BV FUNCTIONS AND NONLOCAL FUNCTIONALS IN METRIC MEASURE SPACES

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ABSTRACT. We study the asymptotic behavior of three classes of nonlocal functionals in complete metric spaces equipped with a doubling measure and supporting a Poincaré inequality. We show that the limits of these nonlocal functionals are comparable to the variation $\|Df\|(\Omega)$ or the Sobolev semi-norm $\int_\Omega g_f^p\,d\mu$, which extends Euclidean results to metric measure spaces. In contrast to the classical setting, we also give an example to show that the limits are not always equal to the corresponding total variation even for Lipschitz functions.

1. Introduction

Consider a sequence $\{\rho_i\}_{i=1}^\infty$ of mollifiers in $L^1_{\mathrm{loc}}(0,\infty)$, $n\geq 1$, for which

(1.1)
$$\int_0^\infty \rho_i(r)r^{n-1}\,dr = 1 \quad \text{for all } i\in\mathbb{N} \quad \text{and} \quad \lim_{i\to\infty}\int_\delta^\infty \rho_i(r)r^{n-1}\,dr = 0 \quad \text{for all } \delta>0.$$

Let $1\leq p<\infty$. For an open set $\Omega\subset\mathbb{R}^n$ and function $f\in W^{1,p}_{\mathrm{loc}}(\Omega)$, we define the Sobolev seminorm by

$$|f|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla f|^p\,dx\right)^{1/p},$$

and if $f\notin W^{1,p}_{\mathrm{loc}}(\Omega)$, then we let $|f|_{W^{1,p}(\Omega)}=\infty$. Given an open set $\Omega\subset\mathbb{R}^n$ and a function f on Ω, we define

$$I_{p,i}(f,\Omega) := \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_i(|x-y|)\,dx\,dy.$$

We denote the energy by

$$E_p(f,\Omega) := \begin{cases} \|Df\|(\Omega) & \text{when } p=1,\\ |f|^p_{W^{1,p}(\Omega)} & \text{when } 1<p<\infty. \end{cases}$$

The so-called BBM formula, shown by Bourgain, Brezis, and Mironescu [4, Theorem 3] states that when $\Omega\subset\mathbb{R}^n$ is a smooth, bounded domain and $1<p<\infty$, then for every $f\in L^p(\Omega)$ we have

$$\lim_{i\to\infty} I_{p,i}(f,\Omega) = K_{p,n}E_p(f,\Omega), \tag{1.2}$$

where $K_{p,n}>0$ is a constant depending only on p and n. Later, Dávila [20] generalized this result to functions of bounded variation (BV functions) f and their variation measures $\|Df\|(\Omega)$. He showed that when $\Omega\subset\mathbb{R}^n$ is a bounded domain with Lipschitz boundary, then for every $f\in L^1(\Omega)$ (we understand $\|Df\|(\Omega)=\infty$ if $f\notin \mathrm{BV}(\Omega)$), the above equality also holds for $p=1$.

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Several different generalizations of these results in Euclidean spaces have been considered e.g. by Ponce [43], Leoni–Spector [35, 36], Nguyen [40], Brezis–Nguyen [7, 8, 9, 10], Brezis–Van Schaftingen–Yung [12, 13, 14], Nguyen–Pinamonti–Vecchi–Squassina [41, 42], Garofalo–Tralli [22, 23], Comi-Stefani [15, 16, 18], Maalaoui–Pinamonti [37], Brena–Pasqualetto–Pinamonti [17] and references therein.

On the other hand, in the past three decades, there has been significant progress in the study of various aspects of first order analysis on metric measure spaces including the theory of first order Sobolev spaces, functions of bounded variation and their relation to variational problems and partial differential equations, see [2, 26, 32, 38] and references therein. In particular, in metric measure spaces (X, d, μ) with the measure being doubling and the space supporting a Poincaré inequality (also known as PI spaces), fruitful results are obtained.

In [6, Remark 6], Brezis raised a question about the relation between the BBM formula and Sobolev spaces on metric measure spaces [2, 26, 33]. Di Marino–Squassina [21] proved new characterizations of Sobolev and BV spaces in PI spaces in the spirit of the BBM formula. A similar result was proved previously in Ahlfors-regular spaces by Munnier [39]. Górny [25] and Han–Pinamonti [28], respectively, studied the problem in certain PI spaces that "locally look like" Euclidean spaces, or respectively finite-dimensional Banach spaces or Carnot groups.

Given a metric measure space (X, d, μ) and an open set $\Omega \subset X$, consider the following functional and energy:

$$I_{p,i}(f,\Omega) := \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y), \tag{1.3}$$

and

$$E_p(f,\Omega) := \begin{cases} \|Df\|(\Omega) & \text{when } p=1, \\ \int_\Omega g_f^p\,d\mu & \text{when } 1<p<\infty, \end{cases}$$

where g_f is the so-called minimal p-weak upper gradient of f which equals $|\nabla f|$ for Sobolev functions $f \in W^{1,p}(\Omega)$ in the Euclidean space. See Section 2 for precise definition. If f is not in $\mathrm{BV}(\Omega)$ when $p=1$, respectively not in $\widehat{N}^{1,p}(\Omega)$ when $1<p<\infty$, then we let $E_p(f,\Omega)=\infty$.

In our previous paper [34], we studied the BBM formula for a more general class of mollifiers ρ_i which contain the mollifiers considered in [21, 25, 28] as special cases. The explicit conditions will be specified in Section 2. Similar to the BBM formula (1.2), we obtained the following [34, Theorem 1.1]:

Theorem 1.4. *Let $1 \le p < \infty$, suppose μ is doubling, and let $\Omega \subset X$ be open. If $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying condition (2.12), and $f \in L^p(\Omega)$, then*

$$\liminf_{i\to\infty} I_{p,i}(f,\Omega) \ge C_1 E_p(f,\Omega).$$

If X supports a (p,p)-Poincaré inequality, $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying condition (2.13), Ω is a strong p-extension domain, and $f \in L^p(\Omega)$, then

$$\limsup_{i\to\infty} I_{p,i}(f,\Omega) \le C_2 E_p(f,\Omega).$$

Here $C_1 \le C_2$ are constants that depend only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

In general, contrary to Euclidean spaces, we cannot expect $C_1 = C_2$ in the above estimates, as we demonstrate in [34, Section 7].

It is natural and interesting to ask whether such asymptotic behaviors of (1.3) still hold for other convex or nonconvex functionals. The pursuit of such question is also driven by applications. For example, Gilboa–Osher [24] consider nonlocal image functionals to improve effectiveness in image denoising and reconstruction. One prototype functional considered in [24] corresponds to a special case with $p = 1$, $q = 2$ of the nonlocal functionals

$$\int_\Omega \left[\int_\Omega \Big(\frac{|f(x)-f(y)|^p}{|x-y|^p}\Big)^q \rho_i(|x-y|)\,dx\right]^{1/q} dy$$

studied by Leoni–Spector [35, 36]. Leoni–Spector also obtain Γ-convergence of the above functional to a constant multiple of $\int_\Omega |\nabla f|^p\,dx$ for all $f \in W^{1,p}(\Omega)$ and $p \in (1,\infty)$ under appropriate assumptions on q and ρ_i [36, Theorem 1.2].

Furthermore, Brezis–Nguyen [7] study the asymptotic behavior of two convex nonlocal functionals which converge formally to the total variation. In [9, 11], Brezis–Nguyen consider approximation of $\int_\Omega |\nabla u|^p$ with $p \geq 1$ for nonconvex nonlocal functionals motivated by questions arising in image processing.

Inspired by the work of Bourgain-Brezis-Mironescu [4], Leoni-Spector [35, 36], and Brezis-Nguyen [7, 9, 10, 11], the main goal of the current paper is to study the asymptotic behavior of certain convex or nonconvex nonlocal functionals in metric measure spaces. In particular, we focus on the following three functionals from the work Leoni-Spector [35] and Brezis–Nguyen [7, 9, 10, 11]:

$$\begin{aligned}
\Psi_{p,i}(f,\Omega) :=&\quad \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}, \quad \varepsilon_i \searrow 0,\\
\Phi_{p,q,i}(f,\Omega) :=&\quad \int_\Omega \left[\int_\Omega \left(\frac{|f(x)-f(y)|^p}{d(x,y)^p}\right)^q \rho_i(x,y)\,d\mu(x)\right]^{1/q} d\mu(y),\\
\Lambda_{p,\delta}(f,\Omega) :=&\quad \int_\Omega\int_\Omega \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(x,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y), \quad \delta>0.
\end{aligned}$$

The conditions on the mollifiers ρ_i and $\varphi\colon [0,\infty) \to [0,\infty)$ will be made explicit in Section 2 and some typical examples of ρ_i and φ will be given in Section 3. In particular, the conditions posed on φ imply that $\Lambda_{p,\delta}$ is generally a nonconvex functional. We obtain the following theorems of the asymptotic behaviors on the above three functionals respectively.

Theorem 1.5. *Let $1 \leq p < \infty$ and suppose μ is doubling, suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying condition (2.12), (2.13), and let $\Omega \subset X$ be open and bounded.*

If $f \in L^p(\Omega)$, then

$$C_1' E_p(f,\Omega) \leq \liminf_{i\to\infty} \Psi_{p,i}(f,\Omega).$$

If, on the other hand, X supports a $(1,p)$–Poincaré inequality and there is $q \in (p,\infty)$ such that $\Omega \subset X$ is a strong p-extension and strong q-extension domain, and $f \in \widehat{N}^{1,q}(\Omega)$, then

$$\limsup_{i\to\infty} \Psi_{p,i}(f,\Omega) \leq C_2' E_p(f,\Omega).$$

Here $C_1' \leq C_2'$ are constants that depend only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

Theorem 1.6. *Let $1 \leq p < \infty$, $1 < q < \infty$, suppose μ is doubling, $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying condition (2.12) or the same condition with p replaced by q, as well as (2.13), and let $\Omega \subset X$ be open and bounded.*

If $f \in L^p(\Omega)$, then

$$C_1'' E_p(f,\Omega) \leq \liminf_{i\to\infty} \Phi_{p,q,i}(f,\Omega).$$

If X supports a $(1,p)$–Poincaré inequality, $\Omega \subset X$ is a bounded pq-extension domain, and $f \in \widehat{N}^{1,pq}(\Omega)$, then

$$\limsup_{i\to\infty} \Phi_{p,q,i}(f,\Omega) \le C_2'' E_p(f,\Omega).$$

Here $C_1'' \le C_2''$ are constants that depend only on p, q, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

Theorem 1.7. *Let $1 \le p < \infty$ and suppose $\varphi\colon [0,\infty) \to [0,\infty)$ satisfies assumptions (2.15)–(2.17), and suppose $\Omega \subset X$ is open and bounded.*

If μ is Ahlfors Q-regular with $1 < Q < \infty$ and $f \in L^p(\Omega)$, then

$$C_1''' E_p(f,\Omega) \le \limsup_{\delta\to 0} \Lambda_{p,\delta}(f,\Omega)$$

for some constant C_1''' depending only on p, the Ahlfors regularity constants, and the constant C_φ associated with the functional $\Lambda_{p,\delta}$.

If, on the other hand, μ is doubling, X supports a $(1,p)$–Poincaré inequality, and $f \in \widehat{N}^{1,1}(X) \cap \widehat{N}^{1,q}(X)$ for some $1 < q < \infty$ in the case $p = 1$, and and $f \in \widehat{N}^{1,p}(X)$ in the case $1 < p < \infty$, then

$$\limsup_{\delta\to 0} \Lambda_{p,\delta}(f,\Omega) \le C_2''' E_{f,p}(\Omega)$$

for some constant C_2''' depending only on p, the doubling constant, the constants in the Poincaré inequality, and the constant C_φ.

Remark 1.8. In general, one cannot have equality of the constants. In particular, in Theorem 1.6 one might hope for the equality $C_1'' = C_2''$. Brezis-Nguyen [7, Proposition 12] give an example where $\Phi_{1,q,i}(f,\Omega)$ defined with radial mollifiers satisfying (1.1) fail to converge to $K_{1,1}E_1(f,\Omega)$ for a Sobolev function $f \in W^{1,1}(\Omega)$ on $\Omega = (-1/2, 1/2)$. If f is further assumed to be in $W^{1,q}(\Omega)$ for some $q > 1$ on an open, bounded and smooth domain in $\mathbb{R}^n$, such convergence holds [7, Proposition 9]. However, we will give an explicit example (Example 6.3) showing that convergence to a constant times $E_1(f,\Omega)$ can fail even for a Lipschitz function on metric measure spaces.

Remark 1.9. It is worth pointing out that in some cases the convergence of the above functionals in the Euclidean space is not yet fully understood, for example $\Phi_{p,q,i}(f,\Omega)$ for a general $1 \le p < \infty$ [7, Section 3]. On the other hand, Γ-convergence of the functionals seems to be more robust and clear. In Euclidean spaces, the Γ-convergence of the functionals $\Phi_{1,q,i}(\cdot,\Omega)$, $\Psi_{1,i}(\cdot,\Omega)$, $\Lambda_{1,\delta}(\cdot,\Omega)$ is proved in [7, 10]. It would be interesting to investigate the Γ-convergence of such functionals in metric measure spaces as well.

The paper is organized as follows. We will review some definitions and introduce the conditions of the mollifiers ρ_i and the function φ in Section 2. Then we give some examples of ρ_i, φ and the related corollaries by applying such mollifiers in the main theorems in Section 3. We prove the lower bounds of our three main theorems in Section 4, and the upper bounds in Section 5. In Section 6 we give an example demonstrating that we do not generally have $C_1'' = C_2''$.

2. Notation and definitions

Throughout this paper, we work in a complete and connected metric measure space (X,d,μ) equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \ge 1$ such that

$$0 < \mu(B(x,2r)) \le C_d\mu(B(x,r)) < \infty$$

for every ball $B(x,r) := \{y \in X\colon\ d(y,x) < r\}$, with $x \in X$, $r > 0$. We assume that X consists of at least two points, that is, $\operatorname{diam} X > 0$. As a complete metric space equipped with a doubling measure, X is proper, meaning that closed and bounded sets are compact. By [3, Corollary 3.8], we know that for every $x \in X$ and every $0 < r \le R < \frac{1}{2} \operatorname{diam} X$, we have

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \le C_0 \left(\frac{r}{R}\right)^{\sigma} \tag{2.1}$$

for constants $C_0 > 0$ and $0 < \sigma < 1$ depending only on C_d. It follows that $\mu(\{x\}) = 0$ for every $x \in X$.

Sometimes we will make the stronger assumption that μ is Ahlfors regular. We say that μ is Ahlfors Q-regular with $1 < Q < \infty$ if for every $x \in X$ and every $0 < r < 2 \operatorname{diam} X$, we have

$$C_A^{-1} r^Q \le \mu(B(x,r)) \le C_A r^Q,$$

for constant $C_A > 0$ depending only on Q. Obviously Ahlfors regularity implies doubling.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into X. The length of a curve γ is denoted by ℓ_γ. We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [26, Theorem 3.2]). A nonnegative Borel function g on X is an upper gradient of a function $f \colon X \to [-\infty, \infty]$ if for all nonconstant curves $\gamma \colon [0, \ell_\gamma] \to X$, we have

$$|f(x) - f(y)| \le \int_\gamma g\, ds := \int_0^{\ell_\gamma} g(\gamma(s))\, ds, \tag{2.2}$$

where x and y are the end points of γ. We interpret $|f(x) - f(y)| = \infty$ whenever at least one of $|f(x)|$, $|f(y)|$ is infinite. Upper gradients were originally introduced in [30].

Let $1 \le p < \infty$. The p-modulus of a family of curves Γ is defined by

$$\operatorname{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho\, ds \ge 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.2) holds for p-almost every curve, we say that g is a p-weak upper gradient of f. By only considering curves γ in a set $A \subset X$, we can talk about a function g being a (p-weak) upper gradient of u in A.

We always let Ω denote an open subset of X. We define the Newton-Sobolev space $N^{1,p}(\Omega)$ to consist of those functions $f \in L^p(\Omega)$ for which there exists a p-weak upper gradient $g \in L^p(\Omega)$ of f in Ω. This space was first introduced in [45]. We write $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every $x \in \Omega$ there exists $r > 0$ such that $f \in N^{1,p}(B(x,r))$; other local function spaces are defined analogously. For every $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ there exists a minimal p-weak upper gradient of f in Ω, denoted by g_f, satisfying $g_f \le g$ μ-almost everywhere (a.e.) in Ω for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of f in Ω, see [3, Theorem 2.25].

Note that Newton-Sobolev functions are understood to be defined at every $x \in \Omega$, whereas the functionals that we consider are not affected by perturbations of f in a set of zero μ-measure. For this reason, we also define

$$\widehat{N}^{1,p}(\Omega) := \{f \colon \Omega \to [-\infty, \infty] \colon f = h \ \mu\text{-a.e. in } \Omega \text{ for some } h \in N^{1,p}(\Omega)\}.$$

For every $f \in \widehat{N}^{1,p}(\Omega)$, we can also define $g_f := g_h$, where g_h is the minimal p-weak upper gradient of any h as above in Ω; this is well defined μ-a.e. in Ω by [3, Corollary 1.49, Proposition 1.59].

Next we give Mazur's lemma, see e.g. [44, Theorem 3.12].

Theorem 2.3. *Let $\{g_i\}_{i=1}^\infty$ be a sequence with $g_i \to g$ weakly in $L^1(\Omega)$. Then there exist convex combinations $\widehat{g}_i := \sum_{j=1}^{N_i} a_{i,j} g_j$, for some $N_i \in \mathbb{N}$, such that $\widehat{g}_i \to g$ in $L^1(\Omega)$.*

By convex combinations we mean that the numbers $a_{i,j}$ are nonnegative and that $\sum_{j=1}^{N_i} a_{i,j} = 1$ for every $i \in \mathbb{N}$.

Let also $1 \le q < \infty$. We say that X supports a (q,p)-Poincaré inequality if there exist constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, every $f \in L^1(X)$, and every upper gradient g of f, we have

$$\left(\fint_{B(x,r)} |f - f_{B(x,r)}|^q \, d\mu\right)^{1/q} \le C_P r \left(\fint_{B(x,\lambda r)} g^p \, d\mu\right)^{1/p}, \tag{2.4}$$

where, as usual,

$$f_{B(x,r)} := \fint_{B(x,r)} f \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu.$$

Next we define functions of bounded variation. Given an open set $\Omega \subset X$ and a function $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of f in Ω by

$$\|Df\|(\Omega) := \inf\left\{\liminf_{i\to\infty} \int_\Omega g_{f_i} \, d\mu \colon \; f_i \in N^{1,1}_{\mathrm{loc}}(\Omega),\, f_i \to f \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{f_i} is the minimal 1-weak upper gradient of f_i in Ω. We say that a function $f \in L^1(\Omega)$ is of bounded variation, and denote $f \in \mathrm{BV}(\Omega)$, if $\|Df\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Df\|(A) := \inf\{\|Df\|(W) \colon \; A \subset W,\, W \subset X \text{ is open}\}.$$

If $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\|Df\|(\cdot)$ is a Borel regular outer measure on Ω by [38, Theorem 3.4].

For a function u defined on an open set $\Omega \subset X$, we abbreviate super-level sets in the form

$$\{u > t\} := \{x \in \Omega \colon \; u(x) > t\}, \quad t \in \mathbb{R}.$$

The following coarea formula is given in [38, Proposition 4.2]: if $\Omega \subset X$ is open and $u \in \mathrm{BV}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt. \tag{2.5}$$

If $f \in N^{1,1}(\Omega)$, then

$$\int_\Omega g_f \, d\mu \le C_* \|Df\|(\Omega), \tag{2.6}$$

where g_f is as usual the minimal 1-weak upper gradient of f in Ω, and $C_* \ge 1$ is a constant depending only on the doubling constant C_d and the constants in the Poincaré inequality C_P, λ; see [27, Remark 4.7].

Definition 2.7. We say that an open set $\Omega \subset X$ is a strong p-extension domain if

- in the case $p = 1$, for every $f \in \mathrm{BV}(\Omega)$ there exists an extension $F \in \mathrm{BV}(X)$ and $\|DF\|(\partial\Omega) = 0$;
- in the case $1 < p < \infty$, for every $f \in \widehat{N}^{1,p}(\Omega)$ there exists an extension $F \in \widehat{N}^{1,p}(X)$ and $\int_{\partial\Omega} g_F^p \, d\mu = 0$.

A p-extension domain is defined similarly, but we omit the conditions $\|DF\|(\partial\Omega) = 0$ resp. $\int_{\partial\Omega} g_F^p \, d\mu = 0$.

For example, in Euclidean spaces, a bounded domain with a Lipschitz boundary is a strong p-extension domain for all $1 \le p < \infty$, see e.g. [1, Proposition 3.21].

The Hardy–Littlewood maximal function of a nonnegative function $g \in L^1_{\mathrm{loc}}(X)$ is defined by

$$\mathcal{M}g(x) := \sup_{r>0} \fint_{B(x,r)} g\,d\mu. \tag{2.8}$$

Given $R > 0$, we define the restricted maximal function $\mathcal{M}_R g(x)$ in a similar way, but we take the supremum over radii $0 < r \le R$.

Given $U \subset X$ and $\delta > 0$, we denote

$$U(\delta) := \{x \in X \colon d(x,U) < \delta\}. \tag{2.9}$$

Let f be a function defined on Ω. We define

$$\operatorname{Lip}_r f(x) := \sup_{y \in \Omega \cap B(x,r)} \frac{|f(y) - f(x)|}{r}, \quad x \in \Omega, \quad r > 0, \tag{2.10}$$

and

$$\operatorname{Lip} f(x) := \limsup_{r \to 0} \operatorname{Lip}_r(x). \tag{2.11}$$

Now we describe the mollifiers that we will use in much of the paper. Let $1 \le p < \infty$. We will consider a sequence of nonnegative $X \times X$-measurable functions $\{\rho_i\}_{i=1}^\infty$ and a fixed constant $1 \le C_\rho < \infty$ satisfying the following conditions:

(1) For every $i \in \mathbb{N}$ and for every $x, y \in X$, we have

$$\rho_i(x,y) \ge C_\rho^{-1} \frac{d(x,y)^p}{r_i^p} \frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))}, \tag{2.12}$$

where $r_i \searrow 0$.

(2) For every $i \in \mathbb{N}$ and every $x, y \in X$, we have

$$\rho_i(x,y) \le \sum_{j \in \mathbb{Z}} d_{i,j} \frac{\chi_{B(y,2^{j+1}) \setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))} \tag{2.13}$$

for numbers $d_{i,j} \ge 0$ for which $\sum_{j \in \mathbb{Z}} d_{i,j} \le C_\rho$ and $\lim_{i \to \infty} \sum_{j \ge M} d_{i,j} = 0$ for all $M \in \mathbb{Z}$.

Remark 2.14. The assumptions (1.1) are ubiquitous in the literature, but depending on the functional under consideration, some additional conditions are needed. For example, the convergence of $\Psi_{1,i}(f,\Omega)$ to $E_1(f,\Omega)$ in a smooth bounded domain in the Euclidean space has been verified only for a special choice of mollifiers [7, Proposition 2]. In our setting, we do not have access to certain Euclidean techniques, especially the Taylor approximation. Our formulation of the assumptions (2.12) and (2.13) is informed by these facts. These assumptions are in any case satisfied by certain typical and important choices of the mollifiers ρ_i, as we will see in Section 3.

The definition of the functional $\Lambda_{p,\delta}(f,\Omega)$ involves a function $\varphi \colon [0,\infty) \to [0,\infty)$. Let $1 \le p < \infty$. Given $b > 0$, we consider the assumptions (we understand "increasing" in the non-strict sense)

$$\varphi(t) \quad \text{is increasing,} \tag{2.15}$$

$$\varphi(t) \le b \quad \text{for } 0 \le t < \infty; \tag{2.16}$$

and

$$C_\varphi^{-1} \le \int_0^\infty \varphi(t) t^{-1-p}\,dt \le C_\varphi \quad \text{for some } 1 \le C_\varphi < \infty. \tag{2.17}$$

Our standing assumptions are the following; note that we do *not* always assume that μ is Ahlfors regular or that X satisfies a Poincaré inequality.

Throughout the paper, (X, d, μ) is a complete and connected metric space equipped with a doubling Borel regular outer measure μ, with $\operatorname{diam} X > 0$. *We always assume $\Omega \subset X$ to be an open set.*

3. Mollifiers and implications

In this section we consider three important examples of mollifiers ρ_i satisfying conditions (2.12) and (2.13). The first mollifier is investigated carefully in our previous work [34] but we add it here to obtain some results needed in our later proofs. We apply the second and third mollifiers in Theorem 1.6 with the choice $p = 1$ to give some interesting corollaries. We also give a simple choice of nonconvex function φ in Theorem 1.7 at the end of this section and compare the result with that in [5].

First we recall one mollifier from [34] which gives important corollaries needed later. Consider

$$\rho_i(x,y) := (1-s_i)\frac{1}{d(x,y)^{p(s_i-1)}\mu(B(y,d(x,y)))}, \quad x, y \in X,$$

where $s_i \nearrow 1$ as $i \to \infty$. One can verify that it satisfies (2.12) and as a result we obtain the following corollary, see [34, Corollary 6.1] for detailed proof.

Corollary 3.1. *Let $1 \le p < \infty$ and let $f \in L^p(\Omega)$. Then*

$$\liminf_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mu(B(y,d(x,y)))}\,d\mu(x)\,d\mu(y) \ge C^{-1}E_p(f,\Omega) \tag{3.2}$$

for some constant $C \ge 1$ depending only on the doubling constant of the measure.

Remark 3.3. Note that in [34, Corollary 6.1] it is also assumed that Ω is a strong p-extension domain and that X supports a (p,p)-Poincaré inequality, but these are only needed for the upper bound given in that corollary.

The following is now an easy consequence of Corollary 3.1.

Corollary 3.4. *Let $1 \le p < \infty$ and suppose that μ is Ahlfors Q-regular and that Ω is bounded. Let $f \in L^p(\Omega)$. Then*

$$\liminf_{\varepsilon\searrow 0}\int_\Omega\int_\Omega \frac{\varepsilon|f(x)-f(y)|^{p+\varepsilon}}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y) \ge C^{-1}E_p(f,\Omega)$$

for some constant $C \ge 1$ depending only on p and the Ahlfors regularity constants of the measure.

Proof. First note that

$$\lim_{\varepsilon\searrow 0}\varepsilon^{p/(p+\varepsilon)}\frac{p(p+\varepsilon)}{\varepsilon(p+Q)} = \frac{p^2}{p+Q}\lim_{\varepsilon\searrow 0}\varepsilon^{-\varepsilon/(p+\varepsilon)} = \frac{p^2}{p+Q}. \tag{3.5}$$

By Hölder's inequality, we have

$$
\begin{aligned}
&\liminf_{\varepsilon\searrow 0}\int_\Omega\int_\Omega \frac{\varepsilon|f(x)-f(y)|^{p+\varepsilon}}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y)\\
&\geq \liminf_{\varepsilon\searrow 0}\varepsilon\mu(\Omega)^{-2\varepsilon/p}\left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^{(Q+p)p/(p+\varepsilon)}}\,d\mu(x)\,d\mu(y)\right)^{(p+\varepsilon)/p}\\
&= \liminf_{\varepsilon\searrow 0}\left(\varepsilon^{p/(p+\varepsilon)}\frac{p(p+\varepsilon)}{\varepsilon(p+Q)}\left(1-\frac{p^2-Q\varepsilon}{p(p+\varepsilon)}\right)\right.\\
&\qquad\left.\times\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^Q d(x,y)^{(p^2-Q\varepsilon)/(p+\varepsilon)}}\,d\mu(x)\,d\mu(y)\right)^{(p+\varepsilon)/p}\\
&\geq \frac{p^2}{p+Q}C^{-1}E_p(f,\Omega)
\end{aligned}
$$

by (3.5) and Corollary 3.1. □

The second mollifier we consider here is defined as

$$\rho_i(x,y)=\frac{d(x,y)^q}{r_i^q}\frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))},$$

where $r_i\searrow 0$ as $i\to\infty$.

It is not hard to check that (2.12) holds. We next verify that it satisfies (2.13). Let

$$d_{i,j}:=\frac{2^{(j+1)q}}{r_i^q}C_d$$

for $j\leq\log_2 r_i$, and $d_{i,j}=0$ otherwise. Now

$$d_{i,j}\geq\sup_{y\in X}\frac{2^{(j+1)q}}{r_i^q}\frac{\mu(B(y,2^{j+1}))}{\mu(B(y,r_i))}$$

for $j\leq\log_2 r_i$, and so for every $x,y\in X$, we have

$$\rho_i(x,y)\leq\sum_{j\in\mathbb{Z}}d_{i,j}\frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}.$$

Moreover,

$$\sum_{j\in\mathbb{Z}}d_{i,j}=C_d r_i^{-q}\sum_{j\leq\log_2 r_i}2^{(j+1)q}\leq 2^{q+1}C_d,$$

and

$$\sum_{j\geq M}d_{i,j}=0$$

when $\log_2 r_i<M$, so that $\lim_{i\to\infty}\sum_{j\geq M}d_{i,j}=0$ for every $M\in\mathbb{Z}$.

The following corollary follows from applying Theorem 1.6 with the choices $p=1$ and the above mollifier.

Corollary 3.6. *Let $1<q<\infty$ and suppose that X supports a $(1,1)$–Poincaré inequality. If $f\in L^q(\Omega)$, then*

$$C^{-1}\|Df\|(\Omega)\leq\liminf_{r\searrow 0}\int_\Omega\left[\frac{1}{\mu(B(y,r))}\int_{B(y,r)\cap\Omega}\frac{|f(x)-f(y)|^q}{r^q}\,d\mu(x)\right]^{1/q}d\mu(y).$$

If Ω is a bounded q-extension domain and $f \in \widehat{N}^{1,q}(\Omega)$, then

$$\limsup_{r \searrow 0} \int_\Omega \left[\frac{1}{\mu(B(y,r))} \int_{B(y,r)\cap\Omega} \frac{|f(x)-f(y)|^q}{r^q} \, d\mu(x) \right]^{1/q} d\mu(y) \le C\|Df\|(\Omega).$$

Here $C \ge 1$ is a constant depending only on q, the doubling constant of the measure, and the constants in the Poincaré inequality.

The third mollifier, studied in the Euclidean setting e.g. by Brezis [6, Eq. (45)], is simple and natural. Consider

$$\rho_i(x,y) = \frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))},$$

where $r_i \searrow 0$ as $i \to \infty$. Again (2.12) can be verified easily. We check (2.13). We can assume that $r_i < \min\{1, \operatorname{diam} X/4\}$ for all $i \in \mathbb{N}$. Letting

$$d_{i,j} := C_0 \left(\frac{2^{j+1}}{r_i} \right)^\sigma,$$

for $j \le \log_2 r_i$ and $d_{i,j} = 0$ otherwise, by (2.1) we have for all $j \le \log_2 r_i$ that

$$d_{i,j} \ge \frac{\mu(B(y,2^{j+1}))}{\mu(B(y,r_i))} \quad \text{for all } y \in X.$$

Then for every $x, y \in X$, we have

$$\rho_i(x,y) \le \sum_{j\in\mathbb{Z}} d_{i,j} \frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}.$$

Finally,

$$\sum_{j\in\mathbb{Z}} d_{i,j} = C_0 \sum_{j\le \log_2 r_i} \left(\frac{2^{j+1}}{r_i} \right)^\sigma \le C,$$

where C depends only on C_0 and σ, and thus in fact only on the doubling constant C_d.

Similar to the case of second mollifier, the following corollary follows from applying Theorem 1.6 with the choice $p = 1$.

Corollary 3.7. *Let $1 < q < \infty$ and suppose that X supports a $(1,1)$–Poincaré inequality. If $f \in L^q(\Omega)$, then*

$$C^{-1}\|Df\|(\Omega) \le \liminf_{r\searrow 0} \int_\Omega \left[\frac{1}{\mu(B(y,r))} \int_{B(y,r)\cap\Omega} \frac{|f(x)-f(y)|^q}{d(x,y)^q} \, d\mu(x) \right]^{1/q} d\mu(y).$$

If $\Omega \subset X$ is a bounded q-extension domain and $f \in \widehat{N}^{1,q}(\Omega)$, then

$$\limsup_{r \searrow 0} \int_\Omega \left[\frac{1}{\mu(B(y,r))} \int_{B(y,r)\cap\Omega} \frac{|f(x)-f(y)|^q}{d(x,y)^q} \, d\mu(x) \right]^{1/q} d\mu(y) \le C\|Df\|(\Omega).$$

Here $C \ge 1$ is a constant depending only on q, the doubling constant of the measure, and the constants in the Poincaré inequality.

Finally, we discuss one corollary of Theorem 1.7. In [9], Brezis and Nguyen gave three examples of φ when studying the asymptotic behavior of $\Lambda_{p,\delta}(f,\Omega)$. Here, we only consider the most simple one, that is, $\varphi(t) = 0$ for $t \in [0,1]$ and $\varphi(t) = 1$ for $t > 1$. Applying Theorem 1.7 with this φ, we obtain the following result.

Corollary 3.8. *Suppose μ is Ahlfors Q-regular with $1 < Q < \infty$, $\Omega \subset X$ is open and bounded, and let $f \in L^1(\Omega)$. Then*

$$C^{-1}\|Df\|(\Omega) \le \limsup_{\delta\to 0} \int_\Omega \int_{\{x\in\Omega:\, |f(x)-f(y)|>\delta\}} \frac{\delta}{d(x,y)^{Q+1}}\, d\mu(x)\, d\mu(y).$$

for some constant $C \ge 1$ depending only on the Ahlfors regularity constants.

Remark 3.9. Let $f \in \widehat{N}^{1,p}(X)$, with $1 < p < \infty$. In [5] it is proved that the inequality in Corollary 3.8 becomes an equality in the setting of metric measure spaces endowed with a doubling measure, supporting a $(1,p)-$Poincaré inequality, and such that at μ-a.e. point the tangent space (in the Gromov-Hausdorff sense) is unique and Euclidean with a fixed dimension.

4. Lower bounds

In this section we prove the lower bounds of our three main theorems. As before, Ω is an open subset of X.

4.1. Lower bound of Theorem 1.5.

The following theorem proves the lower bound of Theorem 1.5; the proof is rather standard and follows mostly as in [7].

Theorem 4.1. *Let $1 \le p < \infty$ and suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying conditions (2.12), (2.13). Let $f \in L^p(\Omega)$. Then*

$$C_1' E_p(f,\Omega) \le \liminf_{i\to\infty} \Psi_{p,i}(f,\Omega),$$

where C_1' is a constant depending only on the doubling constant of the measure and the constant C_ρ associated with the mollifiers.

Proof. We can assume that Ω is nonempty. Consider a nonempty bounded open set $U \subset \Omega$. We apply Hölder's inequality with the exponents

$$\frac{p+\varepsilon_i}{p} \quad \text{and} \quad \frac{p+\varepsilon_i}{\varepsilon_i}$$

to obtain

$$\begin{aligned}
&\int_U \int_U \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \\
&\le \left(\int_U \int_U \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \right)^{p/(p+\varepsilon_i)} \\
&\qquad \times \left(\int_U \int_U \rho_i(x,y)\, d\mu(x)\, d\mu(y) \right)^{\varepsilon_i/(p+\varepsilon_i)} \\
&\le \left(\int_U \int_U \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \right)^{p/(p+\varepsilon_i)} (C_\rho \mu(U))^{\varepsilon_i/(p+\varepsilon_i)} \quad \text{by (2.13).}
\end{aligned}$$

Thus

$$\begin{aligned}
&(C_\rho \mu(U))^{-\varepsilon_i/(p+\varepsilon_i)} \int_U \int_U \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \\
&\qquad \le \left(\int_U \int_U \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \right)^{p/(p+\varepsilon_i)}.
\end{aligned} \tag{4.2}$$

We estimate

$$\begin{aligned}
&\liminf_{i\to\infty}\left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\\
&\quad\ge \liminf_{i\to\infty}\left(\int_U\int_U \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\\
&\quad\ge \liminf_{i\to\infty}\int_U\int_U \frac{|f(x)-f(y)|^{p}}{d(x,y)^{p}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\quad\text{by (4.2)}\\
&\quad\ge C_1 E_p(f,U)\quad\text{by Theorem 1.4.}
\end{aligned}$$

Since this holds for every bounded open $U\subset\Omega$, using the measure property of $\|Df\|$ in the case $p=1$ and [3, Lemma 2.23] in the case $1<p<\infty$, we obtain

$$\liminf_{i\to\infty}\left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\ge C_1E_p(f,\Omega).$$

□

4.2. **Lower bound of Theorem 1.6.** Given a ball $B=B(x,r)$ with a specified center $x\in X$ and radius $r>0$, we denote $2B:=B(x,2r)$. The distance between two sets $A,D\subset X$ is denoted by

$$\operatorname{dist}(A,D):=\inf\{d(x,y)\colon x\in A,\ y\in D\}.$$

The next lemma is standard; for a proof see [34, Lemma 5.1]. Recall the definition (2.9).

Lemma 4.3. *Consider an open set $U\subset\Omega$ with $\operatorname{dist}(U,X\setminus\Omega)>0$, and a scale $0<s<\operatorname{dist}(U,X\setminus\Omega)/10$. Then we can choose an at most countable covering $\{B_j=B(x_j,s)\}_j$ of $U(5s)$ such that $x_j\in U(5s)$, each ball $5B_j$ is contained in Ω, and the balls $\{5B_j\}_{j=1}^\infty$ can be divided into C_d^8 collections of pairwise disjoint balls.*

Given such a covering of $U(5s)$, we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to the covering, such that $0\le\phi_j\le 1$,

$$\text{each } \phi_j \text{ is a } 3C_d^8/s\text{-Lipschitz function,}\tag{4.4}$$

and $\operatorname{spt}(\phi_j)\subset 2B_j$ for each $j\in\mathbb{N}$; see e.g. [32, p. 104]. Finally, we can define a *discrete convolution* h of any $f\in L^1(\Omega)$ with respect to the covering by

$$h:=\sum_j f_{B_j}\phi_j.$$

Clearly $h\in\operatorname{Lip}_{\mathrm{loc}}(U)$.

Now we prove the lower bound of Theorem 1.6.

Theorem 4.5. *Let $1\le p<\infty$, $1\le q<\infty$, and suppose ρ_i is a sequence of mollifiers satisfying either (2.12) or the same condition with p replaced by q. Suppose $f\in L^p(\Omega)$. Then*

$$C_1''E_p(f,\Omega)\le\liminf_{i\to\infty}\int_\Omega\left[\int_\Omega\left(\frac{|f(x)-f(y)|^p}{d(x,y)^p}\right)^q\rho_i(x,y)\,d\mu(x)\right]^{1/q}d\mu(y)\tag{4.6}$$

for some constant C_1'' depending only on p, q, C_ρ, and on the doubling constant of the measure.

Note that here we do not impose any conditions on the open set $\Omega\subset X$.

Proof. We can assume that

$$\liminf_{i\to\infty}\int_\Omega\left[\int_\Omega\left(\frac{|f(x)-f(y)|^p}{d(x,y)^p}\right)^q\rho_i(x,y)\,d\mu(x)\right]^{1/q}\,d\mu(y)=:M<\infty.$$

Fix $0<\varepsilon<1$. Passing to a subsequence (not relabeled), we can assume that

$$\int_\Omega\left[\int_\Omega\left(\frac{|f(x)-f(y)|^p}{d(x,y)^p}\right)^q\rho_i(x,y)\,d\mu(x)\right]^{1/q}\,d\mu(y)\le M+\varepsilon\quad\text{for all } i\in\mathbb{N}.$$

Using the assumption (2.12) (if we have this condition with p replaced by q, the next three lines are similar), we get

$$\int_\Omega\left[\int_\Omega\frac{|f(x)-f(y)|^{pq}}{r_i^p d(x,y)^{pq-p}}\frac{\chi_{B(y,r_i)\cap\Omega}(x)}{\mu(B(y,r_i))}\,d\mu(x)\right]^{1/q}\,d\mu(y)\le(M+\varepsilon)C_\rho^{1/q},$$

and thus

$$\int_\Omega\left[\int_\Omega\frac{|f(x)-f(y)|^{pq}}{r_i^{pq}}\frac{\chi_{B(y,r_i)\cap\Omega}(x)}{\mu(B(y,r_i))}\,d\mu(x)\right]^{1/q}\,d\mu(y)\le(M+\varepsilon)C_\rho^{1/q}. \tag{4.7}$$

Fix $i\in\mathbb{N}$. Let $U\subset\Omega$ be open with $\operatorname{dist}(U,X\setminus\Omega)>r_i$, and let $s:=r_i/10$. Consider a covering $\{B_j\}_{j=1}^\infty$ of $U(5s)$ at scale $s>0$, as described in Lemma 4.3. Then consider the discrete convolution

$$h:=\sum_j f_{B_j}\phi_j.$$

Recall the definition of the Lipschitz number $\operatorname{Lip} h$ from (2.11). Suppose $x\in U$. Then $x\in B_j$ for some $j\in\mathbb{N}$. Consider any other point $y\in B_j$. Denote by I_j those $k\in\mathbb{N}$ for which $2B_k\cap B_j\neq\emptyset$. We estimate

$$\begin{aligned}
|h(x)-h(y)|&=\left|\sum_{k\in I_j}f_{B_k}(\phi_k(x)-\phi_k(y))\right|\\
&=\left|\sum_{k\in I_j}(f_{B_k}-f_{B_j})(\phi_k(x)-\phi_k(y))\right|\\
&\le\frac{3C_d^8 d(x,y)}{s}\sum_{k\in I_j}|f_{B_k}-f_{B_j}|\quad\text{by (4.4)}\\
&\le\frac{3C_d^8 d(x,y)}{s}\left(\sum_{k\in I_j}\fint_{B_k}|f-f_{5B_j}|\,d\mu+\sum_{k\in I_j}\fint_{B_j}|f-f_{5B_j}|\,d\mu\right)\\
&\le\frac{6C_d^{11} d(x,y)}{s}\sum_{k\in I_j}\fint_{5B_j}|f-f_{5B_j}|\,d\mu\\
&\le\frac{6C_d^{19} d(x,y)}{s}\fint_{5B_j}\fint_{5B_j}|f(z)-f(w)|\,d\mu(z)\,d\mu(w),
\end{aligned}\tag{4.8}$$

since by Lemma 4.3 we know that I_j has cardinality at most C_d^8. Letting $y\to x$, we obtain an estimate for Lip_h in the ball B_j. In total, we conclude (we track the constants for a while in order to make the estimates more explicit) that in U it holds that

$$\operatorname{Lip} h\le\frac{6C_d^{19}}{s}\sum_j\chi_{B_j}\fint_{5B_j}\fint_{5B_j}|f(x)-f(y)|\,d\mu(x)\,d\mu(y).$$

Now

$$\begin{aligned}\operatorname{Lip} h &\le \frac{6C_d^{19}}{s}\sum_j \chi_{B_j} \fint_{5B_j}\fint_{5B_j} |f(x)-f(y)|\,d\mu(x)\,d\mu(y)\\ &\le \frac{6C_d^{19}}{s}\sum_j \chi_{B_j}\left[\fint_{5B_j}\left[\fint_{5B_j}|f(x)-f(y)|^{pq}\,d\mu(x)\right]^{1/q} d\mu(y)\right]^{1/p} \quad \text{by Hölder}\\ &\le \frac{6C_d^{21}}{s}\sum_j \chi_{B_j}\left[\fint_{5B_j}\left[\int_{5B_j}|f(x)-f(y)|^{pq}\frac{\chi_{B(y,10s)}(x)}{\mu(B(y,10s))}\,d\mu(x)\right]^{1/q} d\mu(y)\right]^{1/p}.\end{aligned}$$

Since the balls $\{5B_j\}_j$ and so also the balls $\{B_j\}_j$ can be divided into C_d^8 collections of pairwise disjoint balls, in U it holds that

$$(\operatorname{Lip} h)^p \le \left(\frac{6C_d^{29}}{s}\right)^p \sum_j \fint_{5B_j}\left[\int_{5B_j}|f(x)-f(y)|^{pq}\frac{\chi_{B(y,10s)}(x)}{\mu(B(y,10s))}\,d\mu(x)\right]^{1/q} d\mu(y),$$

and so

(4.9)

$$\begin{aligned}\int_U (\operatorname{Lip} h)^p\,d\mu &\le \left(\frac{6C_d^{29}}{s}\right)^p \sum_j \int_{5B_j}\left[\int_{5B_j}|f(x)-f(y)|^{pq}\frac{\chi_{B(y,10s)}(x)}{\mu(B(y,10s))}\,d\mu(x)\right]^{1/q} d\mu(y)\\ &\le \left(\frac{6C_d^{37}}{s}\right)^p \int_\Omega\left[\int_\Omega |f(x)-f(y)|^{pq}\frac{\chi_{B(y,10s)\cap\Omega}(x)}{\mu(B(y,10s))}\,d\mu(x)\right]^{1/q} d\mu(y)\\ &\le C(M+\varepsilon)C_\rho^{1/q}\end{aligned}$$

by (4.7), with $C := (60C_d^{37})^p$. We know that the minimal p-weak upper gradient g_h of h in U satisfies $g_h \le \operatorname{Lip} h$ μ-a.e. in U, see e.g. [3, Proposition 1.14].

Recall that we can do the above for each r_i, and that for a fixed i we denoted $s = r_i/10$. From now on, we can consider any open $U \subset \Omega$ with $\operatorname{dist}(U, X\setminus\Omega) > 0$. We get a sequence of discrete convolutions $\{h_i\}_{i=1}^\infty$ corresponding to scales $s_i \searrow 0$, such that $\{g_{h_i}\}_{i=1}^\infty$ is a bounded sequence in $L^p(U)$. From the properties of discrete convolutions, see e.g. [29, Lemma 5.3], we know that $h_i \to f$ in $L^p(U)$.

Case $p = 1$:
We get

$$\|Df\|(U) \le \liminf_{i\to\infty}\int_U g_{h_i}\,d\mu \le \liminf_{i\to\infty}\int_U \operatorname{Lip} h_i\,d\mu \le C(M+\varepsilon)C_\rho^{1/q},$$

and so $f \in \mathrm{BV}(U)$. Note that $\|Df\|$ is a Radon measure on Ω. Exhausting Ω by sets U, we obtain

$$\begin{aligned}\|Df\|(\Omega) &\le C(M+\varepsilon)C_\rho^{1/q}\\ &= CC_\rho^{1/q}\left(\liminf_{i\to\infty}\int_\Omega\left[\int_\Omega\left(\frac{|f(x)-f(y)|}{d(x,y)}\right)^q \rho_i(x,y)\,d\mu(x)\right]^{1/q} d\mu(y)+\varepsilon\right).\end{aligned}$$

Letting $\varepsilon \to 0$, this proves (4.6).

Case $1 < p < \infty$:
By (4.9), $\{g_{h_i}\}_{i=1}^\infty$ is a bounded sequence in $L^p(U)$. By reflexivity of the space $L^p(U)$, we find a subsequence of $\{g_{h_i}\}_{i=1}^\infty$ (not relabeled) and $g \in L^p(U)$ such that $g_{h_i} \to g$ weakly

in $L^p(U)$ (see e.g. [32, Section 2]). By Mazur's lemma (Theorem 2.3), for suitable convex combinations we get the strong convergence $\sum_{l=i}^{N_i} a_{i,l} g_{h_l} \to g$ in $L^p(U)$. Note that also $\sum_{l=i}^{N_i} a_{i,l} h_l \to f$ in $L^p(U)$. Using e.g. [3, Proposition 2.3], we know that there exists a function $\widehat{f} = f$ μ-a.e. in U such that g is a p-weak upper gradient of $\widehat{f}$ in U. We get

$$E_p(f,U) \le \int_U g^p \, d\mu \le \limsup_{i\to\infty} \int_U (g_{h_i})^p \, d\mu \le \limsup_{i\to\infty} \int_U (\operatorname{Lip} h_i)^p \, d\mu \le C(M+\varepsilon) C_\rho^{1/q}.$$

Exhausting Ω by sets U and using [3, Lemma 2.23], we obtain

$$\begin{aligned} E_p(f,\Omega) &\le C(M+\varepsilon) C_\rho^{1/q} \\ &\le C C_\rho^{1/q} \left(\liminf_{i\to\infty} \int_\Omega \left[\int_\Omega \left(\frac{|f(x)-f(y)|^p}{d(x,y)^p} \right)^q \rho_i(x,y) \, d\mu(x) \right]^{1/q} d\mu(y) + \varepsilon \right). \end{aligned}$$

Letting $\varepsilon \to 0$, this proves (4.6). $\square$

4.3. **Lower bound of Theorem 1.7.** The following theorem proves the lower bound of Theorem 1.7.

Theorem 4.10. *Let $1 \le p < \infty$ and suppose $\varphi\colon [0,\infty) \to [0,\infty)$ satisfies assumptions (2.15) and (2.17). Suppose μ is Ahlfors Q-regular, Ω is bounded, and let $f \in L^p(\Omega)$. Then*

$$E_p(f,\Omega) \le C \limsup_{\delta\to 0} \int_\Omega \int_\Omega \frac{\delta^p \varphi(|f(x)-f(y)|/\delta)}{d(x,y)^{Q+p}} \, d\mu(x) \, d\mu(y),$$

where the constant C depends only on p, C_φ, and the Ahlfors regularity constants.

This proof is very similar to that in the Euclidean setting, see [10, Proposition 2].

Proof. We can assume that

$$\limsup_{\delta\to 0} \int_\Omega \int_\Omega \frac{\delta^p \varphi(|f(x)-f(y)|/\delta)}{d(x,y)^{Q+p}} \, d\mu(x) \, d\mu(y) =: M < \infty.$$

First suppose that $f \in L^\infty(\Omega)$. Fix $\varepsilon > 0$. Also fix $\tau > 0$ and, recalling (2.17), choose $\delta_0 > 0$ sufficiently small that

$$\int_{\delta_0}^\infty \varphi(t) t^{-1-p} \, dt \ge C_\varphi^{-1}(1-\tau) \tag{4.11}$$

and

$$\Lambda_{p,\delta}(f,\Omega) \le \limsup_{\delta\to 0} \Lambda_{p,\delta}(f,\Omega) + \tau \quad \text{for all } 0 < \delta < \delta_0. \tag{4.12}$$

We have by Fubini's theorem

$$\begin{aligned} &C_A \int_0^{\delta_0} \varepsilon \delta^{\varepsilon-1} \Lambda_{p,\delta}(f,\Omega) \, d\delta \\ &\ge \int_0^{\delta_0} \varepsilon \int_\Omega \int_\Omega \frac{\delta^{\varepsilon-1+p} \varphi(|f(x)-f(y)|/\delta)}{d(x,y)^{Q+p}} \, d\mu(x) \, d\mu(y) \, d\delta \\ &= \int_\Omega \int_\Omega \frac{\varepsilon}{d(x,y)^{Q+p}} \int_0^{\delta_0} \delta^{\varepsilon-1+p} \varphi(|f(x)-f(y)|/\delta) \, d\delta \, d\mu(x) \, d\mu(y) \\ &= \int_\Omega \int_\Omega \frac{\varepsilon |f(x)-f(y)|^{p+\varepsilon}}{d(x,y)^{Q+p}} \int_{|f(x)-f(y)|/\delta_0}^\infty \varphi(t) t^{-1-p-\varepsilon} \, dt \, d\mu(x) \, d\mu(y). \end{aligned}$$

Thus

$$\begin{aligned}&C_A\int_0^{\delta_0}\varepsilon\delta^{\varepsilon-1}\Lambda_{p,\delta}(f,\Omega)\,d\delta\\&\ge\int_\Omega\int_{\{x\in\Omega:\,|f(x)-f(y)|<\delta_0^2\}}\frac{\varepsilon|f(x)-f(y)|^{p+\varepsilon}}{d(x,y)^{Q+p}}\int_{\delta_0}^\infty\varphi(t)t^{-1-p-\varepsilon}\,dt\,d\mu(x)\,d\mu(y),\end{aligned}$$

and then

$$\begin{aligned}&C_A\int_0^{\delta_0}\varepsilon\delta^{\varepsilon-1}\Lambda_{p,\delta}(f,\Omega)\,d\delta\\&\quad+\int_\Omega\int_{\{x\in\Omega:\,|f(x)-f(y)|\ge\delta_0^2\}}\frac{\varepsilon(2\|f\|_{L^\infty(\Omega)})^{p+\varepsilon}}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y)\int_{\delta_0}^\infty\varphi(t)t^{-1-p-\varepsilon}\,dt\\&\quad\ge\int_\Omega\int_\Omega\frac{\varepsilon|f(x)-f(y)|^{p+\varepsilon}}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y)\int_{\delta_0}^\infty\varphi(t)t^{-1-p-\varepsilon}\,dt.\end{aligned}\tag{4.13}$$

For $\alpha>0$, by the fact that φ is increasing we have

$$\begin{aligned}\Lambda_{p,\delta}(f,\Omega)&\ge\int_\Omega\int_{\{x\in\Omega:\,|f(x)-f(y)|\ge\alpha\}}\frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y)\\&\ge\delta^p\varphi(\alpha/\delta)\int_\Omega\int_{\{x\in\Omega:\,|f(x)-f(y)|\ge\alpha\}}\frac{1}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y).\end{aligned}$$

Choosing δ sufficiently small such that $\varphi(\alpha/\delta)>0$, we conclude that for every $\alpha>0$, we have

$$\int_\Omega\int_{\{x\in\Omega:\,|f(x)-f(y)|\ge\alpha\}}\frac{1}{d(x,y)^{Q+p}}\,d\mu(x)\,d\mu(y)<\infty.$$

This combined with (4.11), (4.13), and Corollary 3.4 gives

$$C_A\liminf_{\varepsilon\to0}\int_0^{\delta_0}\varepsilon\delta^{\varepsilon-1}\Lambda_{p,\delta}(f,\Omega)\,d\delta\ge(1-\tau)C_\varphi^{-1}C^{-1}E_p(f,\Omega).$$

On the other hand,

$$\begin{aligned}(\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)+\tau)\delta_0^\varepsilon&=(\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)+\tau)\int_0^{\delta_0}\varepsilon(\delta')^{\varepsilon-1}\,d\delta'\\&\ge\int_0^{\delta_0}\varepsilon\delta^{\varepsilon-1}\Lambda_{p,\delta}(f,\Omega)\,d\delta\quad\text{by (4.12)}.\end{aligned}$$

Thus

$$\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)+\tau\ge(1-\tau)C_\varphi^{-1}C^{-1}C_A^{-1}E_p(f,\Omega).$$

Letting $\tau\to0$, we get

$$\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)\ge C_\varphi^{-1}C^{-1}C_A^{-1}E_p(f,\Omega).$$

Finally, we drop the assumption $f\in L^\infty(\Omega)$. Consider the truncations

$$f_M:=\min\{M,\max\{-M,f\}\},\quad M>0.$$

Since φ is increasing, we have

$$\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)\ge\limsup_{\delta\to0}\Lambda_{p,\delta}(f_M,\Omega)\ge C_\varphi^{-1}C^{-1}C_A^{-1}E_p(f_M,\Omega).$$

Letting $M\to\infty$, by using the coarea formula (2.5) in the case $p=1$, and e.g. [3, Proposition 2.3] in the case $1<p<\infty$, we get

$$\limsup_{\delta\to0}\Lambda_{p,\delta}(f,\Omega)\ge C_\varphi^{-1}C^{-1}C_A^{-1}E_p(f,\Omega).$$

□

5. Upper bounds

In this section we prove the upper bounds of our three main theorems. As before, Ω is an open subset of X.

5.1. **Upper bound of Theorem 1.5.**

The following theorem proves the upper bound of Theorem 1.5.

Theorem 5.1. *Suppose $1 \le p < \infty$, $p < q < \infty$, and $\{\rho_i\}_{i=1}^{\infty}$ is a sequence of mollifiers satisfying condition (2.13). Suppose also that X supports a $(1,p)$–Poincaré inequality, $\Omega \subset X$ is a bounded strong p-extension domain and strong q-extension domain, and $f \in \widehat{N}^{1,q}(\Omega)$. Then*

$$\limsup_{i\to\infty} \Psi_{p,i}(f,\Omega) \le C_2' E_p(f,\Omega),$$

where C_2' is a constant depending only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

Proof. The space supports a (p,p)–Poincaré as well as a (q,q)–Poincaré inequality, see e.g. [3, Theorem 4.21]. For sufficiently large i such that $p + \varepsilon_i < q$, we apply Hölder's inequality with the exponents

$$\frac{q-p}{q-p-\varepsilon_i} \quad \text{and} \quad \frac{q-p}{\varepsilon_i}$$

to obtain

$$\begin{aligned}
&\left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\\
&= \left(\int_\Omega\int_\Omega \left(\frac{|f(x)-f(y)|}{d(x,y)}\right)^{\frac{pq-p^2-p\varepsilon_i}{q-p}+\frac{q\varepsilon_i}{q-p}} \rho_i(x,y)^{\frac{q-p-\varepsilon_i}{q-p}+\frac{\varepsilon_i}{q-p}}\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\\
&\qquad \le \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p}}{d(x,y)^{p}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{(q-p-\varepsilon_i)p/((q-p)(p+\varepsilon_i))}\\
&\qquad\qquad \times \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{q}}{d(x,y)^{q}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{\varepsilon_i p/((q-p)(p+\varepsilon_i))}.
\end{aligned}$$

Applying Theorem 1.4 to both factors, we get

$$\begin{aligned}
\limsup_{i\to\infty} &\left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^{p+\varepsilon_i}}{d(x,y)^{p+\varepsilon_i}}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\right)^{p/(p+\varepsilon_i)}\\
&\le C_2E_p(f,\Omega)\times (C_2E_q(f,\Omega))^0\\
&= C_2E_p(f,\Omega).
\end{aligned}$$

□

5.2. **Upper bound of Theorem 1.6.**

Lemma 5.2. *Let $1 \le p < \infty$ and $p < q < \infty$, and X suppose supports a $(1,p)$-Poincaré inequality. Let $f \in N^{1,q}_{\mathrm{loc}}(X)$. Then the minimal p-weak upper gradient $g_{f,p}$ and the minimal q-weak upper gradient $g_{f,q}$ satisfy*

$$g_{f,p} \le g_{f,q} \le \widetilde{C} g_{f,p} \qquad a.e.$$

for a constant $\widetilde{C}$ depending only on the doubling constant of the measure and the constants in the Poincaré inequality.

Proof. For the first inequality, see [3, Proposition 2.44].

Then we prove the second inequality. For the case $1 < p < \infty$, see [3, Corollary A.9]. In the case $p = 1$, we can follow the argument given in the proof of [3, Corollary A.9], but instead of [3, Corollary A.8] we use [19, Proposition 4.26]. □

Now we can prove the upper bound of Theorem 1.6.

Theorem 5.3. *Let $1 \le p < \infty$, $1 < q < \infty$, and suppose X supports a $(1,p)$-Poincaré inequality. Let $\Omega \subset X$ be a bounded pq-extension domain, suppose $\{\rho_i\}_{i=1}^{\infty}$ is a sequence of mollifiers that satisfy (2.13), and let $f \in \widehat{N}^{1,pq}(\Omega)$. Then*

$$\limsup_{i\to\infty} \int_\Omega \left[\int_\Omega \left(\frac{|f(x)-f(y)|^p}{d(x,y)^p} \right)^q \rho_i(x,y)\, d\mu(x) \right]^{1/q} d\mu(y) \le C E_p(f,\Omega), \tag{5.4}$$

where C is a constant depending only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

Proof. Since Ω is a pq-extension domain, we can assume that $f \in \widehat{N}^{1,pq}(X)$. We can further assume that $f \in N^{1,pq}(X)$, since choice of a pointwise μ-representative does not affect either side of (5.4). Fix an upper gradient $g \in L^{pq}(X)$ of f. By Hölder's inequality, X also supports a $(1,pq)$–Poincaré inequality. By Keith-Zhong [31, Theorem 1.0.1], there exists $1 < q' < q$ such that X also supports a $(1,pq')$–Poincaré inequality. Choosing q' sufficiently close to q, we then have that X supports a (pq,pq')–Poincaré inequality; see [3, Theorem 4.21]. Let us denote the constants of the (pq,pq')–Poincaré inequality by C_P', λ'; note that they only depend on the doubling constant of the measure and the constants C_P, λ of the original $(1,p)$-Poincaré inequality.

Recall the definition of the Hardy–Littlewood maximal function from (2.8). For every Lebesgue point $y \in \Omega$ and every $r > 0$, we estimate

$$\begin{aligned} |f(y) - f_{B(y,r)}| &\le \sum_{k=1}^{\infty} |f_{B(y,2^{-k+1}r)} - f_{B(y,2^{-k}r)}| \\ &\le C_d \sum_{k=1}^{\infty} \fint_{B(y,2^{-k+1}r)} \left|f - f_{B(y,2^{-k+1}r)}\right| d\mu \\ &\le C_P' C_d \sum_{k=1}^{\infty} 2^{-k+1} r \left(\fint_{B(y,2^{-k+1}\lambda' r)} g^{pq'}\, d\mu \right)^{1/(pq')} \\ &\le 2 C_P' C_d r \left(\mathcal{M}_{\lambda' r} g^{pq'}(y) \right)^{1/(pq')}. \end{aligned} \tag{5.5}$$

For every $j \in \mathbb{Z}$ and for every $y \in \Omega$, we estimate

$$
\begin{aligned}
&\int_X \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}\, d\mu(x) \\
&\quad \le 2^{pq} \int_X \frac{|f(x)-f_{B(y,2^{j+1})}|^{pq}}{d(x,y)^{pq}} \frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}\, d\mu(x) \\
&\qquad + 2^{pq} \int_X \frac{|f(y)-f_{B(y,2^{j+1})}|^{pq}}{d(x,y)^{pq}} \frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}\, d\mu(x) \\
&\quad \le 2^{-jpq+pq} \fint_{B(y,2^{j+1})} |f(x)-f_{B(y,2^{j+1})}|^{pq}\, d\mu(x) \\
&\qquad + 2^{-jpq+pq} \fint_{B(y,2^{j+1})} |f(y)-f_{B(y,2^{j+1})}|^{pq}\, d\mu(x) \\
&\quad \le 4^{pq}(C_P')^p \left(\fint_{B(y,2^{j+1}\lambda')} g(x)^{pq'}\, d\mu(x)\right)^{q/q'} + (8C_d C_P')^{pq} \left(\mathcal{M}_{2^{j+1}\lambda'} g^{pq'}(y)\right)^{q/q'} \quad \text{by (5.5)} \\
&\quad \le 8^{1+pq}(C_d C_P')^{pq} \left(\mathcal{M}_{2^{j+1}\lambda'} g^{pq'}(y)\right)^{q/q'}.
\end{aligned}
$$

Using (2.13), we can now estimate

$$
\begin{aligned}
&\int_\Omega \left[\int_X \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \rho_i(x,y)\, d\mu(x)\right]^{1/q} d\mu(y) \\
&\quad \le \int_\Omega \left[\sum_{j\in\mathbb{Z}} d_{i,j} \int_X \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \frac{\chi_{B(y,2^{j+1})\setminus B(y,2^j)}(x)}{\mu(B(y,2^{j+1}))}\, d\mu(x)\right]^{1/q} d\mu(y) \\
&\quad \le 8^{1+p}(C_d C_P')^p \int_\Omega \left[\sum_{j\in\mathbb{Z}} d_{i,j} \left(\mathcal{M}_{2^{j+1}\lambda'}(g^{pq'}(y)\right)^{q/q'}\right]^{1/q} d\mu(y).
\end{aligned}
\tag{5.6}
$$

For every $i \in \mathbb{N}$, we can estimate

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} d_{i,j} \left(\mathcal{M}_{2^{j+1}\lambda'} g^{pq'}\right)^{q/q'} &\le \sum_{j\in\mathbb{Z}} d_{i,j} \left(\mathcal{M} g^{pq'}\right)^{q/q'} \\
&\le C_\rho \left(\mathcal{M} g^{pq'}\right)^{q/q'} \\
&\in L^1(X)
\end{aligned}
$$

by the Hardy–Littlewood maximal function theorem, see e.g. [32, Theorem 3.5.6]. Then also

$$
\left[\sum_{j\in\mathbb{Z}} d_{i,j} \left(\mathcal{M}_{2^{j+1}\lambda'}(g^{pq'}(y)\right)^{q/q'}\right]^{1/q} \le \left[C_\rho \left(\mathcal{M} g^{pq'}\right)^{q/q'}\right]^{1/q} \in L^1(\Omega),
\tag{5.7}
$$

since Ω is bounded. Moreover, $\mathcal{M}_{2^{j+1}\lambda'} g^{pq'}(y) \to g^{pq'}(y)$ as $j \to -\infty$ for a.e. $y \in \Omega$, and recall that

$$
\lim_{i\to\infty} \sum_{j\ge M} d_{i,j} = 0
$$

for all $M \in \mathbb{Z}$, as given after (2.13). Thus

$$
\limsup_{i\to\infty} \sum_j d_{i,j} (\mathcal{M}_{2^{j+1}\lambda'} g^{pq'}(y))^{q/q'} \le C_\rho g^{pq}(y)
$$

for a.e. $y \in \Omega$. From (5.6) we get by an application of Lebesgue's dominated convergence theorem, with the majorant given by (5.7),

$$\limsup_{i\to\infty} \int_\Omega \left[\int_X \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \rho_i(x,y)\, d\mu(x)\right]^{1/q} d\mu(y) \le 8^{1+p}(C_d C'_P)^p C_\rho^{1/q} \int_\Omega g^p\, d\mu.$$

Let $g_{f,pq}$ be the minimal pq-weak upper gradient of f, and let $g_{f,p}$ be the minimal p-weak upper gradient of f. Using e.g. [3, Lemma 1.46], we can find a sequence of upper gradients $g_l \ge g$ with $\|g_l - g_{f,pq}\|_{L^{pq}(X)} \to 0$, and then also $\|g_l - g_{f,pq}\|_{L^p(\Omega)} \to 0$. Thus

$$\begin{aligned}\limsup_{i\to\infty} \int_\Omega \left[\int_\Omega \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \rho_i(x,y)\, d\mu(x)\right]^{1/q} d\mu(y) &\le 8^{1+p}(C_d C'_P)^p C_\rho^{1/q} \int_\Omega g_{f,pq}^p\, d\mu \\ &\le 8^{1+p}(C_d C'_P)^p C_\rho^{1/q} \widetilde{C}^p \int_\Omega g_{f,p}^p\, d\mu.\end{aligned}$$

by Lemma 5.2. Finally, applying (2.6) in the case $p=1$, we get for all $1 \le p < \infty$

$$\limsup_{i\to\infty} \int_\Omega \left[\int_\Omega \frac{|f(x)-f(y)|^{pq}}{d(x,y)^{pq}} \rho_i(x,y)\, d\mu(x)\right]^{1/q} d\mu(y) \le 8^{1+p}(C_d C'_P)^p C_\rho^{1/p} C_* E_p(f,\Omega).$$

Note that the above constant $8^{1+p}(C_d C'_P)^p C_\rho^{1/p} C_*$ only depends on $C_d, C_P, \lambda, p, C_\rho$, as desired. □

5.3. **Upper bound of Theorem 1.7.** In the following theorem we prove the upper bound of Theorem 1.7.

Theorem 5.8. *Suppose $\varphi\colon [0,\infty) \to [0,\infty)$ satisfies assumptions (2.15), (2.16), and (2.17). Let $1 \le p < \infty$, suppose X supports a $(1,p)$–Poincaré inequality, let Ω be bounded, and let $f \in \widehat{N}^{1,p}(X)$; in the case $p=1$ suppose additionally that $f \in \widehat{N}^{1,q}(X)$ for some $q>1$. Then*

$$\limsup_{\delta\to 0} \int_\Omega \int_\Omega \frac{\delta^p \varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\, d\mu(x)\, d\mu(y) \le C C_\varphi E_p(f,\Omega),$$

where C depends only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and C_φ.

Proof. When $1 < p < \infty$, by Keith-Zhong [31, Theorem 1.0.1] there exists $1 < p' < p$ such that X supports a $(1,p')$–Poincaré inequality. When $p=1$, we let also $p'=1$. For simplicity, we still denote the constants by C_P, λ.

First note that using (2.15) and (2.17), for any $0 < \alpha < \infty$ we get

$$C_\varphi \ge \int_0^\infty \varphi(t) t^{-1-p}\, dt \ge 2^{-1-p}\alpha^{-p} \sum_{j\in\mathbb{Z}} \varphi(2^j\alpha) 2^{-jp}. \tag{5.9}$$

We can assume that $f \in N^{1,p}(X)$, and in the case $p=1$ we can assume that $f \in N^{1,1}(X) \cap N^{1,q}(X)$; note that the choice of pointwise representatives does not cause problems, by e.g. [3, Propositions 1.59 & 1.61]. For a.e. $x, y \in X$, we have

$$|f(x)-f(y)| \le C' d(x,y)[(\mathcal{M}_{2\lambda d(x,y)} g_f(x)^{p'})^{1/p'} + (\mathcal{M}_{2\lambda d(x,y)} g_f(y)^{p'})^{1/p'}],$$

where g_f is the minimal p-weak upper gradient of f and C' is a constant depending only on C_d, C_P, and λ; see e.g. [3, Proof of Theorem 5.1]. Using this and the fact that φ is increasing, we estimate (we denote briefly $\{d(x,y)<r\}:=\{(x,y)\in\Omega\times\Omega\colon d(x,y)<r\}$)

(5.10)
$$\begin{aligned}
&\iint_{\{d(x,y)<r\}} \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\le \iint_{\{d(x,y)<r\}} \frac{\delta^p\varphi\big(C'd(x,y)((\mathcal{M}_{2\lambda d(x,y)}g_f(x)^{p'})^{\frac{1}{p'}}+(\mathcal{M}_{2\lambda d(x,y)}g_f(y)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\le \int_\Omega\int_{B(y,r)} \frac{\delta^p\varphi\big(2C'd(x,y)(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\qquad+\int_\Omega\int_{B(x,r)} \frac{\delta^p\varphi\big(2C'd(x,y)(\mathcal{M}_{2\lambda r}g_f(x)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(y)\,d\mu(x)
\end{aligned}$$

using again the fact that φ is increasing. Note that

$$\begin{aligned}
&\int_{B(y,r)} \frac{\delta^p\varphi\big(2C'd(x,y)(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\\
&\quad\le \int_X \frac{\delta^p\varphi\big(2C'd(x,y)(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\\
&\quad=\sum_{j\in\mathbb{Z}}\int_{B(y,2^j)\setminus B(y,2^{j-1})} \frac{\delta^p\varphi\big(2C'd(x,y)(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{1}{p'}}/\delta\big)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\\
&\quad\le C_d\delta^p\sum_{j\in\mathbb{Z}}\frac{\varphi(2^{j+1}C'(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{1}{p'}}/\delta)}{2^{(j-1)p}}\\
&\quad\le 2^{3p+1}(C')^pC_dC_\varphi(\mathcal{M}_{2\lambda r}g_f(y)^{p'})^{\frac{p}{p'}}\quad\text{by (5.9)}.
\end{aligned}$$

Estimating the second term of (5.10) analogously, we get

$$\iint_{\{d(x,y)<r\}} \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\le 2^{3p+2}(C')^pC_dC_\varphi\int_\Omega(\mathcal{M}_{2\lambda r}g_f^{p'})^{\frac{p}{p'}}\,d\mu.$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega\setminus B(y,r)} \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\\
&\quad=\sum_{j\in\mathbb{N}}\int_{B(y,2^jr)\setminus B(y,2^{j-1}r)} \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\\
&\quad\le C_d\sum_{j=1}^\infty\frac{\delta^pb}{(2^{j-1}r)^p}\quad\text{by (2.16)}\\
&\quad=\frac{2C_d\delta^pb}{r^p}.
\end{aligned}\tag{5.11}$$

In total, we get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\quad\le 2^{2p+3}(C')^pC_dC_\varphi\int_\Omega(\mathcal{M}_{2\lambda r}g_f^{p'})^{\frac{p}{p'}}\,d\mu+\frac{2C_d\delta^pb}{r^p}\mu(\Omega).
\end{aligned}\tag{5.12}$$

In the case $1<p<\infty$, note that $(\mathcal{M}_{2\lambda r}g_f^{p'})^{\frac{p}{p'}}\in L^1(X)$ by the Hardy–Littlewood maximal theorem, and that $\mathcal{M}_{2\lambda r}g_f(x)\to g_f(x)$ as $r\to 0$ for μ-a.e. $x\in X$. Thus by (5.12) and Lebesgue's dominated convergence,

$$\begin{aligned}
&\limsup_{\delta\to 0}\int_\Omega\int_\Omega \frac{\delta^p\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\quad\le 2^{3p+2}(C')^pC_dC_\varphi\int_\Omega(\mathcal{M}_{2\lambda r}g_f^{p'})^{\frac{p}{p'}}\,d\mu\\
&\quad\to 2^{3p+2}(C')^pC_dC_\varphi\int_\Omega g_f^p\,d\mu\quad\text{as } r\to 0\\
&\quad= 2^{3p+2}(C')^pC_dC_\varphi E_p(f,\Omega).
\end{aligned}$$

Then suppose $p=1$; recall that in this case we assume that $f\in N^{1,1}(X)\cap N^{1,q}(X)$. By Lemma 5.2, the minimal 1-weak and minimal q-weak upper gradients of f satisfy $g_{f,1}\le g_{f,q}$ a.e., and so in (5.12) we can replace $g_{f,1}$ with $g_{f,q}$. Note that $\mathcal{M}_{2\lambda r}g_{f,q}\in L^q(X)$ by the Hardy–Littlewood maximal theorem, so that $\mathcal{M}_{2\lambda r}g_{f,q}\in L^1_{\mathrm{loc}}(X)$ and in particular $\mathcal{M}_{2\lambda r}g_{f,q}\in L^1(\Omega)$, and that $\mathcal{M}_{2\lambda r}g_{f,q}(x)\to g_{f,q}(x)$ as $r\to 0$ for μ-a.e. $x\in X$. From (5.12) and Lebesgue's dominated convergence, we get

$$\begin{aligned}
&\limsup_{\delta\to 0}\int_\Omega\int_\Omega \frac{\delta\varphi(|f(x)-f(y)|/\delta)}{\mu(B(y,d(x,y)))d(x,y)}\,d\mu(x)\,d\mu(y)\\
&\quad\le 32C'C_dC_\varphi\int_\Omega \mathcal{M}_{2\lambda r}g_{f,q}\,d\mu\\
&\quad\to 32C'C_dC_\varphi\int_\Omega g_{f,q}\,d\mu\quad\text{as } r\to 0\\
&\quad\le 32C'C_dC_\varphi\widetilde{C}\int_\Omega g_{f,1}\,d\mu\quad\text{by Lemma 5.2}\\
&\quad\le 32C'C_dC_\varphi\widetilde{C}C_*\|Df\|(\Omega)
\end{aligned}$$

by (2.6). □

6. Example

For a function $f\in\widehat{N}^{1,q}(\Omega)$ with $1<q<\infty$, and with the choice $p=1$, the conclusion of Theorem 1.6 takes the form

(6.1)
$$\begin{aligned}
C_1''\|Df\|(\Omega)&\le\liminf_{i\to\infty}\int_\Omega\left[\int_\Omega\frac{|f(x)-f(y)|^q}{d(x,y)^q}\rho_i(x,y)\,d\mu(x)\right]^{1/q}d\mu(y)\\
&\le\limsup_{i\to\infty}\int_\Omega\left[\int_\Omega\frac{|f(x)-f(y)|^q}{d(x,y)^q}\rho_i(x,y)\,d\mu(x)\right]^{1/q}d\mu(y)\le C_2''\|Df\|(\Omega).
\end{aligned}$$

One natural question is whether $C_1''=C_2''$ might hold. In the Euclidean theory, see Brezis–Nguyen [7, Proposition 9], we know that for every $1<q<\infty$, smooth and bounded $\Omega\subset\mathbb{R}^n$, and every $f\in W^{1,q}(\Omega)$, we indeed have

$$\lim_{i\to\infty}\int_\Omega\left[\int_\Omega\frac{|f(x)-f(y)|^q}{|x-y|^q}\rho_i^*(|x-y|)\,dx\right]^{1/q}dy=K_{q,n}\|Df\|(\Omega),\tag{6.2}$$

where

$$K_{q,n}:=\left(\int_{\mathbb{S}^{n-1}}|\sigma\cdot e|^q\,d\mathcal{H}^{n-1}(\sigma)\right)^{1/q},$$

with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere, $e \in \mathbb{S}^{n-1}$ is arbitrary, and $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure. Here the mollifiers ρ_i^* are required to satisfy (1.1). However, if f is only a BV function, it may happen that (6.2) fails, while a (different) limit exists, see [7, Proposition 10]. Many other counterexamples in the same spirit are given in e.g. [7, 10]. In metric spaces things can go even more seriously wrong, as we will now demonstrate.

Consider the real line equipped with the Euclidean metric and the one-dimensional Lebesgue measure $\mathcal{L}^1$. Consider the sequence of mollifiers

$$\rho_i(x,y) := \rho_i^*(|x-y|) := \frac{\chi_{[0,1/i]}(|x-y|)}{1/i}, \quad x,y \in \mathbb{R}, \quad i \in \mathbb{N}.$$

As noted before Corollary 3.7, these mollifiers satisfy assumptions (2.12) and (2.13). The mollifiers ρ_i^* obviously also satisfy (1.1). The example below, inspired by [27, Example 4.8], shows that even in a compact PI space and for a Lipschitz function f, the desired equality $C_1'' = C_2''$ in (6.1) may fail, and so in particular the Euclidean result (6.2) does not extend to metric spaces.

Example 6.3. Consider the space $X = [0,1]$, equipped with the Euclidean metric and a weighted measure μ that we will next define. First we construct a fat Cantor set A as follows. Let $A_0 := [0,1]$. Then in each step $i \in \mathbb{N}$, we remove from A_{i-1} the set D_i, which consists of 2^{i-1} open intervals of length 2^{-2i}, centered at the middle points of the intervals that make up A_{i-1}. We denote $L_i := \mathcal{L}^1(A_i)$, and we let $A = \bigcap_{i=1}^\infty A_i$. Then we have

$$L := \mathcal{L}^1(A) = \lim_{i\to\infty} L_i = 1/2.$$

Then define the weight

$$w := \begin{cases} 2 & \text{in } A, \\ 1 & \text{in } X \setminus A, \end{cases}$$

and equip the space X with the weighted Lebesgue measure $d\mu := w\, d\mathcal{L}^1$. Obviously the measure is doubling, and X supports a $(1,1)$–Poincaré inequality.

Let

$$g := 2\chi_A \quad \text{and} \quad g_i = \frac{1}{L_{i-1} - L_i}\chi_{D_i}, \quad i \in \mathbb{N}.$$

Then

$$\int_0^1 g(s)\, ds = \int_0^1 g_i(s)\, ds = 1 \quad \text{for all } i \in \mathbb{N}.$$

Next define the function

$$f(x) := \int_0^x g(s)\, ds, \quad x \in [0,1].$$

Now $f \in \mathrm{Lip}(X)$, since g is bounded. Approximate f with the functions

$$f_i(x) := \int_0^x g_i(s)\, ds, \quad x \in [0,1], \quad i \in \mathbb{N}.$$

Now also $f_i \in \mathrm{Lip}(X)$, and $f_i \to f$ uniformly. This can be seen as follows. Given $i \in \mathbb{N}$, the set A_i consists of 2^i intervals of length $L_i/2^i$. If I is one of these intervals, we have

$$2^{-i} = \int_I g(s)\, ds = \int_I g_{i+1}(s)\, ds,$$

and also

$$\int_{X\setminus A_i} g\, d\mathcal{L}^1 = 0 = \int_{X\setminus A_i} g_{i+1}\, d\mathcal{L}^1.$$

Hence $f_{i+1} = f$ in $X \setminus A_i$, and elsewhere $|f_{i+1} - f|$ is at most 2^{-i}. In particular, $f_i \to f$ in $L^1(X)$ and so

$$\|Df\|(X) \le \lim_{i\to\infty} \int_0^1 g_i \, d\mu = \lim_{i\to\infty} \int_0^1 g_i \, d\mathcal{L}^1 = 1.$$

By Rademacher's theorem, for $\mathcal{L}^1$-a.e. $y \in A$, f is differentiable at y and so we have

$$\lim_{i\to\infty} \left(\int_X \frac{|f(x)-f(y)|^q}{|x-y|^q} \rho_i(x,y) \, d\mathcal{L}^1(x) \right)^{1/q} = (2|f'(y)|^q)^{1/q} = 2^{1/q}|f'(y)|.$$

Thus

$$\begin{aligned}
&\liminf_{i\to\infty} \int_X \left[\int_X \frac{|f(x)-f(y)|^q}{|x-y|^q} \rho_i(x,y)\, d\mu(x) \right]^{1/q} d\mu(y) \\
&\quad \ge 2 \liminf_{i\to\infty} \int_A \left[\int_X \frac{|f(x)-f(y)|^q}{|x-y|^q} \rho_i(x,y)\, d\mathcal{L}^1(x) \right]^{1/q} d\mathcal{L}^1(y) \\
&\quad \ge 2 \int_A \liminf_{i\to\infty} \left[\int_X \frac{|f(x)-f(y)|^q}{|x-y|^q} \rho_i(x,y)\, d\mathcal{L}^1(x) \right]^{1/q} d\mathcal{L}^1(y) \quad \text{by Fatou} \\
&\quad = 2^{1+1/q} \int_A |f'(y)| \, d\mathcal{L}^1(y) \\
&\quad = 2^{1+1/q}.
\end{aligned}$$

We conclude that

$$\liminf_{i\to\infty} \int_X \left[\int_X \frac{|f(x)-f(y)|^q}{|x-y|^q} \rho_i(x,y)\, d\mu(x) \right]^{1/q} d\mu(y) \ge 2^{1+1/q} \|Df\|(X). \tag{6.4}$$

On the other hand, consider any nonzero Lipschitz function f_0 supported in $(3/8, 5/8)$. For such a function, from (6.2) we have

$$\lim_{i\to\infty} \int_X \left[\int_X \frac{|f_0(x)-f_0(y)|^q}{|x-y|^q} \rho_i(x,y)\, d\mu(x) \right]^{1/q} d\mu(y) = 2^{1/q} \|Df_0\|(X) \in (0,\infty),$$

since both sides are equal to the classical quantities, that is, the quantities obtained when the measure μ is $\mathcal{L}^1$. This combined with (6.4) shows that we cannot have $C_1'' = C_2''$ in (6.1).

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