Omega$ be a subset of $\partial\Omega$ open in the relative topology. Let $g : [0,T] \to H^1(\Omega)$ be absolutely continuous (see [7] for a precise definition); we indicate the gradient of g at time t by $\nabla g(t)$, and the time derivative of g at time t by $\dot{g}(t)$. For $u \in SBV(\Omega)$, let $S^{g(t)}(u)$ be defined as in (2.3), and for every $A, B \subseteq \mathbb{R}^N$, let $A \,\tilde{\subseteq}\, B$ mean $A \subseteq B$ up to a set of $\mathcal{H}^{N-1}$-measure zero. The main result of [13] is the following theorem.

Theorem 2.2. *Let $\bar{\Gamma}$ be a rectifiable set in $\Omega \cup \partial_D\Omega$ such that $\mathcal{H}^{N-1}(\bar{\Gamma}) < +\infty$. There exists $\{(u(t), \Gamma(t)) \,:\, t \in [0,T]\}$ with $\Gamma(t) \,\tilde{\subseteq}\, \Omega \cup \partial_D\Omega$ rectifiable and $u(t) \in SBV(\Omega)$ with $S^{g(t)}(u(t)) \,\tilde{\subseteq}\, \Gamma(t)$ such that:*

(a) $\bar{\Gamma} \,\tilde{\subseteq}\, \Gamma(s) \,\tilde{\subseteq}\, \Gamma(t)$ *for all* $0 \le s \le t \le T$*;*

(b) $u(0)$ *minimizes*

$$\|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(0)}(v) \setminus \bar{\Gamma})$$

among all $v \in SBV(\Omega)$*;*

(c) *for* $t \in]0,T]$, $u(t)$ *minimizes*

$$\|\nabla v\|^2 + \mathcal{H}^{N-1}\left(S^{g(t)}(v) \setminus \Gamma(t)\right)$$

among all $v \in SBV(\Omega)$.

Furthermore, the total energy

$$\mathcal{E}(t) := \|\nabla u(t)\|^2 + \mathcal{H}^{N-1}(\Gamma(t))$$

is absolutely continuous and satisfies

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau \tag{2.4}$$

for every $t \in [0,T]$.

Condition (a) stands for the *irreversibility* of the crack propagation, conditions (b) and (c) are *minimality* conditions, while (2.4) stands for the *nondissipativity* of the process.

Γ-convergence. Let us recall the definition and some basic properties of De Giorgi's Γ-*convergence* in metric spaces. We refer the reader to [10] for an exhaustive treatment of this subject. Let (X,d) be a metric space. We say that a sequence $F_h : X \to [-\infty,+\infty]$ Γ-converges to $F : X \to [-\infty,+\infty]$ (as $h \to +\infty$) if for all $u \in X$ we have

(i) (Γ-*liminf inequality*) for every sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X,

$$\liminf_{h\to+\infty} F_h(u_h) \geq F(u);$$

(ii) (Γ-*limsup inequality*) there exists a sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_{h\to+\infty} F_h(u_h) \leq F(u).$$

The function F is called the Γ-limit of $(F_h)_{h\in\mathbb{N}}$ (with respect to d), and we write $F = \Gamma\!-\lim_h F_h$. Γ-convergence is a convergence of variational type as explained in the following proposition.

Proposition 2.3. *Assume that the sequence* $(F_h)_{h\in\mathbb{N}}$ Γ-*converges to* F *and that there exists a compact set* $K \subseteq X$ *such that for all* $h \in \mathbb{N}$

$$\inf_{u\in K} F_h(u) = \inf_{u\in X} F_h(u).$$

Then F *admits a minimum on* X, $\inf_X F_h \to \min_X F$, *and any limit point of any sequence* $(u_h)_{h\in\mathbb{N}}$ *such that*

$$\lim_{h\to+\infty}\left(F_h(u_h) - \inf_{u\in X} F_h(u)\right) = 0,$$

is a minimizer of F.

Moreover the following compactness result holds.

Proposition 2.4. *If* (X,d) *is separable, and* $(F_h)_{h\in\mathbb{N}}$ *is a sequence of functionals on* X, *then there exists a subsequence* $(F_{h_k})_{k\in\mathbb{N}}$ *and a function* $F : X \to [-\infty;+\infty]$ *such that* $(F_{h_k})_{k\in\mathbb{N}}$ Γ-*converges to* F.

3. Discrete in time evolution of fractures in the cohesive case

In this section we are interested in generalized antiplanar shear of an elastic body Ω in the context of linearized elasticity and in presence of cohesive fractures.

The notion of *discrete in time evolution* for fractures relative to time dependent boundary displacement $g(t)$ and preexisting crack configuration $(\bar{\Gamma},\bar{\psi})$ has been described in the Introduction. It relies on the minimization of functionals of the form

$$\|\nabla u\|^2 + \int_{\Gamma\cup S^{g(t)}(u)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1}, \tag{3.1}$$

with ψ positive function on Γ. We now define rigorously the functional space to which the displacements belong, and the properties of Ω, Γ, ψ and $g(t)$ in order to prove an existence result for the discrete in time evolution of fractures.

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $\varphi : [0,+\infty[\to [0,+\infty[$ be increasing and concave, $\varphi(0) = 0$ and such that $\lim_{s\to+\infty} \varphi(s) = 1$. If $a := \varphi'(0) < +\infty$, we have

$$\varphi(s) \leq as \qquad \text{for all } s \in [0,+\infty[. \tag{3.2}$$

Let $T > 0$, and let us consider a boundary displacement $g \in AC([0,T]; H^1(\Omega))$ such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. We discretize g in the following way. Given $\delta > 0$, let I_δ be a subdivision of $[0,T]$ of the form $0 = t_0^\delta < t_1^\delta < \cdots < t_{N_\delta}^\delta = T$ such that $\max_i(t_i^\delta - t_{i-1}^\delta) < \delta$. For $0 \leq i \leq N_\delta$ we set $g_i^\delta := g(t_i^\delta)$.

Let $\bar\Gamma \tilde\subseteq \Omega$ be rectifiable, and let $\bar\psi$ be a positive function on $\bar\Gamma$ such that

$$\int_{\bar\Gamma} \varphi(\bar\psi)\, d\mathcal{H}^{N-1} < +\infty. \tag{3.3}$$

Let us extend $\bar\psi$ to $\overline{\Omega}$ setting $\bar\psi = 0$ outside $\bar\Gamma$. As for the space of the displacements, it would be natural following [13] to consider $u \in SBV(\Omega)$. Since $a = \varphi'(0) < +\infty$, we have unfortunately that the minimization of (3.1) is not well posed in $SBV(\Omega)$. Let us in fact consider $(u_n)_{n\in\mathbb{N}}$ minimizing sequence for (3.1): it turns out that we may assume $(u_n)_{n\in\mathbb{N}}$ bounded in $BV(\Omega)$. As a consequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence weakly* convergent in $BV(\Omega)$ to a function $u \in BV(\Omega)$. Then we have that minimizing sequences of (3.1) converge (up to a subsequence) to a minimizer of the relaxation of (3.1) with respect to the weak* topology of $BV(\Omega)$. By Proposition 8.1, the natural domain of this relaxed functional is $BV(\Omega)$, and that its form is

$$\int_\Omega f(\nabla u)\, dx + \int_{\Gamma \cup S^{g(t)}(u)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(\Omega), \tag{3.4}$$

where

$$f(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \leq \frac{a}{2} \\ \\ \frac{a^2}{4} + a(|\xi| - \frac{a}{2}) & \text{if } |\xi| \geq \frac{a}{2}. \end{cases} \tag{3.5}$$

In view of these remarks, we consider $BV(\Omega)$ as the space of displacements u of the body Ω, and a total energy of the form (3.4). The volume part in the energy (3.4) can be interpreted as the contribution of the elastic behavior of the body. The second term represents the work done to create the fracture $\Gamma \cup S^{g(t)}(u)$ with opening given by $|[u]| \vee \psi$. The new term $a|D^c u|$ can be interpreted as the contribute of microcracks in the configuration which are considered as reversible.

Let us define the discrete evolution of the fracture in this new setting. For $i = 0$, let $u_0^\delta \in BV(\Omega)$ be a minimum of

$$\min_{u\in BV(\Omega)} \left\{ \int_\Omega f(\nabla u)\, dx + \int_{S^{g_0^\delta}(u)\cup\bar\Gamma} \varphi(|[u]| \vee \bar\psi)\, d\mathcal{H}^{N-1} + a|D^c u| \right\}. \tag{3.6}$$

We set $\Gamma_0^\delta := S^{g_0^\delta}(u_0^\delta) \cup \bar\Gamma$.

Supposing to have constructed u_j^δ and Γ_j^δ for all $j = 0, \dots, i-1$, let u_i^δ be a minimum of

$$\min_{u\in BV(\Omega)} \left\{ \int_\Omega f(\nabla u)\, dx + \int_{S^{g_i^\delta}(u)\cup\Gamma_{i-1}^\delta} \varphi(|[u]| \vee \psi_{i-1}^\delta)\, d\mathcal{H}^{N-1} + a|D^c u|, \right\} \tag{3.7}$$

where $\psi_{i-1}^\delta := \bar\psi \vee |[u_0^\delta]| \vee \cdots \vee |[u_{i-1}^\delta]|$. We set $\Gamma_i^\delta := \Gamma_{i-1}^\delta \cup S^{g_i^\delta}(u_i^\delta)$.

The following proposition establish the existence of this discrete evolution.

Proposition 3.1. *Let* $I_\delta = \{0 = t_0^\delta < \cdots < t_{N_\delta}^\delta = T\}$ *be a subdivision of* $[0,T]$ *such that* $\max(t_i^\delta - t_{i-1}^\delta) < \delta$, *let* $\bar\Gamma$ *be a preexisting crack, and* $\bar\psi$ *a positive function on* $\bar\Gamma$ *satisfying* (3.3) *and extended to zero outside $\bar\Gamma$. Then for all $i = 0, \dots, N_\delta$ there exists $u_i^\delta \in BV(\Omega)$ such that setting $\Gamma_{-1}^\delta := \bar\Gamma$, $\psi_{-1}^\delta := \bar\psi$ and*

$$\Gamma_i^\delta := \bar\Gamma \cup \bigcup_{j=0}^{i} S^{g_j^\delta}(u_j^\delta), \qquad \psi_i^\delta(x) := \bar\psi(x) \vee |[u_0^\delta]|(x) \vee \cdots \vee |[u_i^\delta]|(x) \tag{3.8}$$

the following holds:

(a) $\|u_i^\delta\|_\infty \le \|g_i^\delta\|_\infty \le C$;

(b) *for all $v \in BV(\Omega)$ we have*

$$\int_\Omega f(\nabla u_i^\delta)\,dx + \int_{\Gamma_i^\delta} \varphi(\psi_i^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_i^\delta|(\Omega) \le \int_\Omega f(\nabla v)\,dx + \int_{S^{g_i^\delta}(v)\cup\Gamma_{i-1}^\delta} \varphi(|[v]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega), \tag{3.9}$$

where $a = \varphi'(0)$ and f is defined in (3.5);

(c) *we have that*

$$\int_\Omega f(\nabla u_i^\delta)\,dx + \int_{\Gamma_i^\delta} \varphi(\psi_i^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_i^\delta|(\Omega) = \inf_{v\in SBV(\Omega)} \left\{ \|\nabla v\|^2 + \int_{S^{g_i^\delta}(v)\cup\Gamma_{i-1}^\delta} \varphi(|[v]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} \right\}. \tag{3.10}$$

Proof. We have to prove that problems (3.6) and (3.7) admit solutions. Let us consider for example problem (3.7), the other being similar. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (3.7). By a truncation argument we may assume that $\|u_n\|_\infty \le \|g_i^\delta\|$. Comparing u_n with g_i^δ, we get for n large

$$\int_\Omega f(\nabla u_n)\,dx + \int_{S^{g_i^\delta}(u_n)\cup\Gamma_{i-1}^\delta} \varphi(|[u_n]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_n|(\Omega) \le \int_\Omega f(\nabla g_0^\delta)\,dx + \int_{\Gamma_{i-1}^\delta} \varphi(\psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + 1 \le C', \tag{3.11}$$

with C' independent of n. Since there exists $d > 0$ such that $a|\xi| - d \le f(\xi)$ for all $\xi \in \mathbb{R}^N$, we deduce that $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega;\mathbb{R}^N)$. Moreover if $\bar s$ is such that $\varphi(\bar s) = \frac{1}{2}$ and $\bar a$ is such that $s \le \bar a\varphi(s)$ for all $s \in [0, \bar s]$, we have

$$\begin{aligned} \int_{S(u_n)} |[u_n]|\,d\mathcal{H}^{N-1} &= \int_{\{|[u_n]|\le\bar s\}} |[u_n]|\,d\mathcal{H}^{N-1} + \|g_i^\delta\|_\infty \mathcal{H}^{N-1}(\{|[u_n]| > \bar s\}) \\ &\le \bar a \int_{|[u_n]|\le\bar s} \varphi(|[u_n]|)\,d\mathcal{H}^{N-1} + 2\|g_i^\delta\|_\infty \int_{|[u_n]|>\bar s} \varphi(|[u_n]|)\,d\mathcal{H}^{N-1} \\ &\le (\bar a + 2\|g_i^\delta\|_\infty)C'. \end{aligned} \tag{3.12}$$

Finally for all n

$$|D^c u_n| \le \frac{C'}{a}.$$

We conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\Omega)$. Then there exists $u \in BV(\Omega)$ such that up to a subsequence $u_n \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega)$ and pointwise almost everywhere. Let us set $u_i^\delta := u$. By

Lemma 8.3 we deduce that

$$\int_\Omega f(\nabla u)\,dx + \int_{S^{g^\delta - i}(u)\cup\Gamma^\delta_{i-1}} \varphi(|[u]|\vee\psi^\delta_{i-1})\,d\mathcal{H}^{N-1} + a|D^c u|(\Omega)$$
$$\le \liminf_n \int_\Omega f(\nabla u_n)\,dx + \int_{S^{g^\delta_i}(u_n)\cup\Gamma^\delta_{i-1}} \varphi(|[u_n]|\vee\psi^\delta_{i-1})\,d\mathcal{H}^{N-1} + a|D^c u_n|(\Omega). \tag{3.13}$$

Setting $\psi^\delta_i := \psi^\delta_{i-1}\vee|[u^\delta_i]|$, we have that point (b) holds. Moreover $\|u^\delta_i\|_\infty \le \|g^\delta_0\|_\infty \le C$, so that point (a) holds. Finally point (c) is a consequence of Proposition 8.1. □

Let us consider now the following piecewise constant interpolation in time:

$$u^\delta(t) := u^\delta_i, \qquad \Gamma^\delta(t) := \Gamma^\delta_i, \qquad \psi^\delta(t) := \psi^\delta_i, \qquad g^\delta(t) := g^\delta_i \qquad t^\delta_i \le t < t^\delta_{i+1} \tag{3.14}$$

with $u^\delta(T) := u^\delta_{N_\delta}$, $\Gamma^\delta(T) := \Gamma^\delta_{N_\delta}$, $\psi^\delta(T) := \psi^\delta_{N_\delta}$, and $g^\delta(T) := g(T)$.

For every $v \in BV(\Omega)$ and for every $t\in[0,T]$ let us set

$$\mathcal{E}^\delta(t,v) := \int_\Omega f(\nabla v)\,dx + \int_{S^{g^\delta(t)}(v)\cup\Gamma^\delta(t)} \varphi(|[v]|\vee\psi^\delta(t))\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega). \tag{3.15}$$

Then the following estimate holds.

Lemma 3.2. *There exists $e^\delta_a \to 0$ for $\delta\to 0$ and $a\to+\infty$ such that for all $t\in[0,T]$ we have*

$$\mathcal{E}^\delta(t,u^\delta(t)) \le \mathcal{E}^\delta(0,u^\delta(0)) + \int_0^{t^\delta_i}\int_\Omega f'(\nabla u^\delta(\tau))\nabla\dot g(\tau)\,dx\,d\tau + e^\delta_a, \tag{3.16}$$

where t^δ_i is the step discretization point such that $t^\delta_i \le t < t^\delta_{i+1}$.

Proof. Comparing u^δ_i with $u^\delta_{i-1} + g^\delta_i - g^\delta_{i-1}$ by means of (3.9) we obtain

$$\mathcal{E}^\delta(t^\delta_i,u^\delta_i) \le \int_\Omega f(\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1})\,dx + \int_{\Gamma^\delta_{i-1}} \varphi(\psi^\delta_{i-1})\,d\mathcal{H}^{N-1} + a|D^c u^\delta_{i-1}|(\Omega).$$

Notice that by the very definition of f the following hold:

1) if $|\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}| \ge \frac{a}{2}$ and $|\nabla u^\delta_{i-1}| \ge \frac{a}{2}$
$$f'(\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}) = f'(\nabla u^\delta_{i-1});$$

2) if $|\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}| < \frac{a}{2}$ and $|\nabla u^\delta_{i-1}| \ge \frac{a}{2}$
$$f(\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}) \le f(\nabla u^\delta_{i-1});$$

3) if $|\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}| \ge \frac{a}{2}$ and $|\nabla u^\delta_{i-1}| < \frac{a}{2}$
$$f(\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}) \le f(\nabla u^\delta_{i-1}) + 2(\nabla u^\delta_{i-1}, \nabla g^\delta_i - \nabla g^\delta_{i-1}) + |\nabla g^\delta_i - \nabla g^\delta_{i-1}|^2;$$

4) if $|\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}| < \frac{a}{2}$ and $|\nabla u^\delta_{i-1}| < \frac{a}{2}$
$$f(\nabla u^\delta_{i-1} + \nabla g^\delta_i - \nabla g^\delta_{i-1}) = f(\nabla u^\delta_{i-1}) + 2(\nabla u^\delta_{i-1}, \nabla g^\delta_i - \nabla g^\delta_{i-1}) + |\nabla g^\delta_i - \nabla g^\delta_{i-1}|^2.$$

Then by convexity of f we deduce

$$\mathcal{E}^\delta(t^\delta_i,u^\delta_i) \le \mathcal{E}^\delta(t^\delta_{i-1},u^\delta_{i-1}) + \int_\Omega f'(\nabla u^\delta_{i-1})(\nabla g^\delta_i - \nabla g^\delta_{i-1})\,dx + R^{\delta,a}_{i-1},$$

where

$$R^{\delta,a}_{i-1} := \int_\Omega |\nabla g^\delta_i - \nabla g^\delta_{i-1}|^2\,dx + \int_{\{|\nabla u^\delta_{i-1}|\ge\frac{a}{2}\}} |f'(\nabla u^\delta_{i-1})|\,|\nabla g^\delta_i - \nabla g^\delta_{i-1}|\,dx.$$

Then summing up from t^δ_i to t^δ_0, and taking into account (3.14) we get

$$\mathcal{E}^\delta(t,u^\delta(t)) \le \mathcal{E}^\delta(0,u^\delta(0)) + \int_0^{t^\delta_i}\int_\Omega f'(\nabla u^\delta(\tau))\nabla\dot g(\tau)\,dx\,d\tau + \int_0^{t^\delta_i} R^{\delta,a}(\tau)\,d\tau,$$

where

$$R^{\delta,a}(\tau) := \sigma(\delta)\|\nabla\dot{g}(\tau)\| + \int_{\{|\nabla u^\delta(\tau)|\geq \frac{a}{2}\}} |f'(\nabla u^\delta(\tau))|\,|\nabla\dot{g}(\tau)|\,dx \tag{3.17}$$

and

$$\sigma(\delta) := \max_{i=1,\dots,N_\delta} \int_{t^\delta_{i-1}}^{t^\delta_i} \|\nabla\dot{g}\|\,d\tau.$$

In order to conclude the proof it is sufficient to see that

$$\int_0^T R^{\delta,a}(\tau)\,d\tau \to 0$$

as $\delta \to 0$ and $a \to +\infty$. Notice that $\sigma(\delta) \to 0$ as $\delta \to 0$ by the absolutely continuity of $\|\nabla\dot{g}\|$. Let us come to the second term. Notice that $|f'(\nabla u^\delta(\tau))| = a$ on $\{|\nabla u^\delta(\tau)| \geq \frac{a}{2}\}$. Then we have to see

$$\int_0^T \int_\Omega a|\nabla\dot{g}(\tau)|\mathbb{1}_{\{|\nabla u^\delta(\tau)|\geq\frac{a}{2}\}}\,dx\,d\tau \to 0 \tag{3.18}$$

as $\delta \to 0$ and $a \to +\infty$. Setting $A^\delta_a(\tau) := \{x \in \Omega \,:\, |\nabla u^\delta(\tau)|(x) \geq \frac{a}{2}\}$ we have by Hölder inequality

$$\int_\Omega a|\nabla\dot{g}(\tau)|\mathbb{1}_{A^\delta_a(\tau)}\,dx \leq a\sqrt{|A^\delta_a(\tau)|}\left(\int_{A^\delta_a(\tau)} |\nabla\dot{g}(\tau)|^2\,dx\right)^{\frac{1}{2}}.$$

Notice that

$$\frac{a^2}{2}|A^\delta_a(\tau)| \leq a\int_{A^\delta_a(\tau)} |\nabla u^\delta(\tau)|\,dx \leq 2\int_{A^\delta_a(\tau)} f(\nabla u^\delta(\tau))\,dx \leq C', \tag{3.19}$$

where C' depends only on g and is obtained comparing $u^\delta(\tau)$ with $g^\delta(\tau)$ by means of (3.9). We deduce that

$$\int_\Omega a|\nabla\dot{g}(\tau)|\mathbb{1}_{A^\delta_a(\tau)}\,dx \leq \sqrt{2C'}\left(\int_{A^\delta_a(\tau)} |\nabla\dot{g}(\tau)|^2\,dx\right)^{\frac{1}{2}} \leq \sqrt{2C'}\|\nabla\dot{g}(\tau)\|. \tag{3.20}$$

As $\delta \to 0$ and $a \to +\infty$, by (3.19) we have that $|A^\delta_a(\tau)| \to 0$. Then by the equicontinuity of $|\nabla\dot{g}(\tau)|^2$ and by the Dominated Convergence Theorem, we deduce that (3.18) holds, and the proof is finished. ☐

4. The main results

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

In this section we consider discrete in time evolution of fractures in a linearly elastic body whose reference configuration is given by $\Omega_h := h\Omega$, where $h > 0$. Let us assume that the cohesive forces on the fractures of Ω_h are given in the sense of Section 3 by a function $\varphi : [0, +\infty[\to [0,1]$ which is increasing, concave, $\varphi(0) = 0$, $\varphi'(0) = a < +\infty$ and such that $\lim_{s\to+\infty}\varphi(s) = 1$. Let us moreover set

$$f(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \leq \frac{a}{2} \\ \\ \frac{a^2}{4} + a(|\xi| - \frac{a}{2}) & \text{if } |\xi| \geq \frac{a}{2}. \end{cases} \tag{4.1}$$

Let us consider on $\partial_D\Omega_h := h\partial_D\Omega$ boundary displacements of the following particular form

$$g_h(t,x) := h^\alpha g\left(t, \frac{x}{h}\right) \tag{4.2}$$

with $g \in AC([0,T]; H^1(\Omega))$ such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. Let moreover $\bar{\Gamma} \tilde{\subseteq} \Omega$ be rectifiable with

$$\mathcal{H}^{N-1}(\bar{\Gamma}) < +\infty \tag{4.3}$$

and let $\bar{\gamma}$ be a positive function defined on $\bar{\Gamma}$. We extend $\bar{\gamma}$ to $\overline{\Omega}$ setting $\bar{\gamma}=0$ outside $\bar{\Gamma}$. Let us consider $(\bar{\Gamma}_h,\bar{\psi}_h)$ as a preexisting crack configuration in Ω_h, where

$$\bar{\Gamma}_h := h\bar{\Gamma}, \qquad \bar{\psi}_h(x) := \bar{\gamma}\left(\frac{x}{h}\right), \quad x\in\Omega_h. \tag{4.4}$$

Given $\delta>0$, let $I_\delta=\{0=t_0^\delta<\cdots<t_{N_\delta}^\delta=T\}$ be a subdivision of $[0,T]$ such that $\max(t_i^\delta-t_{i-1}^\delta)<\delta$, and let $\{t\to(u^{\delta,h}(t),\Gamma^{\delta,h}(t),\psi^{\delta,h}(t))\,:\,t\in[0,T]\}$ be the piecewise constant interpolation in the sense of (3.14) of a discrete in time evolution of fractures in Ω_h relative to the boundary datum g_h, the preexisting crack configuration $(\bar{\Gamma}_h,\bar{\psi}_h)$ and the subdivision I_δ given by Proposition 3.1.

Our aim is to study the asymptotic behavior of $\{t\to(u^{\delta,h}(t),\Gamma^{\delta,h}(t),\psi^{\delta,h}(t))\,:\,t\in[0,T]\}$ as $\delta\to 0$ and $h\to+\infty$. Let us consider $h\in\mathbb{N}$ (we can consider any sequence which diverges to $+\infty$), let us fix $\delta_h\to 0$, and let us set for all $t\in[0,T]$

$$u_h(t) := u^{\delta_h,h}(t), \qquad \Gamma_h(t) := \Gamma^{\delta_h,h}(t), \qquad \psi_h(t) := \psi^{\delta_h,h}(t), \tag{4.5}$$

and let $g_h^\delta(t) := g_h(t_i^\delta)$ where $t_i^\delta\in I_\delta$ is such that $t_i^\delta\le t<t_{i+1}^\delta$. Let us moreover set for every $v\in BV(\Omega)$ and for every $t\in[0,T]$

$$\mathcal{E}_h(t,v) := \int_\Omega f(\nabla v)\,dx + \int_{S^{g_h^\delta(t)}(v)\cup\Gamma_h(t)} \varphi(|[v]|\vee\psi_h(t))\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega). \tag{4.6}$$

The asymptotic of (u_h,Γ_h,ψ_h) depends on α, and we have to distinguish three cases. The first case $\alpha=\frac{1}{2}$ was stated in the Introduction and reveals the prevalence of brittle effects as the size of the body increases. We give here the precise statement we will prove.

Theorem 4.1. *Let $g\in AC(0,T;H^1(\Omega))$ be such that $\|g(t)\|_\infty\le C$ for all $t\in[0,T]$. Let $\{t\to(u_h(t),\Gamma_h(t),\psi_h(t))\,:\,t\in[0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of fractures in Ω_h relative to the preexisting crack configuration $(\bar{\Gamma}_h,\bar{\psi}_h)$ and the boundary data*

$$g_h(x,t) := \sqrt{h}\,g\left(\frac{x}{h},t\right).$$

Then the following facts hold:

(a) *there exists a constant C' dependent only on g such that for all $t\in[0,T]$*

$$\frac{1}{h^{N-1}}\mathcal{E}_h(t,u_h(t))\le C'; \tag{4.7}$$

(b) *for all $t\in[0,T]$*

$$v_h(t,x) := \frac{1}{\sqrt{h}}u_h(t,hx) \qquad \textit{is bounded in } BV(\Omega); \tag{4.8}$$

(c) *there exists a subsequence independent of t and there exists a quasistatic evolution of brittle fractures $\{t\to(v(t),K(t))\,:\,t\in[0,T]\}$ in Ω relative to the preexisting crack $\bar{\Gamma}$ and boundary displacement g in the sense of Theorem 2.2 such that for all $t\in[0,T]$ we have*

$$\nabla v_h(t) \rightharpoonup \nabla v(t) \qquad \textit{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{4.9}$$

and every accumulation point v of $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$ is such that $v\in SBV(\Omega)$, $S^{g(t)}(v)\,\tilde{\subseteq}\,K(t)$ and $\nabla v=\nabla v(t)$. Moreover for all $t\in[0,T]$ we have

$$\frac{1}{h^{N-1}}\mathcal{E}_h(t,u_h(t))\to\|\nabla v(t)\|^2+\mathcal{H}^{N-1}(K(t)); \tag{4.10}$$

in particular $h^{-N+1}|D^c u_h(t)|(\Omega_h)|\to 0$,

$$\frac{1}{h^{N-1}}\int_{\Omega_h} f(\nabla u_h(t))\,dx\to\|\nabla v(t)\|^2, \tag{4.11}$$

and

$$\frac{1}{h^{N-1}}\int_{\Gamma_h(t)}\varphi(\psi_h(t))\,d\mathcal{H}^{N-1}\to\mathcal{H}^{N-1}(K(t)). \tag{4.12}$$

The case $\alpha < \frac{1}{2}$ leads to a problem in elasticity in $\Omega_h \setminus \bar{\Gamma}_h$ in the sense of the following theorem.

Theorem 4.2. *Let $g \in AC(0,T;H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of fractures in Ω_h relative to the initial crack configuration $(\bar{\Gamma}_h, \bar{\psi}_h)$ and the boundary data*

$$g_h(x,t) := h^\alpha g\left(t, \frac{x}{h}\right)$$

with $\alpha < \frac{1}{2}$. Then the following facts hold:

(a) *for all $t \in [0,T]$*

$$v_h(t,x) := \frac{1}{h^\alpha} u_h(t,hx) \qquad \textit{is bounded in } BV(\Omega); \tag{4.13}$$

(b) *for all $t \in [0,T]$ every accumulation point $v(t)$ of $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$ is such that $v(t) \in SBV(\Omega)$ and $S^{g(t)}(v(t)) \tilde{\subseteq} \bar{\Gamma}$;*

(c) *if $\varphi(s) = 1$ for $s \geq \bar{s}$, and $\bar{\gamma} \geq \varepsilon > 0$, then there exists a subsequence independent of t such that for all $t \in [0,T]$*

$$\nabla v_h(t,x) \rightharpoonup \nabla v(t) \qquad \textit{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{4.14}$$

where $v(t)$ is a minimizer of

$$\min\{\|\nabla v\|^2 \,:\, v \in SBV(\Omega),\, S^{g(t)}(v) \tilde{\subseteq} \bar{\Gamma}\}; \tag{4.15}$$

moreover for all $t \in [0,T]$ we have

$$\frac{1}{h^{N+2\alpha-2}} \int_{\Omega_h} f(\nabla u_h(t))\, dx \to \|\nabla v(t)\|^2. \tag{4.16}$$

Finally for the case $\alpha > \frac{1}{2}$ the body goes to *rupture* at time $t = 0$, in the sense of the following theorem.

Theorem 4.3. *Let $g \in AC(0,T;H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of fractures in Ω_h relative to the initial crack configuration $(\bar{\Gamma}_h, \bar{\psi}_h)$ and the boundary data*

$$g_h(x,t) := h^\alpha g\left(\frac{x}{h}, t\right)$$

with $\alpha > \frac{1}{2}$. Let us set $v_h(t,x) := \frac{1}{h^\alpha} u_h(t,hx)$ for all $x \in \Omega$ and for all $t \in [0,T]$.

Then $(v_h(0))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$, and every accumulation point v of $(v_h(0))_{h\in\mathbb{N}}$ in the weak topology of $BV(\Omega)$ is piecewise constant in Ω, that is $v \in SBV(\Omega)$ and $\nabla v = 0$. Moreover*

$$\mathcal{H}^{N-1}(S^{g(0)}(v(0)) \cup \bar{\Gamma}) \leq \mathcal{H}^{N-1}(S^{g(0)}(w) \cup \bar{\Gamma}) \tag{4.17}$$

for all piecewise constant function $w \in SBV(\Omega)$.

5. Proof of Theorem 4.1

In this section we will give the proof of Theorem 4.1. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of cohesive fracture in Ω_h relative the subdivision $I_{\delta_h} := \{0 = t_0^{\delta_h} < \cdots < t_{N_{\delta_h}}^{\delta_h} = T\}$, the preexisting crack configuration $(\bar{\Gamma}_h, \bar{\psi}_h)$ given by (4.4) and the boundary displacement $\sqrt{t} g(t, \frac{x}{h})$.

In order to prove Theorem 4.1, we need some preliminary analysis. First of all, it is convenient to rescale u_h and Γ_h in the following way: for all $t \in [0,T]$ let $v_h(t) \in BV(\Omega)$ and $K_h(t) \tilde{\subseteq} \Omega \cup \partial_D \Omega$ be defined by

$$v_h(t,x) := \frac{1}{\sqrt{h}} u_h\left(t, hx\right), \qquad K_h(t) := \frac{1}{h} \Gamma_h(t). \tag{5.1}$$

Let us moreover set

$$\gamma_h(t,x) := \frac{1}{\sqrt{h}}\psi_h(t,hx) = \max_{s\le t} |[v_h(s)](t,x)| \vee \bar\gamma(x), \quad t\in[0,T], x\in\Omega. \tag{5.2}$$

We notice that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) \,:\, t\in[0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cohesive fractures in Ω relative to the subdivision I_{δ_h}, the preexisting crack configuration $(\bar\Gamma,\bar\gamma)$ and boundary displacement $g(t)$ with respect to the basic total energy

$$\int_\Omega f_h(\nabla v)\,dx + \int_{S^{g^{\delta_h}(t)}(v)\cup K_h(t)} \varphi_h(|[v]|\vee\gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v|(\Omega), \tag{5.3}$$

where

$$\varphi_h(s) := \varphi(\sqrt{h}s), \tag{5.4}$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi|\le \frac{a\sqrt{h}}{2} \\ \frac{a^2h}{4} + a\sqrt{h}(|\xi| - \frac{a\sqrt{h}}{2}) & \text{if } |\xi|\ge\frac{a\sqrt{h}}{2}. \end{cases} \tag{5.5}$$

Let us recall some properties of the evolution $\{t \to (v_h(t), K_h(t), \gamma_h(t)) \,:\, t\in[0,T]\}$ which derive from Proposition 3.1 and that will be employed in the sequel:

(a) for all $t\in[0,T]$

$$\|v_h(t)\|_\infty \le \|g^{\delta_h}(t)\|_\infty; \tag{5.6}$$

(b) for all $w\in BV(\Omega)$ we have

$$\begin{aligned}\int_\Omega f_h(\nabla v_h(0))\,dx &+ \int_{K_h(0)} \varphi_h(|[v_h(0)]|\vee\bar\gamma)\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h(0)|(\Omega) \\ &\le \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(0)}(w)\cup\bar\Gamma} \varphi_h(|[w]|\vee\bar\gamma)\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega);\end{aligned} \tag{5.7}$$

(c) for all $w\in BV(\Omega)$ and for all $t\in]0,T]$ we have

$$\begin{aligned}\int_\Omega f_h(\nabla v_h(t))\,dx &+ \int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h(t)|(\Omega) \\ &\le \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]|\vee\gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega).\end{aligned} \tag{5.8}$$

Let us set for all $v\in BV(\Omega)$ and for all $t\in[0,T]$

$$\mathcal{F}_h(t,w) := \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]|\vee\gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega). \tag{5.9}$$

Notice that for all $t\in[0,T]$

$$\mathcal{F}_h(t,v_h(t)) = \frac{1}{h^{N-1}}\mathcal{E}_h(t,u_h(t)), \tag{5.10}$$

where $\mathcal{E}_h(t,u)$ is defined in (4.6).

Recalling Lemma 3.2, we have that the following holds.

Lemma 5.1. *For all $t\in[0,T]$ we have*

$$\mathcal{F}_h(t,v_h(t)) \le \mathcal{F}_h(0,v_h(0)) + \int_0^{t_h}\int_\Omega f_h'(\nabla v_h(\tau))\nabla\dot g(\tau)\,dx\,d\tau + e(h), \tag{5.11}$$

where $e(h)\to 0$ as $h\to+\infty$, and $t_h := t^{\delta_h}_{i_h}$ is the step discretization point of I_{δ_h} such that $t^{\delta_h}_{i_h} \le t < t^{\delta_h}_{i_h+1}$.

The following corollary provides a bound on the total energy of the discrete in time evolution.

Corollary 5.2. *There exists a constant C' independent of h such that for all $t \in [0,T]$ we have*

$$\mathcal{F}_h(t, v_h(t)) + \|v_h(t)\|_\infty \leq C'. \tag{5.12}$$

Proof. By (5.11) we have

$$\mathcal{F}_h(t, v_h(t)) \leq \mathcal{F}_h(0, v_h(0)) + \int_0^{t_h} \int_\Omega f_h'(\nabla v_h(\tau)) \nabla \dot{g}(\tau)\, dx\, d\tau + e(h),$$

where $e(h) \to 0$ as $h \to +\infty$, and $t_h := t_{i_h}^{\delta_h}$ is such that $t_{i_h}^{\delta_h} \leq t < t_{i_h+1}^{\delta_h}$.

Notice that by (5.7) we have

$$\mathcal{F}_h(0, u_h(0)) \leq \|\nabla g(0)\|^2 + \mathcal{H}^{N-1}(\bar{\Gamma}),$$

and by (5.8) for all $\tau \in [0,T]$

$$\int_\Omega f_h(\nabla v_h(\tau))\, dx \leq \|\nabla g^{\delta_h}(\tau)\|^2.$$

Moreover for all $\tau \in [0,T]$

$$\int_\Omega |f_h'(\nabla v_h(\tau))|^2\, dx \leq 4 \int_\Omega f_h(\nabla v_h(\tau))\, dx$$

and so, taking into account (5.6), we deduce that (5.12) holds. □

As a consequence of Corollary 5.2, we infer a uniform bound on the total variations of $v_h(t)$.

Corollary 5.3. *There exists C'' independent of h such that for all $t \in [0,T]$ we have*

$$|Dv_h(t)|(\Omega) \leq C''. \tag{5.13}$$

Proof. In fact, since for h large we have for all $\xi \in \mathbb{R}^N$

$$|\xi| - 1 \leq f_h(\xi),$$

we deduce that for all $t \in [0,T]$

$$\int_\Omega |\nabla v_h(t)|\, dx \leq \int_\Omega [f_h(\nabla v_h(t)) + 1]\, dx \leq C' + |\Omega|,$$

where C' is given by Corollary 5.2. Moreover if $\bar{s}$ is such that $\varphi(\bar{s}) = \frac{1}{2}$ and $\bar{a}$ is such that $s \leq \bar{a}\varphi(s)$ for all $s \in [0, \bar{s}]$, we have for all h and for all $t \in [0,T]$

(5.14)

$$\int_{S(v_h(t))} |[v_h(t)]|\, d\mathcal{H}^{N-1} = \int_{|[v_h(t)]| \leq \frac{\bar{s}}{\sqrt{h}}} |[v_h(t)]|\, d\mathcal{H}^{N-1} + \|v_h(t)\|_\infty \mathcal{H}^{N-1}\left(\left\{|[v_h(t)]| \geq \frac{\bar{s}}{\sqrt{h}}\right\}\right)$$
$$\leq \bar{a} \int_{|[v_h(t)]| \leq \frac{\bar{s}}{\sqrt{h}}} \varphi_h(|[v_h(t)]|)\, d\mathcal{H}^{N-1} + 2C' \int_{|[v_h(t)]| \geq \frac{\bar{s}}{\sqrt{h}}} \varphi_h(|[v_h(t)]|)\, d\mathcal{H}^{N-1} \leq (\bar{a} + 2C')C'.$$

Finally for all h and for all $t \in [0,T]$

$$|D^c v_h(t)|(\Omega) \leq \frac{C'}{a\sqrt{h}}.$$

We deduce that (5.13) holds, and the proof is concluded. □

In order to construct the quasistatic growth of brittle fractures in the sense of [13] to which $(v_h(t), K_h(t), \gamma_h(t))$ converges, we need the following lemma which employees a Γ-convergence technique (see Section 2).

Lemma 5.4. *Let us fix $t \in [0,T]$, and let us consider the functionals*

$$\mathcal{G}_h(t)(u) := \int_\Omega f_h(\nabla u)\, dx + \int_{K_h(t)} \varphi_h(|[u]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u|(\Omega), \tag{5.15}$$

if $u \in BV(\Omega)$, $S^{g^{\delta_h}(t)}(u) \tilde{\subseteq} K_h(t)$, $|[u]| \leq \gamma_h(t)$ on $K_h(t)$, and $\mathcal{G}_h(t)(u) = +\infty$ otherwise for $u \in BV(\Omega)$.

Let us denote by $\mathcal{G}(t)$ the Γ-limit (up to a subsequence) of $\mathcal{G}_h(t)$ in the weak* topology of $BV(\Omega)$. For all $u \in \mathrm{dom}(\mathcal{G}(t))$ we have $u \in SBV(\Omega)$ and $\nabla u \in L^2(\Omega;\mathbb{R}^N)$. Moreover there exists a countable and dense set $D \subseteq \mathrm{dom}(\mathcal{G}(t))$ such that setting

$$K(t) := \bigcup_{u\in D} S^{g(t)}(u) \tag{5.16}$$

we have

$$S^{g(t)}(u) \,\tilde{\subseteq}\, K(t) \qquad \textit{for all } u \in \mathrm{dom}(\mathcal{G}(t)), \tag{5.17}$$

and

$$\mathcal{H}^{N-1}(K(t)) \le \liminf_h \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1}. \tag{5.18}$$

Proof. In order to deal with $S^g(u)$ as an internal jump, let us consider $\tilde\Omega \subseteq \mathbb{R}^N$ open and bounded, such that $\overline{\Omega} \subseteq \tilde\Omega$, and let us set $\Omega' := \tilde\Omega \setminus \partial_N\Omega$. Let us consider the following functionals $\mathcal{G}_h'(t) : BV(\Omega') \to [0,+\infty]$

$$\mathcal{G}_h'(t)(u) := \int_{\Omega'} f_h(\nabla u)\, dx + \int_{K_h(t)} \varphi_h(|[u]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u|(\Omega'), \tag{5.19}$$

if $u \in BV(\Omega')$, $u = g^{\delta_h}(t)$ on $\Omega'\setminus\Omega$, $S(u) \,\tilde{\subseteq}\, K_h(t)$, $|[u]| \le \gamma_h(t)$ on $K_h(t)$, and $\mathcal{G}_h'(t)(u) = +\infty$ otherwise for $u \in BV(\Omega')$.

By Proposition 2.4, up to a subsequence, $\mathcal{G}_h'(t)$ Γ-converges in the weak* topology of $BV(\Omega')$ to a functional $\mathcal{G}'(t)$. Clearly if $u \in \mathrm{dom}(\mathcal{G}'(t))$, then the restriction of u to Ω belongs to $\mathrm{dom}(\mathcal{G}(t))$. Conversely if $u \in \mathrm{dom}(\mathcal{G}(t))$, the extension of u to Ω' setting $u = g(t)$ on $\Omega'\setminus\Omega$ belongs to $\mathrm{dom}(\mathcal{G}'(t))$. Thus we can use $\mathcal{G}'(t)$ instead of $\mathcal{G}(t)$. Let $u \in \mathrm{dom}(\mathcal{G}'(t))$: clearly we have $u = g(t)$ on $\Omega'\setminus\Omega$. Moreover since

$$\mathcal{G}_h'(t)(u) \ge \int_{\Omega'} f_h(\nabla u)\, dx + \int_{S(u)} \varphi_h(|[u]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u|(\Omega'),$$

if $u \in \mathrm{dom}(\mathcal{G}'(t))$ and $u_h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega')$ with $\mathcal{G}_h'(u_h) \to \mathcal{G}'(u)$, by Proposition 9.1 we deduce that $u \in SBV(\Omega')$ and

$$\|\nabla u\|^2 + \mathcal{H}^{N-1}(S(u)) \le \mathcal{G}'(u). \tag{5.20}$$

So we conclude that $\nabla u \in L^2(\Omega';\mathbb{R}^N)$, and $\mathcal{H}^{N-1}(S(u)) < +\infty$.

Let us now consider

$$\mathrm{epi}(\mathcal{G}'(t)) := \{(u,s) \in BV(\Omega')\times\mathbb{R} \,:\, \mathcal{G}'(t)(u) \le s\},$$

and let $\mathcal{D} \subseteq \mathrm{epi}(\mathcal{G}'(t))$ be countable and dense. If $\pi : BV(\Omega')\times\mathbb{R} \to BV(\Omega')$ denotes the projection on the first factor, let $D := \pi(\mathcal{D})$ and let us set

$$K(t) := \bigcup_{u\in D} S(u). \tag{5.21}$$

Notice that $K(t)$ is precisely of the form (5.16). Let us see that $K(t)$ satisfies the properties of the lemma.

Let us prove (5.18). Let $u_1,\dots,u_k \in D$, and let $u_1^h,\dots,u_k^h \in BV(\Omega')$ be such that $u_i^h \overset{*}{\rightharpoonup} u_i$ weakly* in $BV(\Omega')$ and

$$\lim_h \mathcal{G}_h'(t)(u_i^h) = \mathcal{G}'(t)(u_i), \qquad i = 1,\dots,k. \tag{5.22}$$

Setting $u^h := (u_1^h,\dots,u_k^h)$, by (5.22) we have

$$\sum_{i=1}^k \int_{\Omega'} f_h(\nabla u_i^h)\, dx + \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \cdots \vee |[u_k^h]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u^h|(\Omega') \le \tilde C$$

with $\tilde{C}$ independent of h. By Proposition 9.1 we deduce

$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^{k} S(u_i)\right) \le \liminf_h \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \cdots \vee |[u_k^h]|)\, d\mathcal{H}^{N-1} \le \liminf_h \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1} \le C', \tag{5.23}$$

where C' is given by Corollary 5.2. Taking the sup over all possible $u_1, \dots, u_k$ we get

$$\mathcal{H}^{N-1}(K(t)) \le \liminf_h \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1} \le C', \tag{5.24}$$

so that (5.18) is proved. In particular we have that $\mathcal{H}^{N-1}(K(t)) < +\infty$.

Let us come to (5.17). Let $u \in \mathrm{dom}(\mathcal{G}(t))$and let us extend u to Ω' setting $u = g(t)$ on $\Omega' \setminus \Omega$. We indicate this extension with u'. We have $u' \in \mathrm{dom}(\mathcal{G}'(t))$, and $S(u') = S^{g(t)}(u)$. Let $(u_k, s_k) \in \mathcal{D}$ be such that $u_k \overset{*}{\rightharpoonup} u'$ weakly* in $BV(\Omega')$ and $s_k \to \mathcal{G}'(t)(u')$. By lower semicontinuity of $\mathcal{G}'(t)$ we have

$$\mathcal{G}'(t)(u') \le \liminf_k \mathcal{G}'(t)(u_k).$$

Moreover since $\mathcal{G}'(t)(u_k) \le s_k$, we deduce

$$\limsup_k \mathcal{G}'(t)(u_k) \le \lim_k s_k = \mathcal{G}'(t)(u'),$$

so that we have $\mathcal{G}'(t)(u_k) \to \mathcal{G}'(t)(u')$. By (5.20) we get that $u_k \rightharpoonup u'$ weakly in $SBV(\Omega')$: since $S(u_k) \tilde{\subseteq} K(t)$ for all k, and $\mathcal{H}^{N-1}(K(t)) < +\infty$, by Ambrosio's theorem we get $S(u') \tilde{\subseteq} K(t)$, i.e. $S^{g(t)}(u) \tilde{\subseteq} K(t)$. We conclude that $K(t)$ satisfies (5.17), and the proof is now complete. □

Lemma 5.5. *Let $t \in [0, T]$, and let us consider the subsequence of $(v_h(t), K_h(t), \gamma_h(t))_{h\in\mathbb{N}}$ (which we indicate with the same symbol) and the rectifiable set $K(t)$ given by Lemma 5.4. Then if $v(t) \in SBV(\Omega)$ is an accumulation point for $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$, we have $v(t) \in SBV(\Omega)$, $\nabla v(t) \in L^2(\Omega;\mathbb{R}^N)$,*

$$S^{g(t)}(v(t)) \tilde{\subseteq} K(t), \tag{5.25}$$

and for all $v \in SBV(\Omega)$

$$\|\nabla v(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v) \setminus K(t)). \tag{5.26}$$

Moreover we have

$$\nabla v_h(t) \rightharpoonup \nabla v(t) \qquad \textit{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{5.27}$$

$$\nabla v_h(t) 1\!\!1_{E_h(t)} \to \nabla v(t) \qquad \textit{strongly in } L^2(\Omega;\mathbb{R}^N), \tag{5.28}$$

where

$$E_h(t) := \left\{ x \in \Omega \,:\, |\nabla v_h(t)| \le \frac{a\sqrt{h}}{2} \right\},$$

and

$$\|\nabla v(t)\|^2 = \lim_h \int_\Omega f_h(\nabla v_h(t))\, dx. \tag{5.29}$$

Proof. Let $v(t)$ be an accumulation point of $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$. If $\mathcal{G}(t)$ is the Γ-limit of the functional $\mathcal{G}_h(t)$ defined in Lemma 5.4, by the Γ-liminf inequality and by (5.12) we have

$$\mathcal{G}(t)(v(t)) \le \liminf_k \mathcal{G}_{h_k}(t)(v_{h_k}(t)) \le C',$$

and so $v(t) \in \mathrm{dom}(\mathcal{G}(t))$. By Lemma 5.4 we conclude $v(t) \in SBV(\Omega)$, $\nabla v(t) \in L^2(\Omega;\mathbb{R}^N)$ and $S^{g(t)}(v(t)) \tilde{\subseteq} K(t)$.

In order to prove (5.26) we follow the Transfer of Jump of [13]. In order to deal with $S^g(u)$ as an internal jump, let us consider $\tilde{\Omega} \subseteq \mathbb{R}^N$ open and bounded, and such that $\overline{\Omega} \subseteq \tilde{\Omega}$. Let us set $\Omega' := \tilde{\Omega} \setminus \partial_N \Omega$. By (5.16) we have

$$K(t) = \bigcup_{u \in D} S(u),$$

where $u \in D$ is extended $\Omega' \setminus \Omega$ setting $u = g(t)$ on $\Omega' \setminus \Omega$, so that $S^{g(t)}(u) = S(u)$.

Let $v \in SBV(\Omega)$ with $\nabla v \in L^2(\Omega; \mathbb{R}^N)$ and $\mathcal{H}^{N-1}(S^{g(t)}(v)) < +\infty$. Let us consider $w := v - g(t)$, and let us extend w on Ω' setting $w = 0$ on $\Omega' \setminus \Omega$. Let $\sigma > 0$, and let $u_1, \dots, u_m \in D$ be such that

$$\mathcal{H}^{N-1}\left(S^{g(t)}(v) \setminus K(t)\right) < \mathcal{H}^{N-1}\left(S^{g(t)}(v) \setminus \bigcup_{i=1}^{m} S^{g(t)}(u_i)\right) + \sigma \tag{5.30}$$

Let us fix $G \subseteq \mathbb{R}$ countable and dense: we recall that for all $r = 1, \dots, m$ we have up to a set of $\mathcal{H}^{N-1}$-measure zero

$$S(u_r) = \bigcup_{c_1, c_2 \in G} \partial^* E_{c_1}(u_r) \cap \partial^* E_{c_2}(u_r),$$

where $E_c(u_r) := \{x \in \Omega' \,:\, x \text{ is a Lebesgue point for } u_r,\ u_r(x) > c\}$ and ∂^* denotes the essential boundary (see [4, Definition 3.60]). Let us orient ν_{u_r} in such a way that $u_r^-(x) < u_r^+(x)$ for all $x \in S(u_r)$, $r = 1, \dots, m$, and let us consider

$$J_j := \left\{ x \in \bigcup_{r=1}^{m} S(u_r) \,:\, u_i^+(x) - u_i^-(x) > \frac{1}{j} \text{ for some } i = 1, \dots, m \right\},$$

with j so large that

$$\mathcal{H}^{N-1}\left(\bigcup_{r=1}^{m} S(u_r) \setminus J_j\right) < \sigma.$$

Let U be a neighborhood of $\bigcup_{r=1}^{m} S(u_r)$ such that

$$|U| < \frac{\sigma}{j^2}, \qquad \int_U |\nabla w|^2 \, dx < \sigma. \tag{5.31}$$

Following [13, Theorem 2.1], we can find a finite disjoint collection of closed cubes $\{Q_k\}_{k=1,\dots,n}$ with edge of length $2r_k$, with center $x_k \in S(u_{\tau(k)})$ for some $\tau(k) \in \{1, \dots, m\}$ and oriented as the normal $\nu(x_k)$ to $S(u_{\tau(k)})$ at x_k, such that $\bigcup_{k=1}^{n} Q_k \subseteq U$ and $\mathcal{H}^{N-1}(J_j \setminus \bigcup_{k=1}^{n} Q_k) \le \sigma$. Let us set

$$v_k := u_{\tau(k)} - g(t),$$

and let H_k denote the intersection of Q_k with the hyperplane through x_k orthogonal to $\nu(x_k)$. Following [13] we can suppose that

$$\mathcal{H}^{N-1}\left(\left[\bigcup_{r=1}^{m} S(u_r) \setminus S(v_k)\right] \cap Q_k\right) < \sigma r_k^{N-1},$$

and that the following facts hold:

(a) if $x_k \in \Omega$ then $Q_k \subseteq \Omega$, and if $x_k \in \partial_D \Omega$, then $\partial\Omega \cap Q_k \subseteq \{y + s\nu(x_k) \,:\, y \in H_k,\ s \in [-\frac{\sigma r_k}{2}, \frac{\sigma r_k}{2}]\}$;

(b) $\mathcal{H}^{N-1}(S(v_k) \cap \partial Q_k) = 0$;

(c) $r_k^{N-1} < 2\mathcal{H}^{N-1}(S(v_k) \cap Q_k)$;

(d) $v_k^-(x_k) < c_k^1 < c_k^2 < v_k^+(x_k)$ and $c_k^2 - c_k^1 > \frac{1}{2j}$;

(e) $\mathcal{H}^{N-1}([S(v_k) \setminus \partial^* E_{c_k^s}(v_k)] \cap Q_k) < \sigma r_k^{N-1}$ for $s = 1, 2$;

(f) $\mathcal{H}^{N-1}(\{y \in \partial^* E_{c_k^s}(v_k) \cap Q_k \,:\, \mathrm{dist}(y, H_k) \ge \frac{\sigma}{2} r_k\}) < \sigma r_k^{N-1}$ for $s = 1, 2$;

(g) if $Q_k^+ := \{x \in Q_k \,|\, (x - x_k) \cdot \nu(x_k) > 0\}$ and $s = 1, 2$

$$\|1_{E_{c_k^s}(v_k))\cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} < \sigma^2 r_k^N; \tag{5.32}$$

(h) $\mathcal{H}^{N-1}\left(\left(S(w) \setminus S(v_k)\right) \cap Q_k\right) < \sigma r_k^{N-1}$ and $\mathcal{H}^{N-1}(S(w) \cap \partial Q_k) = 0$.

By definition of D in (5.16), for all $k = 1, \dots, n$ we can find $v_k^h \in BV(\Omega')$ with $v_k^h = 0$ on $\Omega' \setminus \Omega$, $v_k^h \stackrel{*}{\rightharpoonup} v_k$ weakly* in $BV(\Omega')$, $S(v_k^h) \tilde{\subseteq} K_h(t)$, $|[v_k^h]| \le \gamma_h(t)$, and such that

$$\limsup_h \int_{\Omega'} f_h(\nabla v_k^h)\, dx + \int_{K_h(t)} \varphi_h(|[v_k^h]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_k^h|(\Omega') \le \tilde{C}, \tag{5.33}$$

with $\tilde{C} \in]0, +\infty[$. By Proposition 9.1, we have that ∇v_k^h converges weakly in $L^1(\Omega'; \mathbb{R}^N)$, so that we may assume that U is chosen so that for h large

$$\sum_{k=1}^n \int_{Q_k} |\nabla v_k^h|\, dx < \frac{\sigma}{j^2}. \tag{5.34}$$

Let $\eta \in]0, 1[$: we claim that there exists $\delta > 0$ such that for all $k = 1, \dots, n$

$$\limsup_h |Dv_k^h|(\{0 < |[v_k^h]| < \delta\} \cap Q_k) \le \eta |Q_k|. \tag{5.35}$$

In fact let $a' < a$ be such that

$$a's \le \varphi(s) \qquad \text{for all } s \in [0, 1].$$

Then we have

$$\begin{aligned}|Dv_k^h|\left(\left\{0 < |[v_k^h]| < \frac{1}{\sqrt{h}}\right\} \cap Q_k\right) &= \int_{\{0<|[v_k^h]|<\frac{1}{\sqrt{h}}\}\cap Q_k} |[v_k^h]|\, d\mathcal{H}^{N-1} \\ &\le \frac{1}{a'\sqrt{h}} \int_{\{0<|[v_k^h]|<\frac{1}{\sqrt{h}}\}\cap Q_k)} \varphi_h(|[v_k^h]|)\, d\mathcal{H}^{N-1},\end{aligned}$$

so that we conclude for h large

$$\begin{aligned}|Dv_k^h|\left(\left\{0 < |[v_k^h]| < \delta\right\} \cap Q_k\right) &\le \frac{1}{a'\sqrt{h}} \int_{\{0<|[v_k^h]|<\frac{1}{\sqrt{h}}\}\cap Q_k)} \varphi_h(|[v_k^h]|)\, d\mathcal{H}^{N-1} \\ &+ \frac{\delta}{\varphi(1)} \int_{\{\frac{1}{\sqrt{h}}<|[v_k^h]|<\delta\}\cap Q_k)} \varphi_h(|[v_k^h]|)\, d\mathcal{H}^{N-1} \le \left(\frac{1}{a'\sqrt{h}} + \frac{\delta}{\varphi(1)}\right)\tilde{C},\end{aligned} \tag{5.36}$$

where $\tilde{C}$ is defined in (5.33). Taking the limsup in h and choosing δ small enough, we have that (5.35) holds.

Let δ be as in (5.35), and let us set

$$K_h^\delta(t) := \{x \in K_h(t) \,:\, |[v_k^h]|(x) \ge \delta, \text{ for some } k = 1, \dots, n\}.$$

Then in view of (5.34) and of (5.35), by the Coarea formula for BV functions (see [4, Theorem 3.40]) we have for h large enough

$$\begin{aligned}\sum_{k=1}^n \int_{c_k^1}^{c_k^2} \mathcal{H}^{N-1}\left(\partial^* E_c(v_k^h) \cap (Q_k \setminus K_h^\delta(t))\right)\, dc &\le \sum_{k=1}^n |Dv_k^h|(Q_k \setminus K_h^\delta(t)) \\ &= \sum_{k=1}^n \int_{Q_k} |\nabla v_k^h|\, dx + \sum_{k=1}^n |Dv_k^h|(Q_k \cap \{0 < |[v_k^h]| < \delta\}) \le (1+\eta)\frac{\sigma}{j^2}.\end{aligned} \tag{5.37}$$

By the Mean Value Theorem and by property (d) we get that there exist $c_k^1 < c_k^h < c_k^2$ such that

$$\sum_{k=1}^n \mathcal{H}^{N-1}\left(\partial^* E_{c_k^h}(v_k^h) \cap (Q_k \setminus K_h^\delta(t))\right) \le 2(1+\eta)\frac{\sigma}{j}. \tag{5.38}$$

Following [13], by property (g) we have that for h large

$$\|1_{E_{c_k^h}(v_k^h)\cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} \leq \sigma^2 r_k^N.$$

Then by Fubini's Theorem and by the Mean Value Theorem, we can find $s_k^+ \in [\frac{\sigma r_k}{2}, \sigma r_k]$ and $s_k^- \in [-\sigma r_k, -\frac{\sigma r_k}{2}]$ such that setting $H_k^+ := \{x = y + s_k^+ \nu(x_k),\, y \in H_k\}$ and $H_k^- := \{x = y + s_k^- \nu(x_k),\, y \in H_k\}$ we have

$$\mathcal{H}^{N-1}\left(H_k^+ \setminus (E_{c_k^h}(v_k^h) \cap Q_k)\right) + \mathcal{H}^{N-1}\left(H_k^- \cap (E_{c_k^h}(v_k^h) \cap Q_k)\right) \leq 2\sigma r_k^{N-1}. \tag{5.39}$$

Let R_k be the region between H_k^- and H_k^+, i.e.

$$R_k := \{x \in Q_k \,:\, x = y + s\nu(x_k),\, y \in H_k,\, s_k^2 \leq s \leq s_k^1\},$$

and let us indicate by $R_k^+ w$ the reflection in Q_k of $w_{|Q_k^+ \setminus R_k}$ with respect to H_k^+, and by $R_k^- w$ the reflection in Q_k of $w_{|Q_k^- \setminus R_k}$ with respect to H_k^-. We can now consider w_h defined in the following way

$$w_h := \begin{cases} w & \text{on } \Omega' \setminus \bigcup_{k=1}^n R_k \\ R_k^+ w & \text{on } R_k \cap E_{c_k^h}(v_k^h) \\ R_k^- w & \text{on } R_k \setminus E_{c_k^h}(v_k^h). \end{cases} \tag{5.40}$$

w_h is well defined for σ small, and $w_h = 0$ on $\Omega' \setminus \Omega$. Notice that by construction we have

$$\sum_{k=1}^n \mathcal{H}^{N-1}\left(\left(S(w_h) \setminus K_h^\delta(t)\right) \cap Q_k\right) \leq e(\sigma), \tag{5.41}$$

where $e(\sigma) \to 0$ as $\sigma \to 0$.

By (5.8) comparing $v_h(t)$ with $w_h + g^{\delta_h}(t)$ and in view of (5.30) and (5.41) we have

$$\begin{aligned} \int_\Omega f_h(\nabla v_h(t))\, dx \leq & \int_\Omega f_h(\nabla w_h + \nabla g^{\delta_h}(t))\, dx + \mathcal{H}^{N-1}(S^{g(t)}(w) \setminus K(t)) \\ & + \sum_{k=1}^n \left[\int_{K_h^\delta(t)\cap Q_k} \varphi_h(w_h \vee \gamma_h(t)) - \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1}\right] + e(\sigma). \end{aligned} \tag{5.42}$$

Since by construction we have $\gamma_h(t) \geq \delta$ on $K_h^\delta(t)$, we deduce

$$\limsup_h \left[\int_{K_h^\delta(t)\cap Q_k} \varphi_h(w_h \vee \gamma_h(t)) - \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1}\right] \leq 0.$$

Moreover we have that

$$f_h(\nabla w_h + \nabla g^{\delta_h}(t)) \leq |\nabla w_h + \nabla g^{\delta_h}(t)|^2$$

and by (5.31) for h large

$$\|\nabla w_h + \nabla g^{\delta_h}(t)\|^2 \leq \|\nabla w + \nabla g(t)\|^2 + e(\sigma).$$

Since σ is arbitrary, we conclude that

$$\limsup_h \int_\Omega f_h(\nabla v_h(t))\, dx \leq \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v) \setminus K(t)). \tag{5.43}$$

If $v_{h_m}(t) \overset{*}{\rightharpoonup} v(t)$ weakly* in $BV(\Omega)$, by Proposition 9.1 we have that $v(t) \in SBV(\Omega)$, $\nabla v(t) \in L^2(\Omega; \mathbb{R}^N)$,

$$\nabla v_{h_m}(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^N), \tag{5.44}$$

and

$$\|\nabla v(t)\|^2 \leq \liminf_m \int_\Omega f_{h_m}(\nabla v_{h_m}(t))\, dx. \tag{5.45}$$

By (5.43) we obtain

$$\|\nabla v(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(w) \setminus K(t)),$$

so that (5.26) holds.

Let us now come to (5.27), (5.28) and (5.29). Since $S^{g(t)}(v(t)) \tilde{\subseteq} K(t)$ by (5.25), (5.26) implies that $\nabla v(t)$ is a minimum for the problem

$$\min\{\|\nabla v\|^2 \,:\, v \in SBV(\Omega),\, S^{g(t)}(v) \tilde{\subseteq} K(t)\}.$$

Since $\nabla v(t)$ is unique by convexity, by (5.44) we deduce that (5.27) holds. Moreover (5.29) is a direct consequence of (5.45) and (5.43) with $v = v(t)$. Finally, notice that $(\nabla v_h(t) 1\!\!1_{E_h(t)})_{h \in \mathbb{N}}$ is bounded in $L^2(\Omega; \mathbb{R}^N)$. Since $\nabla v_h(t) \rightharpoonup \nabla v(t)$ weakly in $L^1(\Omega; \mathbb{R}^N)$ and $\nabla v(t) \in L^2(\Omega; \mathbb{R}^N)$, we get $\nabla v_h(t) 1\!\!1_{E_h(t)} \rightharpoonup \nabla v(t)$ weakly in $L^2(\Omega; \mathbb{R}^N)$. By (5.43) with $v = v(t)$ we have

$$\limsup_h \|\nabla v_h(t) 1\!\!1_{E_h(t)}\|^2 \le \limsup_h \int_\Omega f_h(\nabla v_h(t))\, dx \le \|\nabla v(t)\|^2,$$

so that (5.28) holds and the proof is concluded. $\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Point (a) is a consequence of Corollary 5.2 and of (5.10). Point (b) comes from Corollary 5.3. Let us come to point (c).

Let us consider $B \subseteq [0,T]$ countable and dense. By a diagonal argument, we may suppose that there exists a unique subsequence of $(v_h(t), K_h(t), \gamma_h(t))_{h \in \mathbb{N}}$ (which we still denote by the same symbol) such that Lemma 5.4 holds for all $t \in B$. For each $t \in B$, let $K(t)$ be the rectifiable set defined in Lemma 5.4.

We notice that $K(t)$ is increasing in t. In fact if $s < t$ and if $u \in \mathrm{dom}(\mathcal{G}(s))$, where $\mathcal{G}(s)$ is defined in Lemma 5.4, there exists $u_h \in BV(\Omega)$ such that $u_h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega)$, $S^{g^{\delta_h}(s)}(u_h) \tilde{\subseteq} K_h(s)$, $|[u_h]| \le \gamma_h(s)$ and for all h

$$\int_\Omega f_h(\nabla u_h)\, dx + \int_{K_h(s)} \varphi_h(|[u_h]|)\, d\mathcal{H}^{N-1} + c\sqrt{h}|D^c u_h|(\Omega) \le \tilde{C}$$

for some $\tilde{C}$ independent of h. Let us set $v_h := u_h - g^{\delta_h}(s) + g^{\delta_h}(t)$. Since $K_h(s) \tilde{\subseteq} K_h(t)$ and $\gamma_h(s) \le \gamma_h(t)$, we have that $S^{g^{\delta_h}(t)}(v_h) \tilde{\subseteq} K_h(t)$, $|[v_h]| \le \gamma_h(t)$; moreover for all h

$$\int_\Omega f_h(\nabla v_h)\, dx + \int_{K_h(t)} \varphi_h(|[v_h]|)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h|(\Omega) \le \tilde{C}'$$

with $\tilde{C}'$ independent of h. We deduce that $u - g(s) + g(t) \in \mathrm{dom}(\mathcal{G}(t))$. Then by definition (5.16) and by (5.17), we obtain $K(s) \tilde{\subseteq} K(t)$.

Since $\{t \to K(t) : t \in [0,T]\}$ is increasing, setting

$$K^-(t) := \bigcup_{s \in B, s \le t} K(s), \qquad K^+(t) := \bigcap_{s \in B, s \ge t} K(s),$$

there exists a countable set $B' \subseteq [0,T] \setminus B$ such that we have $K^-(t) = K^+(t)$ for all $t \in [0,T] \setminus B'$. For all such t's let us set $K(t) := K^-(t) = K^+(t)$.

Clearly Lemma 5.4 and Lemma 5.5 hold for all $t \in [0,T] \setminus (B \cup B')$. In fact, up to a further subsequence, we may apply Lemma 5.4 obtaining $\tilde{K}(t)$ with the required properties and such that $K(s_1) \tilde{\subseteq} \tilde{K}(t) \tilde{\subseteq} K(s_2)$ for all $s_1, s_2 \in B$ and $s_1 < t < s_2$. Then we get $\tilde{K}(t) = K(t)$.

Up to a further subsequence relative to the elements of B', we find

$$\{t \to K(t),\, t \in [0,T]\}$$

such that Lemma 5.4 and Lemma 5.5 hold for every $t \in [0,T]$. Notice that in particular $\mathcal{H}^{N-1}(K(t)) \le C'$, where C' is given by (5.12).

Let $v(t)$ be a minimum for the following problem

$$\min\left\{ \|\nabla v\|^2 \,:\, v \in SBV(\Omega),\, S^{g(t)}(v) \tilde{\subseteq} K(t) \right\}. \tag{5.46}$$

Notice that problem (5.46) is well posed since $K(t)$ has finite $\mathcal{H}^{N-1}$-measure: moreover by strict convexity we have that $\nabla v(t)$ is uniquely determined.

We claim that $\{t \to (v(t), K(t)),\ t \in [0,T]\}$ is a quasistatic growth of brittle fractures in the sense of [13], that is in the sense of Theorem 2.2. In fact $K(t)$ is increasing and satisfies the unilateral minimality property (5.26) by construction. In order to prove the claim, we have just to prove the *nondissipativity* condition

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t (\nabla v(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau, \tag{5.47}$$

where $\mathcal{E}(t) := \|\nabla v(t)\|^2 + \mathcal{H}^{N-1}(K(t))$ for all $t \in [0,T]$. First of all for all $t \in [0,T]$ we have

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau. \tag{5.48}$$

In fact as noticed in [15], using the minimality property (5.26), the map $\{t \to \nabla v(t)\}$ is continuous at all the continuity points of $\{t \to \mathcal{H}^{N-1}(K(t))\}$, in particular it is continuous up to a countable set in $[0,T]$. Given $t \in [0,T]$ and $k > 0$, let us set

$$s_i^k := \frac{i}{k}t, \qquad v^k(s) := v(s_{i+1}^k) \quad \text{ for } s_i^k < s \leq s_{i+1}^k$$

for all $i = 0, 1, \dots, k$. By (5.26), comparing $v(s_i^k)$ with $v(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)$, it is easy to see that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t (\nabla v^k(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau + e(k),$$

where $e(k) \to 0$ as $k \to +\infty$. By the continuity property of ∇v, passing to the limit for $k \to +\infty$ we deduce that (5.48) holds. On the other hand for all $t \in [0,T]$ we have that

$$\mathcal{E}(t) \leq \mathcal{E}(0) + 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau. \tag{5.49}$$

In fact by Proposition 9.2 and by property (5.7) we get

$$\mathcal{F}_h(0, v_h(0)) \to \mathcal{E}(0). \tag{5.50}$$

Moreover by Lemma 5.5 we have that for all $t \in [0,T]$

$$\nabla v_h(t) 1\!\!1_{E_h(t)} \to \nabla v(t) \qquad \text{strongly in } L^2(\Omega; \mathbb{R}^N),$$

where

$$E_h(t) := \left\{ x \in \Omega \,:\, |\nabla v_h(t)| \leq \frac{a\sqrt{h}}{2} \right\}.$$

By (5.11) and by the very definition of f_h we deduce

$$\begin{aligned} \mathcal{F}_h(t, v_h(t)) \leq& \mathcal{F}_h(0, v_h(0)) + 2\int_0^t (\nabla v_h(\tau) 1\!\!1_{E_h(\tau)}, \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau \\ &+ a\sqrt{h}\int_0^t \int_{\Omega\setminus E_h(\tau)} |\nabla \dot{g}(\tau)|\, dx\, d\tau + e(h) \end{aligned} \tag{5.51}$$

where $e(h) \to 0$ as $h \to +\infty$. Notice that by (5.12) we have

$$\frac{a}{2} h|\Omega \setminus E_h(\tau)| \leq \sqrt{h}\int_{\Omega\setminus E_h(\tau)} |\nabla v_h(\tau)|\, dx \leq \frac{2}{a}\int_{\Omega\setminus E_h(\tau)} f_h(\nabla v_h(\tau))\, dx \leq \frac{2}{a} C'.$$

We deduce that

$$\begin{aligned} \sqrt{h}\int_{\Omega\setminus E_h(\tau)} |\nabla \dot{g}(\tau)|\, dx &\leq \left(\int_{\Omega\setminus E_h(\tau)} |\nabla \dot{g}(\tau)|^2\, dx \right)^{\frac{1}{2}} \sqrt{h|\Omega \setminus E_h(\tau)|} \\ &\leq \frac{2\sqrt{C'}}{a} \left(\int_{\Omega\setminus E_h(\tau)} |\nabla \dot{g}(\tau)|^2\, dx \right)^{\frac{1}{2}} \to 0 \end{aligned} \tag{5.52}$$

uniformly in τ as $h \to +\infty$ by equicontinuity of $\nabla \dot{g}(\tau)$. Then passing to the limit for $h \to +\infty$ in (5.51), in view of (5.29), (5.18), (5.28), (5.50) and (5.52) we deduce that (5.49) holds. This proves that (5.47) holds, and so $\{t \to (v(t), K(t)) : t \in [0,T]\}$ is a quasistatic growth of brittle fractures in the sense of [13].

In order to conclude the proof, let us see that (4.9), (4.10), (4.11) and (4.12) hold. By (5.51) we deduce that for all $t \in [0,T]$

$$\mathcal{F}_h(t, v_h(t)) \to \mathcal{E}(t), \tag{5.53}$$

so that by (5.29) and (5.18) we deduce that

$$\mathcal{H}^{N-1}(K(t)) = \lim_h \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1}, \qquad a\sqrt{h}|D^c v_h(t)|(\Omega) \to 0. \tag{5.54}$$

Theorem 4.1 is now completely proved in view of the rescaling (5.1), of (5.2), (5.4) and (5.10). □

6. Proof of Theorem 4.2

In this section we will give the proof of Theorem 4.2. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) : t \in [0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of cohesive fracture in Ω_h relative the subdivision $I_{\delta_h} := \{0 = t_0^{\delta_h} < \cdots < t_{N_{\delta_h}}^{\delta_h} = T\}$, the preexisting crack configuration $(\bar{\Gamma}_h, \bar{\psi}_h)$ given by (4.4) and the boundary displacement $h^\alpha g(t, \frac{x}{h})$ with $\alpha \in]0, \frac{1}{2}[$.

In order to prove Theorem 4.2, proceeding as in Section 5, it is convenient to rescale u_h and Γ_h in the following way: for all $t \in [0,T]$ let $v_h(t) \in BV(\Omega)$ and $K_h(t) \tilde{\subseteq} \Omega \cup \partial_D \Omega$ be given by

$$v_h(t,x) := \frac{1}{h^\alpha} u_h\left(t, hx\right), \qquad K_h(t) := \frac{1}{h}\Gamma_h(t). \tag{6.1}$$

Let us moreover set

$$\gamma_h(t,x) := \frac{1}{h^\alpha}\psi_h(t,hx) = \max_{s \le t} |[v_h(s)](t,x)| \vee \bar{\gamma}(x) \qquad t \in [0,T], x \in \Omega. \tag{6.2}$$

It turns that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) : t \in [0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cohesive fractures in Ω relative to the subdivision I_{δ_h}, the preexisting crack configuration $(\bar{\Gamma}, \bar{\gamma})$ and boundary displacement $g(t)$ with respect to the basic total energy

$$\int_\Omega f_h(\nabla v)\, dx + h^{1-2\alpha} \int_{S^{g^{\delta_h(t)}}(v) \cup K_h(t)} \varphi_h(|[v]| \vee \gamma_h(t))\, d\mathcal{H}^{N-1} + a h^{1-\alpha}|D^c v|(\Omega), \tag{6.3}$$

where

$$\varphi_h(s) := \varphi(h^\alpha s), \tag{6.4}$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{a h^{1-\alpha}}{2} \\ \frac{a^2 h^{2(1-\alpha)}}{4} + a h^{1-\alpha}(|\xi| - \frac{a h^{1-\alpha}}{2}) & \text{if } |\xi| \ge \frac{a h^{1-\alpha}}{2}. \end{cases} \tag{6.5}$$

We have that the following facts hold:

(a) for all $t \in [0,T]$

$$\|v_h(t)\|_\infty \le \|g^{\delta_h}(t)\|_\infty \le C; \tag{6.6}$$

(b) for all $w \in BV(\Omega)$ we have

$$\begin{aligned} &\int_\Omega f_h(\nabla v_h(0))\, dx + h^{1-2\alpha} \int_{K_h(0)} \varphi_h(|[v_h(0)]| \vee \bar{\gamma})\, d\mathcal{H}^{N-1} + a h^{1-\alpha}|D^c v_h(t)|(\Omega) \\ &\quad \le \int_\Omega f_h(\nabla w)\, dx + h^{1-2\alpha} \int_{S^{g^{\delta_h(0)}}(w) \cup \bar{\Gamma}} \varphi_h(|[w]| \vee \bar{\gamma})\, d\mathcal{H}^{N-1} + a h^{1-\alpha}|D^c w|(\Omega); \end{aligned} \tag{6.7}$$

(c) for all $w \in BV(\Omega)$ we have

$$
\begin{aligned}
(6.8) \quad & \int_\Omega f_h(\nabla v_h(t))\,dx + h^{1-2\alpha}\int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v_h(t)|(\Omega) \\
& \leq \int_\Omega f_h(\nabla w)\,dx + h^{1-2\alpha}\int_{S^{g^{\delta_h(t)}}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t)\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c w|(\Omega).
\end{aligned}
$$

Let us set for all $v \in BV(\Omega)$ and for all $t \in [0,T]$

(6.9)

$$\mathcal{F}_h(t,w) := \int_\Omega f_h(\nabla w)\,dx + h^{1-2\alpha}\int_{S^{g^{\delta_h(t)}}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c w|(\Omega).$$

Notice that

$$\mathcal{F}_h(t,v_h(t)) = \frac{1}{h^{N+2\alpha-2}}\mathcal{E}(t,u_h(t)), \tag{6.10}$$

where $\mathcal{E}(t,u_h(t))$ is defined in (4.6). We can now prove Theorem 4.2.

Proof of Theorem 4.2. By Lemma 3.2 we obtain for all $t \in [0,T]$

$$\mathcal{F}_h(t,v_h(t)) \leq \mathcal{F}_h(0,v_h(0)) + \int_0^{t_h}\int_\Omega f_h'(\nabla v_h(\tau))\nabla \dot{g}(\tau)\,dx\,d\tau + e(h), \tag{6.11}$$

where $e(h) \to 0$ as $h \to +\infty$, and $t_h := t_{i_h}^{\delta_h}$ is the step discretization point of I_{δ_h} such that $t_{i_h}^{\delta_h} \leq t < t_{i_h+1}^{\delta_h}$. By (6.7) comparing $v_h(0)$ and $g(0)$ we have

$$
\begin{aligned}
(6.12) \quad & \int_\Omega f_h(\nabla v_h(0))\,dx + h^{1-2\alpha}\left[\int_{S^{g(0)}(v_h(0))\cup\bar{\Gamma}} \varphi_h(|[v_h(0)]| \vee \bar{\gamma}) - \varphi_h(\bar{\gamma})\,d\mathcal{H}^{N-1}\right] \\
& \qquad + ah^{1-\alpha}|D^c v_h(0)|(\Omega) \leq \|\nabla g(0)\|^2.
\end{aligned}
$$

By (6.8) comparing $v_h(t)$ and $g^{\delta_h}(t)$ we obtain

$$\int_\Omega f_h(\nabla v_h(t))\,dx \leq \|\nabla g^{\delta_h}(t)\|^2, \tag{6.13}$$

and since we have

$$\int_\Omega |f_h'(\nabla v_h(\tau))|^2\,dx \leq 4\int_\Omega f_h(\nabla v_h(\tau))\,dx,$$

by (6.11) we deduce that

$$\int_\Omega f_h(\nabla v_h(t))\,dx + h^{1-2\alpha}\int_{K_h(t)\setminus\bar{\Gamma}} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v|(\Omega) \leq C' \tag{6.14}$$

with C' independent of h and of t. By (4.3) and (6.6) and following Corollary 5.3 we deduce that $(v_h(t))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$, and this proves point (a).

Let $v(t)$ be an accumulation point for $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$, and let us consider $\tilde{\Omega} \subseteq \mathbb{R}^N$ open and bounded, and such that $\overline{\Omega} \subseteq \tilde{\Omega}$. Let us set $\Omega' := \tilde{\Omega} \setminus \partial_N\Omega$. Then we can extend $v_h(t)$ and $v(t)$ to Ω' setting $v_h(t) = g^{\delta_h}(t)$ and $v(t) = g(t)$ on $\Omega' \setminus \Omega$ respectively. We have $v_{h_j}(t) \overset{*}{\rightharpoonup} v(t)$ weakly* in $BV(\Omega')$ for a suitable $h_j \nearrow +\infty$, and

$$\int_{\Omega'} f_{h_j}(\nabla v_{h_j}(t))\,dx + h_j^{1-2\alpha}\int_{S(v_{h_j}(t))\setminus\bar{\Gamma}} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1} + ah_j^{1-\alpha}|D^c v_{h_j}(t)|(\Omega') \leq \tilde{C} \tag{6.15}$$

with $\tilde{C}$ independent of j. In particular we have

$$\int_{\Omega'} f_{h_j}(\nabla v_{h_j}(t))\,dx + \int_{S(v_{h_j}(t))} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1} + ah_j^{1-\alpha}|D^c v_{h_j}(t)|(\Omega') \leq \tilde{C}'$$

with $\tilde{C}'$ independent of j. Then by Proposition 9.1 we have that $v(t) \in SBV(\Omega)$,

$$\nabla v_{h_j}(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{6.16}$$

and

$$\|\nabla v(t)\|^2 \le \liminf_j \int_\Omega f_{h_j}(\nabla v_{h_j}(t))\,dx. \tag{6.17}$$

Finally, if we consider for all Borel sets $B \subseteq \Omega'$

$$\lambda_j(B) := \int_{B\cap S(v_{h_j}(t))} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1}$$

and if (up to a subsequence) $\lambda_j \overset{*}{\rightharpoonup} \lambda$ weakly* in the sense of measures, we deduce following Proposition 9.1 that

$$\mathcal{H}^{N-1} \llcorner S(v(t)) \le \lambda \qquad \text{as measures.}$$

Since by (6.15) we have $\lambda(\Omega' \setminus \bar\Gamma) = 0$, then we have $S(v(t)) \tilde\subseteq \bar\Gamma$, that is $S^{g(t)}(v(t)) \tilde\subseteq \bar\Gamma$. Point (b) is thus proved.

Let us come to point (c). Let us suppose that $\varphi(s) = 1$ for $s \ge \bar s$, and $\bar\gamma \ge \varepsilon > 0$. Let us consider $v \in SBV(\Omega)$ with $S^{g(t)}(v) \tilde\subseteq \bar\Gamma$. Comparing $v_h(t)$ with $v - g(t) + g^{\delta_h}(t)$ by minimality property (6.8) we obtain

$$\begin{aligned}\int_\Omega f_h(\nabla v_h(t))\,dx + h^{1-2\alpha}\int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v_h(t)|(\Omega)\\ \le \int_\Omega f_h(\nabla v - \nabla g(t) + \nabla g^{\delta_h}(t))\,dx + h^{1-2\alpha}\int_{K_h(t)} \varphi_h(|[v]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1}.\end{aligned} \tag{6.18}$$

Since for h large we have

$$\varphi_h(\gamma_h(t)) = \varphi_h(|[v]| \vee \gamma_h(t)),$$

we deduce that

$$\begin{aligned}\int_\Omega f_h(\nabla v_h(t))\,dx + ah^{1-\alpha}|D^c v_h(t)|(\Omega) &\le \int_\Omega f_h(\nabla v - \nabla g(t) + \nabla g^{\delta_h}(t))\,dx\\ &\le \|\nabla v - \nabla g(t) + \nabla g^{\delta_h}(t)\|^2.\end{aligned} \tag{6.19}$$

Let $h_j \nearrow +\infty$ such that $v_{h_j}(t) \overset{*}{\rightharpoonup} v(t)$ weakly* in $BV(\Omega)$. By (6.16) we have that $\nabla v_{h_j}(t) \rightharpoonup \nabla v(t)$ weakly in $L^1(\Omega;\mathbb{R}^N)$ and in view of (6.17) we deduce that

$$\|\nabla v(t)\|^2 \le \|\nabla v\|^2,$$

so that $v(t)$ is a minimizer of

$$\min\{\|\nabla v\|^2 \,:\, v \in SBV(\Omega),\, S^{g(t)}(v) \tilde\subseteq \bar\Gamma\}.$$

By strict convexity, we have that $\nabla v(t)$ is uniquely determined, and so we deduce that $\nabla u_h(t) \rightharpoonup \nabla u(t)$ weakly in $L^1(\Omega;\mathbb{R}^N)$. Choosing $v = u(t)$ in (6.19) and taking the limsup in h we have

$$\limsup_h \int_\Omega f_h(\nabla v_h(t))\,dx \le \|\nabla u(t)\|^2,$$

so that

$$\lim_h \int_\Omega f_h(\nabla v_h(t))\,dx = \|\nabla u(t)\|^2.$$

The proof is concluded thank to the rescaling (6.1). □

7. Proof of Theorem 4.3

In this section we will give the proof of Theorem 4.3. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of cohesive fracture in Ω_h relative the subdivision $I_{\delta_h} := \{0 = t_0^{\delta_h} < \cdots < t_{N_{\delta_h}}^{\delta_h} = T\}$, the preexisting crack configuration $(\bar\Gamma_h, \bar\psi_h)$ given by (4.4) and the boundary displacement $h^\alpha g(t, \frac{x}{h})$ with $\alpha \ge \frac12$.

In order to prove Theorem 4.3, proceeding as in Section 5, it is convenient to rescale u_h and Γ_h in the following way: for all $t \in [0,T]$ let $v_h(t) \in BV(\Omega)$ and $K_h(t) \tilde\subseteq \Omega \cup \partial_D\Omega$ be given by

$$v_h(t,x) := \frac{1}{h^\alpha} u_h(t, hx), \qquad K_h(t) := \frac{1}{h}\Gamma_h(t). \tag{7.1}$$

Let us moreover set

$$\gamma_h(t,x) := \frac{1}{h^\alpha}\psi_h(t,hx) = \max_{s\le t}|[v_h(s)](t,x)| \vee \bar\gamma(x), \qquad t\in[0,T], x\in\Omega. \tag{7.2}$$

It turns out that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) \,:\, t\, in[0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cohesive fractures in Ω relative to the preexisting crack configuration $(\bar\Gamma, \bar\gamma)$ and boundary displacement $g(t)$ with respect to the basic total energy

$$h^{2\alpha-1}\int_\Omega f_h(\nabla v)\,dx + \int_{S^{g^{\delta_h}}(v)} \varphi_h(|[v]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + ah^\alpha|D^c v|(\Omega), \tag{7.3}$$

where

$$\varphi_h(s) := \varphi(h^\alpha s), \tag{7.4}$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{ah^{1-\alpha}}{2} \\ \frac{a^2h^{2(1-\alpha)}}{4} + ah^{1-\alpha}(|\xi| - \frac{ah^{1-\alpha}}{2}) & \text{if } |\xi| \ge \frac{ah^{1-\alpha}}{2}. \end{cases} \tag{7.5}$$

Notice that by Proposition 3.1 we have

$$\|v_h(0)\|_\infty \le \|g(0)\|_\infty \le C, \tag{7.6}$$

and for all $w \in BV(\Omega)$ we have

$$\begin{aligned} h^{2\alpha-1}\int_\Omega f_h(\nabla v_h(0))\,dx + \int_{S^{g^{\delta_h(0)}}(v_h(0))\cup\bar\Gamma} \varphi_h(|[v_h(0)]| \vee \bar\gamma)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c v_h(0)|(\Omega) \\ \le h^{2\alpha-1}\int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h(0)}}(w)\cup\bar\Gamma} \varphi_h(|[w]| \vee \bar\gamma)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c w|(\Omega). \end{aligned} \tag{7.7}$$

We can now prove Theorem 4.3.

Proof of Theorem 4.3. Comparing $v_h(0)$ and $w = -C$ by means of (7.7) we have

$$\begin{aligned} h^{2\alpha-1}\int_\Omega f_h(\nabla v_h(0))\,dx + \int_{S^{g^{\delta_h(0)}}(v_h(0))\cup\bar\Gamma} \varphi_h(|[v_h(0)]| \vee \bar\gamma)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c v_h(0)|(\Omega) \\ \le \mathcal{H}^{N-1}(\bar\Gamma \cup \partial_D\Omega). \end{aligned} \tag{7.8}$$

As a consequence, following Corollary 5.3, we obtain

$$|Dv_h(0)|(\Omega) \le C'$$

with C' independent of h. Since moreover $\|v_h(0)\|_\infty \le C$ by (7.6), we deduce that $(v_h(0))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$. Let v be an accumulation point for $(v_h(0))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$. Let us prove that $v \in SBV(\Omega)$ and that $\nabla v = 0$: in fact we have that for all $\xi \in \mathbb{R}^N$

$$\tilde f_h(\xi) \le h^{2\alpha-1} f_h(\xi)$$

where

$$\tilde f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{ah^\alpha}{2} \\ \frac{a^2h^{2\alpha}}{4} + ah^\alpha(|\xi| - \frac{ah^\alpha}{2}) & \text{if } |\xi| \ge \frac{ah^\alpha}{2}. \end{cases} \tag{7.9}$$

We deduce that there exists C'' independent of h such that for all h

$$\int_\Omega \tilde f_h(\nabla v_h(0))\,dx + \int_{S(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + ch^\alpha|D^c v_h(0)|(\Omega) \le C''. \tag{7.10}$$

By Proposition 9.1, we obtain that $v \in SBV(\Omega)$ and that $\nabla v_h(0) \rightharpoonup \nabla v$ weakly in $L^1(\Omega;\mathbb{R}^N)$. By (7.8) we obtain that

$$\|\nabla v_h(0)\|_{L^1(\Omega;\mathbb{R}^N)} \le \frac{\mathcal{H}^{N-1}(\bar\Gamma \cup \partial_D\Omega) + 1}{ah^\alpha},$$

so that we deduce $\nabla v = 0$, that is v is piecewise constant in Ω. Finally taking the limit in (7.7) with w piecewise constant, then we get exactly (4.17), so that the proof of Theorem 4.3 is concluded. $\square$

8. A relaxation result

In this section, we prove a relaxation result we used in order to study the discrete in time evolution of fractures in the cohesive case.

Let $f : \mathbb{R} \to [0, +\infty[$ be convex, $f(0) = 0$ and with superlinear growth, i.e.

$$\limsup_{|\xi| \to +\infty} \frac{f(\xi)}{|\xi|} = +\infty.$$

Let $\varphi : [0, +\infty[\to [0, +\infty[$ be increasing, concave, and such that $\varphi(0) = 0$. Notice that if $a := \varphi'(0) < +\infty$, we have

$$\varphi(s) \le as \quad \text{for all } s \in [0, +\infty[. \tag{8.1}$$

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$, and let $\partial_D \Omega \subseteq \partial\Omega$ be open in the relative topology. Let Γ be a rectifiable set in $\Omega \cup \partial_D \Omega$, and let ψ be a positive function defined on Γ. Let us extend ψ to $\Omega \cup \partial_D \Omega$ setting $\psi = 0$ outside Γ. Let $g \in W^{1,1}(\Omega)$: we may assume that g is extended to the whole $\mathbb{R}^N$, and we indicate this extension still by g.

We will study the following functional

$$F(u) := \begin{cases} \int_\Omega f(|\nabla u|)\, dx + \int_{S^g(u) \cup \Gamma} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} & \text{if } u \in SBV(\Omega) \\ \\ +\infty & \text{otherwise in } BV(\Omega), \end{cases} \tag{8.2}$$

where $S^g(u)$ is defined in (2.3), and $a \vee b := \max\{a, b\}$ for all $a, b \in \mathbb{R}$. The functional (8.2) naturally appears (see Section 3) when dealing with quasistatic growth of fractures in the cohesive case, where one is required to look for its minima. We are led to compute the relaxation of F with respect to the strong topology of $L^1(\Omega)$. The relaxation in the case $\Gamma = \emptyset$ (without boundary conditions but without superlinear growth on f) has been proved in [6]. Let

$$f_1(\xi) := \inf\{f(\xi_1) + a|\xi_2| \, : \, \xi_1 + \xi_2 = \xi\}, \tag{8.3}$$

where $a := \varphi'(0)$. We have that the following result holds.

Proposition 8.1. *The relaxation of the functional* (8.2) *with respect to the weak* topology of* $BV(\Omega)$ *is given by* $\overline{F} : BV(\Omega) \to [0, +\infty]$ *defined as*

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\, dx + \int_{S^g(u) \cup \Gamma} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|, \tag{8.4}$$

where $a = \varphi'(0)$ *and* f_1 *is defined in* (8.3).

In order to prove Proposition 8.1, we need some preliminaries.
Let $I \subseteq \mathbb{R}$ be a finite union of disjoint intervals, and let $J \subseteq I$ be a countable set. Let us consider the functional

$$\mathcal{F}(\mu) := \int_I f_1(|\phi_\mu|)\, dx + \sum_{t \in S_\mu \setminus J} \varphi(|\mu(\{t\})|) + \sum_{t \in J} \varphi(|\mu(\{t\})| \vee \psi(t)) + a|\mu^c|(I) \tag{8.5}$$

defined for all $\mu \in \mathcal{M}_b(I; \mathbb{R}^k)$, i.e. μ is a bounded $\mathbb{R}^k$-valued Radon measure on I. Here ϕ_μ is the density of the absolutely continuous part μ^a of μ, S_μ is the set of atoms of μ, $\mu^c := \mu - \mu^a - \mu \llcorner S_\mu$, ψ is a strictly positive function defined on J, $a = \varphi'(0)$ and f_1 is defined in (8.3).

Lemma 8.2. *The functional* $\mathcal{F}$ *defined in* (8.5) *is lower semicontinuous with respect to the weak* convergence in the sense of measures.*

Proof. Since $\mathcal{F}$ can be obtained as the sup of functionals of the form (8.5) with J finite, we may assume that $J = \{x_1, \dots, x_m\}$. Let $\mu_n \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures, and let λ be the weak* limit (up to a subsequence) of $|\mu_n \llcorner J|$. Let $J := J_1 \cup J_2$, with

$$J_1 := \{t \in J \,:\, |\mu(\{t\})| \geq \psi(t)\}, \qquad J_2 := J \setminus J_1.$$

Let $\varepsilon > 0$ be such that

$$\bigcup_{x_i \in J_2} \bar{B}_\varepsilon(x_i) \subseteq I$$

and such that for all n

$$|\mu_n| \left(\bigcup_{x_i \in J_2} \partial \bar{B}_\varepsilon(x_i) \right) = |\mu| \left(\bigcup_{x_i \in J_2} \partial \bar{B}_\varepsilon(x_i) \right) = 0.$$

Let us set

$$I_1 := I \setminus \bigcup_{x_i \in J_2} \bar{B}_\varepsilon(x_i), \qquad I_2 := \bigcup_{x_i \in J_2} B_\varepsilon(x_i).$$

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the restriction of $\mathcal{F}$ to $\mathcal{M}_b(I_1; \mathbb{R}^k)$ and $\mathcal{M}_b(I_2; \mathbb{R}^k)$ respectively. We have

$$\liminf_n \mathcal{F}(\mu_n) \geq \liminf_n \mathcal{F}_1(\mu_n \llcorner I_1) + \liminf_n \mathcal{F}_2(\mu_n \llcorner I_2).$$

We notice that

$$\mathcal{F}_1(\mu_n \llcorner I_1) \geq \mathcal{G}_1(\mu_n \llcorner I_1)$$

where

$$\mathcal{G}_1(\eta) := \int_{I_1} f_1(|\phi_\eta|)\, dx + \sum_{t \in S_\eta} \varphi(|\eta(\{t\})|) + a|\eta^c|(I_1)$$

for all $\eta \in \mathcal{M}_b(I_1; \mathbb{R}^k)$. By [4, Thorem 5.2] we have that

$$\mathcal{G}_1(\mu \llcorner I_1) \leq \liminf_n \mathcal{G}_1(\mu_n \llcorner I_1),$$

so that

$$\mathcal{F}_1(\mu \llcorner I_1) = \mathcal{G}_1(\mu \llcorner I_1) \leq \liminf_n \mathcal{F}_1(\mu_n \llcorner I_1).$$

On the other hand, we have

$$\mathcal{F}_2(\mu_n \llcorner I_2) = \mathcal{G}_2(\mu_n \llcorner I_2 \setminus J_2) + \sum_{t \in J_2} \varphi(|\mu_n(\{t\})| \vee \psi(t)),$$

where

$$\mathcal{G}_2(\eta) := \int_{I_2} f_1(|\phi_\eta|)\, dx + \sum_{t \in S_\eta} \varphi(|\eta(\{t\})|) + a|\eta^c|(I_2)$$

for all $\eta \in \mathcal{M}_b(I_2; \mathbb{R}^k)$. We have

$$\begin{aligned} \liminf_n \mathcal{F}_2(\mu_n \llcorner I_2) &\geq \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t \in J_2} \varphi(\lambda(\{t\}) \vee \psi(t)) \\ &\geq \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t \in J_2} \varphi(\psi(t)). \end{aligned} \tag{8.6}$$

We deduce

$$\mathcal{F}_2(\mu \llcorner I_2) = \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t \in J_2} \varphi(\psi(t)) \leq \liminf_n \mathcal{F}_2(\mu_n \llcorner I_2),$$

and so we get

$$\mathcal{F}(\mu) = \mathcal{F}_1(\mu \llcorner I_1) + \mathcal{F}_2(\mu \llcorner I_2) \leq \liminf_n \mathcal{F}(\mu_n).$$

The proof is now concluded. ∎

Lemma 8.3. *Let $\bar{F} : BV(\Omega) \to [0,+\infty]$ be defined by*

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\,dx + \int_{S^g(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|,$$

with $a = \varphi'(0)$ and f_1 as in (8.3). Then $\bar{F}$ is lower semicontinuous with respect to the weak topology of $BV(\Omega)$.*

Proof. We may assume without loss of generality that

$$\mathcal{H}^{N-1}(\Gamma) < +\infty, \qquad \int_\Gamma \varphi(\psi)\,d\mathcal{H}^{N-1} < +\infty. \tag{8.7}$$

Firstly consider the case $\varphi(s) > \varepsilon s$ for some $\varepsilon > 0$ and $s \in [0,+\infty[$. Following [4, Theorem 5.4], we use Lemma 8.2 to obtain the lower semicontinuity in the one dimensional case, and we recover the N-dimensional case using a slicing argument.

Let us consider $\tilde{\Omega}$ open and bounded in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \tilde{\Omega}$, and let us set $\Omega' := \tilde{\Omega} \setminus \partial_N \Omega$. The lower semicontinuity of $\bar{F}$ is equivalent to the lower semicontinuity of

$$F'(u) := \int_{\Omega'} f_1(|\nabla u|)\,dx + \int_{S(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(\tilde{\Omega}) \tag{8.8}$$

defined for all $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$. In order to prove the lower semicontinuity of F', it is convenient to localize the functional, i.e. for every open set $A \subseteq \Omega'$, and for every $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$, we consider

$$F'_A(u) := \int_A f_1(|\nabla u|)\,dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(A). \tag{8.9}$$

Let $e \in \mathbb{R}^N$ with $|e| = 1$: for every open set $A \subseteq \Omega'$, and for every $u \in BV(\Omega')$ such that $u = g$ on $\Omega' \setminus \Omega$ let us set

$$F'_{A,e}(u) := \int_A f_1(|\langle \nabla u, e\rangle|)\,dx + \int_{A\cap(S(u)\cup\Gamma)} |\langle \nu, e\rangle|\,\varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|\langle D^c u, e\rangle|(A). \tag{8.10}$$

Here ν denotes the normal to the rectifiable set $S(u) \cup \Gamma$.

By the general theory of slicing (see [4, Section 3.11]), we have that

$$F'_{A,e}(u) = \int_{\pi_e} G_{A^e_y}(u^e_y)\,dy, \tag{8.11}$$

where π_e is the hyperplane through the origin orthogonal to e, $A^e_y := A \cap \{y + te\,,\, t \in \mathbb{R}\}$, $u^e_y(t) := u(y + te)$, and

$$G_{A^e_y}(u^e_y) := \int_{A^e_y} f_1(|(u^e_y)'|)\,dt + \sum_{t\in(A^e_y\cap(S(u^e_y)\cup\Gamma^e_y))} \varphi(|[u]|(y+te) \vee \psi(y+te)) + a|D^c u^e_y|(A^e_y). \tag{8.12}$$

Let us consider $u_n \in BV(\Omega')$ such that $u_n = g$ on $\Omega' \setminus \Omega$, and such that $u_n \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega')$. Then up to a subsequence for a.e. $y \in \pi_e$, we have $(u_n)^e_y \to u^e_y$ strongly in $L^1((\Omega')^e_y)$. For every open set $A \subseteq \Omega'$, we claim that for a.e. $y \in \pi_e$ we have

$$G_{A^e_y}(u^e_y) \le \liminf_n G_{A^e_y}((u_n)^e_y). \tag{8.13}$$

In fact, if $\liminf_n G_{A^e_y}((u_n)^e_y) < +\infty$, we get $(u_n)^e_y \overset{*}{\rightharpoonup} u^e_y$ weakly* in $BV((\Omega')^e_y)$. Moreover notice that for a.e. $y \in \pi_e$ $(\partial_D\Omega)^e_y$ is finite, and by (8.7) Γ^e_y is countable. So in relation with these y's, it is sufficient to set $I := A^e_y$ and $J := (\partial_D\Omega)^e_y \cup \Gamma^e_y$, and to apply Lemma 8.2 to $\mu_n := D(u_n)^e_y$.

Using Fatou's Lemma, by (8.11) and (8.13) we get

$$F'_{A,e}(u) \le \liminf_n F'_{A,e}(u_n). \tag{8.14}$$

Let $\lambda := \mathcal{L}^N + \mathcal{H}^{N-1} \llcorner (S(u) \cup \Gamma) + |D^c u|$, and let $B := \{e_j\}$ be countable and dense in $S^{N-1} := \{x \in \mathbb{R}^N \,:\, |x| = 1\}$. If $E \subseteq \Omega' \setminus (S(u) \cup \Gamma)$ is a $\mathcal{L}^N$ negligible set on which $D^c u$ is concentrated, let us define

$$f'_j(x) := \begin{cases} f_1(|\langle \nabla u(x), e_j \rangle|) & \text{if } x \in \Omega' \setminus (E \cup S(u) \cup \Gamma) \\ \\ |\langle \nu(x), e_j \rangle| \, \varphi(|[u](x)| \vee \psi(x)) & \text{if } x \in S(u) \cup \Gamma \\ \\ a \frac{|\langle D^c u, e_j \rangle|}{|D^c u|}(x) & \text{if } x \in E, \end{cases}$$

where $\nu(x)$ denotes the normal to the rectifiable set $S(u) \cup \Gamma$ at the point x.

For every $A_1, \dots, A_k$ disjoint open subsets of $\tilde{\Omega}$, and for every $e_1, \dots, e_k \in B$, since f_1 is increasing, we obtain that

$$\liminf_n F'(u_n) \geq \sum_{j=1}^k \liminf_n F'_{A_j}(u_n) \geq \sum_{j=1}^k \liminf_n F'_{A_j, e_j}(u_n) \tag{8.15}$$

$$\geq \sum_{j=1}^k F'_{A_j, e_j}(u) = \sum_{j=1}^k \int_{A_j} f'_j \, d\lambda. \tag{8.16}$$

Applying [4, Lemma 2.35], we deduce that

$$\liminf_n F'(u_n) \geq \int_A \sup_j f'_j \, d\lambda = F'(u), \tag{8.17}$$

so that the Lemma is proved under the assumption $\varphi(s) \geq \varepsilon s$.

The general case follows observing that setting $\varphi_\varepsilon(s) := \varphi(s) + \varepsilon s$, and letting $\overline{F}_\varepsilon$ be the functional defined in (8.4) with φ_ε in place of φ, we have

$$\overline{F}(u) \leq \overline{F}_\varepsilon(u) \leq \overline{F}(u) + 2\varepsilon |Du|(\Omega).$$

□

Let us now come to the proof of Proposition 8.1.

Proof of Proposition 8.1. We can assume without loss of generality that

$$\int_\Gamma \varphi(\psi) \, d\mathcal{H}^{N-1} < +\infty. \tag{8.18}$$

Following Lemma 8.3, let us consider $\tilde{\Omega}$ open and bounded in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \tilde{\Omega}$, and let us set $\Omega' := \tilde{\Omega} \setminus \partial_N \Omega$. Let us consider the functional

$$F'(u) := \begin{cases} \int_\Omega f(|\nabla u|) \, dx + \int_{S(u) \cup \Gamma} \varphi(|[u]| \vee \psi) \, d\mathcal{H}^{N-1} & \text{if } u \in SBV(\Omega'), u = g \text{ on } \Omega' \setminus \Omega \\ \\ +\infty & \text{otherwise in } BV(\Omega'). \end{cases} \tag{8.19}$$

The relaxation result of Proposition 8.1 is equivalent to prove that the relaxation of (8.19) under the weak* topology of $BV(\Omega')$ is

$$\overline{F'}(u) := \int_\Omega f_1(|\nabla u|) \, dx + \int_{S(u) \cup \Gamma} \varphi(|[u]| \vee \psi) \, d\mathcal{H}^{N-1} + a|D^c u|(\Omega') \tag{8.20}$$

if $u \in BV(\Omega')$, $u = g$ on $\Omega' \setminus \Omega$, and $\overline{F'}(u) = +\infty$ otherwise in $BV(\Omega')$.

Following [6], it is useful to introduce the localized version of (8.19); namely for all open set $A \subseteq \Omega'$ let us set

$$F'(u, A) := \int_{A \cap \Omega} f(|\nabla u|) \, dx + \int_{A \cap (S(u) \cup \Gamma)} \varphi(|[u]| \vee \psi) \, d\mathcal{H}^{N-1} \tag{8.21}$$

if $u \in SBV(\Omega')$, $u = g$ on $\Omega' \setminus \Omega$, and $F'(u, A) = +\infty$ otherwise in $BV(\Omega')$. Let us indicate by $\overline{F'}(u, A)$ the relaxation of (8.21) under the weak* topology of $BV(\Omega')$.

Arguing as in [6, Proposition 3.3], we have that for every $u \in BV(\Omega')$, $\overline{F'}(u,\cdot)$ is the restriction to the family $\mathcal{A}(\Omega')$ of all open subsets of Ω' of a regular Borel measure. Since for all $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$ and for all $A \in \mathcal{A}(\Omega')$ we have

$$\begin{aligned} \int_{A\cap\Omega} f(|\nabla u|)\,dx + \int_{A\cap S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} &\le F'(u,A) \\ &\le \int_{A\cap\Omega} f(|\nabla u|)\,dx + \int_{A\cap S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} + \int_{A\cap\Gamma} \varphi(\psi)\,d\mathcal{H}^{N-1}, \end{aligned} \tag{8.22}$$

by [6, Theorem 3.1] we obtain that for all $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$ and for all $A \in \mathcal{A}(\Omega')$ with $A \cap \partial_D\Omega = \emptyset$

$$\begin{aligned} \int_{A\cap\Omega} f_1(|\nabla u|)\,dx + \int_{A\cap S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} + a|D^c u|(A) &\le F'(u,A) \\ \le \int_{A\cap\Omega} f_1(|\nabla u|)\,dx + \int_{A\cap S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} &+ a|D^c u|(A) + \int_{A\cap\Gamma} \varphi(\psi)\,d\mathcal{H}^{N-1}. \end{aligned} \tag{8.23}$$

As a consequence of (8.23), we deduce that

$$\overline{F'}(u,\cdot)\llcorner(\Omega' \setminus (S(u)\cup\Gamma\cup\partial_D\Omega)) = f_1(|\nabla u|)\,d\mathcal{L}^N \llcorner \Omega + a|D^c u|.$$

In order to evaluate $\overline{F'}(u,\cdot)\llcorner(S(u)\cup\Gamma\cup\partial_D\Omega)$, we notice that for all $A \in \mathcal{A}(\Omega')$ and for all $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$

$$\int_{A\cap\Omega} f_1(|\nabla u|)\,dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(A) \le F'(u,A),$$

and since the left hand side is lower semicontinuous by Lemma 8.3, we get that for all $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$

$$\int_{A\cap\Omega} f_1(|\nabla u|)\,dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(A) \le \overline{F'}(u,A). \tag{8.24}$$

By outer regularity of $\overline{F'}(u,\cdot)$ we conclude that

$$\overline{F'}(u,E) \ge \int_E \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1}$$

for all Borel sets $E \subseteq S(u)\cup\Gamma\cup\partial_D\Omega$. We have to prove the opposite inequality, namely

$$\overline{F'}(u,E) \le \int_E \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1}$$

for all Borel sets $E \subseteq S(u)\cup\Gamma\cup\partial_D\Omega$. Without loss of generality we may assume that

$$\int_{S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} < +\infty,$$

and by a truncation argument, we can suppose that $u_{|\Omega} \in L^\infty(\Omega)$. Let K be a compact subset of $S(u)\cup\Gamma\cup\partial_D\Omega$, $\varepsilon > 0$, and let A_ε be open with $K \subseteq A_\varepsilon$ and

$$|Du|(A_\varepsilon \setminus K) < \varepsilon, \qquad \int_{(A_\varepsilon\setminus K)\cap\Gamma} \varphi(\psi)\,d\mathcal{H}^{N-1} < \varepsilon.$$

We can find $u_h \in BV(\Omega')$ with $u_h = g$ on $\Omega' \setminus \Omega$ and such that u_h is piecewise constant in Ω (that is $(u_h)_{|\Omega} \in SBV(\Omega)$ with $\nabla u_h = 0$ in Ω), $u_h \to u$ strongly in $L^\infty(\Omega)$, and $|Du_h|(A_\varepsilon \setminus K) < \varepsilon$. Since u_h is piecewise constant in Ω we have for all h

$$\overline{F'}(u_h, A_\varepsilon) \le \int_{A_\varepsilon\cap(S(u_h)\cup\Gamma)} \varphi(|[u_h]| \vee \psi)\,d\mathcal{H}^{N-1}. \tag{8.25}$$

We conclude

$$
\begin{aligned}
(8.26)\quad \overline{F'}(u, A_\varepsilon) &\leq \liminf_h \overline{F'}(u_h, A_\varepsilon) \leq \liminf_h \int_{A_\varepsilon \cap (S(u_h) \cup \Gamma)} \varphi(|[u_h]| \vee \psi)\, d\mathcal{H}^{N-1} \\
&\leq \int_{K \cap (S(u) \cup \Gamma)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|Du_h|(A_\varepsilon \setminus K) + \int_{(A_\varepsilon \setminus K) \cap \Gamma} \varphi(\psi)\, d\mathcal{H}^{N-1} \\
&\leq \int_{K \cap (S(u) \cup \Gamma)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + (a+1)\varepsilon
\end{aligned}
$$

so that, letting $\varepsilon \to 0$ we obtain

$$\overline{F'}(u, K) \leq \int_{K \cap (S(u) \cup \Gamma)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1}.$$

Since K is arbitrary in $S(u) \cup \Gamma \cup \partial_D \Omega$, the proof is concluded. □

9. Two auxiliary propositions

In this section we prove two propositions we used in the proofs of the main theorems of the paper. For all $h \in \mathbb{N}$ let us consider $f_h : \mathbb{R}^N \to [0, +\infty[$ such that for all $\xi \in \mathbb{R}^N$

$$(9.1)\qquad f_h(\xi) \nearrow |\xi|^2, \qquad f_h(\xi) \geq \min\{|\xi|^2 - 1, b_h|\xi|\}$$

with $b_h \to +\infty$, and let $\varphi_h : [0, +\infty[\to [0, 1]$ be such that for all $s \in [0, +\infty[$

$$(9.2)\qquad \varphi_h(s) \geq \min\{c_h s, d_h\}$$

with $c_h \to +\infty$ and $d_h \nearrow 1$ for $h \to +\infty$.

The following result holds.

Proposition 9.1. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, and let us consider the functionals*

$$F_h(u) := \sum_{i=1}^m \int_\Omega f_h(\nabla u_i)\, dx + \int_{S(u)} \varphi_h(|[u_1]| \vee \ldots \vee |[u_m]|)\, d\mathcal{H}^{N-1} + a_h |D^c u|(\Omega)$$

if $u = (u_1, \ldots, u_m) \in BV(\Omega; \mathbb{R}^m)$, and $F_h(u) = +\infty$ otherwise in $BV(\Omega; \mathbb{R}^m)$. Let $a_h \to +\infty$ for $h \to +\infty$. If $F_h(u^h) \leq C$ and $u^h \overset{*}{\rightharpoonup} u$ weakly in $BV(\Omega; \mathbb{R}^m)$, we have $u \in SBV(\Omega; \mathbb{R}^m)$,*

$$(9.3)\qquad \nabla u^h \rightharpoonup \nabla u \qquad \textit{weakly in } L^1(\Omega; \mathbb{R}^{m \times N}),$$

$$(9.4)\qquad \|\nabla u_i\|^2 \leq \liminf_h \int_\Omega f_h(\nabla u_i^h)\, dx, \qquad i = 1, \ldots, m,$$

and

$$(9.5)\qquad \mathcal{H}^{N-1}(S(u)) \leq \liminf_h \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \ldots \vee |[u_m^h]|)\, d\mathcal{H}^{N-1}.$$

Proof. Let us consider $u^h \in BV(\Omega; \mathbb{R}^m)$ such that $F_h(u^h) \leq C$ and $u^h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega; \mathbb{R}^m)$. Notice that (∇u^h) is equintegrable. In fact if r_h is such that for all $|\xi| \leq r_h$

$$|\xi|^2 - 1 \leq b_h |\xi|$$

we get for all $i = 1, \ldots, m$ and for all $E \subseteq \Omega$

$$
\begin{aligned}
(9.6)\quad \int_E |\nabla u_i^h|\, dx &\leq \int_{\{|\nabla u_i^h| \leq r_h\} \cap E} |\nabla u_i^h|\, dx + \int_{\{|\nabla u_i^h| > r_h\} \cap E} |\nabla u_i^h|\, dx \\
&\leq \left(\int_{\{|\nabla u_i^h| \leq r_h\} \cap E} |\nabla u_i^h|^2\, dx \right)^{\frac{1}{2}} |E|^{\frac{1}{2}} + \int_{\{|\nabla u_i^h| > r_h\} \cap E} |\nabla u_i^h|\, dx \\
&\leq \left(\int_\Omega (f_h(\nabla u_i^h) + 1)\, dx \right)^{\frac{1}{2}} |E|^{\frac{1}{2}} + \frac{1}{b_h} \int_\Omega f_h(\nabla u_i^h)\, dx \leq \sqrt{C+1}|E| + \frac{C}{b_h},
\end{aligned}
$$

where $|E|$ denotes the Lebesgue measure of E. This proves that ∇u_h is equintegrable. Up to a subsequence we may suppose that for all $i = 1, \ldots, m$ we have

$$\nabla u_i^h \rightharpoonup g_i \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^N).$$

Since $a_h \to +\infty$, we get $D^c u^h \to 0$ strongly in the sense of measures.

Let us consider $\psi : \mathbb{R} \to \mathbb{R}$ bounded and Lipschitz, and for all $i = 1, \ldots, m$ let us consider the measures

$$\mu_i^h(B) := D\psi(u_i^h)(B) - \int_B \psi'(u_i^h)\nabla u_i^h \, dx, \qquad \lambda_i^h(B) := \int_{S(u_i^h)\cap B} \varphi_h(|[u_i^h]|) \, d\mathcal{H}^{N-1},$$

where B is a Borel set in Ω. We have

$$|D\psi(u_i^h) - \psi'(u_i^h)\nabla u_i^h \, d\mathcal{L}^N| \le ||\psi||_{\varphi_h}\lambda_i^h + ||\psi'||_\infty |D^c u_i^h| \tag{9.7}$$

where

$$||\psi||_{\varphi_h} := \sup\left\{ \frac{\psi(t) - \psi(s)}{\varphi_h(|t-s|)} \, : \, t \neq s \right\}.$$

Up to a subsequence we have

$$\mu_i^h \overset{*}{\rightharpoonup} D\psi(u_i) - \psi'(u)g_i \, d\mathcal{L}^N, \qquad \lambda_i^h \overset{*}{\rightharpoonup} \lambda_i$$

weakly* in the sense of measures, and so by (9.7) and since $D^c u^h \to 0$ strongly in the sense of measures we get

$$|D\psi(u_i) - \psi'(u)g_i \, d\mathcal{L}^N| \le (\sup\psi - \inf\psi)\lambda_i.$$

As a consequence of SBV characterization (see [4, Proposition 4.12]), we get that $u_i \in SBV(\Omega)$, $\nabla u_i = g_i$ and $\mathcal{H}^{N-1} \llcorner S(u_i) \le \lambda_i$ for all $i = 1, \ldots, m$. We deduce that (9.3) holds.

In order to prove (9.4), for every $M > 0$ let g_i^M be the weak limit in $L^1(\Omega)$ (up to a subsequence) of $|\nabla u_i^h| \wedge M$. Since $f_h(\xi) \to |\xi|^2$ uniformly on $[0, M]$, we have

$$||g_i^M||^2 \le \liminf_h \int_\Omega f_h(\nabla u_i^h) \, dx. \tag{9.8}$$

Then as $M \to +\infty$ and summing over i we have that (9.4) holds.

Let us come to (9.5). If λ is the weak limit in the sense of measures of

$$\lambda^h(A) := \int_{S(u^h)\cap A} \varphi_h(|[u_1^h]| \vee \ldots \vee |[u_m^h]|) \, d\mathcal{H}^{N-1},$$

we have that $\lambda_i \le \lambda$ for all $i = 1, \ldots, m$. Since we have $\mathcal{H}^{N-1} \llcorner S(u_i) \le \lambda_i$ for all $i = 1, \ldots, m$, we deduce that $\mathcal{H}^{N-1} \llcorner S(u) \le \lambda$. Then we get that (9.5) holds. $\square$

Let $a_h \to +\infty$, and let us consider $\varphi_h : [0, +\infty[\to [0, 1]$ increasing, concave, $\varphi(0) = 0$, $\varphi'(0) = a_h$ and such that

$$\varphi_h \nearrow 1. \tag{9.9}$$

For all $h \in \mathbb{N}$ let us consider $f_h : \mathbb{R}^N \to [0, +\infty[$ such that for all $\xi \in \mathbb{R}$

$$f_h(\xi) = \begin{cases} |\xi|^2 & |\xi| \le \frac{a_h}{2} \\ \frac{a_h^2}{4} + a_h(|\xi| - \frac{a_h}{2}) & |\xi| \ge \frac{a_h}{2}. \end{cases} \tag{9.10}$$

Let us consider Ω open and bounded in $\mathbb{R}^N$, and let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let us consider $\bar\Gamma \tilde\subseteq \Omega$ with $\mathcal{H}^{N-1}(\bar\Gamma) < +\infty$, and let $\bar\gamma$ be a positive function on $\bar\Gamma$. Let us extend $\bar\gamma$ to $\overline{\Omega}$ setting $\bar\gamma = 0$ outside $\bar\Gamma$. Then the following proposition holds.

Proposition 9.2. *Let $g \in H^1(\Omega)$ with $||g||_\infty < +\infty$, and let us consider the functionals*

$$\mathcal{F}_h(u) := \int_\Omega f_h(\nabla u) \, dx + \int_{\bar\Gamma \cup S^g(u)} \varphi_h(|[u]| \vee \bar\gamma) \, d\mathcal{H}^{N-1} + a_h|D^c u|(\Omega) \tag{9.11}$$

if $u \in BV(\Omega)$, where f_h and φ_h are as in (9.10) and (9.9). Then

$$\inf_{u \in BV(\Omega)} \mathcal{F}_h(u) \to \inf_{u \in BV(\Omega)} \mathcal{F}(u)$$

where

$$\mathcal{F}(u) := \begin{cases} \|\nabla u\|^2 + \mathcal{H}^{N-1}(\bar{\Gamma} \cup S^g(u)) & u \in SBV(\Omega) \\ +\infty & u \in BV(\Omega) \setminus SBV(\Omega). \end{cases} \tag{9.12}$$

Proof. In order to deal with $S^g(u)$ as an internal jump, let us consider $\tilde{\Omega} \subseteq \mathbb{R}^N$ open and bounded, and such that $\overline{\Omega} \subseteq \tilde{\Omega}$. Let us set $\Omega' := \tilde{\Omega} \setminus \partial_N \Omega$. Let us consider the functionals

$$\mathcal{F}'_h(u) := \int_{\Omega'} f_h(\nabla u)\, dx + \int_{\bar{\Gamma} \cup S(u)} \varphi_h(|[u]| \vee \bar{\gamma})\, d\mathcal{H}^{N-1} + a_h |D^c u|(\Omega')$$

for $u \in BV(\Omega')$ such that $u = g$ on $\Omega' \setminus \Omega$, and $\mathcal{F}'_h(u) = +\infty$ otherwise in $BV(\Omega')$. In order to prove the lemma we can equivalently prove that

$$\inf_{u \in BV(\Omega')} \mathcal{F}'_h(u) \to \inf_{u \in BV(\Omega')} \mathcal{F}'(u),$$

where

$$\mathcal{F}'(u) := \begin{cases} \|\nabla u\|^2 + \mathcal{H}^{N-1}(\bar{\Gamma} \cup S(u)) & u \in SBV(\Omega'),\ u = g \text{ on } \Omega' \setminus \Omega \\ +\infty & \text{otherwise in } BV(\Omega'). \end{cases} \tag{9.13}$$

Let us consider

$$\mathcal{G}'_h(u) := \begin{cases} \mathcal{F}'_h(u) & u \in BV(\Omega'),\ \|u\|_\infty \le \|g\|_\infty \\ +\infty & \text{otherwise in } BV(\Omega'). \end{cases} \tag{9.14}$$

We have that $(\mathcal{G}'_h)_{h \in \mathbb{N}}$ is an increasing sequence of functionals which converges pointwise to

$$\mathcal{G}'(u) := \begin{cases} \mathcal{F}'(u) & u \in BV(\Omega'),\ \|u\|_\infty \le \|g\|_\infty \\ +\infty & \text{otherwise in } BV(\Omega'). \end{cases} \tag{9.15}$$

By Lemma 8.3 $\mathcal{G}'_h$ is lower semicontinuous with respect to the weak* convergence in $BV(\Omega')$, and so by [10, Proposition 5.4] we deduce that $\mathcal{G}'_h$ Γ-converges to $\mathcal{G}'$ in the weak* topology of $BV(\Omega')$. As a consequence of Proposition 2.3, we deduce that $\inf_{BV(\Omega')} \mathcal{G}'_h \to \inf_{BV(\Omega')} \mathcal{G}'$. By a truncation argument, we have that $\inf_{BV(\Omega')} \mathcal{G}'_h = \inf_{BV(\Omega')} \mathcal{F}'_h$ and $\inf_{BV(\Omega')} \mathcal{G}' = \inf_{BV(\Omega')} \mathcal{F}'$, so that the proposition is proved. □

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