A HIERARCHY OF PLATEAU PROBLEMS AND THE APPROXIMATION OF PLATEAU'S LAWS VIA THE ALLEN–CAHN EQUATION

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Abstract. We introduce a diffused interface formulation of the Plateau problem, where the Allen–Cahn energy $\mathcal{AC}_\varepsilon$ is minimized under a volume constraint v and a spanning condition on the level sets of the densities. We discuss two singular limits of these Allen–Cahn Plateau problems: when $\varepsilon \to 0^+$, we prove convergence to the Gauss' capillarity formulation of the Plateau problem with positive volume v; and when $\varepsilon \to 0^+$, $v \to 0^+$ and $\varepsilon/v \to 0^+$, we prove convergence to the classical Plateau problem (in the homotopic spanning formulation of Harrison and Pugh). As a corollary of our analysis we resolve the incompatibility between Plateau's laws and the Allen–Cahn equation implied by a regularity theorem of Tonegawa and Wickramasekera. In particular, we show that Plateau-type singularities can be approximated by energy minimizing solutions of the Allen–Cahn equation with a volume Lagrange multiplier and a transmission condition on a spanning free boundary.

Contents

1. Introduction

1.1. **Overview.** The convergence of solutions to the Allen–Cahn equation $\varepsilon^2\,\Delta u = W'(u)$ to limit minimal surfaces is a result of basic importance in the study of the van der Waals-Cahn-Hilliard theory of phase transitions [Gur87, Mod87, Ste88, KS89, HT00]. A regularity result of Tonegawa and Wickramasekera [Ton05, TW12] shows that, in low dimensions, minimal surfaces arising as limits of stable solutions to the Allen–Cahn equation are necessarily *smooth*. While this result makes the Allen–Cahn equation a useful tool for constructing minimal surfaces in Riemannian manifolds, see e.g. [Gua18], it also stands as a limitation to its descriptive power when studying *soap films*. Indeed, according to Plateau's laws, soap films can be modeled as two-dimensional smooth minimal surfaces joining in threes at equal angles along lines of "Y-points", which, in turn, are either closed or meet in fours at isolated "T-points" where they asymptotically form regular tetrahedral angles. The Tonegawa–Wickramasekera theorem implies in particular that no minimal surface with Plateau-type singularities can arise as the limit of stable solutions to the Allen–Cahn equation.

Here we prove that minimal surfaces with Plateau-type singularities can indeed be approximated by energy-minimizing solutions to the Allen–Cahn equation modified by the inclusion of a Lagrange multiplier term corresponding to a *small volume constraint*, and with the introduction of a *transmission condition* along a "spanning" level set. These solutions are constructed as minimizers of a "diffused interface" soap film model $\Upsilon(v,\varepsilon,\delta)$, which is introduced here for the first time. The introduction of a small volume constraint has its origin in the Physics literature, where a distinction between "dry" and "wet" soap films is made [WH99, CCAE$^+$13]. While dry soap films are two dimensional surfaces obeying Plateau's laws, in the wet soap film model Plateau-type singularities are resolved as *Plateau borders* – constant mean curvature channels of liquid developing around lines of Y-points, that are supposed to attach tangentially to smooth interfaces with zero mean curvature; see Figure 1.2 below. In the companion paper [MNR23] we have recently validated the wet soap film model in the framework of Gauss' capillarity theory. The diffused interface soap film model introduced here thus completes a hierarchy of Plateau-type problems including wet and dry soap film models.

The main result of this paper is showing how one can move along this hierarchy of models by taking singular limits. In more concrete terms, and coming back to the problem of approximating Plateau-type singularities by solutions to the Allen–Cahn equation, our main results can be roughly described as follows. *First*, given a compact set $\mathbf{W} \subset \mathbb{R}^{n+1}$ (the "wire frame"), a non-degenerate double-well potential $W : [0,1] \to [0,\infty)$, a related volume potential $V(t) = (\int_0^t \sqrt{W})^{(n+1)/n}$, and interface length scales $\varepsilon_j \to 0^+$ and volumes $v_j \to 0^+$ with $\varepsilon_j/v_j \to 0^+$, we construct energy minimizing solutions $\{u_j\}_j$ to the free boundary problems

$$\begin{cases} 2\,\varepsilon_j^2\,\Delta u_j = W'(u_j) - \varepsilon_j\,\lambda_j\,V'(u_j)\,, & \text{on } \Omega \cap \{u_j < 1\}\,, \\ |\partial_\nu^+ u_j| = |\partial_\nu^- u_j|\,, & \text{on } \Omega \cap \{u_j = 1\}\,, \\ \text{subject to } \int_\Omega V(u_j) = v_j \text{ and } \{u_j = 1\} \text{ spans } \mathbf{W}\,, \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$, $\lambda_j \in \mathbb{R}$ are suitable Lagrange multipliers with $\varepsilon_j\,\lambda_j \to 0$, and $\partial_\nu^\pm$ denote the one-sided directional derivative operators along the hypersurfaces $\{u_j = 1\}$. *Second*, we show that, up to extracting subsequences in j, for every such $\{u_j\}_j$ there is a (possibly singular) minimal surface S, which is area minimizing among surfaces spanning $\mathbf{W}$, and is such that, as $j \to \infty$,

$$\frac{1}{2}\int_\Omega \varphi\left\{\varepsilon_j\,|\nabla u_j|^2 + \frac{W(u_j)}{\varepsilon_j}\right\} \to 2\int_S \varphi\,d\mathcal{H}^n \tag{1.2}$$

for every $\varphi \in C_c^0(\mathbb{R}^{n+1})$; see Figure 1.1. For various choices of $\mathbf{W}$ there will be only one such area minimizing surface S, which will indeed possess Plateau-type singularities. Actually, since our construction passes through the intermediate wet soap film model of [MNR23], in a situation where the Plateau problem defined by $\mathbf{W}$ admits multiple area minimizing surfaces, some smooth and some with Plateau-type singularities, the only possible limits S in (1.2) will be surfaces with Plateau-type singularities.

One can of course think of other possible modifications of the Allen–Cahn equation that lead to a PDE-description of Plateau-type singularities. A well-known possibility consists in working with an Allen–Cahn *system* [Bal90]. From the physical viewpoint this approach corresponds to describing the three regions locally defined by a Y-singularity as occupied by three different immiscible fluids. In this sense, the approach followed here, which insists on the use of a single *scalar* equation and is based on the introduction of a small volume constraint and of a spanning condition, seems more true to the actual nature of soap films. The emergence, in this approach, of the physically meaningful wet soap film model studied in [MNR23], is yet another indication of its naturalness.

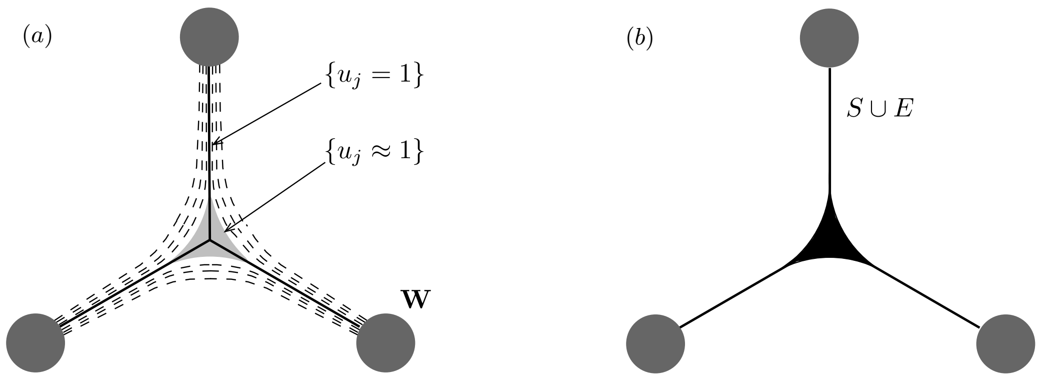

FIGURE 1.1. When $\mathbf{W}$ consists of three disks in the plane, the only possible limit S in (1.2) consists of three segments, each orthogonal to one of the disks, and meeting at a common endpoint at equal angles: (a) Heuristic arguments suggest that, if u_j is a solution to (1.1), then $\{u_j = 1\}$ should be equal to S, with u_j taking values close to 1 on a negatively curvilinear triangle E centered at the triple point of S (depicted in gray), and then sharply transitioning to near zero values on a small neighborhood of $S \cup E$ (depicted by dashed lines). The normal derivatives $\partial_\nu^+ u_j$ and $\partial_\nu^- u_j$ of u_j should take non-zero, non-constant and opposite values along S. (b) As $j \to \infty$, the pointwise limit of u_j should be equal to 1 on $S \cup E$ (depicted in black).

In Section 1.2 we recall the homotopic spanning formulation ℓ of the Plateau problem introduced by Harrison and Pugh in [HP16]. In Section 1.3 and Section 1.4 we introduce, respectively, the capillarity approximation $\Psi_{\rm bk}(v)$ of ℓ studied in [MNR23] and the new diffused interface problems $\Upsilon(v, \varepsilon, \delta)$. In Section 1.5 we state the main result of this paper, Theorem 1.2, where we prove the existence of minimizers of $\Upsilon(v, \varepsilon, \delta)$ and their convergence towards minimizers of $\Psi_{\rm bk}(v)$ and ℓ in the limits as $\varepsilon \to 0^+$, and as $\varepsilon \to 0^+$, $v \to 0^+$ and $\varepsilon/v \to 0^+$, respectively. Additionally, in Theorem 1.3 we derive the distributional form of (1.1), see (1.15), and in Proposition 1.4 we deduce (1.1) from its distributional form under some conditional regularity assumptions.

The results of this paper open the study of Plateau's laws by means of free boundary problems. This point, which seems very interesting, is discussed in detail in Section 1.6.

1.2. **Plateau's laws, the Plateau problem, and homotopic spanning.** The properties of solutions to Plateau's problem of finding area minimizing surfaces with a given boundary depend subtly on the notions of "area" and "boundary" employed. The classical formulation of the Plateau problem based on the theory of currents leads, in physical dimensions, to *smooth* area minimizing surfaces, so that surfaces with Plateau singularities will be "invisible" even when having lower area.

Finding a formulation of the Plateau problem whose minimizers may actually show Plateau-type singularities is a delicate task, with a long history, see [Dav14]. An effective approach has been proposed by Harrison and Pugh in [HP16], with the introduction of the notion of *homotopic spanning*. Following the presentation given in [DLGM17a], given a closed set $\mathbf{W} \subset \mathbb{R}^{n+1}$ (the wire frame to be spanned), and setting $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$, we say that a family $\mathcal{C}$ of smooth embeddings $\gamma : \mathbb{S}^1 \to \Omega$ defines a **spanning class for W** if $\Phi(\cdot, 1) \in \mathcal{C}$ whenever $\Phi \in C^\infty(\mathbb{S}^1 \times [0,1]; \Omega)$, $\Phi(\cdot, t)$ is a smooth embedding of $\mathbb{S}^1$ into Ω for every t, and $\Phi(\cdot, 0) \in \mathcal{C}$. Then a relative closed set $S \subset \Omega$ is said to be **$\mathcal{C}$-spanning W** if

$$S \cap \gamma(\mathbb{S}^1) \neq \varnothing\,, \qquad \forall \gamma \in \mathcal{C}\,, \tag{1.3}$$

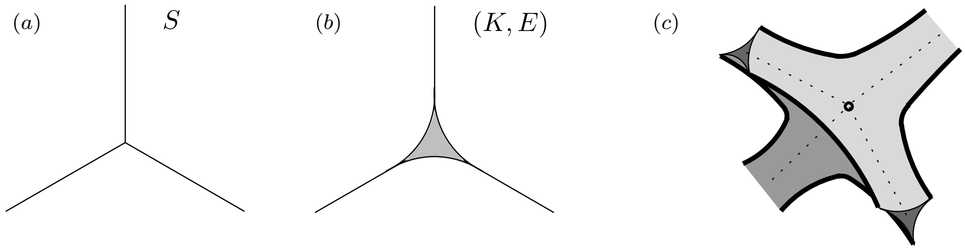

FIGURE 1.2. (a) A "dry" soap film S in $\mathbb{R}^2$ with a Y-type singularity; (b) a corresponding "wet" soap film (K, E): the Y-singularity has been wetted by a negatively curved region E ("planar" Plateau border); (c) in $\mathbb{R}^3$, nearby a T-point of a dry film S, a wet film (K, E) is placing a negatively curved tube-like structure E (Plateau border). Plateau borders are important, for example, to understand drainage phenomena in soap films.

and the following homotopic spanning formulation of **the Plateau problem** is given,

$$\ell = \inf \left\{ \mathcal{H}^n(S) : S \text{ is relatively closed in } \Omega,\ S \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \right\}, \tag{1.4}$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Minimizers of ℓ exist as soon as $\ell < \infty$ [HP16, DLGM17b], and they are **Almgren minimal sets**, that is to say, they satisfy $\mathcal{H}^n(S) \le \mathcal{H}^n(f(S))$ whenever f is a Lipschitz map, not necessarily injective, with $\{f \ne \mathrm{id}\} \subset\subset \Omega$. As proved by Taylor [Tay76], when $n = 2$ an Almgren minimal set S is locally $C^{1,\alpha}$-diffeomorphic either to a plane, or to a Y-cone, or to a T-cone, that is, it obeys Plateau's laws. An analogous result is available, by elementary means, in the other important physical case, $n = 1$; and similar results also hold in dimension $n \ge 3$, see [CES22]. In particular, in physical dimensions $n = 1, 2$, minimizers of ℓ may satisfy Plateau's laws, and, for suitable choices of $\mathbf{W}$ and $\mathcal{C}$, one can prove that this indeed the case when $n = 1$ – see also [BM21] for an analysis of the appearance of singular catenoids when $n = 2$. For all these reasons, our analysis will be based on the Harrison–Pugh formulation of the Plateau problem.

1.3. **Capillarity approximation of the Plateau problem.** In [MSS19, KMS22a] a model for soap films as three-dimensional regions with small but positive volume has been introduced, based on Gauss' capillarity theory. Let us recall that, in Gauss' capillarity theory, one minimizes $\mathcal{H}^n(\Omega \cap \partial E)$ among open sets $E \subset \Omega$ with smooth boundary under a volume constraint $|E| = v$. When v is small such minimizers are close to half-balls [MM16]. To avoid droplet-like minimizers, and actually observe soap film-like minimizers, in [MSS19, KMS22a] the following problem

$$\psi(v) = \inf \left\{ \mathcal{H}^n(\Omega \cap \partial E) : |E| = v \text{ and } \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \right\},$$

where E ranges among subsets of Ω with Lipschitz regular boundary, has been introduced. As proved in [KMS22a, KMS21, KMS22b], $\psi(v)$ admits minimizers only in a generalized sense, and such generalized minimizers converge to minimizers of ℓ, with $\psi(v) \to 2\,\ell$ as $v \to 0^+$. The existence of generalized minimizers in $\psi(v)$ corresponds to the actual physical description [WH99, CCAE$^+$13] of soap films as either "dry" soap films (minimizers of ℓ) or "wet" soap films (minimizers of $\psi(v)$); see Figure 1.2. Establishing the sharp regularity of these generalized minimizers, and in particular the validity of a sort of "third Plateau law" for their characteristic structures known as *Plateau borders*, is the subject of [MNR23]. We now review the approach developed in [MNR23], which is crucial for setting up the Allen–Cahn Plateau problem studied in this paper.

The starting point of [MNR23] is reinterpreting the notion of $\mathcal{C}$-spanning set introduced in (1.3), which is a condition sensitive to pointwise modifications, so to make it stable under modifications by $\mathcal{H}^n$-null sets and under the operation of taking weak limits in the sense of Radon measures. Postponing to Section 2 a detailed discussion of this matter, it suffices to notice here that the work done in [MNR23] gives a meaning to the statement "S is $\mathcal{C}$-spanning $\mathbf{W}$" whenever S is a Borel subset of Ω, and does so in such a way that: (i) if S is relatively closed in Ω, the new condition is equivalent to (1.3); (ii) if S is $\mathcal{C}$-spanning $\mathbf{W}$ and S' is $\mathcal{H}^n$-equivalent to S, then S' is $\mathcal{C}$-spanning $\mathbf{W}$; (iii) if S_j are $\mathcal{H}^n$-finite sets that are $\mathcal{C}$-spanning $\mathbf{W}$, μ is a Radon measure in Ω, and $\mathcal{H}^n \llcorner S_j \stackrel{*}{\rightharpoonup} \mu$ as Radon measures in Ω as $j \to \infty$, then

$$
\begin{aligned}
&S = \{x \in \Omega : \theta_n^*(\mu)(x) \geq 1\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,,\\
&\text{and } \mathcal{H}^n(S) \leq \liminf_{j\to\infty} \mathcal{H}^n(S_j)\,;
\end{aligned}
$$

and (iv) the homotopic spanning Plateau problem ℓ_{B} obtained by minimizing $\mathcal{H}^n(S)$ among Borel sets S that $\mathcal{C}$-spans $\mathbf{W}$ actually coincides with problem ℓ introduced in (1.4), that is, $\ell = \ell_{\mathrm{B}}$ and the two problems have the same minimizers (modulo $\mathcal{H}^n$-equivalence of sets).

Based on this definition we can directly consider Gauss' capillarity energy under homotopic spanning conditions in the class of sets of finite perimeter, and formulate the problem

$$
\psi_{\mathrm{bk}}(v) = \inf\Big\{\mathcal{H}^n(\Omega \cap \partial^* E) : |E| = v \text{ and } \Omega \cap (E^{(1)} \cup \partial^* E) \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,,
$$

where $\partial^* E$ denotes the reduced boundary of a set of locally finite perimeter $E \subset \Omega$ and $E^{(1)}$ is the set of points of density 1 of E. The subscript "bk" stands for "bulk" to reflect the fact that, in formulating $\psi_{\mathrm{bk}}(v)$, we are now imposing the burden of achieving the spanning condition *not* on the boundary of E alone, as done with $\psi(v)$, but rather on the whole bulk of E. The two approaches are evidently related (to the point that one naturally conjectures they should lead to the same minimizers when v is small enough), and we are not aware of a physical reason to prefer one to the other. However, the bulk variant is much more natural to work with in view of the formulation of an Allen–Cahn Plateau problem, which is the decisive reason for us to work with the bulk spanning condition, and to consider $\psi_{\mathrm{bk}}(v)$ in place of $\psi(v)$ in [MNR23], and in what follows.

It is now convenient to recall the main result from [Nov23], a companion paper to [MNR23]. Introducing the class

$$
\mathcal{K}_{\mathrm{B}}
$$

of those pairs (K, E) of Borel subsets of Ω such that

$$
E \text{ is of locally finite perimeter in } \Omega \text{ and } \Omega \cap \partial^* E \text{ is } \mathcal{H}^n\text{-contained in } K\,, \tag{1.5}
$$

and the *relaxed energy*

$$
\mathcal{F}_{\mathrm{bk}}(K, E; A) = \mathcal{H}^n(A \cap \partial^* E) + 2\,\mathcal{H}^n(A \cap K \cap E^{(0)})\,, \qquad \mathcal{F}_{\mathrm{bk}}(K, E) := \mathcal{F}_{\mathrm{bk}}(K, E; \Omega)\,, \tag{1.6}
$$

(where[1] $E^{(0)}$ is the set of points of density 0 of E, and with $A \subset \Omega$), the main result proved in [Nov23] is that $\psi_{\mathrm{bk}}(v) = \Psi_{\mathrm{bk}}(v)$, where

$$
\Psi_{\mathrm{bk}}(v) := \inf\Big\{\mathcal{F}_{\mathrm{bk}}(K, E; \Omega) : (K, E) \in \mathcal{K}_{\mathrm{B}},\ |E| = v,\ K \cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\Big\}\,. \tag{1.7}
$$

Notice that if E is a competitor of $\psi_{\mathrm{bk}}(v)$, then $(\varnothing, E) \in \mathcal{K}_{\mathrm{B}}$ with $\mathcal{F}_{\mathrm{bk}}(\varnothing, E) = \mathcal{H}^n(\Omega \cap \partial^* E)$, so that, trivially $\psi_{\mathrm{bk}}(v) \geq \Psi_{\mathrm{bk}}(v)$. The equality $\psi_{\mathrm{bk}}(v) = \Psi_{\mathrm{bk}}(v)$ thus expresses the fact that minimizers of $\psi_{\mathrm{bk}}(v)$ may fail to exist in a proper sense, and may thus be found only in a relaxed sense as minimizers of $\Psi_{\mathrm{bk}}(v)$. The following theorem summarizes [MNR23, Theorem 1.5, 1.6, B.1]:

[1] It is important to keep in mind that when E is of locally finite perimeter in Ω, then $\{E^{(1)}, E^{(0)}, \Omega \cap \partial^* E\}$ is an $\mathcal{H}^n$-partition of Ω by a theorem of Federer.

Theorem 1.1 (Main results from [MNR23]). *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and $\ell < \infty$, then $\Psi_{\rm bk}(v) \to 2\,\ell = 2\,\ell_{\rm B} = \Psi_{\rm bk}(0)$ as $v \to 0^+$. Moreover:*

(i): *for every $v > 0$ there exist a minimizer (K, E) of $\Psi_{\rm bk}(v)$, and up to an $\mathcal{H}^n$-null modification of K and a Lebesgue null modification of E, K is relatively closed in Ω, E is open with $\Omega \cap {\rm cl}\,(\partial^* E) = \Omega \cap \partial E \subset K$, $K \cup E$ is $\mathcal{C}$-spanning $\mathbf{W}$, and $K \cap E^{(1)} = \varnothing$, so that, with disjoint unions,*

$$K = \big[K \setminus \partial E\big] \cup \big[\Omega \cap (\partial E \setminus \partial^* E)\big] \cup [\Omega \cap \partial^* E]\,;$$

moreover, there exists a closed set $\Sigma \subset K$, with $\Sigma = \varnothing$ if $1 \le n \le 6$, Σ locally finite in Ω if $n = 7$, and $\mathcal{H}^s(\Sigma) = 0$ for every $s > n - 7$ if $n \ge 8$, such that:

(a): *$(K \setminus \partial E) \setminus \Sigma$ is a smooth minimal surface;*

(b): *$\Omega \cap \partial^* E$ is a smooth hypersurface with constant mean curvature denoted by λ if computed with respect to the outer unit normal ν_E to E;*

(c): *if $\Omega \cap (\partial E \setminus \partial^* E) \setminus \Sigma \ne \varnothing$, then $\lambda < 0$, and for every $x \in \Omega \cap (\partial E \setminus \partial^* E) \setminus \Sigma$ there is $r > 0$ such that $K \cap B_r(x)$ is the union of two ordered $C^{1,1}$-graphs which detach tangentially along $\Omega \cap (\partial E \setminus \partial^* E)$; moreover, $\Omega \cap (\partial E \setminus \partial^* E)$ is locally $\mathcal{H}^{n-1}$-rectifiable;*

(ii): *if $v_j \to 0^+$ and (K_j, E_j) are minimizers of $\Psi_{\rm bk}(v_j)$, then, up to extracting subsequences, there is a minimizer S of ℓ such that, as $j \to \infty$,*

$$\int_{\Omega \cap \partial^* E_j} \varphi\, d\mathcal{H}^n + 2 \int_{\Omega \cap K_j \cap E_j^{(0)}} \varphi\, d\mathcal{H}^n \to 2 \int_S \varphi\, d\mathcal{H}^n\,,$$

for every $\varphi \in C^0_c(\mathbb{R}^{n+1})$.

(iii): *if, in addition, $\mathbf{W}$ is the closure of a bounded open set with C^2-boundary, then for every $v > 0$ and every minimizing sequence $\{(K_j, E_j)\}_j$ of $\Psi_{\rm bk}(v)$ there is a minimizer (K, E) of $\Psi_{\rm bk}(v)$ such that K is $\mathcal{H}^n$-rectifiable and, up to extracting subsequences and as $j \to \infty$,*

$$E_j \to E\,, \qquad \mu_j \stackrel{*}{\rightharpoonup} \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) + 2\,\mathcal{H}^n \llcorner (K \cap E^{(0)})\,, \tag{1.8}$$

where $\mu_j = \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j) \cap E_j^{(0)})$.

1.4. **A diffused interface formulation of the Plateau problem.** In the diffused interface approximation of capillarity theory, the position of a liquid at equilibrium is represented, rather than by a set $E \subset \Omega$, by a density function $u : \Omega \to [0, 1]$. Surface tension energy is then represented by the Allen–Cahn energy of u,

$$\mathcal{AC}_\varepsilon(u; \Omega) = \int_\Omega {\rm ac}_\varepsilon(u(x))\, dx\,, \qquad {\rm ac}_\varepsilon(u) = \varepsilon\, |\nabla u|^2 + \frac{W(u)}{\varepsilon}\,,$$

where $\varepsilon > 0$ has the dimensions of a length (in particular, $\mathcal{AC}_\varepsilon(u)$ has the dimensions of surface area), and $W \in C^{2,1}[0, 1]$ is a (dimensionless) double-well potential. We assume W to satisfy the basic structural properties

$$W(0) = W(1) = 0\,, \qquad W > 0 \text{ on } (0, 1)\,, \qquad W''(0), W''(1) > 0\,, \tag{1.9}$$

as well as the normalization

$$\int_0^1 \sqrt{W(t)}\, dt = 1\,. \tag{1.10}$$

We now introduce volume and homotopic spanning constraints on densities u.

Volume constraint: To impose a volume constraint on u, we consider a (dimensionless) "volume density potential" $V : [0, 1] \to [0, \infty)$, with $V(0) = 0$ and V increasing and positive on $(0, 1]$. Given a choice of V, u corresponds to a soap film of volume v if

$$\mathcal{V}(u; \Omega) = v\,, \qquad \text{where} \quad \mathcal{V}(u; \Omega) := \int_\Omega V(u(x))\, dx\,.$$

The choice of V is really a matter of convenience, since any choice of V leads to recover the correct volume constraint in the sharp interface limit $\varepsilon \to 0^+$, and since the model is purely phenomenological. When working on bounded domains Ω, a common choice of V made in the literature is taking $V(t) = t$. This choice does not work well on unbounded domains, since in that case $\mathcal{AC}_\varepsilon(u;\Omega)$ can be made arbitrarily small (while keeping $\int_\Omega u$ fixed) by simply "spreading" u. Following the treatment of the *diffused interface isoperimetric problem on* $\mathbb{R}^{n+1}$ naturally associated with $\mathcal{AC}_\varepsilon$, see [MR22], we will set

$$V(t) = \Phi(t)^{(n+1)/n}\,, \qquad \Phi(t) = \int_0^t \sqrt{W(s)}\,ds\,,$$

for $t \in [0,1]$ and $u \in L^1_{\rm loc}(\Omega)$. This choice is of course motivated by the BV-Sobolev embedding and by the "Modica–Mortola identity"

$$\mathcal{AC}_\varepsilon(u;\Omega) = 2\,|D(\Phi\circ u)|(\Omega) + \int_\Omega \Big(\sqrt{\varepsilon}\,|\nabla u| - \sqrt{W(u)/\varepsilon}\Big)^2 \ge 2\,|D(\Phi\circ u)|(\Omega)\,. \qquad \text{(MM)}$$

Notice also that we have $\Phi(1) = V(1) = 1$ thanks to the normalization (1.10) on W.

Homotopic spanning constraint: Deciding how to impose an homotopic spanning conditions on densities u is of course a delicate choice in the setting of our model. The idea explored here is requiring, given $\delta \in (1/2, 1]$, that all the superlevel sets $\{u \ge t\}$ corresponding to[2] $t \in (1/2, \delta)$ are $\mathcal{C}$-spanning $\mathbf{W}$. Having extended the notion of $\mathcal{C}$-spanning from a pointwise unstable condition to an $\mathcal{H}^n$-stable condition is of course a crucial feature to discuss the existence of minimizers[3]. This kind of stability is natural in our problem since $W^{1,2}(\Omega)$ is the natural energy space for working with the Allen–Cahn energy and since the **Lebesgue representative** u^* of a Sobolev function $u \in W^{1,2}(\Omega)$ is well-defined $\mathcal{H}^n$-a.e. on Ω, so that, given two functions $u_1, u_2 \in W^{1,2}_{\rm loc}(\Omega)$ that are $\mathcal{L}^{n+1}$-equivalent (and thus have same $\mathcal{AC}_\varepsilon$ energy), the Borel sets $\{u_1^* \ge t\}$ and $\{u_2^* \ge t\}$ will be $\mathcal{H}^n$-equivalent for every $t \in [0,1]$.

All this said, we come to introduce the following family of **Allen–Cahn Plateau problems**,

$$\begin{aligned}\Upsilon(v,\varepsilon,\delta) \;=\; \inf\big\{\mathcal{AC}_\varepsilon(u;\Omega)/2 :\;\; & u \in W^{1,2}_{\rm loc}(\Omega)\,,\, \mathcal{V}(u;\Omega) = v\,, \\ & \{u^* \ge t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for every } t \in (1/2,\delta)\big\}\,,\end{aligned} \qquad (1.11)$$

where v and ε are positive parameters and where $\delta \in (1/2, 1]$.

For arbitrary values of (v,ε,δ), we do not expect minimizers of $\Upsilon(v,\varepsilon,\delta)$ to have anything to do with soap films. In other words, we need to identify a **soap film regime** for (v,ε,δ). A first constraint is that v should not be too large with respect to the size of the boundary wire frame $\mathbf{W}$: indeed, we want to avoid the "isoperimetric regime", where minimizers will tend to look like droplets touching $\mathbf{W}$, rather than like soap films (see [MN22]). A second constraint, borne out by heuristic calculations[4] involving the optimal Allen-Cahn profile, and aimed at ensuring the boundedness of the minimum energy at small values of v and ε, is that $\varepsilon \ll v$. Correspondingly, given positive $\tau_0 \ge \tau_1 > 0$, we

[2] The lower bound $t > 1/2$ is assumed here for the sake of definiteness. It could have been replaced by any other positive lower bound since the condition of being $\mathcal{C}$-spanning is monotone by set inclusion.

[3] An alternative approach would have course been working on $W^{1,2} \cap C^0$ and the original definition by Harrison and Pugh. Since this approach requires proving the regularity of minimizers in the process of showing their existence, it seems somehow conceptually less direct and certainly less flexible than first discussing a robust weak formulation, and then proving regularity statements.

[4] In (3.21) it is rigorously proved that $\Upsilon(v(\varepsilon),\varepsilon,\delta) \to +\infty$ if $v(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0^+$, so that one definitely wants to require, to the least, that $\varepsilon \le C\,v$.

introduce the family of triples $(v,\varepsilon,\delta)\in(0,\infty)\times(0,\infty)\times(1/2,1]$ defined by

$$\mathrm{SFR}(\tau_0,\tau_1)=\Big\{(v,\varepsilon,\delta)\ :0<\frac{v}{(\mathrm{diam}\,\mathbf{W})^{n+1}}\le\tau_0\,,\quad 0<\varepsilon\,(\mathrm{diam}\,\mathbf{W})^n\le\tau_1\,v\,,\\ \min\Big\{1-\delta,\frac{v}{(\mathrm{diam}\,\mathbf{W})^{n+1}}\Big\}\le\tau_1\Big\}\,.\tag{1.12}$$

Given $\tau_0>0$ we will work with τ_1 sufficiently small in terms of τ_0 (and $\mathbf{W}$, $\mathcal{C}$ and W). From this viewpoint, the third constraint defining SFR reflects the fact that if we want to keep v "of order one", then, in order to be close to the soap film capillarity model with bulk spanning condition $\Psi_{\rm bk}(v)$, we need δ to be sufficiently close to 1; if, otherwise, we wish to keep the possibility of working with δ close to $1/2$ (thus imposing the spanning condition only on a thin layer of level sets around $t=1/2$), then we will need to work with v sufficiently small.

1.5. **Main results for the diffused interface model.** We are now in the position of formally stating the main results of our paper.

Theorem 1.2. *If $\mathbf{W}\subset\mathbb{R}^{n+1}$ is the closure of an open bounded set with smooth boundary, $\mathcal{C}$ is a spanning class for $\mathbf{W}$ such that $\ell<\infty$, $\tau_0>0$, and $W\in C^{2,1}[0,1]$ satisfies* (1.9) *and* (1.10)*, then there exists $\tau_1>0$, depending on W, $\mathbf{W}$, $\mathcal{C}$, and τ_0 with the following properties:*

(i) Existence of minimizers: *if $(v,\varepsilon,\delta)\in\mathrm{SFR}(\tau_0,\tau_1)$, then there are minimizers u of $\Upsilon(v,\varepsilon,\delta)$, which, for suitable $\lambda\in\mathbb{R}$, satisfy*

$$\int_\Omega \mathrm{ac}_\varepsilon(u)\,\mathrm{div}\,X-2\,\varepsilon\nabla u\,\cdot\,\nabla X[\nabla u]=\lambda\int_\Omega V(u)\,\mathrm{div}\,X\,,\tag{1.13}$$

whenever $X\in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$;

(ii) Convergence to bulk-spanning capillarity: *if $\varepsilon_j\to0^+$, $v_j\to v_0>0$, and $\delta_j\to1^-$ as $j\to\infty$, and if u_j are minimizers of $\Upsilon(v_j,\varepsilon_j,\delta_j)$, then there is a minimizer (K,E) of $\Psi_{\rm bk}(v_0)$ such that, up to extracting subsequences, $u_j\to1_E$ in $L^1(\Omega)$ and*

$$\frac{\mathrm{ac}_{\varepsilon_j}(u_j)}{2}\,\mathcal{L}^{n+1}\llcorner\Omega\overset{*}{\rightharpoonup}2\,\mathcal{H}^n\llcorner\big(K\cap E^{(0)}\big)+\mathcal{H}^n\llcorner\partial^*E$$

as Radon measures in Ω. In particular, for every $v_0>0$,

$$\lim_{\mathrm{SFR}(\tau_0,\tau_1)\ni(v,\varepsilon,\delta)\to(v_0,0,1)}\Upsilon(v,\varepsilon,\delta)=\Psi_{\rm bk}(v_0)\,;$$

(iii) Convergence to the Plateau problem: *if $v_j\to0^+$, $\varepsilon_j/v_j\to0^+$, and $\delta_j\to\delta_0\in[1/2,1]$ as $j\to\infty$, and if u_j are minimizers of $\Upsilon(v_j,\varepsilon_j,\delta_j)$, then there is a minimizer S of $2\,\ell=\Psi_{\rm bk}(0)$ such that, up to extracting subsequences,*

$$\frac{\mathrm{ac}_{\varepsilon_j}(u_j)}{2}\,\mathcal{L}^{n+1}\llcorner\Omega\overset{*}{\rightharpoonup}2\,\Phi(\delta_0)\,\mathcal{H}^n\llcorner S$$

as Radon measures in Ω and $\Upsilon(v_j,\varepsilon_j,\delta_j)\to2\,\Phi(\delta_0)\,\ell$, as $j\to\infty$;

(iv) Equipartition of energy: *in both conclusions (ii) and (iii), we also have*

$$\lim_{j\to\infty}\varepsilon_j\int_\Omega|\nabla u_j|^2=\lim_{j\to\infty}\frac{1}{\varepsilon_j}\int_\Omega W(u_j)\,.\tag{1.14}$$

Theorem 1.2 establishes the existence of minimizers of $\Upsilon(v,\varepsilon,\delta)$ if the soap film regime and organizes problems ℓ, $\Psi_{\rm bk}(v)$, and $\Upsilon(v,\varepsilon,\delta)$ into a *hierarchy of Plateau problems.* The first two problems corresponds to modeling soap films as dry or wet accordingly to the physics descriptions given in [WH99, CCAE+13], while the last problem can be used to provide a diffused interface approximation of both problems which is of definite mathematical interest both from the theoretical and the numerical viewpoint.

Theorem 1.2 does not discuss which Allen–Cahn-type equation is solved by minimizers of $\Upsilon(v,\varepsilon,\delta)$, nor discusses any qualitative property of such minimizers, like their regularity, but for their convergence as Radon measures to minimizers of $\Psi_{\rm bk}(v)$ and ℓ. In the following theorem we answer the first question (at distributional level) and derive some basic regularity properties.

Theorem 1.3 (Euler–Lagrange equation for minimizers of $\Upsilon(v,\varepsilon,\delta)$)**.** *Let $\mathbf{W}\subset\mathbb{R}^{n+1}$ be compact and let $\mathcal{C}$ be a spanning class for $\mathbf{W}$. If u is a minimizer of $\Upsilon(v,\varepsilon,\delta)$ for some $v>0$, $\varepsilon>0$ and $\delta\in(1/2,1]$, then, in the sense of distributions, we have (with λ as in (1.13)),*

$$(\delta-u)\left\{2\varepsilon^2\Delta u-W'(u)-\varepsilon\lambda V'(u)\right\}=0\,,\qquad \textit{in }\Omega\,;\tag{1.15}$$

that is, for every $\varphi\in C^\infty_c(\Omega)$,

$$2\,\varepsilon\int_\Omega|\nabla u|^2\,\varphi=\int_\Omega(\delta-u)\left\{2\,\varepsilon\,\nabla u\cdot\nabla\varphi+\Big(\frac{W'(u)}{\varepsilon}-\lambda\,V'(u)\Big)\,\varphi\right\},\tag{1.16}$$

In particular, u is lower-semicontinuous in Ω. Moreover, if Ω' is a connected component of Ω, then either $u\equiv 0$ on Ω', or $u>0$ in Ω'; and, if $\delta<1$, then $u<1$ in Ω.

The relation between (1.1) and (1.15) is clarified in the following *conditional* regularity statement.

Proposition 1.4 (Strong form of the Euler–Lagrange equation)**.** *Under the assumptions of Theorem 1.3:*

(i): *if u is continuous in Ω, then $u\in C^{3,\alpha}_{\rm loc}(\{u\ne\delta\})$ for every $\alpha<\min\{1,2/n\}$;*

(ii): *if in addition $|\{u=\delta\}|=0$, then*

$$\lim_{t\to0^+}\int_{\partial^*\{u>\delta+t\}}|\nabla u|\,(X\cdot\nabla u)\,d\mathcal{H}^n-\int_{\partial^*\{u<\delta-t\}}|\nabla u|\,(X\cdot\nabla u)\,d\mathcal{H}^n=0\,,\tag{1.17}$$

where the limit is taken along those values of $t>0$ such that $\{u>\delta+t\}$ and $\{u<\delta-t\}$ are sets of finite perimeter (i.e., a.e. $t>0$);

(iii): *if in addition $u(x_0)=\delta$, $\{u=\delta\}$ is a C^1-hypersurface in a neighborhood U of x_0 with unit normal $\nu\in C^0(\{u=\delta\}\cap U;\mathbb{S}^n)$, and $u\in C^1(\{u\le\delta\}\cap U)\cap C^1(\{u\ge\delta\}\cap U)$, then*

$$|\partial^+_\nu u(x)|=|\partial^-_\nu u(x)|\,,\qquad\forall x\in\{u=\delta\}\cap U\,,$$

where we have set

$$\partial^\pm_\nu u(x)=\lim_{t\to0^+}\frac{u(x\pm t\,\nu(x))-u(x)}{t}\,.$$

1.6. **Plateau's laws and free boundary problems.** We finally describe some future directions that naturally stem from the main results of this paper, and that generally concern the study of Plateau's laws in the context of free boundary problems.

A first natural class of problems concerns the regularity of solutions to (1.15) needed to trigger Proposition 1.4. For example, continuity of minimizers (conditional assumption (i)) is expected in general, and, indeed, it is possible to show that minimizers in the planar case $n=1$ are locally Hölder continuous in Ω. The regularity of the free boundaries $\{u=\delta\}$ seems also very interesting. Heuristic considerations (based on the maximum principle) suggest that $\{u=\delta\}$ should always have zero Lebesgue measure (conditional assumption (ii)), but, in general, we definitely do not expect $\{u=\delta\}$ to a be a C^1-hypersurface (conditional assumption (iii)). It is actually natural to conjecture that, in physical dimensions $n=1$ and $n=2$, $\{u=\delta\}$ should obey Plateau's laws. Should this be correct, do solutions u to (1.15) have canonical blow-ups at Y-points and T-points of $\{u=\delta\}$?

A second type of problem concerns the precise description of solutions to (1.15). In this direction, the first problem one wants to solve is the construction, as small modifications of some well-prepared *Ansatz*, of solutions to (1.1) that converge to a Y-cone in $\mathbb{R}^2$, or to a Y-cone or a T-cone in $\mathbb{R}^3$. This kind of result should elucidate several interesting points, like what should be the characteristic length scales of the transition regions of u_j and of the $\{u_j \approx 1\}$-regions depicted in Figure 1.1. In turn, once these fundamental examples have been understood, one would like to prove such qualitative properties for generic minimizers of $\Upsilon(v, \varepsilon, \delta)$.

1.7. **Organization of the paper.** In Section 2 we gather the main results from [MNR23] that concern measure theoretic homotopic spanning. These results are used in Section 3 to prove some closure theorems for densities u satisfying homotopic spanning conditions. In Section 4 we discuss the approximation of "wet soap films", meant as competitors in $\Psi_{\rm bk}(v)$, by competitors in $\Upsilon(v, \varepsilon, \delta)$. Section 5 contains one of the more delicate arguments of the paper, where we prove that the Lagrange multipliers λ_j of minimizers u_j of $\Upsilon(v_j, \varepsilon_j, \delta_j)$ are such that $\varepsilon_j\,\lambda_j \to 0^+$ whenever $v_j \to 0^+$, $\varepsilon_j \to 0^+$, $\varepsilon_j/v_j \to 0^+$ and $\delta_j \to \delta_0 \in [1/2, 1]$. Finally, in Section 6 and Section 7 we prove Theorem 1.2 and Theorem 1.3 (plus Proposition 1.4), respectively.

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2. Measure theoretic homotopic spanning

2.1. **Sets of finite perimeter, rectifiable sets, and essential disconnection.** We generally adopt the (quite common) terminology and notation of [Mag12] for what concerns rectifiable sets and sets of finite perimeter. Given a locally $\mathcal{H}^k$-finite set S in $\mathbb{R}^{n+1}$, we define the **rectifiable part** $\mathcal{R}(S)$ and the **unrectifiable part** $\mathcal{P}(S)$ of S as in [Sim83, 13.1]. Given a Borel set $E \subset \mathbb{R}^{n+1}$, we denote by $E^{(t)}$, $t \in [0,1]$, the **points of density** t of E, by $\partial^* E$ the reduced boundary of E (defined as the largest open set A wherein E is of locally finite perimeter – it could of course be $A = \varnothing$), and by $\nu_E : \partial^* E \to \mathbb{S}^n$ the measure theoretic outer unit normal to E. We shall repeatedly use that if E is of finite perimeter in Ω, then $\Omega \cap \partial^* E \subset E^{(1/2)} \subset \partial^e E$ where $\partial^e E = \mathbb{R}^{n+1} \setminus (E^{(0)} \cup E^{(1)})$ is the essential boundary of E, as well as the theorem by Federer stating that

$$\Omega \text{ is } \mathcal{H}^n\text{-contained in } E^{(0)} \cup E^{(1)} \cup (\Omega \cap \partial^* E)\,, \tag{2.1}$$

and, in particular, that $\Omega \cap \partial^* E$ is $\mathcal{H}^n$-equivalent to $\Omega \cap \partial^e E$.

We also recall the following notion of what it means for a Borel set K to disconnect a Borel set G, originating in the study of rigidity for symmetrization inequalities [CCDPM17, CCDPM14], and lying at the heart of the notion of measure theoretic homotopic spanning. Given Borel sets K and G, we say that K **essentially disconnects** G if there is a Lebesgue partition $\{G_1, G_2\}$ of G (i.e., $|G \Delta (G_1 \cup G_2)| = 0$, $|G_1 \cap G_2| = 0$) which is non-trivial (i.e., $|G_1|\,|G_2| > 0$) and such that

$$G^{(1)} \cap \partial^e G_1 \cap \partial^e G_2 \text{ is } \mathcal{H}^n\text{-contained in } K\,.$$

(Notice that $G^{(1)} \cap \partial^e G_1 \cap \partial^e G_2 = G^{(1)} \cap \partial^e G_k$ for every $k = 1, 2$.) For example, if $J \subset (0,1)$ with $\mathcal{L}^1(J) = 1$, then $K = J \times \{0\}$ essentially disconnects the open unit disk B_1^2 of $\mathbb{R}^2$ (although, evidently, $B_1^2 \setminus K$ will be connected in topological terms as soon as $(0,1) \setminus J \neq \varnothing$). We say that G is **essentially connected** when $\varnothing$ does not essentially disconnect G. In the special case when G is of finite perimeter, being essentially connected is the same as being **indecomposable** (according to the terminology introduced in [ACMM01]).

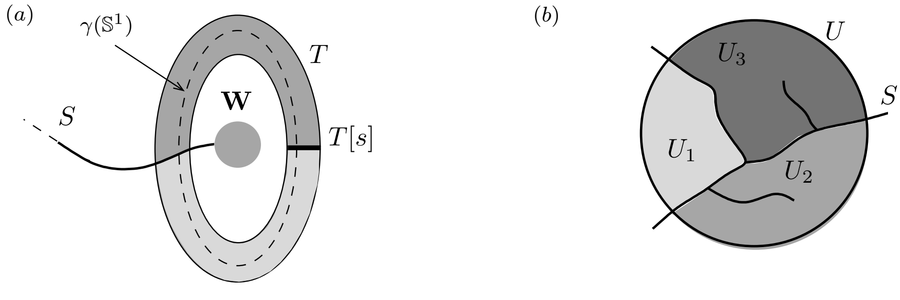

FIGURE 2.1. (a) In the original homotopic spanning condition, S has to intersect $\gamma(\mathbb{S}^1)$; in the new measure theoretic version, $S \cup T[s]$ is (roughly speaking) required to essentially disconnect T (for a.e. $s \in \mathbb{S}^1$); (b) The induced essential partition $\{U_1, U_2, U_3\}$ by S on a disk U. Notice that the tendrils of S that do not contribute to bounding some subset of U do not contribute to the boundaries of the the U_i's, and are thus not part of UBEP$(S; U)$.

2.2. **Homotopic spanning and induced essential partitions.** We now recall the measure theoretic notion of homotopic spanning introduced in [MNR23]; see Figure 2.1-(a) for an illustration. Given a closed set $\mathbf{W} \subset \mathbb{R}^{n+1}$ and a spanning class $\mathcal{C}$ for $\mathbf{W}$, the **tubular spanning class** $\mathcal{T}(\mathcal{C})$ associated to $\mathcal{C}$ is the family of triples (γ, Φ, T) such that $\gamma \in \mathcal{C}$, $T = \Phi(\mathbb{S}^1 \times B_1^n) \subset\subset \Omega$, and (setting $B_1^n = \{x \in \mathbb{R}^n : |x| < 1\}$) $\Phi : \mathbb{S}^1 \times \mathrm{cl}\,(B_1^n) \to \mathrm{cl}\,(T)$ is a diffeomorphism with $\Phi(s, 0) = \gamma(s)$ for every $s \in \mathbb{S}^1$. Given $s \in \mathbb{S}^1$ we set

$$T[s] = \Phi(\{s\} \times B_1^n)$$

for the **slice of** T corresponding to $s \in \mathbb{S}^1$. Finally, we say that a Borel set $S \subset \Omega$ is **$\mathcal{C}$-spanning W** if for each $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$, $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$ has the following property:

$$\begin{aligned} &\text{for } \mathcal{H}^n\text{-a.e. } x \in T[s] \\ &\exists \text{ a partition } \{T_1, T_2\} \text{ of } T \text{ s.t. } x \in \partial^e T_1 \cap \partial^e T_2 \\ &\text{and s.t. } S \cup T[s] \text{ essentially disconnects } T \text{ into } \{T_1, T_2\}\,. \end{aligned} \tag{2.2}$$

As proved in [MNR23, Theorem A.1], as soon as S is closed in Ω, the notion of $\mathcal{C}$-spanning just introduced is equivalent to the one of Harrison and Pugh. The dependency of the partition $\{T_1, T_2\}$ on $x \in T[s]$ has a subtle reason, see [MNR23, Figure A.1].

Now, in the study of soap films, condition (2.2) is only applied to sets S that are either locally $\mathcal{H}^n$-finite in Ω, or that are the bulk $E^{(1)} \cup (\Omega \cap \partial^* E)$ of a set E of finite perimeter in Ω, or are a combination of these two cases, in the sense that $S = K \cup E^{(1)}$ for some $(K, E) \in \mathcal{K}_{\mathrm{B}}$ (the first two cases are then obtained by taking $S = K \cup E^{(1)}$ with either $E = \varnothing$ or $K = \Omega \cap \partial^* E$). In all these cases the geometric meaning of (2.2) can be greatly elucidated using the following results concerning partitions into indecomposable sets of finite perimeter.

Given Borel sets $S, U \subset \mathbb{R}^{n+1}$, a Lebesgue partition $\{U_i\}_i$ of U (that is, $U_i \subset U$ with $|U \setminus \bigcup_i U_i| = 0$ with $|U_i \cap U_j| = \varnothing$ if $i \neq j$) is **induced by** S if, for each i,

$$U^{(1)} \cap \partial^e U_i \text{ is } \mathcal{H}^n\text{-contained in } S\,. \tag{2.3}$$

The following theorem is [MNR23, Theorem 2.1]:

Theorem 2.1 (Induced essential partitions [MNR23])**.** *If $U \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter and $S \subset \mathbb{R}^{n+1}$ is a Borel set with $\mathcal{H}^n(S \cap U^{(1)}) < \infty$, then there exists a partition $\{U_i\}_i$ of U induced by S such that, for every i,*

$$S \text{ does not essentially disconnect } U_i\,. \tag{2.4}$$

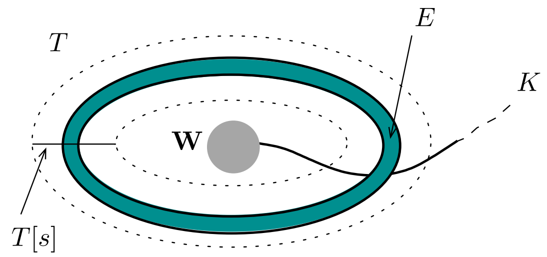

FIGURE 2.2. Condition (2.7) in a situation where $T[s]$ is *not* $\mathcal{H}^n$-contained in $\mathrm{UBEP}(K\cup T[s];T)$ – indeed, E itself is one of the elements of the essential partition of T induced by $K \cup T[s]$ – while at the same time $T[s] \cap E^{(0)}$ (the part of $T[s]$ outside of E) is $\mathcal{H}^n$-contained in $\mathrm{UBEP}(K \cup T[s];T)$. The key feature here is that K contains the boundary of E, but no points inside E, while E contains $\gamma(\mathbb{S}^1)$.

Moreover, if either $S^* = \mathcal{R}(S)$ or S^* is $\mathcal{H}^n$-equivalent to S, and if $\{U_j^*\}_j$ is a partition of U induced by S^* such that S^* does not essentially disconnect U_j^* for every j, then there is a bijection σ such that $|U_i \Delta U^*_{\sigma(i)}| = 0$ for every i. For this reason, $\{U_i\}_i$ is called **the essential partition of U induced by S.**

With S and U as in Theorem 2.1, the **union of the (reduced) boundaries of the essential partition induced by S on U** is uniquely defined as

$$\mathrm{UBEP}(S;U) = U^{(1)} \cap \bigcup_i \partial^* U_i\,, \tag{2.5}$$

see Figure 2.1-(b), and correspondingly we can formulate the following characterization of measure-theoretic homotopic spanning, cf. with [MNR23, Theorem 3.1].

Theorem 2.2 ([MNR23])**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is a closed set in $\mathbb{R}^{n+1}$, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, and $(K,E) \in \mathcal{K}_{\mathrm{B}}$, then*

$$\mathcal{R}(K) \cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \tag{2.6}$$

if and only if, for every $(\gamma,\Phi,T) \in \mathcal{T}(\mathcal{C})$ and $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$,

$$T[s] \cap E^{(0)} \text{ is } \mathcal{H}^n\text{-contained in } \mathrm{UBEP}(K \cup T[s];T)\,; \tag{2.7}$$

see Figure 2.2.

Remark 2.3. An immediate corollary of Theorem 2.2 is that if K is $\mathcal{H}^n$-finite and $(K,E) \in \mathcal{K}_{\mathrm{B}}$ then $K \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$ if and only if $\mathcal{R}(K) \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$. Indeed, $\mathcal{R}(K \cup T[s]) = \mathcal{R}(K) \cup T[s]$, so that, by (2.5), $\mathrm{UBEP}(K \cup T[s]) = \mathrm{UBEP}(\mathcal{R}(K) \cup T[s])$.

2.3. Closure theorems for homotopic spanning.

We finally state the two closure theorems for homotopically spanning sets that make the above definitions useful in the study of minimization problems. The first result corresponds to a particular case of [MNR23, Theorem 1.4]:

Theorem 2.4 ([MNR23])**.** *Let $\mathbf{W}$ be a closed set in $\mathbb{R}^{n+1}$, $\mathcal{C}$ be a spanning class for $\mathbf{W}$, and $\{(K_j,E_j)\}_j$ be a sequence in $\mathcal{K}_{\mathrm{B}}$ such that*

$$K_j \cup E_j^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \qquad \sup_j \mathcal{H}^n(K_j) < \infty\,.$$

Let E be a Borel set and μ_{bk} be a Radon measure in Ω such that, as $j \to \infty$, $E_j \stackrel{\mathrm{loc}}{\to} E$ and

$$\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_j) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_j) \cap E_j^{(0)}) \stackrel{*}{\rightharpoonup} \mu_{\mathrm{bk}}\,,$$

as Radon measures in Ω. Then the set

$$K_{\mathrm{bk}} := \big(\Omega \cap \partial^* E\big) \cup \Big\{x \in \Omega \cap E^{(0)} : \theta^n_*(\mu_{\mathrm{bk}})(x) \geq 2\Big\}\,,$$

is such that $(K_{\rm bk},E)\in\mathcal{K}_{\rm B}$, $K_{\rm bk}\cup E^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*, and*

$$\liminf_{j\to\infty}\mathcal{F}_{\rm bk}(K_j,E_j)\ge\mathcal{F}_{\rm bk}(K_{\rm bk},E)\,.$$

The second closure theorem we shall need from [MNR23] requires the introduction of some additional terminology. Given an open set Ω and an $\mathcal{H}^n$-finite subset S of Ω, we define the **essential spanning part** ${\rm ESP}(S)$ **of** S **in** Ω as the $\mathcal{H}^n$-rectifiable set defined by

$${\rm ESP}(S)=\bigcup_k {\rm UBEP}(S;\Omega_k)=\bigcup_k\Big\{\Omega_k\cap\bigcup_i\partial^*U_i[\Omega_k]\Big\}\,, \tag{2.8}$$

where $\{\Omega_k\}_k$ is the open covering of Ω defined by

$$\{\Omega_k\}_k=\{B_{r_{mh}}(x_m)\}_{m,h}\,, \tag{2.9}$$

where $\{x_m\}_m=\mathbb{Q}^{n+1}\cap\Omega$ and $\{r_{mh}\}_h=\mathbb{Q}\cap(0,{\rm dist}(x_m,\partial\Omega))$, and where $\{U_i[\Omega_k]\}_i$ denotes the essential partition of Ω_k induced by S. In light of Theorem 2.2, the intuition behind this definition is that, by adding up all the unions of boundaries of essential partitions induced by S over smaller and smaller balls we are capturing all the parts of S that may potentially contribute to achieve a spanning condition on $\mathbf{W}=\mathbb{R}^{n+1}\setminus\Omega$. It is thus natural to expect that ${\rm ESP}(S)$ is $\mathcal{C}$-spanning $\mathbf{W}$ whenever S is. This is correct, and follows indeed as a particular case of Theorem 2.5 below. The more general situation considered in Theorem 2.5 requires the introduction of a notion of subsequential limit for $\{{\rm ESP}(S_j)\}_j$. More precisely, given a sequence $\{S_j\}_j$ of Borel subsets of Ω such that $\sup_j\mathcal{H}^n(S_j)<\infty$, we say that S is a **subsequential partition limit of** $\{S_j\}_j$ **in** Ω if

$$S=\bigcup_k\Big\{\Omega_k\cap\bigcup_i\partial^*U_i[\Omega_k]\Big\}\,, \tag{2.10}$$

where $\{U_i[\Omega_k]\}_i$ is a Lebesgue partition of Ω_k such that, denoting by $\{U_i^j[\Omega_k]\}_i$ the essential partition of Ω_k induced by S_j, and up to extracting a subsequence in j, for every i and k we have $|U_i^j[\Omega_k]\Delta U_i[\Omega_k]|\to0$ as $j\to\infty$. The natural expectation is of course that if each S_j is $\mathcal{C}$-spanning $\mathbf{W}$, then every subsequential partition limit S of $\{S_j\}_j$ should be $\mathcal{C}$-spanning $\mathbf{W}$ too. The next theorem, which corresponds to [MNR23, Theorem 5.1], proves this and, actually, an even more general fact:

Theorem 2.5 ([MNR23])**.** *Let* $\mathbf{W}$ *be a closed set in* $\mathbb{R}^{n+1}$, $\mathcal{C}$ *a spanning class for* $\mathbf{W}$*, and* $\{(K_j,E_j)\}_j$ *a sequence in* $\mathcal{K}_{\rm B}$ *such that* $\sup_j\mathcal{H}^n(K_j)<\infty$ *and* $K_j\cup E_j^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for every* j*.*

If S_0 *and* E_0 *are, respectively, a subsequential partition limit of* $\{K_j\}_j$ *in* Ω *and an* L^1*-subsequential limit of* $\{E_j\}_j$ *(corresponding to a same not relabeled subsequence in* j*), then the set*

$$K_0=(\Omega\cap\partial^*E_0)\cup S_0\,,$$

is such that $(K_0,E_0)\in\mathcal{K}_{\rm B}$ *and* $K_0\cup E_0^{(1)}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*. In particular:*

(i): *if* S *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*, then* ${\rm ESP}(S)$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*;*

(ii): *if* S_j *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for each* j *and* S *is a subsequential partition limit of* $\{S_j\}_j$ *in* Ω*, then* S *is* $\mathcal{C}$*-spanning* $\mathbf{W}$*.*

3. Closure theorems for homotopically spanning diffused interfaces

3.1. The precise representative of a Sobolev function.

Given an open set Ω and a Lebesgue measurable function $u:\Omega\to\mathbb{R}\cup\{\pm\infty\}$ the approximate upper and lower limits of u at $x\in\Omega$ are defined by

$$u^+(x)=\inf\big\{t\in\mathbb{R}:x\in\{u>t\}^{(0)}\big\}\,,\qquad u^-(x)=\sup\big\{t\in\mathbb{R}:x\in\{u<t\}^{(0)}\big\}\,,$$

(where $\{u > t\} = \{x \in \Omega : u(x) > t\}$). Both u^+ and u^- are Borel functions on Ω, with values in $\mathbb{R} \cup \{\pm\infty\}$, and their value *at any point in* Ω does not depend on the Lebesgue representative of u. It is easily seen that

$$\begin{aligned} &\{u^+ < t\} \subset \{u < t\}^{(1)} \subset \{u^+ \le t\}\,, \qquad \{u^+ < t\}^{(1)} = \{u < t\}^{(1)}\,,\\ &\{u^- > t\} \subset \{u > t\}^{(1)} \subset \{u^- \ge t\}\,, \qquad \{u^- > t\}^{(1)} = \{u > t\}^{(1)}\,. \end{aligned} \tag{3.1}$$

The approximate jump of u is the Borel function $[u] : \Omega \to [0, \infty]$ defined by $[u] = u^+ - u^-$. Setting $\Sigma_u = \{x \in \Omega : [u](x) > 0\}$, we define the precise representative $u^* : \Omega \setminus \Sigma_u \to \mathbb{R} \cup \{\pm\infty\}$ by taking

$$u^* = u^+ = u^- \qquad \text{on } \Omega \setminus \Sigma_u\,.$$

Since it always hold that $|\Sigma_u| = 0$, it turns out that u^+, u^- and u^* are all Lebesgue representatives of u. If $u \in BV_{\rm loc}(\Omega)$, then the distributional derivative Du of u can be decomposed as $Du = \nabla u\, d\mathcal{L}^{n+1} + [u]\, d\mathcal{H}^n \llcorner \Sigma_u + D^c u$ (where $\nabla u \in L^1_{\rm loc}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ and $D^c u$ is the Cantorian part of Du). In particular, if $u \in W^{1,1}_{\rm loc}(\Omega)$, then $\mathcal{H}^n(\Sigma_u) = 0$. We shall repeatedly use the following fact: if $u \in W^{1,1}_{\rm loc}(\Omega)$, then, for a.e. t, $\{u > t\}$ is a set of finite perimeter in Ω (this is immediate from the coarea formula), is Lebesgue equivalent to $\{u \ge t\}$ (and thus such that $\Omega \cap \partial^*\{u > t\} = \Omega \cap \partial^*\{u \ge t\}$), and satisfies

$$\Omega \cap \partial^*\{u > t\} \text{ is } \mathcal{H}^n\text{-equivalent to } \{u^* = t\}\,. \tag{3.2}$$

To prove (3.2) we notice that if $x \in \partial^*\{u > t\} \subset \{u > t\}^{(1/2)}$, then $x \notin \{u > t\}^{(0)}$, hence $u^+(x) \ge t$; similarly, if $x \in \partial^*\{u < t\} \subset \{u < t\}^{(1/2)}$, then $x \notin \{u < t\}^{(0)}$, and thus $t \ge u^-(x)$; since $\{u > t\}$ is $\mathcal{L}^{n+1}$-equivalent to $\Omega \setminus \{u < t\}$ for a.e. t, we find that $\Omega \cap \partial^*\{u > t\} = \Omega \cap \partial^*\{u < t\}$ for a.e. t, and thus

$$\Omega \cap \partial^*\{u > t\} = \{u^+ \ge t\} \cap \{u^- \le t\}\,, \qquad \text{for a.e. } t\,.$$

In particular, (3.2) follows from $\mathcal{H}^n(\Sigma_u) = 0$. We finally notice that if $u \in W^{1,1}_{\rm loc}(\Omega)$ and E is a set of finite perimeter in Ω, then u^* is such that

$$\int_E u\, \nabla\varphi = -\int_E \varphi\, \nabla u + \int_{\Omega \cap \partial^* E} \varphi\, u^*\, \nu_E\, d\mathcal{H}^n\,, \tag{3.3}$$

for every $\varphi \in C^\infty_c(\Omega)$; see [AFP00].

3.2. **Closure theorems for homotopically spanning densities.** In this section we consider the following setting

$$\mathbf{W} \subset \mathbb{R}^{n+1} \text{ is closed and } \mathcal{C} \text{ is a spanning class for } \mathbf{W}\,, \tag{3.4}$$

$$\{u_j\}_j \subset W^{1,2}(\Omega) \text{ with } \sup_j \mathcal{AC}_{\varepsilon_j}(u_j) < \infty \text{ and } u_j \to u \text{ in } L^1_{\rm loc}(\Omega)\,, \tag{3.5}$$

$$\{u_j^* \ge t\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ for all } t \in (1/2, \delta_j)\,, \tag{3.6}$$

$$\varepsilon_j > 0,\ \delta_j \in (1/2, 1],\ \varepsilon_j \to \varepsilon_0 \ge 0, \text{ and } \delta_j \to \delta_0 \in [1/2, 1]\,, \tag{3.7}$$

where the limits hold as $j \to \infty$ and where $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$. We discuss the problem of showing that the spanning condition (3.6) is transferred from u_j to u. We consider separately the cases when $\varepsilon_0 > 0$ (Theorem 3.2) or $\varepsilon_0 = 0$ (Theorem 3.4). In both cases we use the following lemma.

Lemma 3.1. *Let* $\mathbf{W}$, $\mathcal{C}$, u_j, ε_j, *and* δ_j *satisfy* (3.4), (3.5), (3.6), *and* (3.7).

If $I_0 \subset (0, \delta_0)$ *is a closed interval of positive length, then there is* $\{t_j\}_j \subset I_0$ *such that*

$$t_j \to t_0 \in I_0\,, \qquad E_j := \{u_j > t_j\} \overset{\rm loc}{\to} E_0\,,$$

as $j \to \infty$, *where* E_0 *is a set of finite perimeter in* Ω *and where* $\{\Omega \cap \partial^* E_j\}_j$ *has a subsequential partition limit* S_0 *in* Ω *such that*

$$S_0 \cup (\Omega \cap \partial^* E_0) \cup E_0^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,. \tag{3.8}$$

Proof. Indeed, by the co-area formula and (MM)

$$\int_{\Phi(I_0)} P(\{\Phi\circ u_j > s\};\Omega)\,ds = C(I_0)\int_\Omega |\nabla(\Phi\circ u_j)| \le C(I_0)\,\mathcal{AC}_{\varepsilon_j}(u_j;\Omega)\,,$$

so that $\sup_j \mathcal{AC}_{\varepsilon_j}(u_j;\Omega) < \infty$ and the strict monotonicity of $\Phi(t) = \int_0^t \sqrt{W}$ imply the existence of $\{t_j\}_j \subset I_0$ such that $\{P(E_j;\Omega)\}_j$ is bounded, where $E_j = \{u_j > t_j\}$ and, by (3.1),

$$\{u_j^* > t\} \subset \{u_j^- > t\} \subset \{u_j > t\}^{(1)} \subset E_j^{(1)}\,, \qquad \forall t > t_j\,. \tag{3.9}$$

Since $\sup_j P(E_j,\Omega) < \infty$ we can find $\{t_j\}_j \subset I_0$ and limits t_0, E_0 and S_0 as in the statement, and are left to prove (3.8). Indeed, since $\delta_j \to \delta_0$ and $I_0 \subset (0,\delta_0)$ is closed, we have $t_j < \delta_j$ for j large enough. In particular, by exploiting (3.9) with $t \in (t_j,\delta_j)$ we find that $E_j^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, so that (3.1) implies that $E_j^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$. We can thus apply Theorem 2.5 to $\{(K_j,E_j)\}_j$ with $K_j = \Omega\cap\partial^* E_j$ and conclude the proof. □

Theorem 3.2. *Let* $\mathbf{W}$, $\mathcal{C}$, u_j, ε_j, *and* δ_j *satisfy* (3.4), (3.5), (3.6), *and* (3.7). *If*

$$\varepsilon_0 > 0\,, \qquad \delta_0 > \frac{1}{2}\,,$$

then $u \in W^{1,2}_{\rm loc}(\Omega)$ *and* $\{u^* \ge t\}$ *is* $\mathcal{C}$*-spanning* $\mathbf{W}$ *for all* $t \in (1/2,\delta_0)$.

Remark 3.3. Notice that in the situation of Theorem 3.2 it must be that $\mathcal{V}(u;T) > 0$ for every $(\gamma,\Phi,T) \in \mathcal{T}(\mathcal{C})$. Indeed, $\mathcal{V}(u;T) = 0$ would imply $\{u^* \ge t\}\cap T = \varnothing$ for every $t > 0$, and $\{u^* \ge t\}$ could not be $\mathcal{C}$-spanning $\mathbf{W}$ for any $t \in (1/2,\delta_0)$, a contradiction. As a consequence,

$$\liminf_{(v,\varepsilon,\delta)\to(0,\varepsilon_0,\delta_0)} \Upsilon(v,\varepsilon,\delta) = +\infty\,, \qquad \varepsilon_0 > 0\,, \delta_0 \in (1/2,1]\,, \tag{3.10}$$

that is to say, *the vanishing volume limit of* Υ *with non-vanishing phase transition length is always degenerate.*

Proof of Theorem 3.2. Since $\varepsilon_0 > 0$, (3.5) implies that $u_j \rightharpoonup u$ in $W^{1,2}_{\rm loc}(\Omega)$, and in particular that $u \in W^{1,2}_{\rm loc}(\Omega)$. Given $N \in \mathbb{N}$, let us apply Lemma 3.1 to the interval $I_0 = [\delta_0 - 2/N, \delta_0 - 1/N]$, and correspondingly find $\{t_j\}_j \subset I_0$ such that $t_j \to t_0 \in I_0$, $E_j = \{u_j > t_j\} \overset{\rm loc}{\to} E_0$, $\{\Omega\cap\partial^* E_j\}_j$ has a subsequential partition limit S_0 in Ω, and (3.8) holds. In particular, we can prove the theorem by showing that

$$S_0 \cup (\Omega\cap\partial^* E_0) \cup E_0^{(1)} \overset{\mathcal{H}^n}{\subset} \{u^* \ge t_0\}\,, \tag{3.11}$$

and by then applying the fact that $t_0 = t_0(N) \to \delta_0^-$ as $N \to \infty$. We divide the proof of (3.11) in three parts:

To check that $E_0^{(1)}$ *is* $\mathcal{H}^n$*-contained in* $\{u^* \ge t_0\}$: Up to extract a further subsequence in j, the set

$$E_0^* = \big\{x \in E_0 : 1_{E_j}(x) \to 1 \text{ and } u_j(x) \to u(x) \text{ as } j \to \infty\big\}\,,$$

is Lebesgue equivalent to E_0. If $x \in E_0^*$, then $u_j(x) \ge t_j$ for every $j \ge j(x)$; letting $j \to \infty$ we find $u(x) \ge t_0$, and prove that $E_0^* \subset \{u \ge t_0\}$. In particular, by (3.1),

$$E_0^{(1)} = (E_0^*)^{(1)} \subset \{u \ge t_0\}^{(1)} \subset \{u^- \ge t_0\}$$

and then we find the claimed $\mathcal{H}^n$-containment by intersecting with $\Omega\setminus\Sigma_u$ (and recalling that $\mathcal{H}^n(\Sigma_u) = 0$).

To check that $\Omega\cap\partial^* E_0$ *is* $\mathcal{H}^n$*-contained in* $\{u^* \ge t_0\}$: We combine the general fact that

$$\Omega\cap\partial^* A_2 \overset{\mathcal{H}^n}{\subset} \Omega\cap(\partial^* A_1 \cup A_1^{(1)})\,, \qquad \forall A_1 \subset A_2 \subset \Omega\,,$$

with the inclusion $E_0^{(1)} \subset \{u^* \geq t_0\}$ to find that

$$\Omega \cap \partial^* E_0 = \Omega \cap \partial^* E_0^{(1)} \overset{\mathcal{H}^n}{\subset} \big(\Omega \cap \partial^*\{u^* \geq t_0\}\big) \cup \{u^* \geq t_0\}^{(1)} ,$$

where the latter set is $\mathcal{H}^n$-contained in $\{u^* \geq t_0\}$ thanks to (3.1), (3.2), and $\mathcal{H}^n(\Sigma_u) = 0$.

To check that S_0 is $\mathcal{H}^n$-contained in $\{u^* \geq t_0\}$: Let us recall that given the decomposition $\{\Omega_k\}_k$ of Ω introduced in (2.9), and denoting by $\{U_i^j[\Omega_k]\}_i$ the essential partition of Ω_k induced by $\Omega \cap \partial^* E_j$, the fact that S_0 is a partition limit of $\{\Omega \cap \partial^* E_j\}_j$ means that

$$S_0 = \bigcup_k \Big\{\Omega_k \cap \bigcup_i \partial^* U_i[\Omega_k]\Big\} ,$$

where, for each k, $\{U_i[\Omega_k]\}_i$ is a partition of Ω_k such that $U_i^j[\Omega_k] \to U_i[\Omega_k]$ as $j \to \infty$ and for every i. Thus, if we fix k and consider i such that $\Omega_k \cap \partial^* U_i[\Omega_k] \neq \varnothing$, then it suffices to show that

$$\Omega_k \cap \partial^* U_i[\Omega_k] \text{ is } \mathcal{H}^n\text{-contained in } \{u^* = t_0\}. \tag{3.12}$$

Since $\Omega_k = B_r(x)$ (for some $x \in \Omega$ and $r > 0$) if we set $G_1 = U_i[\Omega_k]$, $G_1^j = U_i^j[\Omega_k]$, $G_2 = B_r(x) \setminus G_1$ and $G_2^j = B_r(x) \setminus G_2^j$, then we see that $\{G_1, G_2\}$ is a non-trivial Borel partition of $B_r(x)$ (indeed $0 < |G_1| < |B_r(x)|$ thanks to $B_r(x) \cap \partial^* G_1 \neq \varnothing$), and $\{G_1^j, G_2^j\}$ is a non-trivial Borel partition of $B_r(x)$ for every j large enough (thanks to $G_1^j, G_2^j \to G_1, G_2$ as $j \to \infty$). In particular,

$$\nu_{G_1} = -\nu_{G_2} \qquad \mathcal{H}^n\text{-a.e. on } B_r(x) \cap \partial^* G_1 = B_r(x) \cap \partial^* G_2 , \tag{3.13}$$

$$\nu_{G_1^j} = -\nu_{G_2^j} \qquad \mathcal{H}^n\text{-a.e. on } B_r(x) \cap \partial^* G_1^j = B_r(x) \cap \partial^* G_2^j . \tag{3.14}$$

Define $L_j : [0,1] \to [0,1]$ by taking $L_j(t) = t$ for $t \in [t_j, 1]$ and L_j to be affine on $[0, t_j]$ with $L_j(0) = 1$ and $L_j(t_j) = t_j$, and similarly $L_0 : [0,1] \to [0,1]$ using t_0 in place of t_j. Since $t_j, t_0 \in I_0 \subset\subset (1/2, \delta_0)$, we have $\mathrm{Lip}(L_0), \mathrm{Lip}(L_j) \leq 1$, and thus $L_0 \circ u, L_j \circ u_j \in W^{1,2}_{\mathrm{loc}}(\Omega)$. If we set

$$z_j = (L_j \circ u_j)\, 1_{G_1^j} + t_j\, 1_{G_2^j} , \qquad z = (L_0 \circ u)\, 1_{G_1} + t_0\, 1_{G_2} , \tag{3.15}$$

then we easily see that $z_j \to z$ in $L^1(B_r(x))$. Moreover, by combining (3.14) and (3.13) with the divergence theorem (3.3) we see that, for every $\varphi \in C_c^\infty(B_r(x))$,

$$\begin{aligned} Dz_j[\varphi] &= -t_j \int_{B_r(x)\cap G_2^j} \nabla\varphi - \int_{B_r(x)\cap G_1^j} (L_j \circ u_j)\, \nabla\varphi \\ &= -t_j \int_{B_r(x)\cap\partial^* G_1^j} \varphi\, \nu_{G_2^j}\, d\mathcal{H}^n - \int_{B_r(x)\cap\partial^* G_1^j} \varphi\, (L_j \circ u_j)^*\, \nu_{G_1^j}\, d\mathcal{H}^n \\ &\quad + \int_{B_r(x)\cap G_1^j} \varphi\, \nabla(L_j \circ u_j) \\ &= \int_{B_r(x)\cap G_1^j} \varphi\, \nabla(L_j \circ u_j) + \int_{B_r(x)\cap\partial^* G_1^j} \varphi \Big\{t_j - (L_j \circ u_j)^*\Big\}\, \nu_{G_1^j}\, d\mathcal{H}^n , \end{aligned} \tag{3.16}$$

and, similarly, that

$$Dz[\varphi] = \int_{B_r(x)\cap G_1} \varphi\, \nabla(L_0 \circ u) + \int_{B_r(x)\cap\partial^* G_1} \varphi \Big\{t_0 - (L_0 \circ u_0)^*\Big\}\, \nu_{G_1}\, d\mathcal{H}^n . \tag{3.17}$$

Now, since, by construction, $K_j = \Omega \cap \partial^* E_j$ essentially disconnects $B_r(x)$ into $\{G_1^j, G_2^j\}$, we have that $B_r(x) \cap \partial^* G_1^j$ is $\mathcal{H}^n$-contained in $B_r(x) \cap \partial^* E_j$, which, in turn, by (3.2), is $\mathcal{H}^n$-contained in $\{u_j^* = t_j\}$; thus, by the Lipschitz continuity of L_j and by $L_j(t_j) = t_j$, we conclude that

$$(L_j \circ u_j)^* = t_j \ \mathcal{H}^n\text{-a.e. on } B_r(x) \cap \partial^* G_1^j . \tag{3.18}$$

Combining (3.16) and (3.18) we conclude that $z_j \in W^{1,2}(B_r(x))$ with

$$\nabla z_j = 1_{G_1^j} \nabla(L_j \circ u_j) = 1_{G_1^j} (L_j' \circ u_j) \nabla u_j \,.$$

As a consequence, $\mathrm{Lip}(L_j) \le 1$, (3.5), and the fact that $\varepsilon_j \to \varepsilon_0 > 0$ combined give $\sup_j \|\nabla z_j\|_{L^2(B_r(x))} < \infty$, and thus, thanks to $z_j \to z$ in $L^1(B_r(x))$, $z \in W^{1,2}(B_r(x))$. In particular $Dz \ll \mathcal{L}^{n+1}$, so that (3.17) implies $(L_0 \circ u)^* = t_0$ $\mathcal{H}^n$-a.e. on $B_r(x) \cap \partial^* G_1$. Since $L_0(t) = t_0$ if and only if $t = t_0$ and L_0 is Lipschitz continuous, this proves that $B_r(x) \cap \partial^* G_1$ is $\mathcal{H}^n$-contained in $\{u^* = t_0\}$. Since this is (3.12), we have concluded the proof of the theorem. □

Theorem 3.4. *Let* $\mathbf{W}$, $\mathcal{C}$, u_j, ε_j, *and* δ_j *satisfy* (3.4), (3.5), (3.6), *and* (3.7), *and assume (as it can always be done up to extracting a further subsequence) that for some* $v_0 \ge 0$ *and* μ *a Radon measure in* Ω, *as* $j \to \infty$, *it holds that*

$$\mathcal{V}(u_j;\Omega) \to v_0 \,, \qquad |\nabla(\Phi \circ u_j)| \, d\mathcal{L}^{n+1} \llcorner \Omega \stackrel{*}{\rightharpoonup} \mu \,.$$

If

$$\varepsilon_0 = 0 \,,$$

then there exists $(K,E) \in \mathcal{K}_{\mathrm{B}}$ *such that*

$$u = 1_E \,, \qquad |E| \le v_0 \,, \qquad K \cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \,,$$

and such that

$$\mu \ge 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner (K \cap E^{(0)}) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) \,. \tag{3.19}$$

Moreover, in the particular case when $v_0 = 0$, *it must be*

$$\liminf_{j\to\infty} \frac{v_j}{\varepsilon_j} > 0 \,. \tag{3.20}$$

Remark 3.5. As a consequence of (3.20) in Theorem 3.4 we see that if $v(\varepsilon)$ and $\delta(\varepsilon)$ are functions of $\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0^+} \frac{v(\varepsilon)}{\varepsilon} = 0 \,, \qquad \lim_{\varepsilon\to 0^+} \delta(\varepsilon) = \delta_0 \in [1/2, 1] \,,$$

then

$$\lim_{\varepsilon\to 0^+} \Upsilon(v(\varepsilon), \varepsilon, \delta(\varepsilon)) = +\infty \,. \tag{3.21}$$

Proof of Theorem 3.4. By $\varepsilon_j \to 0^+$, (MM), and (3.5) there is $E \subset \Omega$ with $|E| \le v_0$ such that $u = 1_E$, $\{u_j > t\} \to E$ as $j \to \infty$ for a.e. $t \in (0,1)$, and

$$\mu \ge \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) \,. \tag{3.22}$$

If we set

$$K = \big(\Omega \cap \partial^* E\big) \cup \Big\{ x \in \Omega \cap E^{(0)} : \theta^n_*(\mu)(x) \ge 2\Phi(\delta_0) \Big\} \,.$$

then [Mag12, Theorem 6.4] implies $\mu \llcorner E^{(0)} \ge 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner (K \cap E^{(0)})$, which combined with (3.22) implies (3.19). We now want to prove that

$$K \cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \,. \tag{3.23}$$

We divide the proof of (3.23) into three steps.

Step one: We prove that for every $N \in \mathbb{N}$ the Borel set

$$K_N = \big(\Omega \cap \partial^* E\big) \cup \Big\{ x \in \Omega \cap E^{(0)} : \theta^n_*(\mu)(x) \ge 2\Phi(\delta_0 - 1/N) \Big\} \,,$$

is such that

$$K_N \cup E^{(1)} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \,. \tag{3.24}$$

To this end, we apply Lemma 3.1 to the interval $I_0 = [\delta_0 - 1/N, \delta_0 - 1/(2N)]$ to find $\{t_j\}_j \subset I_0$ such that $t_j \to t_0 \in I_0$, $E_j = \{u_j > t_j\} \stackrel{\mathrm{loc}}{\to} E_0$, $\{\Omega \cap \partial^* E_j\}_j$ has a partition limit

S_0 in Ω, and (3.8) holds. By monotonicity of $t \mapsto \{u_j > t\}$ and since $\{u_j > t\} \overset{\rm loc}{\to} E$ for a.e. $t \in (0,1)$, we easily see that $E_0 = E$. Hence, we have proved that

$$E^{(1)} \cup (\Omega \cap \partial^* E) \cup S_0 \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,. \tag{3.25}$$

Thanks to Federer's theorem (2.1) and to $\Omega \cap \partial^* E \subset K_N$ we can deduce (3.24) from (3.25) once we prove that

$$E^{(0)} \cap S_0 \text{ is } \mathcal{H}^n\text{-contained in } \big\{\theta^n_*(\mu) \ge 2\,\Phi(\delta_0 - 1/N)\big\}\,. \tag{3.26}$$

We begin the proof of (3.26) by recalling that $S_0 = \bigcup_k \{\Omega_k \cap \bigcup_i \partial^* U_i[\Omega_k]\}$, with $\{\Omega_k\}_k$ as in (2.9), $\{U_i^j[\Omega_k]\}_i$ the essential partition of Ω_k induced by $\Omega \cap \partial^* E_j$, and with $\{U_i[\Omega_k]\}_i$ a Lebesgue partition of Ω_k such that $U_i^j[\Omega_k] \to U_i[\Omega_k]$ as $j \to \infty$ for every k and i. Therefore (3.26) can be further reduced to proving that, for each k and i,

$$\Omega_k \cap E^{(0)} \cap \partial^* U_i[\Omega_k] \text{ is } \mathcal{H}^n\text{-contained in } \big\{\theta^n_*(\mu) \ge 2\,\Phi(\delta_0 - 1/N)\big\}\,. \tag{3.27}$$

Since k will be fixed from now on, we just set for brevity $U_i = U_i[\Omega_k]$, $U_i^j = U_i^j[\Omega_k]$, and consider the sets

$$X_0^j = \{i : |U_i^j| > 0\,,\, (U_i^j)^{(1)} \subset E_j^{(0)}\}\,, \qquad X_1^j = \{i : |U_i^j| > 0\,,\, (U_i^j)^{(1)} \subset E_j^{(1)}\}\,,$$
$$X_0 = \{i : |U_i^j| > 0\,,\, U_i^{(1)} \subset E^{(0)}\}\,, \qquad X_1 = \{i : |U_i| > 0\,,\, U_i^{(1)} \subset E^{(1)}\}\,.$$

Since $\{U_i^j\}_i$ is the essential partition of Ω_k induced by $\Omega \cap \partial^* E_j$, it follows by Federer's theorem and by the $\mathcal{H}^n$-containment of $\Omega_k \cap \partial^* U_i^j$ into $\Omega \cap \partial^* E_j$ that for each i such that $|U_i^j| > 0$ we either have $(U_i^j)^{(1)} \subset E_j^{(1)}$ or $(U_i^j)^{(1)} \subset E_j^{(0)}$. Therefore, if we set

$$X^j := \{i : |U_i^j| > 0\}\,, \qquad X := \{i : |U_i| > 0\}\,,$$

then we have $X^j = X_0^j \cup X_1^j$ (with disjoint union); moreover, by $U_i^j \to U_i$ and $E_j \overset{\rm loc}{\to} E$ for each $i \in X_0$ there exists $j(i)$ such that $i \in X_0^j$ for every $j \ge j(i)$, so that $X = X_0 \cup X_1$ (also with disjoint union), and thus

$$\{U_i\}_{i \in X_0} \text{ is a Lebesgue partition of } \Omega_k \cap E^{(0)}\,,$$

from which we deduce

$$\begin{aligned}\Omega_k \cap E^{(0)} \cap \bigcup_i \partial^* U_i &\overset{\mathcal{H}^n}{=} \Omega_k \cap E^{(0)} \cap \bigcup_{i,i' \in X\,, i \ne i'} \partial^* U_i \cap \partial^* U_{i'} \\ &\overset{\mathcal{H}^n}{=} \Omega_k \cap E^{(0)} \cap \bigcup_{i,i' \in X_0\,, i \ne i'} \partial^* U_i \cap \partial^* U_{i'}\,. \end{aligned} \tag{3.28}$$

By (3.28), the proof of (3.27) can be further reduced to showing that, for every fixed $(i,i') \in X_0 \times X_0$ with $i \ne i'$,

$$\Omega_k \cap \partial^* U_i \cap \partial^* U_{i'} \text{ is } \mathcal{H}^n\text{-contained in } \big\{\theta^n_*(\mu) \ge 2\,\Phi(\delta_0 - 1/N)\big\}\,. \tag{3.29}$$

To prove (3.29), let us fix $i \ne i' \in X_0$, and set

$$G_1 = U_i\,, \qquad G_2 = U_{i'}\,, \qquad G_1^j = U_i^j\,, \qquad G_2^j = U_{i'}^j\,.$$

By (3.2) and the $\mathcal{H}^n$-inclusion of $\Omega_k \cap \partial^* G_m^j$ into $\Omega_k \cap \partial^* E_j$ (recall indeed that $\{U_i^j\}_i$ is the essential partition of Ω_k induced by $K_j = \Omega \cap \partial^* E_j$), it follows that

$$\Omega_k \cap \partial^* G_m^j \text{ is } \mathcal{H}^n\text{-contained in } \{u_j^* = t_j\}\,, \tag{3.30}$$

for each $m = 1, 2$; if, correspondingly, we set

$$u_m^j = u_j\, 1_{\Omega_k \cap G_m^j} + t_j\, 1_{\Omega_k \setminus G_m^j}\,,$$

then by $t_j \to t_0$, $G_m^j \to G_m$ $(m = 1, 2)$, the inclusion $(G_1^j)^{(1)} \cup (G_2^j)^{(1)} \subset E_j^{(0)}$ for $j \ge j(i)$, and the fact that $E_j^{(0)} \to \{u = 0\}$ as $j \to \infty$, we see that

$$u_m^j \to t_0\, 1_{\Omega_k \setminus G_m} \text{ in } L^1_{\rm loc}(\Omega)\,, \qquad \text{as } j \to \infty\,. \tag{3.31}$$

Now, by (3.30) and the divergence theorem (3.3) we have

$$Du_m^j \llcorner \Omega_k = \nabla(\Phi \circ u_j)\, \mathcal{L}^{n+1} \llcorner (\Omega_k \cap G_m^j); \tag{3.32}$$

indeed, for every $\varphi \in C_c^\infty(\Omega_k)$ we have

$$\begin{aligned}\int_{\Omega_k} u_m^j\, \nabla\varphi &= \int_{\Omega_k \cap G_m^j} u_j\, \nabla\varphi + t_j \int_{\Omega_k \setminus G_m^j} \nabla\varphi \\ = \ -\int_{\Omega_k \cap G_m^j} \varphi\, \nabla u_j &+ \int_{\Omega_k \cap \partial^* G_m^j} u_j^* \varphi \nu_{G_m^j} - t_j \int_{\Omega_k \cap \partial^* G_m^j} \varphi \nu_{G_m^j} = -\int_{\Omega_k \cap G_m^j} \varphi\, \nabla u_j\,.\end{aligned}$$

By combining (3.31) and (3.32) with the lower semicontinuity of the total variation we find that, for every open set $A \subset \Omega_k$,

$$\begin{aligned}\liminf_{j\to\infty} \int_{A \cap G_m^j} |\nabla(\Phi \circ u_j)| &= \liminf_{j\to\infty} |D(\Phi \circ u_j)|(A) \\ &\ge \big|D\big(\Phi \circ (t_0\, 1_{A \setminus G_m})\big)\big|(A) = \Phi(t_0)\, P(G_m; A)\,. \end{aligned} \tag{3.33}$$

Adding up over $m = 1, 2$ with $A = B_s(x) \subset\subset \Omega_k$ and recalling that $t_0 \ge \delta_0 - 1/N$ we find

$$\begin{aligned}\mu(\mathrm{cl}\, B_s(x)) &\ge \liminf_{j\to\infty} \int_{B_s(x)} |\nabla(\Phi \circ u_j)| \ge \sum_{m=1}^2 \liminf_{j\to\infty} \int_{B_s(x) \cap G_m^j} |\nabla(\Phi \circ u_j)| \\ &\ge \Phi(\delta_0 - 1/N) \sum_{m=1}^2 P(G_m; B_s(x))\,.\end{aligned}$$

As soon as $x \in \Omega_k \cap \partial^* G_1 \cap \partial^* G_2 = \Omega_k \cap \partial^* U_i \cap \partial^* U_{i'}$, if we divide by $\omega_n\, s^n$, and let $s \to 0^+$ then we conclude that $\theta_*^n(\mu)(x) \ge 2\,\Phi(\delta_0 - 1/N)$, thus proving (3.29).

Step two: We prove that (3.24) implies (3.23). Indeed, let us consider the Radon measures $\lambda_N = \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K_N) \cap E^{(0)})$ and $\lambda = \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) + 2\,\mathcal{H}^n \llcorner (\mathcal{R}(K) \cap E^{(0)})$. Since $\{K_N\}_N$ is decreasing in N and $K = \bigcap_N K_N$ we easily see that $\mathcal{R}(K) = \bigcap_N \mathcal{R}(K_N)$, and thus deduce that $\lambda_N \stackrel{*}{\rightharpoonup} \lambda$ as $N \to \infty$. By (3.24) and Theorem 2.4 we thus conclude that $K_{\rm bk}^\lambda \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, where

$$\begin{aligned}K_{\rm bk}^\lambda &= (\Omega \cap \partial^* E) \cup \big\{x \in \Omega \cap E^{(0)} : \theta_*^n(\lambda) \ge 2\big\} \\ &\overset{\mathcal{H}^n}{=} (\Omega \cap \partial^* E) \cup (\mathcal{R}(K) \cap E^{(0)}) \overset{\mathcal{H}^n}{=} \mathcal{R}(K) \setminus E^{(1)}\,.\end{aligned}$$

We have thus proved that $\mathcal{R}(K) \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, which, by Remark 2.3 implies (3.23).

Step three: We finally prove[5] that (3.20) holds if $v_0 = 0$. We shall actually prove a much stronger property, namely, that for every $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ it holds

$$\liminf_{j\to\infty} \frac{\mathcal{V}(u_j; T)}{\varepsilon_j} > 0\,.$$

If this is not the case then we can find $(\gamma, \Phi, T) \in \mathcal{T}(\mathcal{C})$ and a subsequence in j such that $\mathcal{V}(u_j; T) = \mathrm{o}(\varepsilon_j)$. Let $z_j = u_j \circ \Phi$, so that $z_j \in W^{1,2}(Y)$ where $Y = \mathbb{S}^1 \times B_1^n$. By the slicing theory for Sobolev functions, if we set $z_j^y(s) = z_j(s, y)$ for $(s, y) \in \mathbb{S}^1 \times B_1^n$, then we can find a Borel set F which is $\mathcal{H}^n$-equivalent to B_1^n and is such that, for each $y \in F$ and

[5]This result is not needed in the remaining parts of the paper, and its proof can be omitted on a first reading.

each j, $z_j^y \in W^{1,2}(\mathbb{S}^1)$ with $z_j^y = u_j(\Phi(\cdot,y))$ on a set $S_j(y)$ which is $\mathcal{H}^1$-equivalent to $\mathbb{S}^1$. Notice, in particular, that z_j^y is absolutely continuous on $\mathbb{S}^1$ for each $y \in F$.

We claim that, given j, for $\mathcal{H}^n$-a.e. $y \in F$ it holds

$$\text{there is } s \in \mathbb{S}^1 \text{ such that } z_j^y(s) > 1/2\,. \tag{3.34}$$

If not, there is $F^* \subset F$ with $\mathcal{H}^n(F^*) > 0$ and such that, for every $y \in F^*$, $u_j(\Phi(\cdot,y)) \le 1/2$ on a set $S_j[y]$ which is $\mathcal{H}^1$-equivalent to $\mathbb{S}^1$. We thus find that $\Phi(\mathbb{S}^1 \times F^*)$ is a set of positive volume which is $\mathcal{L}^{n+1}$-contained in $\{u_j \le 1/2\}$. In particular, if $t > 1/2$, then $\{u_j^* > t\}$ is $\mathcal{H}^n$-disjoint from $\Phi(\mathbb{S}^1 \times F^*)$. However, since $\{u_j^* > t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$, for $\mathcal{H}^1$-a.e. s, $T[s] \cup \{u_j^* > t\}$ is essentially disconnecting $T = \Phi(\mathbb{S}^1 \times B_1^n)$; in particular, $\Phi(\mathbb{S}^1 \times F^*)$ is essentially disconnected by $(T[s] \cup \{u_j^* > t\}) \cap \Phi(\mathbb{S}^1 \times F^*)$ which, in turn, is $\mathcal{H}^n$-equivalent to $T[s] \cap \Phi(\mathbb{S}^1 \times F^*) = \Phi(\{s\} \times F^*)$. We have thus concluded that for $\mathcal{H}^1$-a.e. $s \in \mathbb{S}^1$, $\Phi(\{s\} \times F^*)$ is essentially disconnecting $\Phi(\mathbb{S}^1 \times F^*)$, a contradiction.

To conclude, up to modify F on an $\mathcal{H}^n$-null set we can assume that (3.34) holds for every j and every $y \in F$. Next we set

$$F_j = \Big\{ y \in F : \mathcal{H}^1(\{z_j^y > 1/4\}) \le \frac{\mathcal{V}(u_j;T)}{M} \Big\}\,, \tag{3.35}$$

and notice that

$$C(\Phi)\,\mathcal{V}(u_j;T) \ge \int_Y V(z_j) \ge \mathcal{H}^n(B_1^n \setminus F_j)\,V(1/4)\,\frac{\mathcal{V}(u_j;T)}{M}\,,$$

so that, setting $M = \mathcal{H}^n(B_1^n)\,V(1/4)/2\,C(\Phi)$, we find

$$\mathcal{H}^n(F_j) \ge \frac{\mathcal{H}^n(B_1^n)}{2}\,, \qquad \forall j\,. \tag{3.36}$$

Now, if $y \in F_j$, then by (3.34) there is $s_j^y \in \mathbb{S}^1$ such that $z_j^y(s_j^y) \ge 1/2$, and s_j^y must lie in a non-empty connected component I_j^y of $\{z_j^y > 1/4\}$; by (3.35), it must be $\mathcal{H}^1(I_j^y) \le \mathcal{V}(u_j,T)/M$, so that, if j is large enough to ensure $\mathcal{V}(u_j,T)/M < \mathcal{H}^1(\mathbb{S}^1)$, then there must be $t_j^y \in \partial_{\mathbb{S}^1} I_j^y$. In particular, $z_j^y(t_j^y) = 1/4$ and $\mathrm{dist}_{\mathbb{S}^1}(s_j^y, t_j^y) \le \mathcal{H}^1(I_j^y) \le \mathcal{V}(u_j;T)/M$, which, combined with $z_j^y(s_j^y) \ge 1/2$, give

$$\int_{\mathbb{S}^1} |(z_j^y)'|^2 \ge \int_{[s_j(y),t_j(y)]} |(z_j^y)'|^2 \ge \frac{|z_j^y(s_j^y) - z_j^y(t_j^y)|^2}{\mathrm{dist}_{\mathbb{S}^1}(s_j^y, t_j^y)} \ge \frac{M}{16\,\mathcal{V}(u_j;T)}\,.$$

for every $y \in F_j$, and thus, recalling (3.36),

$$\int_Y |\nabla z_j|^2 \ge \int_{F_j} d\mathcal{H}_y^n \int_{\mathbb{S}^1} |(z_j^y)'|^2 \ge \frac{\mathcal{H}^n(B_1^n)}{2}\,\frac{M}{16\,\mathcal{V}(u_j;T)}\,.$$

Finally,

$$\mathcal{AC}_{\varepsilon_j}(u_j;\Omega) \ge \varepsilon_j \int_\Omega |\nabla u_j|^2 \ge \frac{\varepsilon_j}{C(\Phi)} \int_Y |\nabla z_j|^2 \ge c(\Phi, M)\,\frac{\varepsilon_j}{\mathcal{V}(u_j;T)}$$

from which we find $\mathcal{AC}_{\varepsilon_j}(u_j;\Omega) \to \infty$ (a contradiction) as $\mathcal{V}(u_j;T) = \mathrm{o}(\varepsilon_j)$ as $j \to \infty$. $\square$

4. Wet and dry soap films as limits of diffused interface soap films

In the section we address the approximation in Allen–Cahn energy of minimizers (K,E) of $\Psi_{\mathrm{bk}}(v)$ in both the wet ($v > 0$) and dry ($v = 0$) cases. We shall actually be able to work with a slightly more general class of pairs (K,E), namely, we shall work in the class of those $(K,E) \in \mathcal{K}$, with

$$\mathcal{K} = \Big\{ (K,E) : K \text{ is relatively closed and } \mathcal{H}^n\text{-rectifiable in } \Omega,\ E \text{ is open,} \tag{4.1}$$

$$E \text{ has finite perimeter in } \Omega, \text{ and } \Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K \Big\}\,,$$

which also satisfy condition (4.2) and (4.3) below. In Theorem 4.1 we address the wet case ($|E| > 0$), while Theorem 4.3 concerns the dry case ($|E| = 0$).

Theorem 4.1 (Diffused interface approximation of wet soap films)**.** *Let* $\mathbf{W} \subset \mathbb{R}^{n+1}$ *be compact and such that* $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ *has smooth boundary, and let* $\mathcal{C}$ *be a spanning class for* $\mathbf{W}$*. Let* $(K, E) \in \mathcal{K}$ *be such that*

$$|E| > 0\,, \qquad K \cup E \text{ is bounded and } \mathcal{C}\text{-spanning } \mathbf{W}\,, \qquad K \cap E^{(1)} = \varnothing\,, \tag{4.2}$$

and such that there are c *and* r_0 *positive such that*

$$\mathcal{H}^n(K \cap B_r(x)) \geq c\, r^n\,, \tag{4.3}$$

for every $x \in \mathrm{cl}\,(K)$ *and* $r < r_0$*.*

If $\nu : (0,1) \to (0,\infty)$ *and* $\delta : (0,1) \to (1/2,1)$ *are such that, as* $\varepsilon \to 0^+$*,*

$$\nu(\varepsilon) \to |E|\,, \qquad \delta(\varepsilon) \to \delta_0 \in [1/2, 1]\,,$$

then there are $\varepsilon_j \to 0^+$ *and* $\{\{u_\varepsilon^j\}_{\varepsilon<\varepsilon_j}\}_j \subset \mathrm{Lip}(\Omega;[0,1])$ *such that* $\{u_\varepsilon^j > 0\} \subset\subset \mathbb{R}^{n+1}$ *and*

$$\{u_\varepsilon^j \geq \delta(\varepsilon)\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \qquad \forall \varepsilon < \varepsilon_j\,, \tag{4.4}$$

$$\mathcal{V}(u_\varepsilon^j;\Omega) = \nu(\varepsilon)\,, \qquad \forall \varepsilon < \varepsilon_j\,, \tag{4.5}$$

$$\lim_{j\to\infty} \sup_{\varepsilon<\varepsilon_j} \|u_\varepsilon^j - 1_E\|_{L^1(\Omega)} = 0\,, \tag{4.6}$$

$$\lim_{j\to\infty} \sup_{\varepsilon<\varepsilon_j} \frac{\mathcal{AC}_\varepsilon(u_\varepsilon^j;\Omega)}{2} = P(E;\Omega) + 2\Phi(\delta_0)\mathcal{H}^n(K \cap E^{(0)})\,. \tag{4.7}$$

Moreover, if ℓ *is finite, then* $\Upsilon(v,\varepsilon,\delta)$ *is finite for every* $v > 0$*,* $\varepsilon > 0$*, and* $\delta \in (0,1]$*, and*

$$\limsup_{\varepsilon\to 0^+} \sup_{\delta\in(0,1]} \Upsilon(v,\varepsilon,\delta) \leq \Psi_{\mathrm{bk}}(v)\,. \tag{4.8}$$

Remark 4.2 (Choice of spanning set for small ε)**.** Theorem 4.1 shows that given any choice of $\delta(\varepsilon)$ such that $\delta(\varepsilon) \to [1/2, \delta_0) < 1$ as $\varepsilon \to 0^+$ we are bound to find

$$\limsup_{\varepsilon\to 0^+} \Upsilon(v,\varepsilon,\delta(\varepsilon)) \leq P(E;\Omega) + 2\,\Phi(\delta_0)\,\mathcal{H}^n(K \cap E^{(0)}) < \mathcal{F}_{\mathrm{bk}}(K,E) = \Psi_{\mathrm{bk}}(v)\,.$$

where we have applied (4.7) to a minimizer (K,E) of $\Psi_{\mathrm{bk}}(v)$. This explains why, for recovering the expected/correct limit surface tension energy along the collapsed region one has to require $\delta(\varepsilon) \to 1^-$ as $\varepsilon \to 0^+$.

Theorem 4.3 (Diffused interface approximation of dry soap films)**.** *Let* $\mathbf{W} \subset \mathbb{R}^{n+1}$ *be compact and such that* $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$ *has smooth boundary, and let* $\mathcal{C}$ *be a spanning class for* $\mathbf{W}$*. Let* K *relatively closed in* Ω*,* $\mathcal{H}^n$*-rectifiable, bounded,* $\mathcal{C}$*-spanning* $\mathbf{W}$*, and such that there are* c *and* r_0 *positive with*

$$\mathcal{H}^n(K \cap B_r(x)) \geq c\, r^n\,, \tag{4.9}$$

for every $x \in \mathrm{cl}\,(K)$ *and* $r < r_0$*.*

If $\nu : (0,1) \to (0,\infty)$ *and* $\delta : (0,1) \to (1/2,1)$ *are such that, as* $\varepsilon \to 0^+$*,*

$$\nu(\varepsilon) \to 0^+\,, \qquad \delta(\varepsilon) \to \delta_0 \in [1/2,1]\,, \qquad \frac{\varepsilon}{\nu(\varepsilon)} \to 0\,, \tag{4.10}$$

then there are $\varepsilon_j \to 0^+$ *and* $\{\{u_\varepsilon^j\}_{\varepsilon<\varepsilon_j}\}_j \subset (W^{1,2}_{\mathrm{loc}} \cap \mathrm{Lip})(\Omega;[0,1])$ *such that*

$$\{u_\varepsilon^j \geq \delta(\varepsilon)\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,, \qquad \forall \varepsilon < \varepsilon_j\,, \tag{4.11}$$

$$\mathcal{V}(u_\varepsilon^j;\Omega) = \nu(\varepsilon)\,, \qquad \forall \varepsilon < \varepsilon_j\,, \tag{4.12}$$

$$\limsup_{j\to\infty}\sup_{\varepsilon<\varepsilon_j}\frac{\mathcal{AC}_\varepsilon(u^j_\varepsilon;\Omega)}{2}\le 2\Phi(\delta_0)\,\mathcal{H}^n(K)\,. \tag{4.13}$$

Remark 4.4. For the necessity of the third condition in (4.10), see conclusion (3.20) in Theorem 3.4.

In the proof of both theorems, as well as in the sequel, we will make use of the following elementary lemma:

Lemma 4.5. (i): *If $A\subset\mathbb{R}^{n+1}$ is open, $X\in C^\infty_c(A;\mathbb{R}^{n+1})$, and $f_t(x)=x+t\,X(x)$, then there are positive constants t_0 and C_0 depending on X only, such that, for every $|t|<t_0$, $f_t:A\to A$ is a diffeomorphism, and for every $w\in W^{1,2}(A;[0,1])$ we have*

$$\Big|\mathcal{AC}_\varepsilon(w\circ f_t;A)-\mathcal{AC}_\varepsilon(w;A)\\ -t\int_A\varepsilon\,|\nabla w|^2+\frac{W(w)}{\varepsilon}\,\mathrm{div}\,X-2\,\varepsilon\,(\nabla w)\cdot\nabla X[\nabla w]\Big|\le C_0\,\mathcal{AC}_\varepsilon(w;A)\,t^2\,,$$

$$\Big|\mathcal{V}(w\circ f_t;A)-\mathcal{V}(w;A)-t\int_A V(w)\,\mathrm{div}\,X\Big|\le C_0\,\mathcal{V}(w;A)\,t^2\,, \tag{4.14}$$

(ii): *If $u\in L^1(A;[0,1])$ and u is not constant on A, then there are positive constants η_0, β_0 and C_0 (depending on A and u) such that for every $w\in W^{1,2}(A;[0,1])$ with $\|u-w\|_{L^1(A)}\le\beta_0$ and every $|\eta|<\eta_0$ there is a diffeomorphism $f^w_\eta:A\to A$ with $\{f\ne\mathrm{id}\}\subset\subset A$ such that $w_\eta=w\circ f^w_\eta$ satisfies*

$$\mathcal{V}(w_\eta;A)=\mathcal{V}(w;A)+\eta\,,\qquad |\mathcal{AC}_\varepsilon(w_\eta;A)-\mathcal{AC}_\varepsilon(w;A)|\le C_0\,\mathcal{AC}_\varepsilon(w;A)\,|\eta|\,.$$

Proof. Statement (i) is a standard consequence of the area formula. Concerning statement (ii), we notice that since u is not constant in A and V is strictly increasing on $[0,1]$, it follows that $V(u)$ is not constant in A. In turn, this implies the existence of $X\in C^\infty_c(A;\mathbb{R}^{n+1})$ such that $\int_A V(u)\,\mathrm{div}\,X>0$, and then one can argue as in [Mag12, Lemma 29.13, Theorem 29.14]. □

We now prove Theorem 4.1 and Theorem 4.3.

Proof of Theorem 4.1. We start by proving (4.8). By Theorem 1.1, $\ell<\infty$ implies the existence of a minimizer (K,E) of $\Psi_{\rm bk}(v)$ ($|E|=v>0$) which satisfies the assumptions in the first part of the statement. Since u^j_ε (corresponding to $\nu(\varepsilon)\equiv v$ and $\delta(\varepsilon)\equiv 1$) is admissible in $\Upsilon(v,\varepsilon,\delta)$ for every $\delta\in(0,1]$ and $\varepsilon<\varepsilon_j$, we easily deduce (4.8). The rest of the proof is divided in four steps, to which we premise the following result and a preliminary remark related to it:

[Vil09, Proposition 4.13]: *If F is a Borel set in $\mathbb{R}^{n+1}$ such that* (a): *∂F is countably $\mathcal{H}^n$-rectifiable, and* (b): *there are c' and r_0' positive such that*

$$\mathcal{H}^n(B_r(x)\cap\partial F)\ge c'\,r^n\,,\qquad\forall x\in\partial F\,,r<r_0'\,, \tag{4.15}$$

then for every Borel set $A\subset\mathbb{R}^{n+1}$ with

$$\mathcal{H}^n(\partial F\cap\partial A)=0\,, \tag{4.16}$$

it holds

$$\lim_{r\to0^+}\frac{|(I_r(F)\setminus F)\cap A|}{r}=P(F;A)+2\mathcal{H}^n(\partial F\cap F^{(0)}\cap A)\,, \tag{4.17}$$

where $I_r(F)=\{x\in\mathbb{R}^{n+1}:\mathrm{dist}(x,F)<r\}$.

A remark on [Vil09, Proposition 4.13]: In this remark, let us assume F is closed and satisfies (a) and (b). We first point out that the open set F^c satisfies these assumptions also. This is immediate from the fact that a set and its complement share the same topological boundary. We now record three facts to be used later (in each of them, A satisfies (4.16)). First, by (4.17) applied to F and the fact that $|F \setminus \operatorname{int} F| = |\partial F| = 0$,

$$\begin{aligned}\lim_{r\to 0^+}\frac{|(I_r(F)\setminus F)\cap A|}{r} &= P(F;A) + 2\mathcal{H}^n(\partial F\cap F^{(0)}\cap A)\\ &= P(\operatorname{int} F;A) + 2\mathcal{H}^n(\partial F\cap(\operatorname{int} F)^{(0)}\cap A)\,. \qquad (4.18)\end{aligned}$$

Second, applying (4.17) to F^c and again using $|F \setminus \operatorname{int} F| = 0$, we find

$$\begin{aligned}\lim_{r\to 0^+}\frac{|I_r(F^c)\cap(\operatorname{int} F)\cap A|}{r} &= \lim_{r\to 0^+}\frac{|(I_r(F^c)\setminus F^c)\cap A|}{r}\\ &= P(F^c;A) + 2\mathcal{H}^n(\partial(F^c)\cap(F^c)^{(0)}\cap A)\\ &= P(\operatorname{int} F;A) + 2\mathcal{H}^n(\partial F\cap(\operatorname{int} F)^{(1)}\cap A)\,. \qquad (4.19)\end{aligned}$$

Third, setting $\mathrm{sd}_{\partial F}(x) = -\mathrm{dist}(x,\partial F)$ if $x \in F$, $\mathrm{sd}_{\partial F}(x) = \mathrm{dist}(x,\partial F)$ if $x \in F^c$, and $f_\varepsilon(s) = \mathcal{H}^n(A\cap\{\mathrm{sd}_{\partial F} = \varepsilon\, s\})$ for $s \in \mathbb{R}$, we claim that, in the limit as $\varepsilon \to 0^+$ in the sense of Radon measures on $\mathbb{R}$,

$$\begin{aligned}f_\varepsilon\,\mathcal{L}^1 \overset{*}{\rightharpoonup} &\big\{P(\operatorname{int} F;A) + 2\,\mathcal{H}^n(\partial F\cap(\operatorname{int} F)^{(0)}\cap A)\big\}\,\mathcal{L}^1 \llcorner (0,\infty) \qquad (4.20)\\ &+\big\{P(\operatorname{int} F;A) + 2\,\mathcal{H}^n(\partial F\cap(\operatorname{int} F)^{(1)}\cap A)\big\}\,\mathcal{L}^1 \llcorner (-\infty,0)\,.\end{aligned}$$

Indeed, setting for brevity $\alpha = \big\{P(\operatorname{int} F;A) + 2\,\mathcal{H}^n(\partial F\cap(\operatorname{int} F)^{(0)}\cap A)\big\}$, we deduce from (4.19) that, that for every $b > 0$,

$$\begin{aligned}\int_0^b f_\varepsilon &= \int_0^b \mathcal{H}^n(A\cap\{\mathrm{sd}_{\partial F} = \varepsilon\, s\})\,ds = \frac{1}{\varepsilon}\int_0^{\varepsilon\, b}\mathcal{H}^n(A\cap\{\mathrm{sd}_{\partial F} = t\})\,dt\\ &= b\,\frac{|(I_{\varepsilon\, b}(F)\setminus F)\cap A|}{\varepsilon\, b} \to b\,\alpha\,.\end{aligned}$$

In particular, for every $(a,b) \subset (0,\infty)$ we have $\int_a^b f_\varepsilon \to (b-a)\,\alpha$ as $\varepsilon \to 0^+$, and a similar argument based on (4.18) completes the proof of (4.20).

Step one: We prove that $F = \mathrm{cl}\,(E\cup K) = \mathrm{cl}\,E \cup \mathrm{cl}\,K$ satisfies assumptions (a) and (b) of [Vil09, Proposition 4.13]. To prove this, we begin by showing that

$$\partial F = \mathrm{cl}\,K \cup \partial E \subset \mathrm{cl}\,\Omega\,. \qquad (4.21)$$

The containment in $\mathrm{cl}\,\Omega$ is trivial by $K \cup E \subset \Omega$, so we compute ∂F. Towards this end, by the fact (since E is open) that E, ∂E, and $(\mathrm{cl}\,E)^c$ partition $\mathbb{R}^{n+1}$, we decompose

$$\partial F = (\partial F\cap E)\cup(\partial F\cap\partial E)\cup(\partial F\setminus \mathrm{cl}\,E) \qquad (4.22)$$

and evaluate each term individually. First, since E is open and $E \subset F$,

$$\partial F \cap E = \varnothing\,. \qquad (4.23)$$

Second, we claim that $\partial E \subset \partial F$, so that

$$\partial E \cap \partial F = \partial E\,. \qquad (4.24)$$

To prove $\partial E \subset \partial F$, we must show that if $x \in \partial E$, then

$$B_r(x)\cap F \neq \varnothing \quad \text{and} \quad B_r(x)\setminus F \neq \varnothing \quad \forall r > 0\,. \qquad (4.25)$$

Indeed, if $x \in \partial E$, then $B_r(x) \cap E \neq \varnothing$ for all $r > 0$ by definition of ∂E, and so $E \subset F$ gives the first condition in (4.25). For the second, we first claim $x \notin E^{(1)}$. Indeed, since $x \in \partial E \subset \partial\Omega \cup K$ and $\partial\Omega \cap E^{(1)} = \varnothing = K \cap E^{(1)}$ (due to $E \subset \Omega$, the smoothness of $\partial\Omega$, and our assumption on K), we have $x \notin E^{(1)}$. Therefore, noting that $E^{(1)} = F^{(1)}$ (by $|F \setminus E| = |\mathrm{cl}\,K \cup \partial E| \le |K \cup \partial\Omega| = 0$), we find that $x \notin F^{(1)}$. In turn, this implies that $B_r(x) \setminus F \neq \varnothing$ for all $r > 0$ as desired, finishing the proof of (4.25) and thus (4.24). For the last term in (4.22), we claim that

$$\partial F \setminus \mathrm{cl}\,E = \mathrm{cl}\,K \setminus \mathrm{cl}\,E\,. \qquad (4.26)$$

To see this, we note that by the definition of F, $\partial F\setminus \mathrm{cl}\,E=\partial(\mathrm{cl}\,K)\setminus\mathrm{cl}\,E$. Now since $|\mathrm{cl}\,K|\le|K|+|\partial\Omega|=0$, we have $\mathrm{int}\,(\mathrm{cl}\,K)=\varnothing$, and thus

$$\partial(\mathrm{cl}\,K)=\mathrm{cl}\,K\setminus\mathrm{int}\,(\mathrm{cl}\,K)=\mathrm{cl}\,K\,.$$

Thus $\partial F\setminus\mathrm{cl}\,E=\mathrm{cl}\,K\setminus\mathrm{cl}\,E$, which is (4.26). We may now conclude the proof of (4.21). By (4.22), (4.23), (4.24), and (4.26), $\partial F=\partial E\cup(\mathrm{cl}\,K\setminus\mathrm{cl}\,E)=\partial E\cup(\mathrm{cl}\,K\setminus\mathrm{int}\,E)$, so we would be done with (4.21) if $\mathrm{cl}\,K\cap\mathrm{int}\,E=\varnothing$. Now since E is open, $\mathrm{cl}\,K\cap\mathrm{int}\,E=\varnothing$ if and only if $K\cap E=\varnothing$. The latter condition holds since, by the openness of E and our assumption $E^{(1)}\cap K=\varnothing$, $E\cap K\subset E^{(1)}\cap K=\varnothing$.

As a first consequence of (4.21) and $E\cap K=\varnothing$, we have

$$\begin{aligned}\mathrm{int}\,F&=\mathrm{cl}\,F\setminus\partial F=(\mathrm{cl}\,K\cup\mathrm{cl}\,E)\setminus(\mathrm{cl}\,K\cup\partial E)\\&=\mathrm{cl}\,E\setminus(\mathrm{cl}\,K\cup\partial E)=\mathrm{int}\,E\setminus\mathrm{cl}\,K=E\setminus K=E\,.\end{aligned}\tag{4.27}$$

Also (4.21), the relative closedness of K in Ω, and the containment $\Omega\cap\partial E\subset K$ give

$$\Omega\cap\partial F=\Omega\cap(\mathrm{cl}\,K\cup\partial E)=\Omega\cap(K\cup\partial E)=\Omega\cap K\,.\tag{4.28}$$

Since $\partial\Omega$ is $\mathcal{H}^n$-rectifiable, we deduce from (4.28) and the fact that K is $\mathcal{H}^n$-rectifiable that $\partial F=\mathrm{cl}\,K\cup\partial E$ is $\mathcal{H}^n$-rectifiable, and thus satisfies (a). The validity of (4.15) from (b) at every $x\in\partial F\cap\mathrm{cl}\,(K)$ is a consequence of assumption (4.3). The validity of (4.15) at $x\in\partial F\setminus\mathrm{cl}\,(K)\subset\partial E\cap\partial\Omega$ can be deduced as follows: with c and r_0 as in (4.3), if $r<r_0$ and there is $y\in B_{r/2}(x)\cap\mathrm{cl}\,(K)\ne\varnothing$, then by (4.3) and $K\subset\partial F$

$$\mathcal{H}^n(B_r(x)\cap\partial F)\ge\mathcal{H}^n(B_{r/2}(y)\cap K)\ge c\,(r/2)^n\,;$$

if, instead, $B_{r/2}(x)\cap\mathrm{cl}\,(K)=\varnothing$, then $\Omega\cap\partial E\subset K$ and E open imply that $B_{r/2}(x)\cap\partial F=B_{r/2}(x)\cap\partial\Omega$, and we conclude by the fact that $\mathcal{H}^n(B_r(x)\cap\partial\Omega)\ge c_\Omega\,r^n$ for every $r<r_\Omega$, provided r_Ω and c_Ω are suitable positive constants.

Step two: We would like to apply [Vil09, Proposition 4.13] to F and $A=\Omega$, although doing so would require checking that $\mathcal{H}^n(\partial\Omega\cap\partial F)=0$, something that is potentially false (e.g., if $\mathcal{H}^n(\partial\Omega\cap\partial E)>0$). To avoid this difficulty, we "slightly stretch" K and E as follows. Since Ω has a (bounded) smooth boundary, there is $t_0>0$ such that if we define $g_t:\Omega\to\Omega_t:=I_t(\Omega)$, $t\in(0,t_0)$, by setting $g_t(x)=x$ for $x\in\Omega\cap\{\mathrm{dist}_{\partial\Omega}>t\}$ and

$$g_t(x)=x+\big(\mathrm{dist}_{\partial\Omega}(x)-t\big)\,\nabla\mathrm{dist}_{\partial\Omega}(x)\,,\qquad x\in\Omega\cap\{\mathrm{dist}_{\partial\Omega}<t\}\,,$$

then g_t is diffeomorphism with $g_t\to\mathrm{id}$ and $g_t^{-1}\to\mathrm{id}$ as $t\to0^+$ (in every C^k-norm). Setting $K_t=g_t(K)$ and $E_t=g_t(E)$ we see that $F_t=\mathrm{cl}\,(K_t\cup E_t)$ satisfies assumptions (a) and (b) of [Vil09, Proposition 4.13]. Also by (4.21) and the fact that $g_t^{-1}(\Omega)\subset\Omega$,

$$\partial(F_t)\cap\Omega=g_t((\mathrm{cl}\,K\cup\partial E)\cap g_t^{-1}(\Omega))=g_t(K\cap g_t^{-1}(\Omega))=K_t\cap\Omega\,,\tag{4.29}$$

and by (4.27),

$$\mathrm{int}\,F_t=E_t\,.\tag{4.30}$$

Moreover, since $\mathrm{dist}(g_t(x),\partial\Omega)=2\,\mathrm{dist}(x,\partial\Omega)-t$ for every $x\in\Omega\cap\{\mathrm{dist}_{\partial\Omega}<t\}$, we see that

$$g_t^{-1}\big(\partial\Omega\cap\partial(F_t)\big)=g_t^{-1}(\partial\Omega)\cap\partial F\subset\big\{\mathrm{dist}_{\partial\Omega}=t/2\big\}\cap\partial K\,.$$

Since $\mathcal{H}^n(K\cap\{\mathrm{dist}_{\partial\Omega}=t/2\})=0$ for a.e. $t\in(0,t_0)$ and g_t^{-1} is a Lipschitz map we conclude that

$$\mathcal{H}^n\big(\partial\Omega\cap\partial(F_t)\big)=0\,,\qquad\text{for a.e. }t\in(0,t_0)\,.\tag{4.31}$$

As a consequence, we may apply (4.18) to F_t with $A=\Omega$: by using (4.29) and (4.30) to rewrite $\partial F_t\cap\Omega$ and $\mathrm{int}\,F_t$, respectively, for a.e. $t\in(0,t_0)$ and as $r\to0^+$, we obtain

$$\begin{aligned}\big|\big(I_r(F_t)\setminus F_t\big)\cap\Omega\big|&=r\,\big\{P(E_t;\Omega)+2\,\mathcal{H}^n(\Omega\cap K_t\cap E_t^{(0)})\big\}+r\,\mathrm{o}_t(1)\\&=r\,\mathcal{F}_{\mathrm{bk}}(K_t,E_t)+r\,\mathrm{o}_t(1)\end{aligned}$$

$$= \quad r\,\big(1+\omega(t)\big)\mathcal{F}_{\rm bk}(K,E)+r\,\omega_t(r)\,, \tag{4.32}$$

where $\omega(t)\to 0$ as $t\to 0^+$ and $\omega_t(r)\to 0$ as $r\to 0^+$, and where in the last identity we have used the area formula and $\|g_t-{\rm id}\,\|_{C^1}\le C(\Omega)\,t$. By the same logic applied to F_t^c with (4.19) replacing (4.18), we also have

$$\begin{aligned}\big|I_r(F_t^c)\cap F_t\cap\Omega\big| &= r\,\big(1+\omega(t)\big)[P(E;\Omega)+2\,\mathcal{H}^n(K\cap E^{(1)}\cap\Omega)]+r\,\omega_t(r)\\ &= r\,\big(1+\omega(t)\big)P(E;\Omega)+r\,\omega_t(r)\,,\end{aligned} \tag{4.33}$$

where in the second line we have used our assumption that $K\cap E^{(1)}=\varnothing$. Finally, we notice that

$$\Omega\cap F_t=K_t\cup E_t\text{ is }\mathcal{C}\text{-spanning }\mathbf{W}\,; \tag{4.34}$$

indeed, for every $\gamma\in\mathcal{C}$, $\gamma(\mathbb{S}^1)\cap(K_t\cup E_t)=g_t((g_t^{-1}\circ\gamma)(\mathbb{S}^1)\cap(K\cup E))$ where this last set is non-empty since $g_t^{-1}\circ\gamma$ is homotopic to γ relatively to Ω.

Step three: We prove that for a.e. $t<t_0$ (t_0 depending on (K,E)), every $M>0$, and every $\varepsilon>0$ we can define

$$u_\varepsilon^{M,t}\in{\rm Lip}(\Omega;[0,1])\,,\qquad \{u_\varepsilon^{M,t}>0\}\subset\subset\mathbb{R}^{n+1}\,,$$

(see (4.40)) in such a way that $\{u_\varepsilon^{M,t}\ge\delta(\varepsilon)\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ and

$$\big|\mathcal{V}(u_\varepsilon^{M,t};\Omega)-|E|\big| \le C\,\big(t\,P(E)+\varepsilon\,M\big)\,, \tag{4.35}$$

$$\int_\Omega|u_\varepsilon^{M,t}-1_E| \le C\,\big(t\,P(E)+\varepsilon\,M\big)\,, \tag{4.36}$$

$$\Big|\frac{\mathcal{AC}_\varepsilon(u_\varepsilon^{M,t};\Omega)}{2}-P(E;\Omega)-2\,\Phi(\delta_0)\,\mathcal{H}^n(K\cap E^{(0)})\Big| \le \omega(1/M)+\omega(t)+\omega_{t,M}(\varepsilon) \tag{4.37}$$

where C depends on the data of the theorem, and where $\omega(r)$ ($\omega_a(r)$) denotes a generic non-negative increasing function such that the limit $\omega(r)\to 0$ ($\omega_a(r)\to 0$) as $r\to 0^+$ holds at a rate that depends on the data of the theorem (and on the parameter a). The construction goes as follows. By the normalization (1.10) of W, the Allen–Cahn profile $\eta\in C^\infty(\mathbb{R};(0,1))$ defined by $-\eta'=\sqrt{W(\eta)}$ on $\mathbb{R}$, $\eta(0)=1/2$, $\eta(-\infty)=1$ and $\eta(+\infty)=0$, is such that

$$\int_{\mathbb{R}}(\eta')^2+W(\eta)=2\int_{\mathbb{R}}\sqrt{W(\eta)}\,|\eta'|=2\int_0^1\sqrt{W}=2\,.$$

Starting from η, for every $M>0$ we can easily construct $\eta_M\in C^\infty(\mathbb{R};[0,1])$ with $\{\eta_M=1\}=(-\infty,0]$, $\{\eta_M=0\}=[M,\infty)$ and such that

$$\int_{\eta_M^{-1}(\delta_0)}^\infty(\eta_M')^2+W(\eta_M) = 2\Phi(\delta_0)+\omega(1/M) \tag{4.38}$$

$$\int_{-\infty}^{\eta_M^{-1}(\delta_0)}(\eta_M')^2+W(\eta_M) = 2(1-\Phi(\delta_0))+\omega(1/M)\,, \tag{4.39}$$

where $\omega(1/M)\to 0$ as $M\to\infty$. Let $\eta_M^{\delta(\varepsilon)}$ be the translation of η_M such that $\eta_M^{\delta(\varepsilon)}(0)=\delta(\varepsilon)$, and similarly for δ_0. Corresponding to ε, M, and t positive, with $t<t_0$, we now set

$$u_\varepsilon^{M,t}(x)=\eta_M^{\delta(\varepsilon)}\Big(\frac{{\rm sd}_{F_t}(x)}{\varepsilon}\Big)\,,\qquad x\in\Omega\,. \tag{4.40}$$

In this way, $u_\varepsilon^{M,t}\in{\rm Lip}(\Omega;[0,1])$ with compact support on $\mathbb{R}^{n+1}$. Since $\{u_\varepsilon^{M,t}\ge\delta(\varepsilon)\}=\Omega\cap F_t=K_t\cup E_t$, by (4.34) we deduce that $\{u_\varepsilon^{M,t}\ge\delta(\varepsilon)\}$ is $\mathcal{C}$-spanning $\mathbf{W}$. Next we notice that since $0=V(0)\le V(t)\le V(1)=1$ for every $t\in[0,1]$, by combining the area formula (to deduce $|E\Delta E_t|\le C\,P(E)\,t$) with (4.32) we find

$$\begin{aligned}\mathcal{V}(u_\varepsilon^{M,t};\Omega\setminus E_t) &\le \big|\big(I_{\varepsilon\,M}(F_t)\setminus F_t\big)\cap\Omega\big|\\ &\le \varepsilon\,M\,\big(1+\omega(t)\big)\mathcal{F}_{\rm bk}(K,E)+\varepsilon\,M\,\omega_t(\varepsilon\,M)\,.\end{aligned} \tag{4.41}$$

Similarly, using instead (4.33), we find

$$
\begin{aligned}
\mathcal{V}(u_\varepsilon^{M,t};\Omega\cap E_t) &= |E_t| - \mathrm{O}\big(\big|I_{\varepsilon\, M}(F_t^c)\cap F_t\cap\Omega\big|\big)\\
&= |E| - \mathrm{O}\big(C\,P(E)\,t - \varepsilon\, M\,\big(1+\omega(t)\big)P(E;\Omega) - \varepsilon\, M\,\omega_t(\varepsilon\, M)\big)\,. \quad (4.42)
\end{aligned}
$$

Together, (4.41) and (4.42) give (4.35); we deduce (4.36) similarly. Finally, by the coarea formula,

$$
\begin{aligned}
\frac{\mathcal{AC}_\varepsilon(u_\varepsilon^{M,t};\Omega)}{2} &= \frac{1}{2}\int_{\mathbb{R}} \frac{\big[(\eta_M^{\delta(\varepsilon)})'(z/\varepsilon)\big]^2 + W(\eta_M^{\delta(\varepsilon)}(z/\varepsilon))}{\varepsilon}\ \mathcal{H}^n\big(\Omega\cap\{\mathrm{sd}_{F_t}=z\}\big)\,dz\\
&= \frac{1}{2}\int_{\mathbb{R}} \big[\big((\eta_M^{\delta(\varepsilon)})'\big)^2 + W(\eta_M^{\delta(\varepsilon)})\big]\, f_\varepsilon^t
\end{aligned}
$$

where we have set for brevity

$$
f_\varepsilon^t(s) = \mathcal{H}^n\big(\Omega\cap\{\mathrm{sd}_{F_t}=\varepsilon\, s\}\big)\,,\qquad s>0\,.
$$

Since, by (4.20) and (4.29)-(4.30), for a.e. $t\in(0,t_0)$, as $\varepsilon\to 0^+$,

$$
\begin{aligned}
f_\varepsilon^t\,\mathcal{L}^1\llcorner\mathbb{R} &\stackrel{*}{\rightharpoonup} \big\{P(\mathrm{int}\,F_t;\Omega) + 2\,\mathcal{H}^n(\partial F_t\cap(\mathrm{int}\,F_t)^{(0)}\cap\Omega)\big\}\,\mathcal{L}^1\llcorner(0,\infty)\\
&\qquad +\big\{P(\mathrm{int}\,F_t;\Omega) + 2\,\mathcal{H}^n(\partial F_t\cap(\mathrm{int}\,F_t)^{(1)}\cap\Omega)\big\}\,\mathcal{L}^1\llcorner(-\infty,0)\\
&= \big\{P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(0)}\cap\Omega)\big\}\,\mathcal{L}^1\llcorner(0,\infty)\\
&\qquad +\big\{P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(1)}\cap\Omega)\big\}\,\mathcal{L}^1\llcorner(-\infty,0)\quad\text{as }\varepsilon\to 0^+
\end{aligned}
$$

and $\big((\eta_M^{\delta(\varepsilon)})'\big)^2 + W(\eta_M^{\delta(\varepsilon)})$ converges uniformly to $\big((\eta_M^{\delta_0})'\big)^2 + W(\eta_M^{\delta_0})$ in $C_c^\infty(\mathbb{R};[0,1])$ as $\varepsilon\to 0^+$, we find in particular that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \big[\big((\eta_M^{\delta(\varepsilon)})'\big)^2 + W(\eta_M^{\delta(\varepsilon)})\big]\, f_\varepsilon^t\,ds\\
&\quad= \frac{P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(0)}\cap\Omega)}{2}\int_0^\infty \big((\eta_M^{\delta_0})'\big)^2 + W(\eta_M^{\delta_0})\,ds + \omega_{t,M}(\varepsilon)\\
&\qquad + \frac{P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(1)}\cap\Omega)}{2}\int_{-\infty}^0 \big((\eta_M^{\delta_0})'\big)^2 + W(\eta_M^{\delta_0})\,ds + \omega_{t,M}(\varepsilon)\\
&\quad= \mathcal{F}_{\mathrm{bk}}(K_t,E_t)[\Phi(\delta_0) + \omega(1/M) + \omega_{t,M}(\varepsilon)]\\
&\qquad +\big\{P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(1)}\cap\Omega)\big\}[1-\Phi(\delta_0) + \omega(1/M) + \omega_{t,M}(\varepsilon)]\,,
\end{aligned}
$$

where $\omega_{t,M}(\varepsilon)\to 0$ as $\varepsilon\to 0^+$ and we have used (4.38)-(4.39). Since, as noticed in (4.32) and (4.33),

$$
\begin{aligned}
\mathcal{F}_{\mathrm{bk}}(K_t,E_t) &= (1+\omega(t))\,\mathcal{F}_{\mathrm{bk}}(K,E)\quad\text{and}\\
P(E_t;\Omega) + 2\,\mathcal{H}^n(K_t\cap E_t^{(1)}\cap\Omega) &= (1+\omega(t))P(E;\Omega)
\end{aligned}
$$

we conclude that (4.37) holds.

Step four: We conclude the proof. Given $j\in\mathbb{N}$, we can find $t_j\to 0^+$ and $M_j\to\infty$ (as $j\to\infty$) depending on the data of the problem, and then ε_j depending on t_j, M_j and the data of the problem, such that, $\varepsilon_j\to 0^+$ as $j\to\infty$ and, for every $\varepsilon<\varepsilon_j$,

$$
w_\varepsilon^j = u_\varepsilon^{M_j,t_j}\in\mathrm{Lip}(\Omega;[0,1])
$$

with $\{w_\varepsilon^j>0\}\subset\subset\mathbb{R}^{n+1}$, $\{w_\varepsilon^j=1\}$ $\mathcal{C}$-spanning $\mathbf{W}$, and

$$
\begin{aligned}
&\max\Big\{\big|\mathcal{V}(w_\varepsilon^j;\Omega)-|E|\big|\,,\int_\Omega |w_\varepsilon^j - 1_E|\,,\Big|\frac{\mathcal{AC}_\varepsilon(w_\varepsilon^j;\Omega)}{2} - P(E;\Omega) - 2\,\Phi(\delta)\,\mathcal{H}^n(K\cap E^{(0)})\Big|\Big\}\\
&\qquad\le\frac{1}{j}\,. \qquad (4.43)
\end{aligned}
$$

Next, *we use* $|E| > 0$ to deduce that $u = 1_E$ is non-constant in the open set $A = \Omega$: by taking j large enough we can thus apply Lemma 4.5 to $w = w_\varepsilon^j$ and with $\eta = \nu(\varepsilon) - \mathcal{V}(w_\varepsilon^j; \Omega)$ for every $\varepsilon < \varepsilon_j$; denoting by f_j the resulting diffeomorphism, we set $u_\varepsilon^j = w_\varepsilon^j \circ f_j$, and notice that, by (4.43), u_ε^j satisfies (4.4), (4.5), (4.6), and (4.7). $\square$

Next we turn to the proof of Theorem 4.3. This will be the first situation where we make use of the properties of the minimizers in the diffused Euclidean isoperimetric problem $\Theta(v, \varepsilon)$, consisting of the minimization of $\mathcal{AC}_\varepsilon(u) := \mathcal{AC}_\varepsilon(u; \mathbb{R}^{n+1})$ under the volume constraint $\mathcal{V}(u) := \mathcal{V}(u; \mathbb{R}^{n+1}) = v$; see Appendix A for more details.

Proof of Theorem 4.3. Since the assumption "$|E| > 0$" was not used in the proof of Theorem 4.1 until the definition of u_ε^j in step four, we notice that, for a.e. $t \in (0, t_0)$ and for every M and ε positive, if we define $u_\varepsilon^{M,t}$ as in (4.40), then (4.35) and (4.37) combined with $E = \varnothing$ give

$$
\begin{aligned}
\mathcal{V}(u_\varepsilon^{M,t}; \Omega) &\leq C\, \varepsilon\, M\,, && (4.44)\\
\Big| \frac{\mathcal{AC}_\varepsilon(u_\varepsilon^{M,t}; \Omega)}{2} - 2\, \Phi(\delta_0)\, \mathcal{H}^n(K) \Big| &\leq \omega(1/M) + \omega(t) + \omega_{t,M}(\varepsilon)\,, && (4.45)
\end{aligned}
$$

Given $M > 0$, there is $\varepsilon_* = \varepsilon_*(M) > 0$ be such that $C\, \varepsilon\, M < \nu(\varepsilon)$ for every $\varepsilon < \varepsilon_*$. In particular, for a.e. $t \in (0, t_0)$ and every $M > 0$ and $\varepsilon < \varepsilon_*$ we have

$$
\mathcal{V}(u_\varepsilon^{M,t}; \Omega) \leq \nu(\varepsilon) \tag{4.46}
$$

and (4.45). Now, given $w > 0$, let us denote by $\zeta_{\varepsilon,w}$ the unique minimizer of the diffused isoperimetric problem $\Theta(w, \varepsilon)$ (see Appendix A), and let us consider for a.e. $t \in (0, t_0)$, $M > 0$, $\varepsilon < \varepsilon_*(M)$, $\alpha > 0$, and $\beta \in \mathbb{R}$

$$
u_\varepsilon^{M,t,\alpha,\beta} = \max\big\{ u_\varepsilon^{M,t}, \zeta_{\varepsilon,w_*} \circ \lambda_\alpha \circ \tau_\beta \big\}\,, \quad x \in \Omega\,,
$$

where τ is a fixed unit vector in $\mathbb{R}^{n+1}$, $\tau_\beta(x) = x - \beta\, \tau$ and $\lambda_\alpha(x) = x/\alpha$ ($x \in \mathbb{R}^{n+1}$), and where we have set $w_* = w_*(\varepsilon, M, t) = \varepsilon + \nu(\varepsilon) - \mathcal{V}(u_\varepsilon^{M,t}; \Omega)$; we immediately see that

$$
u_\varepsilon^{M,t,\alpha,\beta} \in (W^{1,2}_{\rm loc} \cap {\rm Lip})(\Omega; [0,1])\,,
$$

and that

$$
\{u_\varepsilon^{M,t,\alpha,\beta} \geq \delta(\varepsilon)\} \text{ is } \mathcal{C}\text{-spanning } \mathbf{W}\,,
$$

since it contains $\{u_\varepsilon^{M,t} \geq \delta(\varepsilon)\}$, as well as that

$$
\begin{aligned}
\mathcal{AC}_\varepsilon\big(u_\varepsilon^{M,t,\alpha,\beta}; \Omega\big) &\leq \mathcal{AC}_\varepsilon(u_\varepsilon^{M,t}; \Omega) + \mathcal{AC}_\varepsilon(\zeta_{\varepsilon,w_*} \circ \lambda_\alpha; \Omega)\\
&\leq \mathcal{AC}_\varepsilon(u_\varepsilon^{M,t}; \Omega) + \big\{1 + C(n, W)\, |\alpha - 1|\big\}\, \Theta(\varepsilon, w_*)\,. && (4.47)
\end{aligned}
$$

Now, since $\{u_\varepsilon^{M,t} > 0\}$ is compactly contained in $\mathbb{R}^{n+1}$, we have, uniformly on $|\alpha - 1| < 1/2$,

$$
\begin{aligned}
\lim_{\beta \to \infty} \mathcal{V}(u_\varepsilon^{M,t,\alpha,\beta}; \Omega) &= \mathcal{V}(u_\varepsilon^{M,t}; \Omega) + \mathcal{V}(\zeta_{\varepsilon,w_*} \circ \lambda_\alpha)\\
&= \mathcal{V}(u_\varepsilon^{M,t}; \Omega) + \alpha^n\, \mathcal{V}(\zeta_{\varepsilon,w_*})\\
&= \mathcal{V}(u_\varepsilon^{M,t}; \Omega) + \alpha^n\, \big(\varepsilon + \nu(\varepsilon) - \mathcal{V}(u_\varepsilon^{M,t}; \Omega)\big)\,.
\end{aligned}
$$

This last expression, evaluated at $\alpha = 1$, is an ε above $\nu(\varepsilon)$. In summary, for for a.e. $t \in (0, t_0)$, for every $M > 0$, $\varepsilon < \varepsilon_*(M)$, $\beta > \beta_*(M, t, \varepsilon)$ there is $\alpha(\beta, \varepsilon) \leq 1$, with $\alpha(\beta, \varepsilon) \to 1$ as $\beta \to \infty$ (uniformly in $\varepsilon < \varepsilon_*$), such that

$$
\mathcal{V}(u_\varepsilon^{M,t,\alpha(\beta,\varepsilon),\beta}; \Omega) = \nu(\varepsilon)\,.
$$

We can now pick $t_j \to 0^+$, $M_j \to \infty$, $\varepsilon_j = \min\{\varepsilon_*(M_j), \varepsilon_j^*\} \to 0^+$ (where ε_j^* is such that the error $\omega_{t,M}$ appearing in (4.45) satisfies $\omega_{t_j,M_j}(\varepsilon_j^*) \to 0^+$), $\beta_j = \beta_*(M_j, t_j, \varepsilon_j)$, $\alpha_j(\varepsilon) = \alpha(\beta_j, \varepsilon)$, and define, for every $\varepsilon < \varepsilon_j$,

$$u_\varepsilon^j = u_\varepsilon^{M_j, t_j, \alpha_j(\varepsilon), \beta_j} \in (W^{1,2}_{\rm loc} \cap {\rm Lip})(\Omega; [0,1])$$

so that $\{u_\varepsilon^j \geq \delta(\varepsilon)\}$ is $\mathcal{C}$-spanning $\mathbf{W}$, $\mathcal{V}(u_\varepsilon^j; \Omega) = \nu(\varepsilon)$, and combining (4.45) with (4.47) such that

$$\begin{aligned} \mathcal{AC}_\varepsilon(u_\varepsilon^j; \Omega) &\leq 2\,\mathcal{H}^n(K) + \omega(1/M_j) + \omega(t_j) + \omega_{t_j,M_j}(\varepsilon_j^*) \\ &\quad + \big\{1 + C(n,W)\,|\alpha_j(\varepsilon) - 1|\big\}\,\Theta(\varepsilon, w_j(\varepsilon))\,, \end{aligned}$$

where $w_j(\varepsilon) = w_*(\varepsilon, M_j, t_j) = \varepsilon + \nu(\varepsilon) - \mathcal{V}(u_\varepsilon^{M_j,t_j}; \Omega)$. Now, since [MR22] implies

$$\sup_{0<w\leq 1} \Theta(\varepsilon, w) \leq C(n,W)\,\varepsilon^{n/(n+1)}\,, \qquad \forall \varepsilon < \varepsilon_0(n,W)\,,$$

we conclude from $\omega_{t_j,M_j}(\varepsilon_j^*) \to 0^+$ and $\alpha_j(\varepsilon) \to 1$ (uniformly in $\varepsilon < \varepsilon_*$) that (4.13) holds. □

5. Lagrange multipliers of diffused interface soap films

The following theorem is one of the key results of our analysis, as it provides an upper bound on the size of the Lagrange multipliers λ_j appearing in (1.1) – precisely, we show that $\varepsilon_j\,\lambda_j \to 0$ as $j \to \infty$. This information, which is of course interesting in itself, is also useful in the proof of the existence of minimizers of $\Upsilon(v, \varepsilon, \delta)$ (and the inclusion of the possibility that $\mathbf{v}_j < v_j$ in the statement is needed in that proof). We notice that our analysis does not touch the very interesting problem of understanding the validity of positive lower bounds on the $|\lambda_j|$'s. Intuitively, one would indeed expect that they cannot be too small: indeed, since (1.1) is compatible with convergence to Plateau-type singularities, it should not be possible to identify it as a too close approximation of the standard Allen–Cahn equation (for which convergence to Plateau-type singularities is indeed impossible).

Theorem 5.1 (Lagrange multipliers estimate)**.** *If $\mathbf{W} \subset \mathbb{R}^{n+1}$ is compact, $\mathcal{C}$ is a spanning class for $\mathbf{W}$, v_j, ε_j, and δ_j are sequences with*

$$v_j \to 0^+\,, \qquad \varepsilon_j \to 0^+\,, \qquad \frac{\varepsilon_j}{v_j} \to 0^+\,, \qquad \delta_j \to \delta_0 \in [1/2, 1]\,,$$

as $j \to \infty$, and u_j are minimizers of $\Upsilon(\mathbf{v}_j, \varepsilon_j, \delta_j)$ for some $\mathbf{v}_j \in (0, v_j]$ such that

$$\mathcal{AC}_{\varepsilon_j}(u_j; \Omega) \leq \Upsilon(v_j, \varepsilon_j, \delta_j)\,, \tag{5.1}$$

then

$$\lim_{j\to\infty} \varepsilon_j\,\lambda_j = 0\,, \tag{5.2}$$

where λ_j is the Lagrange multiplier in the inner variation Euler–Lagrange equation for u_j.

Proof. Without loss of generality we can assume that $\varepsilon_j\,\lambda_j$ admits a limit, and then prove (5.2) by finding a subsequence $j_N \to \infty$ as $N \to \infty$ such that $\varepsilon_{j_N}\,\lambda_{j_N} \to 0$ as $N \to \infty$. Setting for the sake of brevity ${\rm ac}_\varepsilon(u) = \varepsilon\,|\nabla u|^2 + W(u)/\varepsilon$, we are going to achieve this by making a suitable choice of $X \in C^\infty_c(\Omega; \mathbb{R}^{n+1})$ in the Euler–Lagrange equation

$$\int_\Omega {\rm ac}_{\varepsilon_j}(u_j)\,{\rm div}\,X - 2\,\varepsilon_j\,\nabla u_j\,\cdot\,\nabla X[\nabla u_j] = \lambda_j \int_\Omega V(u_j)\,{\rm div}\,X \tag{5.3}$$

satisfied by u_j. Since the argument is long, it is convenient to first give an overview of it. In step one we prove the convergence of ${\rm ac}_{\varepsilon_j}j(u_j)\,\mathcal{L}^{n+1} \llcorner \Omega$ to $2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner K$, where K is a minimizer of ℓ, and characterize K as the limit of super/sub-level sets of the u_j's. In step two we blow-up near a regular point of K, say, 0, and identify $r_N \to 0^+$ and $j_N \to \infty$ as $N \to \infty$ such that K, at a scale r_N near 0, is approximately flat and is "sandwiched" between two regions F_N^+ and F_N^- on each of which u_{j_N} concentrates a $\Phi(\delta_0)$-amount of Allen–Cahn energy per unit area (compare with (5.19)). In step three we consider the blow-ups $\tilde{u}_N$ of $u_{\varepsilon_{j_N}}$ near 0 at scale r_N, and prove that they converge to a double transition from 0 to δ_0 on each side of K near 0, along a length scale $\sigma_N \to 0$ (compare with (5.25)). In preparation to further blow-up $\tilde{u}_N$ at the scale σ_N we need to identify first a suitable thin cylinder for performing such blow-up (see the definition of J_N in step four), and then suitable heights that locate the two transitions from 0 to δ_0 (see the definition of s_N and t_N in step five). The resulting blow-ups $\mathbf{u}_N$ of $\tilde{u}_N$ are defined in step five, where their local convergence to on one-dimensional Allen–Cahn profile is proved (see (5.42)), and used to readily infer that $\varepsilon_{j_N}\,\lambda_{j_N} \to 0$ by testing the rescaled Euler–Lagrange equation (5.3) for $\mathbf{u}_N$ on a carefully selected vector field.

Step one: By the assumptions on $\{(v_j, \varepsilon_j, \delta_j)\}_j$, Theorem 4.3 gives

$$2\,\Phi(\delta_0)\,\ell \geq \limsup_{j\to\infty} \Upsilon(v_j, \varepsilon_j, \delta_j) \geq \limsup_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(u_j;\Omega)/2\,. \tag{5.4}$$

In particular, $\sup_j \mathcal{AC}_{\varepsilon_j}(u_j;\Omega) < \infty$, so that if we extract a subsequence and denote by μ the weak-star limit of $|\nabla(\Phi \circ u_j)|\,\mathcal{L}^{n+1} \llcorner \Omega$ as $j \to \infty$, then, by Theorem 3.4, the Borel subset of Ω defined by

$$K = \big\{x \in \Omega : \theta^n_*(\mu)(x) \geq 2\,\Phi(\delta_0)\big\}$$

is such that K is $\mathcal{C}$-spanning $\mathbf{W}$ and $\mu \geq 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner K$. Combining this last inequality with (5.4) and (MM) we find that K is a minimizer of ℓ (thus $\mathcal{H}^n$-equivalent to $S = \mathrm{spt}\,\mathcal{H}^n \llcorner K$) and

$$2\,\Phi(\delta_0)\,\ell = \lim_{j\to\infty} \Upsilon(v_j, \varepsilon_j, \delta_j) = \lim_{j\to\infty} \Upsilon(\mathbf{v}_j, \varepsilon_j, \delta_j) = \lim_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(u_j;\Omega)/2\,; \tag{5.5}$$

$$\mu = 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner K = \mathrm{w}^*\lim_{j\to\infty} |\nabla(\Phi\circ u_j)|\,\mathcal{L}^{n+1} \llcorner \Omega = \mathrm{w}^*\lim_{j\to\infty} \frac{\mathrm{ac}_{\varepsilon_j}(u_j)}{2}\,\mathcal{L}^{n+1} \llcorner \Omega\,. \tag{5.6}$$

We now notice that, thanks to the minimality property in ℓ, K can actually be characterized as a partition limit. To see this, let us recall the construction used in the proof of Theorem 3.4. There, given $N \in \mathbb{N}$ we applied Lemma 3.1 on the interval $I_0^N = [\delta_0 - (1/2N), \delta_0 - (1/N)]$ to find $\{t_j^N\}_j \subset I_0^N$ such that, setting $E_j^N = \{u_j > t_j^N\}$, then $\{\Omega \cap \partial^* E_j^N\}_j$ admitted a partition limit S_0^N with the property that

$$S_0^N \text{ is } \mathcal{C}\text{-spanning } \mathbf{W} \text{ and is } \mathcal{H}^n\text{-contained in } K_N = \{\theta^n_*(\mu) \geq 2\,\Phi(\delta_0 - 1/N)\}\,, \tag{5.7}$$

compare with (3.25) and (3.26). Having proved that in the present case $\mu = 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner K$, we see that $K_N = K$, and therefore, by (5.7), that

$$\ell = \mathcal{H}^n(K) \geq \mathcal{H}^n(S_0^N) \geq \ell\,.$$

Hence $S_0^N \overset{\mathcal{H}^n}{\subset} K$ implies $S_0^N \overset{\mathcal{H}^n}{=} K$, that is, for every $N \in \mathbb{N}$,

$$K \overset{\mathcal{H}^n}{=} \bigcup_k \{\Omega_k \cap \bigcup_i \partial^* U_{N,i}[\Omega_k]\}\,, \tag{5.8}$$

where[6] $\{\Omega_k\}_k$ is as in (2.9), and $\{U_{N,i}[\Omega_k]\}_i$ is the limit of the essential partitions $\{U^j_{N,i}[\Omega_k]\}_i$ of Ω_k induced by $\Omega \cap \partial^* E_j^N$ in the sense that for every k, i, and N, we have $U^j_{N,i}[\Omega_k] \to$

[6]Should $\{\Omega_k\}_k$ be a disjoint family – something it is definitely not! – (5.8) would imply, for each $N \neq N'$ and each k, the existence of a bijection σ so that $U_{N,i}[\Omega_k]$ is Lebesgue equivalent to $U_{N',\sigma(i)}[\Omega_k]$ for every i, and we could thus drop the N-dependency from the following arguments. Quite the opposite happens though, since each Ω_k intersects countably many different $\Omega_{k'}$'s, and it seems there is no obvious way to trivialize the interaction between k and N in the building up of K.

$U_{N,i}[\Omega_k]$ as $j \to \infty$. In particular, since, by (3.2), $\Omega \cap \partial^* E_j^N$ is $\mathcal{H}^n$-equivalent to $\{u_j^* = t_j^N\}$, we find that

$$\Omega_k \cap \partial^* U_{N,i}^j[\Omega_k] \text{ is } \mathcal{H}^n\text{-contained in } \{u_j^* = t_j^N\}\,, \tag{5.9}$$

where $t_j^N \to t_0^N \in I_0^N$ as $j \to \infty$.

Step two: From this step onward, we focus our analysis near a regular point of K. More precisely, since minimizers of ℓ are Almgren minimizing sets in Ω, by [Alm76] K is a smoothly embedded minimal surface in a neighborhood of $\mathcal{H}^n$-a.e. of its points. In particular, setting

$$\begin{aligned}\mathbf{Q}_r &= \big\{x \in \mathbb{R}^{n+1} : |x_i| < r \text{ for } i = 1, ..., n+1\big\} = (-r, r)^{n+1}\,,\\ \mathbf{Q}_r^n &= \mathbf{Q}_r \cap \{x_{n+1} = 0\} = (-r, r)^n \times \{0\}\,,\end{aligned}$$

denoting by $\mathbf{p}$ the projection of $\mathbb{R}^{n+1}$ onto $\{x_{n+1} = 0\}$, and up to a rigid motion, we can assume that $0 \in K$ and that there are $r_0 > 0$ and a smooth solution to the minimal surfaces equation $f \in C^\infty(\mathbf{Q}_{r_0}^n; (-r_0, r_0))$ with $f(0) = 0$, $\nabla f(0) = 0$, and

$$\mathbf{Q}_r \cap K = \big\{x \in \mathbf{Q}_r : x_{n+1} = f(\mathbf{p}(x))\big\}\,, \qquad \forall r \in (0, r_0)\,.$$

Let us now consider the epigraph and subgraph of f in $\mathbf{Q}_r$, that is, let us consider

$$\mathrm{Epi}(f; r) = \big\{x \in \mathbf{Q}_r : x_{n+1} > f(\mathbf{p}(x))\big\}\,, \qquad \mathrm{Sub}(f; r) = \big\{x \in \mathbf{Q}_r : x_{n+1} < f(\mathbf{p}(x))\big\}\,,$$

so that $\{\mathrm{Epi}(f; r_0), \mathrm{Sub}(f; r_0)\}$ is the essential partition of $\mathbf{Q}_{r_0}$ induced by K. By (5.8) and by the smoothness of f, for each N there are k_N and $i_N^+ \neq i_N^-$ such that

$$0 \in \Omega_{k_N} \cap \partial^* U_{N,i_N^\pm}[\Omega_{k_N}]\,.$$

Proceeding inductively in N, we can pick $r_N < \min\{r_0, r_1, ..., r_{N-1}\}$ so that

$$\lim_{N\to\infty} r_N = 0\,, \qquad \mathbf{Q}_{2\,r_N} \subset\subset \Omega_{k_N}\,, \qquad \mathcal{H}^n(K \cap \partial \mathbf{Q}_{r_N}) = 0\,, \tag{5.10}$$

and thus

$$\mathbf{Q}_{r_N} \cap U_{N,i_N^+}[\Omega_{k_N}] = \mathrm{Epi}(f; r_N)\,, \qquad \mathbf{Q}_{r_N} \cap U_{N,i_N^-}[\Omega_{k_N}] = \mathrm{Sub}(f; r_N)\,.$$

In particular, for every N, as $j \to \infty$ we have

$$\mathbf{Q}_{r_N} \cap U^j_{N,i_N^+}[\Omega_{k_N}] \to \mathrm{Epi}(f; r_N)\,, \qquad \mathbf{Q}_{r_N} \cap U^j_{N,i_N^-}[\Omega_{k_N}] \to \mathrm{Sub}(f; r_N)\,. \tag{5.11}$$

We now prove two additional properties of $U^j_{N,i_N^\pm}$, see (5.15) and (5.16) below, which will be used to suitably select $\{j_N\}_N$ such that $j_N \to \infty$ as $N \to \infty$:

First, setting for brevity

$$\alpha^\pm_{N,j} = \int_{\mathbf{Q}_{r_N} \cap U^j_{N,i_N^\pm}[\Omega_{k_N}]} \frac{\mathrm{ac}_{\varepsilon_j}(u_j)}{2}\,,$$

and noticing that, thanks to (5.9), we can argue as in the proof of (3.29) in Theorem 3.4 – see, in particular, (3.33) – we prove that

$$\begin{aligned}\liminf_{j\to\infty} \alpha^\pm_{N,j} &\geq \liminf_{j\to\infty} \int_{\mathbf{Q}_{r_N} \cap U^j_{N,i_N^\pm}[\Omega_{k_N}]} |\nabla(\Phi \circ u_j)| \\ &\geq \Phi(t_0^N)\, P(U_{N,i_N^\pm}[\Omega_{k_N}]; \mathbf{Q}_{r_N}) \geq \Phi\big(\delta_0 - (1/2\,N)\big)\, \mathcal{H}^n(K \cap \mathbf{Q}_{r_N})\,.\end{aligned} \tag{5.12}$$

Since (5.6) and (5.10) give

$$\limsup_{j\to\infty} \alpha^+_{N,j} + \alpha^-_{N,j} \leq 2\, \Phi(\delta_0)\, \mathcal{H}^n(K \cap \mathbf{Q}_{r_N})\,, \tag{5.13}$$

by applying (5.12) with $m=2$ and by (5.13) we find

$$\limsup_{j\to\infty}\alpha_{N,j}^{+} \le 2\,\Phi(\delta_0)\,\mathcal{H}^n(K\cap\mathbf{Q}_{r_N})-\liminf_{j\to\infty}\alpha_{N,j}^{-} \le \Big\{2\,\Phi(\delta_0)-\Phi\Big(\delta_0-\frac{1}{2\,N}\Big)\Big\}\,\mathcal{H}^n(K\cap\mathbf{Q}_{r_N}) \tag{5.14}$$

By similarly applying (5.12) with $m=1$ in combination with (5.13), we conclude that,

$$\Phi\Big(\delta_0-\frac{1}{2\,N}\Big)\,\mathcal{H}^n(K\cap\mathbf{Q}_{r_N}) \le \liminf_{j\to\infty}\alpha_{N,j}^{\pm}\le\limsup_{j\to\infty}\alpha_{N,j}^{\pm} \le \Big\{2\,\Phi(\delta_0)-\Phi\Big(\delta_0-\frac{1}{2\,N}\Big)\Big\}\,\mathcal{H}^n(K\cap\mathbf{Q}_{r_N})\,. \tag{5.15}$$

Second, by (5.9), since each $U_{N,i}^j[\Omega_{k_N}]$ is essentially connected, for each N, j and i,

$$U_{N,i}^j[\Omega_{k_N}]\text{ is }\mathcal{L}^{n+1}\text{-contained either in }\{u_j>t_j^N\}\text{ or in }\{u_j<t_j^N\}\,.$$

Setting

$$\mathbf{Q}_r^+=\mathbf{Q}_r\cap\{x_{n+1}>0\}\,,\qquad \mathbf{Q}_r^-=\mathbf{Q}_r\cap\{x_{n+1}<0\}\,,$$

since $f(0)=0$, $\nabla f(0)=0$, and the smoothness of f imply that $|\mathrm{Epi}(f;r)\Delta\mathbf{Q}_r^+|\le C\,r^{n+2}$ for every $r<r_0$, we see that if $U_{N,i_N^+}^j[\Omega_{k_N}]$ is $\mathcal{L}^{n+1}$-contained in $\{u_j>t_j^N\}$, then, by $\mathcal{V}(u_j;\Omega)=v_j$ and (5.11),

$$|\mathbf{Q}_1^+|\,r_N^{n+1}-C\,r_N^{n+2} \le |\mathbf{Q}_{r_N}\cap\mathrm{Epi}(f;r_N)|\le|\mathbf{Q}_{r_N}\cap U_{N,i_N^+}^j[\Omega_{k_N}]|+\mathrm{o}_j^N \le |\{u_j>t_j^N\}|+\mathrm{o}_j^N\le\frac{\mathcal{V}(u_j;\Omega)}{V(1/4)}+\mathrm{o}_j^N=\mathrm{o}_j^N\,,$$

where $\mathrm{o}_j^N\to0$ as $j\to\infty$ at a rate depending on N. In particular, up to further decrease the value of r_0 (so to have $C\,r_N\le C\,r_0<|\mathbf{Q}_1^+|/2$ for each N), and by repeating the same considerations with $\mathrm{Sub}(f;r_N)$ in place of $\mathrm{Epi}(f;r_N)$, we have proved that for each N, if j is large enough depending on N, then

$$U_{N,i_N^\pm}^j[\Omega_{k_N}]\text{ is }\mathcal{L}^{n+1}\text{-contained in }\{u_j<t_j^N\}\,. \tag{5.16}$$

Step three (selection of $\{j_N\}_N$ and first blow-up): Using the estimates proved in step two, we can diagonally extract a subsequence $j_N\to\infty$ as $N\to\infty$ such that, if we set

$$F_N^\pm=\mathbf{Q}_{r_N}\cap U_{N,i_N^\pm}^{j_N}[\Omega_{k_N}]\,,\qquad \sigma_N=\frac{\varepsilon_{j_N}}{r_N}\,,\qquad w_N=\frac{v_{j_N}}{r_N^{n+1}}\,,$$

then the following holds: first, $\sigma_N\to0$ and $w_N\to0$ as $N\to\infty$; second, by (5.11),

$$\lim_{N\to\infty}\max\Big\{\frac{|F_N^+\Delta\mathrm{Epi}(f;r_N)|}{r_N^{N+1}},\frac{|F_N^-\Delta\mathrm{Sub}(f;r_N)|}{r_N^{N+1}}\Big\}=0\,; \tag{5.17}$$

third, by (5.6) and (5.10) (which yield $\mathcal{AC}_{\varepsilon_j}(u_j;\mathbf{Q}_{r_N})\to2\,\Phi(\delta_0)\,\mathcal{H}^n(K\cap\mathbf{Q}_{r_N})$),

$$\lim_{N\to\infty}\frac{1}{\mathcal{H}^n(K\cap\mathbf{Q}_{r_N})}\int_{\mathbf{Q}_{r_N}}\frac{\mathrm{ac}_{\varepsilon_{j_N}}(u_{j_N})}{2}=2\,\Phi(\delta_0)\,; \tag{5.18}$$

fourth, by (5.15),

$$\lim_{N\to\infty}\frac{1}{\mathcal{H}^n(K\cap\mathbf{Q}_{r_N})}\int_{\mathbf{Q}_{r_N}\cap F_N^\pm}\frac{\mathrm{ac}_{\varepsilon_{j_N}}(u_{j_N})}{2}=\Phi(\delta_0)\,; \tag{5.19}$$

and, finally, by (5.16),

$$\mathbf{Q}_{r_N}\cap\partial^*F_N^\pm\text{ is }\mathcal{H}^n\text{-contained in }\{u_{j_N}^*=t_{j_N}^N\}\,,$$
$$F_N^\pm\text{ is }\mathcal{L}^{n+1}\text{-contained in }\{u_{j_N}^*<t_{j_N}^N\}\,.$$

If we now set $\eta_r(y) = r\,y$ ($y \in \mathbf{Q}_1$) and

$$G_N^\pm = \frac{F_N^\pm}{r_N} \subset \mathbf{Q}_1\,, \qquad \delta_N = t_{j_N}^N \in I_0^N\,, \qquad \tilde u_N = u_{j_N} \circ \eta_{r_N}\,,$$

then $\delta_N \to \delta_0$ as $N \to \infty$, while $|\mathrm{Epi}(f;r)\Delta\mathbf{Q}_r^+| \le C\,r^{n+2}$ and (5.17) give

$$|G_N^+\Delta\mathbf{Q}_1^+| \le \frac{|F_N^+\Delta\mathrm{Epi}(f;r_N)|}{r_N^{n+1}} + \frac{|\mathrm{Epi}(f;r_N)\Delta\mathbf{Q}_{r_N}^+|}{r_N^{n+1}} \le \frac{|F_N^+\Delta\mathrm{Epi}(f;r_N)|}{r_N^{n+1}} + C\,r_N\,,$$

and an analogous estimate for G_N^-, so that,

$$\lim_{N\to\infty} |G_N^\pm\Delta\mathbf{Q}_1^\pm| = 0\,, \tag{5.20}$$

$$\mathbf{Q}_1 \cap \partial^* G_N^\pm \text{ is } \mathcal{H}^n\text{-contained in } \{\tilde u_N^* = \delta_N\}\,, \tag{5.21}$$

$$G_N^\pm \text{ is } \mathcal{L}^{n+1}\text{-contained in } \{\tilde u_N < \delta_N\}\,. \tag{5.22}$$

Similarly, taking into account that $\mathcal{H}^n(K\cap\mathbf{Q}_r)/(2r)^n \to 1$ as $r \to 0^+$, that $r_N \to 0$ as $N \to \infty$, and that

$$\int_{\mathbf{Q}_{r_N}\cap F_N^\pm} \mathrm{ac}_{\varepsilon_{j_N}}(u_{j_N}) = r_N^n \int_{\mathbf{Q}_1\cap G_N^\pm} \mathrm{ac}_{\varepsilon_{j_N}/r_N}(\tilde u_N)$$

we deduce from (5.19) and (5.18) that

$$\lim_{N\to\infty} \Big|\Phi(\delta_0) - 2^{-n}\,\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N; G_N^\pm)}{2}\Big| = \lim_{N\to\infty} \Big|2\,\Phi(\delta_0) - 2^{-n}\,\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N; \mathbf{Q}_1)}{2}\Big| = 0\,. \tag{5.23}$$

Since $V(t) \ge c\,t^{2(n+1)/n}$ for some $c = c(W) > 0$, we have

$$v_{j_N} \ge \int_{\mathbf{Q}_{r_N}} u_{j_N}^{2(n+1)/n} = r_N^{n+1} \int_{\mathbf{Q}_1} \tilde u_N^{2(n+1)/n}\,,$$

so that $w_N \to 0$ implies $\tilde u_N \to 0$ in $L^1(\mathbf{Q}_1)$ as $N \to \infty$; hence, taking also (5.20) and (5.21) into account, we find that, if we set

$$\tilde u_N^\pm := \tilde u_N\,1_{G_N^\pm} + \delta_N\,1_{\mathbf{Q}_1\setminus G_N^\pm}\,, \tag{5.24}$$

then $\tilde u_N^\pm \in W^{1,2}(\mathbf{Q}_1)$, and, moreover,

$$\lim_{N\to\infty} \int_{\mathbf{Q}_1} |\tilde u_N^+ - \delta_0\,1_{\mathbf{Q}_1^-}| = \lim_{N\to\infty} \int_{\mathbf{Q}_1} |\tilde u_N^- - \delta_0\,1_{\mathbf{Q}_1^+}| = 0\,. \tag{5.25}$$

We also record for future use that, thanks to (5.3), it holds

$$\int_{\mathbf{Q}_1} \mathrm{ac}_{\sigma_N}(\tilde u_N)\,\mathrm{div}\,Y - 2\,\sigma_N\,\nabla\tilde u_N \cdot \nabla Y[\nabla\tilde u_N] = \frac{\lambda_{j_N}\,\varepsilon_{j_N}}{\sigma_N} \int_{\mathbf{Q}_1} V(\tilde u_N)\,\mathrm{div}\,Y\,, \tag{5.26}$$

for every $Y \in C_c^\infty(\mathbf{Q}_1;\mathbb{R}^{n+1})$.

Step four (identification of the second blow-up): In this step, we show that for every N large enough, there is $\xi_N \in \mathbf{Q}_1^n$ such that, setting

$$Q_N = \xi_N + \mathbf{Q}_{\sigma_N/2}^n \subset \mathbf{Q}_1^n\,, \qquad J_N = Q_N \times (-1,1) \tag{5.27}$$

we find

$$\lim_{N\to\infty} \max\Big\{\frac{1}{\sigma_N^n}\int_{J_N} |\tilde u_N^+ - \delta_0\,1_{\mathbf{Q}_1^-}|\,, \frac{1}{\sigma_N^n}\int_{J_N} |\tilde u_N^- - \delta_0\,1_{\mathbf{Q}_1^+}|\Big\} = 0\,, \tag{5.28}$$

$$\lim_{N\to\infty} \max\Big\{\Big|2\Phi(\delta_0) - \frac{\mathcal{AC}_{\sigma_N}(\tilde u_N; J_N)}{2\,\sigma_N^n}\Big|, \Big|\Phi(\delta_0) - \frac{\mathcal{AC}_{\sigma_N}(\tilde u_N; J_N \cap G_N^\pm)}{2\,\sigma_N^n}\Big|\Big\} = 0\,. \tag{5.29}$$

(We shall actually prove the existence of *many* ξ_N with these properties). To begin with, let us define a sequence $\beta_N \to 0$ as $N \to \infty$ by setting

$$\beta_N = \max\Big\{ \int_{\mathbf{Q}_1} |\tilde u_N^+ - \delta_0\, 1_{\mathbf{Q}_1^-}|\,, \int_{\mathbf{Q}_1} |\tilde u_N^- - \delta_0\, 1_{\mathbf{Q}_1^+}|\Big\}\,,$$

and, denoting by M_N the integer part of $2/\sigma_N$, we consider a collection

$$\{Q_{N,i}\}_{i=1}^{M_N^n}\,,\qquad Q_{N,i} = \xi_{N,i} + \mathbf{Q}^n_{\sigma_N/2}\,,\qquad \xi_{N,i}\in\mathbf{Q}_1^n\,,$$

of disjoint open cubes contained in $\mathbf{Q}_1^n$, with side length σ_N, and such that $\mathcal{H}^n(\mathbf{Q}_1^n\setminus\bigcup_i Q_{N,i}) \le C(n)\,\sigma_N$ (of course, if $2/\sigma_N$ itself is an integer, then such cubes can be chosen so that $\mathcal{H}^n(\mathbf{Q}_1^n\setminus\bigcup_i Q_{N,i})=0$). Noticing that

$$\Big|1-(M_N\,\sigma_N)^n/\mathcal{H}^n(\mathbf{Q}_1^n)\Big| = \Big|1-(M_N\,\sigma_N)^n/2^n\Big| \le C(n)\,\sigma_N\,, \tag{5.30}$$

we consider the open cylinders $\{J_{N,i}\}_{i=1}^{M_N^n}$, $J_{N,i} := Q_{N,i}\times(-1,1)\subset\mathbf{Q}_1$, so that

$$|J_{N,i}| = 2\,\sigma_N^n\,,\qquad \Big|\mathbf{Q}_1\setminus\bigcup_i J_{N,i}\Big| \le C(n)\,\sigma_N\,,$$

and let

$$\mathcal{G}_N^1 = \Big\{1\le i\le M_N^n : \int_{J_{N,i}} |\tilde u_N^+ - \delta_0\,1_{\mathbf{Q}_1^-}| \le \sqrt{\beta_N}\,\sigma_N^n\Big\}\,,$$

$$\mathcal{G}_N^2 = \Big\{1\le i\le M_N^n : \int_{J_{N,i}} |\tilde u_N^- - \delta_0\,1_{\mathbf{Q}_1^+}| \le \sqrt{\beta_N}\,\sigma_N^n\Big\}\,.$$

On combining (5.30) with

$$(M_N^n - \#\,\mathcal{G}_N^1)\,\sigma_N^n\,\sqrt{\beta_N} \le \sum_{i\notin\mathcal{G}_N^1}\int_{J_{N,i}} |\tilde u_N^+ - \delta_0\,1_{\mathbf{Q}_1^-}| \le \beta_N\,,$$

we find

$$0 \le 1 - \frac{\#\mathcal{G}_N^m}{M_N^n} \le \frac{\sqrt{\beta_N}}{M_N^n\sigma_N^n} \le C(n)\,\sqrt{\beta_N}\,,\qquad m=1,2\,, \tag{5.31}$$

provided N is large enough. In particular,

$$\lim_{N\to\infty}\frac{\#(\mathcal{G}_N^1\cap\mathcal{G}_N^2)}{M_N^n} = 1\,, \tag{5.32}$$

so that, for N large, the vast majority of the cubes $\{Q_{N,i}\}_{i=1}^{M_N^n}$ satisfies (5.28). This suggests to consider a class $\mathcal{G}_{N,\mathrm{good}}$ of cubes in $\mathcal{G}_N^1\cap\mathcal{G}_N^2$ such that (5.29) holds: more precisely, we set $\mathcal{G}_{N,\mathrm{good}} = (\mathcal{G}_N^1\cap\mathcal{G}_N^2)\setminus\mathcal{G}_{N,\mathrm{bad}}$, where

$$\mathcal{G}_{N,\mathrm{bad}} = \Big\{i\in\mathcal{G}_N^1\cap\mathcal{G}_N^2 : \frac{\mathcal{AC}_{\sigma_N}(\tilde u_N; J_{N,i})}{2\,\sigma_N^n} \ge 2\,\Phi(\delta_0) + \sqrt{\alpha_N}\Big\}\,,$$

is defined in dependence of the quantity

$$\begin{aligned}\alpha_N \quad:=\quad \max\Big\{&\Big|\#(\mathcal{G}_N^1\cap\mathcal{G}_N^2)\,\sigma_N^n\,2\,\Phi(\delta_0) - \frac{\mathcal{AC}_{\sigma_N}(\tilde u_N,\mathbf{Q}_1)}{2}\Big|\,,\\ &\Big[\Phi(\delta_0) - \inf_{i\in\mathcal{G}_N^1\cap\mathcal{G}_N^2}\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N^\pm; J_{N,i}\cap G_N^\pm)}{2\,\sigma_N^n}\Big]_+\Big\}\,,\end{aligned}$$

(where $t_+ := \max\{t,0\}$, $t\in\mathbb{R}$). We now claim that

$$\lim_{N\to\infty}\alpha_N = 0\,, \tag{5.33}$$

$$\lim_{N\to\infty}\frac{\#\mathcal{G}_{N,\mathrm{good}}}{M_N^n} = 1\,, \tag{5.34}$$

and that *any choice* of $Q_N \in \mathcal{G}_{N,\rm good}$ satisfies (5.28) and (5.29).

To prove (5.33): By (5.30), (5.31) we have $\#(\mathcal{G}_N^1 \cap \mathcal{G}_N^2)\,\sigma_N^n \to \mathcal{H}^n(\mathbf{Q}_1^n) = 2^n$ as $N \to \infty$, so that the first quantity in the definition of α_N is vanishing as $N \to \infty$ thanks to (5.23). We are thus left to prove that, if we set

$$L := \liminf_{N\to\infty} \inf_{i \in \mathcal{G}_N^1 \cap \mathcal{G}_N^2} \frac{\mathcal{AC}_{\sigma_N}(\tilde{u}_N^\pm; J_{N,i} \cap G_N^\pm)}{2\,\sigma_N^n}\,, \tag{5.35}$$

then $L \ge \Phi(\delta_0)$. To prove this, let us consider $N_k \to \infty$, $i_k \in \mathcal{G}_{N_k}^1 \cap \mathcal{G}_{N_k}^2$, and $m_k \in \{+,-\}$ such that

$$L = \lim_{k\to\infty} \frac{\mathcal{AC}_{\sigma_{N_k}}(\tilde{u}_{N_k}^{m_k}; J_{N_k,i_k} \cap G_{N_k}^{m_k})}{2\,\sigma_{N_k}^n}\,.$$

Up to extracting a subsequence we can assume that either $m_k = +$ or $m_k = -$ for every k, and we can assume without loss of generality to be in the first case. Recalling that $Q_{N,i} = \xi_{N,i} + \mathbf{Q}^n_{\sigma_N/2}$ and $J_{N,i} = Q_{N,i} \times (-1,1)$, if we set

$$J_k^* := [(J_{N_k,i_k} - \xi_{N_k,i_k})/\sigma_{N_k}] = \mathbf{Q}^n_{1/2} \times \Big(-1/\sigma_{N_k}, 1/\sigma_{N_k}\Big)\,,$$

and define

$$G_k^* = [(G_{N_k}^+ - \xi_{N_k,i_k})/\sigma_{N_k}]\,, \qquad \mathbf{u}_k(z) = \tilde{u}_{N_k}^+(\sigma_{N_k}\,z + \xi_{N_k,i_k})\,, \qquad z \in J_k^*\,,$$

then we have

$$\frac{\mathcal{AC}_{\sigma_{N_k}}(\tilde{u}_{N_k}^{m_k}; J_{N_k,i_k} \cap G_{N_k}^{m_k})}{2\,\sigma_{N_k}^n} = \frac{\mathcal{AC}_1(\mathbf{u}_k; J_k^* \cap G_k^*)}{2}\,. \tag{5.36}$$

Now, since $\tilde{u}_N^+ \in W^{1,2}(\mathbf{Q}_1)$ implies $\mathbf{u}_k \in W^{1,2}(J_k^*)$, and, by definition of $\mathcal{G}_N^1$,

$$\begin{aligned} C(n)\,\sqrt{\beta_{N_k}} &\ge \frac{1}{\sigma_{N_k}^n} \int_{J_{N_k,i_k}} |\tilde{u}_{N_k}^+ - \delta_0\,1_{\mathbf{Q}_1^-}| = \sigma_{N_k} \int_{J_k^*} |\mathbf{u}_k - \delta_0 1_{\mathbf{Q}^n_{1/2}\times(-1/\sigma_{N_k},0)}| \\ &= 2 \int_{\mathbf{Q}^n_{1/2}} d\mathcal{H}^n_y \fint_{-1/\sigma_{N_k}}^{1/\sigma_{N_k}} |\mathbf{u}_k(y,t) - \delta_0 1_{(-1/\sigma_{N_k},0)}(t)|\,dt\,, \end{aligned}$$

we find that, for $\mathcal{H}^n$-a.e. $y \in \mathbf{Q}^n_{1/2}$, $t \mapsto \mathbf{u}_k(y,t)$ is absolutely continuous on $(-1/\sigma_{N_k}, 1/\sigma_{N_k})$ and there are a_k^y and b_k^y in $(-1/\sigma_{N_k}, 1/\sigma_{N_k})$ such that

$$\begin{aligned} &\lim_{k\to\infty} a_k^y = -\infty\,, \quad \lim_{k\to\infty} \mathbf{u}_k(y, a_k^y) = \delta_0\,, \\ &\lim_{k\to\infty} b_k^y = +\infty\,, \quad \lim_{k\to\infty} \mathbf{u}_k(y, b_k^y) = 0\,; \end{aligned}$$

as a consequence, by (MM), by the fact that $\mathbf{u}_k$ is constant on $J_k^* \setminus G_k^*$, by Fubini's theorem, and by Fatou's lemma, we find that

$$\begin{aligned} \lim_{k\to\infty} \frac{\mathcal{AC}_1(\mathbf{u}_k; J_k^* \cap G_k^*)}{2} &\ge \liminf_{k\to\infty} \int_{J_k^* \cap G_k^*} |\partial_{x_{n+1}}(\Phi \circ \mathbf{u}_k)| = \liminf_{k\to\infty} \int_{J_k^*} |\partial_{x_{n+1}}(\Phi \circ \mathbf{u}_k)| \\ &\ge \int_{\mathbf{Q}^n_{1/2}} \liminf_{k\to\infty} \int_{(a_k^y, b_k^y)} |\partial_{x_{n+1}}(\Phi \circ \mathbf{u}_k)| \ge \Phi(\delta_0)\,, \end{aligned}$$

which, combined with (5.36), proves that $L \ge \Phi(\delta_0)$, and thus that (5.33) holds.

To prove (5.34): We can estimate that

$$\frac{\#\mathcal{G}_{N,\rm bad}}{M_N^n} \le \frac{1}{M_N^n\,\sqrt{\alpha_N}} \sum_{i \in \mathcal{G}_{N,\rm bad}} \frac{\mathcal{AC}_{\sigma_N}(\tilde{u}_N; J_{N,i})}{\sigma_N^n} - 2\,\Phi(\delta_0) = A_N + B_N\,,$$

where, by definition of α_N and since $(M_N\,\sigma_N)^n\to\mathcal{H}^n(\mathbf{Q}_1^n)$ as $N\to\infty$,

$$
\begin{aligned}
A_N &:= \frac{1}{M_N^n\sqrt{\alpha_N}}\sum_{i\in\mathcal{G}_N^1\cap\mathcal{G}_N^2}\Big\{\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;J_{N,i})}{\sigma_N^n}-2\,\Phi(\delta_0)\Big\}\\
&\le \frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;\mathbf{Q}_1)-\#(\mathcal{G}_N^1\cap\mathcal{G}_N^2)\,\sigma_N^n\,2\Phi(\delta_0)}{M_N^n\,\sigma_N^n\,\sqrt{\alpha_N}}\le C(n)\,\sqrt{\alpha_N}\,,
\end{aligned}
$$

and, again by definition of α_N,

$$
\begin{aligned}
B_N &:= -\frac{1}{M_N^n\sqrt{\alpha_N}}\sum_{i\in\mathcal{G}_{N,\mathrm{good}}}\Big\{\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;J_{N,i})}{\sigma_N^n}-2\,\Phi(\delta_0)\Big\}\\
&\le \frac{2}{M_N^n\sqrt{\alpha_N}}\sum_{i\in\mathcal{G}_{N,\mathrm{good}}}\Big\{\Phi(\delta_0)-\sum_{m\in\{+,-\}}\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;J_{N,i}\cap G_N^m)}{2\,\sigma_N^n}\Big\}\le C(n)\,\sqrt{\alpha_N}\,.
\end{aligned}
$$

This shows that $\#\mathcal{G}_{N,\mathrm{bad}}/M_N^n\to 0$ as $N\to\infty$, and since $\#(\mathcal{G}_N^1\cap\mathcal{G}_N^2)/M_N^n\to 1$ as $N\to\infty$ by (5.31), we conclude the proof of (5.34).

To conclude the proof of (5.28) *and* (5.29): For an arbitrary choice of $i(N)\in\mathcal{G}_{N,\mathrm{good}}$, let $Q_N:=Q_{N,i(N)}$ and $J_N:=Q_N\times(-1,1)$. Since $\mathcal{G}_{N,\mathrm{good}}\subset\mathcal{G}_N^1\cap\mathcal{G}_N^2$ we deduce the validity of (5.28). At the same time, by $L\ge\Phi(\delta_0)$, by $G_N^+\cap G_N^-=\varnothing$, by $\tilde u_N^m=\tilde u_N$ on G_N^m for $m\in\{+,-\}$, and by the very definition of $\mathcal{G}_{N,\mathrm{good}}$, we see that

$$
\begin{aligned}
\Phi(\delta_0) &\le \liminf_{N\to\infty}\min_{m\in\{+,-\}}\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N^m;J_N\cap G_N^m)}{2\,\sigma_N^n}\le\liminf_{N\to\infty}\frac12\sum_{m\in\{+,-\}}2\,\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N^m;J_N\cap G_N^m)}{2\,\sigma_N^n}\\
&= \frac12\liminf_{N\to\infty}\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;J_N\cap(G_N^+\cup G_N^-))}{2\,\sigma_N^n}\le\frac12\liminf_{N\to\infty}\frac{\mathcal{AC}_{\sigma_N}(\tilde u_N;J_N)}{2\,\sigma_N^n}\\
&\le \Phi(\delta_0)+\liminf_{N\to\infty}\frac{\sqrt{\alpha_N}}{2}=\Phi(\delta_0)\,,
\end{aligned}
$$

which readily implies (5.29).

Step five (analysis of the second blow-up): With $Q_N=\xi_N+\mathbf{Q}^n_{\sigma_N/2}$ and $J_N=Q_N\times(-1,1)$ as in step five, if we now set

$$
\begin{aligned}
\mathbf{J}_N &= (J_N-\xi_N)/\sigma_N=\mathbf{Q}^n_{1/2}\times(-1/\sigma_N,1/\sigma_N)\,,\\
\mathbf{J}_N^\pm &= \mathbf{J}_N\cap\{x_{n+1}\gtrless 0\}\,,\\
\mathbf{G}_N^\pm &= [(J_N\cap G_N^\pm)-\xi_N]/\sigma_N\,,\\
\mathbf{u}_N(z) &= \tilde u_N(\xi_N+\sigma_N\,z)\,,\\
\mathbf{u}_N^\pm(z) &= \tilde u_N^\pm(\xi_N+\sigma_N\,z)\\
&= \mathbf{u}_N(z)\,1_{\mathbf{G}_N^\pm}(z)+\delta_N\,1_{\mathbf{J}_N\setminus\mathbf{G}_N^\pm}(z)\,,
\end{aligned}
$$

(where $m\in\{+,-\}$ and $z\in\mathbf{J}_N$), then by (5.28), (5.29), (5.21), and (5.22), we find that

$$
\lim_{N\to\infty}\max\Big\{\sigma_N\int_{\mathbf{J}_N}|\mathbf{u}_N^+-\delta_0\,1_{\mathbf{J}_N^-}|\,,\sigma_N\int_{\mathbf{J}_N}|\mathbf{u}_N^--\delta_0\,1_{\mathbf{J}_N^+}|\Big\}=0\,, \tag{5.37}
$$

$$
\lim_{N\to\infty}\max\Big\{\Big|2\,\Phi(\delta_0)-\frac{\mathcal{AC}_1(\mathbf{u}_N;\mathbf{J}_N)}{2}\Big|,\Big|\Phi(\delta_0)-\frac{\mathcal{AC}_1(\mathbf{u}_N;\mathbf{G}_N^\pm)}{2}\Big|\Big\}=0\,, \tag{5.38}
$$

$$
\mathbf{J}_N\cap\partial^*\mathbf{G}_N^\pm \text{ is } \mathcal{H}^n\text{-contained in } \{\mathbf{u}_N^*=\delta_N\}\,, \tag{5.39}
$$

$$
\mathbf{G}_N^\pm \text{ is } \mathcal{L}^{n+1}\text{-contained in } \{\mathbf{u}_N<\delta_N\}\,, \tag{5.40}
$$

where $\sigma_N \to 0$ and $\delta_N \to \delta_0$ as $N \to \infty$. We now formulate a clam, where, for $a \in \mathbb{R}$, we use the notation,

$$\mathbf{T}_a(x) = x - a\,e_{n+1}\,, \qquad \mathbf{t}(x) = x_{n+1}\,, \qquad \forall x \in \mathbb{R}^{n+1}\,. \tag{5.41}$$

Claim: there exists a sequence $\{a_N\}_N$ with $\sigma_N\,|a_N| \le 1/2$ such that

$$\lim_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times Z} |\mathbf{u}_N \circ \mathbf{T}_{a_N} - q_0 \circ \mathbf{t}|^2 + |(\nabla \mathbf{u}_N) \circ \mathbf{T}_{a_N} - (q_0' \circ \mathbf{t})\,e_{n+1}|^2\,dx = 0\,, \tag{5.42}$$

for every $Z \subset\subset (-\infty, Q_0]$. Here we denote by $q : \mathbb{R} \to (0,1)$ the unique solution to

$$q' = \sqrt{W(q)} \quad \text{on } \mathbb{R}\,, \qquad q(0) = \frac{1}{4}\,, \qquad \lim_{t\to-\infty} q(t) = 0\,, \tag{5.43}$$

and we define $q_0 : \mathbb{R} \to (0, \delta_0]$ as

$$q_0 = 1_{(-\infty,Q_0]}\,q + 1_{(Q_0,+\infty)}\,\delta_0\,, \qquad Q_0 = q^{-1}(\delta_0)\,. \tag{5.44}$$

(In particular, $Q_0 > 0$ by $\delta_0 \ge 1/2$, and $Q_0 = +\infty$ if and only if $\delta_0 = 1$.)

Conclusion of the theorem from the claim: By (5.26), if we set $\mathbf{\Lambda}_N := \lambda_{j_N}\,\varepsilon_{j_N}$, then

$$\int_{\mathbf{J}_N} \mathrm{ac}_1(\mathbf{u}_N)\,\mathrm{div}\,Z - 2\,\nabla\mathbf{u}_N \cdot \nabla Z[\nabla\mathbf{u}_N] = \mathbf{\Lambda}_N \int_{\mathbf{J}_N} V(\mathbf{u}_N)\,\mathrm{div}\,Z\,,$$

for every $Z \in C^\infty_c(\mathbf{J}_N;\mathbb{R}^{n+1})$. In fact, setting $\mathbf{U}_N(x) = \mathbf{u}_N(x - a_N\,e_{n+1})$ and noticing that by $\sigma_N\,|a_N| \le 1/2$ it holds that

$$B_1(z_0) \subset \mathbf{Q}^n_{1/2} \times (-1/2\,\sigma_N, 1/2\,\sigma_N) \subset \mathbf{J}_N - a_N\,e_{n+1}$$

where $z_0 = (Q_0 - 1)\,e_{n+1}$, we conclude that

$$\int_{B_1(z_0)} \mathrm{ac}_1(\mathbf{U}_N)\,\mathrm{div}\,Z - 2\,\nabla\mathbf{U}_N \cdot \nabla Z[\nabla\mathbf{U}_N] = \mathbf{\Lambda}_N \int_{B_1(z_0)} V(\mathbf{U}_N)\,\mathrm{div}\,Z\,, \tag{5.45}$$

for every $Z \in C^\infty_c(B_1(z_0);\mathbb{R}^{n+1})$. Let $\varphi \in C^\infty_c(B_1(z_0))$ be radially symmetric decreasing with respect to z_0, and set $Z(x) = \varphi(x)\,e_{n+1}$. In this way, denoting by ρ the reflection of $\mathbb{R}^{n+1}$ with respect to $\{x_{n+1} = Q_0 - 1\}$ and noticing that $\partial_{n+1}\varphi$ is odd with respect to such reflection, we deduce by (5.42) that

$$\begin{aligned}\lim_{N\to\infty} \int_{B_1(z_0)} V(\mathbf{U}_N)\,\mathrm{div}\,Z &= \int_{B_1(z_0)} V(q_0(x_{n+1}))\,\partial_{n+1}\varphi \\ &= \int_{B_1(z_0)\cap\{x_{n+1}>Q_0-1\}} \big\{V(q_0(x_{n+1})) - V\big(q_0(\rho(x)_{n+1})\big)\big\}\,\partial_{n+1}\varphi\,,\end{aligned}$$

Now, since q_0 is strictly increasing on $(-\infty, Q_0]$, and V is strictly increasing on $[0,1]$, we have $V(q_0(x_{n+1})) > V(q_0(\rho(x)_{n+1}))$ for every $x \in B_1(z_0) \cap \{x_{n+1} > Q_0 - 1\}$. Since φ being radially symmetric decreasing with respect to z_0 implies that $\partial_{n+1}\varphi \le 0$ on $\{x_{n+1} > Q_0 - 1\}$, the choice of φ can thus be arranged so that

$$\lim_{N\to\infty} \int_{B_1(z_0)} V(\mathbf{U}_N)\,\mathrm{div}\,Z < 0\,. \tag{5.46}$$

At the same time, by (5.42) and $q_0' = \sqrt{W(q_0)}$ we find that

$$\begin{aligned}&\lim_{N\to\infty} \int_{B_1(z_0)} \mathrm{ac}_1(\mathbf{U}_N)\,\mathrm{div}\,Z - 2\,\nabla\mathbf{U}_N \cdot \nabla Z[\nabla\mathbf{U}_N] \\ &= \int_{B_1(z_0)} \big[W(q_0) - (q_0')^2\big](x_{n+1})\,\partial_{n+1}\varphi = 0\,,\end{aligned}$$

which combined with (5.45) and (5.46) implies that $\mathbf{\Lambda}_N \to 0$ as $N \to \infty$, and completes the proof of the theorem.

Proof of the claim: We begin by reducing the proof of (5.42) to showing the existence of $\{b_N\}_N$ and $\{c_N\}_N$ with $\max\{|b_N|,|c_N|\}\,\sigma_N \le 1/2$ such that for every $W \subset\subset [P_0,\infty)$ and $Z \subset\subset (-\infty, Q_0]$ we have

$$\lim_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times W} |\mathbf{u}_N^+\circ \mathbf{T}_{b_N} - p_0\circ \mathbf{t}|^2 + |(\nabla \mathbf{u}_N^+\circ \mathbf{T}_{b_N}) - (p_0'\circ \mathbf{t})\, e_{n+1}|^2\, dx = 0\,, \tag{5.47}$$

$$\lim_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times Z} |\mathbf{u}_N^-\circ \mathbf{T}_{c_N} - q_0\circ \mathbf{t}|^2 + |(\nabla \mathbf{u}_N^-\circ \mathbf{T}_{c_N}) - (q_0'\circ \mathbf{t})\, e_{n+1}|^2\, dx = 0\,, \tag{5.48}$$

where we set denote by $p:\mathbb{R}\to(0,1)$ the unique solution to

$$p' = -\sqrt{W(p)} \quad \text{on } \mathbb{R}\,, \qquad p(0) = \frac{1}{4}\,, \qquad \lim_{t\to+\infty} p(t) = 0\,, \tag{5.49}$$

and we define $p_0:\mathbb{R}\to(0,\delta_0]$ as

$$p_0 = 1_{[P_0,+\infty)}\, p + 1_{(-\infty,P_0)}\, \delta_0\,, \qquad P_0 = p^{-1}(\delta_0)\,. \tag{5.50}$$

To deduce (5.42) from (5.48) and (5.47), we first note that for any $Z \subset\subset (-\infty,Q_0]$ and $W \subset\subset [P_0,\infty)$, since $Z \subset \{q_0 < \delta_0\}$ and $\{\mathbf{u}_N^- < \delta_N\} = \mathbf{G}_N^-$ (which follows from the definition (5.24) of $\tilde u_N^-$), (5.48) and the analogous statements for $\mathbf{u}_N^+$ imply that

$$\lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times Z)\setminus \mathbf{T}_{c_N}(\mathbf{G}_N^-)| = 0 = \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times W)\setminus \mathbf{T}_{b_N}(\mathbf{G}_N^+)|\,. \tag{5.51}$$

As a consequence, the moving rectangles in $\mathbf{J}_N$ on which $\mathbf{u}_N^-$ and $\mathbf{u}_N^+$ are locally converging to translations of $q_0\circ\mathbf{t}$ and $p_0\circ\mathbf{t}$, respectively, cannot overlap too much: more precisely, given $C_1, C_2 > 0$ with $-2Q_0 + C_1 < C_2$, there exists $N_0(C_1,C_2)$ such that

$$c_N - b_N \notin [-2Q_0 + C_1, C_2] \quad \forall N \ge N_0\,. \tag{5.52}$$

Indeed, if (5.52) did not hold for some C_1 and C_2, then, up to a subsequence which we do not notate, we would have $c_N - b_N \to C' \in [-2Q_0 + C_1, C_2]$. Then testing (5.51) with $W = [P_0 + C_1/3, C_2]$ and $Z = [-C_2, Q_0 - C_1/3]$ would give

$$\begin{aligned}
0 &= \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times[P_0 + C_1/3, C_2])\setminus \mathbf{T}_{b_N}(\mathbf{G}_N^+)|\\
&= \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times[b_N + P_0 + C_1/3, b_N + C_2])\setminus \mathbf{G}_N^+| = \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times(b_N + W))\setminus \mathbf{G}_N^+|\,,\\
0 &= \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times[-C_2, Q_0 - C_1/3])\setminus \mathbf{T}_{b_N}(\mathbf{G}_N^-)|\\
&= \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times[c_N - C_2, c_N + Q_0 - C_1/3])\setminus \mathbf{G}_N^-| = \lim_{N\to\infty} |(\mathbf{Q}^n_{1/2}\times(c_N + Z))\setminus \mathbf{G}_N^-|\,.
\end{aligned}$$

Now since $b_N + W$ and $c_N + Z$ are intervals of equal length (by $P_0 = -Q_0$), we may bound the length of their intersection from below by subtracting endpoints as follows:

$$\begin{aligned}
&\liminf_{N\to\infty} \mathcal{L}^1\big((b_N + W)\cap(c_N + Z)\big)\\
&\quad\ge \liminf_{N\to\infty} \min\{c_N + Q_0 - C_1/3 - (b_N + P_0 + C_1/3), b_N + C_2 - (c_N - C_2)\}\\
&\quad= \liminf_{N\to\infty} \min\{c_N - b_N + 2Q_0 - 2C_1/3, b_N - c_N + 2C_2\}\\
&\quad= \min\{C' - (-2Q_0 + 2C_1/3), 2C_2 - C')\} \ge \min\{C_1/3, C_2\} > 0\,.
\end{aligned}$$

But this contradicts $|\mathbf{G}_N^+\cap\mathbf{G}_N^-| = 0$, since the previous two estimates yield

$$0 = \lim_{N\to\infty} |\mathbf{G}_N^+\cap\mathbf{G}_N^-| \ge \liminf_{N\to\infty} \mathcal{H}^n(\mathbf{Q}^n_{1/2})\times\mathcal{L}^1\big((b_N + W)\cap(c_N + Z)\big) > 0\,.$$

Moving on to proving (5.42), since $\mathbf{u}_N = \mathbf{u}_N^- 1_{\mathbf{G}_N^-} + \mathbf{u}_N 1_{\mathbf{J}_N\setminus\mathbf{G}_N^-}$, (5.51) and (5.48) imply that for any $Z' \subset\subset (-\infty, Q_0]$,

$$\lim_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times Z'} |\mathbf{u}_N\circ\mathbf{T}_{c_N} - q_0\circ\mathbf{t}|^2 dx = 0\,. \tag{5.53}$$

To finish proving (5.42), we fix $Z \subset\subset (-\infty, Q_0]$. By (5.53), it suffices to show that, for $Z \subset\subset Z' \subset\subset (-\infty, Q_0]$ to be chosen shortly,

$$\lim_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times Z'} |\nabla(\mathbf{u}_N \circ \mathbf{T}_{c_N}) - q_0' \circ \mathbf{t}\, e_{n+1}|^2 dx = 0\,. \tag{5.54}$$

We observe that as a consequence of the uniform energy bound (5.38), any subsequence $\{\mathbf{u}_{N_k} \circ \mathbf{T}_{c_{N_k}}\}_k$ of $\{\mathbf{u}_N \circ \mathbf{T}_{c_N}\}_N$ has a further subsequence with a weak $W^{1,2}(\mathbf{Q}^n_{1/2} \times Z')$ limit, which by (5.53), must be $q_0 \circ \mathbf{t}$. Therefore, the entire sequence $\{\mathbf{u}_N \circ \mathbf{T}_{c_N}\}_N$ weakly converges in $W^{1,2}(\mathbf{Q}^n_{1/2} \times Z')$ to $q_0 \circ \mathbf{t}$; in particular $\nabla(\mathbf{u}_N \circ \mathbf{T}_{c_N}) \rightharpoonup q_0' \circ \mathbf{t}\, e_{n+1}$ in $L^2(\mathbf{Q}^n_{1/2} \times Z'; \mathbb{R}^{n+1})$. To upgrade this weak convergence to (5.54), it is enough to show

$$\lim_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times Z'} |\nabla(\mathbf{u}_N \circ \mathbf{T}_{c_N})|^2 dx = \int_{\mathbf{Q}^n_{1/2}\times Z'} |q_0' \circ \mathbf{t}\, e_{n+1}|^2 dx\,. \tag{5.55}$$

Assuming for contradiction that (5.55) were false, then by the lower-semicontinuity of norms under weak convergence, we would have

$$\liminf_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times Z'} |\nabla(\mathbf{u}_N \circ \mathbf{T}_{c_N})|^2 dx \geq \tau + \int_{\mathbf{Q}^n_{1/2}\times Z'} |q_0' \circ \mathbf{t}\, e_{n+1}|^2 dx\,, \quad \tau > 0\,. \tag{5.56}$$

Since q and p are optimal Allen-Cahn profiles, we may now choose $0 < C_1 < 2Q_0$ small enough and $C_2 > 0$ large enough such that the set $Z' = [-C_2/2, Q_0 - C_1/2] \subset\subset (-\infty, Q_0]$ compactly contains Z and such that Z' and $W' = [P_0 + C_1/2, C_2/2]$ satisfy

$$\mathcal{AC}_1(q_0 \circ \mathbf{t}; \mathbf{Q}^n_{1/2} \times Z')/2 + \mathcal{AC}_1(p_0 \circ \mathbf{t}; \mathbf{Q}^n_{1/2} \times W')/2 \geq 2\,\Phi(\delta_0) - \tau/2\,. \tag{5.57}$$

By (5.52), for each large N, either $c_N - b_N > C_2$ or $b_N - c_N > 2Q_0 - C_1$. This implies that the intervals $c_N + Z'$ and $b_N + W'$ are disjoint for large N: indeed, two closed intervals $[\alpha_1, \alpha_2]$ and $[\alpha_3, \alpha_4]$ are disjoint if and only if $\max\{\alpha_3 - \alpha_2, \alpha_1 - \alpha_4\} > 0$, and

$$\begin{aligned}&\max\{c_N - C_2/2 - (b_N + C_2/2), b_N + P_0 + C_1/2 - (c_N + Q_0 - C_1/2)\}\\ &= \max\{c_N - b_N - C_2, b_N - c_N - 2Q_0 + C_1\} > 0 \quad \text{for large } N\,.\end{aligned}$$

Then using in order (5.38) and the disjointness of $c_N + Z'$ and $b_N + W'$; (5.56), the lower semicontinuity of norms under weak convergence, and Fatou's lemma; and (5.57), we may compute

$$\begin{aligned}2\,\Phi(\delta_0) &\geq \liminf_{N\to\infty} \mathcal{AC}_1(\mathbf{u}_N; \mathbf{Q}_{1/2} \times (c_N + Z'))/2 + \mathcal{AC}_1(\mathbf{u}_N; \mathbf{Q}_{1/2} \times (b_N + W'))/2\\ &\geq \tau + \mathcal{AC}_1(q_0 \circ \mathbf{t}; \mathbf{Q}_{1/2} \times Z')/2 + \mathcal{AC}_1(p_0 \circ \mathbf{t}; \mathbf{Q}_{1/2} \times W')/2\\ &\geq 2\,\Phi(\delta_0) + \tau/2\,.\end{aligned}$$

This is a contradiction since $\tau > 0$. Thus (5.55) holds and gives (5.42).

The rest of the proof is thus devoted to showing the validity of (5.48); the proof of (5.47) is the same. To begin with, we show that, setting $\nabla' = (\partial_1, ..., \partial_n)$, we have

$$\lim_{N\to\infty} \int_{\mathbf{J}_N} |\nabla' \mathbf{u}_N^-|^2 = 0\,, \qquad \limsup_{N\to\infty} \int_{\mathbf{J}_N} |\nabla(\Phi \circ \mathbf{u}_N^-)| \leq \Phi(\delta_0)\,. \tag{5.58}$$

(In particular, local limits of $\mathbf{u}_N^\pm$ will depend only on the x_{n+1}-variable.) Indeed, by (5.37), for $\mathcal{H}^n$-a.e. $y \in \mathbf{Q}^n_{1/2}$ we can find

$$\begin{aligned}s_N^y &\in \Big(-\frac{1}{\sigma_N}, -\frac{2}{3\,\sigma_N}\Big)\,, \quad \text{s.t.} \quad \lim_{N\to\infty} \mathbf{u}_N^-(y, s_N^y) = 0\,,\\ t_N^y &\in \Big(\frac{2}{3\,\sigma_N}, \frac{1}{\sigma_N}\Big)\,, \quad \text{s.t.} \quad \lim_{N\to\infty} \mathbf{u}_N^-(y, t_N^y) = \delta_0\,.\end{aligned} \tag{5.59}$$

Since, for $\mathcal{H}^n$-a.e. $y \in \mathbf{Q}^n_{1/2}$,

$$t \mapsto \mathbf{u}_N^-(y,t) \text{ is absolutely continuous on } (-1/\sigma_N, 1/\sigma_N) \text{ for every } N\,, \tag{5.60}$$

by Fubini's theorem and Fatou's lemma,

$$\liminf_{N\to\infty} \int_{\mathbf{J}_N} |\partial_{n+1}(\Phi \circ \mathbf{u}_N^-)| \ge \int_{\mathbf{Q}^n_{1/2}} \liminf_{N\to\infty} \int_{(s_N^y, t_N^y)} |\partial_{n+1}(\Phi\circ\mathbf{u}_N^-)| \ge \Phi(\delta_0)\,. \tag{5.61}$$

Combining (5.61) with (5.38) and with $\nabla(\Phi\circ\mathbf{u}_N^-) = 0$ a.e. on $\mathbf{G}_N^-$, we find

$$\begin{aligned}\Phi(\delta_0) &\le \liminf_{N\to\infty} \int_{\mathbf{J}_N} |\partial_{n+1}(\Phi\circ\mathbf{u}_N^-)| = \liminf_{N\to\infty}\int_{\mathbf{G}_N^-} |\partial_{n+1}(\Phi\circ\mathbf{u}_N^-)| \\ &\le \liminf_{N\to\infty} \int_{\mathbf{G}_N^-} \big(|\partial_{n+1}\mathbf{u}_N^-|^2 + W(\mathbf{u}_N^-)\big)/2 \le \liminf_{N\to\infty}\int_{\mathbf{G}_N^-} \mathrm{ac}_1(\mathbf{u}_N^-)/2 \le \Phi(\delta_0)\,.\end{aligned} \tag{5.62}$$

from which we immediately deduce the first conclusion in (5.58); the limsup inequality in (5.58) is of course derived along similar lines.

Next, we claim that for every $\eta \in (0,\delta_0)$, we can find sequences $\{c_N^-\}_N$ and $\{c_N^+\}_N$ with $-1/2\sigma_N \le c_N^- < c_N^+ < 1/2\sigma_N$ such that

$$\int_{\mathbf{Q}^n_{1/2}\times(c_N^-, c_N^-+1)} \mathbf{u}_N^- = \eta\,, \qquad \int_{\mathbf{Q}^n_{1/2}\times(c_N^+, c_N^++1)} \mathbf{u}_N^- = \delta_0 - \eta\,. \tag{5.63}$$

To prove this, let us consider the continuous function f_N defined by

$$f_N(t) = \int_{\mathbf{Q}^n_{1/2}\times(t,t+1)} \mathbf{u}_N^-\,, \qquad |t| < \frac{1}{\sigma_N} - 1\,.$$

Denoting by k_N the integer part of $1/(2\,\sigma_N)$, we have that

$$\begin{aligned}\sigma_N \int_{\mathbf{J}_N} |\mathbf{u}_N^- - \delta_0\, 1_{\mathbf{J}_N^+}| \;&\ge\; \sigma_N \Big\{ \sum_{k=0}^{k_N} \int_{\mathbf{Q}^n_{1/2}\times(k,k+1)} |\mathbf{u}_N^- - \delta_0| + \sum_{h=0}^{k_N} \int_{\mathbf{Q}^n_{1/2}\times(-h-1,-h)} \mathbf{u}_N^- \Big\} \\ &\ge\; \sigma_N \Big\{ \sum_{k=0}^{k_N} |f_N(k) - \delta_0| + \sum_{h=0}^{k_N} f_N(-h) \Big\}\,,\end{aligned}$$

so that, by $\sigma_N\, k_N \ge (1/2) - \sigma_N \ge 1/4$ (N large), we see that

$$\lim_{N\to\infty} \Big\{ \inf_{0\le k\le k_N} |f_N(k) - \delta_0| + \inf_{0\le h\le k_N} f_N(-h) \Big\} = 0\,.$$

In particular, for every N large enough depending on η, we can find integers $k,h \in (0, 1/(2\sigma_N))$ such that $f_N(k) > \delta_0 - \eta$ and $f_N(-h) < \eta$, and c_N^+ and c_N^- satisfying (5.63) are then found by the intermediate value theorem.

By (5.38), both $\{\mathbf{u}_N^- \circ \mathbf{T}_{c_N^-}\}_N$ and $\{\mathbf{u}_N^-\circ\mathbf{T}_{c_N^+}\}_N$ are bounded in $W^{1,2}(\mathbf{Q}^n_{1/2}\times Z)$ for every $Z \subset\subset \mathbb{R}$. Thus, up to extracting a not relabeled subsequence and taking into account (5.58), we can find $q_0^-, q_0^+ \in (W^{1,2}_{\mathrm{loc}} \cap C^{0,1/2}_{\mathrm{loc}})(\mathbb{R};[0,\delta_0])$ such that, as $N\to\infty$,

$$\begin{cases} \mathbf{u}_N^- \circ \mathbf{T}_{c_N^-} \rightharpoonup q_0^- \circ \mathbf{t}\,, \\ \mathbf{u}_N^- \circ \mathbf{T}_{c_N^+} \rightharpoonup q_0^+ \circ \mathbf{t}\,, \qquad \text{weakly in } W^{1,2}(\mathbf{Q}^n_{1/2}\times Z),\ \forall\, Z \subset\subset \mathbb{R}\,, \end{cases} \tag{5.64}$$

$$\begin{cases} \mathbf{u}_N^- \circ \mathbf{T}_{c_N^-} \to q_0^- \circ \mathbf{t}\,, \\ \mathbf{u}_N^- \circ \mathbf{T}_{c_N^+} \to q_0^+ \circ \mathbf{t}\,, \qquad \text{a.e. on } \mathbf{Q}^n_{1/2}\times\mathbb{R}\,. \end{cases} \tag{5.65}$$

In particular, by (5.63) and (5.64), we have that

$$\int_{(0,1)} q_0^- = \eta\,,\quad \int_{(0,1)} q_0^+ = \delta_0-\eta\,,\quad \exists\, s_*,t_*\in(0,1)\text{ s.t.}\begin{cases} q_0^-(s_*)=\eta\,,\\ q_0^+(t_*)=\delta_0-\eta\,.\end{cases} \tag{5.66}$$

On combining (5.66), (5.65) and (5.60) we conclude that, for $\mathcal{H}^n$-a.e. $y\in\mathbf{Q}^n_{1/2}$,

$$\eta=\lim_{N\to\infty}\mathbf{u}_N^-(y,c_N^-+s_*)\,,\qquad \delta_0-\eta=\lim_{N\to\infty}\mathbf{u}_N^-(y,c_N^++t_*)\,. \tag{5.67}$$

Since $(q_0^\pm)'\in L^2(\mathbb{R})$, the limits

$$q_0^\pm(+\infty):=\lim_{t\to+\infty}q_0^\pm(t)\,,\qquad q_0^\pm(-\infty):=\lim_{t\to-\infty}q_0^\pm(t)\,,$$

exist, and we can prove that we always have

$$\{q_0^+(\pm\infty),q_0^-(\pm\infty)\}\subset\{0,\delta_0\}\,. \tag{5.68}$$

Indeed, by (5.38), by $\mathbf{G}_N^-=\{\mathbf{u}_N^-<\delta_N\}$ and $\mathbf{u}_N=\mathbf{u}_N^-$ on $\mathbf{G}_N^-$, by Fatou's lemma, and by (5.65),

$$\begin{aligned}\Phi(\delta_0)&=\lim_{N\to\infty}\frac{\mathcal{AC}_1(\mathbf{u}_N;\mathbf{G}_N^-)}{2}\ge\limsup_{N\to\infty}\frac12\int_{\{\mathbf{u}_N^-<\delta_N\}}W(\mathbf{u}_N^-)\\ &=\limsup_{N\to\infty}\frac12\int_{\{\mathbf{u}_N^-\circ\mathbf{T}_{c_N^-}<\delta_N\}}W\big(\mathbf{u}_N^-\circ\mathbf{T}_{c_N^-}\big)\\ &\ge\frac12\int_{\{\mathbf{u}_N^-\circ\mathbf{T}_{c_N^-}<\delta_N\}}\liminf_{N\to\infty}W\big(\mathbf{u}_N^-\circ\mathbf{T}_{c_N^-}\big)\ge\int_{\{q_0^-<\delta_0\}}W(q_0^-)\,,\end{aligned}$$

and, similarly, we prove that $\int_{\{q_0^+<\delta_0\}}W(q_0^+)<\infty$; since $W>0$ on $(0,\delta_0)$, we deduce (5.68). We now need to split the proof of (5.48) depending on the value of the limit inferior of $c_N^+-c_N^-\ge0$.

Proof of (5.48) *in the case when*

$$\liminf_{N\to\infty}c_N^+-c_N^-\ge1\,. \tag{5.69}$$

In this case we must have $c_N^-+s_*<c_N^++t_*$ for every N large enough. Therefore, by (5.67), we thus find

$$\liminf_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times[c_N^-+s_*,c_N^++t_*]}|\nabla(\Phi\circ\mathbf{u}_N^-)|\ge\Phi(\delta_0-\eta)-\Phi(\eta)\,. \tag{5.70}$$

We can now improve (5.68) by showing that, if η is sufficiently small in terms of δ_0 and W, then

$$q_0^+(+\infty)=\delta_0\,,\qquad q_0^-(-\infty)=0\,. \tag{5.71}$$

Indeed, by using, in the order, (5.66) and the absolute continuity of $\Phi\circ q_0^\pm$, (5.64) and the lower semicontinuity of the total variation, the second conclusion in (5.58), and (5.70), we find

$$\begin{aligned}|\Phi(q_0^-(-\infty))-\Phi(\eta)|+|\Phi(q_0^+(+\infty))-\Phi(\delta_0-\eta)|&\le\int_{-\infty}^{s_*}|(\Phi\circ q_0^-)'|+\int_{t_*}^{\infty}|(\Phi\circ q_0^+)'|\\ &\le\liminf_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times[(-1/\sigma_N,c_N^-+s_*)\cup(c_N^++t_*,1/\sigma_N)]}|D(\Phi\circ\mathbf{u}_N^-)|\\ &\le\Phi(\delta_0)-\liminf_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}\times(c_N^-+s_*,c_N^++t_*)}|D(\Phi\circ\mathbf{u}_N^-)|\\ &\le\Phi(\delta_0)-\big(\Phi(\delta_0-\eta)-\Phi(\eta)\big)\,.\end{aligned} \tag{5.72}$$

This inequality, combined with (5.68) and considered with η small enough in terms of δ_0 and W, implies (5.71).

Armed with (5.71), we prove that q_0^+ is strictly increasing on $\{0 < q_0^+ < \delta_0\}$. Indeed, if this were not the case, then

$$\exists\, t_1 < t_2 \text{ s.t. } q_0^+(t_1) \geq q_0^+(t_2) \in (0,\delta_0)\,. \tag{5.73}$$

Setting for brevity $I_N[t_1,t_2] = \mathbf{Q}^n_{1/2} \times [t_1 + c_N^+, t_2 + c_N^+]$ we could then estimate

$$\begin{aligned}\mathcal{A}_1(\mathbf{u}_N^-;\mathbf{G}_N^-) &\geq \mathcal{A}_1(\mathbf{u}_N^-;\mathbf{G}_N^- \setminus I_N[t_1,t_2]) + \mathcal{A}_1(\mathbf{u}_N^-;\mathbf{G}_N^- \cap I_N[t_1,t_2]) \\ &\geq 2\int_{\mathbf{J}_N\setminus I_N[t_1,t_2]} |D(\Phi\circ\mathbf{u}_N^-)| + \int_{\mathbf{G}_N^-\cap I_N[t_1,t_2]} W(\mathbf{u}_N^-)\,. \end{aligned}\tag{5.74}$$

Now, with s_N^y and t_N^y as in (5.59), we have

$$s_N^y < -\frac{2}{3\,\sigma_N} < -\frac{1}{2\sigma_N} < c_N^- < c_N^+ < \frac{1}{2\,\sigma_N} < \frac{2}{3\,\sigma_N} < t_N^y\,,$$

so that, thanks to $\sigma_N \to 0^+$ as $N\to\infty$, for N large enough we find $s_N^y < c_N^+ + t_1$ and $t_N^y > c_N^+ + t_2$ $\mathcal{H}^n$-a.e. on $\mathbf{Q}^n_{1/2}$. In particular,

$$\begin{aligned}&\liminf_{N\to\infty}\int_{\mathbf{J}_N\setminus I_N[t_1,t_2]} |D(\Phi\circ\mathbf{u}_N^-)| \\ &\geq \liminf_{N\to\infty}\int_{\mathbf{Q}^n_{1/2}} d\mathcal{H}^n_y \int_{c_N^+ + t_2}^{t_N^y} |D(\Phi\circ\mathbf{u}_N^-)| + \int_{\mathbf{Q}^n_{1/2}} d\mathcal{H}^n_y \int_{s_N^y}^{c_N^+ + t_1} |D(\Phi\circ\mathbf{u}_N^-)| \\ &\geq \int_{\mathbf{Q}^n_{1/2}} \liminf_{N\to\infty} |\Phi(\mathbf{u}_N^-(y,t_N^y)) - \Phi(\mathbf{u}_N^-(y,c_N^+ + t_2))| \\ &+ \int_{\mathbf{Q}^n_{1/2}} \liminf_{N\to\infty} |\Phi(\mathbf{u}_N^-(y,c_N^+ + t_1)) - \Phi(\mathbf{u}_N^-(y,s_N^y))|\,d\mathcal{H}^n_y \\ &= |\Phi(\delta_0) - \Phi(q_0^+(t_2))| + |\Phi(q_0^+(t_1)) - \Phi(0)| \\ &= \Phi(\delta_0) - \Phi(q_0^+(t_2)) + \Phi(q_0^+(t_1)) \geq \Phi(\delta_0)\,, \end{aligned}\tag{5.75}$$

where we have used $q_0^+(t_2) \leq q_0^+(t_1)$ and the fact that Φ is increasing on $[0,1]$. Concerning the second term in (5.74) we notice that, thanks to $\mathbf{G}_N^- = \{\mathbf{u}_N^- < \delta_N\}$ we find

$$\int_{\mathbf{G}_N^-\cap I_N[t_1,t_2]} W(\mathbf{u}_N^-) = \int_{\{\mathbf{u}_N^-\circ\mathbf{T}_{c_N^+} < \delta_N\}\cap[\mathbf{Q}^n_{1/2}\times(t_1,t_2)]} W(\mathbf{u}_N^-\circ\mathbf{T}_{c_N^+})\,,$$

so that

$$\liminf_{N\to\infty}\int_{\mathbf{G}_N^-\cap I_N[t_1,t_2]} W(\mathbf{u}_N^-) \geq \int_{\{q_0^+<\delta_0\}\cap(t_1,t_2)} W(q_0^+) =: c\,, \tag{5.76}$$

where $c > 0$ since $q_0^+(t_1), q_0^+(t_2) \in (0,\delta_0)$, $W > 0$ on $(0,\delta_0)$, and q_0^+ is continuous. By combining (5.38) with (5.74), (5.75) and (5.76) we conclude that

$$\Phi(\delta_0) = \lim_{N\to\infty}\mathcal{A}_1(\mathbf{u}_N^-;\mathbf{G}_N^-) \geq \Phi(\delta_0) + c > \Phi(\delta_0)\,,$$

thus obtaining a contradiction with (5.73).

Having proved that q_0^+ is strictly increasing on the (possibly unbounded) interval $(a,b) = \{0 < q_0^+ < \delta_0\}$, we see that it must be $q_0^+(t) \to 0^+$ as $t\to a^+$. Since $q_0^+(+\infty) = \delta_0$ implies $q_0^+(t)\to\delta_0$ as $t\to b^-$, we find

$$\Phi(\delta_0) = \int_{\{0<q_0^+<\delta_0\}} |(\Phi\circ q_0^+)'| \leq \int_{\{0<q_0^+<\delta_0\}} \mathcal{AC}_1(q_0^+\circ\mathbf{t}) \leq \Phi(\delta_0)\,,$$

so that $(q_0^+)' = \sqrt{W(q_0)}$ on $\{0 < q_0^+ < \delta_0\}$. By the Cauchy uniqueness theorem and by definition (5.43) of the Allen–Cahn one-dimensional profile q, since q_0^+ is not identically equal to 0, there exists $c_0 \in \mathbb{R}$ such that

$$q_0^+(t) = q(t - c_0)\,, \qquad \forall t \in \{0 < q_0^+ < \delta_0\} = \big(-\infty, c_0 + Q_0\big)\,,$$

where, we recall, $Q_0 := q^{-1}(\delta_0)$. Therefore, if we set $c_N = c_N^+ + c_0$, then we conclude that

$$\mathbf{u}_N^- \circ \mathbf{T}_{c_N} \rightharpoonup q_0 \circ \mathbf{t} \text{ weakly in } W^{1,2}(\mathbf{Q}^n_{1/2} \times Z),\, \forall\, Z \subset\subset \mathbb{R}\,, \tag{5.77}$$

where $q_0 = 1_{(-\infty,Q_0]}\, q + 1_{(Q_0,\infty)}\, \delta_0$ is defined as in (5.44). The fact that

$$\begin{aligned}\Phi(\delta_0) &= \int_{\mathbf{Q}^n_{1/2}\times(-\infty,Q_0)} |\nabla(q_0\circ\mathbf{t})|^2 = \frac{\mathcal{AC}_1(q_0\circ\mathbf{t};\mathbf{Q}^n_{1/2}\times(-\infty,Q_0))}{2}\\ &\le \liminf_{N\to\infty} \frac{\mathcal{AC}_1(\mathbf{u}_N^-\circ\mathbf{T}_{c_N};\mathbf{Q}^n_{1/2}\times(-\infty,Q_0))}{2} \le \Phi(\delta_0)\,,\end{aligned}$$

implies that

$$\lim_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times(-\infty,Q_0)} |\nabla(\mathbf{u}_N^-\circ\mathbf{T}_{c_N})|^2 = \int_{\mathbf{Q}^n_{1/2}\times(-\infty,Q_0)} |\nabla(q_0\circ\mathbf{t})|^2\,,$$

and thus to improve the weak convergence stated in (5.77) to the strong convergence claimed in (5.48). This complete the proof of (5.48) in the case when (5.69) holds.

Proof of (5.48) *in the case when*

$$\liminf_{N\to\infty} c_N^+ - c_N^- < 1\,. \tag{5.78}$$

In this case, up to extracting a subsequence in N, we can assume that $c_N^+ - c_N^- \to d_0 \in [0,1)$. In particular, we find

$$q_0^+(t) = q_0^-(t - d_0)\,, \qquad \forall t \in \mathbb{R}\,,$$

so that, by (5.66),

$$\int_{(-d_0,-d_0+1)} q_0^+ = \eta\,, \qquad \int_{(0,1)} q_0^+ = \delta_0 - \eta\,, \tag{5.79}$$

and, in particular, $d_0 \in (0,1)$ by taking $\eta < \delta_0/2$. With respect to the case when (5.69) holds, we now define s_* and t_* differently, by claiming that

$$\exists\, s_* < t_* \text{ s.t. } q_0^+(s_*) = \eta \text{ and } q_0^+(t_*) = \delta_0 - 2\,\eta\,. \tag{5.80}$$

Indeed, by (5.79) and the continuity of q_0^+, there is $s_* \in (-d_0, -d_0+1)$ such that $q_0^+(s_*) = \eta$, while again by (5.80),

$$\int_{-d_0+1}^1 q_0^+ = \int_0^1 q_0^+ - \int_0^{-d_0+1} q_0^+ = \delta_0 - \eta - \int_0^{-d_0+1} q_0^+ \ge \delta_0 - 2\,\eta\,,$$

so that there is also $r_* \in (-d_0+1, 1)$ (and thus, such that $r_* > s_*$) with the property that

$$q_0^+(r_*) = \frac{1}{1-(-d_0+1)} \int_{-d_0+1}^1 q_0^+ \ge \frac{\delta_0 - 2\,\eta}{d_0} \ge \delta_0 - 2\,\eta\,.$$

Since $q_0^+(s_*) = \eta$, we can find $t_* \in (s_*, r_*)$ such that $q_0^+(t_*) = \delta_0 - 2\,\eta$ and conclude the proof of (5.80).

Having proved (5.80), we find

$$\liminf_{N\to\infty} \int_{\mathbf{Q}^n_{1/2}\times[c_N^+ + s_*, c_N^+ + t_*]} |\nabla(\Phi\circ\mathbf{u}_N^-)| \ge \Phi(\delta_0 - 2\,\eta) - \Phi(\eta)\,, \tag{5.81}$$

which we can use in place of (5.70) to argue as in (5.72) to prove (5.71) (that is, $q_0^+(+\infty) = \delta_0$ and $q_0^-(-\infty) = 0$). Without modifications one repeats the rest of the proof, from establishing that q_0^+ is strictly increasing on $\{0 < q_0^+\}$ to showing that q_0^+ is a translation of q and a strong $W^{1,2}$-limit of $\mathbf{T}_{c_N} \circ \mathbf{u}_N^-$ for some finite translation c_N of c_N^+. The proof of the theorem is thus complete. $\square$

6. A hierarchy of Plateau problems (Proof of Theorem 1.2)

This section is devoted the proof of Theorem 1.2. We premise a simple lemma that will be used in the course of the proof.

Lemma 6.1. *If* $\mathbf{W} \subset \mathbb{R}^{n+1}$ *is compact,* $\Omega = \mathbb{R}^{n+1} \setminus \mathbf{W}$*,* $\varepsilon_j \to 0^+$ *as* $j \to \infty$*,* $u_j \in W^{1,2}_{\rm loc}(\Omega; [0,1])$*,* $u_j \to u$ *in* $L^1_{\rm loc}(\Omega)$ *with* $\sup_j \mathcal{AC}_{\varepsilon_j}(u_j; \Omega) < \infty$*,* $\mathcal{V}(u_j; \Omega) \to v > 0$*, and*

$$\mathrm{ac}_{\varepsilon_j}(u_j)\, \mathcal{L}^{n+1} \llcorner \Omega \overset{*}{\rightharpoonup} \mu\,, \qquad \textit{as Radon measures in } \Omega\,,$$

as $j \to \infty$*, then*

$$\liminf_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(u_j;\Omega) \geq \mu(\Omega) + \Theta(v,\varepsilon)\,,$$

where $\Theta(v,\varepsilon)$ *is the diffused Euclidean isoperimetric profile if* $\varepsilon > 0$ *(see Appendix A) and* $\Theta(v,0) = (n+1)\,\omega_{n+1}^{1/(n+1)}\, v^{n/(n+1)}$.

Proof of Lemma 6.1. This can be proved by combining a localization argument which is quite common in the theory of concentration-compactness (and whose details are therefore omitted) with (MM) and the Euclidean isoperimetric inequality in its sharp and diffused versions. $\square$

Proof of Theorem 1.2. Momentarily assuming the validity of conclusion (i), we prove conclusions (ii), (iii), and (iv). To this end, let $\{u_j\}_j$ be a sequence of minimizers for $\Upsilon(v_j, \varepsilon_j, \delta_j)$, where, after achieving $\operatorname{diam} \mathbf{W} = 1$ by scaling, we can assume that, as $j \to \infty$,

$$v_j \to v_0 \in [0, \tau_0]\,, \qquad \frac{\varepsilon_j}{v_j} \to 0^+\,, \tag{6.1}$$

$$\delta_j \to \delta_0 \in \left[\frac{1}{2}, 1\right]\,, \qquad \min\{1-\delta_j, v_j\} \to 0\,. \tag{6.2}$$

In this way, the situation of conclusion (ii) is met when $v_0 > 0$ (in which case (6.2) forces $\delta_0 = 1$) and the situation of conclusion (iii) is met when $v_0 = 0$; and, in both cases, (6.1) implies that $\varepsilon_j \to 0^+$. Since $\ell < \infty$, Ω has smooth boundary, and $\varepsilon_j/v_j \to 0^+$, we can apply Theorem 4.1 (if $v_0 > 0$) and Theorem 4.3 (if $v_0 = 0$), to find

$$\limsup_{j\to\infty} \Upsilon(v_j,\varepsilon_j,\delta_j) \leq \begin{cases} \Psi_{\rm bk}(v_0)\,, & \text{if } v_0 > 0\,, \\ 2\,\Phi(\delta_0)\,\ell\,, & \text{if } v_0 = 0\,. \end{cases} \tag{6.3}$$

Therefore, by Theorem 3.4, up to extracting subsequences, there exist μ a Radon measure on Ω and $(K,E) \in \mathcal{K}_{\rm B}$ with $|E| \leq v_0$ and $K \cup E^{(1)}$ is $\mathcal{C}$-spanning, such that μ is the weak-star limit of $\{|D(\Phi\circ u_j)|\}_j$, 1_E is $L^1_{\rm loc}(\Omega)$-limit of $\{u_j\}_j$, and

$$\mu \geq \begin{cases} 2\mathcal{H}^n \llcorner (K \cap E^{(0)}) + \mathcal{H}^n \llcorner (\partial^* E \cap \Omega)\,, & \text{if } v_0 > 0\,, \\ 2\,\Phi(\delta_0)\,\mathcal{H}^n \llcorner K\,, & \text{if } v_0 = 0\,. \end{cases} \tag{6.4}$$

We claim that $|E| = v_0$. If $v_0 = 0$ this is trivial, while if $v_0 > 0$ we can combine (6.4) with Lemma 6.1 to conclude that

$$\Psi_{\rm bk}(v_0) \geq \mathcal{F}_{\rm bk}(K,E) + (n+1)\,\omega_{n+1}\,(v - |E|)^{n/(n+1)}\,.$$

In particular, if $|E| < v_0$, then we can construct a minimizing sequence for $\Psi_{\rm bk}(v_0)$ with positive volume loss at infinity, thus contradicting Theorem 1.1-(iii). Having proved $|E| =$

v_0, and thus that (K,E) is a minimizer of $\Psi_{\rm bk}(v_0)$, we can combine (6.3), (MM) and (6.4) to find, in the case $v_0>0$,

$$\begin{aligned}\Psi_{\rm bk}(v_0) &\ge \limsup_{j\to\infty}\frac{\mathcal{AC}_{\varepsilon_j}(u_j;\Omega)}{2}\ge\limsup_{j\to\infty}|D(\Phi\circ u_j)|(\Omega) \\ &\ge \mu(\Omega)\ge\mathcal{F}_{\rm bk}(K,E)=\Psi_{\rm bk}(v_0)\,;\end{aligned} \tag{6.5}$$

and, in the case $v_0=0$,

$$2\,\Phi(\delta_0)\,\ell\ge\limsup_{j\to\infty}\frac{\mathcal{AC}_{\varepsilon_j}(u_j;\Omega)}{2}\ge\limsup_{j\to\infty}|D(\Phi\circ u_j)|(\Omega)\ge\mu(\Omega)\ge 2\,\Phi(\delta_0)\,\ell\,. \tag{6.6}$$

From (6.5) and (6.6) we find

$$\lim_{j\to\infty}\Upsilon(v_j,\varepsilon_j,\delta_j)=\begin{cases}\Psi_{\rm bk}(v_0)\,, & \text{if } v_0>0\,,\\ 2\,\Phi(\delta_0)\,\ell\,, & \text{if } v_0=0\,,\end{cases}$$

thus proving conclusions (ii) and (iii), as well as, looking back at (MM),

$$\lim_{j\to\infty}\int_\Omega\Big(\sqrt{\varepsilon_j}\,|\nabla u_j|-\sqrt{W(u_j)/\varepsilon_j}\Big)^2=0\,,$$

from which conclusion (iv) follows immediately.

Proof of conclusion (i): Since the validity of the Euler–Lagrange equation in inner variation form is immediate from Lemma 4.5-(i), it is really a matter of proving that for every positive τ_0 there is a positive τ_1 (depending on the data of the problem) such that $\Upsilon(v,\varepsilon,\delta)$ admits a minimizer for every $(v,\varepsilon,\delta)\in{\rm SFR}(\tau_0,\tau_1)$. We shall actually prove that for every such (v,ε,δ), every minimizing sequence of $\Upsilon(v,\varepsilon,\delta)$ converge (modulo extracting subsequences) to a minimizer. To do this, after a rescaling that sets ${\rm diam}\,\mathbf{W}=1$, we argue by contradiction. This means assuming that there are a sequence $\{(v_j,\varepsilon_j,\delta_j)\}_j$ satisfying (6.1) and (6.2), and, for each j, a minimizing sequence $\{u_j^k\}_k$ for $\Upsilon(v_j,\varepsilon_j,\delta_j)$, such that, up to extracting a diagonal subsequence, for each j there is $u_j^0\in W^{1,2}_{\rm loc}(\Omega;[0,1])$ such that $u_j^k\to u_j^0$ in $L^1_{\rm loc}(\Omega)$ as $k\to\infty$, *but, for no index j, u_j^0 is a minimizer of* $\Upsilon(v_j,\varepsilon_j,\delta_j)$.

Now, by Theorem 3.2 and Theorem 3.4, we know that

$$\{(u_j^0)^*\ge t\}\text{ is }\mathcal{C}\text{-spanning }\mathbf{W}\text{ for every }t\in(1/2,\delta_j)\,, \tag{6.7}$$

for every j, while by lower semicontinuity of the Allen–Cahn energy we have

$$\Upsilon(v_j,\varepsilon_j,\delta_j)\ge\frac{\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)}{2}\,,\qquad\forall j\,. \tag{6.8}$$

Therefore the only possibility for u_j^0 not to be a minimizer of $\Upsilon(v_j,\varepsilon_j,\delta_j)$ is that

$$\mathbf{v}_j^\infty:=v_j-\mathcal{V}(u_j^0;\Omega)>0\,,\qquad\forall j\,. \tag{6.9}$$

We shall conclude the proof of the theorem by exploiting (6.9) to identify a subsequence $\{j(i)\}_{i\in\mathbb{N}}$ such that, for every i large enough,

$$\lim_{i\to\infty}\frac{\mathcal{AC}_{\varepsilon_{j(i)}}(u^k_{j(i)})}{2}>\Upsilon(v_{j(i)},\varepsilon_{j(i)},\delta_{j(i)})\,, \tag{6.10}$$

thus obtaining the desired contradiction.

The idea behind the proof of (6.10) under (6.9) is to "bring back from infinity" the lost volume $\mathbf{v}_j^\infty$ in the form of a "half-bubble" that touches the wire frame, and then to exploit the fact that the isoperimetric profile of half-spaces is strictly less than the isoperimetric profile of $\mathbb{R}^{n+1}$. There are several issues that must be addressed to make this approach work. First is the fact that we do not yet know that the sharp interface limit is a good approximation for the behavior of the escaping volume at infinity, since the sharp interface problem is, roughly speaking, closely describing the escaping volume only if $\varepsilon_j/(\mathbf{v}_j^\infty)^{1/(n+1)} \to 0$. Second, unlike sharp interface problems, in which escaping volumes can be "brought back" and placed somewhere that does not disturb the rest of the configuration, Allen-Cahn minimizers will always have interacting tails. Without the regularity of minimizers (or even of limits of minimizing sequences), controlling these tail interactions will require care. Last but not least is the fact that the energy corresponding to a volume escaping at infinity corresponds to a lower order energy term whose presence does not contradict the leading order convergence of $\Upsilon(v_j, \varepsilon_j, \delta_j)$ to, respectively, $\Psi_{\rm bk}(v_0)$ or to $2\,\Phi(\delta_0)\,\ell$. Any sort of analysis leading to the existence of minimizers must therefore be fine enough to detect vanishingly small inefficiencies in minimizing sequences with escaping volumes.

We divide the argument into steps. In step one, we sharply improve (6.8) to (6.11) and establish minimality and criticality properties of u_j^0. In step two, we conclude the contradiction in the case when $v_0 > 0$ by using Theorem 1.1-(iii). Step three deals with the case $v_0 = 0$, and it is in this case that the difficulties described in the previous paragraph are carefully addressed.

Step one: Denoting by ζ_j a radial minimizer with maximum at the origin for the diffused Euclidean isoperimetric problem $\Theta(\mathbf{v}_j^\infty, \varepsilon_j)$ and by Λ_j the Lagrange multiplier of ζ_j, so that $2\,\varepsilon_j^2\,\Delta\zeta_j = W'(\zeta_j) - \varepsilon_j\,\Lambda_j\,V'(\zeta_j)$ on $\mathbb{R}^{n+1}$ (see Appendix A), we claim that,

$$\Upsilon(v_j, \varepsilon_j, \delta_j) = \big(\mathcal{AC}_\varepsilon(u_j^0;\Omega)/2\big) + \Theta(\mathbf{v}_j^\infty, \varepsilon_j)\,, \tag{6.11}$$

$$u_j^0 \text{ is a minimizer of } \Upsilon(v_j - \mathbf{v}_j^\infty, \varepsilon_j, \delta_j)\,, \tag{6.12}$$

for every j, where u_j^0 satisfies (1.13) with $\lambda = \Lambda_j$, that is

$$\int_\Omega \Big(\varepsilon_j\,|\nabla u_j^0|^2 + \frac{W(u_j^0)}{\varepsilon_j}\Big)\operatorname{div} X - 2\,\varepsilon_j\,\nabla u_j^0 \cdot \nabla X[\nabla u_j^0] = \Lambda_j \int_\Omega V(u_j^0)\operatorname{div} X\,, \tag{6.13}$$

whenever $X \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega = 0$ on $\partial\Omega$.

Indeed, we can prove the $\le$-part of (6.11) by constructing a competitor for $\Upsilon(v_j, \varepsilon_j, \delta_j)$ obtained as a slight modification via Lemma 4.5-(ii) of the *Ansatz* $x \mapsto u_j^0(x) + \zeta_j(x - k\,e)$ ($e \in \mathbb{S}^n$, k large). The matching lower bound is obtained by applying Lemma 6.1 to the minimizing sequence $\{u_j^k\}_k$ of $\Upsilon(v_j, \varepsilon_j, \delta_j)$ (notice that the lemma is applied here "at fixed ε"). Having proved (6.11), we notice that, again by the construction of Theorem 4.3,

$$\Upsilon(v_j, \varepsilon_j, \delta_j) \le \Upsilon(v_j - \mathbf{v}_j^\infty, \varepsilon_j, \delta_j) + \Theta(\mathbf{v}_j^\infty, \varepsilon_j)\,. \tag{6.14}$$

This inequality, combined with (6.11), implies (6.12). By (6.12), (6.13) holds indeed with some Lagrange multiplier λ_j: the fact that $\lambda_j = \Lambda_j$ thus follows by a standard first variation argument (it they were different, we could violate (6.11)).

Step two: We conclude the proof in the case when $v_0 > 0$. Indeed, up to extracting a further subsequence there is $\mathbf{v}_0^\infty \in [0, v_0]$ such that $\mathbf{v}_j^\infty \to \mathbf{v}_0^\infty$ as $j \to \infty$. Since $\mathcal{V}(u_j^0;\Omega) = v_j - \mathbf{v}_j^\infty \to v_0 - \mathbf{v}_0^\infty$ and (thanks to (6.3)) $\sup_j \mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega) < \infty$, up to extracting subsequences we can apply Theorem 3.4 to conclude the existence of $(K, E) \in \mathcal{K}_{\rm B}$ such that $u_j^0 \to 1_E$ in $L^1_{\rm loc}(\Omega)$, $K \cup E^{(1)}$ is $\mathcal{C}$-spanning $\mathbf{W}$, $|E| \le v_0 - \mathbf{v}_0^\infty$, and $\liminf_j \mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)/2 \ge \mathcal{F}_{\rm B}(K, E)$ (when exploiting (3.19), recall that in the present argument $v_0 > 0$ implies $\delta_j \to 1^-$). We can actually improve on this lower bound by using Lemma 6.1, thus concluding that

$$\liminf_{j\to\infty} \frac{\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)}{2} \ge \mathcal{F}_{\rm B}(K, E) + (n+1)\,\omega_{n+1}\,\big(v_0 - \mathbf{v}_0^\infty - |E|\big)^{n/(n+1)}\,. \tag{6.15}$$

By (6.3), (6.11), (6.15), the continuity of Θ (see Theorem A.1-(ii)), and the concavity of the Euclidean isoperimetric profile, we find that

$$
\begin{aligned}
\Psi_{\rm B}(v_0) &\ge \limsup_{j\to\infty}\Upsilon(v_j,\varepsilon_j,\delta_j)=\liminf_{j\to\infty}\big(\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)/2\big)+\Theta(\mathbf{v}_j^\infty,\varepsilon_j)\\
&\ge \mathcal{F}_{\rm B}(K,E)+(n+1)\,\omega_{n+1}^{1/(n+1)}\Big\{(\mathbf{v}_0^\infty)^{n/(n+1)}+\big(v_0-\mathbf{v}_0^\infty-|E|\big)^{n/(n+1)}\Big\}\\
&\ge \mathcal{F}_{\rm B}(K,E)+(n+1)\,\omega_{n+1}^{1/(n+1)}\,2\,(v_0-|E|)^{n/(n+1)}\,. \qquad (6.16)
\end{aligned}
$$

Now, thanks to Theorem 1.1-(iii), no minimizing sequence in $\Psi_{\rm B}(v_0)$ can lose volume at infinity. Therefore (6.16) implies that $|E|=v_0$ and $\mathbf{v}_0^\infty=0$.

We can use the latter information to obtain a contradiction by arguing as follows. Since $|E|=v_0>0$ implies that 1_E is not constant in Ω and $u_j^0\to 1_E$ in $L^1_{\rm loc}(\Omega)$, by Lemma 4.5-(ii) and $\mathcal{V}(u_j^0;\Omega)=v_j\to v_0$ we can find diffeomorphisms $f_j:\Omega\to\Omega$ such that $\mathcal{V}(u_j^0\circ f_j;\Omega)=v_j$ and

$$
|\mathcal{AC}_{\varepsilon_j}(u_j^0\circ f_j;\Omega)-\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)|\le C_0\,\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)\,|\mathcal{V}(u_j^0;\Omega)-v_j|\le C\,\mathbf{v}_j^\infty\,,
$$

where C_0 depends on Ω and E as in Lemma 4.5-(ii), and $C=C_0\sup_j\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)$. Since the homotopic spanning constraint is preserved under composition with a diffeomorphism of Ω, we find that $u_j^0\circ f_j$ is a competitor in $\Upsilon(v_j,\varepsilon_j,\delta_j)$, so that

$$
2\,\Upsilon(v_j,\varepsilon_j,\delta_j)\le\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)+C\,\mathbf{v}_j^\infty\,; \qquad (6.17)
$$

at the same time, by (6.11), we have

$$
2\,\Upsilon(v_j,\varepsilon_j,\delta_j)=\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)+2\,\Theta(\mathbf{v}_j^\infty,\varepsilon_j)\ge\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)+2\,(n+1)\,\omega_{n+1}^{1/(n+1)}\,(\mathbf{v}_j^\infty)^{n/(n+1)} \qquad (6.18)
$$

where the combination of (6.17) and (6.18) leads to a contradiction since $\mathbf{v}_j^\infty\to 0$.

Step three: We are now left to consider the case when $v_0=0$. Thanks to (6.12), (6.11), $v_0=0$, and (6.1), $\{u_j^0\}_j$ is a sequence of minimizers for $\Upsilon(\mathbf{v}_j,\varepsilon_j,\delta_j)$ for some

$$
\mathbf{v}_j:=v_j-\mathbf{v}_j^\infty\in(0,v_j)
$$

such that $\mathcal{AC}_{\varepsilon_j}(u_j^0;\Omega)\le\Upsilon(v_j,\varepsilon_j,\delta_j)$, $v_j\to0^+$, and $\varepsilon_j/v_j\to0^+$. Therefore, by Theorem 5.1 and taking (6.13) into account, we have that

$$
\lim_{j\to\infty}\varepsilon_j\,\Lambda_j=0\,, \qquad (6.19)
$$

as well as that (compare with (5.6))

$$
2\,\Phi(\delta_0)\,\mathcal{H}^n\llcorner K=\mathrm{w}^*\lim_{j\to\infty}|\nabla(\Phi\circ u_j^0)|\,\mathcal{L}^{n+1}\llcorner\Omega=\mathrm{w}^*\lim_{j\to\infty}\frac{\mathrm{ac}_{\varepsilon_j}(u_j^0)}{2}\,\mathcal{L}^{n+1}\llcorner\Omega\,, \qquad (6.20)
$$

where K is a minimizer of ℓ.

Recalling that ζ_j is a minimizer in $\Theta(\mathbf{v}_j^\infty,\varepsilon_j)$ and that $\Lambda_j=\Lambda(\zeta_j)$, by (A.4) (see Theorem A.1-(iii)) we have that

$$
\Lambda_j\ge\frac{c(n,W)}{(\mathbf{v}_j^\infty)^{1/(n+1)}}\,. \qquad (6.21)
$$

This inequality, combined with (6.19), gives in particular

$$
\lim_{j\to\infty}\frac{\varepsilon_j}{(\mathbf{v}_j^\infty)^{1/(n+1)}}=0\,. \qquad (6.22)
$$

Hence, for j large enough, we have $\varepsilon_j<\sigma_0\,(2\,\mathbf{v}_j^\infty)^{1/(n+1)}$ for $\sigma_0=\sigma_0(n,W)>0$ as in Theorem A.1-(iv). Setting $\sigma_j=\varepsilon_j/(2\,\mathbf{v}_j^\infty)^{1/(n+1)}$, and recalling that $\zeta_{v,\varepsilon}$ denotes the unique modulo translations radially symmetric decreasing minimizer of $\Theta(v,\varepsilon)$ when $\varepsilon < \sigma_0\, v^{1/(n+1)}$, let us now consider that

$$\zeta_{2\mathbf{v}_j^\infty,\varepsilon_j}(x)=\zeta_{1,\sigma_j}\Big(x/(2\,\mathbf{v}_j^\infty)^{1/(n+1)}\Big)\,,\qquad \forall x\in\mathbb{R}^{n+1}\,,$$

where we have set $\pi_j(x)=x/(2\,\mathbf{v}_j^\infty)^{1/(n+1)}$ ($x\in\mathbb{R}^{n+1}$); denoting by $B_{r(n)}$ the unit volume ball with center at the origin, the fact that $\sigma_j\to 0^+$ guarantees that

$$\lim_{j\to\infty}\int_{\mathbb{R}^{n+1}}|\zeta_{1,\sigma_j}-1_{B_{r(n)}}|=0\,,\qquad \gamma_j:=\max\Big\{\mathcal{V}(\zeta_{1,\sigma_j};B^c_{2\,r(n)}),\mathcal{AC}_{\sigma_j}(\zeta_{1,\sigma_j};B^c_{2\,r(n)})\Big\}\to 0\,.$$

Hence, by Lemma 4.5, there exist $\eta_0>0$ such that for every j large enough and every $|\eta|<\eta_0$ there is a radially symmetric diffeomorphism $f_j^\eta:B_{2\,r(n)}\to B_{2\,r(n)}$ with $\{f_j^\eta\neq \mathrm{id}\}\subset\subset B_{2\,r(n)}$ and

$$\begin{aligned}&\mathcal{V}(\zeta_{1,\sigma_j}\circ f_j^\eta)=\mathcal{V}(\zeta_{1,\sigma_j})+\eta=1+\eta\,,\\ &\big|\mathcal{AC}_{\sigma_j}(\zeta_{1,\sigma_j}\circ f_j^\eta)-\mathcal{AC}_{\sigma_j}(\zeta_{1,\sigma_j})\big|\le C(n)\,\Theta(1,\sigma_j)\,|\eta|\,.\end{aligned}$$

By radial symmetry of $\zeta_{1,\sigma_j}\circ f_j^\eta$ and ζ_{1,σ_j}, if we set $H=\{x_{n+1}>0\}$, then, for every $|\eta|<\eta_0$ and j large enough,

$$\begin{aligned}&\mathcal{V}(\zeta_{1,\sigma_j}\circ f_j^\eta;H)=\frac{1}{2}+\eta\,,\\ &\mathcal{AC}_{\sigma_j}(\zeta_{1,\sigma_j}\circ f_j^\eta;H)\le\big(1+C(n)\,|\eta|\big)\,\Theta(1,\sigma_j)\,.\end{aligned}$$

Then the functions $\zeta_j^\eta=\zeta_{1,\sigma_j}\circ f_j^\eta\circ\pi_j$ satisfy

$$\begin{aligned}&\mathcal{V}(\zeta_j^\eta;H)=2\,\mathbf{v}_j^\infty\,\mathcal{V}(\zeta_{1,\sigma_j}\circ f_j^\eta;H)=\mathbf{v}_j^\infty+\eta\,\mathbf{v}_j^\infty\,,\\ &\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta;H)=(2\,\mathbf{v}_j^\infty)^{n/(n+1)}\,\mathcal{AC}_{\sigma_j}(\zeta_j^\eta;H)\le\big(1+C(n)\,|\eta|\big)\,\Theta(2\,\mathbf{v}_j^\infty,\varepsilon_j)\,.\end{aligned}$$

At the same time, by (6.22) we can use (A.5) to find

$$\lim_{j\to\infty}\frac{\Theta(2\,\mathbf{v}_j^\infty,\varepsilon_j)}{\Theta(\mathbf{v}_j^\infty,\varepsilon_j)}=2^{n/(n+1)}\,,\tag{6.23}$$

so that there are $\beta(n)\in(0,1)$ and $J_0\in\mathbb{N}$ such that

$$\frac{\Theta(2\,\mathbf{v}_j^\infty,\varepsilon_j)}{2}\le(1-\beta(n))\,\Theta(\mathbf{v}_j^\infty,\varepsilon_j)\,,\qquad\forall j\ge J_0\,,$$

and, in summary,

$$\frac{\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta;H)}{2}\le\big(1-\beta(n)+C(n)|\eta|\big)\,\Theta(\mathbf{v}_j^\infty,\varepsilon_j)\,,\qquad\forall j\ge j_0\,,|\eta|<\eta_0\,.\tag{6.24}$$

We next notice that, by the smoothness of $\partial\mathbf{W}$ and up to a rigid motion that takes $0\in\partial\mathbf{W}$ and $\nu_{\mathbf{W}}(0)=e_{n+1}$, we can find positive constant C and r' depending on $\mathbf{W}$ such that

$$\mathcal{H}^n((H\Delta\Omega)\cap\partial B_r)\le C\,r^{n+1}\,,\qquad\forall r<r'\,,\tag{6.25}$$

where $x=(x',x_{n+1})\in\mathbb{R}^{n+1}\equiv\mathbb{R}^n\times\mathbb{R}$. Therefore, if we set for brevity

$$r_j:=(\mathbf{v}_j^\infty)^{1/(n+1)}\,2\,r(n)$$

them,, up to increase the value of J_0 so to have $r_j<r'$ when $j\ge J_0$, and noticing that $\mathrm{ac}_{\varepsilon_j}(\zeta_j^\eta)$ is a radial function, by $\zeta_{1,\sigma_j}=\zeta_{1,\sigma_j}\circ f_j^\eta$ on $B^c_{2\,r(n)}$ and by definition of γ_j,

$$\begin{aligned}&\Big|\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta;\Omega)-\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta;H)\Big|\le\int_{(\Omega\Delta H)\cap B_{r_j}}\mathrm{ac}_{\varepsilon_j}(\zeta_j^\eta)+2\,\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta;B^c_{r_j})\\ &=\int_0^{r_j}\mathcal{H}^n((\Omega\Delta H)\cap\partial B_r)\,\mathrm{ac}_{\varepsilon_j}(\zeta_j^\eta)\,dr+2\,(\mathbf{v}_j^\infty)^{n/(n+1)}\,\mathcal{AC}_{\sigma_j}(\zeta_{1,\sigma_j}\circ f_j^\eta;B^c_{2\,r(n)})\end{aligned}$$

$$\leq \sup_{0<r<r_j} \Big\{ \frac{\mathcal{H}^n((\Omega \Delta H) \cap \partial B_r)}{\mathcal{H}^n(H \cap \partial B_r)} \Big\} \mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta; H) + 2\, (\mathbf{v}_j^\infty)^{n/(n+1)}\, \gamma_j$$
$$\leq C(n, \mathbf{W})\, r_j\, \Theta(\mathbf{v}_j^\infty, \varepsilon_j) + 2\, (\mathbf{v}_j^\infty)^{n/(n+1)}\, \gamma_j = \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)} .$$

where in the last two inequalities we have used, in the order, by (6.25), (6.24), (A.5), and $\gamma_j \to 0$. By combining this inequality with (6.24) we thus conclude that

$$\frac{\mathcal{AC}_{\varepsilon_j}(\zeta_j^\eta; \Omega)}{2} \leq \Big(1 - \beta(n) + C(n)\, |\eta|\Big)\, \Theta(\mathbf{v}_j^\infty, \varepsilon_j) + \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)} , \tag{6.26}$$

(where we recall that $\Theta(\mathbf{v}_j^\infty, \varepsilon_j)/(\mathbf{v}_j^\infty)^{n/(n+1)} \to c(n) > 0$ as $j \to \infty$ by (A.5).) By an identical argument we also find that

$$\mathcal{V}(\zeta_j^\eta; \Omega) = \big(1 + \eta\big)\, \mathbf{v}_j^\infty + \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big) .$$

In summary, we can claim the existence of $J_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that for each $x \in \partial\mathbf{W}$ we can find $\zeta_j^{x,\eta}$ with the properties that

$$\mathcal{V}(\zeta_j^{x,\eta}; B_{r_j}(x)^c) = \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big) , \qquad \mathcal{AC}_\varepsilon(\zeta_j^{x,\eta}; B_{r_j}(x)^c) = \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)} , \tag{6.27}$$
$$\mathcal{V}(\zeta_j^{x,\eta}; \Omega) = \big(1 + \eta\big)\, \mathbf{v}_j^\infty + \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big) \tag{6.28}$$
$$\frac{\mathcal{AC}_{\varepsilon_j}(\zeta_j^{x,\eta}; \Omega)}{2} \leq \Big(1 - \beta(n) + C(n)\, |\eta|\Big)\, \Theta(\mathbf{v}_j^\infty, \varepsilon_j) + \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)} , \tag{6.29}$$

where $r_j = (\mathbf{v}_j^\infty)^{1/(n+1)}\, 2\, r(n)$.

The final step in the construction is making a choice of $x = x_j \in \partial\mathbf{W}$ such that the interaction between $\zeta_j^{x,\eta}$ and u_0^j in minimized. We claim that indeed $x_j \in \partial\mathbf{W}$ can be found such that

$$\mathcal{V}(u_0^j; \Omega \cap B_{r_j}(x_j)) = \mathrm{o}(\mathbf{v}_j^\infty) . \tag{6.30}$$

We now show, first, how to derive a contradiction from (6.27), (6.28), (6.29), and (6.30); and, finally, how to prove (6.30). In this way the proof of the theorem will be complete.

Derivation of a contradiction from (6.27), (6.28), (6.29), *and* (6.30): Let us consider the functions

$$h_j^\eta = \max\big\{u_0^j, \zeta_j^{x_j,\eta}\big\} , \qquad j \geq J_0 \, , |\eta| < \eta_0 .$$

Since $h_j^\eta \geq u_0^j$ on Ω, we have that $\{(h_j^\eta)^* \geq t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in [1/2, \delta_j)$. We claim that we can find $|\eta_j| < \eta_0$ such that

$$\mathcal{V}(h_j^{\eta_j}; \Omega) = v_j \, , \qquad \lim_{j\to\infty} \eta_j = 0 . \tag{6.31}$$

We start noticing that by (6.27), (6.30), $V(u_0^j) \leq V(\zeta_j^{x_j,\eta})$ on $\{u_0^j \leq \zeta_j^{x_j,\eta}\}$, and $V(u_0^j) \geq V(\zeta_j^{x_j,\eta})$ on $\{u_0^j \geq \zeta_j^{x_j,\eta}\}$, we find that

$$\mathcal{V}(h_j^\eta; \Omega) = \mathcal{V}(u_j^0; \Omega) + \mathcal{V}(\zeta_j^{x_j,\eta}; \Omega) + \mathrm{o}(\mathbf{v}_j^\infty) .$$

Therefore, by (6.28), and recalling that, by definition $\mathcal{V}(u_j^0; \Omega) = v_j - \mathbf{v}_j^\infty$, we find that

$$\mathcal{V}(h_j^\eta; \Omega) = v_j + \eta\, \mathbf{v}_j^\infty + \mathrm{o}(\mathbf{v}_j^\infty) . \tag{6.32}$$

In particular, up to increase the value of J_0, if $j \geq J_0$ we have $\mathcal{V}(h_j^{\eta_0/2}; \Omega) > v_j$ and $\mathcal{V}(h_j^{-\eta_0/2}; \Omega) < v_j$. By continuity of $\big(|\eta| \leq \eta_0/2\big) \mapsto \mathcal{V}(h_j^\eta; \Omega)$ we find $|\eta_j| \leq \eta_0/2$ such that $\mathcal{V}(h_j^{\eta_j}; \Omega) = v_j$. Plugging this information back into (6.32) we find that $\eta_j \to 0$ as $j \to \infty$.

To derive a contradiction we notice that, being $h_j^{\eta_j}$ admissible in $\Upsilon(v_j, \varepsilon_j, \delta_j)$, by (6.11) and (6.29)

$$\mathcal{AC}_\varepsilon(u_j^0; \Omega) + 2\, \Theta(\mathbf{v}_j^\infty, \varepsilon_j) = 2\, \Upsilon(v_j, \varepsilon_j, \delta_j) \leq \mathcal{AC}_{\varepsilon_j}\big(h_j^{\eta_j}; \Omega\big)$$

$$= \mathcal{AC}_{\varepsilon_j}\big(u_j^0; \Omega \cap \{u_j^0 \ge \zeta_j^{x_j,\eta_j}\}\big) + \mathcal{AC}_{\varepsilon_j}\big(\zeta_j^{x_j,\eta_j}; \Omega \cap \{\zeta_j^{x_j,\eta_j} \ge u_0^j\}\big)$$
$$\le \mathcal{AC}_{\varepsilon_j}(u_j^0; \Omega) + \mathcal{AC}_{\varepsilon_j}(\zeta_j^{x_j,\eta_j}; \Omega)$$
$$\le \mathcal{AC}_{\varepsilon_j}(u_j^0; \Omega) + 2\Big(1 - \beta(n) + C(n)\,|\eta_j|\Big)\,\Theta(\mathbf{v}_j^\infty, \varepsilon_j) + \mathrm{o}_{n,W}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)},$$

which leads to a contradiction with $\beta(n) \in (0,1)$ as soon as j is large enough.

Proof of (6.30)*:* We finally prove the existence of $x_j \in \partial\mathbf{W}$ such that (6.30) holds. To this end, we recall the validity of (6.20), where K is a minimizer of ℓ. By exploiting this minimality property as done in [KMS22a, Proof of Theorem 1.4, Step 6], we see that K does not concentrate area near $\partial\mathbf{W}$, that is

$$\mathcal{H}^n\big(K \cap \{x : \mathrm{dist}(x, \partial\mathbf{W}) < r\}\big) \le C\,r\,, \qquad \forall r > 0\,, \tag{6.33}$$

with some C depending on K. Moreover, as shown for example in [KMS22a, Appendix B],

$$\mathcal{H}^n(K \cap B_r(x)) \ge c_0 r^n \quad \forall x \in \mathrm{cl}\,(K)\,,\ r \in (0, r_0)\,. \tag{6.34}$$

By combining (6.33) and (6.34) we see that K is not "wetting" the whole $\partial\mathbf{W}$, that is

$$(\partial\mathbf{W}) \setminus \mathrm{cl}\,(K) \ne \varnothing\,. \tag{6.35}$$

In particular, there are $x_0 \in \partial\mathbf{W}$ and $r_0 > 0$ such that $\mathrm{cl}\,B_{r_0}(x_0) \cap K = \varnothing$, so that, by (6.20), and taking also into account that $u_j^0 \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$, we find

$$\lim_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(u_j^0; B_{r_0}(x_0) \cap \Omega) = 0\,. \tag{6.36}$$

Correspondingly to x_0, and up to decreasing the value of r_0, we can find a cube $Q \subset \mathbb{R}^n$ and an embedding g of $Q \times [0, r_0)$ into $\overline{\Omega} \cap B_{r_0}(x_0)$ so that g embeds $Q \times \{0\}$ into $(\partial\Omega) \cap B_{r_0}(x_0)$ and g is arbitrarily C^1-close to an isometry, in such a way that if Q' is a cube contained in Q with side length $2\,s$ and center z, then

$$\Omega \cap B_s(g(z)) \subset g(Q' \times (0, s))\,. \tag{6.37}$$

If j is large enough, then we can find a partition $\mathcal{F}_j = \{Q_j^k\}_{k=1}^{N(j)}$ of Q into subcubes of sidelength $s_j > r_j$ for some s_j and N_j satisfying

$$s_j = \mathrm{O}(\mathbf{v}_j^\infty)^{1/(n+1)}\,, \qquad N_j = \mathrm{O}(\mathbf{v}_j^\infty)^{-n/(n+1)}\,. \tag{6.38}$$

The subfamily $\mathcal{G}_j$ defined by

$$\mathcal{G}_j = \Big\{ Q_j^k : \mathcal{AC}_{\varepsilon_j}\big(u_0^j; g(Q_j^k \times (0, s_j))\big) \le \frac{\mathcal{AC}_{\varepsilon_j}(u_j^0; B_{r_0}(x_0) \cap \Omega)^{1/2}}{N(j)} \Big\}$$

is of course such that

$$0 \le 1 - \frac{\#\,\mathcal{G}_j}{N(j)} \le \mathcal{AC}_{\varepsilon_j}(u_j^0; B_{r_0}(x_0) \cap \Omega)^{1/2}\,, \tag{6.39}$$

and by (6.36) and (6.38) we have

$$\mathcal{AC}_{\varepsilon_j}\big(u_0^j; g(Q_j^k \times (0, s_j))\big) = \mathrm{o}\big(\mathbf{v}_j^\infty\big)^{n/(n+1)}\,, \qquad \forall Q_j^k \in \mathcal{G}_j\,. \tag{6.40}$$

We now claim that, for every j large enough, there is $Q_j^{k(j)} \in \mathcal{G}_j$ such that

$$\mathcal{V}\big(u_0^j; g(Q_j^{k(j)} \times (0, s_j))\big) = \mathrm{o}\big(\mathbf{v}_j^\infty\big) \tag{6.41}$$

Denoting by z_j the center of $Q_j^{k(j)}$ and setting $x_j = g(z_j)$, and by applying (6.37) with $s = s_j > r_j$, we conclude that

$$\mathcal{V}(u_0^j; \Omega \cap B_{r_j}(x_j)) \le \mathcal{V}\big(u_0^j; g(Q_j^{k(j)} \times (0, s_j))\big) = \mathrm{o}\big(\mathbf{v}_j^\infty\big)\,,$$

and complete the proof of (6.30). To prove (6.41), we argue by contradiction. Should (6.41) fail, then, up to extracting a subsequence in j and for some constant $c_0>0$, we would have that

$$\mathcal{V}(u_0^j; g(Q_j^k\times(0,s_j)))\ge c_0\,s_j^{n+1}\,,\qquad \forall Q_j^k\in\mathcal{G}_j\,,\tag{6.42}$$

so that, for some $c_1\in(0,1)$,

$$c_1\le \fint_{Q_j^k\times(0,s_j)} V(u_0^j\circ g)\,,\qquad \forall Q_j^k\in\mathcal{G}_j\,.$$

Since $V(u_0^j\circ g)$ takes values in $[0,1]$ we find that

$$\begin{aligned}\frac{c_1}{2}\,|Q_j^k\times(0,s_j)| &\le \Big|\Big\{(y,s)\in Q_j^k\times(0,s_j): V(u_0^j\circ g)\ge\frac{c_1}{2}\Big\}\Big|\\ &\le \mathcal{H}^n\Big(\Big\{y\in Q_j^k:\sup_{(0,s_j)}V((u_0^j)^*(g(y,\cdot)))\ge\frac{c_1}{2}\Big\}\Big)\,s_j\,,\end{aligned}$$

that is, for some $c\in(0,1)$ and recalling that V is strictly increasing on $(0,1)$,

$$c\,\mathcal{H}^n(Q_j^k)\le\mathcal{H}^n\Big(\Big\{y\in Q_j^k:\sup_{(0,s_j)}(u_0^j)^*(g(y,\cdot))\ge c\Big\}\Big)\,,\qquad\forall Q_j^k\in\mathcal{G}_j\,.$$

Adding up over all the cubes in $\mathcal{G}_j$, and recalling (6.39) and that $\mathcal{F}_j$ is a partition of Q, we thus conclude (up to further decrease the value of c) that

$$\mathcal{H}^n\Big(\Big\{y\in Q:\sup_{(0,s_j)}(u_0^j)^*(g(y,\cdot))\ge c\Big\}\Big)\ge c\,\mathcal{H}^n(Q)\,,\qquad\forall j\,.\tag{6.43}$$

However, by (6.36), (MM), Fubini's theorem, the area formula, and the slicing theory for Sobolev functions (see, e.g. [EG92, Section 4.9.2]), there is a set $Z\subset Q$ with full $\mathcal{H}^n$-measure in Q such that, for every $y\in Z$, $(u_0^j)^*(g(y,\cdot))$ is absolutely continuous on $(0,r_0)$, $(u_0^j)^*(g(y,\cdot))\to0$ $\mathcal{H}^1$-a.e. on $(0,r_0)$ as $j\to\infty$ (recall indeed that $\mathcal{V}(u_0^j;\Omega)\to0$), and

$$2\int_{g(\{y\}\times(0,r_0))}|\nabla[\Phi\circ(u_0^j)^*]|\,d\mathcal{H}^1\le\int_{g(\{y\}\times(0,r_0))}\varepsilon_j|\nabla(u_0^j)^*|^2+\frac{W((u_0^j)^*)}{\varepsilon_j}\,d\mathcal{H}^1\to0\,.$$

In particular, we find that

$$\lim_{j\to\infty}\sup_{(0,r_0)}(u_0^j)^*(g(y,\cdot))=0\,,\qquad\forall y\in Z\,,$$

and obtain a contradiction with (6.43). □

7. Euler-Lagrange equation and regularity (Proof of Theorem 1.3)

We finally prove Theorem 1.3 and Proposition 1.4.

Proof of Theorem 1.3. By Theorem 1.2.(i), there is $\lambda\in\mathbb{R}$ such that

$$\int_\Omega\Big(\varepsilon|\nabla u|^2+\frac{W(u)}{\varepsilon}\Big)\mathrm{div}\,X-2\,\varepsilon\nabla u\,\cdot\,\nabla X[\nabla u]=\lambda\int_\Omega V(u)\,\mathrm{div}\,X\,,\tag{7.1}$$

whenever $X\in C_c^\infty(\Omega;\mathbb{R}^{n+1})$; also, (7.1) extends to $X\in C_c^1(\Omega;\mathbb{R}^{n+1})$ by density.

Step one: We claim that if $\varphi\in C_c^1(\Omega)$ and $h\in\mathrm{Lip}_c([0,1))$ with $\varphi\ge0$ and $h\ge0$, then

$$0\le\int_\Omega\varphi\,h'(u)\,|\nabla u|^2+h(u)\,\nabla u\cdot\nabla\varphi+\varphi\,h(u)\,F_\varepsilon'(u)\,,\tag{7.2}$$

where

$$F_\varepsilon(t)=\frac{1}{2\,\varepsilon}\Big\{\frac{W(t)}{\varepsilon}-\lambda\,V(t)\Big\}\,,\qquad t\in[0,1]\,.$$

To prove this, we start by noticing that there is $\sigma_0 > 0$ small enough (depending on h and φ) such that, if $\sigma \in [0, \sigma_0)$, then

$$u_\sigma = u + \sigma\, h(u)\, \varphi \text{ takes values in } [0,1]\,.$$

Since $u_\sigma \ge u$ implies that $\{u^*_\sigma \ge t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in (1/2, \delta)$, in order to make u_σ admissible in $\Upsilon(v, \varepsilon, \delta)$ we just need to compose with a diffeomorphism in order to restore the volume constraint. To this end, given $X \in C^\infty_c(\Omega; \mathbb{R}^{n+1})$ with

$$\int_\Omega V(u)\, \mathrm{div}\ X = 1\,, \tag{7.3}$$

let $\tau_0 > 0$ be small enough so that, defining $\Phi \in C^\infty((-\tau_0, \tau_0) \times \Omega; \mathbb{R}^{n+1})$ by $\Phi(\tau, x) = \Phi_\tau(x) = x + \tau\, X(x)$, we have that Φ_τ is a diffeomorphism of Ω for every $|\tau| < \tau_0$. Denoting by Ψ_τ the inverse of Φ_τ, and letting $g(\sigma, \tau) = \mathcal{V}\big(u_\sigma \circ \Psi_\tau\big)$, we observe that $g(0,0) = v$ with $\partial_\tau g(0,0) = \int_\Omega V(u)\, \mathrm{div}\, X = 1$ by (7.3). Therefore, by the implicit function theorem, up to decreasing the value of σ_0, we can find $m(\sigma)$ with $m(0) = 0$, $|m(\sigma)| < \tau_0$, and $g(\sigma, m(\sigma)) = v$ for every $\sigma \in [0, \sigma_0)$ – in particular, differentiating $g(\sigma, m(\sigma)) = v$ and recalling (7.3), we find

$$0 = m'(0) + \int_\Omega h(u)\, \varphi\, V'(u)\,. \tag{7.4}$$

Since $v_\sigma = u_\sigma \circ \Psi_{m(\sigma)}$ is admissible in $\Upsilon(v, \varepsilon, \delta)$ for every $\sigma \in [0, \sigma_0)$, the minimality of $u = u_{\sigma=0}$ in $\Upsilon(v, \varepsilon, \delta)$ implies that $f(\sigma) = \mathcal{AC}_\varepsilon(u_\sigma \circ \Psi_{m(\sigma)})$ has a minimum on $[0, \sigma_0)$ at $\sigma = 0$. By combining $f'(0) \ge 0$ with (7.4), (7.1) and (7.3) we thus find

$$\begin{aligned} 0 \ &\le\ m'(0) \int_\Omega \Big\{\varepsilon\, |\nabla u|^2 + \frac{W(u)}{\varepsilon}\Big\}\, \mathrm{div}\, X - 2\, \nabla u \cdot \nabla X[\nabla u] \\ &\quad + \int_\Omega 2\, \varepsilon\, \nabla u \cdot \nabla[h(u)\, \varphi] + \frac{W'(u)}{\varepsilon}\, h(u)\, \varphi \\ &=\ -\lambda \int_\Omega h(u)\, \varphi\, V'(u) + \int_\Omega 2\, \varepsilon\, \varphi\, h'(u) |\nabla u|^2 + 2\, \varepsilon\, h(u)\, \nabla u \cdot \nabla \varphi + \frac{W'(u)}{\varepsilon}\, h(u)\, \varphi\,, \end{aligned}$$

that is (7.2) by definition of $F_\varepsilon(t)$.

Step two: We prove that

$$2\, \varepsilon^2\, \Delta u \le W'(u) - \lambda\, \varepsilon\, V'(u) \quad \text{as distributions on } \Omega\,. \tag{7.5}$$

Indeed, let $\{h_k\}_k \subset C^1_c([0,1))$ be a sequence such that $0 \le h_k(t) \le h_{k+1}(t) \to 1$ for every $t \in [0,1)$ and such that $h'_k \le 0$ on $[0,1)$. Then for any $\varphi \in C^\infty_c(\Omega; [0,\infty))$, we have $\varphi h'_k(u) |\nabla u|^2 \le 0$. Therefore, letting $k \to \infty$ and applying (7.2) we can deduce

$$0 \le \int_{\Omega \cap \{u<1\}} \nabla \varphi \cdot \nabla u \cdot + \varphi\, F'_\varepsilon(u) \qquad \forall \varphi \in C^1_c(\Omega; [0,\infty))$$

by means of the dominated convergence theorem. Since $F'_\varepsilon(1) = 0$ and $\nabla u = 0$ a.e. on $\{u = 1\}$, we immediately deduce (7.5).

Step three: We prove that, if $x_0 \in \Omega$ and $r_0 = \mathrm{dist}(x_0, \partial\Omega)$, then the function

$$g(r) = e^{-kr}\, \phi(r)\,, \quad \text{where} \quad \phi(r) = \fint_{B_r(x_0)} u\,, \qquad k = \sup_{[0,1]} |F''_\varepsilon|\,,$$

is decreasing on $(0, r_1)$ where $r_1 = \min\{r_0, 2\}$. Indeed, assuming without loss of generality that $x_0 = 0$ and testing (7.5) with a sequence $\{\varphi_k\} \subset C^1_c(\Omega; [0, \infty))$ such that $\varphi_k(x) \to [(r^2 - |x|^2)/2]_+$ uniformly and $\nabla \varphi_k \to \nabla[(r^2 - |x|^2)/2]_+$ in L^2 yields

$$\int_{B_r} x \cdot \nabla u \le \int_{B_r} \frac{r^2 - |x|^2}{2} |F'_\varepsilon(u)| \le k\, \frac{r^2}{2} \int_{B_r} u\,, \qquad \forall r < r_0\,, \tag{7.6}$$

where we have used $F'_\varepsilon(0) = 0$ and the fundamental theorem of calculus to bound $|F'_\varepsilon(u)| \le ku$. Now, for a.e. $r \in (0, r_0)$ we have that

$$\phi'(r) = \fint_{B_1} (y \cdot \nabla u(r\,y))\,dy = \frac{1}{r}\fint_{B_r} (x \cdot \nabla u(x))\,dx\,. \tag{7.7}$$

By combining (7.6) with (7.7) and using $r_1 \le 2$ we find $\phi'(t) \le k\,\phi(r)$ for a.e. $r < r_1$ and conclude.

Step four: We prove that u is (Lebesgue equivalent to) a lower semicontinuous function on Ω. Indeed, by step three, we can define $\tilde u : \Omega \to [0,1]$ by setting

$$\tilde u(x) = \lim_{r\to 0^+} e^{-k\,r}\fint_{B_r(x)} u\,, \qquad x \in \Omega\,.$$

Denoting by $\tilde u$ this limit, we have $\tilde u = u$ a.e. on Ω by the Lebesgue points theorem. We conclude by proving that $\tilde u$ is lower semicontinuous: indeed, if $x_j \to x \in \Omega$ as $j \to \infty$, then

$$e^{-k\,r}\fint_{B_r(x)} u = \lim_{j\to\infty} e^{-k\,r}\fint_{B_r(x_j)} u \le \liminf_{j\to\infty} \tilde u(x_j)\,,$$

where we have used $e^{-k\,r}\fint_{B_r(x_j)} u \le \tilde u(x_j)$ for every $r < \min\{2, \operatorname{dist}(x_j, \partial\Omega)\}$. The conclusion follows by letting $r \to 0^+$.

Step five: We prove that for every Ω' connected component of Ω, either $u \equiv 0$ on Ω' or $u > 0$ on Ω'; and that, if $\delta < 1$, then $u < 1$ on Ω.

To prove the first assertion we notice that the lower semicontinuity and the non-negativity of u imply that $\{u = 0\}$ is relatively closed in Ω. At the same time, if $x \in \{u = 0\}$, then by step three $0 = u(x) \ge e^{-k\,r}\fint_{B_r(x)} u \ge 0$ implies that $u \equiv 0$ on $B_r(x)$ for every $r \le \min\{\operatorname{dist}(x, \partial\Omega), 2\}$; in particular, $\{u = 0\}$ is open. Since $\{u = 0\}$ is both open and relatively closed in Ω, we conclude that $u \equiv 0$ on Ω' or $u > 0$ on Ω' for any given connected component of Ω.

Next, we show that if $\delta < 1$, then $\{u > \delta\}$ is open and

$$2\varepsilon\Delta u = \frac{1}{\varepsilon}W'(u) - \lambda V'(u)\,, \quad \text{as distributions on } \{u > \delta\}\,. \tag{7.8}$$

By a standard application of the strong maximum principle [MR22, Theorem 6.2], (7.8) allows us to show that $u < 1$ on Ω if $\delta < 1$, finishing the proof of Theorem 1.3.(iii). We first notice that $\{u > \delta\}$ is open by step four. To prove (7.8), the inequality (7.5) reduces our task to showing that for every $B_r(x) \subset\subset \{u > \delta\}$ and every $\varphi \in C^\infty_c(B_r(x); [0,\infty))$,

$$0 \ge \int_\Omega \nabla u \cdot \nabla\varphi + \varphi\,F'_\varepsilon(u)\,. \tag{7.9}$$

And indeed, by lower semicontinuity of u, there is $\delta_0 > 0$ such that $u \ge \delta + \delta_0$ on $\operatorname{cl}(B_r(x))$. In particular, for every $\varphi \in C^\infty_c(B_r(x))$ with $\varphi \ge 0$ there is $\sigma_0 > 0$ such that, for every $\sigma \in (0, \sigma_0]$, $u_\sigma = u - \sigma\,\varphi$ takes values in $(\delta, 1]$ on $B_r(x)$, and agrees with u on $\Omega \setminus B_r(x)$. It is therefore immediate to check that $\{u^*_\sigma \ge t\} = \{u^* \ge t\}$ for every $t \in (1/2, \delta)$, so that $\{u^*_\sigma \ge t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in (1/2, \delta)$. We can then repeat the volume-fixing argument of step one and prove (7.9), as desired.

Step six: We claim that for every $\varphi \in C^1_c(\Omega)$ and $h \in \operatorname{Lip}_c([0,1] \setminus \{\delta\})$, it holds

$$0 = \int_\Omega \varphi\,h'(u)\,|\nabla u|^2 + h(u)\,\nabla u \cdot \nabla\varphi + \varphi\,h(u)\,F'_\varepsilon(u). \tag{7.10}$$

By virtue of step 5, we can assume without loss of generality that φ is supported in a connected component of Ω where $u > 0$. Since u is lower semicontinuous, then $\inf_{\mathrm{supp}(\varphi)} u > 0$. So, for σ_0 small enough, if $|\sigma| \le \sigma_0$ then

$$u_\sigma = u + \sigma\, h(u)\, \varphi \quad \text{takes values in } [0,1]\,.$$

We wish to test the minimality of u against u_σ, which requires verifying the spanning condition and then fixing volumes and testing as in step one. Regarding the spanning condition, if $\delta = 1$, then $h \in \mathrm{Lip}_c([0,\delta'))$ for some $\delta' < \delta$, and so $\{u \ge \gamma\} = \{u_\sigma \ge \gamma\}$ for $\gamma \in [\delta', 1]$ and small enough σ, implying that $\{u^*_\sigma \ge t\}$ is $\mathcal{C}$-spanning $\mathbf{W}$ for every $t \in (1/2, \delta)$. If $\delta < 1$, then $(\delta - \eta, \delta + \eta) \subset \mathrm{supp}(h)^c$ for some $\eta > 0$, and so again for small enough η depending on σ we find that $\{u \ge \gamma\} = \{u_\sigma \ge \gamma\}$ for $\delta - \eta/2 \le \gamma \le \delta + \eta/2$. So we can repeat the volume-fixing argument of step one to obtain (7.10).

Step seven: We finally show that u satisfies (1.16). By the coarea formula and since $u \in W^{1,2}(\Omega)$, $\mathcal{L}^1$-a.e. $t_0 \in (0, \delta_0)$ is a Lebesgue point of $t \to \int_{\{u=t\}} |\nabla u| d\mathcal{H}^n$. For such a value of t_0, let us consider the functions

$$h_k(r) = \begin{cases} 1, & r \in [0, t_0 - 2^{-k}], \\ 2^k(t_0 - r), & r \in [t_0 - 2^{-k}, t_0], \\ 0, & r \in [t_0, 1]. \end{cases}$$

By plugging h_k into (7.10), taking $k \to \infty$, and using the coarea formula, we deduce

$$\int_{\{u=t_0\}} |\nabla u|\, \varphi\, d\mathcal{H}^n = \int_{\{u<t_0\}} \nabla u \cdot \nabla \varphi + \varphi F'_\varepsilon(u)\,.$$

Integrating between 0 and δ, using the coarea formula and Fubini's theorem, we find

$$\begin{aligned} \int_{\{u<\delta\}} |\nabla u|^2\, \varphi &= \int_0^\delta dt \int_{\{u<t\}} \nabla u \cdot \nabla \varphi + \varphi F'_\varepsilon(u) \\ &= \int_{\{u<\delta\}} (\delta - u)\Big(\nabla u \cdot \nabla \varphi + \varphi F'_\varepsilon(u)\Big)\,. \end{aligned} \tag{7.11}$$

By analogous reasoning, we deduce

$$-\int_{\{u=t_0\}} |\nabla u|\, \varphi\, d\mathcal{H}^n = \int_{\{u>t_0\}} \nabla u \cdot \nabla \varphi + \varphi F'_\varepsilon(u)$$

for a.e. $t_0 \in (\delta, 1)$ and thus

$$\int_{\{u>\delta\}} |\nabla u|^2\, \varphi = \int_{\{u>\delta\}} (\delta - u)\Big(\nabla u \cdot \nabla \varphi + \varphi F'_\varepsilon(u)\Big)\,. \tag{7.12}$$

By combining (7.11) and (7.12), we obtain (1.16), and complete the proof of the theorem. □

We finally prove Proposition 1.4.

Proof of Proposition 1.4. To prove statement (i), let us assume that u is continuous in Ω. This implies that $\{u = \delta\}$ is closed, and we can thus proceed as in step five to deduce that

$$\Delta u = F'_\varepsilon(u) \qquad \text{on } \{u \neq \delta\}\,. \tag{7.13}$$

Since $W \in C^{2,1}[0,1]$ implies $V \in C^{2,\gamma(n)}[0,1]$ with $\gamma(n) = \min\{1, 2/n\}$ (see [MR22, Appendix 3]), we have $F'_\varepsilon \in C^{1,\gamma(n)}[0,1]$. The continuity of u implies that Δu is continuous on $\{u \neq \delta\}$, hence that $u \in C^{1,\alpha}_{\mathrm{loc}}(\{u \neq \delta\})$ for every $\alpha < 1$. Hence $F'_\varepsilon(u) \in C^{1,\alpha}_{\mathrm{loc}}(\{u \neq \delta\})$ for every $\alpha < \gamma(n)$, and thus $u \in C^{3,\alpha}_{\mathrm{loc}}(\{u \neq \delta\})$ for every $\alpha < \gamma(n)$, as claimed.

Since statement (iii) follows easily from statement (ii), we give a detailed proof of the latter only. We employ an argument similar to [ACF84, Theorem 2.4]. Given $X \in C_c^\infty(\Omega;\mathbb{R}^{n+1})$, let us set

$$Y = \big\{|\nabla u|^2 + 2\,F_\varepsilon(u)\big\}\,X - 2\,(X\cdot\nabla u)\,\nabla u\,.$$

In this way $Y \in C^1(\{u\neq\delta\};\mathbb{R}^{n+1})$, and by direct computation we find that, on $\{u \neq \delta\}$,

$$\begin{aligned}
\mathrm{div}\ Y &= \big\{|\nabla u|^2 + 2\,F_\varepsilon(u)\big\}\,\mathrm{div}\ X - 2\,\nabla u\cdot\nabla X[\nabla u] + 2\,(X\cdot\nabla u)\,(F_\varepsilon'(u) - \Delta u)\\
&= \big\{|\nabla u|^2 + 2\,F_\varepsilon(u)\big\}\,\mathrm{div}\ X - 2\,\nabla u\cdot\nabla X[\nabla u]\,, \qquad (7.14)
\end{aligned}$$

where in the second identity we have used (7.13). Now, let us set

$$\mathcal{L}[X] = \big\{|\nabla u|^2 + 2\,F_\varepsilon(u)\big\}\,\mathrm{div}\ X - 2\,\nabla u\cdot\nabla X[\nabla u]\,,$$

so that, thanks to $\nabla u = 0$ $\mathcal{L}^{n+1}$-a.e. on $\{u=\delta\}$, we have

$$\mathcal{L}[X] \in L^1(\Omega)\,,\qquad \mathcal{L}[X] = 2\,F_\varepsilon(\delta)\,\mathrm{div}\ X \qquad \mathcal{L}^{n+1}\text{-a.e. on } \{u=\delta\}\,. \qquad (7.15)$$

If we set $S_t = \{u > \delta + t\}\cup\{u<\delta - t\}$, $t>0$, then, by the inner variation critical point condition (7.1), which gives $\int_\Omega \mathcal{L}[X] = 0$, and by $\mathrm{div}\,Y = \mathcal{L}[X]$ on $\{u\neq\delta\}$, we find

$$\int_{S_t}\mathrm{div}\,Y = \int_{S_t}\mathcal{L}[X] = -\int_{\{|u-\delta|<t\}}\mathcal{L}[X]$$

where, by (7.15),

$$\lim_{t\to 0^+}\int_{\{|u-\delta|<t\}}\mathcal{L}[X]\,d\mathcal{L}^{n+1} = \int_{\{u=\delta\}}\mathcal{L}[X]\,d\mathcal{L}^{n+1} = 2\,F_\varepsilon(\delta)\int_{\{u=\delta\}}\mathrm{div}\,X\,d\mathcal{L}^{n+1}\,.$$

We now write $Y = Y_1 + Y_2$ with

$$Y_1 = |\nabla u|^2\,X - 2\,(X\cdot\nabla u)\,\nabla u\,,\qquad Y_2 = 2\,F_\varepsilon(u)\,X\,.$$

Since $u\in W^{1,2}(\Omega)$ and $X\in C_c^\infty(\Omega)$ it turns out that $Y_2 \in W^{1,1}(\Omega;\mathbb{R}^{n+1})$ with

$$\lim_{t\to 0^+}\int_{S_t}\mathrm{div}\ Y_2 = \int_\Omega \mathrm{div}\ Y_2 = 0\,.$$

We have thus proved that

$$\lim_{t\to 0^+}\int_{S_t}\mathrm{div}\,Y_1 = 2\,F_\varepsilon(\delta)\int_{\{u=\delta\}}\mathrm{div}\,X\,d\mathcal{L}^{n+1}\,, \qquad (7.16)$$

where we are stressing that the integral over $\{u=\delta\}$ is respect with the Lebesgue measure. Now, for a.e. $t>0$, we have that S_t is a set of finite perimeter in Ω, with $\partial^* S_t = \partial^*\{u > \delta + t\}\cup\partial^*\{u<\delta - t\}$ and

$$\begin{aligned}
\nu_{S_t} &= -\frac{\nabla u}{|\nabla u|}\,, &\qquad \mathcal{H}^n\text{-a.e. on } \partial^*\{u>\delta+t\}\,,\\
\nu_{S_t} &= \frac{\nabla u}{|\nabla u|}\,, &\qquad \mathcal{H}^n\text{-a.e. on } \partial^*\{u<\delta-t\}\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{S_t}\mathrm{div}\ Y_1 &= \int_{\partial^*\{u<\delta-t\}} Y_1\cdot\frac{\nabla u}{|\nabla u|} - \int_{\partial^*\{u>\delta+t\}} Y_1\cdot\frac{\nabla u}{|\nabla u|}\\
&= -\int_{\partial^*\{u<\delta-t\}}(X\cdot\nabla u)\,|\nabla u| + \int_{\partial^*\{u>\delta+t\}}(X\cdot\nabla u)\,|\nabla u|\,.
\end{aligned}$$

By (7.16), if we assume that $|\{u=\delta\}| = 0$, we conclude that

$$\lim_{t\to 0^+}\int_{\partial^*\{u<\delta-t\}}(X\cdot\nabla u)\,|\nabla u| - \int_{\partial^*\{u>\delta+t\}}(X\cdot\nabla u)\,|\nabla u| = 0\,, \qquad (7.17)$$

with the limit taken with t such that S_t has finite perimeter. This is (1.17), and the proof of the proposition is complete. □

Appendix A. The diffused interface Euclidean isoperimetric problem

In this appendix we collect some important properties of the diffused interface Euclidean isoperimetric problem considered in [MR22], i.e.

$$\Theta(v,\varepsilon):=\inf\Big\{\frac{\mathcal{AC}_\varepsilon(u)}{2}:\mathcal{V}(u)=v\Big\}\,, \tag{A.1}$$

(where $\mathcal{V}(u)=\mathcal{V}(u;\mathbb{R}^{n+1})$ and $\mathcal{AC}_\varepsilon(u)=\mathcal{AC}_\varepsilon(u;\mathbb{R}^{n+1})$), including the uniqueness of minimizers and the characterization of minimizers as the only critical points in the "geometric regime" where $\varepsilon\ll v^{1/(n+1)}$, which is one the main results proved in [MR22].

Theorem A.1. *If $W\in C^{2,1}[0,1]$ satisfies (1.9) and (1.10), then the following holds:*

(i): *for every v and ε positive, there exists a radial decreasing symmetric minimizer ζ of $\Theta(v,\varepsilon)$ such that $\zeta\in C^{2,\alpha}_{\rm loc}(\mathbb{R}^{n+1};(0,1))$ for some $\alpha\in(0,1)$;*

(ii): *Θ is continuous on $(0,\infty)\times(0,\infty)$ with $\Theta(v,\varepsilon)=r^n\,\Theta(v/r^{n+1},\varepsilon/r)$ for every $r>0$; moreover, for every $\varepsilon>0$, the function $v>0\mapsto\Theta(v,\varepsilon)/v$ is strictly decreasing;*

(iii): *if ζ is a minimizer of $\Theta(v,\varepsilon)$, then there exists $\Lambda(\zeta)\in\mathbb{R}$ such that for all $X\in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$*

$$\int_{\mathbb{R}^{n+1}}{\rm ac}_\varepsilon(\zeta)\,{\rm div}\,X-2\,\varepsilon\nabla\zeta\cdot\nabla X[\nabla\zeta]=\Lambda(\zeta)\int_{\mathbb{R}^{n+1}}V(\zeta)\,{\rm div}\,X\,, \tag{A.2}$$

as well as

$$2\,\varepsilon^2\,\Delta\zeta=W'(\zeta)-\varepsilon\,\Lambda(\zeta)\,V'(\zeta)\,,\qquad\text{on }\mathbb{R}^{n+1}\,. \tag{A.3}$$

Moreover, for some positive constant $c=c(n,W)$, we have

$$\Lambda(\zeta)\ge\frac{c}{v^{1/(n+1)}}\,. \tag{A.4}$$

(iv): *there is $\sigma_0=\sigma_0(n,W)>0$ such that if $0<\varepsilon<\sigma_0\,v^{1/(n+1)}$, then there is a unique modulo translation radial decreasing symmetric minimizer $\zeta_{v,\varepsilon}$ of $\Theta(v,\varepsilon)$ with maximum at the origin, which satisfies*

$$\Theta(v,\varepsilon)=v^{n/(n+1)}\Big\{(n+1)\,\omega_{n+1}^{1/(n+1)}+{\rm O}_{n,W}\Big(\frac{\varepsilon}{v^{1/(n+1)}}\Big)\Big\}\,, \tag{A.5}$$

as $\varepsilon/v^{1/(n+1)}\to0$.

Proof. It is convenient to notice that by (1.9) there are $\beta_0\in(0,1)$ and $c_0>0$ (depending on W) such that

$$\frac{t^2}{c_0}\ge W(t)\ge c_0\,t^2\,,\qquad\frac{V(t)}{c_0}\ge t\,V'(t)\qquad\forall t\in(0,\beta_0)\,. \tag{A.6}$$

Step one: We prove the existence of $c=c(n,W)$ such that, if u is a competitor of $\Theta(v,\varepsilon)$ and $\beta\in(0,\beta_0)$, then

$$\frac{\mathcal{AC}_\varepsilon(u;A)}{\mathcal{V}(u;A)}\ge\frac{c}{\varepsilon\,\beta^{2/n}}\,,\qquad\forall A\subset\{0\le u\le\beta\}\,. \tag{A.7}$$

Indeed, by (A.6), if $A\subset\{0\le u\le\beta\}$ for some $\beta\in(0,\beta_0)$, then

$$\mathcal{AC}_\varepsilon(u;A)\ge\frac{1}{\varepsilon}\int_A W(u)\ge\frac{c_0}{\varepsilon}\int_A u^2\,dx\,,$$

while $V(t)=(\int_0^t\sqrt{W})^{(n+1)/n}\le Ct^{2(n+1)/n}$ ($t<\beta_0$) implies $\mathcal{V}(u;A)\le C\,\beta^{2/n}\int_A u^2$.

Step two: We prove[7] conclusion (i). By the Pólya-Szegö inequality [BZ88] we can consider a minimizing sequence $\{u_j\}_j$ of $\Theta(v,\varepsilon)$ such that each u_j is radial decreasing symmetric with respect to the origin. Up to extracting subsequences we can assume that $u_j \to \zeta$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$, where ζ is radial decreasing symmetric with respect to the origin and such that $\mathcal{AC}_\varepsilon(\zeta) \le 2\,\Theta(v,\varepsilon)$. We prove that $\mathcal{V}(\zeta)=v$, and thus that ζ is a minimizer of $\Theta(v,\varepsilon)$, by showing that

$$\lim_{R\to\infty}\sup_j \mathcal{V}(u_j;\mathbb{R}^{n+1}\setminus B_R)=0\,. \tag{A.8}$$

To prove (A.8), let us set $u_j(x)=g_j(|x|)$ and $\zeta(x)=g(|x|)$, and notice that, since $g_j\to g$ a.e. on $(0,\infty)$ with g_j and g decreasing on $(0,\infty)$ and $g(R)\to 0$ as $R\to\infty$, it holds that $\sup_j g_j(R)\to 0$ as $R\to\infty$. In particular, for every R large enough to ensure $\sup_j g_j(R)\le\beta_0$ we can apply (A.7) to conclude that

$$\mathcal{V}(u_j;\mathbb{R}^{n+1}\setminus B_R)\le\frac{\varepsilon\,g_j(R)^{2/n}}{c_0}\,\sup_i \mathcal{AC}_\varepsilon(u_i)\,,$$

which implies (A.8) thanks (again) to $\sup_j g_j(R)\to 0$ as $R\to\infty$. The fact that $\zeta\in C^{2,\alpha}_{\rm loc}(\mathbb{R}^{n+1};(0,1))$ for some $\alpha\in(0,1)$ is proved as in [MR22, Proof of Theorem 2.1, Step four].

Step three: We prove conclusion (ii). Since the scaling property and the continuity of Θ can be proved as in [MR22, Appendix A] and [MR22, Step 3, Proof of Theorem 2.1], we focus on showing that, for $\varepsilon>0$ fixed, $v\mapsto\Theta(v,\varepsilon)/v$ is strictly decreasing on $(0,\infty)$. Indeed, by Fubini's theorem, if ζ is a minimizer for $\Theta(v,\varepsilon)$ and we set

$$Z(t)=\frac{(1/2)\int_{\{x_1=t\}}\mathrm{ac}_\varepsilon(\zeta)\,d\mathcal{H}^n}{\int_{\{x_1=t\}}V(\zeta)\,d\mathcal{H}^n}\,,\qquad t\in\mathbb{R}\,,$$

then, trivially, $\Theta(v,\varepsilon)\ge v\,\inf_{\mathbb{R}}Z$, with equality if and only if Z is constant on $\mathbb{R}$. Since ζ is radial decreasing symmetric, we have $\beta(t)=\sup_{\{x_1=t\}}\zeta\to 0$ as $t\to\infty$. Since $\mathcal{V}(\zeta)<\infty$, we can find $t_j\to\infty$ with $\int_{\{x_1=t_j\}}V(\zeta)\,d\mathcal{H}^n\to 0$ as $j\to\infty$. Correspondingly, $\beta(t_j)\to 0^+$ and, by (A.7), $Z(t_j)\to+\infty$ as $j\to\infty$. In particular, Z is not constant on $(0,\infty)$, so that $\Theta(v,\varepsilon)>v\,\inf_{\mathbb{R}}Z$, i.e., there is $t_0\in\mathbb{R}$ such that

$$\Theta(v,\varepsilon)>v\,Z(t_0)\,. \tag{A.9}$$

Now, given $\tilde v>v$, if we pick $\delta=(\tilde v-v)/(2\int_{\{x_1=t_0\}}V(\zeta))$, decompose $x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^n\equiv\mathbb{R}^{n+1}$, and set

$$u(x)=\begin{cases}\zeta(t_0,x')\,, & \text{if } t_0-\delta\le x_1\le t_0+\delta\,,\\ \zeta(x_1+\delta,x')\,, & \text{if } x_1\le t_0-\delta\,,\\ \zeta(x_1-\delta,x')\,, & \text{if } x_1\ge t_0+\delta\,,\end{cases}$$

then

$$\mathcal{AC}_\varepsilon\big(u;\{|x_1-t_0|<\delta\}\big)=2\,\delta\,Z(t_0)\int_{\{x_1=t_0\}}V(\zeta)=Z(t_0)\,(\tilde v-v)\,, \tag{A.10}$$

and, similarly, $\mathcal{V}(u)=\mathcal{V}(\zeta)+2\,\delta\int_{\{x_1=t_0\}}V(\zeta)=\tilde v$. Since u is admissible in $\Theta(\tilde v,\varepsilon)$, by (A.10) we find, as desired,

$$\frac{\Theta(\tilde v,\varepsilon)}{\tilde v}\le\frac{\mathcal{AC}_\varepsilon(u)/2}{\tilde v}\le\frac{\Theta(v,\varepsilon)+Z(t_0)\,(\tilde v-v)}{\tilde v}<\frac{\Theta(v,\varepsilon)}{v}\,,$$

where in the last inequality we have used (A.9).

[7] We notice that the analysis performed in [MR22], which is focused on uniqueness and stability issues, is limited to the regime where $\varepsilon/v^{1/(n+1)}$ is small enough in terms of n and W.

Step four: The validity of (A.2) is immediate from Lemma 4.5, while (A.3) can be deduced from (A.2) via integration by parts. To prove (A.4) (and thus complete the proof of conclusion (iii)) it is enough to show that, for some constant $C = C(n, W)$, it holds

$$C\,\Lambda(\zeta)\,v \geq \mathcal{AC}_\varepsilon(\zeta)\,. \tag{A.11}$$

(Indeed, $\mathcal{AC}_\varepsilon(\zeta)/v = \Theta(v,\varepsilon)/v \geq c(n)\,v^{1/(n+1)}$ thanks to $\Theta(v,\varepsilon) \geq c(n)\,v^{n/(n+1)}$.) To prove (A.11) we first notice that testing (A.3) with suitable radial vector fields as done in [MR22, Equation (2.32)] one finds

$$\begin{aligned}(n+1)\,\Lambda(\zeta)\,v &= n\,\mathcal{AC}_\varepsilon(\zeta) + \int_{\mathbb{R}^{n+1}} \frac{W(\zeta)}{\varepsilon} - \varepsilon\,|\nabla\zeta|^2 \\ &\geq (n-1)\,\mathcal{AC}_\varepsilon(\zeta) + \int_{\mathbb{R}^{n+1}} \frac{W(\zeta)}{\varepsilon}\,. \end{aligned} \tag{A.12}$$

Clearly (A.12) implies (A.11) when $n \geq 2$, but leaves open the case $n = 1$; however, it always ensures that $\Lambda(\zeta) \geq 0$. Next, we notice that by testing (A.3) with $\varphi = \zeta\,\phi_k^2$ for $\phi_k \in C_c^\infty(B_{k+1};[0,1])$ with $\phi_k = 1$ on B_k and $\mathrm{Lip}(\phi_k) \leq 2$ for every k, and keeping in mind that $V' \geq 0$ and that $\Lambda(\zeta) \geq 0$, we find

$$2\,\varepsilon \int_{\mathbb{R}^{n+1}} \phi_k^2\,|\nabla\zeta|^2 \leq 2\,\varepsilon \int_{\mathbb{R}^{n+1}} \phi_k\,\zeta\,|\nabla\zeta|\,|\nabla\phi_k| + \int_{\mathbb{R}^{n+1}} \frac{W'(\zeta)\,\zeta\,\phi_k}{\varepsilon} + \Lambda(\zeta) \int_{\mathbb{R}^{n+1}} V'(\zeta)\zeta\,.$$

Since for $k \geq k(\zeta)$ we have $\zeta \leq \beta_0$ on $\mathbb{R}^{n+1} \setminus B_k$, and we can thus use (A.6) and the Cauchy–Schwartz inequality to deduce, for some $C = C(W)$,

$$\varepsilon \int_{\mathbb{R}^{n+1}} \phi_k^2|\nabla\zeta|^2 \leq C\,\varepsilon \int_{\{\zeta<\beta_0\}} \zeta^2 + \int_{\mathbb{R}^{n+1}} \frac{[W'(\zeta)]^+\,\zeta}{\varepsilon} + C\,\Lambda(\zeta)\,\mathcal{V}(\zeta)\,, \tag{A.13}$$

where we have used $c_0\,t\,V'(t) \leq t^{2(n+1)/n} \leq V(t)/c_0$ for every $t \in (0,1)$ (which, in turns, easily follows from $W(t) \geq c_0\,t^2$ on $(0,\beta_0)$ and $W(t) \leq t^2/c_0$ on $(0,1)$). Finally, by (1.9), and up to decreasing the value of β_0, we have $W' < 0$ on $(1-\beta_0, 1)$. Using this fact in combination with $\inf_{(\beta_0,1-\beta_0)} W > 0$ and $t\,W'(t) \leq W(t)/c_0$ for $t \in (0,\beta_0)$, we see that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} [W'(\zeta)]^+\,\zeta &\leq \int_{\{\zeta<\beta_0\}} W'(\zeta)\,\zeta + \mathrm{Lip}(W) \int_{\{\beta_0\leq\zeta\leq 1-\beta_0\}} \zeta \\ &\leq \left\{\frac{1}{c_0} + \frac{\mathrm{Lip}(W)\,(1-\beta_0)}{\inf_{(\beta_0,1-\beta_0)} W}\right\} \int_{\mathbb{R}^{n+1}} W(\zeta)\,,\end{aligned}$$

and thus conclude from (A.13) that

$$\varepsilon \int_{\mathbb{R}^{n+1}} \phi_k^2\,|\nabla\zeta|^2 \leq C \int_{\mathbb{R}^{n+1}} \frac{W(\zeta)}{\varepsilon} + C\,\Lambda(\zeta)\,v\,.$$

By letting $k \to \infty$, by adding $\int_{\mathbb{R}^{n+1}} W(\zeta)/\varepsilon$ to both sides of this inequality, and by noticing that (A.12) implies $\int_{\mathbb{R}^{n+1}} W(\zeta)/\varepsilon \leq C(n)\,\Lambda(\zeta)\,v$ for every $n \geq 1$, we conclude the proof of (A.11).

Step five: The outer form of the Euler–Lagrange equation (A.3) follows from Lemma 4.5, and a classical computation (based on integration by parts made possible by the C^2-regularity of ζ) allows one to derive (A.2) from (A.3). This completes the proof of conclusion (iii). Conclusions (iv) is contained in [MR22, Theorem 1.1]. □

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