Almansi decomposition and expansion of a polyharmonic function near a crack-tip

Michele Carriero, Antonio Leaci and Franco Tomarelli

Abstract. We study polyharmonic functions in 2-dimensional open sets with a flat crack: for these functions we show a decomposition of Almansi-type and make explicit the coefficients of a strongly converging expansion near the crack-tip.

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Keywords. Crack-tip, crease-tip, Almansi decomposition, power series expansion, biharmonic functions, polyharmonic functions, image analysis, smooth cracks in elasticity, free discontinuity, calculus of variations.

To our friend Giuseppe Buttazzo on his 70^{th} *Birthday*

Contents

1. Introduction

A classical result by Emilio Almansi ([1]) provides a decomposition of polyharmonic functions in starshaped domains. This result states, in modern language: assume that $u \in C^{2m}(\Omega)$, $\mathbf{x}_0 \in \Omega \subset \mathbb{R}^2$ and Ω is an open set star-shaped with respect to $\mathbf{x}_0$, then u fulfils $\Delta^m_{\mathbf{x}} u = 0$ on Ω if and only if there are harmonic functions ψ_j such that

$$u(\mathbf{x}) = \psi_0(\mathbf{x}) + \|\mathbf{x}-\mathbf{x}_0\|^2\psi_1(\mathbf{x}) + ... + \|\mathbf{x}-\mathbf{x}_0\|^{2(m-1)}\psi_{m-1} \quad \text{on } \Omega\,.$$

In present paper we extend this classical result by relaxing both assumptions on domain topology and the function regularity. The aim is considering open sets with a crack: for simplicity we refer to a disk with center at $\mathbf{x}_0 = \mathbf{0}$ and a straight crack with endpoint at the origin. This geometry introduces several difficulties, since some singularity is allowed at the crack-tip which affects the decomposition focussed on that point: here and in the sequel, the term crack-tip denotes the endpoint of either a discontinuity line or a crease line. In this setting we make explicit the operators related to an Almansi type decomposition for polyharmonic functions. The decomposition that we obtain turns out to be essentially unique, up to the fact that any linear function $w \in L^2$ of two variables can be represented in two ways: either as w itself which is harmonic, or as $w = \|\mathbf{x}\|^2\,(\|\mathbf{x}\|^{-2}\,w)$, where $\|\mathbf{x}\|^{-2}\,w$ is harmonic off the origin.
In addition to the decomposition we look for a compatible power series expansion around the crack-tip. In this perspective partial results were stated in [21] for biharmonic functions only.
Results of this kind are relevant in the study of Blake & Zisserman functional ([10, 14, 17, 18, 21, 23, 44]). Moreover the analysis of crack-tip is met in the study of variational models for image segmentation, inpainting and denoising ([2, 6, 13, 15, 16, 21, 24, 25, 30, 31, 36, 37, 45]).
Here we show several results which may have a wider range of application than image analysis, namely the analysis of singularities for free discontinuity problems and crack-tips in planar elasticity ([2, 4, 10]), polyharmonic functions in open sets ([11, 12, 34]). We mention [35, 38, 39, 42] as basic references about power series expansions for variational solution of elliptic problems in an open set with a concave corner, and [4, 5, 26, 27, 28, 33] for approaches related to a crack tip in planar elasticity and fracture.
Our main results are stated in Theorems 3.1, 4.4, 4.9 and 5.5.
Theorem 3.1 provides the decomposition in the natural framework of biharmonic functions $v \in L^2(B_\varrho \setminus \Gamma)$ with $\Delta v \in L^2(B_\varrho \setminus \Gamma)$ together with their asymptotic expansion converging in $L^2(B_\varrho \setminus \Gamma)$, where B_ϱ is the disk with center $\mathbf{0}$ and radius $0 < \varrho < +\infty$ in $\mathbb{R}^2$ and Γ is the closed negative real axis. Theorem 4.4 provides the decomposition for biharmonic functions in $H^2(B_\varrho \setminus \Gamma)$ which are orthogonal to restrictions on $B_\varrho \setminus \Gamma$ of functions in $H^2(B_\varrho)$ and the explicit form of coefficients for a series expansion in terms of non-integer powers converging in $H^2(B_\varrho \setminus \Gamma)$. Theorem 4.9 deals with the Almansi decomposition of an $H^2(B_\varrho \setminus \Gamma)$ function which is biharmonic on $B_\varrho \setminus \Gamma$. Theorem 5.5 and Corollary 5.7 show the Almansi decomposition and a description of the basis for a power series expansion converging in L^2 in the case of polyharmonic functions on $B_\varrho \setminus \Gamma$.
Our choice of basis functions (see (2.1)) for the expansion is motivated by

the Euler conditions at the free discontinuity for local minimizers of Blake & Zisserman functional ([18]) and natural boundary conditions for the free Kirchhoff plate ([35]).
In all cases we write explicitly the operators Φ, Ψ (see (3.3),(3.4)) providing the explicit decomposition for biharmonic functions, the operators Φ_j providing the explicit decomposition for polyharmonic functions (see (5.16)) and the coefficients of the related power series expansions strongly converging in respective frameworks: either (3.8)-(3.9), or (4.19)-(4.20), or (5.14)-(5.16).
Starting point of the analysis are Lemmas 2.5 and 5.4 which provide an L^2 converging expansion for functions which are either harmonic or harmonic times $\|x\|^{2j}$, $j \in \mathbb{N}$, on a disk with crack-tip at the center of the disk; this is achieved by suitably tuned systems of orthogonal basis, made of functions with discontinuity or crease along Γ.
Notation: The whole paper deals with functions of two variables.
$\mathbf{x}=(x,y)$ denotes coordinates of the points $\mathbf{x}\in\mathbb{R}^2$; Γ is the closed negative real axis in $\mathbb{R}^2$ and $B_\varrho = \{\mathbf{x}\in\mathbb{R}^2 : \|\mathbf{x}\|<\varrho\}$ for $0<\varrho<+\infty$; whenever the polar reference in 2-d is exploited, we refer to the polar coordinates (r,θ) centered at $\mathbf{0}\in\mathbb{R}^2$; $L^2_r(a,b)$ denote the weighted L^2 space of measurable functions v in (a,b) fulfilling $\int_a^b |v(r)|^2\, r\, dr \ < +\infty$; the Laplacean operator is denoted by $\Delta_{\mathbf{x}} := \partial^2/\partial_x^2 + \partial^2/\partial_y^2$; for every open set A, $H^k(A)$ denotes the Sobolev space of the scalar functions with domain in A and their distributional partial derivatives in $L^2(A)$ for all orders not exceeding k.
We set $M\!:\!N=\sum_{i,j} M_{i,j}N_{i,j}$ for every pair of square matrices M, N.
$U+W$ denotes the algebraic sum of generic vector spaces U, W.
Referring to Section II.4 in [43], if U and W are Hilbert spaces then we denote by $U\oplus W$ and $U\otimes W$ respectively the *direct sum* and the *tensor product* of U and W; $\operatorname{span} U$ denotes the completion of finite linear combinations of elements in the vector space U.
Definitions of the function spaces and the basis functions tuned for Almansi type decomposition and expansion series are postponed in Section 2 and 5, respectively for biharmonic and polyharmonic functions.

2. Preliminary results

Here we provide the proofs of statements concerning harmonic functions with all details, since these issues were partly announced in [21] and [23] but the second one is rather concise, while here we provide a self-contained analysis.

Definition 2.1. Assuming $0<\varrho<+\infty$, we list some function spaces which are relevant for the aimed decomposition:

$$\begin{aligned}
A^1_\varrho &:= \{v\in L^2(B_\varrho) \text{ s.t. } \Delta_{\mathbf{x}} v = 0 \text{ in } B_\varrho\setminus\Gamma\ \},\\
A^2_\varrho &:= \{z\in L^2(B_\varrho) \text{ s.t. } \Delta^2_{\mathbf{x}} z = 0 \text{ in } B_\varrho\setminus\Gamma\ \},\\
Z_\varrho &:= \ r^2 A^1_\varrho = \{w \text{ s.t. } w = r^2\varphi,\ \varphi\in L^2(B_\varrho),\ \Delta_{\mathbf{x}}\varphi = 0 \text{ in } B_\varrho\setminus\Gamma\ \},\\
\mathcal{Z}_\varrho &:= \{w\in L^2(B_\varrho) \text{ s.t. } w = r^2\varphi,\ \Delta_{\mathbf{x}}\varphi = 0 \text{ in } B_\varrho\setminus\Gamma\ \}.
\end{aligned}$$

We label the complex functions v_k, z_k (and their real counterparts) which are relevant in the expansion we are looking for (here $r > 0$, $|\vartheta| < \pi$) by setting:

$$\begin{aligned} R_n(r) &:= \{r^{n-3/2}\}, && n = 1, \dots, \\ \varphi_k(\vartheta) &:= \exp\big(i(k - 3/2 \text{ sign } k)\,\vartheta\big), && k \in \mathbb{Z} \setminus \{0\} \end{aligned}$$

and define

$$\begin{aligned} v_k(r,\vartheta) &:= r^{|k|-3/2}\varphi_k(\vartheta) = \\ &= r^{|k|-3/2}\exp\big(i(k - 3/2 \text{ sign } k)\,\vartheta\big), && k \in \mathbb{Z} \setminus \{0\}, \\ z_k(r,\vartheta) &:= r^2 v_k = r^{|k|+1/2}\varphi_k(\vartheta) = \\ &= r^{|k|+1/2}\exp\big(i(k - 3/2 \text{ sign } k)\,\vartheta\big), && k \in \mathbb{Z} \setminus \{0\}, \\ f_k^1(r,\vartheta) &:= r^{(k+3/2)}\cos((k+3/2)\vartheta), && k = -2, -1, 0, 1, \dots \\ f_k^2(r,\vartheta) &:= r^{(k+3/2)}\sin((k+3/2)\vartheta), && k = -2, -1, 0, 1, \dots \\ f_k^3(r,\vartheta) &:= r^{(k+3/2)}\cos((k-1/2)\vartheta), && k = 0, 1, 2, \dots \\ f_k^4(r,\vartheta) &:= r^{(k+3/2)}\sin((k-1/2)\vartheta), && k = 0, 1, 2, \dots \end{aligned} \tag{2.1}$$

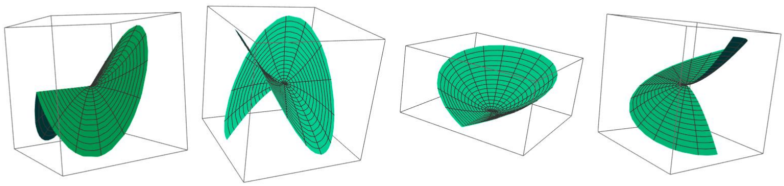

FIGURE 1. Graphs of $f_0^1, f_0^2, f_0^3, f_0^4$, exhibiting respectively crease, jump, crease, jump.

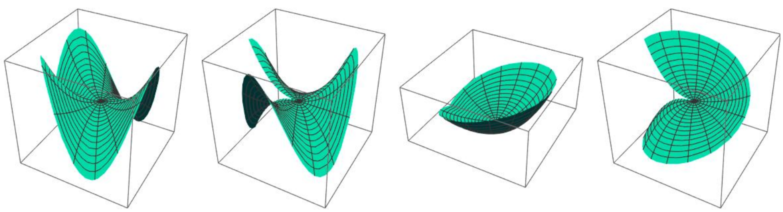

FIGURE 2. Graphs of $f_1^1, f_1^2, f_1^3, f_1^4$, exhibiting respectively crease, jump, crease, jump.

Remark 2.2. A_ϱ^1, A_ϱ^2 and $\mathcal{Z}_\varrho$ are closed subspaces of $L^2(B_\varrho)$.
$\mathcal{Z}_\varrho$ is the completion of Z_ϱ in $L^2(B_\varrho)$, moreover

$$\begin{aligned} f_k^1 &= \operatorname{Re} v_{k+3} = (v_{k+3} + v_{-k-3})/2, && k = -2, -1, 0, 1, \dots \\ f_k^2 &= \operatorname{Im} v_{k+3} = (v_{k+3} - v_{-k-3})/2i, && k = -2, -1, 0, 1, \dots \\ f_k^3 &= \operatorname{Re} z_k = (z_{k+1} + z_{-k-1})/2, && k = 1, 2, \dots \\ f_k^4 &= \operatorname{Im} z_k = (z_{k+1} - z_{-k-1})/2i, && k = 1, 2, \dots . \end{aligned}$$

We want to describe the harmonic functions with separate variables without any *a priori* assumption concerning the kind of functional dependance on r because the aim is to study solutions of PDEs in $L^2(B_\varrho)$ and their asymptotic expansions without prescribing boundary values; nevertheless, at higher regularity level, the natural boundary conditions on the crack play a relevant role, according to choice of angular dependance: see Remarks 3.3, 4.12.
We refer both to cartesian coordinates x, y and polar coordinates r, ϑ, and work in the space $L^2(B_\varrho)$ when referring to x, y and $L^2(-\pi,\pi) \otimes L^2_r(0,\varrho)$ when referring to r, ϑ, by letting free all boundary conditions: we will fix arbitrarily the dependance on angle referring to the system $\{\varphi_k\}_{|k|\neq 0}$ in (2.1). This arbitrary choice is possible in the L^2 framework, where boundary traces are meaningless; but when higher regularity is asked for, e.g. H^1 or H^2, only some kinds of boundary conditions on Γ are allowed by this choice. Actually the basis we choose in (2.1) is motivated by the boundary conditions for the linear plate ([35]) and also fulfils several additional Euler conditions at the free discontinuity for local minimizers of Blake & Zisserman functional ([18], Theorems 3.4, 4.3, 4.4).

First, we recall an elementary Lemma about harmonic functions with separated variables.

Lemma 2.3. *By assuming that*

$$u \in L^2(B_\varrho(\mathbf{0}))\,, \qquad \Delta_{\mathbf{x}} u = 0 \quad in\ B_\varrho(\mathbf{0})\,, \quad 0 < \varrho < +\infty$$

and there exist $R \in L^2_r(0,1)$ and $k \in \mathbb{Z} \setminus \{0\}$ such that, referring to (2.1), u has the following representation with separate variables in polar coordinates

$$u(x,y) \;=\; \varphi_k(\vartheta)R(r) \;\in\; L^2(-\pi,\pi) \otimes L^2_r(0,\varrho) \quad a.e.\ (x,y) \in B_\varrho(\mathbf{0})\,,$$

one obtains:

1. *if $|k| > 2$ then $u \;=\; R_{|k|}\,\varphi_k(\vartheta) \;=\; v_k$;*
2. *if $k = 2$ then u is a linear combination of*
   $r^{1/2}\varphi_2 = R_2\varphi_2$ *and* $r^{-1/2}\varphi_2 = R_1\varphi_2 = R_1\varphi_{-1}$ *;*
3. *if $k = -2$ then u is a linear combination of*
   $r^{1/2}\varphi_{-2} = R_2\varphi_{-2}$ *and* $r^{-1/2}\varphi_{-2} = R_1\varphi_{-2} = R_1\varphi_1$ *;*
4. *if $k = 1$ then u is a linear combination of*
   $r^{-1/2}\varphi_1 = R_1\varphi_1$ *and* $r^{1/2}\varphi_1 = R_2\varphi_1 = R_2\varphi_{-2}$ *;*
5. *if $k = -1$ then u is a linear combination of*
   $r^{-1/2}\varphi_{-1} = R_1\varphi_{-1}$ *and* $r^{1/2}\varphi_{-1} = R_2\varphi_{-1} = R_2\varphi_2$ *.*

Proof. By separation of variables in the Laplace equation, if $\Delta_{\mathbf{x}}(\varphi_k R) = 0$ on $B_\varrho(\mathbf{0})$, then R must be a solution in $L^2_r(0,\varrho)$ of an Euler-type O.D.E.

$$r^2R'' + rR' - \big(k - 3/2\,\mathrm{sign}(k)\big)^2 R = 0$$

say, due to $|k| > 0$ and the squared power,

$$r^2R'' + rR' - \big(|k| - 3/2\big)^2 R = 0\;.$$

Then

- $k \in \mathbb{Z} \setminus \{0, \pm1, \pm2\}$ entails
  either $R(r) = R_{|k|}(r) = r^{|k|-3/2}$, or $R(r) = R_{3-|k|}(r) = r^{3/2-|k|}$, actually the second choice $R(r) = r^{3/2-|k|}$ must be always discarded since it never belongs to L^2_r.
- $k = 2$ entails
  either $R(r) = R_2(r) = r^{1/2}$, or $R(r) = R_1(r) = r^{-1/2}$ and in that case also the second choice belongs to L^2_r;
- $k = -2$ entails
  either $R(r) = R_2(r) = r^{1/2}$, or $R(r) = R_1(r) = r^{-1/2}$ and in that case also the second choice belongs to L^2_r.
- $k = 1$ entails
  either $R(r) = R_1(r) = r^{-1/2}$, or $R(r) = R_2(r) = r^{1/2}$;
- $k = -1$ entails
  either $R(r) = R_1(r) = r^{-1/2}$, or $R(r) = R_2(r) = r^{1/2}$. □

Remark 2.4. Under the assumption of Lemma 2.3 all choices of the kind $R_{|k|}\varphi_k$ are allowed as k varies in $\mathbb{Z}\setminus\{0\}$ and no other function is admissible. Precisely, only 4 double appearances of admissible functions are present in the list: the ones mentioned in the items $2, 3, 4, 5$ of the list above.

Lemma 2.5. *Referring to Definition 2.1, for every $\varrho > 0$: the system*

$$\left\{ \ v_k(r,\vartheta), \ k \in \mathbb{Z}\setminus\{0\} \ \right\} \tag{2.2}$$

is orthogonal in $L^2(B_\varrho)$ and dense in A^1_ϱ with respect to $L^2(B_\varrho)$ norm; the system

$$\left\{ \ z_k(r,\vartheta), \ k \in \mathbb{Z}\setminus\{0\} \ \right\} \tag{2.3}$$

is orthogonal in $L^2(B_\varrho)$ and dense in Z_ϱ and $\mathcal{Z}_\varrho$ with respect to $L^2(B_\varrho)$ norm; the system

$$\left\{ \ f^1_k\,, \ f^2_k \ \right\}_{k=-2,-1,0,1,2,\dots} \tag{2.4}$$

is orthogonal in $L^2(B_\varrho)$ and dense in A^1_ϱ with respect to $L^2(B_\varrho)$ norm; the system

$$\left\{ \ f^3_k\,, \ f^4_k \ \right\}_{k=0,1,2,\dots} \tag{2.5}$$

is orthogonal in $L^2(B_\varrho)$ and dense in Z_ϱ and $\mathcal{Z}_\varrho$ with respect to $L^2(B_\varrho)$ norm. Four systems above lead to uniqueness of related expansion series in $L^2(B_\varrho)$.

Proof. All the statements about real systems follow as soon as the ones about complex systems are proved, since (2.4),(2.5) correspond to the real form of (2.2),(2.3), respectively f^1_k, f^2_k for $k = -2, -1, 0, 1, \dots$ and f^3_k, f^4_k for $k = 0, 1, 2, \dots$ (see Remark 2.2).

All the claims about orthogonality stated in the sequel follow by integration with respect to θ since the variables are separated.
We are left only to show in every case that the related system is dense.

First we consider the statements concerning A^1_ϱ.
It is well known that the system $\{e^{i(k-1/2)\theta}\}_{k\in\mathbb{Z}}$ is dense in $L^2(-\pi,\pi)$.

For our purposes we have rearranged and relabelled previous dense system as in (2.1): $\left\{ \varphi_k(\vartheta) := \exp\left(i(k - 3/2 \text{ sign } k)\,\vartheta\right) \right\}_{k\in\mathbb{Z}\setminus\{0\}}$.

Notice that $\varphi_1 = \varphi_{-2}$ and $\varphi_{-1} = \varphi_2$, while no other duplication occurs.

Legendre polynomials $\{P_n\}_n$ are an orthogonal and dense system in $L^2(-1,1)$:

$$P_n(x) \;=\; \frac{1}{2^n\, n!}\,\frac{d^n}{dx^n}\Big((x^2-1)^n\Big), \qquad n = 0,1,2,\dots\,. \tag{2.6}$$

By the change of coordinates $x = 2r-1$, $d/dx = 2\,d/dr$, we get the orthonormnal system of the *shifted Legendre polynomials* $\widetilde{P}_n(r)$, explicitely

$$\widetilde{P}_n(r) \;=\; \sqrt{2n+1}\; P_n(2r-1), \quad \|\widetilde{P}_n\|_{L^2(0,1)} = 1\,, \quad n = 0,1,2,\dots\,. \tag{2.7}$$

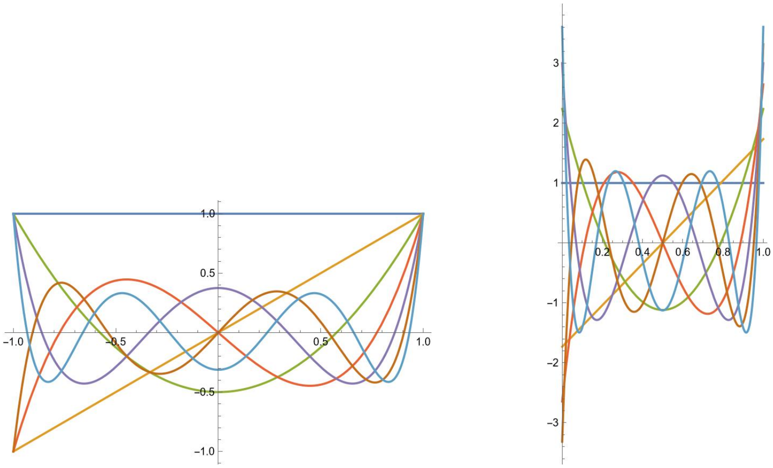

FIGURE 3. Graphs of Legendre polynomials P_n in $L^2(-1,1)$ (left) and graphs of rescaled and normalized shifted Legendre polynomials $\widetilde{P}_n$ in $L^2(0,1)$ (right), $n = 0,1,2,3,4,5,6$.

Hence the system

$$\{\,\mathcal{P}_n(r)\,\}_{n=0,1,\dots} \;\stackrel{\text{def}}{=}\; \{\,\widetilde{P}_n(r)\, r^{-1/2}\,\}_{n=0,1,\dots} \tag{2.8}$$

is orthonormal and dense in

$$L^2_r(0,1) := \left\{ R(r);\; \int_0^1 |R(r)|^2\, r\, dr < +\infty \right\},$$

since $(\mathcal{P}_n, \mathcal{P}_m)_{L^2_r(0,1)} = (\widetilde{P}_n, \widetilde{P}_m)_{L^2(0,1)}$ for every n, m.
Moreover, due to Stone-Weierstrass Theorem,

$$\{R_n(r)\}_{n=1,2,\dots} = \{r^{n-3/2}\}_{n=1,2,\dots} \quad \text{is dense in } L^2_r(0,1).$$

We emphasize that $\{\mathcal{P}_n(r)\}_{n=0,1,\dots}$ is an orthonormal basis in $L^2_r(0,1)$, while $\{R_n(r)\}_{n\in\mathbf{N}}$ is a neither orthogonal nor normalized basis. Therefore we set

$$\psi_k(\vartheta) := \varphi_k(\vartheta)\,/\,\|\varphi_k(\vartheta)\|_{L^2(-\pi,\pi)} = \varphi_k(\vartheta)\,/\,\sqrt{2\pi}\,, \quad k\in\mathbb{Z}\setminus\{0\}$$

so that

$$\|\,\mathcal{P}_n(r)\,\psi_k(\vartheta)\,\|_{L^2(B_1(0))} \;=\; \|\,\mathcal{P}_n(r)\|_{L^2_r(0,1)}\,\|\psi_k(\vartheta)\,\|_{L^2(-\pi,\pi)} \;=\; 1\,. \qquad (2.9)$$

All the subsequent statements about orthogonality follow either by integration with respect to θ when φ_k and φ_h are present with $k\neq h$, or integration with respect to r when $\mathcal{P}_n$ and $\mathcal{P}_m$ are present with $n\neq m$, since the variables are always separated.

Linear combinations of characteristic functions of sets

$$\{(r,\theta)\colon\; 0\le r_1\le r\le r_2\le 1,\; -\pi<\theta_1\le\theta\le\theta_2<\pi\}$$

are dense in

$$L^2(B_1) =$$

$$\left\{v(r,\theta):\; 0<r<1,\; |\vartheta|<\pi, \int_{B_1}|v|^2\,dxdy = \int_{-\pi}^{\pi}\int_0^1|v(r,\theta)|^2\,r\,dr\,d\theta<+\infty\right\};$$

hence the functions with separated variables $\{\mathcal{P}_n(r)\psi_k(\vartheta)\}_{n=0,1,\dots,|k|\neq 0}$ are dense in $L^2(B_1)$. For any fixed $k\in\mathbb{Z}\setminus\{0\}$, we define

$$W_k := \operatorname{span}\{\,\mathcal{P}_n(r)\,\psi_k(\vartheta);\; n=0,1,\dots\} = \operatorname{span}\{\,R_n(r)\,\varphi_k(\vartheta);\; n=1,2,\dots\}\,,$$

where span denotes the completion in $L^2(B_1)$ of the set of linear combinations. Then, by taking into account that $\varphi_1=\varphi_{-2}$ and $\varphi_{-1}=\varphi_2$, we summarize

$$W_k\perp W_h \text{ in } L^2(B_1),\; k\neq h, |k|>2, |h|>2\,;$$
$$W_j\perp W_h \text{ in } L^2(B_1),\; j=\pm1,\pm2,\; h>2\,;$$
$$L^2(B_1) \;=\; \Big(\bigoplus_{|k|>2} W_k\Big)\oplus\Big\{W_1+W_2+W_{-1}+W_{-2}\Big\}$$

where $\oplus$ denotes a direct and orthogonal sum. Thus every function $v\in L^2(B_1)$ can be represented by an $L^2(B_1)$ converging series

$$v(r,\vartheta) \;=\; \sum_{k\in\mathbb{Z}\setminus\{0\}}\,\sum_{n=1}^{+\infty} c_{n,k}\,\mathcal{P}_n(r)\,\psi_k(\vartheta)\,. \qquad (2.10)$$

Precisely, due to Lemma 2.3, since W_k is a space of functions with separate variables, once k is fixed:

- any finite (or possibly infinite but $L^2(B_1)$ convergent) linear combination of functions of the kind $R_n(r)\,\varphi_k(\vartheta)$ with $n\neq|k|$ cannot belong to the closed subspace $W_k\cap A^1_1$ for $|k|>2$, since it has separate variables;
- any finite (or possibly infinite but $L^2(B_1)$ convergent) linear combination of functions of the kind $R_n(r)\,\varphi_1(\vartheta)$ with $n\neq 1,2$ cannot belong to the closed subspace $W_k\cap A^1_1$ for $k=1,2$, since it has separate variables;
- any finite (or possibly infinite but L^2_r convergent) linear combination of functions of the kind $R_n(r)\,\varphi_{-1}(\vartheta)$ with $n\neq 1,2$ cannot belong to the closed subspace $W_k\cap A^1_1$ for $k=-1,-2$, since it has separate variables.

So we get

$$
\begin{array}{lcll}
A_1^1 \cap W_1 & = & \mathrm{span}\,\{\ R_1\varphi_1,\ R_2\varphi_1\ \}\ , & \\
A_1^1 \cap W_{-1} & = & \mathrm{span}\,\{\ R_1\varphi_{-1},\ R_2\varphi_{-1}\}\ , & \\
A_1^1 \cap W_2 & = & \mathrm{span}\,\{\ R_2\varphi_2,\ R_1\varphi_2\ \}\ , & \\
A_1^1 \cap W_{-2} & = & \mathrm{span}\,\{\ R_2\varphi_{-2},\ R_1\varphi_{-2}\ \}\ , & \\
A_1^1 \cap W_k & = & \mathrm{span}\,\{\ R_k\varphi_k\ \} & \forall k\in\mathbb{Z}:\ k>2\,, \\
A_1^1 \cap W_k & = & \mathrm{span}\,\{\ R_{|k|}\varphi_k\ \} & \forall k\in\mathbb{Z}:\ k<-2\,.
\end{array}
$$

Hence, by $\varphi_1=\varphi_{-2},\ \varphi_2=\varphi_{-1}$ we get
$R_2\varphi_1 = R_2\varphi_{-2},\ R_2\varphi_{-1}=R_2\varphi_2,\ R_1\varphi_2=R_1\varphi_{-1},\ R_1\varphi_{-2}=R_1\varphi_1,$ and

$$
\begin{aligned}
&\Big(\textstyle\bigoplus_{|k|>2} W_k\cap A_1^1\Big)\oplus\Big\{\big(W_1+W_2+W_{-1}+W_{-2}\big)\cap A_1^1\Big\} = \\
&\quad = \mathrm{span}\,\{R_n\,\varphi_n,\ n=1,2,\dots;\ R_n\,\varphi_{-n},\ n=1,2,\dots\ \}
\end{aligned}
\tag{2.11}
$$

where $\bigoplus$ denotes a direct and L^2 orthogonal sum of Hilbert spaces.
Summarizing:

$$
\dim W_k = 2 \quad (k=\pm 1,\ \pm 2)\,,\qquad \dim W_k = 1 \quad (|k|>2)\,.
$$

$$
L^2(B_1)\ =\ \mathrm{span}\,\{\mathcal{P}_n(r)\,\psi_k(\vartheta)\}_{\substack{n=1,2,\dots\\ k\in\mathbb{Z}\setminus\{0\}}}\ =\ \mathrm{span}\,\{R_n(r)\,\psi_k(\vartheta)\}_{\substack{n=1,2,\dots\\ k\in\mathbb{Z}\setminus\{0\}}}\ .
$$

Thus by taking into account that the set A_1^1 is a Hilbert space when endowed with the $L^2(B_1)$ norm, we obtain that the smaller system of unitary functions

$$
\left\{\ w_k = \sqrt{\frac{2|k-1|+1}{2\pi}}\ v_k\ \right\}_{k\in\mathbb{Z}\setminus\{0\}}
$$

is not only orthonormal in $L^2(B_1)$ (due to orthogonality of spaces W_k) but also dense in A_1^1 with respect to $L^2(B_1)$, as we prove in the sequel.

We denote by $\mathbb{P}$ the $L^2(B_1)$-orthogonal projection on the closed linear subspace A_1^1 of the L^2 and harmonic functions on $B_1\backslash\Gamma$.
Since $W_k\cap A_1^1$ is a closed subspace of A_1^1, the restriction of $\mathbb{P}$ to $W_k\cap A_1^1$ is the identity operator:

$$
\mathbb{P}\,(W_k\cap A_1^1)\ =\ W_k\cap A_1^1 \qquad \forall\,k\in\mathbb{Z}\backslash\{0\},
\tag{2.12}
$$

and, referring to the representation (2.10), $v\in W_k\cap A_1^1$ implies $\mathbb{P}(v) = c_{|k|,k}R_{|k|}\psi_k$; therefore the fact that the system (2.2) is dense in A_1^1 easily follows from (2.11) and Lemma 2.3.

By dilation we recover also the general case of (2.12):

$$
\text{the restriction of } \mathbb{P}_\varrho \text{ to }\ W_k\cap A_\varrho^1 \text{ is the identity operator,}
\tag{2.13}
$$

where $\mathbb{P}_\varrho$ denotes the $L^2_r(B_\varrho)$–orthogonal projection on the closed linear subspace A_ϱ^1 of harmonic functions in $B_\varrho\setminus\Gamma$. Indeed, when $0<\varrho<+\infty$,

$$
\text{system }\ \{\,R_n(r)\varphi_k(\vartheta)\,;\ k\in\mathbb{Z}\backslash\{0\},\ n=0,1,\dots\}\text{ is dense in } L^2(B_\varrho):
\tag{2.14}
$$

$$
L^2(B_\varrho)\ =\ \Big(\bigoplus_{|k|>2} W_k\Big)\oplus\Big\{W_1+W_2+W_{-1}+W_{-2}\Big\}\,,
$$

$$\left(\bigoplus_{|k|>2} W_k \cap A^1_\varrho\right) \oplus \left\{ (W_1 + W_2 + W_{-1} + W_{-2}) \cap A^1_\varrho \right\} = \\ = \operatorname{span}\left\{ R_n\, \varphi_n,\ n = 1, 2, \dots;\ R_n\, \varphi_{-n},\ n = 1, 2, \dots \right\},$$

where W_k is defined as above, up substituting span in $L^2(B_\varrho)$ in place of $L^2(B_1)$.

The set A^1_ϱ is a Hilbert space endowed with the $L^2(B_\varrho)$ norm, the dilated set w_k (orthonormal in $L^2(B_\varrho)$, dense in A^1_ϱ w.r.t. $L^2(B_\varrho)$) reads as follows:

$$\left\{ \begin{aligned} w_k(r) &= \sqrt{\frac{2|k-1|+1}{2\pi\varrho}}\, v_k(r/\varrho, \vartheta) = \\ &= \sqrt{\frac{2|k-1|+1}{2\pi\varrho}} \left(\frac{r}{\varrho}\right)^{|k|-3/2} \exp(i(k - 3/2\, \operatorname{sign} k)\vartheta) \end{aligned} \right\}_{k\in\mathbb{Z}\setminus\{0\}} . \tag{2.15}$$

Eventually, the statements about Z_ϱ: by (2.1) and Definition 2.1, $z_k = r^2 v_k$ belongs to Z_ϱ for any k and every $f \in Z_\varrho$ fulfils $f = r^2 g$ for suitable $g \in A^1_\varrho$. By the density of (2.2) in A^1_ϱ the claims about density of (2.3) in Z_ϱ (hence in $\mathcal{Z}_\varrho$) easily follow. As usual, orthogonality follows by integration in ϑ. □

3. Decomposition and expansion of a biharmonic function near the tip of a flat crack

We introduce the space A^Δ_ϱ as the natural framework to achieve an Almansi decomposition of biharmonic functions under weak regularity assumptions:

$$\begin{aligned} A^\Delta_\varrho &:= \left\{ u \in L^2(B_\varrho\setminus\Gamma),\ \Delta u \in L^2(B_\varrho\setminus\Gamma),\ \Delta^2_{\mathbf{x}}\, u = 0 \text{ on } B_\varrho\setminus\Gamma \right\} = \\ &= A^2_\varrho \cap \{ v \,:\, \Delta v \in L^2(B_\varrho \setminus\Gamma) \}. \end{aligned} \tag{3.1}$$

Theorem 3.1. Almansi decomposition in A^Δ_ϱ and expansion near crack-tip *Assume $0<\varrho<+\infty$. Then, $u \in A^\Delta_\varrho$ if and only if*

$$\exists\ \varphi, \psi \in A^1_\varrho : \qquad u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x}), \ \ , \forall\, \mathbf{x} \in B_\varrho \setminus \Gamma. \tag{3.2}$$

Decomposition (3.2) is unique up to possible linear terms in ψ.
The decomposition in A^Δ_ϱ is made explicit by the operators Φ and Ψ which, in polar coordinates, read as follows:

$$\Phi : A^\Delta_\varrho \to A^1_\varrho, \quad \Phi[u] = \frac{1}{4r}\int_0^r \Delta_{\mathbf{x}} u(t,\vartheta)\, dt\,, \tag{3.3}$$

$$\Psi : A^\Delta_\varrho \to A^1_\varrho, \quad \Psi[u] \ = \ u - r^2\, \Phi[u], \tag{3.4}$$

$$u(r,\vartheta) \ = \ \Psi[u] + r^2\, \Phi[u]\,, \tag{3.5}$$

The (non-orthogonal) system formed by (2.2) and (2.3) together is dense in A^Δ_ϱ with respect to $L^2(B_\varrho)$.
The (non-orthogonal) system formed by (2.4) and (2.5) together is dense in A^Δ_ϱ with respect to $L^2(B_\varrho)$.
Moreover we have the relationships

$$A^\Delta_\varrho = A^1_\varrho + (\mathcal{Z}_\varrho \cap A^\Delta_\varrho) \ = A^1_\varrho + Z_\varrho\,, \tag{3.6}$$

$$A^1_\varrho \cap \mathcal{Z}_\varrho = \{\, \textit{linear functions}\, \}, \tag{3.7}$$

all terms in the neither orthogonal nor direct sum (3.6) *are Hilbert spaces, except* Z_ϱ *which is pre-Hilbert.*
Moreover, we obtain these expansions strongly converging in L^2

$$\begin{cases} \Psi[u] = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} c_k\, v_k\,, \quad \forall u \in A^\Delta_\varrho\,, \qquad \textit{where} \\[2ex] c_k := \dfrac{\left(u - r^2\Phi[u], v_k\right)_{L^2(B_\varrho\setminus\Gamma)}}{\left(\|v_k\|_{L^2(B_\varrho\setminus\Gamma)}\right)^2} = \dfrac{\left(\Psi[u], v_k\right)_{L^2(B_\varrho\setminus\Gamma)}}{\left(\|v_k\|_{L^2(B_\varrho\setminus\Gamma)}\right)^2}\,, \end{cases} \tag{3.8}$$

$$\begin{cases} \Phi[u] = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} e_k\, v_k\,, \quad \forall u \in A^\Delta_\varrho\,, \qquad \textit{where} \\[2ex] e_k := \dfrac{\left((u - \Psi[u])\,, z_k\right)_{L^2(B_\varrho\setminus\Gamma)}}{\left(\|z_k\|_{L^2(B_\varrho\setminus\Gamma)}\right)^2} = \dfrac{\left(r^2\,\Phi[u]\,,\, z_k\right)_{L^2(B_\varrho\setminus\Gamma)}}{\left(\|z_k\|_{L^2(B_\varrho\setminus\Gamma)}\right)^2}\,. \end{cases} \tag{3.9}$$

Proof. We use these identities, for smooth scalar p, q or vector-valued $\mathbf{p}, \mathbf{q}$:

$$\Delta_{\mathbf{x}}(pq) \;=\; p\,\Delta_{\mathbf{x}} q + q\,\Delta_{\mathbf{x}}\, p + 2\,\nabla_{\mathbf{x}}\, p \cdot \nabla_{\mathbf{x}}\, q \qquad \forall p, q \in C^2(B_\varrho\setminus\Gamma), \tag{3.10}$$

$$\Delta_{\mathbf{x}}(\mathbf{p}\cdot\mathbf{q}) = \mathbf{p}\cdot\Delta_{\mathbf{x}}\mathbf{q} + \mathbf{q}\cdot\Delta_{\mathbf{x}}\mathbf{p} + 2\,(\nabla_{\mathbf{x}}\mathbf{p}) : (\nabla_{\mathbf{x}}\mathbf{q}) \;\; \forall \mathbf{p}, \mathbf{q} \in C^2(B_\varrho\setminus\Gamma, \mathbb{R}^2). \tag{3.11}$$

$$\Delta_{\mathbf{x}}\big(\|\mathbf{x}\|^2\varphi\big) \;=\; \|\mathbf{x}\|^2\Delta_{\mathbf{x}}\varphi + 4\varphi + 4\mathbf{x}\cdot\nabla_{\mathbf{x}}\varphi \;=\; 4\,\varphi \,+\, 4\,r\,\frac{\partial\varphi}{\partial r} \qquad \forall\varphi\in C^2(B_\varrho\setminus\Gamma)\,, \tag{3.12}$$

$$\Delta^2_{\mathbf{x}}(\|\mathbf{x}\|^2\varphi) \;=\; \|\mathbf{x}\|^2\Delta^2_{\mathbf{x}}\varphi + 16\Delta_{\mathbf{x}}\varphi + 8\mathbf{x}\cdot(\nabla_{\mathbf{x}}\Delta_{\mathbf{x}}\varphi) \qquad \forall\varphi\in C^4(B_\varrho\setminus\Gamma)\,. \tag{3.13}$$

"If part": (3.2) entails $\varphi, \psi, u \in C^\infty(B_\varrho\setminus\Gamma)$ and $u \in A^\Delta_\varrho$. If $\exists\ \varphi, \psi \in A^1_\varrho$: $u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2\varphi(\mathbf{x})\,,\ \Delta_{\mathbf{x}}\,\varphi \,=\, \Delta_{\mathbf{x}}\,\psi \,\equiv\, 0$ on $B_\varrho\setminus\Gamma$, we can set $p = \|\mathbf{x}\|^2$, $q = \varphi$, $\varrho = \|\mathbf{x}\|$: we get $\nabla_{\mathbf{x}}\, p = 2\mathbf{x}$, $\Delta_{\mathbf{x}} p = 4$, thus by (3.13) $\Delta_{\mathbf{x}}\varphi = 0$ entails $\Delta^2_{\mathbf{x}}\big(\|\mathbf{x}\|^2\varphi\big) = 0$, hence $\Delta^2_{\mathbf{x}} u = 0$. Eventually $\varphi \in A^1_\varrho$ entails $\Delta_{\mathbf{x}}(\|\mathbf{x}\|^2\varphi) = 4\varphi + 4r\varphi_r \in L^2(B_\varrho\setminus\Gamma)$ by density of the orthonormal system v_k in A^1_ϱ. Thus (3.2) entails $u \in A^\Delta_\varrho$.

"Only if part": $u \in A^\Delta_\varrho$ entails (3.2).
In fact, by setting $\sigma := \frac{1}{4}\,\Delta_{\mathbf{x}}\, u$, if we find a solution φ of

$$\varphi \in A^1_\varrho \;: \qquad r\,\frac{\partial\varphi}{\partial r} + \varphi \;=\; \sigma \quad \text{on } B_\varrho\setminus\Gamma\,, \tag{3.14}$$

then the function $\psi = u - r^2\varphi$ is harmonic (thanks to (3.12)) and φ, ψ together will match the claim thank to (3.13).
Actually, it is enough showing that an explicit solution of (3.14) in polar coordinates is given by

$$\varphi(r,\vartheta) \;=\; r^{-1}\int_0^r \sigma(t,\vartheta)\, dt \qquad 0 < r < \varrho\,,\ |\vartheta| < \pi\,. \tag{3.15}$$

By Lemma 2.5 and referring to the spaces definition (2.1), we know that the systems (2.2) and (2.4) are dense in $L^2(B_\varrho\backslash\Gamma)$ for every $\varrho>0$, moreover $u\in A^{\Delta}_\varrho$ entails that σ belongs to A^1_ϱ; by Lemma 2.5 and referring to the basis w_k defined in (2.15), we get this expansion with uniquely defined coefficients and strongly converging in $L^2(B_\varrho\backslash\Gamma)$:

$$\sigma(r,\vartheta) \;=\; \sum_{k\in\mathbb{Z}\backslash\{0\}} c_k\,\sqrt{\frac{2|k-1|+1}{2\pi\varrho}}\;\left(\frac{r}{\varrho}\right)^{|k|-3/2}\exp\big(i(k-3/2\ \mathrm{sign}\,k)\,\vartheta\big). \tag{3.16}$$

Hence, by the orthogonality in $L^2(B_\varrho\backslash\Gamma)$ of the system $\{\,v_k\,\}_{k\in\mathbb{Z}\backslash\{0\}}$ and by Parseval identity (taking into account of the 2-d jacobian r):

$$\frac{1}{\varrho^2}\sum_{k\in\mathbb{Z}\backslash\{0\}}\frac{2|k-1|+1}{2|k|-1}\left(\frac{r}{\varrho}\right)^{2|k|-1}|c_k|^2 <\; +\infty\,,\qquad \forall r\le\varrho\,.$$

Moreover, for every $u\in A^{\Delta}_\varrho$, $0<r<\varrho$ and $|\vartheta|<\pi$, the operator image $\Phi[u](r,\vartheta)$ can be evaluated termwise via (3.16), due to Lebesgue dominated convergence Theorem:

$$\begin{aligned}\varphi(r,\vartheta) \;&=\; \Phi[u](r,\vartheta)=\frac{1}{r}\int_0^r\sigma(t,\vartheta)\,dt=\\ &=\sum_{k\in\mathbb{Z}\backslash\{0\}}\frac{c_k}{|k|-1/2}\sqrt{\frac{2|k-1|+1}{2\pi\varrho}}\;\left(\frac{r}{\varrho}\right)^{|k|-3/2}\exp\big(i(k-3/2\ \mathrm{sign}\,k)\,\vartheta\big).\end{aligned}$$

Hence, $\sigma=\frac{1}{4}\Delta_{\mathbf{x}}u$ is a real analytic function of two variables in $B_\varrho\backslash\Gamma$ which belongs to $L^2(B_\varrho)$ and to $L^1(\Sigma_{\vartheta,s})$ for every radius $\Sigma_{\vartheta,s}$ of length $s<\varrho$ from the origin with frozen ϑ, $|\vartheta|<\pi$.
Thus the right-hand side in (3.15) is well defined function for every $r<\varrho$ and $|\vartheta|<\pi$ and both $\Phi[u]=r^{-1}\int_0^r\sigma$ and $\Psi[u]=u-r^2\Phi[u]$ belong to $L^2(B_\varrho\backslash\Gamma)$.
Therefore, if we prove $\Delta_{\mathbf{x}}\Phi[u]=0$ on $B_\varrho\backslash\Gamma$, then, by (3.14) we deduce that both operators Φ, Ψ have range in A^1_ϱ.
By (3.16), we get

$$r\,\sigma_r\in L^1(\Sigma_{\vartheta,s})\,,\qquad r\,\left(\,r\,\sigma_r\,\right)_r\in L^1(\Sigma_{\vartheta,s})\,,$$

while, by definition of σ and $u\in A^{\Delta}_\varrho$, $\Delta_{\mathbf{x}}\sigma=\frac{1}{4}\Delta^2_{\mathbf{x}}u=0$. Thus

$$\sigma_{\vartheta\vartheta} \;=\; -\,r\,\left(\,r\,\sigma_r\,\right)_r \tag{3.17}$$

say $\sigma_{\vartheta\vartheta}\in L^1(\Sigma_{\vartheta,s})$.
Summarizing, (3.15) is well defined and $r\to r\,\varphi(r,\vartheta)$ is absolutely continuous on any radius of length $s<\varrho$ starting at the origin.

First we prove that $r\,\varphi_r+\varphi=\sigma$ on $B_\varrho\backslash\Gamma$.
By (3.15) and integrability of σ on $\Sigma_{\vartheta,s}$ we get, $\forall\vartheta\in(-\pi,\pi)$ and $\forall r\in[0,s]$:

$$(\partial/\partial r)\int_0^r\sigma\,dt\;=\;\sigma\,,$$

$$\frac{\partial\varphi}{\partial r}\;=\;r^{-1}\,\sigma(r,\theta)\;-\;r^{-2}\int_0^r\sigma(t,\theta)\,dt\,,$$

hence

$$r\,\frac{\partial\varphi}{\partial r}+\varphi \;=\; \sigma(r,\theta)\;-\;r^{-1}\int_0^r\sigma(t,\theta)\,dt\;+\;r^{-1}\int_0^r\sigma(t,\theta)\,dt\;=\;\sigma\,.$$

Second, we prove $\Delta_{\mathbf{x}}\varphi=0$: we compute, for any $r\le s$:

$$\Delta_{\mathbf{x}}\,\varphi(r,\vartheta)\;=\;\frac{1}{r^3}\left(-r\,\sigma(r,\vartheta)\;+\;r^2\,\sigma_r(r,\vartheta)\;+\;\int_0^r\big(\sigma(t,\vartheta)\;+\sigma_{\vartheta\vartheta}(t,\vartheta)\big)\,dt\right).$$

The map

$$r\;\rightarrow\;\left(-r\,\sigma\;+\;r^2\,\sigma_r\;+\;\int_0^r\;(\sigma\;+\sigma_{\vartheta\vartheta})\;dt\right)\;=\;r^3\,\Delta_{\mathbf{x}}\varphi \qquad (3.18)$$

is absolutely continuous on $(0,s)$ for any $\vartheta\in(-\pi,\pi)$ and any $s<\varrho$ and vanishes at $r=0$ since σ and $\sigma_{\vartheta\vartheta}$ belong to $L^1(\Sigma_{\vartheta,s})$. By taking into account (3.17) and applying $\partial/\partial r$ to (3.18):

$$-\,\sigma\;-r\,\sigma_r\;+\;2\,r\,\sigma_r+r^2\sigma_{rr}+\sigma\;+\;\sigma_{\vartheta\vartheta}\;=\;r\,(r\,\sigma_r)_r\;+\;\sigma_{\vartheta\vartheta}\;=\;0\qquad\forall r\le s\,.$$

Summarizing $\varphi\in A^1_\varrho$ and $r\,\dfrac{\partial\varphi}{\partial r}+\varphi=\sigma$ on $B_s\backslash\Gamma$, $\forall s<\varrho$, say φ solves (3.14).

About uniqueness of Almansi-type decomposition it is enough showing that, if $\sigma\equiv 0$ then the solution of linear problem (3.14) is a linear combination of $\cos\vartheta$ and $\sin\vartheta$. Indeed the linear ODE in (3.14) entails that $r\,\varphi$ is absolutely continuous on $\Sigma_{\vartheta,s}$; moreover $\sigma\equiv 0$ imply $r\,\varphi$ is constant on every radius $\Sigma_{\vartheta,s}$ (the constant may depend on ϑ), hence it entails $\varphi(r,\vartheta)=C(\vartheta)/r$; then by $\Delta_{\mathbf{x}}\varphi=0$, $C(\vartheta)$ must be a linear combination of $\cos\vartheta$ and $\sin\vartheta$ in φ, which can be replaced by $C(\vartheta)\,r=r^2\varphi$ in ψ.

Now, we prove the statements concerning decomposition (3.6) of A^{Δ}_ϱ.
By direct computation every function $z_k=r^2v_k$ is biharmonic in $B_\varrho\backslash\Gamma$, moreover $z_k\in A^2_\varrho$ for any k. Due to the equivalence proved above, A^{Δ}_ϱ is expressed by a (non orthogonal) algebraic sum: $A^{\Delta}_\varrho\;=\;A^1_\varrho+Z_\varrho$. More precisely, since $A^1_\varrho\subset A^{\Delta}_\varrho$, by difference $A^{\Delta}_\varrho\;=\;A^1_\varrho+(\mathcal{Z}_\varrho\cap A^{\Delta}_\varrho)$. Thus (3.6) is proved and by Lemma 2.5 we achieve the density in A^{Δ}_ϱ of the whole system $\{v_k,z_k\}_{k\in\mathbb{Z}\backslash\{0\}}$ defined by (2.2) and (2.3).

In order to show formula (3.7) about the intersection $A^1_\varrho\cap\mathcal{Z}_\varrho$, which prevents the algebraic sum of Hilbert spaces in (3.6) to be a direct sum, we notice that $u\in A^1_\varrho\cap\mathcal{Z}_\varrho$ if and only if

$$u\in L^2(B_\varrho\backslash\Gamma),\quad \Delta u=0 \text{ on } B_\varrho\backslash\Gamma,\quad u=r^2\varphi \text{ with } \Delta\varphi=0 \text{ on } B_\varrho\backslash\Gamma\,,$$

thus

$$0=\Delta u=\Delta(r^2\varphi)=2\,\varphi+2\,\mathbf{x}\cdot\nabla_{\mathbf{x}}\varphi=2\,(\varphi+r\frac{\partial\varphi}{\partial r})$$

the general solution of $\varphi=-\,r\,\dfrac{\partial\varphi}{\partial r}$ is $\varphi=C(\vartheta)\,r^{-1}$ and since φ is harmonic we get $\varphi=\big(A\cos(\vartheta)+B\sin(\vartheta)\big)/r$ and $u=r^2\varphi=Ax+By$.

By exploiting the density of system (2.2) in A^1_ϱ and of the system (2.3) in Z_ϱ (Lemma 2.5), we obtain the expansions (3.8) and (3.9), strongly convergent in L^2. The last equality in (3.9) follows by $r^2\sum e_kv_k=\sum c_kz_k$. $\square$

Remark 3.2. The intersection at middle term of equalities (3.6) is mandatory, since $A^1_\varrho \subset A^2_\varrho$ and $\mathcal{Z}_\varrho \backslash A^\Delta_\varrho \neq \emptyset$, for instance $r^{1/2}\exp(-3i\vartheta/2) \in \mathcal{Z}_\varrho \backslash A^\Delta_\varrho$.

Remark 3.3. We emphasize that functions in A^1_ϱ and Z_ϱ "are not required to solve a prescribed boundary value problem", say they are not required to fulfil any kind of boundary conditions on Γ. Hence the convergence of series associated to the basis (2.2), or (2.3), may be very slow even in L^2; obviously they fit better with some kind of boundary conditions: in such a case the convergence is faster and possibly stronger, as is the case of the space V, introduced and studied in the next section.
The choice of the basis (2.2) and (2.3) correspond to fix a priori the basis in the angular variable in such a way that the associated functions depending on the radial coordinate are in $L^2_r(0,1)$ (where the weight r is the 2-d jacobian) and have an homogeneity property tightly dependent on the choice of the angular basis whenever the condition of vanishing Laplacean is imposed.

Remark 3.4. The closed space A^Δ_ϱ is smaller than A^2_ϱ and is wider than $A^1_\varrho \cap H^2(B_\varrho \setminus \Gamma)$ and $A^1_\varrho \cap H^1(B_\varrho \setminus \Gamma)$. Indeed

$$r^{1/2}\exp(-3i\vartheta/2) \in \left(A^2_\varrho \setminus A^\Delta_\varrho\right) \cap \left(A^2_\varrho \setminus A^1_\varrho\right) \tag{3.19}$$

entails

$$A^\Delta_\varrho \underset{\neq}{\subset} A^2_\varrho\,, \tag{3.20}$$

whereas $v_{\pm 1} = r^{-1/2}\exp(\mp i\vartheta/2) \in A^\Delta_\varrho \backslash H^1(B_\varrho \backslash \Gamma)$ (by $|\nabla v_{\pm 1}| \sim a\, r^{-3/2}$) and $v_{\pm 2} = r^{+1/2}\exp(\pm i\vartheta/2) \in \left(A^\Delta_\varrho \cap H^1(B_\varrho \backslash \Gamma)\right) \backslash H^2(B_\varrho \backslash \Gamma)$
(by $|\nabla v_{\pm 2}| \sim a r^{-1/2}$, $|D^2 v_{\pm 2}| \sim r^{-3/2}$), together with $v_{\pm 1}, v_{\pm 2} \in A^1_\varrho \subset A^2_\varrho$ entail

$$A^2_\varrho \cap H^2(B_\varrho \backslash \Gamma) \underset{\neq}{\subset} A^2_\varrho \cap H^1(B_\varrho \backslash \Gamma) \underset{\neq}{\subset} A^\Delta_\varrho\,. \tag{3.21}$$

Thus Theorem 3.1 shows that A^Δ_ϱ is the widest framework where the decomposition (3.2) of a biharmonic function is achieved with summands having both terms φ and ψ in $L^2(B_\varrho \backslash \Gamma)$. On the other hand, the four biharmonic functions $r^{1/2}\exp(\pm 3i\vartheta/2)$ and $r^{-1/2}\exp(\pm 5i\vartheta/2)$, though outside this framework, still have an Almansi type decomposition; actually they are already decomposed: they reduce to a single term belonging to $\mathcal{Z}_\varrho \backslash Z_\varrho \subset A^2_\varrho \backslash A^\Delta_\varrho$.
In general, if one drops the request of summands having both terms φ and ψ in $L^2(B_\varrho \backslash \Gamma)$ in the decomposition, still a decomposition can be achieved as stated in the subsequent Corollary, extending the results in Theorem 3.1 about framework A^Δ_ϱ to the wider A^2_ϱ framework. To this aim, we add another space to the list in Definition 2.1:

$$F_\varrho := \{ w \in \mathcal{D}'(B_\varrho \backslash \Gamma) \text{ s.t. } r^2 w \in L^2(B_\varrho \backslash \Gamma),\ \Delta_{\mathbf{x}} w = 0 \text{ on } B_\varrho \setminus \Gamma \}\,, \tag{3.22}$$

so that

$$\mathcal{Z}_\varrho = r^2 F_\varrho\,, \tag{3.23}$$

and

$$\eta_k(\vartheta) \;:=\; \exp\big(i(k - 7/2 \,\mathrm{sign}\, k)\,\vartheta\big), \qquad k \in \mathbb{Z} \setminus \{0\} \tag{3.24}$$

$$\begin{aligned} w_k(r,\vartheta) \;&:=\; r^{|k|-7/2}\eta_k(\vartheta) = \\ &=\; r^{|k|-7/2} \exp\big(i(k - 7/2 \,\mathrm{sign}\, k)\,\vartheta\big), \quad k \in \mathbb{Z} \setminus \{0\}, \end{aligned} \tag{3.25}$$

Notice that

$$\{\, w_k,\, k\in\mathbb{Z}\setminus\{0\}\,\} \;=\; \{\, v_k,\, k\in\mathbb{Z}\setminus\{0\}\,\} \cup \{\, r^{-3/2}e^{\pm i3/2\,\vartheta}\} \cup \{\, r^{-5/2}e^{\pm i5/2\,\vartheta}\}, \tag{3.26}$$

though

$$A^2_\varrho + \{\, r^{-3/2}e^{\pm i3/2\,\vartheta}\} + \{\, r^{-5/2}e^{\pm i5/2\,\vartheta}\} \underset{\neq}{\subset} F_\varrho\,,$$

due to differently weighted L^2 spaces.

Corollary 3.5. *If $0<\varrho<+\infty$, then the system (3.25) is orthogonal and dense in F_ϱ with respect to $L^2_{r^4}(B_\varrho\setminus\Gamma)$ norm, moreover $w \in A^2_\varrho$ if and only if*

$$\exists\; \psi\in A^1_\varrho,\; \varphi \in \mathcal{Z}_\varrho : \quad w(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2\varphi(\mathbf{x}), \;\; , \forall\, \mathbf{x}\in B_\varrho\setminus\Gamma\,, \tag{3.27}$$

say,

$$A^2_\varrho \;=\; A^1_\varrho + \mathcal{Z}_\varrho\,. \tag{3.28}$$

Given $w\in A^2_\varrho$ the representation (3.27) is achieved by setting $\varphi = \widetilde{\Phi}[w]$ and $\psi = \widetilde{\Psi}[w]$, with

$$\begin{aligned} &\widetilde{\Phi} : A^2_\varrho \to F_\varrho\,, \quad \widetilde{\Psi} : A^2_\varrho \to A^1_\varrho\,, \quad \textit{where} \\ &\textit{for}\;\; \Delta w = \sum_{k\neq 0} a_k w_k = v \;\textit{in}\; F_\varrho \quad \textit{(series converging in } L^2_{r^4}(B_\varrho\setminus\Gamma)\textit{)} \\ &\widetilde{\Phi}[w] \;:=\; \sum_{k\neq 0} \frac{a_k}{4\,(|k|-7/2)}\, w_k \qquad \textit{and} \qquad \widetilde{\Psi}[w] := w - r^2\widetilde{\Phi}[w]\,. \end{aligned} \tag{3.29}$$

Proof. Notice that $F_\varrho \subset L^2_{r^4}(B_\varrho\setminus\Gamma) = \{w : \int_{B_\varrho\setminus\Gamma} |w|^2\, r^4\, d\mathbf{x}\}$
The proof of Lemma 2.5 can be easily adapted to the present case in order to show

$$\{\, \mathfrak{P}_n(r)\,\}_{n=0,1,\dots} \overset{\mathrm{def}}{=} \{\, \widetilde{P}_n(r)\, r^{-5/2}\,\}_{n=0,1,\dots}$$

is orthonormal and dense in

$$L^2_{r^5}(0,1) := \left\{ R(r);\; \int_0^1 |R(r)|^2\, r^5\, dr < +\infty \right\},$$

since $(\mathcal{P}_n, \mathcal{P}_m)_{L^2_{r^5}(0,1)} = (\widetilde{P}_n, \widetilde{P}_m)_{L^2(0,1)}$ for every n, m; as usual orthogonality properties are plain consequences of separate variables and dependance on ϑ; moreover, due to Stone-Weierstrass Theorem,

$$\{\mathfrak{R}_n(r)\}_{n=1,2,\dots} = \{r^{n-5/2}\}_{n=1,2,\dots} \quad \text{is dense in } L^2_{r^5}(0,1)\,;$$

$\{\mathfrak{P}_n(r)\}_{n=0,1,\dots}$ is an orthonormal basis in $L^2_r(0,1)$, while $\{\mathfrak{R}_n(r)\}_{n\in\mathbf{N}}$ is a neither orthogonal nor normalized basis. Summarizing, the functions with separated variables $\{\mathfrak{P}_n(r)\eta_k(\vartheta)\}_{n=0,1,\dots,|k|\neq 0}$ are dense in F_ϱ, say the system (3.25) is orthogonal and dense in F_ϱ with respect to $L^2_{r^4}(B_\varrho\setminus\Gamma)$ norm.

Moreover the proof of Theorem 3.1 can be repeated in the present case with no change, up to one point: here the whole Laplacean of w is not integrable near the origin along the radii and this makes meaningless an integral representation of $\Phi[w]$ of the kind (3.3). However $w \in A^2_\varrho$ entails $\Delta w \in F_\varrho$; thus Δw has an expansion in terms of w_k and converging in $L^2_{r^4}$; moreover φ can be equivalently achieved by termwise computation along the expansion of Δw referrred to the dense system w_k and replacing each term $\frac{1}{4r}\int_0^r a_k w_k(r,t)\,dt$ (in the formal expansion of $\frac{1}{4r}\int_0^r \Delta_{\mathbf{x}} w\,dt$) respectively with $\frac{a_k\, w_k(r,\vartheta)}{4\,r\,(|k|-7/2)}$, $0<r<\varrho$. Thus the series $\sum_{k\neq 0} a_k w_k$ in (3.29) is converging in $L^2_{r^4}(B_\varrho\setminus\Gamma)$ to Δw, whence $r^2 \sum_{k\neq 0} \frac{a_k\, w_k(r,\vartheta)}{4\,r\,(|k|-7/2)}$ is converging in $L^2(B_\varrho)$ to $r^2\varphi$.
Eventually, both $\widetilde{\Phi}$ and $\widetilde{\Psi}$ map A^2_ϱ to harmonic functions in F_ϱ, $r^2\widetilde{\Phi}$ maps A^2_ϱ to $L^2(B_\varrho\backslash\Gamma)$: summarizing $\widetilde{\Psi}$ maps A^2_ϱ to $F_\varrho \cap L^2(B_\varrho\backslash\Gamma) = A^1_\varrho$. □

Remark 3.6. If $w \in A^2_\varrho$ and $\Delta w = \sum_{k\neq 0} a_k w_k$ in F_ϱ and in addition with convergence of $\sum_{|k|>2} a_k w_k$ to some v in $L^2(B_\varrho)$ too, then

$$\widetilde{\Phi}[w] = \Phi[v] - \tfrac{2}{20} r^{-5/2}(a_1 e^{i\frac{5}{2}\vartheta} + a_{-1} e^{-i\frac{5}{2}\vartheta}) - \tfrac{2}{12} r^{-3/2}(a_2 e^{i\frac{3}{2}\vartheta} + a_{-2} e^{-i\frac{3}{2}\vartheta}).$$

4. Decomposition and expansion of a biharmonic function near the tip of a flat crack in the H^2 framework

In this Section we verify that all nonaffine functions in $A^1_\varrho \cap H^2(B_\varrho\backslash\Gamma)$ are orthogonal in $H^2(B_\varrho\backslash\Gamma)$ to nonaffine functions belonging to $Z_\varrho \cap H^2(B_\varrho\backslash\Gamma)$; moreover any function in $A^2_\varrho \cap H^2(B_\varrho\backslash\Gamma)$ can be expanded in power series in terms of these four systems

$$\{\, v_k \,\}_{k\in\mathbb{Z}\backslash\{0\}},\ \{\, z_k \,\}_{k\in\mathbb{Z}\backslash\{0\}} \tag{4.1}$$

$$\{\, r^{|k|}\exp(i\,k\vartheta) \,\}_{k\in\mathbb{Z}},\ \{\, r^{|k|+2}\exp(i\,k\vartheta) \,\}_{k\in\mathbb{Z}}, \tag{4.2}$$

altogether, to take into account of functions which are smooth across Γ; Therefore, we exploit the splitting $A^\Delta_\varrho = A^1_\varrho + Z_\varrho$, which allows us to get rid of redundancy related to functions without jump or crease on Γ and write expansion series based on nothing more than the system (4.1), as it is clarified by Lemma 4.3, which refers to the strong $H^2(B_\varrho\backslash\Gamma)$ topology in the subspace V of functions orthogonal to ones which are smooth across Γ (see Definition 4.2). Eventually, Theorems 4.4 and 4.9 describe the restriction of operators Φ and Ψ for decomposition, providing decomposition summands in the whole $H^2(B_\varrho\backslash\Gamma)$ together with explicit power series expansions converging strongly in this space.

Definition 4.1. We introduce the sesquilinear form

$$(\varphi,\psi)_{2,\varrho} \ := \ a_{B_\varrho\backslash\Gamma}(\varphi,\psi) \ = \ \int_{B_\varrho\backslash\Gamma} D^2\varphi \ : \ \overline{D^2\psi}\ dx\,dy\ .$$

which induces a semi-norm in $H^2(B_\varrho \setminus \Gamma)$:

$$|v|_{2,\varrho} = (v,v)^{1/2}_{2,\varrho}\,.$$

Set

$$\|v\|^2_{H^2(B_\varrho\setminus\Gamma)} = \|v\|^2_{L^2(B_\varrho)} + \|D^2 v\|^2_{L^2(B_\varrho\setminus\Gamma)} = \|v\|^2_{L^2(B_\varrho)} + (v,v)_{2,\varrho} \quad (4.3)$$

Definition 4.2. Let V be the space of biharmonic functions that are orthogonal to the smooth functions in the whole B_ϱ with respect to the scalar product in $H^2(B_\varrho\setminus\Gamma)$ associated to the semi-norm $|\cdot|_{2,\varrho}$ and orthogonal to affine functions with respect to the $L^2(B_\varrho)$ scalar product; precisely, referring to Definition 2.1, we set:

$$V := A^2_\varrho \bigcap \{\ H^2(B_\varrho)\ \}^{\perp\, H^2(B_\varrho\setminus\Gamma)} \bigcap \{\ \text{affine functions}\ \}^{\perp\, L^2(B_\varrho)} \quad (4.4)$$

where

$$\left\{H^2(B_\varrho)\right\}^{\perp\, H^2(B_\varrho\setminus\Gamma)} = \left\{v\in H^2(B_\varrho\setminus\Gamma):\ (v,w)_\varrho = 0\ \ \forall w\in H^2(B_\varrho)\right\}$$

is the Hilbert space orthogonal to $H^2(B_\varrho)$ in $H^2(B_\varrho\setminus\Gamma)$, and

$$\{\ \text{affine functions}\ \}^{\perp\, L^2(B_\varrho)} = \left\{v\in H^2(B_\varrho\setminus\Gamma):\ \int_{B_\varrho} v\,\overline{w}\,dxdy = 0\ \forall\ \text{affine } w\right\}.$$

Note that V is a Hilbert space when endowed with the norm $|\cdot|_{2,\varrho}$, which turns out to be equivalent to $\|v\|^2_{L^2(B_\varrho)} + \|D^2 v\|^2_{L^2(B_\varrho\setminus\Gamma)}$ in V.

Lemma 4.3. *Each one of the two systems*

$$\left\{ \{\ v_k\ \}_{k\in Z,\,|k|>2}\ ,\ \{\ z_k\ \}_{k\in\mathbb{Z},\,k\neq 0} \right\} \qquad \textit{(complex basis)} \quad (4.5)$$

and

$$\left\{ \{\ f^1_k\ ,\ f^2_k\ ,\ f^3_k\ ,\ f^4_k\ \}_{k=0,1,\ldots} \right\} \qquad \textit{(real basis)} \quad (4.6)$$

is an orthogonal system with respect to the scalar product $(\cdot,\cdot)_\varrho$; moreover they are both dense in V with respect to $H^2(B_\varrho\setminus\Gamma)$ norm.
Therefore the terms (4.2) (corresponding to H^2 functions smooth across Γ) are redundant and unnecessary here: precisely, for any v in V there exists a unique expansion converging to v in $H^2(B_\varrho\setminus\Gamma)$, with respect to both systems (either the complex one (4.5) or its real counterpart (4.6)), as follows

$$v = \sum_{h\in\mathbb{Z}\setminus\{0,\pm1,\pm2\}} C_h v_h + \sum_{h\in\mathbb{Z}\setminus\{0\}} E_h z_h \quad (4.7)$$

and

$$\begin{aligned} v = \sum_{h=0}^{+\infty} r^{h+\frac{3}{2}} \Bigg(& c^1_h \cos\left(\left(h+\frac{3}{2}\right)\vartheta\right) + c^2_h \sin\left(\left(h+\frac{3}{2}\right)\vartheta\right) + \\ & + c^3_h \cos\left(\left(h-\frac{1}{2}\right)\vartheta\right) + c^4_h \sin\left(\left(h-\frac{1}{2}\right)\vartheta\right)\Bigg) \end{aligned} \quad (4.8)$$

where $\forall v\in V$ the coefficients $C_h, E_h, c^1_h, c^2_h, c^3_h, c^4_h$ are uniquely defined by

$$C_h = \frac{(v,v_h)_{2,\varrho}}{(|v_h|_{2,\varrho})^2} \qquad h\in\mathbb{Z},\ |h|>2, \quad (4.9)$$

$$E_h = \frac{(v,z_h)_{2,\varrho}}{(|z_h|_{2,\varrho})^2} \qquad h\in\mathbb{Z}\setminus\{0\}, \quad (4.10)$$

$$c_h^j = \frac{(v, f_h^j)_{2,\varrho}}{(|f_h^j|_{2,\varrho})^2} \qquad h = 0, 1, 2, \dots , \; j = 1, 2, 3, 4. \tag{4.11}$$

Both expansions (4.7) and (4.8) are strongly convergent in $H^2(B_\varrho \setminus \Gamma)$.

Proof. Assume $u_h \in V$ and $u_h \to u$ strongly in $H^2(B_\varrho\setminus\Gamma)$ then $\Delta^2_{\mathbf{x}} u = 0$ in $B_\varrho\setminus\Gamma$ and $(u, w)_{L^2(B_\varrho)} = 0$ for any affine w and $(u, w)_{2,\varrho} = 0$. Thus V is complete with respect to the norm induced by the scalar product $(\,.\,,\,.\,)_{2,\varrho}$. Lemma 2.5 entails that the system $\{v_k\}$ is orthogonal and dense with respect to $L^2(B_\varrho)$ norm in A^1_ϱ and the system $\{z_k\}$ is dense with respect to $L^2(B_\varrho)$ norm in $\mathcal{Z}_\varrho$. Therefore, by $V \subset A^\Delta_\varrho$ and (3.6) in Theorem 3.1, both systems (4.5) and (4.6) are dense in V with respect to the $L^2(B_\varrho)$ norm. We have only to show orthogonality and density in $H^2(B_\varrho\setminus\Gamma)$ of the whole set (4.5). By performing long computations, checked with software Wolfram Mathematica© 13.2 too, we obtain this list of orthogonality relationships:

$$\begin{cases} (v_k, v_h)_{2,\varrho} = 0, & k, h \in \mathbb{Z}\setminus\{0, \pm1, \pm2\}, \;\; k \neq h, \\ (z_k, z_h)_{2,\varrho} = 0, & k, h \in \mathbb{Z}\setminus\{0\}, \;\; k \neq h, \\ (v_k, z_h)_{2,\varrho} = 0, & k, h \in \mathbb{Z}\setminus\{0\}, \\ (f_k^i, f_l^j)_{2,\varrho} = 0, & k, l = 0,1, \dots , \; i, j = 1,2,3,4, \text{ either } i \neq j \text{ or } k \neq l, \\ (|v_k|_{2,\varrho})^2 \;=\; \dfrac{\pi}{2}\,(2k-5)\,(2k-3)^2\,\varrho^{2k-5} & k \in \mathbb{Z}\setminus\{0, \pm1, \pm2\} \\ (|z_k|_{2,\varrho})^2 \;=\; \frac{\pi}{2}\,(2k-1)\,(4k^2-12k+17)\,\varrho^{2k-1} & k \in \mathbb{Z}\setminus\{0\} \end{cases} \tag{4.12}$$

Therefore the two sets $\{v_k, z_k\}_{k\in\mathbb{Z}}$ and $\left\{f_k^1,\, f_k^2,\, f_k^3,\, f_k^4,\,\right\}_{k=0,1,\dots}$ are built with independent functions mutually orthogonal with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_{2,\varrho}$.
By Theorem 3.1 we know that an expansion of type (4.7) exists with coefficients c_h and e_h (a priori, possibly different from C_h and E_h evaluated by (4.9),(4.10)) and is strongly convergent at least in L^2 for any $v \in V \subset A^\Delta_\varrho$. The orthogonality relationship (4.12) entails pairwise orthogonality in V of terms in the expansion hence uniqueness of expansion (4.7) (if it exists); so we are left to show the existence of such expansion for any $v \in V$, or equivalently the $H^2(B_\varrho\setminus\Gamma)$ density in V of the whole system $\Big\{ \{\, v_k \,\}_{k\in Z,\; k\neq 0,\pm1,\pm2} \,,\; \{\, z_k \,\}_{k\in Z\setminus\{0\}} \Big\}$.
A preliminary remark is that $z_k \in V \;\forall\, k \in \mathbb{Z}\setminus\{0\}$, but $v_k \in V \;\forall\, k \in \mathbb{Z}\setminus\{0, \pm1, \pm2\}$.
For any fixed $v \in V$, by uniqueness of projections and Parseval inequality there exist coefficients C_h, E_h and a function $w \in H^2(B_\varrho \setminus \Gamma)$ such that

$$w \;=\; \sum_{h\in\mathbb{Z}\setminus\{0,\pm1,\pm2\}} C_h v_h \;+\; \sum_{h\in\mathbb{Z}\setminus\{0\}} E_h z_h$$

where both series are strongly converging in $H^2(B_\varrho\setminus\Gamma)$, hence in $L^2(B_\varrho)$ and the coefficients C_h,E_h are defined by (4.9),(4.10). We will show that $w = v$.

By (3.6), $A^\Delta_\varrho = A^1_\varrho + Z_\varrho$, the systems $\{v_h\}_{h\in\mathbb{Z}\setminus\{0\}}$, $\{z_h\}_{h\in\mathbb{Z}\setminus\{0\}}$ are separately orthogonal in $L^2(B_\varrho)$, they are dense with respect to $L^2(B_\varrho)$ norm respectively in the sets A^1_ϱ and Z_ϱ, moreover, neither of them contains $\ln r$ or $r^2 \ln r$

and the affine functions do not belong to $V \subset A^2_\varrho \cap H^2(B_\varrho \backslash \Gamma)$;
in addition $v_j \notin V$, $j = \pm 1, \pm 2$.
Then, by taking into account that $v_k \notin V$ if $|k| = 1, 2$, every $v \in V$ is represented by a uniquely defined expansion strongly converging in $L^2(B_\varrho \backslash \Gamma)$:

$$v \;=\; \sum_{h \in \mathbb{Z} \backslash \{0, \pm 1, \pm 2\}} c_h v_h \;+\; \sum_{h \in \mathbb{Z} \backslash \{0\}} e_h z_h\,. \tag{4.13}$$

In the expansion (4.13) $(c_h\, v_h \,+\, e_h\, z_h)$ is the unique $L^2(B_\varrho)$ projection of v either on 2 dimensional spaces $V_h := \text{span}\{v_h, z_h\}$ if $h \in \mathbb{Z}$, $|h| > 2$, or on the 1 dimensional spaces $V_h := \text{span}\{z_h\}$ if $|h| =, \pm 1, \pm 2$.
We emphasize that the orthogonality $V_h \perp V_l$ for $h \neq l$ holds true with respect to both scalar products $(\cdot, \cdot)_{L^2(B_\varrho)}$ and $(\cdot, \cdot)_{2,\varrho}$; moreover the coefficients c_h, e_h are not evaluated by scalar products of v with v_h and with z_h since v_h, z_h are not $L^2(B_\varrho \backslash \Gamma)$ mutually orthogonal: nevertheless, they can obtained by scalar products of u with v_h and of w with z_h, if $v = u + w$, $u \in A^1_\varrho$, $w \in \mathcal{Z}_\varrho$.

Eventually, we claim that the expansion (4.13) is strongly converging to v in the norm $H^2(B_\varrho \backslash \Gamma)$ too; hence $c_h = C_h$, $e_h = E_h$, $w = v$ thanks to (4.12).

Indeed: $(v_h, v_l)_{L^2(B_\varrho)} = (z_h, z_l)_{L^2(B_\varrho)} = 0$, $\forall h \neq l$; $(v_h, z_l)_{L^2(B_\varrho)} = 0$, $\forall h, l$. Thus v is obtained in (4.13) as an infinite sum of terms belonging to a sequence of 2 dimensional subspaces V_h (each one spanned by v_h and z_h for any fixed h in $\mathbb{Z}$ with $|h| > 2$, or spanned by z_h if $h = \pm 1, \pm 2$) and all these finite dimensional spaces V_h are pairwise orthogonal in $L^2((B_\varrho \backslash \Gamma))$.

Since $v \in V \subset H^2(B_\varrho \backslash \Gamma)$ and every finite truncated sum from (4.13) belongs to $H^2(B_\varrho \backslash \Gamma)$, by subtraction the sum of any residual series belongs to $H^2(B_\varrho \backslash \Gamma)$ too; precisely, the sum of the residual series (obtained by truncation at $N > 2$) belongs to the space orthogonal to every V_k in $H^2(B_\varrho \backslash \Gamma)$, $0 < |k| \leq N$.
Though the residual series is a priori converging in L^2 only, the sum of the residual series actually belongs to the closed space $\bigoplus_{|h|>N} V_h$.

We prove now that the whole expansion (4.13) converges in $H^2(B_\varrho \setminus \Gamma)$ too: this follows from the uniform boundedness in $H^2((B_\varrho \setminus \Gamma))$ of finite truncated sums of (4.13):

$$\exists\, C \;\; s.t. \;\forall\, N \qquad \left| \sum_{\substack{h=-N \\ c_h = 0 \;\; if \;\; |h| \leq 2}}^{N} (c_h\, v_h \,+\, e_h\, z_h) \right|^2_{2,\varrho} \leq C \;<\; +\infty \tag{4.14}$$

since this boundedness, together with $V_h \perp V_l$ in $H^2(B_\varrho \setminus \Gamma)$, implies

$$\exists\, C \;\; s.t. \;\forall\, N \qquad \sum_{\substack{h=-N \\ c_h = 0 \;\; if \;\; |h| \leq 2}}^{N} |c_h\, v_h \,+\, e_h\, z_h|^2_{2,\varrho} \;\leq C \;<\; +\infty\,,$$

hence exists $w \in H^2(B_\varrho \setminus \Gamma)$ s.t.

$$w \;=\; \sum_{\substack{h \in \mathbf{Z}, \\ c_h = 0 \;\; if \;\; |h| \leq 2}} (c_h\, v_h \,+\, e_h\, z_h) \qquad \text{with strong } H^2(B_\varrho \setminus \Gamma) \text{ convergence}$$

and this w must coincide with v for uniqueness of limit in L^2.
Assuming by contradiction that (4.14) is false, say the uniform boundedness in $H^2(B_\varrho \backslash \Gamma)$ of truncated sums of (4.13) does not hold true, we would obtain

$$\sum_{\substack{h\in\mathbf{Z},\\ c_h=0 \ \ if \ \ |h|\leq 2}} |\, c_h\, v_h \, + \, e_h\, z_h \,|^2_{2,\varrho} \; = \; +\infty \, ,$$

thus, applying Parseval inequality and the orthogonality relationships (4.12) to (4.13), leads to a contradiction with $v \in H^2(B_\varrho \backslash \Gamma)$:

$$|v|^2_{2,\varrho} \; \geq \sum_{\substack{h\in\mathbf{Z},\\ c_h=0 \ \ if \ \ |h|\leq 2}} \left(|\, c_h\, v_h|^2_{2,\varrho} + |e_h\, z_h\,|^2_{2,\varrho} \right) = \left| \sum_{\substack{h\in\mathbf{Z},\\ c_h=0 \ \ if \ \ |h|\leq 2}} c_h\, v_h \, + \, e_h\, z_h \right|^2_{2,\varrho} = \; +\infty \, .$$

Then we have proved that (4.13) is strongly converging to v in $H^2(B_\varrho \setminus \Gamma)$ and hence for any fixed $k \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$ and $N \in \mathbb{N}$, $N \geq k$, we have

$$C_k \; = \; \frac{1}{(|v_k|_{2,\varrho})^2} \, (v, v_k)_\varrho \; =$$

$$= \; \frac{1}{(|v_k|_{2,\varrho})^2} \left(\sum_{h\in\mathbb{Z}, \ h\leq N} (c_h v_h + e_h z_h) \, , \, v_k \right)_{2,\varrho} \; +$$

$$+ \; \frac{1}{(|v_k|_{2,\varrho})^2} \left(\sum_{h\in\mathbb{Z}, \ h> N} (c_h v_h + e_h z_h) \, , \, v_k \right)_{2,\varrho} \; = \; c_k \, ,$$

where last equality holds true since the first sum is a finite sum of $H^2(B_\varrho \backslash \Gamma)$ functions so we can exploit (4.12), while the second one (infinite sum, a priori converging only in L^2) is a function belonging to the space orthogonal to V_k in $H^2(B_\varrho \backslash \Gamma)$, hence with vanishing scalar product against v_k.
By the same procedure exploited above one can show that $E_k = e_k$, $\forall k \neq 0$.
Hence the system $\left\{ \{ \, v_k \, \}_{k\in Z, \ k\neq 0, \pm 1} \, , \, \{ \, z_k \, \}_{k\in\mathbb{Z}} \right\}$ is dense and (thanks to orthogonality relationships in (4.12)) has no redundancy.
The density and non redundancy of $\{f^1_k, f^2_k, f^3_k, f^4_k\}$ follows by considering real and imaginary parts of $\left\{ \{ \, v_k \, \}_{k\in Z, \ k\neq 0, \pm 1} \, , \, \{ \, z_k \, \}_{k\in\mathbb{Z}} \right\}$ (see Remark 2.2). $\square$

Without relabeling, we consider the operators Φ, Ψ defined by (3.3),(3.4) with domain restricted on V and related decomposition (3.2), which for this restriction is uniquely defined, since affine functions do not belong to V.
Next statement shows that both restrictions in V of Φ, Ψ map V on $A^1_\varrho \cap V$.

Theorem 4.4. -
Almansi Decomposition near a crack-tip of a biharmonic function in V.
Referring to Definition 2.1, (4.4) *and* (4.13) *we have*

$$V \;\; = \;\; \left(A^1_\varrho \cap V \right) \, \oplus \, \left(\mathcal{Z}_\varrho \cap V \right) \;\; \underset{\neq}{\subset} \;\; H^2(B_\varrho \backslash \Gamma) \cap A^\Delta_\varrho \qquad (4.15)$$

where $\oplus$ denotes the topological direct sum in $H^2(B_\varrho \setminus \Gamma)$ of two orthogonal subspaces (orthogonality refers to the scalar product $(\cdot,\cdot)_\varrho$, whereas the decomposition is not orthogonal in $L^2(B_\varrho)$). The restrictions on V of Φ, Ψ and the related decomposition, in polar coordinates, are given by

$$\Phi : V \to A^1_\varrho \cap V, \quad \Phi[u] = \frac{1}{4r}\int_0^r \Delta_{\mathbf{x}} u(t,\vartheta)\, dt\,, \tag{4.16}$$

$$\Psi : V \to A^1_\varrho \cap V, \quad \Psi[u] \;=\; u - r^2\,\Phi[u], \tag{4.17}$$

$$u(r,\vartheta) \;=\; \Psi[u] + r^2\,\Phi[u]\,, \tag{4.18}$$

and can be represented by these expansion series, both strongly converging in $H^2(B_\varrho \setminus \Gamma)$:

$$\begin{cases} \Psi[u] = \displaystyle\sum_{|k|>2} C_k\, v_k\,, \quad \forall u \in V\,, \quad \textit{where, for } |k|>2, \\[2ex] C_k := \dfrac{(u,v_k)_{2,\varrho}}{(|v_k|_{2,\varrho})^2} = \dfrac{(u - r^2\,\Phi[u], v_k)_{2,\varrho}}{(|v_k|_{2,\varrho})^2} = \dfrac{\big(\,\Psi[u]\,,\, v_k\big)_{L^2(B_\varrho\setminus\Gamma)}}{\big(\|v_k\|_{L^2(B_\varrho\setminus\Gamma)}\big)^2}\,. \end{cases} \tag{4.19}$$

$$\begin{cases} \Phi[u] = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} E_k\, v_k\,, \quad \forall u \in V\,, \quad \textit{where, for } |k|>0, \\[2ex] E_k := \dfrac{(u,z_k)_{2,\varrho}}{(|z_k|_{2,\varrho})^2} = \dfrac{(u - \Psi[u], z_k)_{2,\varrho}}{(|z_k|_{2,\varrho})^2} = \dfrac{\Big(r^2\,\Phi[u]\,,\, z_k\Big)_{L^2(B_\varrho\setminus\Gamma)}}{\|z_k\|^2_{L^2(B_\varrho\setminus\Gamma)}}\,. \end{cases} \tag{4.20}$$

Proof. By Theorem 3.1 we know that an Almansi decomposition (3.2) holds true also in $V \subset \big(A^2_\varrho \cap H^2(B_\varrho \setminus \Gamma)\big) \subset A^\Delta_\varrho$ (a priori with summands having both terms φ and ψ in A^1_ϱ only), moreover affine functions do not belong to V; hence uniqueness of decomposition and the equality in (4.15) hold true, thanks to orthogonality relationships in (4.12). Embedding in (4.15) is a straightforward consequence of Definitions 4.1, 4.2, and Lemma 4.3; this embedding is strict since smooth functions are missing in V.
By exploiting the density in V of the system (4.5) stated by Lemma 4.3, we find the first equalities (coefficient defining) in the expansions (4.19),(4.20), which are strongly converging in $H^2(B_\varrho \setminus \Gamma)$ to the restrictions in V of operators Φ and Ψ: the second equalities in (4.19),(4.20) follow from $(v_k, z_h)_{2,\varrho} = 0\,,\ \forall\, k,h \neq 0$; the third equalities in (4.19),(4.20) follows from the fact that

$$\Psi[u] = u - r^2\Phi[u] \in A^1_\varrho\ , \qquad r^2\Phi[u] = \big(u - \Psi[u]\big) \in Z_\varrho\ ,$$

hence they have a unique expansion respectively in terms of the systems $\{v_k\}_{k\in\mathbb{Z}\setminus\{0,\pm1,\pm2\}}$ and $\{z_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ with coefficients $c_k = \big(\Psi[u], v_k\big)_{L^2}/\|v_k\|^2_{L^2}$, $e_k = \big(r^2\Phi[u], z_k\big)_{L^2}/\|z_k\|^2_{L^2}$. Such expansions are not only strongly converging in L^2 (by Lemma 2.5) but also strongly in $H^2(B_\varrho \setminus \Gamma)$, by uniqueness of the expansion in V (by Lemma 4.3).
Eventually, the strong convergence in $H^2(B_\varrho \setminus \Gamma)$ entails that the restriction to V of operators Φ and Ψ have range in $H^2(B_\varrho \setminus \Gamma)$; precisely Φ maps V in

V, since, referring to the expansion (3.16) converging in $H^1(B_\varrho\backslash\Gamma)$ to σ and the operator representation (3.3) we get the expansion

$$\Phi[v] = \sum_{k\in\mathbb{Z}\backslash\{0\}} \frac{c_k}{4(|k|-1/2)} \sqrt{\frac{2|k-1|+1}{2\pi}} \left(\frac{r}{\varrho}\right)^{|k|-3/2} \exp\left(i(k-3/2 \operatorname{sign} k)\,\vartheta\right)$$

which is strongly converging in $H^2(B_\varrho\backslash\Gamma)$, so that the sum belongs to the closed subspace V. Thus, the Almansi decomposition (3.3),(3.4),(3.5) already proved in A^Δ_ϱ, here is achieved in V too, but with summands fulfilling natural higher regularity: $\Phi[u]$ and $\Psi[u]$ in $A^1_\varrho\cap V$. Eventually (4.21) follows by the proof of Lemma 4.3. □

Remark 4.5. The operators Φ and Ψ (a priori defined in domain A^Δ_ϱ with range in A^1_ϱ) when restricted to V have range in $A^1_\varrho\cap V$. We remind that the decomposition of Theorem 3.1 for a function in A^Δ_ϱ is uniquely defined through the operators Φ and Ψ by (3.3),(3.4), nevertheless it is not unique in the Almansi sense due to the twofold Almansi representation of linear functions. On the other hand, the decomposition of Theorem 4.4 is unique in both ways, since linear functions are missing in V.

Remark 4.6. Formulas (4.19) and (4.20) are coherent with (4.9), (4.10) and (4.11) due to the orthogonality relationships
$(r^2\Phi[u],v_k)_{2,\varrho}=0 \quad \forall k\in\mathbb{Z}\backslash\{0,\pm1,\pm2\}, \qquad (\Psi[u],z_k)_{2,\varrho}=0 \quad \forall k\in\mathbb{Z}\backslash\{0\}.$

Remark 4.7. As usual, the expansions (3.8) and (3.9) have a real counterpart, referring to the basis $\{f^1_k,f^2_k\}_{k=-2,-1,0,1,\dots}$ and $\{f^3_k,f^4_k\}_{k=0,1,\dots}$ defined in (2.1) by replacing v_k with f^1_k,f^2_k, and replacing z_k with $f^3_k\,f^4_k$.

Remark 4.8. The relationships (4.19)-(4.20) altogether, make clear that for every h the term $(C_hv_h+E_hz_h)=(c_hv_h+e_hz_h)$ of the expansion (4.7) in the space V is the unique $L^2(B_\varrho)$ projection of v on the 2-d space $V_h:=\operatorname{span}\{v_h,z_h\}$, if $h\in\mathbb{Z}$, $|h|>2$, or on the 1-d space $V_h:=\operatorname{span}\{v_h\}$, if $h=0,\pm1,\pm2$. The orthogonality $V_h\perp V_k$, $\forall h\neq k$ hold true with respect to both scalar product $(\cdot,\cdot)_{L^2(B_\varrho)}$ and $(\cdot,\cdot)_{2,\varrho}$. Notice that coefficients c_h and e_h cannot be obtained by straightforward L^2 scalar products with v_h, z_h, since v_h and z_h are not mutually orthogonal in L^2 (actually they are orthogonal with respect to $(\cdot,\cdot)_{2,\varrho}$) : therefore the preliminary corrections by subtraction in (3.8), (3.9) are mandatory. By inspection of the proof of Theorem 4.4

$$\forall\, u\in V: \quad c_k=C_k\ \forall|k|>2,\ c_k=0 \text{ if } 0<|k|\le 2, \quad e_k=E_k\ \forall k\neq 0. \tag{4.21}$$

Notice that if $0\not\equiv u\in H^2(B_\varrho\backslash\Gamma)\backslash V$ then the equalities $C_k=c_k,\ E_k=e_k$ must fail for some index; moreover $C_{\pm1},C_{\pm2}$ are always 0 in the expansions (4.19) and (4.20) of any nontrivial $u\in V$, whereas $c_0,c_{\pm1},c_{\pm2}$ and e_0 may be non trivial.

Eventually the whole Sobolev space $H^2(B_r\backslash\Gamma)$ is examined in the next statement.

Theorem 4.9. -
Almansi Decomposition of biharmonic $H^2(B_\varrho\setminus\Gamma)$ functions near a crack-tip. *Let $u \in H^2(B_\varrho \setminus \Gamma)$, $0 < \varrho < +\infty$. Then*

$$\Delta^2_{\mathbf{x}}\, u = 0 \quad \textit{in } B_\varrho \setminus \Gamma \tag{4.22}$$

if and only if

$$\begin{aligned} &\exists\, \varphi, \psi \in H^2(B_\varrho \setminus \Gamma) : \\ &u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x}),\ \ \Delta_{\mathbf{x}}\varphi(\mathbf{x}) = \Delta_{\mathbf{x}}\psi(\mathbf{x}) \equiv 0\,,\ \forall\, \mathbf{x} \in B_\varrho \setminus \Gamma. \end{aligned} \tag{4.23}$$

Decomposition (4.23) is unique up to possible linear terms in ψ, of the kind $A\varrho\cos\vartheta = Ax$ or $B\varrho\sin\vartheta = By$, which can be replaced in φ by respectively $A\varrho^{-1}\cos\vartheta$ and $B\varrho^{-1}\sin\vartheta$.
By denoting $A^j_\varrho := \{v \in L^2(B_\varrho)$ s.t. $\Delta^j_{\mathbf{x}} v = 0$ in $B_\varrho\setminus\Gamma\ \}$ the L^2 j-harmonic functions for $j = 1, 2$, the decomposition can be made explicit by introducing the operators Φ and Ψ that act on every $u \in A^2_\varrho \cap H^2(B_\varrho\setminus\Gamma)$ and are expressed in polar coordinates as follows:

$$\Phi : A^2_\varrho \cap H^2(B_\varrho\setminus\Gamma) \to A^1_\varrho \cap H^2(B_\varrho\setminus\Gamma), \quad \Phi[u] = \frac{1}{4r}\int_0^r \Delta_{\mathbf{x}} u(t,\vartheta)\, dt\,, \tag{4.24}$$

$$\Psi : A^2_\varrho \cap H^2(B_\varrho\setminus\Gamma) \to A^1_\varrho \cap H^2(B_\varrho\setminus\Gamma), \quad \Psi[u] \;=\; u - r^2\,\Phi[u], \tag{4.25}$$

$$u(r,\vartheta) \;=\; \Psi[u] + r^2\,\Phi[u]\,. \tag{4.26}$$

Proof. The claim that decomposition (4.23) entails u is biharmonic follows by a straightforward computation.
Opposite inference is a straightforward consequence of Theorem 4.4 together with classical decomposition ([1]) of biharmonic functions in the disk B_ϱ.
Moreover, both operators Φ and Ψ map $H^2(B_\varrho \setminus \Gamma)$ into $A^1_\varrho \cap H^2(B_\varrho \setminus \Gamma)$: indeed this is already proved for their restriction in V (Theorem 4.4) and is well known in $H^2(B_\varrho)$, namely in the space orthogonal to V. □

Remark 4.10. As a consequence of previous theorem, one recovers the power series expansion of a generic function u which is biharmonic in $B_\varrho\setminus\Gamma$ and belongs to $H^2(B_\varrho\setminus\Gamma)$.
Every function $u \in H^2(B_\varrho\setminus\Gamma) \cap A^2_\varrho$ can be expanded in the form

$$\begin{aligned} u \;=\; &\sum_{h=0}^{+\infty} r^{h+\frac{3}{2}} \left(c^1_h \cos\left(\left(h+\frac{3}{2}\right)\vartheta\right) + c^2_h \sin\left(\left(h+\frac{3}{2}\right)\vartheta\right) + \right. \\ &\qquad\qquad \left. + c^3_h \cos\left(\left(h-\frac{1}{2}\right)\vartheta\right) + c^4_h \sin\left(\left(h-\frac{1}{2}\right)\vartheta\right)\right) + \\ &+ \sum_{n=0}^{+\infty} \Big(\big(a_n \cos(n\vartheta) + b_n \sin(n\vartheta)\big)\varrho^n + \big(\alpha_n \cos(n\vartheta) + \beta_n \sin(n\vartheta)\big)\varrho^{n+2}\Big), \end{aligned} \tag{4.27}$$

where the coefficients are uniquely defined first by detecting the component of v in V (space of biharmonic functions that are orthogonal to the smooth functions in B_ϱ with respect to the scalar product in $H^2(B_\varrho \setminus \Gamma)$ associated to the semi-norm $|\cdot|_{2,\varrho}$ and orthogonal to affine functions with respect to the $L^2(B_\varrho)$ scalar product, see Definition 4.2); then by expanding such v

with Lemma 4.3 and finding explicitly the coefficients $c_h^1, c_h^2, c_h^3, c_h^4$ via (4.11), possibly by expanding in the standard way the remainder $u-v$, which actually turns out to be smooth in the whole B_ϱ and can be split via the classical Almansi Theorem.

Remark 4.11. The first part of the claim (equivalence) in Theorem 4.9, without explicit mention of operators Φ and Ψ, was stated in [21].
We emphasize that the nontrivial candidate local minimizers of Blake & Zisserman functional (see [16],[17],[21]) fulfil the assumption of Theorem 4.9: that is, for any $\varrho > 0$, it belongs to $H^2(B_\varrho \setminus \Gamma)$ and is biharmonic in $B_\varrho \setminus \Gamma$.

Remark 4.12. In the evaluation of the coefficients of the expansion for a function in $A^2_\varrho \cap H^2(B_\varrho \setminus \Gamma)$ one has to take into account the warnings in Remarks 4.8 and 4.10. For instance, consider
$w = r^3(\cos(\vartheta) - \cos(3\vartheta)) = (x^2+y^2)x + (3x^2y - x^3) = r^2\Phi[v] + \Psi[v] \in A^\Delta_\varrho$,
which is biharmonic but it is not harmonic. Figure 4 shows an approximation in L^2 of the function w by the series (3.8),(3.9) in Theorem 3.1: the first 100 coefficients are computed with software Wolfram Mathematica© 13.2 in the real form according to Remark 4.7.
Though the 100 terms approximation is very good, the spike at the origin and poor accuracy along the boundary of $B_\varrho \setminus \Gamma$ are unavoidable, due to the behavior of the basis functions; actually the truncated expansion has a negligible error in L^2 norm which is obviously vanishing in the limit.
Notice that w belongs to $H^2(B_\varrho) \subset A^\Delta_\varrho$ but w is orthogonal to V in $H^2(B_\varrho \setminus \Gamma)$. Actually, according to Remark 4.10 the appropriate expansion is obtained by preliminary evaluation of the component of w in V, which actually is 0 since w is smooth, then by evaluation of the expansion with integer powers only ($\cos(n\vartheta)\varrho^n$, $\sin(n\vartheta)\varrho^n$, $\cos(n\vartheta)\varrho^{2n}$, $\sin(n\vartheta)\varrho^{2n}$) : thus providing a finite expansion which is coincident with w.

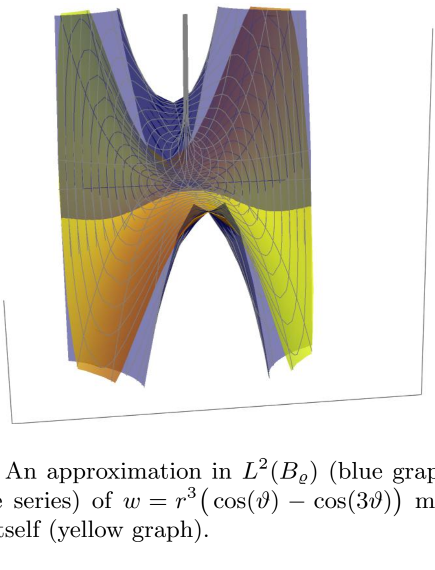

FIGURE 4. An approximation in $L^2(B_\varrho)$ (blue graph: first 100 terms in the series) of $w = r^3\big(\cos(\vartheta) - \cos(3\vartheta)\big)$ matching the function w itself (yellow graph).

5. Decomposition and expansion of a polyharmonic function near the tip of a flat crack

In this section we consider the polyharmonic functions near a flat cracktip.

Definition 5.1. For $0<\varrho<+\infty$, we introduce some spaces of polyharmonic functions indexed by $j=0,1,2,\ldots$:

$$A^0_\varrho := L^2(B_\varrho\backslash\Gamma),\ A^j_\varrho := \{v\in L^2(B_\varrho) : \Delta^j_{\mathbf{x}} v = 0 \text{ in } B_\varrho\backslash\Gamma\},\ j=1,2,\ldots \tag{5.1}$$

$$\mathcal{A}^0_\varrho := L^2(B_\varrho\backslash\Gamma),\ \mathcal{A}^j_\varrho := \{v\in A^j_\varrho : \Delta^i_{\mathbf{x}} v\in L^2(B_\varrho\backslash\Gamma), 0<i\le j\},\ j=1,2\ldots \tag{5.2}$$

$$Z^j_\varrho := r^{2j}A^1_\varrho = \{w:\ w=r^{2j}\varphi,\ \varphi\in L^2(B_\varrho\backslash\Gamma),\ \Delta_{\mathbf{x}}\varphi=0 \text{ in } B_\varrho\backslash\Gamma\} \tag{5.3}$$

$$\mathcal{Z}^j_\varrho := \{w\in L^2(B_\varrho\backslash\Gamma) \text{ s.t. } w=r^{2j}\varphi,\ \Delta_{\mathbf{x}}\varphi=0 \text{ in } B_\varrho\backslash\Gamma\}. \tag{5.4}$$

Definition 5.2. We introduce the complex fuctions z^j_k, $j=0,1,2,\ldots$ (and their real counterparts) (here $r>0$, $|\vartheta|<\pi$) by setting

$$\begin{aligned} z^j_k(r,\vartheta) &:= r^{2j}v_k = r^{|k|+2j-3/2}\varphi_k(\vartheta) = \\ &= r^{|k|+2j-3/2}\exp\big(i(k-3/2\ \mathrm{sign}\,k)\,\vartheta\big),\ k\in\mathbb{Z}\setminus\{0\}, \\ &\text{and for } k=2j-2, 2j-1, 2j, 2j+1,\ldots \\ f^{2j+1}_k(r,\vartheta) &:= r^{(k+3/2)}\cos((k+3/2-2j)\vartheta), \\ f^{2j+2}_k(r,\vartheta) &:= r^{(k+3/2)}\sin((k+3/2-2j)\vartheta). \end{aligned} \tag{5.5}$$

Remark 5.3. The sets A^j_ϱ, $\mathcal{A}^j_\varrho$ and $\mathcal{Z}^j_\varrho$ are closed subspaces of $L^2(B_\varrho\backslash\Gamma)$ and $\mathcal{Z}^j_\varrho$ is the completion of Z^j_ϱ in $L^2(B_\varrho\backslash\Gamma)$. Notice that
$\mathcal{Z}^0_\varrho = Z^0_\varrho = A^1_\varrho = \mathcal{A}^1_\varrho$, $\mathcal{Z}^1_\varrho = \mathcal{Z}_\varrho \neq Z_\varrho = Z^1_\varrho$ and $\mathcal{Z}^0_\varrho \cap \mathcal{Z}^1_\varrho = \{\text{linear functions}\}$, $A^j_\varrho \subset A^{j+1}_\varrho$, $\mathcal{A}^j_\varrho \subset \mathcal{A}^{j+1}_\varrho$ for $j=0,1,2,\ldots$, and, referring to (2.1) and Definition 2.1, we have $z^0_k = v_k$, $z^1_k = z_k$;

$$\mathcal{Z}^j_\varrho \subset \mathcal{A}^m_\varrho \qquad 0\le j\le m-1\ ; \tag{5.6}$$

$$\mathcal{A}^j_\varrho \underset{\neq}{\subset} \mathcal{A}^{j+1}_\varrho, \quad j\ge 2\ , \tag{5.7}$$

since $(x^2+y^2)^{j-1} \in A^j_\varrho\backslash\mathcal{A}^j_\varrho$ for $j\ge 2$; moreover

$$\mathcal{A}^j_\varrho \underset{\neq}{\subset} \mathcal{A}^{j+1}_\varrho, \quad j\ge 2\ , \tag{5.8}$$

since $(x^2+y^2)^{j-1} \in A^j_\varrho\backslash\mathcal{A}^j_\varrho$; eventually (see (3.19))

$$r^{1/2}\cos\left(\frac{3}{2}\vartheta\right) \in A^2_\varrho\backslash A^\Delta_\varrho = A^2_\varrho\backslash\mathcal{A}^2_\varrho \subset A^j_\varrho\backslash\mathcal{A}^j_\varrho \qquad j\ge 2$$

entails

$$\mathcal{A}^j_\varrho \underset{\neq}{\subset} A^j_\varrho \qquad j\ge 2. \tag{5.9}$$

Moreover notation f^l_k (5.5) are coherent with (2.1), for $l=1,2,3,4$.

Lemma 5.4. *Referring to Definitions 5.1,5.2, for any $\varrho > 0$ and $j = 0, 1, 2, \dots,$ the system*

$$\left\{ \; z_k^j(r,\vartheta) \; \right\}_{k \in \mathbb{Z}\setminus\{0\}} \tag{5.10}$$

is orthogonal in $L^2(B_\varrho)$ and dense in Z_ϱ^j and $\mathcal{Z}_\varrho^j$ with respect to $L^2(B_\varrho)$ norm; the system

$$\left\{ \; f_k^{2j+1} \, , \; f_k^{2j+2} \; \right\}_{k=0,1,2,\dots} \tag{5.11}$$

is orthogonal in $L^2(B_\varrho)$ and dense in Z_ϱ^j and $\mathcal{Z}_\varrho^j$ with respect to $L^2(B_\varrho)$ norm.

Proof. Due to Remark 5.3, the cases $j = 0, 1$ are already proved in Lemma 2.5. The general case $j > 1$ is achieved as like as in the proof of density of system (2.3) in $\mathcal{Z}_\varrho^j$ (Lemma 2.5): by definitions (5.3) and (5.4), $z_k^j = r^{2j} z_k^0$ belongs to Z_ϱ^j for any k, and every $f \in Z_\varrho^j$ fulfils $f = r^{2j} g$ for suitable $g \in Z_\varrho^0$. By the density in A_ϱ^1 of system $\{z_k^0\}_{k\in\mathbb{Z}\setminus\{0\}}$ we deduce the density of $\{z_k^j\}_{k\in\mathbb{Z}\setminus\{0\}}$ in Z_ϱ^j and hence in $\mathcal{Z}_\varrho^j$. As usual, the orthogonality property follows by integration in ϑ. $\square$

Theorem 5.5. Decomposition near a crack-tip of a polyharmonic function. *Assume $m \geq 2$ is an integer and $0 < \varrho < +\infty$; then*

$$u \in \mathcal{A}_\varrho^m \tag{5.12}$$

if and only if

$$\begin{cases} \exists\, \psi_0, \psi_1, \dots, \psi_{m-1} \in \mathcal{A}_\varrho^1 \quad \text{such that} \\ u(\mathbf{x}) = \psi_0(\mathbf{x}) + \|\mathbf{x}\|^2 \psi_1(\mathbf{x}) + \ldots + \|\mathbf{x}\|^{2(m-1)} \psi_{m-1}(\mathbf{x}) \quad \text{on } B_\varrho \setminus \Gamma. \end{cases} \tag{5.13}$$

Precisely, given $u \in \mathcal{A}_\varrho^m$, $m \geq 2$, the functions ψ_j in (5.13) are provided by:

$$\psi_j := \eta_j - r^2 \eta_{j+1} \qquad j = 0, \dots, m-2\,, \qquad \psi_{m-1} = \eta_{m-1}, \tag{5.14}$$

where

$$\eta_0 := u, \qquad \eta_{j+1} := \left(\Delta_{\mathbf{x}}^{m-j-1}\right)^{-1} [\Phi_j[\eta_j]] \qquad j = 0, \dots, m-2\,, \tag{5.15}$$

$$\Phi_j[\eta] := \frac{1}{4\,(m-j)\, r^{m-j}} \int_0^r t^{j-1} \Delta_{\mathbf{x}}^{m-j} \eta(t,\vartheta)\, dt \qquad j = 1, \dots, m-1\,. \tag{5.16}$$

For every $0 < \varrho < +\infty$ we have

$$\mathcal{A}_\varrho^m = \left(\mathcal{Z}_\varrho^0 + \mathcal{Z}_\varrho^1 \cdots + \mathcal{Z}_\varrho^{m-1}\right) \cap \mathcal{A}_\varrho^m \; = Z_\varrho^0 + Z_\varrho^1 \cdots + Z_\varrho^{m-1} \qquad \forall\, m \geq 1 \tag{5.17}$$

where the algebraic sum is neither orthogonal nor direct in $L^2(B_\varrho \setminus \Gamma)$ and every space appearing in the identity is a Hilbert space, except the Z_ϱ^j which are only pre-Hilbert when $j > 0$.
Moreover $\psi_j \in H_0^{2m-2j-2}(B_\varrho \setminus \Gamma)$ for $j = 1, \dots, m-1$, whereas in general ψ_0 cannot have more regularity than u.

Proof. By (3.10)-(3.13), we get

$$\begin{aligned} \Delta_{\mathbf{x}}^3 \left(\|\mathbf{x}\|^2 \varphi\right) \;&=\; \|\mathbf{x}\|^2\, \Delta_{\mathbf{x}}^3 \varphi \;+ 4\Delta_{\mathbf{x}}^2 \varphi \;+ 4\mathbf{x} \cdot \nabla_{\mathbf{x}} \left(\Delta_{\mathbf{x}}^2 \varphi\right) \\ &\qquad\qquad\qquad + 16 \Delta_{\mathbf{x}}^2 \varphi \\ &\qquad\qquad\qquad + 16 \Delta_{\mathbf{x}}^2 \varphi \;+ 8\, \mathbf{x} \cdot \nabla_{\mathbf{x}} \left(\Delta_{\mathbf{x}}^2 \varphi\right) \\ &=\; \|\mathbf{x}\|^2\, \Delta_{\mathbf{x}}^3 \varphi \;+ 36 \Delta_{\mathbf{x}}^2 \varphi \;+ 12\, \mathbf{x} \cdot \nabla_{\mathbf{x}} \left(\Delta_{\mathbf{x}}^2 \varphi\right)\,. \end{aligned}$$

By iterating we find, for suitable integer constants C_j, E_j, $j=1,2,\ldots$,

$$\begin{aligned}\Delta_{\mathbf{x}}^{j}\left(\|\mathbf{x}\|^2\varphi\right) &= \|\mathbf{x}\|^2\Delta_{\mathbf{x}}^{j}\varphi + C_j\Delta_{\mathbf{x}}^{j-1}\varphi + E_j\mathbf{x}\cdot\nabla_{\mathbf{x}}\Delta_{\mathbf{x}}^{j-1}\varphi \\ \Delta_{\mathbf{x}}^{j+1}\left(\|\mathbf{x}\|^2\varphi\right) &= \|\mathbf{x}\|^2\,\Delta_{\mathbf{x}}^{j+1}\varphi + 4\Delta_{\mathbf{x}}^{j}\varphi + 4\mathbf{x}\cdot\nabla_{\mathbf{x}}\left(\Delta_{\mathbf{x}}^{j}\varphi\right) \\ &\quad + C_j\Delta_{\mathbf{x}}^{j}\varphi \\ &\quad + 2E_j\Delta_{\mathbf{x}}^{j}\varphi + E_j\,\mathbf{x}\cdot\nabla_{\mathbf{x}}\left(\Delta_{\mathbf{x}}^{j}\varphi\right) \\ &= \|\mathbf{x}\|^2\Delta_{\mathbf{x}}^{j+1}\varphi + C_{j+1}\Delta_{\mathbf{x}}^{j}\varphi + E_{j+1}\mathbf{x}\cdot\nabla_{\mathbf{x}}\Delta_{\mathbf{x}}^{j}\varphi\end{aligned}$$

By induction, we get

$$C_{j+1}=C_j+2E_j+4\,,\quad E_{j+1}=E_j+4,\quad j=1,2,\ldots\ \text{ with }\ C_1=E_1=4. \tag{5.18}$$

The recursive relationships (5.18) are solved by $E_j=4\,j$, $C_j=4\,j^2$. By inserting these values we get the next identities valid for every positive integer j :

$$\Delta_{\mathbf{x}}^{j}\left(\|\mathbf{x}\|^2\varphi\right) = \|\mathbf{x}\|^2\Delta_{\mathbf{x}}^{j}\varphi + 4\,j^2\,\Delta_{\mathbf{x}}^{j-1}\varphi + 4\,j\,\mathbf{x}\cdot\nabla_{\mathbf{x}}\Delta_{\mathbf{x}}^{j-1}\varphi \tag{5.19}$$

$$= \varrho^2\Delta_{\mathbf{x}}^{j}\varphi + 4\,j^2\,\Delta_{\mathbf{x}}^{j-1}\varphi + 4\,j\,r\,\frac{\partial}{\partial r}\Delta_{\mathbf{x}}^{j-1}\varphi \tag{5.20}$$

By (5.19) with $j=2$, we get:

$$\Delta\varphi=0 \quad\Rightarrow\quad \Delta_{\mathbf{x}}^{2}\left(\|\mathbf{x}\|^2\varphi\right)=0\,. \tag{5.21}$$

By (5.19) with $j=3$ and (5.21), we get:

$$\begin{aligned}&\Delta\varphi=0 \quad\Rightarrow \\ &\Delta_{\mathbf{x}}^{3}\left(\|\mathbf{x}\|^4\varphi\right)=\Delta_{\mathbf{x}}^{3}\Big(\|\mathbf{x}\|^2(\|\mathbf{x}\|^2\varphi)\Big)=\|\mathbf{x}\|^2\Delta_{\mathbf{x}}^{3}\Big(\|\mathbf{x}\|^2\varphi)\Big)=0\,.\end{aligned} \tag{5.22}$$

By iteration of (5.19) and (5.21), we get:

$$\begin{aligned}&\Delta\varphi=0 \quad\Rightarrow \\ &\Delta_{\mathbf{x}}^{j}\left(\|\mathbf{x}\|^{2(j-1)}\varphi\right)=\Delta_{\mathbf{x}}^{j}\Big(\|\mathbf{x}\|^2(\|\mathbf{x}\|^{2(j-2)}\varphi)\Big)=\|\mathbf{x}\|^2\Delta_{\mathbf{x}}^{j}\Big(\|\mathbf{x}\|^2\varphi)\Big)=0\,.\end{aligned} \tag{5.23}$$

In particular $\Delta_{\mathbf{x}}^{m-1}\Big(\|\mathbf{x}\|^{2(m-1)}\varphi)\Big)=\Delta_{\mathbf{x}}^{m-1}\Big(\|\mathbf{x}\|^2(\|\mathbf{x}\|^{2(m-2)}\varphi)\Big)=0$.
Summarizing (5.13) entails that $\Delta_{\mathbf{x}}^{m}u=0$, hence the "if part" is proved.

We are left to show the "only if" part, that is: $\Delta_{\mathbf{x}}^{m}u=0$ with $u\in\mathcal{A}_{\varrho}^{m}$ entails that u has a decomposition of the kind (5.13).

We prove decomposition (5.13) for every $u\in\mathcal{A}_{\varrho}^{m}$ by induction. We already know the claim of the theorem for $m=2$ (by Theorem 3.1 about biharmonic functions in A_{ϱ}^{Δ}, which is denoted here by $\mathcal{A}_{\varrho}^{2}$); thus, if we prove that the validity of function decomposition stated by the theorem for all j with $2\le j\le m-1$, entails the function decomposition validity at m, then the proof is achieved.
For any integer $j>2$ we formulate this indexed claim:

$$\text{claim with index } j\text{:}\quad u\in\mathcal{A}_{\varrho}^{j} \;\Rightarrow\; \exists\,\xi\in\mathcal{A}_{\varrho}^{j-1},\ \eta\in\mathcal{A}_{\varrho}^{j-1}:\ u=\xi+\|\mathbf{x}\|^2\eta\,. \tag{5.24}$$

From now on we assume that claim (5.24) holds true at every integer index s with $2\le s\le j-1$ and that the function u belongs to $\mathcal{A}_{\varrho}^{j}$ for some $j>2$: we show that these assumptions entail that claim (5.24) is true at value j of the index too.

The implication above together with the assumption $u \in \mathcal{A}^m_\varrho$ and the validity of the decomposition stated by the theorem at $m = 2$ provides the aimed decomposition for u.
We are left only to prove the implication about the claim (5.24): this will be done as soon as, given $u \in \mathcal{A}^j_\varrho$ we are able to find $\eta \in \mathcal{A}^{j-1}_\varrho$ fulfilling

$$\Delta^{j-1}_{\mathbf{x}} \eta = 0 \quad \text{and} \quad \Delta^{j-1}_{\mathbf{x}} \left(\|\mathbf{x}\|^2 \, \eta \right) = \Delta^{j-1}_{\mathbf{x}} u \quad \text{on } B_\varrho \setminus \Gamma \,, \tag{5.25}$$

by setting $\xi = u - \|\mathbf{x}\|^2 \eta$ which belongs to $\mathcal{A}^{j-1}_\varrho$, thanks to (5.25).
By (5.20), the problem (5.25) is equivalent to:

$$\begin{cases} \text{given } u \in \mathcal{A}^j_\varrho \text{ find } \eta \in \mathcal{A}^{j-1} \text{ solving} \\ \Delta_{\mathbf{x}} \left(\Delta^{j-2}_{\mathbf{x}} \eta \right) \;=\; 0 \quad \text{on } B_\varrho \setminus \Gamma \;\; \text{and} \\ (j-1) \left(\Delta^{j-2}_{\mathbf{x}} \eta \right) \,+\, r \, \dfrac{\partial}{\partial r} \left(\Delta^{j-2}_{\mathbf{x}} \eta \right) \;=\; \dfrac{\Delta^{j-1}_{\mathbf{x}} u}{4(j-1)} \quad \text{on } B_\varrho \setminus \Gamma \,, \end{cases} \tag{5.26}$$

which, in turn, is equivalent to

$$\begin{cases} \text{given } u \in \mathcal{A}^j_\varrho \text{ find } \eta \in \mathcal{A}^{j-1}_\varrho \text{ s.t. } \varphi = \Delta^{j-2}_{\mathbf{x}} \eta, \text{ solves} \\ \Delta_{\mathbf{x}} \varphi = 0 \;\; \text{on } B_\varrho \setminus \Gamma \;\; \text{and} \\ (j-1) \, \varphi \,+\, r \, \dfrac{\partial}{\partial r} \varphi \;=\; \dfrac{\Delta^{j-1}_{\mathbf{x}} u}{4(j-1)} \quad \text{on } B_\varrho \setminus \Gamma \,, \end{cases} \tag{5.27}$$

The solution of (5.27) can be achieved by a procedure similar to the one in proof of Theorem 3.1 (proof of (3.2), "only if" part), with these differences only: we still have $\varphi \in L^2(B_\varrho \setminus \Gamma)$ and $\Delta \varphi = 0$ on $B_\varrho \setminus \Gamma$, hence φ belongs to $\mathcal{A}^1_\varrho$, but here the first order ODE reads

$$r \, \frac{\partial \varphi}{\partial r} \,+\, (j-1) \varphi \;=\; \sigma \,, \quad \text{with } \sigma = \frac{1}{4(j-1)} \, \Delta^{j-1}_{\mathbf{x}} \, u \,.$$

The fact that u belongs to $\mathcal{A}^j_\varrho$ entails that σ belongs to $\mathcal{A}^1_\varrho$, hence σ has an expansion with respect to the basis $\{z^0_k\}_{k \in \mathbb{Z} \setminus \{0\}} = \{v_k\}_{k \in \mathbb{Z} \setminus \{0\}}$, with uniquely defined coefficients and strongly converging in $L^2(B_\varrho \setminus \Gamma)$:

$$\sigma(r, \vartheta) \;=\; \sum_{n=1}^{+\infty} \left(c_n \, e^{(n-3/2) i \vartheta} \,+\, c_{-n} \, e^{(-n+3/2) i \vartheta} \right) r^{n-3/2}.$$

Thus we can solve (5.27) in the unknown φ, obtaining

$$\Delta^{j-2}_{\mathbf{x}} \, \eta(r, \vartheta) \;=\; \varphi(r, \vartheta) \;:=\; \frac{1}{4\,(j-1)\, r^{j-1}} \int_0^r t^{j-2} \, \Delta^{j-1}_{\mathbf{x}} u(t, \vartheta) \, dt \,. \tag{5.28}$$

Therefore

$$\eta(r, \vartheta) \;=\; \left(\Delta^{j-2}_{\mathbf{x}} \right)^{-1} \left(\frac{1}{4\,(j-1)\, r^{j-1}} \int_0^r t^{j-2} \, \Delta^{j-1}_{\mathbf{x}} u(t, \vartheta) \, dt \right) \tag{5.29}$$

where $\left(\Delta^{j-2}_{\mathbf{x}} \right)^{-1}$ is a resolvent (whose definition is postponed) for the operator $\Delta^{j-2}_{\mathbf{x}}$ over $B_\varrho \setminus \Gamma$, with values in $H^{m-2}_0(B_\varrho \setminus \Gamma)$.

From now on we assume (5.12) and prove the explicit representation of summands $\|\mathbf{x}\|^{2j} \psi_j$ in (5.13) via (5.14)-(5.16).

We start from $\eta_0 = u$ and look for a preliminary decomposition: $u = \psi_0 + r^2\eta_1$, with η_1 solution of

$$\begin{cases} \text{given } \eta_0 := u \in \mathcal{A}^m_\varrho \text{ find } \eta_1 \in H_0^{2m-4}(B_\varrho \backslash \Gamma) \text{ s.t. } \varphi_1 := \Delta_{\mathbf{x}}^{m-2}\eta_1 \text{ fulfils} \\ \varphi_1 \in \mathcal{A}^1_\varrho \text{ and } \quad (m-1)\varphi_1 + r\dfrac{\partial \varphi_1}{\partial r} = \dfrac{\Delta_{\mathbf{x}}^{m-1}\eta_0}{4(m-1)} \qquad \text{on } B_\varrho \backslash \Gamma\,. \end{cases} \tag{5.30}$$

Arguing as like as in the proof of Theorem 3.1, the solution to (5.30) is:

$$\eta_1 \;=\; (\Delta_{\mathbf{x}}^{m-2})^{-1}\, \Phi_1[\eta_0]\,,$$

and we recover $\psi_0 = \eta_0 - r^2\eta_1 = u - r^2\eta_1$.
We proceed by looking for a decomposition of η_2: $\eta_1 = \psi_1 + r^2\eta_2$, with η_2 solution of

$$\begin{cases} \text{given } \eta_1 \in H_0^{2m-4}(B_\varrho \backslash \Gamma) \text{ find } \eta_2 \in H_0^{2m-6}(B_\varrho \backslash \Gamma) \text{ s.t } \varphi_2 := \Delta_{\mathbf{x}}^{m-3}\eta_1 \\ \text{fulfils } \varphi_2 \in \mathcal{A}^1_\varrho \text{ and } \quad (m-2)\varphi_2 + r\dfrac{\partial \varphi_2}{\partial r} = \dfrac{\Delta_{\mathbf{x}}^{m-2}\eta_1}{4(m-2)} \qquad \text{on } B_\varrho \backslash \Gamma\,. \end{cases} \tag{5.31}$$

The solution to (5.31) is:

$$\eta_2 \;=\; (\Delta_{\mathbf{x}}^{m-3})^{-1}\, \Phi_2[\eta_1]$$

and we get $\psi_1 = \eta_1 - r^2\eta_2$.
By iteration of the same procedure we get ψ_j for $j = 0, \ldots, m-2$ and eventually setting $\psi_{m-1} = \eta_{m-1}$, we achieve (5.14)-(5.16).

Eventually we show that $\psi_j \in H_0^{2m-2j-2}(B_\varrho \backslash \Gamma)$, $j = 1, \ldots, m-2$.
Since $v \in \mathcal{A}^j_\varrho$ entails $\Delta_{\mathbf{x}}^{j-1} v \in L^2(B_\varrho \backslash \Gamma)$, for every $j > 2$, as a resolvent $(\Delta_{\mathbf{x}}^{j-2})^{-1}$ for the operator $\Delta_{\mathbf{x}}^{j-2}$ on $B_\varrho \setminus \Gamma$ we choose the unique variational solution in $H_0^{j-2}(B_\varrho \backslash \Gamma)$ of $\Delta_{\mathbf{x}}^{j-2}\eta = 0$, obtained via minimization on $H_0^{j-2}(B_\varrho \backslash \Gamma)$ of $\int_{B_\varrho \backslash \Gamma} |D_{\mathbf{x}}^{j-2} w|^2\, d\mathbf{x}$ (this functional is equivalent to the squared $H_0^{j-2}(B_\varrho \setminus \Gamma)$ Sobolev norm, by Poincaré inequality).
Thus, (5.14) entails, for $j = 1, ..., m-1$,
$\psi_j = \eta_j - r^2\eta_{j+1} \in H_0^{2m-2j}(B_\varrho \backslash \Gamma) + H_0^{2m-2j-2}(B_\varrho \backslash \Gamma) \subset H_0^{2m-2j-2}(B_\varrho \backslash \Gamma)$.
The space decomposition (5.17) is a straightforward consequence of function decomposition (5.13). $\square$

Remark 5.6. Obviously the decomposition (5.13) is not unique due to the twofold Almansi representation in L^2 of linear functions times $\|\mathbf{x}\|^{2j}$, for $j = 0, 1, ..., m-2$, whenever the open set has a crack reaching the origin. Nevertheless it has an essentially unique representation, as soon as we initialize by setting first $\eta_0 = u$ and iteratively evaluate ψ_j via (5.14)-(5.16).

Corollary 5.7. Asymptotic expansion of a polyharmonic function at the tip of a flat crack
Referring to Definitions 5.1,5.2, for any $\varrho > 0$ and $m = 1, 2, \ldots,$ the system

$$\{\, z_k^0\,, z_k^1\,, \ldots\,, z_k^{m-1} \,\}_{k \neq 0} \tag{5.32}$$

is dense in $\mathcal{A}^m_\varrho$ with respect to $L^2(B_\varrho)$, but non orthogonal.
The system

$$\{\, \{f_n^1\}_{n=0,1,\ldots}\,, \{f_n^2\}_{n=0,1,\ldots}\,, \ldots\,, \{f_n^{2m-1}\}_{n=0,1,\ldots,\ldots}\,, \{f_n^{2m}\}\}_{n=0,1,\ldots} \,\} \tag{5.33}$$

is dense in A^m_ϱ with respect to $L^2(B_\varrho \backslash \Gamma)$, but non orthogonal.
Therefore for every $u \in A^m_\varrho$, relying on the non orthogonal decomposition (5.13), the coefficients of a strongly $L^2(B_\varrho \backslash \Gamma)$ converging expansion can be evaluated by scalar products in $L^2(B_\varrho \backslash \Gamma)$ of $\|\mathbf{x}\|^{2j}\psi_j$ with z^j_ϱ, $j = 0, 1, \ldots$.

Proof. First and third claim are straightforward consequences of decomposition (5.17) and the fact that $\{z^j_k\}_{|k|\neq 0}$ is an orthogonal dense system in $\mathcal{Z}^j_\varrho$ (by Lemma 5.4). Second claim follows by Remark 2.2 □

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