BEYOND BV: NEW PAIRINGS AND GAUSS-GREEN FORMULAS FOR MEASURE FIELDS WITH DIVERGENCE MEASURE

GIOVANNI E. COMI, VIRGINIA DE CICCO, AND GIOVANNI SCILLA

Abstract. A new notion of pairing between measure vector fields with divergence measure and scalar functions, which are not required to be weakly differentiable, is introduced. In particular, in the case of essentially bounded divergence-measure fields, the functions may not be of bounded variation. This naturally leads to the definition of BV-like function classes on which these pairings are well defined. Despite the lack of fine properties for such functions, our pairings surprisingly preserve many features of the recently introduced λ-pairings [19], as coarea formula, lower semicontinuity, Leibniz rules, and Gauss-Green formulas. Moreover, in a natural way new anisotropic "degenerate" perimeters are defined, possibly allowing for sets with fractal boundary.

Contents

2010 *Mathematics Subject Classification.* 26B30, 49Q15.

Key words and phrases. Divergence-measure fields, functions of bounded variation, coarea formula, Gauss–Green formula, normal traces, perimeter.

Acknowledgments. The authors are members of the Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), and are partially supported by the INdAM–GNAMPA 2022 Project *Alcuni problemi associati a funzionali integrali: riscoperta strumenti classici e nuovi sviluppi*, codice CUP_E55F22000270001 and the INdAM–GNAMPA 2023 Project *Problemi variazionali degeneri*, codice CUP_E53C22001930001. Part of this work was undertaken while the first author was visiting the Università Sapienza and SBAI Department in Rome. He would like to thank these institutions for the support and warm hospitality during the visits. The authors would like to warmly thank Professor Gianni Dal Maso for his insightful comments on a preliminary version of this paper. The authors would also like to thank Doctor Anna Nikishova for Figure 1.

1. Introduction

Starting from the seminal paper of Anzellotti [5], in the last decades there has been a growing interest in giving a well posed definition of *pairing*, i.e. a scalar product between a vector field and the weak gradient of a suitably regular function, motivated by the integration by parts. Indeed, many mathematicians have made effort to look for formulas of this type in increasingly general contexts and to establish the validity of the Gauss-Green formula under very weak regularity assumptions (see [1, 6, 9, 13, 15–17, 19, 41–43]). The motivation of this large interest is that the notion of pairing is a fundamental tool and is widely used in several contexts, e.g. in the study of Dirichlet problems for the Prescribed Mean Curvature and 1-Laplace equations, as well as related topics (see [3, 4, 8, 29–32, 35, 38, 39]). In addition, we point to some recent extensions of the notion of pairing to non-Euclidean frameworks [7, 14, 27, 28].

The aim of this paper is to introduce a very general pairing between measure vector fields with divergence measure and distributional gradients of functions which can be beyond the class of functions of bounded variation, by extending all the notions of pairings given previously in the literature, and possibly paving the way to integration by parts formulas on sets with fractal boundary. More precisely, taking inspiration from the original definition of Anzellotti [5, Definition 1.4] we look for the minimal conditions under which the pairing is still well-posed.

We briefly review the classical notion of pairing. To this purpose, we recall the definition of *divergence-measure fields*. We consider vector fields $\boldsymbol{A} \in L^p(\Omega;\mathbb{R}^N)$, for some open set $\Omega \subset \mathbb{R}^N$ and $p \in [1,+\infty]$, with distributional divergences $\operatorname{div}\boldsymbol{A}$ represented by scalar valued measures belonging to $\mathcal{M}(\Omega)$, that is, the space of finite Radon measures on Ω. Their space is denoted by $\mathcal{DM}^p(\Omega)$, and its local version by $\mathcal{DM}^p_{\rm loc}(\Omega)$. Then, the pairing between $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and Du, for a given function $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ is the distribution given by

$$\varphi \in C^\infty_c(\Omega) \to \langle (\boldsymbol{A}, Du), \varphi\rangle := -\int_\Omega \varphi u^*\, d\operatorname{div}\boldsymbol{A} - \int_\Omega u\boldsymbol{A}\cdot\nabla\varphi\, dx, \tag{1.1}$$

where u^* is the precise representative of the BV-function u. Since the measure $\operatorname{div}\boldsymbol{A}$ is absolutely continuous with respect to the $(N-1)$-Hausdorff measure $\mathscr{H}^{N-1}$, and u^* is well defined $\mathscr{H}^{N-1}$-almost everywhere, this definition is well-posed. As proven in [10], the vector field $u\boldsymbol{A}$ belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and the pairing is a measure satisfying the following Leibniz-type formula:

$$(\boldsymbol{A}, Du) = \operatorname{div}(u\boldsymbol{A}) - u^*\operatorname{div}\boldsymbol{A}.$$

A decisive step in the study of the pairing was done in [16]: here the authors, redefining the pairing distribution for functions $u \in BV_{\rm loc}(\Omega)$ such that $u^* \in L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$ (as it was firstly done in [21], thus requiring a dependence between the scalar functions and the vector field), proved a coarea formula and a Leibniz rule, and finally achieved generalizations of the Gauss–Green formulas.

Later on, motivated by obstacle problems in BV and semicontinuity issues, for which the standard pairing is not adequate, in [19] the authors introduced a new family of pairings, depending on the choice of the pointwise representative of u. More precisely, they proved that, given $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega)$ such that $u^* \in L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$, for every Borel function $\lambda\colon \Omega \to [0,1]$ there exists a measure $(\boldsymbol{A}, Du)_\lambda$ defined as

$$\varphi \in C^1_c(\Omega) \to \langle (\boldsymbol{A}, Du)_\lambda, \varphi\rangle := -\int_\Omega \varphi u^\lambda\, d\operatorname{div}\boldsymbol{A} - \int_\Omega u\boldsymbol{A}\cdot\nabla\varphi\, dx, \tag{1.2}$$

which then satisfies the Leibniz rule

$$\operatorname{div}(u\boldsymbol{A}) = u^\lambda \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)_\lambda \,, \tag{1.3}$$

where

$$u^\lambda := (1-\lambda)u^- + \lambda u^+, \tag{1.4}$$

and $u^\mp$ are the approximate liminf and limsup of u. Indeed, the approximate liminf and limsup of a functions with bounded variation are well defined $\mathscr{H}^{N-1}$-almost everywhere. In particular, if $\lambda(x) \equiv \frac{1}{2}$, we get $u^{\frac{1}{2}}(x) = u^*(x)$ for $\mathscr{H}^{N-1}$-a.e. $x \in \Omega$.

As for divergence-measure fields not in L^∞, problems related to the foundations of continuum mechanics (especially concerning the representation of Cauchy fluxes) naturally led to weaker versions of the Gauss–Green formulas for $\mathcal{DM}^1_{\rm loc}$-fields and sets satisfying suitable measure-theoretic assumptions (see [9,11,22,40–43]). As a further step in this direction, it was introduced the larger space $\mathcal{DM}(\Omega)$ of measure-valued vector fields $\boldsymbol{A} \in \mathcal{M}(\Omega;\mathbb{R}^N)$ whose distributional divergences $\operatorname{div} \boldsymbol{A}$ belongs to $\mathcal{M}(\Omega)$; and analogously the local version $\mathcal{DM}_{\rm loc}(\Omega)$. In this general setting, another definition of pairing may be found in [42]: given $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and $u \in \operatorname{Lip}_{\rm loc}(\Omega)$, we have $u\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and

$$\operatorname{div}(u\boldsymbol{A}) = u \operatorname{div} \boldsymbol{A} + \langle\langle \nabla u, \boldsymbol{A} \rangle\rangle \,, \tag{1.5}$$

where $\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle$ is a Radon measure. In particular, thanks to [11], if $u \in C^1(\Omega)$, we see that

$$\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle = \nabla u \cdot \boldsymbol{A},$$

thus showing that this is a generalization of the scalar product between a continuous function and a vector valued measure.

The key point of our research is to weaken the regularity assumption on the scalar function u. We started with the case $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, in order to check whether the pairing could de defined outside of the class of BV functions. Then, we noticed that our approach did not actually depend on the summability of the vector fields, and that there was no reason not to directly consider measure valued divergence-measure fields.

Given a vector field $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and a Borel function $\lambda \colon \Omega \to [0,1]$, we define the class $X^{\boldsymbol{A},\lambda}(\Omega)$ of those equivalence classes of Borel functions u such that

$$u^\lambda \in L^1(\Omega, |\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|) \,,$$

and we denote by $X^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$ its local version. Here, $u^\lambda(x)$ is defined as in (1.4) for all $x \in \Omega \setminus Z_u$, where Z_u is the set where both u^+ and u^- are infinite (with different sign). This definition means that u^λ is the convex combination of u^+ and u^-: in order to extend it even on Z_u, we set

$$u^\lambda(x) = (2\lambda(x) - 1) \cdot (+\infty) \quad \text{if } x \in Z_u$$

where we convene that $0 \cdot (\pm\infty) = 0$ (see (2.10) and the subsequent comments). Note that u does not need to be an L^1 function and $u^+(x)$, $u^-(x)$ are defined *for every* $x \in \Omega$ through the densities of sublevels of u, for which we refer (2.3) and the remarks below it. However, if $u \in L^1_{\rm loc}(\Omega)$, we could define the general λ-pairings by exploiting the representatives of u given by the limit of averages on balls and half-balls. This was the approach followed in [19], where $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$: indeed, in such a case we have $\mathscr{H}^{N-1}(Z_u) = 0$ and $\operatorname{div} \boldsymbol{A}$ is absolutely continuous with respect to $\mathscr{H}^{N-1}$, so that our definition coincides with (1.4) up to an $|\operatorname{div} \boldsymbol{A}|$-negligible set (see Remark 3.7 for details).

Then, for $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, we define the λ-pairing between $\boldsymbol{A}$ and u as the distribution $(\boldsymbol{A}, Du)_\lambda$ acting as

$$\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle := - \int_\Omega u^\lambda \, \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u^\lambda \nabla \varphi \cdot d\boldsymbol{A} \quad \text{for} \;\; \varphi \in C^\infty_c(\Omega) \, . \tag{1.6}$$

We note that, unless otherwise specified, Du is only a distributional gradient.

In analogy with the classical theory of functions of bounded variation, we introduce a version of BV-type classes, with respect to the variation given by this generalized λ-pairing, by setting

$$BV^{\boldsymbol{A},\lambda}(\Omega) := \Big\{ u \in X^{\boldsymbol{A},\lambda}(\Omega) : (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega) \Big\} \, ,$$

and analogously for the local classes $BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$.

For $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, we see that $u^\lambda \boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and we have the following Leibniz rule

$$(\boldsymbol{A}, Du)_\lambda = -u^\lambda \operatorname{div} \boldsymbol{A} + \operatorname{div}(u^\lambda \boldsymbol{A}) \;\; \text{on} \;\; \Omega,$$

in the sense of Radon measures (Proposition 3.5).

To the best of our knowledge, (1.6) is the first definition of pairing which extends to a more general setting all the notions of pairing and scalar product available in the literature, often not comparable each other. Indeed, we explicitly mention that

- (i) if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in BV_{\mathrm{loc}}(\Omega)$ is such that $u^* \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$, then $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ and $(\boldsymbol{A}, Du)_\lambda$ coincides with the one defined in [19], see (1.2);
- (ii) if $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$, $1 \le p \le +\infty$, and $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, then $(\boldsymbol{A}, Du)_\lambda = \boldsymbol{A} \cdot \nabla u \, \mathscr{L}^N$, where q is the conjugate exponent of p;
- (iii) if $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, then $(\boldsymbol{A}, Du)_\lambda$ coincides with the pairing $\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle$ of (1.5).

In particular, in the case (i) from [19] it is well known that we have the absolute continuity estimate $|(\boldsymbol{A}, Du)_\lambda| \ll |Du| \ll \mathscr{H}^{N-1}$, due to well known properties of BV functions. We stress the fact that such absolute continuity of the λ-pairing with respect to $\mathscr{H}^{N-1}$ still holds true for $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ (see Proposition 3.14(ii)). In our general setting, as far as we know, the problem of the absolute continuity of the pairing is still largely unexplored. Clearly, a strong estimate as $|(\boldsymbol{A}, Du)_\lambda| \ll \mathscr{H}^{N-1}$ would not be reasonable. With Proposition 3.15(ii) we establish a result for $\mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ fields, $p \in \left[\frac{N}{N-1}, +\infty\right)$, as we show that, if q is the conjugate exponent of p and $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, then $|(\boldsymbol{A}, Du)_\lambda|(B) = 0$ for every Borel set $B \subset \Omega$ which is σ-finite with respect to the measure $\mathscr{H}^{N-q}$. In the case $p \in \left[1, \frac{N}{N-1}\right)$ the λ-pairing may not be absolutely continuous with respect to $\mathscr{H}^\alpha$ for all $\alpha > 0$, since the singularities of $\operatorname{div} \boldsymbol{A}$ can be arbitrary (Remark 3.17). Finally, if $\boldsymbol{A} \in \mathcal{DM}^p(\Omega)$, it is natural to characterize the case of absolute continuity $|(\boldsymbol{A}, Du)_\lambda| \ll \mathscr{L}^N$ as some sort of "$(\boldsymbol{A}, \lambda)$-Sobolev class" $W^{\boldsymbol{A},\lambda}(\Omega)$, which is clearly a subset of $BV^{\boldsymbol{A},\lambda}(\Omega)$ and it is easily proved to contain $W^{1,q}(\Omega) \cap L^\infty(\Omega)$, where q is the conjugate exponent of p (Proposition 3.22).

In analogy with the classical theory of the calculus of variations, we investigated whether our objects enjoy the necessary topological properties to obtain the existence of minimizers of functionals involving the total variation of the λ-pairings. We show in Theorem 4.3 that a suitable notion of convergence in $BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ which ensures the lower semicontinuity of the pairing is the "$(\boldsymbol{A}, \lambda)$-convergence":

$$u_n^\lambda \rightharpoonup u^\lambda \text{ in } L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|), \quad u_n^\lambda \rightharpoonup u^\lambda \text{ in } L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|).$$

It seems to be quite natural that this convergence involves the Borel function λ, given that $(\boldsymbol{A}, Du)_\lambda$ is affected by the pointwise values of u^λ. In addition, with Theorem 4.4 we prove the existence of smooth approximations for the λ-pairing with respect to such convergence in some particular cases.

That said, we have to notice that $BV^{\boldsymbol{A},\lambda}(\Omega)$ is not a linear space in general, since the map

$$u \to (\boldsymbol{A}, Du)_\lambda$$

is not linear (see Remarks 3.4 and 7.5). This fact is not surprising, since it was already noticed in [19, Remark 4.6] that, in the setting $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ such that $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, this is the general picture, except the case $\lambda(x) = \frac{1}{2}$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$, when the λ-pairing coincides with the standard one. Indeed, in Remark 7.5, we point out that the choice $\lambda \equiv \frac{1}{2}$ loses its privileged role as soon as the field $\boldsymbol{A}$ is singular (say, its divergence is a Dirac delta), since the corresponding pairing fails to be linear. However, if we require

$$|u|^+ \in L^1(\Omega, |\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|) \text{ and } (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega) \text{ for every Borel } \lambda : \Omega \to [0, 1],$$

we identify a subclass $BV^{\boldsymbol{A},+}(\Omega) \subseteq BV^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel λ, which is a linear space. Moreover, if $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ is such that $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, then $BV^{\boldsymbol{A},+}(\Omega)$ coincides with $BV^{\boldsymbol{A},\frac{1}{2}}(\Omega) =: BV^{\boldsymbol{A}}(\Omega)$ and is endowed with the natural seminorm

$$\|u\|_{BV^{\boldsymbol{A}}(\Omega)} := \|u\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)} + \|u\|_{L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)} + |\operatorname{div}(u\boldsymbol{A})|(\Omega).$$

which, under the additional assumption $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$, turns out to be indeed a norm. The corresponding normed space $BV^{\boldsymbol{A}}(\Omega)$ is a Banach space (Proposition 5.1). On the other hand, in general, $BV^{\boldsymbol{A}}(\Omega)$ fails to be locally compact with respect to $(\boldsymbol{A}, \frac{1}{2})$-convergence, at least in dimension $N \geq 2$ (see Example 5.5). This prevents the use of the direct method to prove the existence of minimizers for functionals of the type

$$\mathcal{E}(u) := |(\boldsymbol{A}, Du)|(\Omega), \quad u \in BV^{\boldsymbol{A}}(\Omega), \ \boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega),$$

so that the addition of "fidelity terms" of the form $\|u - g\|_{L^p(\Omega, |\boldsymbol{A}|\mathscr{L}^N)}$ for $1 \leq p \leq +\infty$, $g \in L^p(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$, together with suitable additional natural assumptions on the vector field, are needed to force an L^p-compactness (see Theorem 5.6).

In analogy with the results established in [19] for the λ-pairing, we look for some type of coarea formula. As showed with Theorem 6.1, our generalized λ-pairing complies with the following *coarea inequality*:

$$|(\boldsymbol{A}, Du)_\lambda| \leq \int_{-\infty}^{+\infty} |(\boldsymbol{A}, D\chi_{\{u>t\}})_\lambda| \, dt$$

in the sense of measures (whenever the right hand side is well posed), under suitable technical assumptions on both u and $\boldsymbol{A}$ (see (6.1) and (6.2)). These assumptions are always satisfied in the situations available in literature (see Remark 6.2), hence they seem to be quite natural in our more general setting.

The notion of λ-pairing suggests in a natural way the definition of an "anisotropic degenerate" perimeter, the $(\boldsymbol{A}, \lambda)$-*perimeter*, as the total variation of the pairing; i.e.,

$$P_{\boldsymbol{A},\lambda}(E, \Omega) = |(\boldsymbol{A}, D\chi_E)_\lambda|(\Omega).$$

We immediately notice that a set of finite (Euclidean) perimeter has finite $(\boldsymbol{A}, \lambda)$-perimeter, but the converse is, in general, not true. Indeed, we can construct a fractal Borel set E such that $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\mathbb{R}^N) \setminus BV_{\rm loc}(\mathbb{R}^N)$ and $P_{\boldsymbol{A},\lambda}(E, \Omega) = 0$, while still being nonnegligible with respect to the measure $|\boldsymbol{A}|$ (see Remark 7.3).

Before we list the relevant features of the $(\boldsymbol{A}, \lambda)$-perimeter, some clarifications on the chosen terminology are in order. Differently from the classical notion of anisotropic perimeters (which are obtained, for instance, by taking another norm on the measure theoretic interior normal ν_E), this type of perimeter can be zero for nontrivial sets (as already noticed), thus being degenerate at least for some choices of the divergence-measure field $\boldsymbol{A}$. Indeed, let $\boldsymbol{A} \equiv v$, for some constant vector $v \neq 0$, and let $E_\nu = \{x \in \mathbb{R}^N : x \cdot \nu > 0\}$ for some other constant vector $\nu \neq 0$ such that $\nu \cdot v = 0$. Hence, it is easy to see that

$$(\boldsymbol{A}, D\chi_{E_\nu})_\lambda = D_v \chi_{E_\nu} = 0,$$

so that $P_{\boldsymbol{A},\lambda}(E, \Omega) = |(\boldsymbol{A}, D\chi_{E_\nu})_\lambda|(\Omega) = 0$ for all open sets Ω. This degeneracy creates some difficulties when one wants to follow in the footsteps of the classical theory of sets of finite perimeter and functions of bounded variation (indeed, it means that $|(\boldsymbol{A}, Du)_\lambda|(\Omega) = 0$ does not imply u to be constant in the connected components of Ω, in general). On the other hand, we believe it to be relevant for many applications in which one is indeed not interested in the growth of some quantity u outside of a selected field of directions.

With the case of the classical Euclidean perimeter in mind, a natural question is to ask whether the $(\boldsymbol{A}, \lambda)$-perimeter is concentrated on some type of generalized boundary of the set. To this purpose, we recall the definition of another type of Lebesgue measure-invariant boundary of a Borel set E:

$$\partial^- E := \{x \in \mathbb{R}^N : 0 < \mathscr{L}^N(E \cap B_r(x)) < \mathscr{L}^N(B_r(x)) \ \text{ for all } r > 0\}.$$

Then we have $\operatorname{supp}((\boldsymbol{A}, D\chi_E)_\lambda) \subseteq \partial^- E$ (see Proposition 7.2), even though such a control on the size of the support of the pairing distribution is in general too large (see Remark 7.3).

The $(\boldsymbol{A}, \lambda)$-perimeter enjoys some absolute continuity properties both in the case of essentially bounded (Proposition 7.8) and unbounded (Proposition 7.11) fields $\boldsymbol{A}$. More precisely, if $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$, we obtain the expected estimate

$$P_{\boldsymbol{A},\lambda}(E, \cdot) \leq C_N \|\boldsymbol{A}\|_{L^\infty(\Omega;\mathbb{R}^N)} \mathscr{H}^{N-1} \llcorner \partial^- E \ \text{ on } \Omega.$$

In particular, if E is a set of finite perimeter in Ω, we retrieve the representation formula

$$(\boldsymbol{A}, D\chi_E)_\lambda = \big((1-\lambda)\mathrm{Tr}^i(\boldsymbol{A}, \partial^* E) + \lambda \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E)\big) \mathscr{H}^{N-1} \llcorner \partial^* E,$$

where $\partial^* E$ is the *measure theoretic boundary* of E[1] and

$$\mathrm{Tr}^i(\boldsymbol{A}, \partial^* E), \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E) \in L^\infty(\partial^* E, \mathscr{H}^{N-1})$$

are the *interior and exterior normal traces* of $\boldsymbol{A}$ on $\partial^* E$, see (2.25).

Furthermore, $P_{\boldsymbol{A},\lambda}$ enjoys some typical properties of the classical perimeter: locality and additivity on suitably "disjoint" sets (Proposition 7.4) and the lower semicontinuity with respect to the $(\boldsymbol{A}, \lambda)$-convergence (Proposition 7.13). On the contrary, in the case of dimension $N \geq 2$,

[1]We point out that in the literature [24, 33] it is sometimes De Giorgi's *reduced boundary* to be denoted as $\partial^* E$; however, given that we do not employ it in this paper, we use this notation for Federer's measure theoretic boundary.

we provide a counterexample to the compactness for family of sets with uniformly bounded $(\boldsymbol{A}, \frac{1}{2})$-perimeter (Example 7.14).

For sets E with finite $(\boldsymbol{A}, \lambda)$-perimeter, we prove a general Gauss-Green formula, Theorem 8.1. In order to make a comparison with the classical results, we only mention here that, for Borel sets $E \Subset \Omega$, if $\lambda \equiv 0$ and $\chi_E \in BV^{\boldsymbol{A},0}(\Omega)$, then

$$\operatorname{div} \boldsymbol{A}(E^1) = -\int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_0 \,,$$

while if $\lambda \equiv 1$ and $\chi_E \in BV^{\boldsymbol{A},1}(\Omega)$, then

$$\operatorname{div} \boldsymbol{A}(E^1 \cup \partial^* E) = -\int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_1 \,.$$

These are the counterpart in our general setting of the well-known Gauss-Green formulas

$$\operatorname{div} \boldsymbol{A}(E^1) = -\int_{\partial^* E} \mathrm{Tr}^i(\boldsymbol{A}, \partial^* E)\, d\mathscr{H}^{N-1} \quad \text{and} \quad \operatorname{div} \boldsymbol{A}(E^1 \cup \partial^* E) = -\int_{\partial^* E} \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E)\, d\mathscr{H}^{N-1},$$

established for $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $\chi_E \in BV(\Omega)$ such that $E \Subset \Omega$. In particular, we can integrate on sets which do not have local finite perimeter, given that the only assumptions are $(\boldsymbol{A}, D\chi_E)_0, (\boldsymbol{A}, D\chi_E)_1 \in \mathcal{M}(\Omega)$. In particular, we deduce a representation formula for $\operatorname{div} \boldsymbol{A}$ on $\partial^* E$ (see Corollary 8.2). As a further immediate consequence, we deduce integration by parts formulas (Theorem 8.6), which provide an extension of [19, Theorem 6.3].

Finally, in Section 9 we give an insight in the one-dimensional case $N = 1$, which usually represents a toy-model for understanding the theory in higher dimension. That is not the case since, quite surprisingly, in dimension one some stronger results hold which are characteristic of this setting and cannot be extended to higher dimension, since, if $N = 1$, $\mathcal{DM}(\Omega) = BV(\Omega)$.

Outline of the paper: In Section 2 we fix the basic notation and recall some definitions and preliminary results about measures and distributions, approximate limits and representatives, and divergence-measure fields. In Section 3 we introduce the classes $BV^{\boldsymbol{A},\lambda}$ and the new notion of λ-pairing, then we investigate its basic properties, in particular the absolute continuity, and we define a related "degenerate" Sobolev class. In Section 4 we prove the lower semicontinuity of the λ-pairing functional with respect to the $(\boldsymbol{A}, \lambda)$-convergence. In Section 5 we identify a linear space contained in $BV^{\boldsymbol{A},\lambda}$ and briefly investigate its properties. Section 6 focuses on a coarea-type formula for the λ-pairing, while in Section 7 we introduce a corresponding notion of perimeter and prove some of its features. In Section 8 we establish general Gauss-Green and integration by parts formulas. Eventually, the last Section 9 deals with the case $N = 1$.

2. Notation and preliminary results

In the following we denote by Ω a nonempty open subset of $\mathbb{R}^N$, and for every set $E \subset \mathbb{R}^N$ we denote by χ_E its characteristic function. For $x \in \mathbb{R}^N$ and $r > 0$, we denote by $B_r(x)$ the ball centered in x with radius r, and we set $B_1 := B_1(0)$. Given a set U, we denote its closure by $\overline{U}$. We say that a set E is compactly contained in Ω, and we write $E \Subset \Omega$, if $\overline{E}$ is a bounded set and $\overline{E} \subset \Omega$.

2.1. **Measures and distributions.** The following definitions and basic facts about measures can be found, e.g., in [2, Chapter 1].

We denote by $\mathscr{L}^N$ and $\mathscr{H}^\alpha$ the Lebesgue measure and the α-dimensional Hausdorff measure in $\mathbb{R}^N$ for some $\alpha \in [0, N]$, respectively. Unless otherwise stated, a measurable set is a $\mathscr{L}^N$-measurable set. We set $\omega_N := \mathscr{L}^N(B_1) = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$, where Γ is Euler's Gamma function.

Following the notation of [2], we denote by $\mathcal{M}_{\rm loc}(\Omega)$ the space of Radon measures on Ω, and by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω.

Given $\mu \in \mathcal{M}_{\rm loc}(\Omega)$ and a μ-measurable set E, the *restriction* $\mu \llcorner E$ is the Radon measure defined by

$$\mu \llcorner E(B) = \mu(E \cap B), \qquad \forall\ B\ \mu\text{-measurable},\ B \subset \Omega.$$

The *total variation* $|\mu|$ of $\mu \in \mathcal{M}_{\rm loc}(\Omega)$ is the nonnegative Radon measure defined by

$$|\mu|(E) := \sup \left\{ \sum_{h=0}^{\infty} |\mu(E_h)| : \ E_h\ \mu\text{-measurable sets, pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h \right\},$$

for every μ-measurable set $E \Subset \Omega$. If $\mu \in \mathcal{M}(\Omega)$, then $|\mu|(\Omega) < \infty$.

A measure $\mu \in \mathcal{M}_{\rm loc}(\Omega)$ is *absolutely continuous* with respect to a given nonnegative measure ν (notation: $\mu \ll \nu$) if $|\mu|(B) = 0$ for every Borel set B such that $\nu(B) = 0$. Two positive measures $\nu_1, \nu_2 \in \mathcal{M}_{\rm loc}(\Omega)$ are *mutually singular* (notation: $\nu_1 \perp \nu_2$) if there exists a Borel set E such that $|\nu_1|(E) = 0$ and $|\nu_2|(\Omega \setminus E) = 0$. Given $\mu \in \mathcal{M}_{\rm loc}(\Omega)$, the *Lebesgue decomposition* of μ with respect to $\mathscr{L}^N$ is

$$\mu = \mu^a + \mu^s\,,$$

where μ^a is the *absolutely continuous part*, satisfying $\mu^a \ll \mathscr{L}^N$, and μ^s is the *singular part*, satisfying $\mu^s \perp \mathscr{L}^N$.

A measure μ in Ω is *concentrated on* $E \subset \Omega$ if $\mu(\Omega \backslash E) = 0$. The intersection of the closed sets $E \subset \Omega$ such that μ is concentrated on E is called the *support of* μ and is denoted by $\operatorname{supp}(\mu)$. In particular,

$$\Omega \backslash \operatorname{supp}(\mu) = \{x \in \Omega : \ \mu(B_r(x)) = 0 \text{ for some } r > 0\}\,.$$

We define a space of equivalence classes of Borel measure functions in the following way:

$$\mathscr{B}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ Borel measurable}\}/\sim,$$

where $\sim$ is the equivalence relation given by the almost everywhere equality; that is,

$$u \sim v \iff \mathscr{L}^N(\{x \in \Omega : u(x) \neq v(x)\}) = 0.$$

Now, we recall some basic definitions about distributions. We refer the interested reader to [37, Chapter 6] for a detailed treatment of this topic. We denote by $\mathscr{D}(\Omega)$ the space of *test functions*; i.e., $\varphi \in \mathscr{D}(\Omega)$ if and only if $\varphi \in C^\infty(\Omega)$ and the support of φ is a compact subset of Ω. We consider the norms

$$\|\varphi\|_n := \max\{|D^\alpha \varphi(x)| : \ x \in \Omega,\ \alpha \in \mathbb{N}^n,\ |\alpha| \le n\}$$

for $\varphi \in \mathscr{D}(\Omega)$ and $n \in \mathbb{N}$, where, corresponding to the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, D^α denotes the differential operator of order $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$ defined by $D^\alpha := (\partial_{x_1})^{\alpha_1}(\partial_{x_2})^{\alpha_2} \ldots (\partial_{x_n})^{\alpha_n}$. It is well-known (see [37, Theorem 6.4 and 6.5]) that $\mathscr{D}(\Omega)$ is a topological vector space when equipped with a suitable topology τ for which all Cauchy sequences do converge. A linear functional Λ on $\mathscr{D}(\Omega)$ which is continuous (with respect to the topology τ) is called a *distribution* in Ω. The space of all distributions in Ω is denoted by $\mathscr{D}'(\Omega)$. The smallest integer $n \in \mathbb{N}$ such that

$$|\Lambda(\varphi)| \leq C\|\varphi\|_n$$

for every $\varphi \in \mathscr{D}(\Omega)$, if exists, is called the *order* of Λ.

Remark 2.1. *Given a $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, setting*

$$\Lambda_\mu(\varphi) := \int_\Omega \varphi\, d\mu\,, \quad \varphi \in \mathscr{D}(\Omega)\,,$$

defines a distribution Λ_μ in Ω of order zero. Conversely, given a distribution Λ in Ω of order zero, by the Riesz Representation Theorem there exists $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ such that $\Lambda = \Lambda_\mu$.

Let $\Lambda \in \mathscr{D}'(\Omega)$, and set

$$W := \bigcup \{U \subset \Omega\,,\ U \text{ open: } \Lambda(\varphi) = 0\,,\ \text{ for all } \varphi \in \mathscr{D}(U)\}\ .$$

Then, the *support of* Λ is defined as $\mathrm{supp}(\Lambda) := \Omega \backslash W$. In the case of distributions of order zero, this definition coincides with the one of the support of a measure; that is, $\mathrm{supp}(\Lambda_\mu) = \mathrm{supp}(\mu)$.

2.2. **Approximate limits and λ-representatives.** The following basic definitions and results can be found, e.g., in [2, Sections 3.6 and 4.5].

We say that a function $u \in L^1_{\mathrm{loc}}(\Omega)$ has an *approximate limit* $z \in \mathbb{R}$ at $x \in \Omega$ if

$$\lim_{r \to 0^+} \frac{1}{\mathscr{L}^N(B_r(x))} \int_{B_r(x)} |u(y) - z|\, dy = 0\,; \tag{2.1}$$

in this case we say that x is a *Lebesgue point* of u. The set $S_u \subset \Omega$ of points where this property does not hold is called the *approximate discontinuity set* of u, and, thanks to Lebesgue's differentiation theorem, we know that $\mathscr{L}^N(S_u) = 0$. For any $x \in \Omega \setminus S_u$ the approximate limit z is uniquely determined and is denoted by $z =: \tilde{u}(x)$. Note that Chebychev inequality and 2.1 imply

$$\lim_{r \to 0^+} \frac{\mathscr{L}^N(\{y \in B_r(x) :\ |u(y) - \tilde{u}(x)| > \varepsilon\})}{\mathscr{L}^N(B_r(x))} = 0 \tag{2.2}$$

for every $\varepsilon > 0$. In fact, (2.2) provides an alternative (weaker) definition of approximate limit for a Borel measurable (even non locally summable) function (see [2, Remark 4.29] and (see [25, §2.9.12])), and the two definitions are equivalent for locally bounded functions (see [2, Proposition 3.65]).

If $u = \chi_E$, for a measurable set $E \subset \mathbb{R}^N$, then the approximate limit at a point $x \in \mathbb{R}^N$ is also called *density* of E at x, and it is given by

$$D(E; x) := \lim_{r \to 0^+} \frac{\mathscr{L}^N(E \cap B_r(x))}{\mathscr{L}^N(B_r(x))}$$

whenever this limit exists. We call *measure theoretic interior* of E the set of points with density 1, and we denote it by

$$E^1 := \{x \in \mathbb{R}^N : D(E; x) = 1\}.$$

We call *measure theoretic boundary* of E the approximate discontinuity set of χ_E, and we denote it by $\partial^* E := S_{\chi_E}$, which also satisfies $\partial^* E = \mathbb{R}^N \setminus (E^1 \cup E^0)$.

Set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Given a Borel measurable function $u : \Omega \to \mathbb{R}$, we denote the *sublevel and superlevel sets* of u as

$$\{u < t\} = \{x \in \Omega : u(x) < t\} \text{ and } \{u > t\} = \{x \in \Omega : u(x) > t\},$$

and we recall the definition of the *approximate liminf and limsup* at a point $x \in \Omega$ for $u \in \mathscr{B}(\Omega)$:

$$u^-(x) := \sup\left\{t \in \overline{\mathbb{R}} \colon D(\{u < t\}; x) = 0\right\}, \quad u^+(x) := \inf\left\{t \in \overline{\mathbb{R}} \colon D(\{u > t\}; x) = 0\right\} \tag{2.3}$$

(see [2, Definition 4.28]). We notice that $u^+, u^- : \Omega \to [-\infty, +\infty]$ are Borel measurable functions and the set $S_u^* := \{x \in \Omega : u^-(x) < u^+(x)\}$ satisfies

$$\mathscr{L}^N(S_u^*) = 0, \tag{2.4}$$

so that $u^+(x) = u^-(x)$ for $\mathscr{L}^N$-a.e. $x \in \Omega$, by [2, Definition 4.28] and the comments below. In the particular case $u \in L^1_{\mathrm{loc}}(\Omega)$, (2.2) implies that

$$u^+(x) = u^-(x) = \tilde{u}(x) \text{ for all } x \in \Omega \setminus S_u, \tag{2.5}$$

which implies $S_u^* \subset S_u$. Therefore, in $\Omega \setminus S_u^*$ we shall write $\tilde{u}(x) := u^+(x) = u^-(x)$, with a little abuse of notation. If $u \in \mathscr{B}(\Omega)$, we tacitly identify the canonical representative of the class with $\tilde{u}$, by setting $\tilde{u} = 0$ on S_u^*. However, by definition of $\mathscr{B}(\Omega)$ it is clear that $u = \tilde{u}$ with respect to the Lebesgue measure, so that, whenever dealing with $\mathscr{L}^N$, we shall simply write u as the representative of the class.

Given $u \in L^1_{\mathrm{loc}}(\Omega)$, we say that $x \in \Omega$ is an *approximate jump point* of u if there exist $a, b \in \mathbb{R}$, $a \neq b$, and a unit vector $\nu \in \mathbb{R}^N$ such that

$$\begin{aligned} \lim_{r \to 0^+} \frac{1}{\mathscr{L}^N(B_r^i(x))} \int_{B_r^i(x)} |u(y) - a| \, dy = 0, \\ \lim_{r \to 0^+} \frac{1}{\mathscr{L}^N(B_r^e(x))} \int_{B_r^e(x)} |u(y) - b| \, dy = 0, \end{aligned} \tag{2.6}$$

where $B_r^i(x) := \{y \in B_r(x) : \ (y - x) \cdot \nu > 0\}$, and $B_r^e(x) := \{y \in B_r(x) : \ (y - x) \cdot \nu < 0\}$. The triplet (a, b, ν), uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^i(x), u^e(x), \nu_u(x))$. The set of approximate jump points of u is denoted by J_u, and it clearly satisfies $J_u \subset S_u$. We notice that, even for a function $u \in L^1_{\mathrm{loc}}(\Omega)$, the jump set J_u is $(N-1)$-rectifiable (see [23]), while S_u can have high Hausdorff dimension, even equal to the space dimension N.

We point out that (cfr. [2, Definition 4.30] and the comments below therein)

$$u^-(x) = \min\{u^i(x), u^e(x)\} \text{ and } u^+(x) = \max\{u^i(x), u^e(x)\} \quad \text{for all } x \in J_u. \tag{2.7}$$

Finally, for $u \in L^1_{\mathrm{loc}}(\Omega)$ we define the *precise representative* of u in $x \in \Omega$ as

$$u^*(x) := \lim_{r \to 0^+} \frac{1}{\mathscr{L}^N(B_r(x))} \int_{B_r(x)} u(y) \, dy, \tag{2.8}$$

whenever the limit exists. It is then clear that

$$u^*(x) = \begin{cases} \tilde{u}(x) & x \in \Omega \setminus S_u, \\ \dfrac{u^i(x) + u^e(x)}{2} & x \in J_u. \end{cases} \tag{2.9}$$

A priori, it is not clear whether u^* is well posed in $S_u \setminus J_u$, in general. However, for sufficiently regular functions it is known that $S_u \setminus J_u$ is suitably small. To this purpose, we recall that u is a *function of bounded variation*, and we write $u \in BV(\Omega)$, if $u \in L^1(\Omega)$ and its distributional gradient Du belongs to $\mathcal{M}(\Omega;\mathbb{R}^N)$; and we denote by $BV_{\rm loc}(\Omega)$ the local version of the space. For a detailed treatment of the theory of BV functions, we refer the reader to the monography [2]. Therefore, for $u \in BV_{\rm loc}(\Omega)$, it is well known that we have $\mathscr{H}^{N-1}(S_u \setminus J_u) = 0$, so that $u^*(x)$ exists for $\mathscr{H}^{N-1}$-a.e $x \in \Omega$ and, up to a $\mathscr{H}^{N-1}$-negligible set, is given by (2.9). However, in the rest of the paper we are mostly dealing with functions not in $BV_{\rm loc}(\Omega)$ nor even locally summable, so that we will have to consider another kind of representative.

For every function $u \in \mathscr{B}(\Omega)$ and every Borel function $\lambda : \Omega \to [0,1]$, we define the *$\lambda$–representative* $u^\lambda : \Omega \to \overline{\mathbb{R}}$ as

$$u^\lambda(x) := \begin{cases} (1-\lambda(x))u^-(x) + \lambda(x)u^+(x) & \text{if } x \in \Omega \setminus Z_u \\ +\infty & \text{if } x \in Z_u \text{ and } \lambda(x) > \frac{1}{2} \\ 0 & \text{if } x \in Z_u \text{ and } \lambda(x) = \frac{1}{2} \\ -\infty & \text{if } x \in Z_u \text{ and } \lambda(x) < \frac{1}{2} \end{cases} \tag{2.10}$$

where $Z_u := \{x \in \Omega : u^+(x) = +\infty \text{ and } u^-(x) = -\infty\}$. In the particular cases $\lambda \equiv 1$ and $\lambda \equiv 0$, we simply have $u^1 := u^+$ and $u^0 := u^-$, respectively. We notice that, if $u \in L^1_{\rm loc}(\Omega)$, then $Z_u \subseteq S_u \setminus J_u$, and, if $u \in L^\infty_{\rm loc}(\Omega)$, then $Z_u = \emptyset$. This definition can be seen as a generalization of the [19, Eq. (2.4)], where the authors considered BV functions (see Remark 3.7 below). Indeed, in the particular case of $u \in BV_{\rm loc}(\Omega)$, we see that $\mathscr{H}^{N-1}(Z_u) \le \mathscr{H}^{N-1}(S_u \setminus J_u) = 0$ (see [2, Theorem 3.78]).

If $u \in L^1_{\rm loc}(\Omega)$, we notice that $u^\lambda(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus S_u$. More in general, if $u \in \mathscr{B}(\Omega)$, we get

$$u^\lambda(x) = \tilde{u}(x) \quad \text{for all } x \in \Omega \setminus S^*_u, \tag{2.11}$$

so that, by (2.4), we deduce that

$$u^\lambda(x) = u(x) \quad \text{for } \mathscr{L}^N\text{-a.e. } x \in \Omega. \tag{2.12}$$

In addition, if $u \in L^1_{\rm loc}(\Omega)$ and $\lambda \equiv \frac{1}{2}$ on J_u, we get $u^{\frac{1}{2}}(x) = u^*(x)$ for all $x \in \Omega \setminus (S_u \setminus J_u)$, but we might have $u^*(x) \neq u^{\frac{1}{2}}(x)$ for some $x \in S_u \setminus J_u$, where the limit in (2.8) exists. In particular, if $x \in Z_u$, we might have $u^*(x) \neq 0$: consider for instance the case $N = 1$, $\Omega = (-1,1)$ and

$$u(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{\sqrt[3]{x}} & \text{if } x < 0 \end{cases},$$

for which we have $u^+(0) = +\infty$, $u^-(0) = -\infty$ and $u^*(0) = +\infty$.

Arguing as in the proof of [20, Lemma 2.2], we can characterize the approximate liminf and limsup of the characteristic functions of the superlevel sets of u, outside of some essential discontinuity set.

Lemma 2.2. *Let $u\colon \Omega \to \mathbb{R}$ be a measurable function, $t \in \mathbb{R}$ and let*

$$N_t := \{u^- \le t < u^+\} \setminus \{u > t\}^{1/2}.$$

Then

$$\chi_{\{u^{\pm}>t\}}(x) = \chi^{\pm}_{\{u>t\}}(x) \qquad \forall x \in \Omega \setminus N_t. \tag{2.13}$$

Remark 2.3. *We recall that, if $u \in BV(\Omega)$, then formula* (2.13) *holds $\mathscr{H}^{N-1}$-a.e. in Ω, since*

$$\mathscr{H}^{N-1}(N_t) = 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in \mathbb{R},$$

(see [2, Theorems 3.40 and 3.61]*).*

Proof. Thanks to [20, Lemma 2.2, eq. (2.11)], we know that for all $t \in \mathbb{R}$ and $x \in \Omega$

$$u^-(x) > t \implies \chi^*_{\{u>t\}}(x) = 1, \qquad u^+(x) \le t \implies \chi^*_{\{u>t\}}(x) = 0\,,$$

which easily implies

$$\chi_{\{u^->t\}}(x) = 1 \implies \chi^-_{\{u>t\}}(x) = 1, \qquad \chi_{\{u^+>t\}}(x) = 0 \implies \chi^+_{\{u>t\}}(x) = 0\,. \tag{2.14}$$

Let now x be a point in $\Omega \setminus N_t$ such that $\chi^-_{\{u>t\}}(x) = 1$. We notice that, for $\tau \in \mathbb{R}$, we have

$$\big\{\chi_{\{u>t\}} < \tau\big\} = \begin{cases} \Omega & \text{if } \tau > 1 \\ \{u \le t\} & \text{if } 0 < \tau \le 1\,. \\ \emptyset & \text{if } \tau \le 0 \end{cases}$$

Hence, thanks to the definitions (2.3), $\chi^-_{\{u>t\}}(x) = 1$ implies

$$D\big(\{u \le t\}; x\big) = 0, \text{ and so } D\big(\{u > t\}; x\big) = 1.$$

All in all, this yields $u^-(x) \ge t$. However, if $u^-(x) = t$, then x should satisfy $u^-(x) \le t < u^+(x)$, which, given that $x \notin N_t$, would imply $x \in \{u > t\}^{1/2}$, but this contradicts the fact that $D\big(\{u > t\}; x\big) = 1$. Therefore, we conclude that $u^-(x) > t$, and so we obtain

$$\chi^-_{\{u>t\}}(x) = 1 \implies \chi_{\{u^->t\}}(x) = 1\,.$$

In a similar way it is possible to prove that

$$\chi^+_{\{u>t\}}(x) = 0 \implies \chi_{\{u^+>t\}}(x) = 0\,,$$

so that we conclude

$$\chi_{\{u^->t\}}(x) = 1 \iff \chi^-_{\{u>t\}}(x) = 1, \qquad \chi_{\{u^+>t\}}(x) = 0 \iff \chi^+_{\{u>t\}}(x) = 0\,. \tag{2.15}$$

If $u^-(x) \le t < u^+(x)$ and $x \notin N_t$, then necessarily $x \in \{u > t\}^{1/2}$, which means $\chi^*_{\{u>t\}}(x) = 1/2$. Hence, we get

$$\chi^+_{\{u>t\}}(x) + \chi^-_{\{u>t\}}(x) = 1,$$

and, since $\chi^{\pm}_{\{u>t\}}(x) \in \{0, 1\}$, we conclude that

$$\chi^+_{\{u>t\}}(x) = 1 \quad \text{and} \quad \chi^-_{\{u>t\}}(x) = 0.$$

Then, we exploit (2.15) to conclude. $\square$

2.3. **Divergence-measure fields.** Given a measure valued vector field $\boldsymbol{A} \in \mathcal{M}(\Omega;\mathbb{R}^N)$, we say that $\boldsymbol{A}$ is a *divergence-measure field* if its divergence in the sense of distributions is a finite Radon measure in Ω, acting as

$$\int_\Omega \varphi \, d \operatorname{div} \boldsymbol{A} = - \int_\Omega \nabla \varphi \cdot d\boldsymbol{A} \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{2.16}$$

We denote by $\mathcal{DM}(\Omega)$ the space of all such vector fields. Analogously, we define the local spaces $\mathcal{DM}_{\mathrm{loc}}(\Omega)$, as the sets of all vector fields $\boldsymbol{A} \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, such that $\operatorname{div} \boldsymbol{A} \in \mathcal{M}_{\mathrm{loc}}(\Omega)$.

We exhibit here a family of $\mathcal{DM}$-fields whose divergence is a non-trivial measure. We fix $y \in \Omega$ and we consider

$$\boldsymbol{A} := (a_1, a_2, \dots, a_{N-1}, a_N \mathscr{L}^N),$$

where

$$a_j = \mathscr{H}^1 \llcorner \{x \in \Omega : x_k = y_k \text{ for all } k \in \{1,2,\dots,N\}, k \neq j\} \quad \text{for } j \in \{1,\dots,N-1\}$$

and $a_N \in BV_{\mathrm{loc}}(\Omega)$. By construction, $\boldsymbol{A} \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $\operatorname{div} \boldsymbol{A} = D_{x_N} a_N \in \mathcal{M}_{\mathrm{loc}}(\Omega)$.

Given $p \in [1,+\infty]$, by $\mathcal{DM}^p(\Omega)$ we denote the space of all vector fields $\boldsymbol{A} \in L^p(\Omega;\mathbb{R}^N)$ whose divergence in the sense of distributions is a finite Radon measure in Ω, again acting as in (2.16). Similarly, we define the local spaces $\mathcal{DM}^p_{\mathrm{loc}}(\Omega)$, as the sets of all vector fields $\boldsymbol{A} \in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, such that $\operatorname{div} \boldsymbol{A} \in \mathcal{M}_{\mathrm{loc}}(\Omega)$. With a little abuse of notation, even in the case $\boldsymbol{A} \in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for some $p \in [1,+\infty]$, we shall sometimes write $\boldsymbol{A}$ to denote the measure $\boldsymbol{A}\mathscr{L}^N$.

For a more detailed exposition on the properties of these vector fields, we refer the reader to [5,10,11,13,15–19,41–43].

We recall that the divergence measure of a field $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ enjoys absolute continuity properties with respect to suitable Hausdorff measures, depending on the value of $p \in [1,+\infty]$. More precisely, we have the following cases:

(1) if $p = +\infty$, then $\operatorname{div} \boldsymbol{A} \ll \mathscr{H}^{N-1}$ ([10, Proposition 3.1] and [41, Theorem 3.2]) and, if $\boldsymbol{A} \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, then $|\operatorname{div} \boldsymbol{A}| \le c_N \|\boldsymbol{A}\|_{L^\infty(\Omega;\mathbb{R}^N)} \mathscr{H}^{N-1}$, where $c_N > 0$ is a constant depending only on the space dimension ([43, Proposition 3.1]);

(2) if $p \in \left[\frac{N}{N-1}, +\infty\right)$, then $|\operatorname{div} \boldsymbol{A}|(B) = 0$ for all Borel sets B of σ-finite $\mathscr{H}^{N-\frac{p}{p-1}}$ measure ([41, Theorem 3.2]);

(3) if $p \in \left[1, \frac{N}{N-1}\right)$, then the singularities of $\operatorname{div} \boldsymbol{A}$ may be arbitrary ([41, Example 3.3]).

Actually, in case (2) a slightly stronger result holds; that is, $|\operatorname{div} \boldsymbol{A}|$ vanishes on Borel sets with zero $\frac{p}{p-1}$-Sobolev capacity ([36, Theorem 2.8]), but this goes beyond the scope of our paper.

For the purposes of calculus, many different Leibniz rules for $\mathcal{DM}$ vector fields and suitably regular scalar functions have been discovered in the past years (see [10–12,26,42]). We collect here a list of such rules. In the most general case of $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, if $u \in C^1_c(\Omega)$, then we have $u\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and

$$\operatorname{div}(u\boldsymbol{A}) = u \operatorname{div} \boldsymbol{A} + \nabla u \cdot \boldsymbol{A} \quad \text{on } \Omega, \tag{2.17}$$

which is a particular case of [11, Theorem 3.2]. More in general, even if $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, it is possible to prove that $u\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, with

$$\operatorname{div}(u\boldsymbol{A}) = u \operatorname{div} \boldsymbol{A} + \langle\langle \nabla u, \boldsymbol{A} \rangle\rangle \quad \text{on } \Omega, \tag{2.18}$$

where $\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle$ is the weak*-limit of the family of measures $\nabla u_\rho \cdot \boldsymbol{A}$, for any standard mollification u_ρ of u, which is a Radon measure satisfying

$$|\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle| \le \|\nabla u\|_{L^\infty(\Omega';\mathbb{R}^N)} |\boldsymbol{A}| \quad \text{on } \Omega' \tag{2.19}$$

for every open set $\Omega' \Subset \Omega$ (see [42, Proposition 2.2]).

If instead $\boldsymbol{A} \in \mathcal{DM}^p_{\rm loc}(\Omega)$ for some $p \in [1, +\infty]$, then, under the following set of assumptions:

(1) if $p \in [1, +\infty)$ and $u \in L^\infty_{\rm loc}(\Omega) \cap W^{1,q}_{\rm loc}(\Omega)$, where $q = \frac{p}{p-1}$ is the conjugate exponent of p,
(2) if $p = +\infty$ and $u \in L^\infty_{\rm loc}(\Omega) \cap BV_{\rm loc}(\Omega)$,

it holds that $u\boldsymbol{A} \in \mathcal{DM}^r_{\rm loc}(\Omega)$ for all $r \in [1, p]$, with

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + \begin{cases} \boldsymbol{A} \cdot \nabla u \, \mathscr{L}^N, & \text{if } p < +\infty, \text{ or } p = +\infty \text{ and } u \in L^\infty_{\rm loc}(\Omega) \cap W^{1,1}_{\rm loc}(\Omega), \\ \\ (\boldsymbol{A}, Du), & \text{if } p = +\infty \text{ and } u \in L^\infty_{\rm loc}(\Omega) \cap (BV_{\rm loc}(\Omega) \backslash W^{1,1}_{\rm loc}(\Omega)), \end{cases} \tag{2.20}$$

where u^* is the precise representative of u, which satisfies $u^*(x) = \tilde{u}(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$ if u is a Sobolev function, and $(\boldsymbol{A}, Du)$ is the standard *pairing measure* between $\boldsymbol{A}$ and Du introduced in [5] (see [16] for more details). For such results, we refer the reader to [10, Theorem 3.1], [12, Theorem 3.2.3], [26, Theorem 2.1]. The pairing measure can be also characterized as the weak*-limit of the measures $\boldsymbol{A} \cdot \nabla(u * \rho_\varepsilon) \mathscr{L}^N$ as $\varepsilon \to 0$, for any standard mollifier ρ. In addition, for every open set $\Omega' \Subset \Omega$ we have

$$|(\boldsymbol{A}, Du)| \le \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} |Du| \quad \text{on } \Omega' \tag{2.21}$$

by [13, Proposition 3.4].

Finally, in [19, Proposition 4.4] the notion of λ-pairings is introduced in the case $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$. More precisely, given $u \in BV_{\rm loc}(\Omega)$ such that $u^* \in L^1(\Omega; |\operatorname{div} \boldsymbol{A}|)$ and a Borel function $\lambda : \Omega \to [0, 1]$, then (1.2) defines a family of λ-pairings, which provide the following Leibniz rule

$$\operatorname{div}(u\boldsymbol{A}) = u^\lambda \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)_\lambda \,, \tag{2.22}$$

where u^λ is given by (2.10) (see the subsequent discussion for the particular case of BV functions). In addition, analogously to (2.21), for every open set $\Omega' \Subset \Omega$ we have

$$|(\boldsymbol{A}, Du)_\lambda| \le \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} |Du| \quad \text{on } \Omega' \tag{2.23}$$

If $\lambda \equiv \frac{1}{2}$, then we retrieve the standard pairing given by (1.1) (see also [16, Theorem 4.12]). If instead $\lambda \equiv 0$ or $\lambda \equiv 1$, we simply write $(\boldsymbol{A}, Du)_0$ and $(\boldsymbol{A}, Du)_1$, respectively.

We state now a technical result which can be seen as a basic version of a Gauss–Green formula. It has been proved already in some special cases (see for instance [15, Lemma 3.1]), while we were not able to find it in literature in the most general form.

Lemma 2.4. *Let* $\boldsymbol{A} \in \mathcal{DM}(\Omega)$ *be such that* $\operatorname{supp}(|\boldsymbol{A}|) \Subset \Omega$. *Then* $\operatorname{div} \boldsymbol{A}(\Omega) = 0$.

Proof. Let $V \Subset \Omega$ be an open set such that $\operatorname{supp}(|\boldsymbol{A}|) \subset V$. It is well known (see, e.g., [15, Remark 2.21]) that $\operatorname{supp}(\operatorname{div} \boldsymbol{A}) \subset \operatorname{supp}(|\boldsymbol{A}|)$. Therefore, $\operatorname{div} \boldsymbol{A} = 0$ on $\Omega \setminus \overline{V}$. Then, we can apply the definition of weak divergence (2.16) to a function $\eta \in C^\infty_c(\Omega)$ such that $\eta \equiv 1$ on a neighborhood of V: we get

$$\operatorname{div} \boldsymbol{A}(\overline{V}) = \int_{\overline{V}} \eta \, d \operatorname{div} \boldsymbol{A} = \int_{\Omega} \eta \, d \operatorname{div} \boldsymbol{A} = -\int_{\Omega} \nabla \eta \cdot d\boldsymbol{A} = -\int_{\Omega \setminus V} \nabla \eta \cdot d\boldsymbol{A} = 0.$$

Thus, we get $\operatorname{div} \boldsymbol{A}(\Omega) = \operatorname{div} \boldsymbol{A}(\overline{V}) + \operatorname{div} \boldsymbol{A}(\Omega \setminus \overline{V}) = 0$. $\square$

We conclude this section with the Gauss–Green formulas for essentially bounded divergence-measure fields and sets of finite perimeter, for which we refer the reader to [15, Theorem 3.2]. To this purpose, we recall that a measurable $E \subset \Omega$ is a *set of finite perimeter* if $\chi_E \in BV(\Omega)$; and, in such a case, we have $|D\chi_E| = \mathscr{H}^{N-1} \llcorner \partial^* E$, due to De Giorgi's and Federer's Theorems (see for instance [2, Theorems 3.59 and 3.61]). Then, given $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $E \Subset \Omega$ of finite perimeter, we have

$$\operatorname{div} \boldsymbol{A}(E^1) = -\int_{\partial^* E} \operatorname{Tr}^i(\boldsymbol{A}, \partial^* E) \, d\mathscr{H}^{N-1} \quad \text{and} \quad \operatorname{div} \boldsymbol{A}(E^1 \cup \partial^* E) = -\int_{\partial^* E} \operatorname{Tr}^e(\boldsymbol{A}, \partial^* E) \, d\mathscr{H}^{N-1}, \tag{2.24}$$

where $\operatorname{Tr}^i(\boldsymbol{A}, \partial^* E), \operatorname{Tr}^e(\boldsymbol{A}, \partial^* E) \in L^\infty(\partial^* E, \mathscr{H}^{N-1})$ are the *interior and exterior normal traces* of $\boldsymbol{A}$ on $\partial^* E$. In particular, the normal traces can be characterized as the densities of the measures $(\boldsymbol{A}, D\chi_E)_0$ and $(\boldsymbol{A}, D\chi_E)_1$, respectively, with respect to the measure $|D\chi_E|$, thanks to [19, Proposition 4.7] applied to the case $u = \chi_E$. More precisely, if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $E \subset \Omega$ is a set of locally finite perimeter, we have

$$(\boldsymbol{A}, D\chi_E)_0 = \operatorname{Tr}^i(\boldsymbol{A}, \partial^* E) \, \mathscr{H}^{N-1} \llcorner \partial^* E \quad \text{and} \quad (\boldsymbol{A}, D\chi_E)_1 = \operatorname{Tr}^e(\boldsymbol{A}, \partial^* E) \, \mathscr{H}^{N-1} \llcorner \partial^* E \tag{2.25}$$

and, for all open sets $\Omega' \Subset \Omega$,

$$\begin{aligned} \|\operatorname{Tr}^i(\boldsymbol{A}, \partial^* E)\|_{L^\infty(\Omega' \cap \partial^* E, \mathscr{H}^{N-1})} &\le \|\boldsymbol{A}\|_{L^\infty(\Omega' \cap E; \mathbb{R}^N)}, \\ \|\operatorname{Tr}^e(\boldsymbol{A}, \partial^* E)\|_{L^\infty(\Omega' \cap \partial^* E, \mathscr{H}^{N-1})} &\le \|\boldsymbol{A}\|_{L^\infty(\Omega' \setminus E; \mathbb{R}^N)}. \end{aligned} \tag{2.26}$$

For these results we refer the reader to [19, Proposition 4.7] and [15, Theorem 4.2].

3. The class $BV^{\boldsymbol{A},\lambda}(\Omega)$

In this section we introduce our new general definition of λ-pairing. First of all, we define good ambient classes of summable functions, inevitably depending on the chosen field and the Borel function which defines the representatives.

Given a vector field $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and a Borel function $\lambda\colon \Omega \to [0,1]$, we set

$$X^{\boldsymbol{A},\lambda}(\Omega) := \{u \in \mathscr{B}(\Omega) : u^\lambda \in L^1(\Omega, |\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)\},$$

$$X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) := \{u \in \mathscr{B}(\Omega) : u^\lambda \in L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|) \cap L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)\}.$$

It is interesting to notice that these sets of functions are not linear spaces, in general, due to the fact that the λ-representative of a sum is not the sum of λ-representatives, see Remark 7.5 for some examples.

We define now a λ-pairing for functions in $X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$.

Definition 3.1 (General λ-pairing)**.** *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, $\lambda\colon \Omega \to [0,1]$ be a Borel function and $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$. We define the λ-pairing between $\boldsymbol{A}$ and u as the distribution*

$$(\boldsymbol{A}, Du)_\lambda : C^\infty_c(\Omega) \to \mathbb{R}$$

acting as

$$\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle := - \int_\Omega u^\lambda \, \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u^\lambda \nabla \varphi \cdot d\boldsymbol{A} \quad \text{for } \varphi \in C_c^\infty(\Omega) \, . \tag{3.1}$$

We emphasize that, for a given function $u \in X_{\rm loc}^{\boldsymbol{A},\lambda}(\Omega)$, the derivative Du may not exist as a Radon measure, but it is always well defined as a distribution (of order one), so that we are justified in choosing the standard λ-pairing notation as in the case of BV functions, [19, Definition 4.1].

In addition, if $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$, then (3.1) becomes

$$\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle := - \int_\Omega u^\lambda \, \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u \nabla \varphi \cdot \boldsymbol{A} \, dx \quad \text{for } \varphi \in C_c^\infty(\Omega) \, , \tag{3.2}$$

since $u^\lambda(x) = \tilde{u}(x) = u(x)$ for $\mathscr{L}^N$-a.e. $x \in \Omega$, where we denote by u one of the elements of its equivalence class, with a little abuse of notation, following the convention adopted in Section 2.2.

We provide an alternative equivalent formulation of the definition of λ-pairing (3.1), which is often used in the following.

Lemma 3.2. *Let $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$, $\lambda \colon \Omega \to [0,1]$ be a Borel function and $u \in X_{\rm loc}^{\boldsymbol{A},\lambda}(\Omega)$. Then, the distribution $(\boldsymbol{A}, Du)_\lambda$ is of order 1 and satisfies*

$$\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle = - \int_\Omega u^\lambda \, d \operatorname{div}(\varphi \boldsymbol{A}) \quad \text{for all } \varphi \in C_c^1(\Omega) \, . \tag{3.3}$$

In particular, if $u \equiv c$ for some $c \in \mathbb{R}$, then $(\boldsymbol{A}, Dc)_\lambda = 0$.

Proof. It is clear that

$$|\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle| \le \|\varphi\|_{L^\infty(\Omega)} \int_\Omega |u^\lambda| \, d|\operatorname{div} \boldsymbol{A}| + \|\nabla \varphi\|_{L^\infty(\Omega;\mathbb{R}^N)} \int_\Omega |u^\lambda| d|\boldsymbol{A}|,$$

so that the distribution $(\boldsymbol{A}, Du)_\lambda$ can be extended to C_c^1 test functions. Then, (3.3) is a straightforward consequence of (2.17). Indeed, for all $\varphi \in C_c^1(\Omega)$, we can rewrite (3.1) as

$$\langle (\boldsymbol{A}, Du)_\lambda \, , \, \varphi \rangle := - \int_\Omega u^\lambda \left(\varphi \, d \operatorname{div} \boldsymbol{A} + \nabla \varphi \cdot d\boldsymbol{A} \right),$$

and now we apply the Leibniz rule (2.17) to $\boldsymbol{A}$ and φ. Finally, if u is constant and equal to $c \in \mathbb{R}$, then we have

$$\langle (\boldsymbol{A}, Dc)_\lambda \, , \, \varphi \rangle := - \int_\Omega c \, d \operatorname{div}(\varphi \boldsymbol{A}) = -c \operatorname{div}(\varphi \boldsymbol{A})(\Omega) = 0,$$

thanks to Lemma 2.4. □

Arguing in analogy with the classical theory of functions of bounded variation, we introduce a version of BV-type classes with respect to the variation given by this generalized λ-pairing.

Definition 3.3. *Given $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and a Borel function $\lambda \colon \Omega \to [0,1]$, we define the classes*

$$BV^{\boldsymbol{A},\lambda}(\Omega) := \left\{ u \in X^{\boldsymbol{A},\lambda}(\Omega) : (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega) \right\} ,$$

$$BV_{\rm loc}^{\boldsymbol{A},\lambda}(\Omega) := \left\{ u \in X_{\rm loc}^{\boldsymbol{A},\lambda}(\Omega) : (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}_{\rm loc}(\Omega) \right\} .$$

Remark 3.4. *We point out that $BV^{\boldsymbol{A},\lambda}(\Omega)$ is not a linear space, in general, due to the fact that the λ-pairing is not linear in the second component. This was already noticed in* [19, Remark 4.6] *in the classical case of $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, whenever $\lambda(x) \neq \frac{1}{2}$ for all $x \in B$, for some Borel set $B \subseteq J_u$ with $|\operatorname{div} \boldsymbol{A}|(B) > 0$. In our setting, however, the map $u \to (\boldsymbol{A}, Du)_\lambda$ may fail to be linear even in the case $\lambda \equiv \frac{1}{2}$, as showed by the second example in Remark 7.5. It is also relevant to notice that the linearity may be ensured in some particular cases, which are explored in Section 5.*

Whenever λ is constant; that is, $\lambda \equiv t$ for some $t \in [0, 1]$, we simply write $BV^{\boldsymbol{A},t}(\Omega)$ (and analogously for the local classes). In the extreme cases $\lambda \equiv 0$ and $\lambda \equiv 1$, we therefore have $BV^{\boldsymbol{A},0}(\Omega)$ and $BV^{\boldsymbol{A},1}(\Omega)$, respectively. In the particular case $\lambda \equiv \frac{1}{2}$, we use the following shorthand notation:

$$X^{\boldsymbol{A}}(\Omega) := X^{\boldsymbol{A},\frac{1}{2}}(\Omega), \quad BV^{\boldsymbol{A}}(\Omega) := BV^{\boldsymbol{A},\frac{1}{2}}(\Omega), \quad (\boldsymbol{A}, Du) := (\boldsymbol{A}, Du)_{\frac{1}{2}},$$

and analogously for the local classes. This case is indeed relevant due to the fact that, in the classical theory of pairings, for $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in BV_{\mathrm{loc}}(\Omega)$, we have $u^{\frac{1}{2}}(x) = u^*(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$, and, as long as $u^* \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$, the pairing which we obtain coincides with the one defined by Anzellotti. We discuss this equivalence more in detail in the following proposition.

Proposition 3.5. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and $\lambda : \Omega \to [0, 1]$ be a Borel function.*

(1) *For all $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ we have*

$$\langle (\boldsymbol{A}, Du)_\lambda \,, \varphi \rangle = - \int_\Omega \varphi \, u^\lambda \, d \operatorname{div} \boldsymbol{A} + \Big\langle \operatorname{div}(u^\lambda \boldsymbol{A}), \varphi \Big\rangle \quad \text{for} \ \ \varphi \in C^1_c(\Omega) \tag{3.4}$$

in the sense of distributions. Hence, $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ if and only if $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ and $\operatorname{div}(u^\lambda \boldsymbol{A}) \in \mathcal{M}(\Omega)$, and, for all $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, we get

$$(\boldsymbol{A}, Du)_\lambda = -u^\lambda \operatorname{div} \boldsymbol{A} + \operatorname{div}(u^\lambda \boldsymbol{A}) \quad \text{on} \ \ \Omega, \tag{3.5}$$

in the sense of Radon measures. Therefore,

$$\begin{aligned} BV^{\boldsymbol{A},\lambda}(\Omega) &= \Big\{ u \in X^{\boldsymbol{A},\lambda}(\Omega) : \operatorname{div}(u^\lambda \boldsymbol{A}) \in \mathcal{M}(\Omega) \Big\} \,, \\ BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) &= \Big\{ u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) : \operatorname{div}(u^\lambda \boldsymbol{A}) \in \mathcal{M}_{\mathrm{loc}}(\Omega) \Big\} \,. \end{aligned}$$

(2) *If $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$, then for all $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ we have $\operatorname{div}(u^\lambda \boldsymbol{A}) = \operatorname{div}(u \boldsymbol{A})$ in the sense of distributions. Hence, $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ if and only if $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ and $\operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)$, and, for all $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, we get*

$$(\boldsymbol{A}, Du)_\lambda = -u^\lambda \operatorname{div} \boldsymbol{A} + \operatorname{div}(u \boldsymbol{A}) \quad \text{on} \ \ \Omega, \tag{3.6}$$

in the sense of Radon measure. Therefore,

$$\begin{aligned} BV^{\boldsymbol{A},\lambda}(\Omega) &= \Big\{ u \in X^{\boldsymbol{A},\lambda}(\Omega) : \operatorname{div}(u \boldsymbol{A}) \in \mathcal{M}(\Omega) \Big\} \,, \\ BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) &= \Big\{ u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) : \operatorname{div}(u \boldsymbol{A}) \in \mathcal{M}_{\mathrm{loc}}(\Omega) \Big\} \,. \end{aligned}$$

(3) *If* $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$*, then we have*
$$BV(\Omega) \cap BV^{\boldsymbol{A},\lambda}(\Omega) = BV(\Omega) \cap X^{\boldsymbol{A},\lambda}(\Omega) = \{u \in BV(\Omega) : u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)\},$$
and, if $u \in BV(\Omega)$ *and* $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$*, then the* λ*-pairing* $(\boldsymbol{A}, Du)_\lambda$ *coincides with the one defined in* [19]*.*

(4) *If* $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$*, then for every two Borel functions* $\lambda_1, \lambda_2 \colon \Omega \to [0,1]$*, we have*
$$BV^{\boldsymbol{A},\lambda_1}(\Omega) \cap L^\infty(\Omega) = BV^{\boldsymbol{A},\lambda_2}(\Omega) \cap L^\infty(\Omega).$$
In particular, if $u \in BV^{\boldsymbol{A},\lambda_1}(\Omega) \cap L^\infty(\Omega)$*, then*
$$(\boldsymbol{A}, Du)_{\lambda_1} - (\boldsymbol{A}, Du)_{\lambda_2} = (u^{\lambda_2} - u^{\lambda_1}) \operatorname{div} \boldsymbol{A} \llcorner S_u^* = (\lambda_2 - \lambda_1)(u^+ - u^-) \operatorname{div} \boldsymbol{A} \llcorner S_u^*.$$

Proof. In order to prove (1), we observe that, given $\boldsymbol{A} \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and $v \in L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|)$, we clearly have
$$\langle \operatorname{div}(v\boldsymbol{A}), \varphi \rangle = -\int_\Omega v \nabla\varphi \cdot d\boldsymbol{A} \quad \text{for} \quad \varphi \in C_c^\infty(\Omega)$$
in the sense of distributions. Hence, for $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ (3.1) can be rewritten as (3.4). Given that both distributions are clearly of order 1, we can test them against functions in $C^1_c(\Omega)$. Therefore, (3.5) follows as long as either $(\boldsymbol{A}, Du)_\lambda$ or $\operatorname{div}(u^\lambda \boldsymbol{A})$ are Radon measures. This concludes the proof of point (1).

Point (2) follows immediately from point (1) by noticing that, if $\boldsymbol{A} \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, then $u^\lambda(x) = u(x)$ for $|\boldsymbol{A}|\mathscr{L}^N$-a.e. $x \in \Omega$.

As for point (3), we start by observing that Definition 3.1 extends the standard definition of λ-pairing [19, Definition 4.1], given that the equation satisfied by $(\boldsymbol{A}, Du)_\lambda$ is the same in both cases. Now, if $u \in BV(\Omega)$, in particular u is a class of Lebesgue measurable functions, and therefore, as an easy consequence of Lusin's Theorem, it admits a Borel representative. Therefore, we see that $u \in \mathscr{B}(\Omega)$. Then, we recall that, if $u \in BV(\Omega)$, we have $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ (for instance, see [2, Theorem 3.78]) and so $|\operatorname{div} \boldsymbol{A}|(S_u \setminus J_u) = 0$, since $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{H}^{N-1}$ for $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, see Section 2.3. This means that $u^*(x) = u^{\frac{1}{2}}(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$ (see Section 2.2). Moreover, as proven in [19, Lemma 3.2] for every $u \in BV(\Omega)$ and for every two Borel functions $\lambda_1, \lambda_2 \colon \Omega \to [0,1]$, $u^{\lambda_1} \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$ if and only if $u^{\lambda_2} \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Hence, if $u \in BV(\Omega) \cap X^{\boldsymbol{A},\lambda}(\Omega)$ for some Borel function $\lambda : \Omega \to [0,1]$, then we get $u^{\frac{1}{2}} \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, which allows to conclude that $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. This proves the inclusions
$$BV(\Omega) \cap BV^{\boldsymbol{A},\lambda}(\Omega) \subseteq BV(\Omega) \cap X^{\boldsymbol{A},\lambda}(\Omega) \subseteq \{u \in BV(\Omega) : u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)\}.$$
As for the second inclusion, the opposite one can be proved by reversing this argument. Then, we notice that [19, Lemma 3.2 and Proposition 4.4] imply that, if $u \in BV(\Omega)$ is such that $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$, then $(\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega)$: this proves the remaining opposite inclusion.

Finally, in dealing with point (4) we start by observing that, if $u \in L^\infty(\Omega)$, we have
$$u^{\lambda_1}, u^{\lambda_2} \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|).$$
Hence, $u \in X^{\boldsymbol{A},\lambda_1}(\Omega) \cap X^{\boldsymbol{A},\lambda_2}(\Omega)$. Then, we notice that
$$u^{\lambda_1}\boldsymbol{A} = u^{\lambda_2}\boldsymbol{A} = u\boldsymbol{A} \text{ up to a } \mathscr{L}^N\text{-negligible set},$$
so that, if $u \in BV^{\boldsymbol{A},\lambda_1}(\Omega) \cap L^\infty(\Omega)$, then point (2) implies $\operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)$, and thus $u \in BV^{\boldsymbol{A},\lambda_2}(\Omega)$. The opposite inclusion follows by exchanging the roles of λ_1 and λ_2. Then, if

$u \in BV^{\boldsymbol{A},\lambda_1}(\Omega) \cap L^\infty(\Omega)$, we know that it also belongs to $BV^{\boldsymbol{A},\lambda_2}(\Omega) \cap L^\infty(\Omega)$, so that we obtain (3.6) in both cases, and we subtract the equation with $\lambda = \lambda_2$ from the one with $\lambda = \lambda_1$. Thus, it is enough to exploit (2.11) and the fact that $Z_u = \emptyset$ to end the proof. □

Remark 3.6. *We point out that the "if and only if" condition in point (1) of Proposition 3.5 is actually equivalent to saying that, given $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$, $\lambda : \Omega \to [0,1]$ Borel and $u \in X^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$, we have*

$$u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) \iff u^\lambda \boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega).$$

Analogously, the necessary and sufficient condition in point (2), for which we have $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$, can be rewritten as

$$u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) \iff u\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega).$$

We stress the fact that no equivalent statement holds for higher summability of the field $\boldsymbol{A}$, since $u \in X^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$ only implies that $u\boldsymbol{A} \in L^1_{\rm loc}(\Omega;\mathbb{R}^N)$.

Remark 3.7. *We point out that, if $u \in L^1_{\rm loc}(\Omega)$, we could define the general λ-pairings by exploiting the representatives of u given by the limit of averages on balls and half-balls; that is,* (2.1) *and* (2.6)*, respectively. This is indeed the approach followed in the classical setting of* [5] *for $\lambda \equiv \frac{1}{2}$, and for a general Borel function $\lambda : \Omega \to [0,1]$ in* [19]*: therein, the field $\boldsymbol{A}$ is $\mathcal{DM}^\infty$ and the scalar function u is BV, and therefore it is well known that the function*

$$u^\lambda(x) = \begin{cases} \tilde{u}(x)\,, & \text{if } x \in \Omega \setminus S_u\,, \\ \\ (1-\lambda(x))\min\{u^i(x), u^e(x)\} + \lambda(x)\max\{u^i(x), u^e(x)\}\,, & \text{if } x \in J_u, \end{cases} \tag{3.7}$$

is well defined $|\operatorname{div}\boldsymbol{A}|$-a.e. (see the argument in the proof of point (3) in Proposition 3.5). We point out that, if $u \in BV(\Omega)$, then $\mathscr{H}^{N-1}(Z_u) \le \mathscr{H}^{N-1}(S_u \setminus J_u) = 0$, so that our definition (2.10) *coincides with* (3.7) *up to an $\mathscr{H}^{N-1}$-negligible set, due to* (2.5) *and* (2.7)*. However, as soon as $u \notin BV_{\rm loc}(\Omega)$ or $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega) \setminus L^\infty_{\rm loc}(\Omega;\mathbb{R}^N)$, in general the function u^λ given by* (3.7) *is not well defined $|\boldsymbol{A}|$-a.e. or $|\operatorname{div}\boldsymbol{A}|$-a.e., since we have no control, a priori, on the set $S_u \setminus J_u$, except that $\mathscr{L}^N(S_u \setminus J_u) = 0$, obviously. An alternative possible approach would be to consider functions $u \in L^1_{\rm loc}(\Omega)$ such that*

$$|\boldsymbol{A}^s|(S_u \setminus J_u) = |\operatorname{div}^s \boldsymbol{A}|(S_u \setminus J_u) = 0 \tag{3.8}$$

and $u^\lambda \in L^1_{\rm loc}(\Omega, |\boldsymbol{A}|) \cap L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$. Under these assumptions, the λ-pairing can be defined as done in (3.1)*, and therefore we can provide alternative definitions of $X^{\boldsymbol{A},\lambda}(\Omega)$ and $BV^{\boldsymbol{A},\lambda}(\Omega)$ as the classes*

$$\widetilde{X}^{\boldsymbol{A},\lambda}(\Omega) := \Big\{u \in L^1(\Omega) : (|\boldsymbol{A}^s| + |\operatorname{div}^s \boldsymbol{A}|)(S_u \setminus J_u) = 0, u^\lambda \in L^1(\Omega, |\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)\Big\},$$

$$\widetilde{BV}^{\boldsymbol{A},\lambda}(\Omega) := \Big\{u \in \widetilde{X}^{\boldsymbol{A},\lambda}(\Omega) : (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega)\Big\}$$

and analogously for the corresponding local versions. Due to the additional condition on $S_u \setminus J_u$, there are no obvious inclusions between $X^{\boldsymbol{A},\lambda}(\Omega)$ and $BV^{\boldsymbol{A},\lambda}(\Omega)$ and these alternative versions, unless $(|\boldsymbol{A}| + |\operatorname{div}\boldsymbol{A}|) \ll \mathscr{L}^N$, in which case $\widetilde{X}^{\boldsymbol{A},\lambda}(\Omega) \subseteq X^{\boldsymbol{A},\lambda}(\Omega)$ and $\widetilde{BV}^{\boldsymbol{A},\lambda}(\Omega) \subseteq BV^{\boldsymbol{A},\lambda}(\Omega)$. Furthermore, in the particular case $\lambda \equiv \frac{1}{2}$, we could completely identify $u^{\frac{1}{2}}$ with the precise representative u^*, in such way removing the problematic related to the set $S_u \setminus J_u$. Therefore, we *could drop* (3.8)*, and replace it with the requirement that the limit* (2.8) *exists for* $(|\boldsymbol{A}|+|\operatorname{div}\boldsymbol{A}|)$-*a.e.* $x \in \Omega$*. This is indeed coherent with the fact that the pointwise limit of any mollification* $(u*\rho_\varepsilon)(x)$ *is* $u^*(x)$*, whenever the precise representative is well defined (see point (1) in Theorem 4.4 below). This approach takes also in account the possibility of having a precise representative which does not necessarily coincide with* $\frac{u^+(x)+u^-(x)}{2}$ *for* $x \in S_u \setminus J_u$*: consider for instance* $u = \chi_{(0,1)^N}$ *and* $x = 0$*.*

Nevertheless, such an analysis would be out the scope of the present paper, so we preferred to exploit the fact that the approximate liminf and limsup $u^\pm$ *are always well defined, in order to circumvent all these difficulties involving the well-posedness of the representatives and to avoid any assumption on the set* $S_u \setminus J_u$*. Furthermore, in the classical setting there are no substantial differences.*

Remark 3.8. *Point (4) in Proposition 3.5 does not hold if* $\boldsymbol{A} \in \mathcal{DM}(\Omega)\setminus\mathcal{DM}^1_{\mathrm{loc}}(\Omega)$*, in general. Indeed, let us consider* $N \geq 2$*,* $\Omega = (-1,1)^N$*,* $\boldsymbol{A} = \big(\mathscr{H}^1 \llcorner J, 0, \dots, 0\big)$*, where* J *is the segment*

$$J = \{x \in \Omega : x_j = 0 \textit{ for all } j = 2, \dots, N\} = \{x \in \Omega : x = (t, 0, \dots, 0) \textit{ for } t \in (-1,1)\},$$

so that $\operatorname{div}\boldsymbol{A} = 0$ *and* $\boldsymbol{A} \in \mathcal{DM}(\Omega) \setminus \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$*. Then, we choose* $u = \chi_{(-1,1)\times(0,1)^{N-1}}$*: it is clear that* $u \in L^\infty(\Omega)$ *and*

$$u^-(x) = 0 \quad \textit{and} \quad u^+(x) = 1 \quad \textit{for all } x \in J.$$

In particular, we see that $u \in BV^{\boldsymbol{A},0}(\Omega)$*. Indeed,* $u^0(x) = u^-(x) = 0$ *for* $|\boldsymbol{A}|$*-a.e.* $x \in \Omega$*, so that* $u^- \in L^1(\Omega, |\boldsymbol{A}|)$*, obviously* $u^- \in L^1(\Omega, |\operatorname{div}\boldsymbol{A}|)$*, and, thanks to* (3.5)*,*

$$(\boldsymbol{A}, Du)_0 = -u^- \operatorname{div}\boldsymbol{A} + \operatorname{div}(u^-\boldsymbol{A}) = 0.$$

Let now $\overline{\lambda}(x) = \chi_F(x_1)$*, for some Borel set* $F \subset (-1,1)$ *such that* $\chi_F \notin BV_{\mathrm{loc}}((-1,1))$ *(for instance,* F *could be any fat Cantor set). Then we have*

$$u^{\overline{\lambda}}(x) = (1-\overline{\lambda}(x))u^-(x) + \overline{\lambda}(x)u^+(x) = \chi_F(x_1) \quad \textit{for all } x \in J.$$

Hence, $u \in X^{\boldsymbol{A},\overline{\lambda}}(\Omega)$*, and for all* $\varphi \in C^1_c(\Omega)$ *we get*

$$\begin{aligned} \big\langle (\boldsymbol{A}, Du)_{\overline{\lambda}}, \varphi \big\rangle &= -\int_\Omega \varphi u^{\overline{\lambda}} \, d\operatorname{div}\boldsymbol{A} - \int_\Omega u^{\overline{\lambda}} \nabla\varphi \cdot d\boldsymbol{A} = -\int_J \chi_F(x_1) \frac{\partial\varphi(x)}{\partial x_1} \, d\mathscr{H}^1 \\ &= -\int_{-1}^{-1} \chi_F(x_1) \frac{\partial\varphi(x)}{\partial x_1} \, dx_1. \end{aligned}$$

Therefore, $(\boldsymbol{A}, Du)_{\overline{\lambda}} \notin \mathcal{M}_{\mathrm{loc}}(\Omega)$*, given that* $\chi_F \notin BV_{\mathrm{loc}}((-1,1))$*, and so* $u \notin BV^{\boldsymbol{A},\overline{\lambda}}_{\mathrm{loc}}(\Omega)$*. Thus, we conclude that*

$$BV^{\boldsymbol{A},0}(\Omega) \cap L^\infty(\Omega) \neq BV^{\boldsymbol{A},\overline{\lambda}}(\Omega) \cap L^\infty(\Omega).$$

As a consequence of the Leibniz-type rule (3.5), we obtain an integration by parts formula for $BV^{\boldsymbol{A},\lambda}$ functions with compact support.

Lemma 3.9. *Let* $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$*,* $\lambda : \Omega \to [0,1]$ *be a Borel function and* $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ *be such that* $\operatorname{supp}(u) \Subset \Omega$*. Then we have*

$$\int_\Omega u^\lambda \, d\operatorname{div}\boldsymbol{A} = -\int_\Omega d(\boldsymbol{A}, Du)_\lambda \,. \tag{3.9}$$

Proof. Since $\mathrm{supp}(u) \Subset \Omega$, it is clear that $\mathrm{supp}(u^\lambda |\boldsymbol{A}|) \Subset \Omega$, and therefore, since $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$, we conclude that $u^\lambda \boldsymbol{A} \in \mathcal{DM}(\Omega)$. Hence, thanks to Lemma 2.4, we see that $\mathrm{div}(u^\lambda \boldsymbol{A})(\Omega) = 0$. Thus, (3.9) follows by evaluating (3.5) over Ω. □

An immediate consequence of point (3) Proposition 3.5 is that, if $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, then

$$\{u \in BV(\Omega) : u^* \in L^1(\Omega, |\,\mathrm{div}\,\boldsymbol{A}|)\} \subseteq BV^{\boldsymbol{A},\lambda}(\Omega) \tag{3.10}$$

for all Borel functions $\lambda : \Omega \to [0, 1]$. In general, the inclusion is strict, as shown in the following remark.

Remark 3.10. *Let $N \geq 2$ and $\lambda : \Omega \to [0, 1]$ be a Borel function. There exist a field $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and a function $v \in BV^{\boldsymbol{A},\lambda}(\Omega)$ such that $v \notin BV_{\mathrm{loc}}(\Omega)$. Let us consider*

$$\boldsymbol{A}(x) := (a_1(\hat{x}_1), a_2(\hat{x}_2), \dots, a_{N-1}(\hat{x}_{N-1}), 0)\,,$$

where

$$\hat{x}_j := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N),\ a_j \in L^\infty(\Omega_j) \ \text{ and } \ \Omega_j = \{y \in \mathbb{R}^{N-1} : y = \hat{x}_j \text{ for some } x \in \Omega\}$$

for every $j \in \{1, 2, \dots, N-1\}$. By construction, $\mathrm{div}\,\boldsymbol{A} = 0$. Then, thanks to (3.6), a function $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ belongs to $BV^{\boldsymbol{A},\lambda}(\Omega)$ if and only if $(\boldsymbol{A}, Du)_\lambda = \mathrm{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)$.

Now, let $y \in \Omega$, $r > 0$ be such that $B_{2r}(y) \subset \Omega$ and $\eta_{r,y} \in C_c^\infty(B_{2r}(y))$ be such that $\eta_{r,y} \equiv 1$ on $B_r(y)$. Then we set

$$v(x) := \log(|x_N - y_N|)\eta_{r,y}(x).$$

It is clear that $v \in L^1(\Omega)$, which implies $v \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$, since $|\boldsymbol{A}| \in L^\infty(\Omega)$, while trivially $v^\lambda \in L^1(\Omega, |\,\mathrm{div}\,\boldsymbol{A}|)$, being $\mathrm{div}\,\boldsymbol{A} = 0$. Hence, $v \in X^{\boldsymbol{A},\lambda}(\Omega)$. On the other hand, $v \notin BV_{\mathrm{loc}}(\Omega)$ given that

$$D_{x_N} v = \mathrm{p.v.}\frac{1}{((\cdot)_N - y_N)} \quad \text{on } B_r(y),$$

so that the distributional gradient of v cannot be a vector valued Radon measure on Ω. However, we have

$$\mathrm{div}(v\boldsymbol{A}) = \sum_{j=1}^{N-1} D_j(va_j) = \left(\log(|(\cdot)_N - y_N|)a_j \sum_{j=1}^{N-1} \partial_{x_j}\eta_{r,y}\right)\mathscr{L}^N$$

being va_j not depending on x_j, for every $j \in \{1, 2, \dots, N-1\}$. Thus, as remarked above,

$$(\boldsymbol{A}, Dv)_\lambda = \mathrm{div}(v\boldsymbol{A}) \in \mathcal{M}(\Omega),$$

so that $v \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$.

In addition, we point out that, if $\boldsymbol{A} \in \mathcal{DM}(\Omega) \setminus \mathcal{DM}^\infty(\Omega)$, then the inclusion (3.10) may fail to hold true for every λ. We provide now an example of such a case.

Example 3.11. *Let $N = 2$, $\Omega = (-1, 1)^2$,*

$$\boldsymbol{A}(x_1, x_2) = \frac{(-x_2, x_1)}{x_1^2 + x_2^2}$$

and $u = \chi_E$, where $E = (-1, 1) \times (-1, 0)$. It is clear that $\boldsymbol{A} \in \mathcal{DM}^p(\Omega)$ for all $p \in [1, 2)$, $u \in BV(\Omega)$ and $u^\lambda \in L^1(\Omega, |\,\mathrm{div}\,\boldsymbol{A}|)$ trivially, since $\mathrm{div}\,\boldsymbol{A} = 0$. In particular, we easily deduce *that* $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ *for all Borel functions* $\lambda : \Omega \to [0,1]$*. However, as it was showed in* [9, Remark 4.8] *(see also* [42, Example 2.5]*), we have*

$$\operatorname{div}(u\boldsymbol{A}) = \text{p.v.}\left(\frac{1}{x_1}\right)\mathscr{H}^1 \llcorner (-1,1) \times \{0\},$$

where p.v. *stands for principal value integral. Hence,* $\operatorname{div}(u\boldsymbol{A})$ *is not a Radon measure, and therefore, by Proposition 3.5, this means that* $u \notin BV^{\boldsymbol{A},\lambda}(\Omega)$ *for any Borel function* $\lambda : \Omega \to [0,1]$*.*

Intuitively, the cases $\lambda \equiv 0$ and $\lambda \equiv 1$ represent the extreme choices for λ, and therefore it is natural to expect that the λ-pairing is a convex combination of the 0-pairing and the 1-pairing. We prove this idea in the following lemma. To this purpose, we notice that, if $u \in BV^{\boldsymbol{A},0}(\Omega)$ or $u \in BV^{\boldsymbol{A},1}(\Omega)$, then by (3.5) we have

$$(\boldsymbol{A}, Du)_0 = -u^- \operatorname{div} \boldsymbol{A} + \operatorname{div}(u^-\boldsymbol{A}) \quad \text{or} \quad (\boldsymbol{A}, Du)_1 = -u^+ \operatorname{div} \boldsymbol{A} + \operatorname{div}(u^+\boldsymbol{A}), \tag{3.11}$$

respectively.

Lemma 3.12. *Let* $\boldsymbol{A} \in \mathcal{DM}_{\text{loc}}(\Omega)$*.*

(i) $u \in X^{\boldsymbol{A},0}_{\text{loc}}(\Omega) \cap X^{\boldsymbol{A},1}_{\text{loc}}(\Omega)$ *if and only if* $u \in X^{\boldsymbol{A},\lambda}_{\text{loc}}(\Omega)$ *for every Borel function* $\lambda : \Omega \to [0,1]$*.*

(ii) If $u \in BV^{\boldsymbol{A},0}_{\text{loc}}(\Omega) \cap BV^{\boldsymbol{A},1}_{\text{loc}}(\Omega)$ *and furthermore* $\boldsymbol{A} \in \mathcal{DM}^1_{\text{loc}}(\Omega)$*, then we have* $u \in BV^{\boldsymbol{A},\lambda}_{\text{loc}}(\Omega)$ *for every Borel function* $\lambda : \Omega \to [0,1]$*, and it holds*

$$(\boldsymbol{A}, Du)_\lambda = (1-\lambda)(\boldsymbol{A}, Du)_0 + \lambda(\boldsymbol{A}, Du)_1 \quad \text{on } \Omega. \tag{3.12}$$

(iii) If $u \in BV^{\boldsymbol{A},0}_{\text{loc}}(\Omega) \cap BV^{\boldsymbol{A},1}_{\text{loc}}(\Omega)$*, then we have* $u \in BV^{\boldsymbol{A},t}_{\text{loc}}(\Omega)$ *for every* $t \in [0,1]$*, and* (3.12) *holds for* $\lambda \equiv t$*.*

Proof. (i) If $u \in X^{\boldsymbol{A},0}_{\text{loc}}(\Omega) \cap X^{\boldsymbol{A},1}_{\text{loc}}(\Omega)$, by the boundedness of λ we have

$$\lambda u^+, (1-\lambda)u^- \in L^1_{\text{loc}}(\Omega, |\boldsymbol{A}|) \cap L^1_{\text{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|) \tag{3.13}$$

and so

$$u^\lambda \in L^1_{\text{loc}}(\Omega, |\boldsymbol{A}|) \cap L^1_{\text{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|).$$

The opposite implication is trivial.

(ii) If $u \in BV^{\boldsymbol{A},0}_{\text{loc}}(\Omega) \cap BV^{\boldsymbol{A},1}_{\text{loc}}(\Omega)$, by (i) we have (3.13) and $u \in X^{\boldsymbol{A},\lambda}_{\text{loc}}(\Omega)$ for every Borel function $\lambda : \Omega \to [0,1]$. Therefore, since $\boldsymbol{A} \in \mathcal{DM}^1_{\text{loc}}(\Omega)$, by (3.6) and (3.11) we obtain

$$\begin{aligned}
(\boldsymbol{A}, Du)_\lambda &= -u^\lambda \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) \\
&= -(1-\lambda)u^- \operatorname{div} \boldsymbol{A} - \lambda u^+ \operatorname{div} \boldsymbol{A} + (1-\lambda) \operatorname{div}(u\boldsymbol{A}) + \lambda \operatorname{div}(u\boldsymbol{A}) \\
&= (1-\lambda)\left(-u^- \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A})\right) + \lambda\left(-u^+ \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A})\right) \\
&= (1-\lambda)(\boldsymbol{A}, Du)_0 + \lambda(\boldsymbol{A}, Du)_1 \quad \text{on } \Omega.
\end{aligned}$$

Hence, the conclusion follows, since $(\boldsymbol{A}, Du)_\lambda \in \mathcal{M}_{\text{loc}}(\Omega)$ if $(\boldsymbol{A}, Du)_0, (\boldsymbol{A}, Du)_1 \in \mathcal{M}_{\text{loc}}(\Omega)$.

(iii) Thanks to (i) with $\lambda \equiv t$ for $t \in [0,1]$, we have $u \in X^{\boldsymbol{A},t}_{\text{loc}}(\Omega)$. Then, we argue as in the proof of point (ii), this time exploiting (3.5) and noticing that, for $\lambda \equiv t$, we have $\operatorname{div}(u^\lambda \boldsymbol{A}) = (1-\lambda)\operatorname{div}(u^-\boldsymbol{A}) + \lambda \operatorname{div}(u^+\boldsymbol{A})$, since λ is constant.

□

Remark 3.13. *We notice that in point (ii) of Lemma 3.12 we cannot drop the assumption $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$. Indeed, if we choose $\boldsymbol{A} \in \mathcal{DM}(\Omega) \setminus \mathcal{DM}^1_{\rm loc}(\Omega)$ and u as in Remark 3.8, we know that $u \in BV^{\boldsymbol{A},0}(\Omega)$, with $(\boldsymbol{A}, Du)_0 = 0$, and it is not difficult to check that actually $u \in BV^{\boldsymbol{A},0}(\Omega) \cap BV^{\boldsymbol{A},1}(\Omega)$, with*

$$(\boldsymbol{A}, Du)_1 = \operatorname{div}(u^+ \boldsymbol{A}) = \operatorname{div} \boldsymbol{A} = 0.$$

However, whenever $\overline{\lambda}(x) = \chi_F(x_1)$, for some Borel set $F \subset (-1, 1)$ such that $\chi_F \notin BV((-1, 1))$, Remark 3.8 shows that $(\boldsymbol{A}, Du)_{\overline{\lambda}}$ is not a Radon measure, and so $u \notin BV^{\boldsymbol{A},\overline{\lambda}}_{\rm loc}(\Omega)$.

Instead, point (iii) of Lemma 3.12 ensures that, under the general assumption $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$, for all $t \in [0, 1]$ the t-pairing is the convex combination of the 0-pairing and the 1-pairing. In particular, if $u \in BV^{\boldsymbol{A},0}_{\rm loc}(\Omega) \cap BV^{\boldsymbol{A},1}_{\rm loc}(\Omega)$, then $u \in BV^{\boldsymbol{A}}_{\rm loc}(\Omega) = BV^{\boldsymbol{A},\frac{1}{2}}_{\rm loc}(\Omega)$, with

$$(\boldsymbol{A}, Du) = \frac{(\boldsymbol{A}, Du)_0 + (\boldsymbol{A}, Du)_1}{2} \quad \text{on } \Omega.$$

As recalled in Section 2.3, the pairing measure between a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega)$ with $u^* \in L^1_{\rm loc}(\Omega, |\operatorname{div} \boldsymbol{A}|)$ enjoys the quite natural absolute continuity property given by (2.21). With the following proposition, we analize the absolute continuity of the general λ-pairing, in the case of an essentially bounded divergence-measure field.

Proposition 3.14. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, $\lambda : \Omega \to [0, 1]$ be a Borel function and $\Omega' \Subset \Omega$ be an open set.*

(i) If $u \in BV_{\rm loc}(\Omega)$ is such that $u^* \in L^1_{\rm loc}(\Omega, |\operatorname{div} \boldsymbol{A}|)$, then we have

$$|(\boldsymbol{A}, Du)_\lambda| \le \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} |Du| \quad \text{on } \Omega'.$$

(ii) If $u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$, then we have

$$|(\boldsymbol{A}, Du)_\lambda| \le 2c_N \|u\|_{L^\infty(\Omega')} \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} \mathscr{H}^{N-1} \quad \text{on } \Omega',$$

where

$$c_N = N \left(\frac{2N}{N+1} \right)^{\frac{N-1}{2}} \frac{\omega_N}{\omega_{N-1}}. \tag{3.14}$$

In addition, if $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ in (i), and $u \in BV^{\boldsymbol{A},\lambda}(\Omega) \cap L^\infty(\Omega)$ in (ii), then the respective statements hold true with Ω instead of Ω'.

Proof. Property (i) follows from point (3) of Proposition 3.5 applied to Ω' and (2.23) (see also [19, Proposition 4.4]). In order to prove (ii) it suffices to note that, if $u \in L^\infty(\Omega')$, then $u\boldsymbol{A} \in L^\infty(\Omega'; \mathbb{R}^N)$. By equation (3.6) and by point (2) of Proposition 3.5, we deduce that $\operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega')$, and so $u\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega')$. Hence, thanks to [43, Proposition 3.1], we get

$$|\operatorname{div} \boldsymbol{A}| \le c_N \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} \mathscr{H}^{N-1} \text{ on } \Omega' \text{ and } |\operatorname{div}(u\boldsymbol{A})| \le c_N \|u\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} \mathscr{H}^{N-1} \text{ on } \Omega',$$

where c_N is as in (3.14). Thanks to (3.6) and the fact that $|u^\lambda| \le \|u\|_{L^\infty(\Omega')}$ on Ω', the conclusion follows. Finally, it is clear that, under global assumptions, we can simply repeat on Ω the argument above. □

Arguing analogously, we can characterize the absolute continuity properties of the pairing measure in the case $\boldsymbol{A} \in \mathcal{DM}^p_{\rm loc}(\Omega)$ for $p \in [1, +\infty]$.

Proposition 3.15. *Let $p, q \in [1, +\infty]$ be conjugate exponents; that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ and $\lambda : \Omega \to [0, 1]$ be a Borel function.*

(i) If $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$, then

$$(\boldsymbol{A}, Du)_\lambda = \boldsymbol{A} \cdot \nabla u \, \mathscr{L}^N.$$

(ii) If $N \geq 2$, $p \in \left[\frac{N}{N-1}, +\infty\right)$ and $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, then $|(\boldsymbol{A}, Du)_\lambda|(B) = 0$ for every Borel set $B \subset \Omega$ which is σ-finite with respect to the measure $\mathscr{H}^{N-q}$.

Proof. Assertion (i) follows from the Leibniz rules recalled in Section 2.3, see (2.20) and the comments afterwards. Indeed, it is clear that $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$. In addition, if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, we know that either u admits a continuous representative and so S_u is empty for $q > N$ (by Morrey's inequality), or $u^*(x) = \tilde{u}(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$, by [12, Theorem 1.1.24 and Theorem 3.2.2]. All in all, this implies that $|\operatorname{div} \boldsymbol{A}|(S_u) = 0$, and so $u^\lambda(x) = \tilde{u}(x) = u^*(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$. Therefore, we exploit (2.20) and (3.6) to obtain

$$\boldsymbol{A} \cdot \nabla u \, \mathscr{L}^N = \operatorname{div}(u\boldsymbol{A}) - u^* \operatorname{div} \boldsymbol{A} = \operatorname{div}(u\boldsymbol{A}) - u^\lambda \operatorname{div} \boldsymbol{A} = (\boldsymbol{A}, Du)_\lambda$$

As for (ii), we notice that $u\boldsymbol{A} \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Since $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, then $(\boldsymbol{A}, Du)_\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and so, thanks to (3.6), we conclude that $u\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$. Hence, thanks to the absolute continuity properties recalled in Section 2.3 (see also [41, Theorem 3.2]), we exploit again (3.6) to conclude. □

Remark 3.16. *In case (ii) of Proposition 3.15, we can exploit* [36, Theorem 2.8] *to conclude that $(A, Du)_\lambda$ vanishes on Borel sets with zero q-Sobolev capacity, and the proof is completely analogous.*

Remark 3.17. *As for the subcritical case $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ for $p \in \left[1, \frac{N}{N-1}\right)$, we point out that we cannot expect any absolute continuity property for the λ-pairing. Indeed, we provide an example of a field $\boldsymbol{A}$, a Borel function $\lambda : \Omega \to [0, 1]$ and a function $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ such that $(\boldsymbol{A}, Du)_\lambda$ involves a Dirac delta. Let $N \geq 2$, $\Omega = \mathbb{R}^N$ and*

$$\boldsymbol{A}(x) = \frac{1}{N\omega_N} \frac{x}{|x|^N},$$

where $\omega_N = \mathscr{L}^N(B_1)$, so that $\mathscr{H}^{N-1}(\partial B_1) = N\omega_N$, in particular. Clearly, $\boldsymbol{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ for all $p \in \left[1, \frac{N}{N-1}\right)$, and, by a standard calculation, we see that $\operatorname{div} \boldsymbol{A} = \delta_0$, which is the Dirac's delta measure centered in the origin . We choose $u = \chi_{(0,1)^N}$. Arguing as it was done in [9, Example 3.1] *for the special case $N = 2$, we can show that $\operatorname{div}(\chi_{(0,1)^N} \boldsymbol{A}) \in \mathcal{M}(\mathbb{R}^N)$ and*

$$\operatorname{div}(\chi_{(0,1)^N} \boldsymbol{A}) = \frac{1}{2^N} \delta_0 + \overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}, \tag{3.15}$$

where $\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}$ *is the measure acting as*

$$\int_{\mathbb{R}^N} \varphi\, d\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})} = \frac{1}{N\omega_N} \int_{\partial(0,1)^N \cap \{x_j>0,\ \forall j \in \{1,\dots,N\}\}} \varphi(x) \frac{x \cdot \nu_{(0,1)^N}(x)}{|x|^N} \, d\mathscr{H}^{N-1}(x)$$

$$= -\frac{1}{N\omega_N} \sum_{j=1}^{N} \int_{\partial(0,1)^N \cap \{x_j=1\}} \varphi(x) \frac{1}{(1+|\hat{x}_j|^2)^{\frac{N}{2}}} \, d\mathscr{H}^{N-1}(x) \tag{3.16}$$

for all $\varphi \in C_c(\mathbb{R}^N)$*, where* $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$*. Thanks to point (2) in Proposition 3.5, this fact implies that* $\chi_{(0,1)^N} \in BV^{\boldsymbol{A},\lambda}(\mathbb{R}^N)$ *for all Borel functions* $\lambda : \mathbb{R}^N \to [0,1]$*, with*

$$(\boldsymbol{A}, D\chi_{(0,1)^N})_\lambda = \left(\frac{1}{2^N} - \lambda(0)\right) \delta_0 + \overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}. \tag{3.17}$$

Hence, as long as $\lambda(0) \neq 2^{-N}$*, the measure* $(\boldsymbol{A}, D\chi_{(0,1)^N})_\lambda$ *cannot be absolutely continuous with respect to* $\mathscr{H}^\alpha$ *for all* $\alpha \in (0, N]$*. In order to prove* (3.15)*, we take* $\varphi \in C^1_c(\mathbb{R}^N)$*, and we notice that* $\boldsymbol{A} \in C^1(\mathbb{R}^N \setminus B_\delta(0))$ *and* $\operatorname{div} \boldsymbol{A} = 0$ *on* $\mathbb{R}^N \setminus B_\delta(0)$ *for all* $\delta > 0$*. We set* $H^+ = \{x_j > 0,\ \forall j \in \{1, \dots, N\}\}$ *and* $Q = (0,1)^N$ *for brevity. Hence, we can integrate by parts in the following way:*

$$\int_{\mathbb{R}^N} \chi_Q \boldsymbol{A} \cdot \nabla\varphi \, dx = \lim_{\varepsilon \to 0^+} \int_{Q \setminus B_\varepsilon(0)} \boldsymbol{A} \cdot \nabla \varphi \, dx$$
$$= -\lim_{\varepsilon \to 0^+} \Big(\int_{\partial Q \setminus B_\varepsilon(0)} \varphi \boldsymbol{A} \cdot \nu_Q \, d\mathscr{H}^{N-1} + \int_{\partial B_\varepsilon(0) \cap H^+} \varphi \boldsymbol{A} \cdot \nu_{\mathbb{R}^N \setminus B_\varepsilon(0)} \, d\mathscr{H}^{N-1} \Big)$$
$$= -\frac{1}{N\omega_N} \lim_{\varepsilon \to 0^+} \Big(\int_{(\partial Q \setminus B_\varepsilon(0)) \cap H^+} \varphi(x) \frac{x \cdot \nu_Q(x)}{|x|^N} \, d\mathscr{H}^{N-1}(x) + \int_{\partial B_\varepsilon(0) \cap H^+} \frac{\varphi(x)}{|x|^{N-1}} \, d\mathscr{H}^{N-1}(x) \Big)$$
$$= -\frac{1}{N\omega_N} \int_{\partial Q \cap H^+} \varphi(x) \frac{x \cdot \nu_Q(x)}{|x|^N} \, d\mathscr{H}^{N-1}(x) - \frac{1}{N\omega_N} \lim_{\varepsilon \to 0^+} \int_{\partial B_1(0) \cap H^+} \varphi(\varepsilon x) \, d\mathscr{H}^{N-1}(x)$$
$$= -\int_{\mathbb{R}^N} \varphi \, d\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})} - \frac{1}{N\omega_N} \mathscr{H}^{N-1}(\partial B_1(0) \cap H^+) \varphi(0),$$

and this implies (3.15)*, due to the fact that*

$$\mathscr{H}^{N-1}(\partial B_1(0) \cap \{x_j > 0,\ \forall j \in \{1, \dots, N\}\}) = \frac{N\omega_N}{2^N}.$$

All in all, (3.6) *and* (3.15) *imply* (3.17)*.*

Remark 3.18. *If* $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ *and* $u \in \mathrm{Lip}_{\rm loc}(\Omega)$*, then* $u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$ *for all Borel function* $\lambda : \Omega \to [0,1]$*, and*

$$(\boldsymbol{A}, Du)_\lambda = \langle\langle \nabla u, \boldsymbol{A} \rangle\rangle \quad \text{on } \Omega,$$

where $\langle\langle \nabla u, \boldsymbol{A} \rangle\rangle$ *was recalled in* (2.18)*. Indeed, u is continuous, so that*

$$u^\lambda(x) = \tilde{u}(x) = u(x) \quad \text{for all } x \in \Omega.$$

The assertion then follows combining (3.5) *and* (2.18)*. Furthermore, if* $\boldsymbol{A} \in \mathcal{DM}(\Omega)$ *and* $u \in \mathrm{Lip}(\Omega) \cap X^{\boldsymbol{A}}(\Omega)$*, then* $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ *for all Borel function* $\lambda : \Omega \to [0,1]$*.*

Proposition 3.19. *Let $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$ be such that $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$. Then, for every Borel function $\lambda : \Omega \to [0,1]$, we have*

$$BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) = BV^{\boldsymbol{A}}_{\rm loc}(\Omega) = \{u \in \mathscr{B}(\Omega) : \ u \in L^1_{\rm loc}(\Omega, |\boldsymbol{A}| + |\operatorname{div} \boldsymbol{A}|), \operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}_{\rm loc}(\Omega)\} \quad (3.18)$$

and, given $u \in BV^{\boldsymbol{A}}_{\rm loc}(\Omega)$

$$(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du) = -u \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) \quad \text{on } \Omega. \quad (3.19)$$

In particular, the operator $BV^{\boldsymbol{A}}_{\rm loc}(\Omega) \ni u \to (\boldsymbol{A}, Du) \in \mathcal{M}_{\rm loc}(\Omega)$ is linear.

Proof. Thanks to (2.4) and (2.12), we know that, for every $u \in \mathscr{B}(\Omega)$ and every Borel function $\lambda : \Omega \to [0,1]$,

$$|\operatorname{div} \boldsymbol{A}|(S^*_u) = 0, \quad \text{ and } \quad u^\lambda(x) = u^{\frac{1}{2}}(x) = u(x) \quad \text{for } |\operatorname{div} \boldsymbol{A}|\text{-a.e } x \in \Omega,$$

Thus, due to (3.6), we get

$$(\boldsymbol{A}, Du)_\lambda = -u \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) = (\boldsymbol{A}, Du) \quad \text{ for every Borel function } \lambda : \Omega \to [0,1],$$

and this implies (3.18), which in turn easily yields the linearity of the operator $u \to (\boldsymbol{A}, Du)$. □

Remark 3.20. *Due to* (2.11)*, we notice that, under the assumption $|\operatorname{div} \boldsymbol{A}|(S^*_u) = 0$ for a given $u \in \mathscr{B}(\Omega)$, we have $u \in BV^{\boldsymbol{A},\lambda_1}_{\rm loc}(\Omega)$ if and only $u \in BV^{\boldsymbol{A},\lambda_2}_{\rm loc}(\Omega)$ for all Borel functions $\lambda_1, \lambda_2 : \Omega \to [0,1]$, with*

$$(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du) = -\tilde{u} \operatorname{div} \boldsymbol{A} + \operatorname{div}(u\boldsymbol{A}) \quad \text{on } \Omega,$$

for every Borel function $\lambda : \Omega \to [0,1]$, which is (3.19) *with the representative $\tilde{u}$. However, in general we cannot weaken the absolute continuity assumption $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, since the set S^*_u could have Hausdorff dimension equal to N.*

Remark 3.21. *If $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$ satisfies $\operatorname{div} \boldsymbol{A} = 0$, then the results of Proposition 3.19 can be simplified, since the condition $u \in L^1_{\rm loc}(\Omega, |\operatorname{div} \boldsymbol{A}|)$ is always satisfied, so that* (3.18) *is improved to*

$$BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) = BV^{\boldsymbol{A}}_{\rm loc}(\Omega) = \{u \in \mathscr{B}(\Omega) : \ u \in L^1_{\rm loc}(\Omega, |\boldsymbol{A}|), \ \operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}_{\rm loc}(\Omega)\}$$

for every Borel function $\lambda : \Omega \to [0,1]$, and

$$(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A}, Du) = \operatorname{div}(u\boldsymbol{A}). \quad (3.20)$$

As a particular case of divergence-free vector fields, let us choose a fixed direction $\nu \in \mathbb{S}^{N-1}$, and consider the constant field $\boldsymbol{A} := \nu$. Then, by virtue of (3.20)*, for all $u \in BV^\nu_{\rm loc}(\Omega)$ we have*

$$(\boldsymbol{A}, Du) = \operatorname{div}(u\nu) = D_\nu u\,,$$

where $D_\nu u$ is nothing else than the distributional derivative of u in the direction ν. Note that, if $N = 1$, then $\nu \in \{\pm 1\}$, and so, up to a sign, we simply get the first distributional derivative of u: therefore, in the one-dimensional case, if $\boldsymbol{A}$ is a nonzero constant, we obtain $BV^{\boldsymbol{A}}(\Omega) = BV(\Omega)$. If instead $N \geq 2$, by the characterization of the differentiability of the precise representative (see [2, Theorem 3.107]*) we then have*

$$D_\nu u = \mathscr{L}^{N-1} \llcorner \Omega_\nu \otimes Du^\nu_y\,, \quad (3.21)$$

where $\Omega_\nu = \pi_\nu(\Omega)$, $u^\nu_y(t) = u(y + t\nu)$ for $t \in \mathbb{R}$ and $y \in \Omega_\nu$ such that $y + t\nu \in \Omega$. Although [2, Theorem 3.107] *requires $u \in BV(\Omega)$, an inspection to the proof, based on Fubini's Theorem, shows that $D_\nu u \in \mathcal{M}_{\rm loc}(\Omega)$; i.e., the weak differentiability of u in a fixed direction ν, will suffice* *to get* (3.21) *(see also the remarks in* [2, Section 3.11]*). We claim that* (3.21) *and general results on the disintegration of measures (see* [2, Section 2.5]*) imply*

$$|(\nu, Du)| = |D_\nu u| \ll \mathcal{H}^{N-1}\,.$$

Indeed, we first note that, as a consequence of [24, Theorem 28] *applied to π_ν, $\mathcal{H}^{N-1}(B) = 0$ implies $\mathscr{L}^{N-1}(\pi_\nu(B)) = 0$ for every B Borel set, $B \subset \Omega$. Now, for any such set B,*

$$|D_\nu u|(B) = \int_{B_\nu} \left(\int_{B_y^\nu} dDu_y^\nu \right) d\mathscr{L}^{N-1}(y) = 0\,,$$

since $\mathscr{L}^{N-1}(B_\nu) = 0$ and $|Du_y^\nu|(B_y^\nu) < +\infty$ for $\mathscr{L}^{N-1}$-a.e. $y \in B_\nu$.

We end this section with the definition of a “degenerate” Sobolev-type class.
Let $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$ and $\lambda : \Omega \to [0,1]$ be a Borel function. We define

$$W^{\boldsymbol{A},\lambda}(\Omega) := \{u \in BV^{\boldsymbol{A},\lambda}(\Omega) :\ |(\boldsymbol{A}, Du)_\lambda| \ll \mathscr{L}^N\}\,.$$

$$W^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) := \{u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega) :\ |(\boldsymbol{A}, Du)_\lambda| \ll \mathscr{L}^N\}\,.$$

Analogously to the case of $BV^{\boldsymbol{A}}(\Omega)$, we set $W^{\boldsymbol{A}}(\Omega) := W^{\boldsymbol{A},\frac{1}{2}}(\Omega)$, and analogously for the local class.

We point out that the assumption $\boldsymbol{A} \in L^1_{\rm loc}(\Omega; \mathbb{R}^N)$ is not restrictive. Indeed, given $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and $u \in {\rm Lip}_{\rm loc}(\Omega)$, we have only $|(\boldsymbol{A}, Du)| \ll |\boldsymbol{A}|$, thanks to Remark 3.18 and (2.19) (see also [42, Propositions 2.1 and 2.2]), which implies that $(\boldsymbol{A}, Du)_\lambda$ may not be absolutely continuous with respect to the Lebesgue measure even for a regular function u, as long as $\boldsymbol{A}$ is a singular measure.

In addition, we note that, analogously to the inclusion (3.10) for $BV^{\boldsymbol{A},\lambda}$, classical Sobolev functions with suitable summability for their precise representatives belong indeed to $W^{\boldsymbol{A},\lambda}$, as long as $\boldsymbol{A}$ satisfies a natural summability assumption.

Proposition 3.22. *Let $p, q \in [1, +\infty]$ be conjugate exponents; that is, $\frac{1}{p} + \frac{1}{q} = 1$, and $\boldsymbol{A} \in \mathcal{DM}^p(\Omega)$. Then for all Borel functions $\lambda : \Omega \to [0,1]$ we have*

$$W^{1,q}(\Omega) \cap X^{\boldsymbol{A}}(\Omega) \subset W^{\boldsymbol{A},\lambda}(\Omega)$$

with strict inclusion, and $(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A} \cdot \nabla u)\mathscr{L}^N$ for all $u \in W^{1,q}(\Omega) \cap X^{\boldsymbol{A}}(\Omega)$.

Proof. Let $\lambda : \Omega \to [0,1]$ be a Borel function. Arguing as in the proof of Proposition 3.15, we see that, if $u \in W^{1,q}(\Omega)$, then $|\operatorname{div} \boldsymbol{A}|(S_u) = 0$ and $u^\lambda(x) = \tilde{u}(x) = u^*(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$. Hence, $W^{1,q}(\Omega) \cap X^{\boldsymbol{A}}(\Omega) = W^{1,q}(\Omega) \cap X^{\boldsymbol{A},\lambda}(\Omega)$. Let now $u \in W^{1,q}(\Omega) \cap X^{\boldsymbol{A}}(\Omega)$. For $k \in \mathbb{N}$, we set $T_k : \mathbb{R} \to \mathbb{R}$ to be the truncation map

$$T_k(u) = \begin{cases} k & \text{if } u > k, \\ u & \text{if } |u| \le k, \\ -k & \text{if } u < -k. \end{cases} \tag{3.22}$$

It is clear that $T_k(u) \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$ for all $k \in \mathbb{N}$. By Proposition 3.15, we have

$$(\boldsymbol{A}, DT_k(u))_\lambda = (\boldsymbol{A} \cdot \nabla T_k(u))\mathscr{L}^N = (\boldsymbol{A} \cdot \nabla u)\mathscr{L}^N \llcorner \{|u| < k\},$$

where $\boldsymbol{A} \cdot \nabla u \in L^1(\Omega)$. Hence, for all $\varphi \in C^1_c(\Omega)$, thanks to Lebesgue's Dominated Convergence Theorem with respect to the measure $|\operatorname{div} \boldsymbol{A}|$, we get

$$\int_\Omega \varphi(\boldsymbol{A} \cdot \nabla u)\, dx = \lim_{k \to +\infty} \int_\Omega \varphi\, d(\boldsymbol{A}, DT_k(u))_\lambda = - \lim_{k \to +\infty} \int_\Omega \widetilde{T_k(u)}\, d \operatorname{div} \boldsymbol{A} + \int_\Omega T_k(u) \boldsymbol{A} \cdot \nabla \varphi\, dx$$
$$= - \int_\Omega \tilde{u}\, d \operatorname{div} \boldsymbol{A} - \int_\Omega u \boldsymbol{A} \cdot \nabla \varphi\, dx.$$

Thus, by Definition 3.1, we conclude that $(\boldsymbol{A}, Du)_\lambda = (\boldsymbol{A} \cdot \nabla u) \mathscr{L}^N$, and so $u \in W^{\boldsymbol{A},\lambda}(\Omega)$. Finally, in order to prove that the inclusion is in general strict, we consider the example given in Remark 3.10. For such choices of the vector field $\boldsymbol{A}$ and the function v, we clearly have $v \notin W^{1,q}_{\mathrm{loc}}(\Omega)$ and $v \in W^{\boldsymbol{A},\lambda}(\Omega)$ for all Borel functions $\lambda : \Omega \to [0, 1]$, given that $v \in BV^{\boldsymbol{A},\lambda}(\Omega)$ and it satisfies $|(\boldsymbol{A}, Dv)_\lambda| \ll \mathscr{L}^N$. This ends the proof. □

Remark 3.23. *If $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ satisfies $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, then, arguing as in the proof of Proposition 3.19, we get $W^{\boldsymbol{A},\lambda}(\Omega) = W^{\boldsymbol{A}}(\Omega)$ for every Borel function $\lambda : \Omega \to [0, 1]$. In particular, for $u \in BV^{\boldsymbol{A}}(\Omega)$ (3.19) implies*

$$|(\boldsymbol{A}, Du)| \ll \mathscr{L}^N \iff |\operatorname{div}(u\boldsymbol{A})| \ll \mathscr{L}^N,$$

so that we get

$$W^{\boldsymbol{A}}(\Omega) = \{u \in X^{\boldsymbol{A}}(\Omega) : \operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega) \text{ and } |\operatorname{div}(u\boldsymbol{A})| \ll \mathscr{L}^N\}.$$

4. Lower semicontinuity properties and approximations results

Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and $\lambda \colon \Omega \to [0, 1]$ be a Borel function. In this section we will study some lower semicontinuity and continuity properties of the λ-pairing and its total variation in the class $BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, with respect to a suitable notion of convergence. Since $(\boldsymbol{A}, Du)_\lambda$ is affected by the pointwise value of u^λ, the natural notion of convergence in $BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ involves the function λ.

Definition 4.1. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and $\lambda : \Omega \to [0, 1]$ be a Borel function. We say that a sequence $(u_n)_n \subset X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ $(\boldsymbol{A}, \lambda)$-converges to $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ if*

(1) $u^\lambda_n \rightharpoonup u^\lambda$ in $L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|)$,
(2) $u^\lambda_n \rightharpoonup u^\lambda$ in $L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$.

Remark 4.2. *We list some particular cases in which the $(\boldsymbol{A}, \lambda)$-convergence is easier to check.*

i) If $(|\boldsymbol{A}| + |\operatorname{div} \boldsymbol{A}|) \ll \mathscr{L}^N$, then λ does not play any role in the convergence, given that $u^\lambda(x) = u(x)$ for $(|\boldsymbol{A}| + |\operatorname{div} \boldsymbol{A}|)$-a.e. $x \in \Omega$, due to (2.12). In such a case, we omit the λ in the notation for the $(\boldsymbol{A}, \lambda)$-convergence and simply refer to it as $\boldsymbol{A}$-convergence.
ii) If $\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^N)$ and $|\boldsymbol{A}| \geq c$ for some $c > 0$, then condition (1) is equivalent to the weak convergence in $L^1_{\mathrm{loc}}(\Omega)$.
iii) If $\operatorname{div} \boldsymbol{A} = 0$, then condition (2) can be dropped, so that the $(\boldsymbol{A}, \lambda)$-convergence reduces to the weak convergence in $L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|)$.

We have the following lower semicontinuity and continuity results.

Theorem 4.3. *Let $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$ and $\lambda : \Omega \to [0,1]$ be a Borel function. Then for every sequence $(u_n)_n \subset X^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$ and for every $u \in X^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$ and such that $(u_n)_n$ $(\boldsymbol{A},\lambda)$-converges to u, it holds*

$$\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle = \lim_{n\to+\infty} \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \qquad \forall \varphi \in C^1_c(\Omega) \tag{4.1}$$

in the sense of distributions. In addition, if $u, u_n \in BV^{\boldsymbol{A},\lambda}(\Omega)$ for all $n \in \mathbb{N}$, then

$$|\,(\boldsymbol{A}, Du)_\lambda\,|(\Omega) \le \liminf_{n\to+\infty} |\,(\boldsymbol{A}, Du_n)_\lambda\,|(\Omega). \tag{4.2}$$

and, if $\sup_{n\in\mathbb{N}} |(\boldsymbol{A}, Du_n)_\lambda|(\Omega) < +\infty$, we get

$$(\boldsymbol{A}, Du_n)_\lambda \rightharpoonup (\boldsymbol{A}, Du)_\lambda \tag{4.3}$$

weakly in the sense of measures.

Proof. We see (4.1) is a consequence of the definition of $(\lambda, \boldsymbol{A})$-convergence; indeed

$$\begin{aligned}
\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi\rangle &= -\int_\Omega \varphi u^\lambda \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u^\lambda \, \nabla\varphi \cdot d\boldsymbol{A} \\
&= \lim_{n\to+\infty} \left\{ -\int_\Omega \varphi u_n^\lambda \, d\operatorname{div} \boldsymbol{A} - \int_\Omega u_n^\lambda \, \nabla\varphi \cdot d\boldsymbol{A} \right\} \\
&= \lim_{n\to+\infty} \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi\rangle \,.
\end{aligned}$$

Then, if $u, u_n \in BV^{\boldsymbol{A},\lambda}(\Omega)$ for all $n \in \mathbb{N}$, we exploit the fact that the λ-pairings are all finite Radon measures to take the supremum in φ with $\|\varphi\|_{L^\infty(\Omega)} \le 1$ to get (4.2) from (4.1). Finally, if the sequence $(\boldsymbol{A}, Du_n)_\lambda$ is equibounded in total variation, then, given $\psi \in C^0_c(\Omega)$, for all $\varepsilon > 0$ we choose $\varphi_\varepsilon \in C^1_c(\Omega)$ such that $\|\psi - \varphi_\varepsilon\|_{L^\infty(\Omega)} < \varepsilon$ in order to get

$$\begin{aligned}
|\langle (\boldsymbol{A}, Du)_\lambda \,,\, \psi\rangle - \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \psi\rangle| &\le |\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi_\varepsilon\rangle - \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi_\varepsilon\rangle| + |\langle (\boldsymbol{A}, Du)_\lambda \,,\, \psi - \varphi_\varepsilon\rangle| \\
&\quad + |\langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \psi - \varphi_\varepsilon\rangle| \\
&\le |\langle (\boldsymbol{A}, Du)_\lambda \,,\, \varphi_\varepsilon\rangle - \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \varphi_\varepsilon\rangle| + \varepsilon |(\boldsymbol{A}, Du)_\lambda|(\Omega) \\
&\quad + \varepsilon \sup_{n\in\mathbb{N}} |(\boldsymbol{A}, Du_n)_\lambda|(\Omega).
\end{aligned}$$

Therefore, by (4.1) we deduce that

$$\limsup_{n\to+\infty} |\langle (\boldsymbol{A}, Du)_\lambda \,,\, \psi\rangle - \langle (\boldsymbol{A}, Du_n)_\lambda \,,\, \psi\rangle| \le C\varepsilon,$$

for some $C > 0$. Since $\varepsilon > 0$ is arbitrary, this proves (4.2). □

With the following theorem, we establish the existence of smooth approximations for the λ-pairing. As for the case $\lambda \equiv \frac{1}{2}$ [16], we adapt a standard mollification technique to our general setting, by adding a suitable assumption on the concentration of the measure $|\operatorname{div} \boldsymbol{A}|$. As for the λ-pairing functional, instead, we take advantage of a more refined approximation result, recently obtained in [30].

Theorem 4.4. *The following hold true:*

(1) if $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$*, for every* $u \in BV^{\boldsymbol{A}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ *such that* $|\operatorname{div} \boldsymbol{A}|(S_u \setminus J_u) = 0$ *there exists a sequence* $(u_\varepsilon)_{\varepsilon>0} \subset BV^{\boldsymbol{A}}(\Omega) \cap L^\infty(\Omega) \cap C^\infty(\Omega)$ *converging to* u *in* $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$ *and to* u^* *in* $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$*, such that*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \boldsymbol{A} \cdot \nabla u_\varepsilon \, dx = \int_\Omega \varphi \, d\,(\boldsymbol{A}, Du) \qquad \forall \varphi \in C^1_c(\Omega); \tag{4.4}$$

(2) if $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$*, for every* $u \in BV(\Omega)$ *with* $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ *there exists a sequence* $(u^\lambda_\varepsilon)_{\varepsilon>0} \subset BV(\Omega) \cap C^\infty(\Omega)$ *converging to* u *in* $L^1(\Omega)$ *such that*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \, u^\lambda_\varepsilon \, d \operatorname{div} \boldsymbol{A} = \int_\Omega \varphi \, u^\lambda \, d \operatorname{div} \boldsymbol{A} \qquad \forall \varphi \in C^1_c(\Omega) \tag{4.5}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \boldsymbol{A} \cdot \nabla u^\lambda_\varepsilon \, dx = \int_\Omega \varphi \, d\,(\boldsymbol{A}, Du)_\lambda \qquad \forall \varphi \in C^1_c(\Omega). \tag{4.6}$$

Proof. In order to prove (4.4), we first notice that $u^*(x) = u^{\frac{1}{2}}(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$, thanks to $|\operatorname{div} \boldsymbol{A}|(S_u \setminus J_u) = 0$, see Section 2.2. We start by assuming $u \in L^\infty(\Omega)$, and we set $u_\varepsilon = \rho_\varepsilon * u$, for some standard mollifier ρ. It is clear that $u_\varepsilon \to u$ in $L^1(\Omega)$, $u_\varepsilon(x) \to u^*(x)$ for all $x \in \Omega \setminus (S_u \setminus J_u)$, so that $u_\varepsilon(x) \to u(x)$ for $\mathscr{L}^N$-a.e. $x \in \Omega$, and $|u_\varepsilon(x)| \le \|u\|_{L^\infty(\Omega)}$ for all $x \in \Omega$ and $\varepsilon > 0$. Hence, thanks to Lebesgue's Dominated Convergence Theorem with respect to the measures $|\boldsymbol{A}|\mathscr{L}^N$ and $|\operatorname{div} \boldsymbol{A}|$, we see that

$$\|u_\varepsilon - u\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)} + \|u_\varepsilon - u^*\|_{L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)} \to 0.$$

Therefore, (4.4) immediately follows from Definition 3.1 for $\lambda \equiv \frac{1}{2}$. In the general case of $u \in BV^{\boldsymbol{A}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, we consider the truncation of u, $T_k(u)$, defined as in (3.22). Since $u^*(x) = u^{\frac{1}{2}}(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$, we see that $u^* \in L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$ and $T_k(u)^*(x) \to u^*(x)$ as $k \to +\infty$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \Omega$. By Lebesgue's Dominated Convergence Theorem with respect to the measure $|\operatorname{div} \boldsymbol{A}|$, $T_k(u)^* \to u^*$ in $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Analogously, we know that $u \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$, so that $T_k(u) \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$ again by Lebesgue's Dominated Convergence Theorem. Now, we consider $u_{\varepsilon,k} = (T_k(u))_\varepsilon$ and we see that

$$\lim_{k \to +\infty} \lim_{\varepsilon \to 0} \int_\Omega |u_{\varepsilon,k} - u||\boldsymbol{A}| \, dx + \int_\Omega |u_{\varepsilon,k} - u^*| d|\operatorname{div} \boldsymbol{A}| = 0.$$

Hence, via a diagonal argument, we may find a sequence (u_{ε_k}) such that $u_{\varepsilon_k} \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$ and $u_{\varepsilon_k} \to u^*$ in $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$. Thus, (4.4) immediately follows again from Definition 3.1 in the case $\lambda \equiv \frac{1}{2}$.

As for point (2), by [30, Theorem 3.3 and Lemma 4.11] there exists $\{u^\lambda_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega) \cap L^\infty(\Omega) \cap C^\infty(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ (so that $u_\varepsilon \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$) and

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \, d\mu = \int_\Omega u^\lambda \, d\mu,$$

for all Radon measures μ such that $|\mu|$ belongs to the dual of $BV(\Omega)$, which in turn implies $\mu = \operatorname{div} \mathbf{B}$ for some $\mathbf{B} \in \mathcal{DM}^\infty(\Omega)$ [30, Remark 4.8]. Hence, for all $\varphi \in C^1_c(\Omega)$ by (3.3) we get

$$\langle (\boldsymbol{A}, Du)_\lambda \,, \varphi \rangle = - \int_\Omega u^\lambda \, d \operatorname{div}(\varphi \boldsymbol{A}) = \lim_{\varepsilon \to 0} - \int_\Omega u^\lambda_\varepsilon \, d \operatorname{div}(\varphi \boldsymbol{A}) = \lim_{\varepsilon \to 0} \int_\Omega \varphi \boldsymbol{A} \cdot \nabla u_\varepsilon \, dx,$$

since $\varphi \boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ (see (2.20) and the discussion therein). This proves (4.6). Finally, (4.5) follows by combining (4.6), the convergence $u_\varepsilon \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$, Lemma 3.2 and (3.2). $\square$

Remark 4.5. *We notice that point (i) of Theorem 4.4 actually implies that* $(u_\varepsilon)_{\varepsilon>0}$ $(\boldsymbol{A}, \frac{1}{2})$*-converges to* u*. In addition, if in point (ii) of Theorem 4.4 we also assume that* $u \in L^\infty(\Omega)$*, then we obtain* $u_\varepsilon^\lambda \to u^\lambda$ *in* $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$*, so that the sequence* $(u_\varepsilon^\lambda)_{\varepsilon>0}$ $(\boldsymbol{A}, \lambda)$*-converges to* u*. Indeed, the convergence in* $L^1(\Omega)$ *implies the one in* $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$*, given that* $\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^N)$*. As for the convergence in* $L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)$*, it follows by Lebesgue's Dominated Convergence Theorem, since, by* [30, Theorem 3.3]*, we have* $\|u_\varepsilon^\lambda\|_{L^\infty(\Omega)} \le 2\|u\|_{L^\infty(\Omega)}$ *and* $u_\varepsilon(x) \to u^\lambda(x)$ *for* $\mathscr{H}^{N-1}$*-a.e.* $x \in \Omega$ *as* $\varepsilon \to 0$*, and so* $u_\varepsilon(x) \to u^\lambda(x)$ *for* $|\operatorname{div} \boldsymbol{A}|$*-a.e.* $x \in \Omega$*, given that* $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{H}^{N-1}$ *(see Section 2.3).*

5. A linear space contained in $BV^{\boldsymbol{A},\lambda}(\Omega)$

Let $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$. We define a new subclass of functions summable with respect to the measures $|\boldsymbol{A}|$ and $|\operatorname{div} \boldsymbol{A}|$:

$$X^{\boldsymbol{A},+}(\Omega) := \{u \in \mathscr{B}(\Omega) : |u|^+ \in L^1(\Omega, |\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div} \boldsymbol{A}|)\}.$$

We notice that

$$X^{\boldsymbol{A},+}(\Omega) \subseteq X^{\boldsymbol{A},\lambda}(\Omega) \tag{5.1}$$

for every Borel function $\lambda \colon \Omega \to [0, 1]$. Indeed, thanks to the inequality

$$|u|^- = \min\{|u^+|, |u^-|\} \le \max\{|u^+|, |u^-|\} = |u|^+,$$

it is easy to see that

$$|u^\lambda| \le \lambda |u^+| + (1-\lambda)|u^-| \le \lambda |u|^+ + (1-\lambda)|u|^+ \le |u|^+$$

for every Borel function $\lambda \colon \Omega \to [0, 1]$.

Moreover, we note that $X^{\boldsymbol{A},+}(\Omega)$ is a linear space. Indeed, for every $u, v \in X^{\boldsymbol{A},+}(\Omega)$ we see that

$$\int_\Omega |u+v|^+ \, d\mu \le \int_\Omega (|u| + |v|)^+ \, d\mu \le 2 \int_{\Omega \cap \{|v| \le |u|\}} |u|^+ \, d\mu + 2 \int_{\Omega \cap \{|u| < |v|\}} |v|^+ \, d\mu < +\infty\,,$$

for $\mu = |\boldsymbol{A}|$ and $\mu = |\operatorname{div} \boldsymbol{A}|$, thanks to the fact that $|f|^+ \le |g|^+$, if $|f| \le |g|$.

At this point, we can define the related $BV^{\boldsymbol{A}}$-like class:

$$BV^{\boldsymbol{A},+}(\Omega) := \left\{u \in X^{\boldsymbol{A},+}(\Omega) : \ (\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega) \ \text{ for every Borel } \lambda : \Omega \to [0,1]\right\}.$$

It is not difficult to see that, whenever $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$, we get

$$BV^{\boldsymbol{A},+}(\Omega) = \left\{u \in X^{\boldsymbol{A},+}(\Omega) : \ \operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)\right\}. \tag{5.2}$$

Indeed, given $u \in X^{\boldsymbol{A},+}(\Omega)$, by (5.1) we know that $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel function $\lambda : \Omega \to [0, 1]$. Hence, thanks to point (2) of Proposition 3.5, we see that $(\boldsymbol{A}, Du)_\lambda \in \mathcal{M}(\Omega)$ if and only if $\operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)$.

We employ all these remarks in order to explore the case in which $BV^{\boldsymbol{A},+}(\Omega)$ enjoys a linear structure.

Proposition 5.1. *Let* $\boldsymbol{A} \in \mathcal{DM}_{\rm loc}(\Omega)$*.*

(1) We have $BV^{\boldsymbol{A},+}(\Omega) \subseteq BV^{\boldsymbol{A},\lambda}(\Omega)$ *for every Borel function* $\lambda \colon \Omega \to [0, 1]$*.*

(2) *If* $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$*, then* $BV^{\boldsymbol{A},+}(\Omega)$ *is a linear space.*
(3) *If* $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ *with* $\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} \mathscr{L}^N$ *for some* $\operatorname{div}^a \boldsymbol{A} \in L^1_{\mathrm{loc}}(\Omega)$*, then*

$$BV^{\boldsymbol{A},+}(\Omega) = BV^{\boldsymbol{A}}(\Omega),$$

so that $BV^{\boldsymbol{A}}(\Omega)$ *is a linear space satisfying*

$$BV^{\boldsymbol{A}}(\Omega) = \{u \in \mathscr{B}(\Omega) : \|u\|_{BV^{\boldsymbol{A}}(\Omega)} < +\infty\},$$

where $\|\cdot\|_{BV^{\boldsymbol{A}}(\Omega)}$ *is a seminorm defined as*

$$\|u\|_{BV^{\boldsymbol{A}}(\Omega)} := \|u\|_{L^1(\Omega,|\boldsymbol{A}|\mathscr{L}^N)} + \|u\|_{L^1(\Omega,|\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)} + |\operatorname{div}(u\boldsymbol{A})|(\Omega). \tag{5.3}$$

If in addition $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$*, then* $BV^{\boldsymbol{A}}(\Omega)$ *is a Banach space, endowed with the norm given by* (5.3)*.*

Proof. Point (1) is a trivial consequence of the definition of $BV^{\boldsymbol{A},+}(\Omega)$. As for point (2), the linearity follows from (5.2) and the fact $X^{\boldsymbol{A},+}(\Omega)$ is a linear space. Concerning point (3), the absolute continuity $(|\boldsymbol{A}| + |\operatorname{div} \boldsymbol{A}|) \ll \mathscr{L}^N$ implies

$$|u|^+(x) = |u(x)| \ \text{ for } (|\boldsymbol{A}| + |\operatorname{div} \boldsymbol{A}|)\text{-a.e. } x \in \Omega,$$

due to (2.12), and therefore $BV^{\boldsymbol{A},+}(\Omega) = BV^{\boldsymbol{A}}(\Omega)$ (see Proposition 3.19), with $u \in BV^{\boldsymbol{A}}(\Omega)$ if and only if $u \in \mathscr{B}(\Omega)$ and $\|u\|_{BV^{\boldsymbol{A}}(\Omega)} < +\infty$. Finally, it is easy to check that (5.3) is a norm, under the assumption $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$, so that we are left to show the completeness. Let $(u_n)_n$ be a Cauchy sequence in $BV^{\boldsymbol{A}}(\Omega)$, i.e. for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\|u_n - u_{n+1}\|_{BV^{\boldsymbol{A}}(\Omega)} < \varepsilon. \tag{5.4}$$

Then $(u_n)_n$ is a Cauchy sequence in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$ and in $L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)$. Since these spaces are Banach, there exists two functions $u \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)$ and $v \in L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)$ such that

$$\lim_{n\to+\infty} \int_\Omega |u_n - u|\,|\boldsymbol{A}|\,dx = 0\,, \qquad \lim_{n\to+\infty} \int_\Omega |u_n - v||\operatorname{div}^a \boldsymbol{A}|\,dx = 0\,.$$

Hence, there exists a subsequence (u_{n_k}) such that $u_{n_k}(x) \to u(x)$ for $|\boldsymbol{A}|\mathscr{L}^N$-a.e. $x \in \Omega$, and, thanks to the assumption $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$, this implies that $u_{n_k}(x) \to u(x)$ for $\mathscr{L}^N$-a.e. $x \in \Omega$. Therefore, we exploit Fatou's Lemma with respect to $\mathscr{L}^N$ to get

$$\int_\Omega |u - v||\operatorname{div}^a \boldsymbol{A}|\,dx \leq \int_\Omega \liminf_{k\to+\infty} |u_{n_k} - v||\operatorname{div}^a \boldsymbol{A}|\,dx \leq \liminf_{k\to+\infty} \int_\Omega |u_{n_k} - v||\operatorname{div}^a \boldsymbol{A}|\,dx = 0,$$

and so we conclude that $u = v$ for $|\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N$-a.e. $x \in \Omega$. Thus, $u_n \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N) \cap L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)$. Moreover, by (5.4) the sequence of measures $\mu_n := \operatorname{div}(u_n\boldsymbol{A})$ is a Cauchy sequence, and therefore it is uniformly bounded, so that there exists a measure μ and a subsequence (μ_{n_j}) such that μ_{n_j} weakly converges to μ in $\mathcal{M}(\Omega)$. It remains to prove that $\mu = \operatorname{div}(u\boldsymbol{A})$. We recall that, since $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, all pairings are identical, see (3.19), so that we just consider $(\boldsymbol{A}, Du)$. By (4.1) for all $\varphi \in C^1_c(\Omega)$ we have

$$\langle (\boldsymbol{A}, Du)\,, \varphi\rangle = \lim_{n\to+\infty} \langle (\boldsymbol{A}, Du_n)\,, \varphi\rangle = \lim_{n\to+\infty} \int_\Omega \varphi\, d\,(\boldsymbol{A}, Du_n)\,. \tag{5.5}$$

Hence, we get

$$\begin{aligned}\int_\Omega \varphi \, d \operatorname{div}(u\boldsymbol{A}) &= -\int_\Omega \varphi u \operatorname{div}^a \boldsymbol{A} \, dx + \langle (\boldsymbol{A}, Du), \varphi \rangle \\ &= \lim_{j\to+\infty} \left(-\int_\Omega \varphi u_{n_j} \operatorname{div}^a \boldsymbol{A} \, dx + \int_\Omega \varphi \, d\left(\boldsymbol{A}, Du_{n_j}\right)\right) \\ &= \lim_{j\to+\infty} \int_\Omega \varphi \, d \operatorname{div}(u_{n_j}\boldsymbol{A}) = \int_\Omega \varphi \, d\mu.\end{aligned}$$

Therefore $\mu = \operatorname{div}(u\boldsymbol{A})$, so that $\operatorname{div}(u\boldsymbol{A}) \in \mathcal{M}(\Omega)$, and we conclude that $u \in BV^{\boldsymbol{A}}(\Omega)$. □

Corollary 5.2. *Let* $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$ *be such that* $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$*, with* $\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} \mathscr{L}^N$ *for some* $\operatorname{div}^a \boldsymbol{A} \in L^1_{\rm loc}(\Omega)$*. Then the functional*

$$BV^{\boldsymbol{A}}(\Omega) \ni u \to |(\boldsymbol{A}, Du)|(\Omega)$$

is a seminorm, and an equivalent seminorm on $BV^{\boldsymbol{A}}(\Omega)$ *is given by*

$$BV^{\boldsymbol{A}}(\Omega) \ni u \to \|u\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)} + \|u\|_{L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)} + |(\boldsymbol{A}, Du)|(\Omega).$$

Proof. Thanks to Proposition 3.19, we know that the operator $BV^{\boldsymbol{A}}(\Omega) \ni u \to (\boldsymbol{A}, Du)$ is linear. In particular, this immediately implies the total variation of the pairing is 1-homogeneous and satisfies the triangle inequality. Then, given $u \in BV^{\boldsymbol{A}}(\Omega)$, we exploit (3.19) twice, in order to get

$$\begin{aligned}|(\boldsymbol{A}, Du)|(\Omega) &\le \|u\|_{L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)} + |\operatorname{div}(u\boldsymbol{A})|(\Omega), \\ |\operatorname{div}(u\boldsymbol{A})|(\Omega) &\le \|u\|_{L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)} + |(\boldsymbol{A}, Du)|(\Omega).\end{aligned}$$

The rest of the proof follows from point (3) of Proposition 5.1. □

Under the assumption that both the vector field $\boldsymbol{A}$ and its divergence are locally summable we prove that also the Sobolev-like class $W^{\boldsymbol{A}}(\Omega)$ is indeed a linear space (in such a case, λ does not play any role, as noticed in Remark 3.23).

Corollary 5.3. *Let* $\boldsymbol{A} \in \mathcal{DM}^1_{\rm loc}(\Omega)$ *be such that* $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$*, with* $\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} \mathscr{L}^N$ *for some* $\operatorname{div}^a \boldsymbol{A} \in L^1_{\rm loc}(\Omega)$*. Then* $W^{\boldsymbol{A}}(\Omega)$ *is a linear space on which we define the seminorm*

$$\|u\|_{W^{\boldsymbol{A}}(\Omega)} := \|u\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^N)} + \|u\|_{L^1(\Omega, |\operatorname{div}^a \boldsymbol{A}|\mathscr{L}^N)} + \|(\boldsymbol{A}, Du)^a\|_{L^1(\Omega)}, \tag{5.6}$$

where $(\boldsymbol{A}, Du) = (\boldsymbol{A}, Du)^a \mathscr{L}^N$ *for some* $(\boldsymbol{A}, Du)^a \in L^1(\Omega)$*. If in addition* $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$*, then* $W^{\boldsymbol{A}}(\Omega)$ *is a Banach space, endowed with the norm given by* (5.6)*.*

Proof. Thanks to Corollary 5.2, we know that the pairing is linear in the second component. This implies that $W^{\boldsymbol{A}}(\Omega)$ is a linear space. Then, it is easy to check that $\|\cdot\|_{W^{\boldsymbol{A}}(\Omega)}$ is a seminorm, and a norm whenever $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$. It remains to prove the completeness. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{\boldsymbol{A}}(\Omega)$. Since $W^{\boldsymbol{A}}(\Omega) \subset BV^{\boldsymbol{A}}(\Omega)$, and, by Proposition 5.1, $BV^{\boldsymbol{A}}(\Omega)$ is a Banach space, we know that there exists $u \in BV^{\boldsymbol{A}}(\Omega)$ such that

$$\lim_{n\to+\infty} \|u_n - u\|_{BV^{\boldsymbol{A}}(\Omega)} = 0.$$

Hence, we need to check that $|(\boldsymbol{A}, Du)| \ll \mathscr{L}^N$. To this purpose, we notice that, for all $n \in \mathbb{N}$, there exists $(\boldsymbol{A}, Du_n)^a \in L^1(\Omega)$ such that $(\boldsymbol{A}, Du_n) = (\boldsymbol{A}, Du_n)^a \mathscr{L}^N$. Hence, $((\boldsymbol{A}, Du_n)^a)_{n\in\mathbb{N}}$

is a Cauchy sequence in $L^1(\Omega)$, and therefore it admits a limit $\xi_{\boldsymbol{A},u} \in L^1(\Omega)$. Due to (3.19) and the fact that $u_n \to u$ in $L^1(\Omega, (|\boldsymbol{A}| + |\operatorname{div}^a \boldsymbol{A}|)\mathscr{L}^N)$, we see that

$$\begin{aligned}\int_\Omega \varphi\, \xi_{\boldsymbol{A},u}\, dx &= \lim_{n\to+\infty} \int_\Omega \varphi\, (\boldsymbol{A}, Du_n)^a\, dx = - \lim_{n\to+\infty} \left(\int_\Omega \varphi\, u_n \operatorname{div}^a \boldsymbol{A}\, dx + \int_\Omega u_n\, \boldsymbol{A} \cdot \nabla\varphi\, dx \right) \\ &= - \int_\Omega \varphi\, u \operatorname{div}^a \boldsymbol{A}\, dx - \int_\Omega u\, \boldsymbol{A} \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, d(\boldsymbol{A}, Du)\end{aligned}$$

for all $\varphi \in C^1_c(\Omega)$. This proves that $(\boldsymbol{A}, Du) = \xi_{\boldsymbol{A},u}\mathscr{L}^N$, and therefore $u \in W^{\boldsymbol{A}}(\Omega)$. □

Remark 5.4. *We point out the assumption $\mathscr{L}^N(\Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)) = 0$ in point (3) of Proposition 5.1 is necessary in order to avoid the degeneracy of the seminorm $\|\cdot\|_{BV^{\boldsymbol{A}}(\Omega)}$. Indeed, consider $\boldsymbol{A} \in C^1_c(\Omega; \mathbb{R}^N)$ such that $V = \Omega \setminus \operatorname{supp}(|\boldsymbol{A}|)$ is a non-empty open set. Then, given any nontrivial function $u \in C^1_c(\Omega)$ such that $\operatorname{supp}(u) \subset V$, we clearly have $u \in BV^{\boldsymbol{A}}(\Omega)$, with $\|u\|_{BV^{\boldsymbol{A}}(\Omega)} = 0$, since u and $\boldsymbol{A}$ are both regular and have disjoint supports, so that, in particular*

$$(\boldsymbol{A}, Du) = \boldsymbol{A} \cdot \nabla u \mathscr{L}^N = 0.$$

Thanks to Proposition 5.1, we know that, if $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ is such that $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, then $BV^{\boldsymbol{A}}(\Omega)$ is a linear space endowed with a seminorm. Hence, it is natural to ask whether, under such conditions, it enjoys some local compactness with respect to the $\boldsymbol{A}$-convergence (see Remark 4.2). This would be relevant since the seminorm given by the total variation of the pairing is lower semicontinuous with respect to the $\boldsymbol{A}$-convergence (Theorem 4.3). However, it is important to point out that $BV^{\boldsymbol{A}}(\Omega)$ is not locally compact with respect to such convergence, at least in dimension $N \geq 2$. In other words, we can find a field $\boldsymbol{A}$ and a sequence (u_k) which is uniformly bounded in $BV^{\boldsymbol{A}}(\Omega)$, but does not admit an $\boldsymbol{A}$-converging subsequence. The counterexample below shows the occurrence of this pathological phenomenon even for a “smooth” transversal vector field.

Example 5.5. *Let $N \geq 2$, $\Omega = (-1,1)^N$, $f \in C^1_c(\mathbb{R})$ such that $f(0) \neq 0$, $\boldsymbol{A}(x) = (f(x_N), 0, \dots, 0)$. It is clear that $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $\operatorname{div} \boldsymbol{A} = 0$. For $k \in \mathbb{N}$, $k \geq 1$, we define*

$$u_k(x) = k\chi_{(-1,1)^{N-1}\times(0,\frac{1}{k})}(x),$$

so that $D_j u_k = 0$ for all $j \in \{1, \dots, N-1\}$ and

$$D_N u_k = k \left(\mathscr{H}^{N-1} \llcorner (-1,1)^{N-1} \times \{0\} - \mathscr{H}^{N-1} \llcorner (-1,1)^{N-1} \times \left\{ \frac{1}{k} \right\} \right).$$

Therefore, by (3.20) we get

$$(\boldsymbol{A}, Du_k) = \operatorname{div}(u_k \boldsymbol{A}) = D_1(u_k \boldsymbol{A}_1) = 0,$$

since u_k and $\boldsymbol{A}_1$ are constant in x_1. Hence, we get $u_k \in BV^{\boldsymbol{A}}(\Omega) = BV^{\boldsymbol{A},\lambda}(\Omega)$ for all Borel functions $\lambda : \Omega \to [0,1]$, due to Proposition 3.19. In addition, we have

$$\begin{aligned}\|u_k\|_{BV^{\boldsymbol{A}}(\Omega)} &= \|u_k\|_{L^1(\Omega,|\boldsymbol{A}|)} + \|u_k^{\frac{1}{2}}\|_{L^1(\Omega,|\operatorname{div}\boldsymbol{A}|)} + |(\boldsymbol{A}, Du_k)|(\Omega) = 2^{N-1}k \int_0^{\frac{1}{k}} |f(x_N)|\, dx_N \\ &\leq 2^{N-1}\|f\|_{L^\infty(\mathbb{R})}\end{aligned}$$

for all $k \in \mathbb{N}, k \geq 1$. Hence, $(u_k)_{k \in \mathbb{N}}$ is a uniformly bounded sequence in $BV^{\boldsymbol{A}}(\Omega)$, while it is clearly not so in $BV(\Omega)$. However, it cannot admit a converging subsequence in $L^1(\Omega, |\boldsymbol{A}|)$, since we have $u_k(x) \to 0$ for all $x \in \Omega$, so that the only limit could be $u = 0$, but we have

$$\|u_k\|_{L^1(\Omega, |\boldsymbol{A}|)} = 2^{N-1} k \int_0^{\frac{1}{k}} |f(x_N)| \, dx_N \to 2^{N-1} |f(0)| \neq 0.$$

In particular, this rules out the existence of any $\boldsymbol{A}$-converging subsequences, even in the case in which $BV^{\boldsymbol{A}}(\Omega)$ is a Banach space; that is, whenever $\mathscr{L}^1((-1,1) \setminus \operatorname{supp}(f)) = 0$ (by point (3) of Proposition 5.1).

Furthermore, we notice that we cannot have even the weak convergence $u_k \rightharpoonup 0$ in $L^1(\Omega, |\boldsymbol{A}|)$: indeed, it is not difficult to see that

$$u_k \mathscr{L}^N \rightharpoonup \mathscr{H}^{N-1} \llcorner (-1,1)^{N-1} \times \{0\} \quad \text{in} \quad \mathcal{M}(\Omega),$$

so that, for all $\phi \in C_c(\Omega)$, we get

$$\begin{aligned} \int_\Omega \phi(x) u_k(x) |\boldsymbol{A}(x)| \, dx \to \int_\Omega \phi(x) |f(x_N)| \, d\mathscr{H}^{N-1} \llcorner (-1,1)^{N-1} \times \{0\} \\ = |f(0)| \int_{(-1,1)^{N-1}} \phi(x_1, \ldots, x_{N-1}, 0) \, dx_1 \ldots dx_{N-1}. \end{aligned}$$

Nevertheless, we can prove the existence of minimizers for functionals involving the pairing and a forcing term. For instance, we consider the following family of functionals:

$$\mathcal{E}_p(u) := |(\boldsymbol{A}, Du)|(\Omega) + \|u - g\|_{L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)}, \quad u \in BV^{\boldsymbol{A}}(\Omega) \cap L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N), \tag{5.7}$$

for $1 \leq p \leq +\infty$, $g \in L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$ and $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ with $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, and under other suitable additional assumptions on the vector field.

Theorem 5.6. *Let $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$ be such that $|\boldsymbol{A}| \geq c$ for some $c > 0$ and $\operatorname{div} \boldsymbol{A} \in L^{p'}(\Omega)$. Let $g \in L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$ for some $1 \leq p \leq +\infty$. Then the functional $\mathcal{E}_p$ admits a minimizer in $BV^{\boldsymbol{A}}_{\mathrm{loc}}(\Omega)$.*

Proof. Let $(u_k)_k \subset BV^{\boldsymbol{A}}_{\mathrm{loc}}(\Omega)$ be a minimizing sequence. In particular, $(u_k)_k$ is equibounded in $L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$. Therefore, we can find $u \in L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$ such that $u_k \rightharpoonup u$ in $L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$, up to a subsequence. Hence, we see that $u_k \rightharpoonup u$ in $L^1(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$, since $L^\infty(\Omega, |\boldsymbol{A}| \mathscr{L}^N) \subset L^r(\Omega, |\boldsymbol{A}| \mathscr{L}^N)$ for all $r \geq 1$, due to the fact that $|\boldsymbol{A}| \in L^1(\Omega)$. In addition, since $|\boldsymbol{A}| \geq c > 0$, we get

$$\|u_k - g\|_{L^p(\Omega, |\boldsymbol{A}| \mathscr{L}^N)} \geq c \|u_k - g\|_{L^p(\Omega)},$$

so that we conclude that $(u_k)_k$ is equibounded in $L^p(\Omega)$ and therefore $u_k \rightharpoonup u$ in $L^p(\Omega)$, up to a further subsequence. In particular, this implies $u_k \rightharpoonup u$ in $L^1(\Omega, |\operatorname{div} \boldsymbol{A}| \mathscr{L}^N)$, up to another subsequence. From Theorem 4.3 and the lower semicontinuity of the norm in $L^p(\Omega)$ with respect to the weak-L^p convergence, we have

$$\mathcal{E}_p(u) \leq \liminf_{k \to +\infty} \mathcal{E}_p(u_k). \tag{5.8}$$

Therefore, u is a minimizer of $\mathcal{E}$ and the proof is concluded. $\square$

We notice that the assumptions on the vector field in Theorem 5.6 allow for fields with some vanishing components. Indeed, the transversal vector field $\boldsymbol{A}$ used in the counterexample to the compactness of $BV^{\boldsymbol{A}}(\Omega)$, Example 5.5, satisfies these assumptions as long as we require $|f(x_N)| \geq c > 0$ for $x_N \in (-1, 1)$. We also note that such assumptions imply that $BV^{\boldsymbol{A}}(\Omega)$ is a Banach space, thanks to point (3) of Proposition 5.1, and still we need a forcing term in $\mathcal{E}_p$ to achieve the existence of a minimizer, due to the lack of local compactness in $BV^{\boldsymbol{A}}(\Omega)$ (see Example 5.5).

6. A coarea formula

Theorem 6.1. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, let $\lambda : \Omega \to [0,1]$ be a Borel function and let $u \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ be such that*

$$\lambda u^+, (1-\lambda)u^- \in L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|) \cap L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|) \tag{6.1}$$

and

$$|\boldsymbol{A}|(N_t) + |\operatorname{div} \boldsymbol{A}|(N_t) = 0 \text{ for } \mathscr{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{6.2}$$

where $N_t := \{u^- \leq t < u^+\} \setminus \{u > t\}^{1/2}$. Then we have

$$\int_\Omega \varphi \, d(\boldsymbol{A}, Du)_\lambda = \int_{-\infty}^{+\infty} \Big\langle (\boldsymbol{A}, D\chi_{\{u>t\}})_\lambda, \varphi \Big\rangle \, dt \qquad \forall \varphi \in C^1_c(\Omega) \tag{6.3}$$

in the sense of distributions. In particular, we get

$$|(\boldsymbol{A}, Du)_\lambda| \leq \int_{-\infty}^{+\infty} |(\boldsymbol{A}, D\chi_{\{u>t\}})_\lambda| \, dt \tag{6.4}$$

in the sense of measures, whenever the right hand size is well posed.

Proof. For all $\varphi \in C^1_c(\Omega)$, by (3.3) for every $t \in \mathbb{R}$ we have

$$\Big\langle (\boldsymbol{A}, D\chi_{\{u>t\}})_\lambda, \varphi \Big\rangle := - \int_\Omega \chi^\lambda_{\{u>t\}} \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega \chi^\lambda_{\{u>t\}} \nabla \varphi \cdot d\boldsymbol{A} = - \int_\Omega \chi^\lambda_{\{u>t\}} \, d \operatorname{div}(\varphi \boldsymbol{A})$$

in the sense of distributions. Thanks to Lemma 2.2, we have that

$$\chi^\lambda_{\{u>t\}}(x) = (1-\lambda(x))\chi_{\{u^->t\}}(x) + \lambda(x)\chi_{\{u^+>t\}}(x) \quad \forall x \in \Omega \setminus N_t, \tag{6.5}$$

where $N_t := \{u^- \leq t < u^+\} \setminus \{u > t\}^{1/2}$, and we also have

$$|\operatorname{div}(\varphi \boldsymbol{A})|(N_t) \leq \int_{N_t} |\varphi| \, d|\operatorname{div} \boldsymbol{A}| + \int_{N_t} |\nabla \varphi| \, d|\boldsymbol{A}| = 0$$

for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$, thanks to (2.17) and (6.2). Therefore, (6.5) holds for $|\operatorname{div}(\varphi \boldsymbol{A})|$-a.e. $x \in \Omega$. In addition, Lemma 2.4 implies that $\operatorname{div}(\varphi \boldsymbol{A})(\Omega) = 0$, since φ has compact support. Hence, by exploiting these facts together with (3.3), Cavalieri formula and Fubini's Theorem, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} \Big\langle (\boldsymbol{A}, D\chi_{\{u>t\}})_\lambda, \varphi \Big\rangle \, dt &= -\int_{-\infty}^{+\infty} \int_\Omega \chi^\lambda_{\{u>t\}} \, d \operatorname{div}(\varphi \boldsymbol{A}) \, dt \\
&= -\int_0^{+\infty} \int_\Omega \big((1-\lambda)\chi_{\{u^- > t\}} + \lambda \chi_{\{u^+>t\}} \big) \, d\operatorname{div}(\varphi \boldsymbol{A}) \, dt \\
&\quad + \int_{-\infty}^0 \int_\Omega \big(1 - (1-\lambda)\chi_{\{u^->t\}} - \lambda\chi_{\{u^+>t\}} \big) \, d\operatorname{div}(\varphi\boldsymbol{A}) \, dt \\
&= -\int_\Omega \int_0^{+\infty} (1-\lambda)\chi_{\{u^->t\}} \, dt \, d\operatorname{div}(\varphi\boldsymbol{A}) \\
&\quad + \int_\Omega \int_{-\infty}^0 (1-\lambda) \big(1 - \chi_{\{u^->t\}}\big) \, dt \, d\operatorname{div}(\varphi\boldsymbol{A}) \\
&\quad - \int_\Omega \int_0^{+\infty} \lambda \chi_{\{u^+>t\}} \, dt \, d\operatorname{div}(\varphi\boldsymbol{A}) + \\
&\quad + \int_\Omega \int_{-\infty}^0 \lambda(1 - \chi_{\{u^+>t\}}) \, dt \, d\operatorname{div}(\varphi\boldsymbol{A}) \\
&= -\int_\Omega (1-\lambda) u^- \, d\operatorname{div}(\varphi\boldsymbol{A}) - \int_\Omega \lambda u^+ \, d\operatorname{div}(\varphi\boldsymbol{A}) \\
&= -\int_\Omega u^\lambda \, d\operatorname{div}(\varphi\boldsymbol{A}) = \int_\Omega \varphi \, d(\boldsymbol{A}, Du)_\lambda.
\end{aligned}$$

This proves (6.3), while (6.4) follows immediately by taking the supremum in $\varphi \in C^1_c(\Omega)$ with $\|\varphi\|_{L^\infty(\Omega)} \le 1$. □

Remark 6.2. *We notice that the assumptions* (6.1) *and* (6.2) *in Theorem 6.1 are always satisfied in the case* $|\boldsymbol{A}| \ll \mathscr{L}^N$ *and* $|\operatorname{div}\boldsymbol{A}| \ll \mathscr{L}^N$*, since* $N_t \subset S^*_u$ *and* $\mathscr{L}^N(S^*_u) = 0$ *by* (2.4)*; and* $u^+(x) = u^-(x) = u(x)$ *for* $\mathscr{L}^N$*-a.e.* $x \in \Omega$*, so that* (6.1) *is implied by* $u \in L^1_{\rm loc}(\Omega, |\boldsymbol{A}|) \cap L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$*, which is one of the conditions for having* $u \in BV^{\boldsymbol{A},\lambda}_{\rm loc}(\Omega)$*. Actually, in such a case, by Proposition 3.19 all the* λ*-pairing coincide, so that, the coarea formula can be rewritten in the following way:*

$$\int_\Omega \varphi \, d(\boldsymbol{A}, Du) = \int_{-\infty}^{+\infty} \Big\langle (\boldsymbol{A}, D\chi_{\{u>t\}}), \varphi \Big\rangle \, dt, \qquad \forall \varphi \in C^1_c(\Omega).$$

If instead we have $|\boldsymbol{A}^s|(N_t) + |\operatorname{div}^s \boldsymbol{A}|(N_t) = 0$*, we can also drop* (6.2)*, while we still need* (6.1)*.*

Alternatively, if $\boldsymbol{A} \in L^\infty_{\rm loc}(\Omega; \mathbb{R}^N)$ *and we take* $u \in BV_{\rm loc}(\Omega)$*, then* (6.2) *follows from to the absolute continuity* $|\operatorname{div}\boldsymbol{A}| \ll \mathscr{H}^{N-1}$ *and the fine properties of BV functions with respect to the* $\mathscr{H}^{N-1}$*-measure (see for instance* [20, Lemma 2.2]*). As for* (6.1)*, we see that* $u^\lambda \in L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$ *implies* $u^\pm \in L^1_{\rm loc}(\Omega, |\operatorname{div}\boldsymbol{A}|)$*, thanks to* [19, Lemma 3.2]*, so that we recover* [19, Theorem 5.1]*, where* (6.3) *holds in the sense of Radon measures (that is, for test functions* $\varphi \in C_c(\Omega)$*). If* $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ *and* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ *with* $\operatorname{div}\boldsymbol{A} \in L^1_{\rm loc}(\Omega)$*, then the assumptions* (6.1) *and* (6.2) *are satisfied and* (6.4) *holds as an equality (see* [18, Theorem 4.4]*).*

7. The $(\boldsymbol{A}, \lambda)$-perimeter

In this section, we focus ourselves with the particular case in which $u \in X^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ is the characteristic function of a Borel set (up to Lebesgue negligible sets). We notice that, given any Borel set E and Borel function $\lambda : \Omega \to [0,1]$, we get

$$\chi_E^\lambda = (1-\lambda)\chi_{E^1} + \lambda \chi_{E^1 \cup \partial^* E} = \chi_{E^1} + \lambda \chi_{\partial^* E}, \tag{7.1}$$

since $\chi_E^- = \chi_{E^1}$ and $\chi_E^+ = \chi_{E^1 \cup \partial^* E}$.

Definition 7.1. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, $\lambda : \Omega \to [0,1]$ be Borel function and let E be a Borel subset of Ω. We define the $(\boldsymbol{A}, \lambda)$-perimeter of E in Ω, denoted by $P_{\boldsymbol{A},\lambda}(E, \Omega)$, as the following variation*

$$P_{\boldsymbol{A},\lambda}(E,\Omega) := \sup \left\{ \int_\Omega \chi_E^\lambda \nabla \varphi \cdot d\boldsymbol{A} + \int_\Omega \chi_E^\lambda \varphi \, d \operatorname{div} \boldsymbol{A} : \varphi \in C^1_c(\Omega; \mathbb{R}^N), \ \|\varphi\|_\infty \le 1 \right\}. \tag{7.2}$$

We say that E is a set of finite $(\boldsymbol{A}, \lambda)$-perimeter in Ω if $P_{\boldsymbol{A},\lambda}(E,\Omega) < +\infty$. Moreover, E is a set of locally finite $(\boldsymbol{A}, \lambda)$-perimeter in Ω if $P_{\boldsymbol{A},\lambda}(E,\Omega') < +\infty$ for any open set $\Omega' \Subset \Omega$.

By the definition of λ-pairing (Definition 3.1), it is immediate to see that

$$P_{\boldsymbol{A},\lambda}(E,\Omega) = |(\boldsymbol{A}, D\chi_E)_\lambda|(\Omega).$$

It is an interesting question to ask whether the $(\boldsymbol{A}, \lambda)$-perimeter of a set is concentrated on some type of generalized boundary of the set. To this purpose, we recall the definition of another type of Lebesgue measure-invariant boundary of a Borel set E:

$$\partial^- E := \{x \in \mathbb{R}^N : 0 < \mathscr{L}^N(E \cap B_r(x)) < \mathscr{L}^N(B_r(x)) \ \text{ for all } r > 0\}.$$

We point out that $\partial^- E$ is a closed set and $\partial^* E \subseteq \partial^- E \subseteq \partial E$, possibly with strict inclusions. We prove below that this boundary contains the support of the pairing distribution $(\boldsymbol{A}, D\chi_E)_\lambda$, thus covering also the case of Borel sets of infinite $(\boldsymbol{A}, \lambda)$-perimeter.

Proposition 7.2. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, $\lambda : \Omega \to [0,1]$ be Borel function and $E \subset \Omega$ be a Borel set. Then we have $\operatorname{supp}((\boldsymbol{A}, D\chi_E)_\lambda) \subseteq \partial^- E$.*

Proof. Since $\partial^- E$ is a closed set, then $\Omega \setminus \partial^- E$ is an open set. Therefore, let $\varphi \in C^1_c(\Omega)$ be such that $\operatorname{supp}(\varphi) \Subset \Omega \setminus \partial^- E$. By (3.3) we get

$$\langle (\boldsymbol{A}, D\chi_E)_\lambda \, , \, \varphi \rangle = - \int_\Omega \chi_E^\lambda \, d \operatorname{div}(\varphi \boldsymbol{A}) = - \int_{E^1} d \operatorname{div}(\varphi \boldsymbol{A}) - \int_{\partial^* E} \lambda \, d \operatorname{div}(\varphi \boldsymbol{A}).$$

Since $\partial^* E \subseteq \partial^- E$ and $\operatorname{supp}(\operatorname{div}(\varphi \boldsymbol{A})) \subset \operatorname{supp}(\varphi \boldsymbol{A}) \Subset \Omega \setminus \partial^- E$, we can conclude that the second term must be zero. As for the first one, we notice that

$$\Omega \setminus \partial^- E = E^{1,-} \cup E^{0,-},$$

where

$$E^{1,-} = \{x \in \mathbb{R}^N : \text{ there exists } r > 0 \text{ such that } \mathscr{L}^N(E \cap B_r(x)) = \mathscr{L}^N(B_r(x))\},$$
$$E^{0,-} = \{x \in \mathbb{R}^N : \text{ there exists } r > 0 \text{ such that } \mathscr{L}^N(E \cap B_r(x)) = 0\}.$$

It is easy to check that $E^{1,-} \subseteq E^1$, so that $E^1 \cap (\Omega \setminus \partial^- E) = E^{1,-}$. Now, since $\operatorname{supp}(\varphi) \Subset \Omega \setminus \partial^- E$, then there exists an open set $V \Subset E^{1,-} \cup E^{0,-}$ such that $\operatorname{supp}(\varphi) \subset V$. Since $E^{1,-} \cap E^{0,-} = \emptyset$, then $V_0 = V \cap E^{0,-}$ and $V_1 = V \cap E^{1,-}$ are open sets satisfying $V_j \Subset E^{j,-}$ for $j = 0, 1$. All in all, we get

$$\int_{E^1} d\,\mathrm{div}(\varphi \boldsymbol{A}) = \int_{E^{1,-}} d\,\mathrm{div}(\varphi \boldsymbol{A}) = \int_{E^{1,-}\cap V} d\,\mathrm{div}(\varphi \boldsymbol{A}) = \int_{V_1} d\,\mathrm{div}(\varphi \boldsymbol{A}) = 0,$$

thanks to Lemma 2.4, since $\varphi \in C^1_c(V_1 \cup V_0)$, and so $\varphi \in C^1_c(V_1)$, in particular. Therefore,

$$\langle (\boldsymbol{A}, D\chi_E)_\lambda\,,\, \varphi\rangle = 0$$

for all $\varphi \in C^1_c(\Omega)$ be such that $\mathrm{supp}(\varphi) \Subset \Omega \setminus \partial^- E$, and this ends the proof. □

Remark 7.3. *We point out that the control on the size of the support of the pairing distribution $(\boldsymbol{A}, D\chi_E)_\lambda$ given in Proposition 7.2 is in general too large. Indeed, we can find $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\mathbb{R}^N)$ and a Borel set E such that $(\boldsymbol{A}, D\chi_E) = 0$, while $\partial^- E$ has Hausdorff dimension equal to N. To this purpose, we let $N \geq 2$ and consider the set $F \subset \mathbb{R}$ defined in* [13, Example 3.9]*, satisfying $\dim_{\mathscr{H}}(\partial^* F) = 1$. Arguing analogously as it was done in* [13, Example 3.9]*, we define*

$$E = F \times \mathbb{R}^{N-1},$$

and we conclude that

$$\dim_{\mathscr{H}}(\partial^* E) = N = \dim_{\mathscr{H}}(\partial^- E),$$

since $\partial^* E \subseteq \partial^- E$. It is clear that χ_E is constant in all variables except for x_1. Hence, if now we set $\boldsymbol{A} = (0, 0, \ldots, 0, \chi_E(x))$, we see that $\chi_E \boldsymbol{A} = \boldsymbol{A}$ and so $\mathrm{div}\, \boldsymbol{A} = \mathrm{div}(\chi_E \boldsymbol{A}) = 0$. Thanks to Proposition 3.19, this implies $\chi_E \in BV^{\boldsymbol{A}}(\Omega) = BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$ for all Borel functions $\lambda : \Omega \to [0, 1]$, with $(\boldsymbol{A}, D\chi_E)_\lambda = 0$. In addition, this provides an example of a Borel set E such that $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\mathbb{R}^N) \setminus BV_{\mathrm{loc}}(\mathbb{R}^N)$ and $P_{\boldsymbol{A},\lambda}(E, \Omega) = 0$, while still being nonnegligible with respect to the measure $|\boldsymbol{A}|$; thus showing the degeneracy of the $(\boldsymbol{A}, \lambda)$-perimeter.

In analogy with the classical notion of perimeter, it is interesting to check whether the $(\boldsymbol{A}, \lambda)$-perimeter satisfies locality, additivity and similar properties.

Proposition 7.4. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$, $E, F \subset \Omega$ be Borel sets, and $\lambda : \Omega \to [0, 1]$ be a Borel function.*

(1) If $\mathscr{L}^N(E \Delta F) = 0$, then $(\boldsymbol{A}, D\chi_E)_\lambda = (\boldsymbol{A}, D\chi_F)_\lambda$ in the sense of distributions.

(2) In the sense of distributions, we have

$$(\boldsymbol{A}, D\chi_E)_\lambda = -(\boldsymbol{A}, D\chi_{\Omega\setminus E})_{1-\lambda}, \tag{7.3}$$

so that $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$ if and only if $\chi_{\Omega\setminus E} \in BV^{\boldsymbol{A},1-\lambda}(\Omega)$, with

$$P_{\boldsymbol{A},\lambda}(E, \Omega) = P_{\boldsymbol{A},1-\lambda}(\Omega \setminus E, \Omega).$$

In particular, if $\lambda \equiv \frac{1}{2}$, then we see that $(\boldsymbol{A}, D\chi_E) = -(\boldsymbol{A}, D\chi_{\Omega\setminus E})$ in the sense of distributions, so that $\chi_E \in BV^{\boldsymbol{A}}(\Omega)$ if and only if $\chi_{\Omega\setminus E} \in BV^{\boldsymbol{A}}(\Omega)$, with

$$P_{\boldsymbol{A},\frac{1}{2}}(E, \Omega) = P_{\boldsymbol{A},\frac{1}{2}}(\Omega \setminus E, \Omega).$$

(3) If $\mathscr{L}^N(E \cap F) = 0$ and $(\partial^* E) \cap F^1 = (\partial^* F) \cap E^1 = \partial^* E \cap \partial^* F = \emptyset$, then

$$(\boldsymbol{A}, D\chi_{E\cup F})_\lambda = (\boldsymbol{A}, D\chi_E)_\lambda + (\boldsymbol{A}, D\chi_F)_\lambda, \tag{7.4}$$

in the sense of distributions. In addition, if $\chi_E, \chi_F \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$*, then* $\chi_{E\cup F} \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$*, with*

$$|(\boldsymbol{A}, D\chi_{E\cup F})_\lambda| \le |(\boldsymbol{A}, D\chi_E)_\lambda| + |(\boldsymbol{A}, D\chi_F)_\lambda| \quad \text{on } \Omega. \tag{7.5}$$

Furthermore, if $\partial^- E \cap \partial^- F = \emptyset$*, then*

$$|(\boldsymbol{A}, D\chi_{E\cup F})_\lambda| = |(\boldsymbol{A}, D\chi_E)_\lambda| + |(\boldsymbol{A}, D\chi_F)_\lambda| \quad \text{on } \Omega. \tag{7.6}$$

Proof. Clearly, $\mathscr{L}^N(E\Delta F) = 0$ implies that $\chi_E(x) = \chi_F(x)$ for $\mathscr{L}^N$-a.e. $x \in \Omega$, and so

$$\chi^\lambda_E(x) = \chi^\lambda_F(x) \quad \text{for all } x \in \Omega.$$

Hence, the equality of the λ-pairing distributions in point (1) follows immediately from Definition 3.1. Then, we note that $\chi_{\Omega\setminus E} = 1 - \chi_E$ implies

$$\chi^+_{\Omega\setminus E} = 1 - \chi^-_E = 1 - \chi_{E^1}$$

and

$$\chi^-_{\Omega\setminus E} = 1 - \chi^+_E = 1 - \chi_{E^1\cup\partial^* E}.$$

All in all, we get

$$\chi^\lambda_{\Omega\setminus E} = 1 - \chi_{E^1} - (1-\lambda)\chi_{\partial^* E} = 1 - \chi^{1-\lambda}_E.$$

Therefore, (7.3) is an easy consequence of Lemma 3.2; and the rest of point (2) follows immediately. Finally, under the assumptions of point (3), we notice that

$$\partial^*(E\cup F) = \partial^* E \cup \partial^* F \quad \text{and} \quad (E\cup F)^1 = E^1 \cup F^1,$$

which follow from [34, Proposition 2.1] and a straightforward computation. Therefore, we get $\chi^\lambda_{E\cup F} = \chi^\lambda_E + \chi^\lambda_F$, so that (7.4) follows from Definition 3.1. Then, if $\chi_E, \chi_F \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, then (7.4) easily implies (7.5), so that $\chi_{E\cup F} \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$. Finally, if $\partial^- E \cap \partial^- F = \emptyset$, thanks to Proposition 7.2 we see that the pairings $(\boldsymbol{A}, D\chi_E)_\lambda$ and $(\boldsymbol{A}, D\chi_F)_\lambda$ have disjoint supports; and so, by taking the total variations in (7.4), we deduce (7.6). □

Remark 7.5. *We point out that only the assumption* $\mathscr{L}^N(E\cap F) = 0$ *is not enough in point (3) of Proposition 7.4. Indeed, let us consider* $\Omega = \mathbb{R}^N, E = (-1,0)\times(0,1)^{N-1}$ *and* $F = (0,1)^N$*. In this case, we have* $E^1 = E$*,* $F^1 = F$*,* $\partial^* E = \partial E$*,* $\partial^* F = \partial F$ *and*

$$(E\cup F)^1 = E^1 \cup F^1 \cup L \quad \text{and } \partial^*(E\cup F) = (\partial^* E \cup \partial^* F)\setminus L,$$

where $L = \{0\}\times(0,1)^{N-1}$*; so that we have*

$$\begin{aligned} \chi^\lambda_{E\cup F} = \chi_{E^1\cup F^1\cup L} + \lambda\chi_{(\partial^* E\cup\partial^* F)\setminus L} &= \chi_{E^1} + \chi_{F^1} + \lambda\chi_{\partial^* E} + \lambda\chi_{\partial^* F} + (1-2\lambda)\chi_L \\ &= \chi^\lambda_E + \chi^\lambda_F + (1-2\lambda)\chi_L. \end{aligned} \tag{7.7}$$

Therefore, in general we cannot have (7.4) *whenever the set* $L\cap\left\{\lambda \ne \frac{1}{2}\right\}$ *is not negligible with respect to the measure* $|\boldsymbol{A}| + |\operatorname{div}\boldsymbol{A}|$*. This happens for instance if we choose*

$$\boldsymbol{A}(x) = (\chi_{\{x_1>0\}}, 0, \dots, 0),$$

in which case we have $\operatorname{div}\boldsymbol{A} = \mathscr{H}^{N-1} \llcorner \{x_1 = 0\}$*. Due to the fact that* $\boldsymbol{A} \in \mathcal{DM}^\infty(\mathbb{R}^N)$ *and* E, F *are sets of finite perimeter, we exploit point (3) of Proposition 3.5 to conclude that*

$\chi_E, \chi_F, \chi_{E\cup F} \in BV^{\boldsymbol{A},\lambda}(\mathbb{R}^N)$ *for all Borel functions* $\lambda : \mathbb{R}^N \to [0,1]$. *Hence, by* (3.6) *and* (7.7) *we obtain*

$$\begin{aligned}(\boldsymbol{A}, D\chi_{E\cup F})_\lambda &= -\chi^\lambda_{E\cup F} \operatorname{div} \boldsymbol{A} + \operatorname{div}(\chi_{E\cup F}\boldsymbol{A}) \\ &= -\left(\chi^\lambda_E + \chi^\lambda_F + (1-2\lambda)\chi_L\right) \operatorname{div} \boldsymbol{A} + \operatorname{div}(\chi_E \boldsymbol{A}) + \operatorname{div}(\chi_F \boldsymbol{A}) \\ &= (\boldsymbol{A}, D\chi_E)_\lambda + (\boldsymbol{A}, D\chi_F)_\lambda - (1-2\lambda)\mathscr{H}^{N-1} \llcorner L.\end{aligned}$$

Hence, the λ*-pairing is not additive, as soon as* $\lambda(x) \neq \frac{1}{2}$ *for* $\mathscr{H}^{N-1}$*-a.e.* $x \in L$. *It is important to notice that the case* $\lambda \equiv \frac{1}{2}$ *is not privileged: to see this, we consider* $E = (0,1)^N$ *and* $F = (-1,0)^N$. *Again,* $E^1 = E$, $F^1 = F$, $\partial^* E = \partial E$, $\partial^* F = \partial F$ *and*

$$\begin{aligned}\chi^\lambda_{E\cup F} = \chi_{E^1\cup F^1} + \lambda\chi_{\partial^* E\cup\partial^* F} &= \chi_{E^1} + \chi_{F^1} + \lambda\chi_{\partial^* E} + \lambda\chi_{\partial^* F} - \lambda\chi_{\{0\}} \\ &= \chi^\lambda_E + \chi^\lambda_F - \lambda\chi_{\{0\}},\end{aligned} \tag{7.8}$$

where we denote by 0 *the origin of* $\mathbb{R}^N$. *In this case,* (7.4) *fails to hold if we choose*

$$\boldsymbol{A}(x) = \frac{1}{N\omega_N}\frac{x}{|x|^N} \quad \text{and} \quad \lambda(0) \neq 0.$$

In this case, we have $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $\operatorname{div} \boldsymbol{A} = \delta_0$, *and so, arguing as in Remark 3.17 with minor changes, we get* $\chi_E, \chi_F, \chi_{E\cup F} \in BV^{\boldsymbol{A},\lambda}(\mathbb{R}^N)$ *for all Borel functions* $\lambda : \mathbb{R}^N \to [0,1]$. *Thus we exploit* (3.6) *and* (7.8) *to see that*

$$\begin{aligned}(\boldsymbol{A}, D\chi_{E\cup F})_\lambda &= -\chi^\lambda_{E\cup F} \operatorname{div} \boldsymbol{A} + \operatorname{div}(\chi_{E\cup F}\boldsymbol{A}) \\ &= -\left(\chi^\lambda_E + \chi^\lambda_F - \lambda\chi_{\{0\}}\right) \operatorname{div} \boldsymbol{A} + \operatorname{div}(\chi_E \boldsymbol{A}) + \operatorname{div}(\chi_F \boldsymbol{A}) \\ &= (\boldsymbol{A}, D\chi_E)_\lambda + (\boldsymbol{A}, D\chi_F)_\lambda + \lambda(0)\delta_0.\end{aligned}$$

In analogy with point (4) of Proposition 3.5, we investigate the relation between $(\boldsymbol{A}, \lambda)$-perimeters for different choices of λ, as long as $\boldsymbol{A}$ is a summable divergence-measure field.

Proposition 7.6. *If* $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ *and* $E \subset \Omega$ *is a Borel set, then for every couple of Borel functions* $\lambda_1, \lambda_2 : \Omega \to [0,1]$ *we have*

$$(\boldsymbol{A}, D\chi_E)_{\lambda_1} - (\boldsymbol{A}, D\chi_E)_{\lambda_2} = (\lambda_2 - \lambda_1) \operatorname{div} \boldsymbol{A} \llcorner \partial^* E \tag{7.9}$$

in the sense of distributions on Ω. *In particular, if* $|\operatorname{div} \boldsymbol{A}|(\partial^* E) = 0$, *then*

$$(\boldsymbol{A}, D\chi_E)_{\lambda_1} = (\boldsymbol{A}, D\chi_E)_{\lambda_2}$$

in the sense of distributions on Ω. *Therefore, if* $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$, *we get*

$$P_{\boldsymbol{A},\lambda_1}(E,\Omega) < +\infty \quad \text{if and only if} \quad P_{\boldsymbol{A},\lambda_2}(E,\Omega) < +\infty,$$

and, as long as one of these conditions holds, then (7.9) *holds in the sense of Radon measures on* Ω.

Proof. We apply the definition of the λ-pairing distribution (3.2) to $u = \chi_E$ and $\lambda = \lambda_1$, and $u = \chi_E$ and $\lambda = \lambda_2$, respectively. Then, we take the difference between the two formulas and for all $\varphi \in C^1_c(\Omega)$ we get

$$\left\langle (\boldsymbol{A}, D\chi_E)_{\lambda_1} - (\boldsymbol{A}, D\chi_E)_{\lambda_2}, \varphi \right\rangle = -\int_\Omega \varphi \left(\chi^{\lambda_1}_E - \chi^{\lambda_2}_E\right) d \operatorname{div} \boldsymbol{A}.$$

Therefore, (7.9) follows by exploiting (7.1), where the right hand side clearly vanishes as soon as $\partial^* E$ is $|\operatorname{div} \boldsymbol{A}|$-negligible. Finally, if $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$, we see that the right hand side of (7.9) is always a finite Radon measure, so that $(\boldsymbol{A}, D\chi_E)_{\lambda_1} \in \mathcal{M}(\Omega)$ if and only if $(\boldsymbol{A}, D\chi_E)_{\lambda_2} \in \mathcal{M}(\Omega)$. ☐

Remark 7.7. *In light of* (7.9)*, if $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$ and $\lambda(x) = \frac{1}{2}$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \partial^* E$, then we obtain $(\boldsymbol{A}, D\chi_E)_\lambda = (\boldsymbol{A}, D\chi_E)$. Therefore, by point (2) of Proposition 7.4, these assumptions are also sufficient to ensure*

$$(\boldsymbol{A}, D\chi_{\Omega\setminus E})_\lambda = (\boldsymbol{A}, D\chi_{\Omega\setminus E}) = -(\boldsymbol{A}, D\chi_E) = -(\boldsymbol{A}, D\chi_E)_\lambda,$$

and therefore $P_{\boldsymbol{A},\lambda}(E, \Omega) = P_{\boldsymbol{A},\lambda}(\Omega \setminus E, \Omega)$.

In analogy with Proposition 3.14, we list the absolute continuity properties of the $(\boldsymbol{A}, \lambda)$-perimeter.

Proposition 7.8. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, $\lambda : \Omega \to [0, 1]$ be Borel function, E be a Borel set and $\Omega' \Subset \Omega$ be an open set.*

i) If E is a set of locally finite perimeter in Ω, then $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ and we have

$$(\boldsymbol{A}, D\chi_E)_\lambda = \big((1-\lambda)\mathrm{Tr}^i(\boldsymbol{A}, \partial^* E) + \lambda \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E)\big)\, \mathscr{H}^{N-1} \llcorner \partial^* E, \tag{7.10}$$

where $\mathrm{Tr}^i(\boldsymbol{A}, \partial^* E), \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E) \in L^\infty_{\mathrm{loc}}(\partial^* E, \mathscr{H}^{N-1})$.

ii) If $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$, then

$$|(\boldsymbol{A}, D\chi_E)_\lambda| \le 2c_N \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} \mathscr{H}^{N-1} \llcorner \partial^- E \quad \text{on } \Omega', \tag{7.11}$$

where c_N is as in Proposition 3.14.

In addition, if $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$, E is a set of finite perimeter in Ω in (i), and $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$ in (ii), then the respective statements hold true globally.

Proof. We notice that clearly $\chi_E^\lambda \in L^1_{\mathrm{loc}}(\Omega, |\operatorname{div} \boldsymbol{A}|)$, so that, by point (3) of Proposition 3.5, we conclude that $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$. Hence, by (3.12), we get

$$(\boldsymbol{A}, D\chi_E)_\lambda = (1-\lambda)(\boldsymbol{A}, D\chi_E)_0 + \lambda(\boldsymbol{A}, D\chi_E)_1.$$

Hence, the result is a consequence of (2.25) and (2.26). As for point (ii), thanks to Proposition 3.14 (ii) we see that

$$|(\boldsymbol{A}, D\chi_E)_\lambda| \le 2c_N \|\boldsymbol{A}\|_{L^\infty(\Omega';\mathbb{R}^N)} \mathscr{H}^{N-1} \quad \text{on } \Omega'.$$

Hence, it is enough to apply Proposition 7.2 to conclude. Finally, it is clear that, under global assumptions, the normal traces are in $L^\infty(\partial^* E, \mathscr{H}^{N-1})$ in point (i) and (7.11) holds on Ω. ☐

Remark 7.9. *We point out that the inclusion $\partial^* E \subseteq \partial^- E$ might be strict, since $\partial^- E$ can have positive Lebesgue measure (see* [33, Proposition 12.19 and Example 12.25]*). In addition, $\mathscr{H}^{N-1} \llcorner \partial^- E$ is a Radon measure if and only if E is a set of locally finite perimeter, in which case $\mathscr{H}^{N-1}(\partial^- E \setminus \partial^* E) = 0$. Indeed, if $\mathscr{H}^{N-1} \llcorner \partial^- E$ is a Radon measure, then for every compact set K we get*

$$\mathscr{H}^{N-1}(\partial^* E \cap K) \le \mathscr{H}^{N-1}(\partial^- E \cap K) < +\infty.$$

Hence, Federer's Theorem [24, Theorem 5.23] *implies that E is a set of locally finite perimeter. Therefore, we can apply De Giorgi's and Federer's Theorems to conclude that the perimeter* *measure satisfies* $|D\chi_E| = \mathscr{H}^{N-1} \llcorner \partial^* E$ *(see for instance* [2, Theorem 3.59 and 3.61]*). Since the perimeter measure is supported on* $\partial^- E$*, thanks to* [33, Proposition 12.19]*, we conclude that* $\partial^- E \setminus \partial^* E$ *is* $\mathscr{H}^{N-1}$*-negligible. The reverse implication is instead trivial.*

Remark 7.10. *It is interesting to notice that a representation analogous to* (7.10) *may hold even for sets which do not have locally finite perimeter. To this purpose, in the case* $N \geq 2$ *we provide an example of a field* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *and of a Borel set* F *satisfying* $\chi_F \notin BV_{\mathrm{loc}}(\mathbb{R}^N)$ *such that*

$$(\boldsymbol{A}, D\chi_F)_\lambda = h\, \mathscr{H}^{N-1} \llcorner L$$

for every Borel function $\lambda : \mathbb{R}^N \to [0,1]$*, where* $h \in L^\infty_{\mathrm{loc}}(L, \mathscr{H}^{N-1})$ *and* $L \subset \partial^* F$ *is a Borel set such that* $\mathscr{H}^{N-1}(L) < +\infty$*. We argue similarly as in* [9, Remark 4.9]*: we consider the set* E *defined therein; that is, the open bounded set in* $\mathbb{R}^2$ *whose boundary is given by*

$$\partial E = \big(\{0\} \times [0,1]\big) \cup \big([0,1] \times \{0\}\big) \cup \big([0, 1 + \log 2] \times \{1\}\big) \cup S,$$

where

$$\begin{aligned} S = & \left(\{1\} \times \left[0, \frac{1}{2}\right]\right) \bigcup \left([1,2] \times \left\{\frac{1}{2}\right\}\right) \bigcup \left(\bigcup_{n \geq 1} \left\{ 1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \right\} \times \left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right] \right) \\ & \bigcup \left(\bigcup_{n \geq 1} \left[1 + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}, 1 + \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k} \right] \times \left\{ 1 - \frac{1}{2^{2n+1}} \right\} \right) \\ & \bigcup \left(\bigcup_{n \geq 1} \left[1 + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}, 1 + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \right] \times \left\{ 1 - \frac{1}{2^{2n}} \right\} \right). \end{aligned}$$

It is clear that $\mathscr{H}^1(S) = +\infty$*. Then, we set*

$$F = E \times (0,1)^{N-2},$$

which clearly satisfies $\chi_F \notin BV_{\mathrm{loc}}(\mathbb{R}^N)$*, and, in particular,* $\mathscr{H}^{N-1}(\partial^* F) = \mathscr{H}^{N-1}(\partial^- F) = +\infty$*. However, we can exploit the same approach as in* [9, Remark 4.9] *to show that* $D_{x_1}\chi_F \in \mathcal{M}(\mathbb{R}^N)$*, with*

$$\begin{aligned} D_{x_1}\chi_F = & \mathscr{H}^{N-1} \llcorner \big(\{0\} \times (0,1)^{N-1}\big) - \mathscr{H}^{N-1} \llcorner \left(\{1\} \times \left(0, \frac{1}{2}\right) \times (0,1)^{N-2}\right) \\ & - \mathscr{H}^{N-1} \llcorner \left(\bigcup_{n \geq 1} \left\{ 1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \right\} \times \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right) \times (0,1)^{N-2} \right). \end{aligned}$$

Let now $\boldsymbol{A}(x) = (f(\hat{x}_1) g(x_1), 0, \ldots, 0)$*, where* $\hat{x}_1 = (x_2, x_3, \ldots, x_N)$*, for some* $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{N-1})$ *and* $g \in C^1_c(\mathbb{R})$*. It is immediate to see that* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\mathbb{R}^N)$*,*

$$\operatorname{div} \boldsymbol{A} = f(\hat{x}_1) g'(x_1) \mathscr{L}^N, \tag{7.12}$$

and

$$\operatorname{div}(\chi_F \boldsymbol{A}) = D_{x_1}(f(\hat{x}_1) g(x_1) \chi_F(x)) = f(\hat{x}_1) g(x_1) D_{x_1}\chi_F + \chi_F(x) f(\hat{x}_1) g'(x_1) \mathscr{L}^N,$$

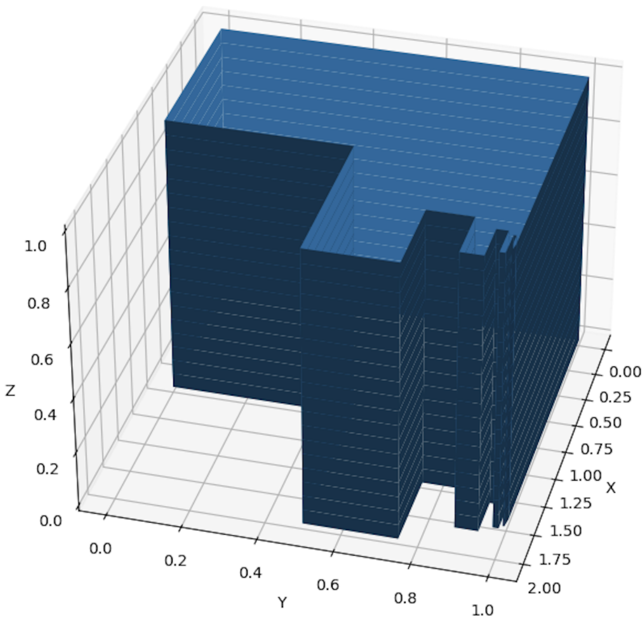

FIGURE 1. The set F in the case $N = 3$

since f is constant in x_1 and $g \in C^1_c(\mathbb{R})$ (see [9, Remark 4.9] *for more details). Now, since $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$, thanks to Proposition 3.19 we know that, for every Borel function $\lambda : \mathbb{R}^N \to [0,1]$, $BV^{\boldsymbol{A},\lambda}(\mathbb{R}^N) = BV^{\boldsymbol{A}}(\mathbb{R}^N)$ and $\chi_F^\lambda(x) = \chi_F(x)$ for $|\operatorname{div} \boldsymbol{A}|$-a.e. $x \in \mathbb{R}^N$. Therefore, we get*

$$(\boldsymbol{A}, D\chi_F)_\lambda = (\boldsymbol{A}, D\chi_F) = -\chi_F \operatorname{div} \boldsymbol{A} + \operatorname{div}(\chi_F \boldsymbol{A}) = f(\hat{x}_1) g(x_1) D_{x_1} \chi_F = h \, \mathscr{H}^{N-1} \llcorner L,$$

where

$$\begin{aligned} L = & \big(\{0\} \times (0,1)^{N-1}\big) \cup \left(\{1\} \times \left(0, \frac{1}{2}\right) \times (0,1)^{N-2}\right) \cup \\ & \cup \left(\bigcup_{n \geq 1} \left\{ 1 + \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \right\} \times \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right) \times (0,1)^{N-2} \right) \end{aligned} \tag{7.13}$$

and

$$h(x) = f(\hat{x}_1) g(x_1) \left(\chi_{\{0\} \times (0,1)^{N-1}}(x) - \chi_{L \setminus (\{0\} \times (0,1)^{N-1})}(x) \right). \tag{7.14}$$

In particular, we conclude that $\chi_F \in BV^{\boldsymbol{A}}(\mathbb{R}^N) \setminus BV_{\mathrm{loc}}(\mathbb{R}^N)$. One could even define the normal trace of $\boldsymbol{A}$ on $\partial^* F$ by setting

$$\operatorname{Tr}(\boldsymbol{A}, \partial^* F)(x) := \begin{cases} h(x) & \text{if } x \in L \\ 0 & \text{if } x \in \partial^* F \setminus L \end{cases},$$

in this way recovering the representation for the pairing

$$(\boldsymbol{A}, D\chi_F) = \operatorname{Tr}(\boldsymbol{A}, \partial^* F) \mathscr{H}^{N-1} \llcorner \partial^* F, \tag{7.15}$$

which is an extension of (7.10) *in the case* $|\operatorname{div}\boldsymbol{A}| \ll \mathscr{L}^N$*, for which interior and exterior normal traces on the measure theoretic boundary of sets of locally finite perimeters coincide (this fact is a simple consequence of* [15, Theorem 4.2]*, for instance).*

Finally, we notice that this example works even for $f \in L^p_{\mathrm{loc}}(\mathbb{R}^{N-1})$ *for any* $p \in [1, +\infty]$*, which gives us* $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $h \in L^p_{\mathrm{loc}}(L, \mathscr{H}^{N-1})$*.*

As an immediate consequence of Proposition 3.15 and Proposition 7.2, we get the following absolute continuity property in the case $\boldsymbol{A}$ is not essentially bounded.

Proposition 7.11. *Let* $N \geq 2$*,* $p \in \left[\frac{N}{N-1}, +\infty\right)$ *and* $q \in (1, N]$ *be conjugate exponents; that is, satisfying* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* $\boldsymbol{A} \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$*,* $\lambda : \Omega \to [0,1]$ *be Borel function and* $E \subset \Omega$ *be a Borel set. If* $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$*, then* $|(\boldsymbol{A}, D\chi_E)_\lambda|(B) = 0$ *for every Borel set* $B \subset \Omega$ *which is* σ*-finite with respect to the measure* $\mathscr{H}^{N-q} \llcorner \partial^- E$*.*

As for $BV^{\boldsymbol{A},\lambda}$-functions, we consider a notion of $(\boldsymbol{A}, \lambda)$-convergence for sets of (locally) finite $(\boldsymbol{A}, \lambda)$-perimeter.

Definition 7.12. *Let* $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$*. We say that a sequence of Borel sets* $(E_n)_n$ $(\boldsymbol{A}, \lambda)$*-converges to a Borel set* E *if*

(1) $\chi^\lambda_{E_n} \rightharpoonup \chi^\lambda_E$ *in* $L^1_{\mathrm{loc}}(\Omega, |\boldsymbol{A}|)$*,*

(2) $\chi^\lambda_{E_n} \rightharpoonup \chi^\lambda_E$ *in* $L^1_{\mathrm{loc}}(\Omega, |\operatorname{div}\boldsymbol{A}|)$*.*

In the particular case $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ with $|\operatorname{div}\boldsymbol{A}| \ll \mathscr{L}^N$, we refer to this convergence as $\boldsymbol{A}$-convergence, as established in Remark 4.2.

As an immediate consequence of Theorem 4.3, we get the following lower semicontinuity property of the $(\boldsymbol{A}, \lambda)$-perimeter.

Proposition 7.13. *The function* $E \mapsto P_{\boldsymbol{A},\lambda}(E, \Omega)$ *is lower semicontinuous with respect to the* $(\boldsymbol{A}, \lambda)$*-convergence.*

The lower semicontinuity of the $(\boldsymbol{A}, \lambda)$-perimeter naturally suggests the question whether there is any sort of compactness for families of sets with uniformly bounded $(\boldsymbol{A}, \lambda)$-perimeter. This of course requires first that such perimeter defines some sort of seminorm: due to Corollary 5.2, we see that this happens as soon as $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ satisfies $|\operatorname{div}\boldsymbol{A}| \ll \mathscr{L}^N$. However, similarly to the case of $BV^{\boldsymbol{A}}$-functions (Example 5.5), in the case of dimension $N \geq 2$, we can find a counterexample to the compactness with respect to the $\boldsymbol{A}$-convergence.

Example 7.14. *For* $k \geq 1$ *we set*

$$F_k = \bigcup_{j=0}^{2^{k-1}-1} \left(\frac{2j}{2^k}, \frac{2j+1}{2^k}\right).$$

It is not difficult to see that $\mathscr{L}^1(F_k) = \frac{1}{2}$ *for all* $k \geq 1$*, and that*

$$\mathscr{L}^1 \llcorner F_k \rightharpoonup \frac{1}{2}\mathscr{L}^1 \llcorner (0,1) \quad \text{in } \mathcal{M}(\mathbb{R}).$$

In particular, this means that the sequence of sets $(F_k)_{k\geq 1}$ *does not admit any subsequence converging in measure. Let now* $N \geq 2$*,* $\Omega = (-1,1)^N$ *and* $\boldsymbol{A}(x) = (1, 0, \ldots, 0)$*. It is clear that*

$\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ and $\operatorname{div} \boldsymbol{A} = 0$. We set

$$E_k = (-1,1)^{N-1} \times F_k,$$

and it is easy to see that $(\boldsymbol{A}, D\chi_{E_k}) = 0$, so that $P_{\boldsymbol{A},\lambda}(E_k, \Omega) = 0$ for all $k \geq 1$. Actually, $\chi_{E_k} \in BV^{\boldsymbol{A}}(\Omega) = BV^{\boldsymbol{A},\lambda}(\Omega)$ for all Borel functions $\lambda : \Omega \to [0,1]$, due to Proposition 3.19. In addition, by Proposition 5.1, we know that $BV^{\boldsymbol{A}}(\Omega)$ is a Banach space, and we see that

$$\|\chi_{E_k}\|_{BV^{\boldsymbol{A}}(\Omega)} = \|\chi_{E_k}\|_{L^1(\Omega, |\boldsymbol{A}|)} = 2^{N-1} \mathscr{L}^1(F_k) = 2^{N-2} \quad \text{for all } k \geq 1.$$

On the other hand, we notice that

$$\begin{aligned} \chi_{E_k} |\boldsymbol{A}| \mathscr{L}^N = \mathscr{L}^{N-1} \llcorner (-1,1)^{N-1} \otimes \mathscr{L}^1 \llcorner F_k \rightharpoonup \mathscr{L}^{N-1} \llcorner (-1,1)^{N-1} \otimes \frac{1}{2} \mathscr{L}^1 \llcorner (0,1) \\ = \frac{|\boldsymbol{A}|}{2} \mathscr{L}^N \llcorner (-1,1)^{N-1} \times (0,1) \quad \text{in } \mathcal{M}(\Omega). \end{aligned}$$

Thus, the sequence of sets $(E_k)_{k \geq 1}$ is uniformly bounded in $BV^{\boldsymbol{A}}(\Omega)$, but it is not weakly compact with respect to the $\boldsymbol{A}$-convergence.

8. Gauss-Green and integration by parts formulas

As a remarkable consequence of the previous sections, we establish Gauss-Green and integration by parts formulas in our framework.

Theorem 8.1. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and let $E \Subset \Omega$ be a Borel set. If $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$ for some Borel function $\lambda : \Omega \to [0,1]$, then we have*

$$\operatorname{div} \boldsymbol{A}(E^1) + \int_{\partial^* E} \lambda \, d \operatorname{div} \boldsymbol{A} = - \int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_\lambda \, . \tag{8.1}$$

If $\chi_E \in BV^{\boldsymbol{A},0}(\Omega)$, then

$$\operatorname{div} \boldsymbol{A}(E^1) = - \int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_0 \, ; \tag{8.2}$$

if instead $\chi_E \in BV^{\boldsymbol{A},1}(\Omega)$, then

$$\operatorname{div} \boldsymbol{A}(E^1 \cup \partial^* E) = - \int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_1 \, . \tag{8.3}$$

Proof. It is easy to see that (8.1) follows by applying Lemma 3.9 to $u = \chi_E$ and exploiting (7.1). Then, (8.2) and (8.3) are particular cases of (8.1) for $\lambda \equiv 0$ and $\lambda \equiv 1$, respectively. □

Corollary 8.2. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and let $E \Subset \Omega$ be a Borel set. If $\chi_E \in BV^{\boldsymbol{A},0}(\Omega) \cap BV^{\boldsymbol{A},1}(\Omega)$, then we have*

$$\operatorname{div} \boldsymbol{A}(\partial^* E) = \int_{\partial^- E} d\big((\boldsymbol{A}, D\chi_E)_0 - (\boldsymbol{A}, D\chi_E)_1\big) \, . \tag{8.4}$$

In addition, if $\boldsymbol{A} \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ and $E \subset \Omega$ is a Borel set satisfying $\chi_E \in BV^{\boldsymbol{A},\lambda}_{\mathrm{loc}}(\Omega)$ for some Borel function $\lambda : \Omega \to [0,1]$, then

$$\operatorname{div} \boldsymbol{A} \llcorner \partial^* E = (\boldsymbol{A}, D\chi_E)_0 - (\boldsymbol{A}, D\chi_E)_1 \quad \text{on } \Omega. \tag{8.5}$$

Proof. We obtain (8.4) by subtracting (8.2) from (8.3). As for (8.5), it is clear that $\boldsymbol{A} \in \mathcal{DM}^1(\Omega')$ for every open set $\Omega' \Subset \Omega$, therefore the result follows by applying Proposition 7.6 to the couple $\lambda_1 = 0$ and $\lambda_2 = 1$. □

Remark 8.3. *In the case* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ *and* $\chi_E \in BV(\Omega)$*, then we know that* $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$ *for every Borel function* $\lambda : \Omega \to [0,1]$*, by Proposition 7.8. Hence,* $\chi_E \in BV^{\boldsymbol{A},0}(\Omega) \cap BV^{\boldsymbol{A},1}(\Omega)$*, and so, taking into account* (7.10)*, from the Gauss–Green formulas* (8.2) *and* (8.3) *we retrieve* (2.24)*. Analogously, Corollary* (8.2) *is a generalization of*

$$\operatorname{div} \boldsymbol{A} \llcorner \partial^* E = \big(\mathrm{Tr}^i(\boldsymbol{A}, \partial^* E) - \mathrm{Tr}^e(\boldsymbol{A}, \partial^* E)\big)\, \mathscr{H}^{N-1} \llcorner \partial^* E,$$

for which we refer to [15, Corollary 3.5 and Theorem 4.2]*.*

We exploit two examples seen in the previous sections to show some applications of our general Gauss–Green formulas.

Example 8.4. *Let* $\boldsymbol{A}(x) = (f(\hat{x}_1)g(x_1), 0, \ldots, 0)$*, where* $\hat{x}_1 = (x_2, x_3, \ldots, x_N)$*, for some* $f \in L^\infty_{\rm loc}(\mathbb{R}^{N-1})$ *and* $g \in C^1_c(\mathbb{R})$*. Let* F *be the Borel set in Remark 7.10. Then, we know that* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\mathbb{R}^N)$ *with* $|\operatorname{div} \boldsymbol{A}| \ll \mathscr{L}^N$*,* $\chi_F \in BV^{\boldsymbol{A}}(\mathbb{R}^N) = BV^{\boldsymbol{A},\lambda}(\mathbb{R}^N)$ *and* $(\boldsymbol{A}, D\chi_F) = (\boldsymbol{A}, D\chi_F)_\lambda$ *for every Borel function* $\lambda : \Omega \to [0,1]$*. Thus, we can apply* (8.1)*, or equivalently* (8.2) *or* (8.3)*, to* $\boldsymbol{A}$ *and* F *to get*

$$\int_F f(\hat{x}_1) g'(x_1)\, dx = \operatorname{div} \boldsymbol{A}(F^1) = -\int_{\partial^- F} d(\boldsymbol{A}, D\chi_F) = -\int_L h \, d\mathscr{H}^{N-1}, \tag{8.6}$$

thanks to (7.12) *and* (7.15)*, where* L *is given by* (7.13) *and* h *by* (7.14)*. We point out that* (8.6) *cannot be derived directly from the standard Gauss–Green formula for sets of locally finite perimeter, since* $\chi_F \notin BV_{\rm loc}(\mathbb{R}^N)$*.*

Example 8.5. *Let* $N \geq 2$*,* $\Omega = \mathbb{R}^N$*,* $\lambda : \mathbb{R}^N \to [0,1]$ *be a Borel function and*

$$\boldsymbol{A}(x) = \frac{1}{N\omega_N} \frac{x}{|x|^N},$$

be as in Remark 3.17. We apply (8.1) *to* $\boldsymbol{A}$ *and* $E = (0,1)^N$*, and exploit* (3.17) *to get*

$$\lambda(0) = \operatorname{div} \boldsymbol{A}(E^1) + \int_{\partial^* E} \lambda \, d \operatorname{div} \boldsymbol{A} = -\int_{\partial^- E} d(\boldsymbol{A}, D\chi_E)_\lambda = -\frac{1}{2^N} + \lambda(0) - \overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}(\partial(0,1)^N),$$

which easily implies

$$\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}(\partial(0,1)^N) = -\frac{1}{2^N}.$$

On the other hand, Proposition 7.2 ensures that $(\boldsymbol{A}, D\chi_E)_\lambda$ *is supported on* $\partial^- E = \partial(0,1)^N$*, so that also* $\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}$ *is supported on* $\partial(0,1)^N$*. Hence, by taking* $\varphi \equiv 1$ *on* $(-2,2)^N$ *in* (3.16)*, we see that*

$$\begin{aligned}\overline{(\boldsymbol{A}, D\chi_{(0,1)^N})}(\partial(0,1)^N) &= -\frac{1}{N\omega_N} \sum_{j=1}^N \int_{\partial(0,1)^N \cap \{x_j = 1\}} \frac{1}{(1 + |\hat{x}_j|^2)^{\frac{N}{2}}} \, d\mathscr{H}^{N-1}(x) \\ &= -\frac{1}{\omega_N} \int_{(0,1)^{N-1}} \frac{1}{(1+|y|^2)^{\frac{N}{2}}} \, d\mathscr{L}^{N-1}(y),\end{aligned}$$

since the integrals on each face of the cube are clearly equal. All in all, by setting $n = N-1$*, we deduce directly the following nice identity:*

$$\int_{(0,1)^n} \frac{1}{(1+|y|^2)^{\frac{n+1}{2}}} \, dy = \frac{\omega_{n+1}}{2^{n+1}} \quad \textit{for all} \;\; n \geq 1.$$

We were not able to find it in literature, and we do believe that a direct computation of such integrals would be a hard task.

As an immediate consequence of Theorem 8.1, we deduce the following general version of the integration by parts formula. For the reader's convenience, we clarify the notation adopted below. Namely,

$$BV^{u^{\lambda_1}\boldsymbol{A},\lambda_2}(\Omega) = \left\{ v \in X^{u^{\lambda_1}\boldsymbol{A},\lambda_2}(\Omega) : (u^{\lambda_1}\boldsymbol{A}, Dv)_{\lambda_2} \in \mathcal{M}(\Omega) \right\},$$

where $\lambda_1, \lambda_2 : \Omega \to [0,1]$ are Borel functions and

$$X^{u^{\lambda_1}\boldsymbol{A},\lambda_2}(\Omega) = \{ v \in \mathscr{B}(\Omega) : v^{\lambda_2} \in L^1(\Omega, |u^{\lambda_1}\boldsymbol{A}|) \cap L^1(\Omega, |\operatorname{div}(u^{\lambda_1}\boldsymbol{A})|) \}.$$

Theorem 8.6. *Let $\boldsymbol{A} \in \mathcal{DM}_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a Borel set. Let $\lambda_1, \lambda_2 : \Omega \to [0,1]$ be Borel functions. If $u \in BV^{\boldsymbol{A},\lambda_1}(\Omega)$ and $\chi_E \in BV^{u^{\lambda_1}\boldsymbol{A},\lambda_2}(\Omega)$ satisfy* $\operatorname{supp}(\chi_E^{\lambda_2} u^{\lambda_1}|\boldsymbol{A}|) \Subset \Omega$*, then we have*

$$\int_{E^1} u^{\lambda_1} \operatorname{div} \boldsymbol{A} + \int_{\partial^* E} \lambda_2 u^{\lambda_1} \operatorname{div} \boldsymbol{A} + \int_{E^1} d(\boldsymbol{A}, Du)_{\lambda_1} + \int_{\partial^* E} \lambda_2 \, d(\boldsymbol{A}, Du)_{\lambda_1} = - \int_{\partial^- E} d(u^{\lambda_1}\boldsymbol{A}, D\chi_E)_{\lambda_2}. \tag{8.7}$$

In particular, if $\lambda : \Omega \to [0,1]$ is Borel function, $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$, $\chi_E \in BV^{u^{\lambda}\boldsymbol{A},0}(\Omega)$ and they satisfy $\operatorname{supp}(\chi_E^{-} u^{\lambda}|\boldsymbol{A}|) \Subset \Omega$*, then we have*

$$\int_{E^1} u^{\lambda} \operatorname{div} \boldsymbol{A} + \int_{E^1} (\boldsymbol{A}, Du)_{\lambda} = - \int_{\partial^- E} d(u^{\lambda}\boldsymbol{A}, D\chi_E)_0 ; \tag{8.8}$$

while, if $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$, $\chi_E \in BV^{u^{\lambda}\boldsymbol{A},1}(\Omega)$ and they satisfy $\operatorname{supp}(\chi_E^{+} u^{\lambda}|\boldsymbol{A}|) \Subset \Omega$*, then we have*

$$\int_{E^1 \cup \partial^* E} u^{\lambda} \operatorname{div} \boldsymbol{A} + \int_{E^1 \cup \partial^* E} (\boldsymbol{A}, Du)_{\lambda} = - \int_{\partial^- E} d(u^{\lambda}\boldsymbol{A}, D\chi_E)_1 . \tag{8.9}$$

Proof. Since $\operatorname{supp}(\chi_E^{\lambda_2} u^{\lambda_1}|\boldsymbol{A}|) \Subset \Omega$, we get $\operatorname{div}(\chi_E^{\lambda_2} u^{\lambda_1}\boldsymbol{A})(\Omega) = 0$ thanks to Lemma 2.4. Then, we apply (3.5), to the scalar function χ_E, the field $u^{\lambda_1}\boldsymbol{A}$ and the Borel function λ_2, obtaining

$$\operatorname{div}(\chi_E^{\lambda_2} u^{\lambda_1}\boldsymbol{A}) = \chi_E^{\lambda_2} \operatorname{div}(u^{\lambda_1}\boldsymbol{A}) + (u^{\lambda_1}\boldsymbol{A}, D\chi_E)_{\lambda_2}.$$

Now, we apply again (3.5), this time to the scalar function u, the field $\boldsymbol{A}$ and the Borel function λ_1, and we get

$$\operatorname{div}(\chi_E^{\lambda_2} u^{\lambda_1}\boldsymbol{A}) = \chi_E^{\lambda_2} u^{\lambda_1} \operatorname{div} \boldsymbol{A} + \chi_E^{\lambda_2} (\boldsymbol{A}, Du)_{\lambda_1} + (u^{\lambda_1}\boldsymbol{A}, D\chi_E)_{\lambda_2}.$$

Therefore, by evaluating this identity of measures over Ω and exploiting (7.1), we obtain (8.7). Finally, (8.8) and (8.9) are the cases $\lambda_1 = \lambda$ and $\lambda_2 \equiv 0$ and $\lambda_2 \equiv 1$, respectively. □

Remark 8.7. *The formulas* (8.8) *and* (8.9) *are generalizations of the integration by parts formulas in* [19, Theorem 6.3]*, respectively.*

9. The one-dimensional case

The case $N = 1$ presents essential differences in some instances, and therefore we consider it separately. First of all, for all open sets $\Omega \subset \mathbb{R}$ we have the following identifications:

$$\mathcal{DM}(\Omega) = \mathcal{DM}^1(\Omega) = BV(\Omega),$$

and analogously for the local versions. Indeed, if $N = 1$, then the operator div reduces only to the first (distributional) derivative, which we denote by D; so that the equivalence between $\mathcal{DM}^1(\Omega)$ and $BV(\Omega)$ is trivial. In addition, $\boldsymbol{A} \in \mathcal{DM}(\Omega)$ if and only if $\boldsymbol{A}, D\boldsymbol{A} \in \mathcal{M}(\Omega)$: this implies that $|\boldsymbol{A}| \ll \mathscr{L}^1$, with $\boldsymbol{A} = \overline{\boldsymbol{A}}\mathscr{L}^1$ for some $\overline{\boldsymbol{A}} \in BV(\Omega)$ (see for instance [2, Exercise 3.2]). Clearly, the opposite is also true; that is, for all $\mathbf{B} \in BV(\Omega)$ the measure $\mathbf{B}\mathscr{L}^1$ belongs to $\mathcal{DM}(\Omega)$. In other words, there is a bijection between $\mathcal{DM}(\Omega)$ and $BV(\Omega)$. Analogously, we see that

$$\mathcal{DM}^p(\Omega) = \{\boldsymbol{A} \in L^p(\Omega) : D\boldsymbol{A} \in \mathcal{M}(\Omega)\} \subset L^p(\Omega) \cap BV_{\mathrm{loc}}(\Omega)$$

for all $p \in (1, +\infty]$. Clearly, if $\mathscr{L}^1(\Omega) < +\infty$, then $\mathcal{DM}^p(\Omega) \subseteq \mathcal{DM}^1(\Omega) = BV(\Omega) \subseteq \mathcal{DM}^\infty(\Omega)$, due to the embedding $BV(\Omega) \subset L^\infty(\Omega)$, so that, in this case,

$$\mathcal{DM}^p(\Omega) = BV(\Omega) \ \text{ for all } p \in [1, +\infty].$$

It is also interesting to point out that in general we have $\mathcal{DM}^p(\Omega) \subset L^\infty(\Omega)$ for all $p \in [1, +\infty]$, given that, if $\boldsymbol{A} \in L^1_{\mathrm{loc}}(\Omega)$ is such that $D\boldsymbol{A} \in \mathcal{M}(\Omega)$, then $\boldsymbol{A} \in L^\infty(\Omega)$ (see [2, Sect. 3.2]). Because of these facts, in this section we shall choose $\boldsymbol{A} \in BV(\Omega)$.

We gather in the following proposition the main basic properties of $BV^{\boldsymbol{A},\lambda}$ in the case $N = 1$.

Proposition 9.1. *Let $\boldsymbol{A} \in BV(\Omega)$ and $\lambda : \Omega \to [0, 1]$ be a Borel function. Then we have $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ if and only if $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ and $u\boldsymbol{A} \in BV(\Omega)$, in which case we have*

$$D(u\boldsymbol{A}) = u^\lambda D\boldsymbol{A} + (\boldsymbol{A}, Du)_\lambda \quad \text{on } \Omega$$

and

$$\|u\boldsymbol{A}\|_{BV(\Omega)} \le \|u\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)} + \|u^\lambda\|_{L^1(\Omega, |D\boldsymbol{A}|)} + |(\boldsymbol{A}, Du)_\lambda|(\Omega). \tag{9.1}$$

In particular, if $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$, then $u\boldsymbol{A} \in L^\infty(\Omega)$.

Proof. Given that $N = 1$, we know that $\operatorname{div}(u\boldsymbol{A}) = D(u\boldsymbol{A})$, so that the first part of the statement follows from point (2) of Proposition 3.5. The second part is a consequence of the embedding $BV(\Omega) \subset L^\infty(\Omega)$. □

As in the higher dimensional case (without additional assumptions on $\boldsymbol{A}$, given that it is always locally essentially bounded), we point out that the inclusion of $BV(\Omega)$ in $BV^{\boldsymbol{A},\lambda}(\Omega)$ is strict.

Example 9.2. *Let $\Omega = (-1, 1)$, $\boldsymbol{A}(x) = \chi_{(\frac{1}{2},1)}(x)$ and $u(x) = \log(|x|)$. Then $u \in BV^{\boldsymbol{A},\lambda}(\Omega) \setminus BV_{\mathrm{loc}}(\Omega)$ for every Borel function $\lambda : \Omega \to [0, 1]$. Indeed, $D\boldsymbol{A} = \delta_{\frac{1}{2}}$, and so $u \in X^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel function $\lambda : \Omega \to [0, 1]$, given that $u \in C(\Omega \setminus \{0\})$, and so $u^\lambda(x) = u(x)$ for all $x \in \Omega \setminus \{0\}$. In addition, we see that $u\boldsymbol{A} \in BV(\Omega)$, with*

$$D(u\boldsymbol{A}) = \log\left(\frac{1}{2}\right)\delta_{\frac{1}{2}} + \frac{1}{x}\mathscr{L}^1 \llcorner \left(\frac{1}{2}, 1\right).$$

All in all, by Proposition 9.1, we obtain $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ with

$$(\boldsymbol{A}, Du)_\lambda = -u\left(\frac{1}{2}\right)\delta_{\frac{1}{2}} + \log\left(\frac{1}{2}\right)\delta_{\frac{1}{2}} + \frac{1}{x}\mathscr{L}^1 \llcorner \left(\frac{1}{2}, 1\right) = \frac{1}{x}\mathscr{L}^1 \llcorner \left(\frac{1}{2}, 1\right).$$

The peculiarity of the one dimensional case lies in the fact that we achieve compactness in a fashion similar to the classical BV space, under suitable assumptions.

Proposition 9.3. *Let $\boldsymbol{A} \in BV(\Omega)$ and $\overline{\lambda} : \Omega \to [0,1]$ be a Borel function. Let $(u_k) \subset BV^{\boldsymbol{A},\overline{\lambda}}(\Omega)$ be a sequence of functions such that*

$$\sup_{k \in \mathbb{N}} \|u_k\|_{L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)} + \|u_k^{\overline{\lambda}}\|_{L^1(\Omega, |D\boldsymbol{A}|)} + |(\boldsymbol{A}, Du_k)_{\overline{\lambda}}|(\Omega) < +\infty. \tag{9.2}$$

Then there exist $u \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$ such that $u\boldsymbol{A} \in BV(\Omega)$ and a subsequence $(u_{k_j})_{j \in \mathbb{N}}$ such that $u_{k_j} \to u \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$. In addition, assume that at least one of the following conditions is satisfied:

i) there exists $c > 0$ such that $|\boldsymbol{A}(x)| > c$ for $\mathscr{L}^1$-a.e. $x \in \Omega$,
ii) $\boldsymbol{A} \in W^{1,1}(\Omega)$,
iii) the sequence $(u_k)_{k \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$.

Then $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel function $\lambda : \Omega \to [0,1]$. Finally, if $\boldsymbol{A} \in W^{1,1}(\Omega)$ and the sequence $(u_k)_{k \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, then, possibly up to a further subsequence, $u_{k_j} \to u$ in $L^1(\Omega, |D\boldsymbol{A}|)$; so that $(u_{k_j})_{j \in \mathbb{N}}$ $\boldsymbol{A}$-converges to u.

Proof. By combining to (9.1) and (9.2), we see that the sequence $(u_k\boldsymbol{A})_{k \in \mathbb{N}}$ is uniformly bounded in $BV(\Omega)$. Thanks to the compactness theorem in BV [2, Theorem 3.3], we deduce that there exists $w_{\boldsymbol{A}} \in BV(\Omega)$ and a subsequence $(u_{k_j}\boldsymbol{A})_{j \in \mathbb{N}}$ such that $u_{k_j}\boldsymbol{A} \to w_{\boldsymbol{A}}$ in $L^1(\Omega)$. With a little abuse of notation, we still denote by $\boldsymbol{A}$ and $w_{\boldsymbol{A}}$ the Borel representatives of these two functions. Then, we set

$$u(x) = \begin{cases} \frac{w_{\boldsymbol{A}}(x)}{\boldsymbol{A}(x)} & \text{if } \boldsymbol{A}(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, we clearly get $u \in L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$, $u\boldsymbol{A} \in BV(\Omega)$ and $u_{k_j} \to u$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$. We assume now that there exists $c > 0$ such that $|\boldsymbol{A}(x)| > c$ for $\mathscr{L}^1$-a.e. $x \in \Omega$. Then, $u \in L^\infty(\Omega)$ by definition, given that $w_{\boldsymbol{A}} \in BV(\Omega) \subset L^\infty(\Omega)$. Hence, we immediately obtain $u^\lambda \in L^1(\Omega, |D\boldsymbol{A}|)$ for every Borel function $\lambda : \Omega \to [0,1]$, and so, by Proposition 9.1, we get $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$. If instead $\boldsymbol{A} \in W^{1,1}(\Omega)$, then $BV^{\boldsymbol{A},\lambda}(\Omega) = BV^{\boldsymbol{A}}(\Omega)$ for every Borel function $\lambda : \Omega \to [0,1]$. Then we notice that, up to extracting a further subsequence, we have $u_{k_j}(x) \to u(x)$ for $|\boldsymbol{A}|\mathscr{L}^1$-a.e. $x \in \Omega$, and therefore $u_{k_j}(x) \to u(x)$ for $\mathscr{L}^1$-a.e. $x \in \mathrm{supp}(|\boldsymbol{A}|)$. Given that $\mathrm{supp}(|D\boldsymbol{A}|) \subset \mathrm{supp}(|\boldsymbol{A}|)$, we conclude that $u_{k_j}(x) \to u(x)$ for $\mathscr{L}^1$-a.e. $x \in \mathrm{supp}(|D\boldsymbol{A}|)$. Thus, since $|D\boldsymbol{A}| \ll \mathscr{L}^1$, this implies

$$u_{k_j}(x) \to u(x) \quad \text{for} \quad |D\boldsymbol{A}|\text{-a.e. } x \in \Omega, \tag{9.3}$$

and so by Fatou's Lemma we obtain

$$\int_\Omega |u| \, d|D\boldsymbol{A}| = \int_\Omega \liminf_{j \to +\infty} |u_{k_j}| \, d|D\boldsymbol{A}| \leq \liminf_{j \to +\infty} \int_\Omega |u_{k_j}| \, d|D\boldsymbol{A}| < +\infty.$$

All in all, we get $u \in L^1(\Omega, |D\boldsymbol{A}|)$, and so $u \in BV^{\boldsymbol{A}}(\Omega)$, by Proposition 9.1. As for condition (iii), we claim that it entails $u \in L^\infty(\Omega)$. To see this, we let $C > 0$ be such that $\|u_k\|_{L^\infty(\Omega)} \le C$ for all $k \in \mathbb{N}$. As noted above, up to a further subsequence, we have $u_{k_j}(x) \to u(x)$ for $|\boldsymbol{A}|\mathscr{L}^1$-a.e. $x \in \Omega$, and so

$$|u(x)| = \lim_{j\to+\infty} |u_{k_j}(x)| \le \liminf_{j\to+\infty} \|u_{k_j}\|_{L^\infty(\Omega)} \le C \quad \text{for } |\boldsymbol{A}|\mathscr{L}^1\text{-a.e. } x \in \Omega.$$

This implies that $u \in L^\infty(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$ with $\|u\|_{L^\infty(\Omega, |\boldsymbol{A}|\mathscr{L}^1)} \le C$, which means

$$\int_{\{|u|>t\}} |\boldsymbol{A}| \, dx = 0 \quad \text{for all } t > C.$$

By definition, $u(x) = 0$ for $\mathscr{L}^1$-a.e. $x \in \Omega$ such that $\boldsymbol{A}(x) = 0$. Therefore, by Chebyshev's inequality, for all $t > C$ we obtain

$$\begin{aligned}
\mathscr{L}^1(\{|u| > t\}) &= \mathscr{L}^1(\{|u| > t\} \cap \{|\boldsymbol{A}| > 0\}) \\
&= \mathscr{L}^1\left(\{|u| > t\} \cap \left(\{|\boldsymbol{A}| \ge 1\} \cup \bigcup_{k=1}^{+\infty} \left\{\frac{1}{k} > |\boldsymbol{A}| \ge \frac{1}{k+1}\right\}\right)\right) \\
&\le \mathscr{L}^1\left(\{|u| > t\} \cap \{|\boldsymbol{A}| \ge 1\}\right) + \sum_{k=1}^{+\infty} \mathscr{L}^1\left(\{|u| > t\} \cap \left\{\frac{1}{k} > |\boldsymbol{A}| \ge \frac{1}{k+1}\right\}\right) \\
&\le \int_{\{|u|>t\}} |\boldsymbol{A}| \, dx + \sum_{k=1}^{+\infty} \int_{\{|u|>t\}} (k+1)|\boldsymbol{A}| \, dx = 0.
\end{aligned}$$

All in all, this entails that $\|u\|_{L^\infty(\Omega)} \le C$, and so, as above, we have $u^\lambda \in L^1(\Omega, |D\boldsymbol{A}|)$ for every Borel function $\lambda : \Omega \to [0,1]$, which in turn, by Proposition 9.1, implies $u \in BV^{\boldsymbol{A},\lambda}(\Omega)$. Finally, let $\boldsymbol{A} \in W^{1,1}(\Omega)$ and $C > 0$ be such that $\|u_k\|_{L^\infty(\Omega)} \le C$. Arguing as above, up to extracting possibly a further subsequence, (9.3) holds, and therefore, given that

$$|u_{k_j} - u| \le 2C \in L^1(\Omega, |D\boldsymbol{A}|),$$

we exploit Lebesgue's Dominated Convergence Theorem to conclude that $u_{k_j} \to u$ in $L^1(\Omega, |D\boldsymbol{A}|)$, and so $(u_{k_j})_{j\in\mathbb{N}}$ $\boldsymbol{A}$-converges to u. □

Remark 9.4. *We point out that in the last part of Proposition 9.3 we cannot remove the assumption that $A \in W^{1,1}(\Omega)$ and still achieve the convergence of the λ-representatives in $L^1(\Omega, |D\boldsymbol{A}|)$. Indeed, let $\Omega = (-1, 1)$, $\boldsymbol{A}(x) = \chi_{(0,1)}(x)$,*

$$u_k(x) = \begin{cases} a \arctan(kx) & \text{if } x \ge 0 \\ b \arctan(kx) & \text{if } x < 0 \end{cases},$$

for some $a, b \ge 0$, and $\lambda : \Omega \to [0,1]$ be any Borel function. Since $D\boldsymbol{A} = \delta_0$ and $u_k^\lambda(0) = u_k(0) = 0$, we see that

$$D(u_k\boldsymbol{A}) = \frac{ak}{1 + k^2x^2} \mathscr{L}^1 \llcorner (0,1),$$

and so $u_k \in BV^{\boldsymbol{A},\lambda}(\Omega)$, by Proposition 9.3, with

$$(\boldsymbol{A}, Du_k)_\lambda = -u_k^\lambda(0)\delta_0 + D(u_k\boldsymbol{A}) = \frac{ak}{1 + k^2x^2} \mathscr{L}^1 \llcorner (0,1).$$

It is clear that the sequence $(u_k)_{k\in\mathbb{N}}$ *satisfies* (9.2)*, is uniformly bounded in* $L^\infty(\Omega)$ *and converges to*

$$u(x) = \begin{cases} a\frac{\pi}{2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -b\frac{\pi}{2} & \text{if } x < 0 \end{cases}$$

pointwise and in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$*. However,*

$$u^\lambda(0) = (1-\lambda(0))\left(-b\frac{\pi}{2}\right) + \lambda(0)a\frac{\pi}{2} = ((a+b)\lambda(0)-b)\frac{\pi}{2},$$

so that u_k^λ *does not converge to* u^λ *in* $L^1(\Omega, |D\boldsymbol{A}|)$*, as long as* $\lambda(0) \neq \frac{b}{a+b}$ *and* $a+b \neq 0$*. Furthermore, under such conditions we do not even have any lower semicontinuity with respect to the* $L^1(\Omega, |D\boldsymbol{A}|)$ *norm of the* λ*-representatives.*

Corollary 9.5. *Let* $\boldsymbol{A} \in BV(\Omega)$ *and* $\overline{\lambda} : \Omega \to [0,1]$ *be a Borel function. Let* (E_k) *be a family of Borel sets such that*

$$\sup_{k\in\mathbb{N}} P_{\boldsymbol{A},\overline{\lambda}}(E_k, \Omega) < +\infty.$$

Then there exist a Borel set E *such that* $P_{\boldsymbol{A},\lambda}(E,\Omega) < +\infty$ *for every Borel function* $\lambda : \Omega \to [0,1]$ *and a subsequence* $(E_{k_j})_{j\in\mathbb{N}}$ *such that* $\chi_{E_{k_j}} \to \chi_E$ *in* $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$*. If* $\boldsymbol{A} \in W^{1,1}(\Omega)$*, we also have* $\chi_{E_{k_j}} \to \chi_E$ *in* $L^1(\Omega, |D\boldsymbol{A}|)$*, possibly up to a further subsequence, which entails the* $\boldsymbol{A}$*-convergence.*

Proof. Arguing as in the proof of Proposition 9.3, we see that there exist $w_{\boldsymbol{A}} \in BV(\Omega)$ and a subsequence $(\chi_{E_{k_j}}\boldsymbol{A})_{j\in\mathbb{N}}$ such that $\chi_{E_{k_j}}\boldsymbol{A} \to w_{\boldsymbol{A}}$ in $L^1(\Omega)$. Therefore, $\chi_{E_{k_j}} \to \frac{w_{\boldsymbol{A}}}{\boldsymbol{A}}$ in $L^1(\Omega, |\boldsymbol{A}|\mathscr{L}^1)$, and this readily implies that $\frac{w_{\boldsymbol{A}}(x)}{\boldsymbol{A}(x)} \in \{0,1\}$ for $|\boldsymbol{A}|\mathscr{L}^1$-a.e. $x \in \Omega$. Thus, there exists a Borel set E such that $\frac{w_{\boldsymbol{A}}(x)}{\boldsymbol{A}(x)} = \chi_E(x)$ for $|\boldsymbol{A}|\mathscr{L}^1$-a.e. $x \in \Omega$. It is plain to see that $\chi_E\boldsymbol{A} \in BV(\Omega)$ and $\chi_E \in L^\infty(\Omega) \subset X^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel function $\lambda : \Omega \to [0,1]$, so that Proposition 9.1 implies $\chi_E \in BV^{\boldsymbol{A},\lambda}(\Omega)$ for every Borel function $\lambda : \Omega \to [0,1]$. Let now $\boldsymbol{A} \in W^{1,1}(\Omega)$. Given that $\|\chi_{E_k}\|_{L^\infty(\Omega)} \le 1$, we just need to employ Proposition 9.3 to conclude that, up to extracting a further subsequence, $\chi_{E_{k_j}} \to \chi_E$ in $L^1(\Omega, |D\boldsymbol{A}|)$, and so $(E_{k_j})_{j\in\mathbb{N}}$ $\boldsymbol{A}$-converges to E. ☐

Remark 9.6. *Proposition 9.3 implies that, if* $\boldsymbol{A} \in \mathcal{DM}^1(\Omega)$ *with* $|D\boldsymbol{A}| \ll \mathscr{L}^1$ *and* $\mathscr{L}^1(\Omega \setminus \mathrm{supp}(|\boldsymbol{A}|)) = 0$*, we have a weak compactness result in* $BV^{\boldsymbol{A}}(\Omega)$ *(which is actually a Banach space, due to Proposition 5.1), as well as Corollary 9.5 implies a weak compactness in the class of sets with finite* $\boldsymbol{A}$*-perimeter. However, if* $N \ge 2$*, analogous results fail to hold true due to new degrees of freedom, even if* $\operatorname{div} \boldsymbol{A} = 0$ *(see Examples 5.5 and 7.14).*

References

[1] L. Ambrosio, G. Crippa, and S. Maniglia, *Traces and fine properties of a BD class of vector fields and applications*, Ann. Fac. Sci. Toulouse Math. (6) **14** (2005), no. 4, 527–561.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[3] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón, *The Dirichlet problem for the total variation flow*, J. Funct. Anal. **180** (2001), no. 2, 347–403, DOI 10.1006/jfan.2000.3698.

[4] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, vol. 223, Birkhäuser Verlag, Basel, 2004.

[5] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) **135** (1983), 293–318 (1984).
[6] ———, *Traces of bounded vector–fields and the divergence theorem*, 1983. Unpublished preprint.
[7] V. Buffa, G. E. Comi, and M. Jr. Miranda, *On BV functions and essentially bounded divergence-measure fields in metric spaces*, Rev. Mat. Iberoam. **38** (2022), no. 3, 883–946, DOI 10.4171/rmi/1291.
[8] V. Caselles, *On the entropy conditions for some flux limited diffusion equations*, J. Differential Equations **250** (2011), no. 8, 3311–3348.
[9] G.-Q. Chen, G. E. Comi, and M. Torres, *Cauchy fluxes and Gauss-Green formulas for divergence-measure fields over general open sets*, Arch. Ration. Mech. Anal. **233** (2019), no. 1, 87–166.
[10] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **147** (1999), no. 2, 89–118.
[11] ———, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Comm. Math. Phys. **236** (2003), no. 2, 251–280.
[12] G. E. Comi, *Refined Gauss–Green formulas and evolution problems for Radon measures*, Scuola Normale Superiore, Pisa (2020). Ph.D. Thesis. Available at cvgmt.sns.it/paper/4579/.
[13] G. E. Comi, G. Crasta, V. De Cicco, and A. Malusa, *Representation formulas for pairings between divergence-measure fields and BV functions*, J. Funct. Anal. **286** (2024), no. 1, 110192.
[14] G. E. Comi and V. Magnani, *The Gauss-Green theorem in stratified groups*, Adv. Math. **360** (2020), 106916, 85, DOI 10.1016/j.aim.2019.106916.
[15] G. E. Comi and K. R. Payne, *On locally essentially bounded divergence measure fields and sets of locally finite perimeter*, Adv. Calc. Var. **13** (2020), no. 2, 179–217, DOI 10.1515/acv-2017-0001.
[16] G. Crasta and V. De Cicco, *Anzellotti's pairing theory and the Gauss–Green theorem*, Adv. Math. **343** (2019), 935–970.
[17] G. Crasta and V. De Cicco, *An extension of the pairing theory between divergence-measure fields and BV functions*, J. Funct. Anal. **276** (2019), no. 8, 2605–2635.
[18] G. Crasta and V. De Cicco, *On the variational nature of the Anzellotti pairing*, Preprint (2022).
[19] G. Crasta, V. De Cicco, and A. Malusa, *Pairings between bounded divergence-measure vector fields and BV functions*, Adv. Calc. Var. **15** (2022), no. 4, 787-810.
[20] V. De Cicco, N. Fusco, and A. Verde, *A chain rule formula in BV and application to lower semicontinuity*, Calc. Var. Partial Differential Equations **28** (2007), no. 4, 427–447.
[21] V. De Cicco, D. Giachetti, and S. Segura de León, *Elliptic problems involving the 1-Laplacian and a singular lower order term*, J. Lond. Math. Soc. (2) **99** (2019), no. 2, 349–376, DOI 10.1112/jlms.12172.
[22] M. Degiovanni, A. Marzocchi, and A. Musesti, *Cauchy fluxes associated with tensor fields having divergence measure*, Arch. Ration. Mech. Anal. **147** (1999), no. 3, 197–223.
[23] G. Del Nin, *Rectifiability of the jump set of locally integrable functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **XXII** (2021), no. 5, 1233–1240.
[24] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015.
[25] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
[26] H. Frid, *Divergence-measure fields on domains with Lipschitz boundary*, Hyperbolic conservation laws and related analysis with applications, vol. 49, Springer Proc. Math. Stat., Springer, Heidelberg, (2014).
[27] W. Górny, *Local and nonlocal 1-Laplacian in Carnot groups*, Ann. Fenn. Math. **47** (2022), no. 1, 427–456, DOI 10.54330/afm.114742.
[28] W. Górny and J. M. Mazón, *The Anzellotti-Gauss-Green Formula and Least Gradient Functions in Metric Measure Spaces*, Communications in Contemporary Mathematics (2023).
[29] B. Kawohl and F. Schuricht, *Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem*, Commun. Contemp. Math. **9** (2007), no. 4, 515–543, DOI 10.1142/S0219199707002514.
[30] G. P. Leonardi and G. E. Comi, *The prescribed mean curvature measure equation in non-parametric form*, Preprint (2023).
[31] G. P. Leonardi and G. Saracco, *The prescribed mean curvature equation in weakly regular domains*, NoDEA Nonlinear Differential Equations Appl. **25** (2018), no. 2, Art. 9, 29.

[32] G.P. Leonardi and G. Saracco, *Rigidity and trace properties of divergence-measure vector fields*, Adv. Calc. Var. **15** (2022), no. 1, 133–149, DOI 10.1515/acv-2019-0094.
[33] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.
[34] A. Marzocchi and A. Musesti, *Decomposition and integral representation of Cauchy interactions associated with measures*, Contin. Mech. Thermodyn. **13** (2001), no. 3, 149–169, DOI 10.1007/s001610100046.
[35] A. Mercaldo, S. Segura de León, and C. Trombetti, *On the solutions to 1-Laplacian equation with L^1 data*, J. Funct. Anal. **256** (2009), no. 8, 2387–2416, DOI 10.1016/j.jfa.2008.12.025.
[36] N. C. Phuc and M. Torres, *Characterizations of signed measures in the dual of BV and related isometric isomorphisms*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 1, 385–417.
[37] W. Rudin, *Functional Analysis*, International series in pure and applied mathematics, Tata McGraw-Hill, 1974.
[38] C. Scheven and T. Schmidt, *BV supersolutions to equations of 1-Laplace and minimal surface type*, J. Differential Equations **261** (2016), no. 3, 1904–1932, DOI 10.1016/j.jde.2016.04.015.
[39] ———, *On the dual formulation of obstacle problems for the total variation and the area functional*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 5, 1175–1207, DOI 10.1016/j.anihpc.2017.10.003.
[40] F. Schuricht, *A new mathematical foundation for contact interactions in continuum physics*, Arch. Ration. Mech. Anal. **184** (2007), no. 3, 495–551.
[41] M. Šilhavý, *Divergence measure fields and Cauchy's stress theorem*, Rend. Sem. Mat. Univ. Padova **113** (2005), 15–45.
[42] M. Šilhavý, *The divergence theorem for divergence measure vectorfields on sets with fractal boundaries*, Math. Mech. Solids **14** (2009), no. 5, 445–455, DOI 10.1177/1081286507081960.
[43] ———, *The Gauss-Green theorem for bounded vector fields with divergence measure on sets of finite perimeter*, Indiana Univ. Math. J. **72** (2023), no. 1, 29–42.

(G. E. Comi) Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna (BO), Italy
Email address: `giovannieugenio.comi@unibo.it`

(V. De Cicco) Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via A. Scarpa 10 – I-00185 Roma (Italy)
Email address: `virginia.decicco@uniroma1.it`

(G. Scilla) Dipartimento di Matematica ed Applicazioni "R.Caccioppoli", Università degli Studi di Napoli Federico II, Via Cintia, Monte Sant'Angelo – I-80126 Napoli (Italy)
Email address: `giovanni.scilla@unina.it`