CONSTRUCTIVE NONHOLONOMIC CONTROLLABILITY OF CONTROL AFFINE SYSTEMS ON COMPACT MANIFOLDS

MAREK IZYDOREK, JOANNA JANCZEWSKA, SERGEY KRYZHEVICH, AND EUGENE STEPANOV

ABSTRACT. We prove the general version of point-to-point constructive controllability results for control affine systems on compact manifolds with drift under nonholonomic constraints on controls. Namely, we find the sufficient (and almost necessary) conditions when arbitrarily small controls satisfying such constraints can be found to steer the ODE from one given state to another. This result provides controllability for systems having invariant measures supported on the whole phase space and being finite (for example, for divergence-free drifts) or, more generally, for nonwandering flows.

1. INTRODUCTION

Consider the ODE over a C^p – smooth ($p \in \mathbb{N} \bigcup \{\infty\}$) closed manifold M

$$\dot{x} = V(x), \tag{1.1}$$

where V is a given sufficiently smooth vector field on M. We assume that M is a Riemannian manifold with the norm in each tangent space T_zM at $z \in M$ denoted by $|\cdot|_z$. Recall some classical definitions.

Definition 1.1. *Let $\varepsilon > 0$, $T > 0$. We say that a continuous function $\xi : [0,T] \to M$ with a piecewise continuous derivative $\dot{\xi}$ is an ε – solution of* (1.1) *if $|\dot{\xi}(t) - V(\xi(t))|_{\xi(t)} \le \varepsilon$ for all t where the derivative exists.*

Definition 1.2. *The ODE* (1.1) *is called controllable by arbitrary small controls or chain transitive, if for every couple of points $x_0, x_1 \in M$ and every $\varepsilon > 0$ there exists a $T \ge 1$ and an ε – solution $\xi : [0,T] \to M$ such that $\xi(0) = x_0$ and $\xi(T) = x_1$.*

Of course, such a property implies that the manifold M is connected. On the other hand, for flows on connected manifolds, controllability is equivalent to chain recurrence. Finally, controllability implies that for any $\varepsilon > 0$ an any points $x_0, x_1 \in M$ there exists a $T \ge 1$, and a *control function* $u(t,x)$ such that $u(t,x)$ is ε – small, i.e. $\|u\|_\infty := \sup_{z\in M} \sup_{t\in[0,T]} |u(t,z)|_z \le \varepsilon$ and there is a solution of system $\dot{x} = V(x) + u(t,x)$ such that $x(0) = x_0$, $x(T) = x_1$.

Key words and phrases. global controllability, control affine system, nonholonomic control, Hörmander condition.

In this paper, we consider the classical problem of controllability of (1.1) under nonholonomic constraints. Namely, given the distribution of planes provided by a finite number of smooth vector fields X_j $j = 1, \dots, m$ (which do not necessarily form a basis of T_xM at any point x), we prove the existence of a sufficiently small control steering the state variable from one given point to another. In other words, the control problem becomes, given $x_0 \in M$ and $x_1 \in M$, to find $u_j \colon [0, T] \to \mathbb{R}$, $j = 1, \dots, m$ such that the boundary value problem

$$\begin{gathered} \dot{x} = V(x) + u_1(t)X_1(x(t)) + \ldots + u_m(t)X_m(x(t)), \\ x(0) = x_0, \quad x(T) = x_1 \end{gathered} \tag{1.2}$$

is solvable while

$$u(t, x) := u_1(t)X_1(x(t)) + \ldots + u_m(t)X_m(x(t)) \tag{1.3}$$

is ε – small.

1.0.1. *Background.* In the driftless case $V = 0$ the controllability result is given by the Chow-Rashevskiĭ theorem [4, 8], see also [6]. Namely, in this case, the control problem (1.2) is solvable when the vector fields X_j, $j = 1, \dots, m$ satisfy Hörmander (also called Lie bracket generating) condition, which means that the distribution of planes given by the Lie algebra generated by all X_j in every $x \in M$ is the full tangent space T_xM. In other words, the rank of this distribution at every point is equal to the dimension of M (see Section 3 for precise definition). When the vector fields X_j are real analytic, this is also necessary for nonholonomic controllability by the Nagano-Sussmann theorem (also known as Orbit Theorem) [9], see also [10].

For the case $V \neq 0$, the situation is much more subtle. If the distribution of planes generated by all X_j spans the whole tangent space T_xM for every $x \in M$, controllability is provided when all the points of M are chain recurrent for the flow generated by V, see [2]. In particular, this is true if V is divergence-free (see Theorem 4.2.7 in [1] where it is formulated for real analytic vector fields).

In the presence of nonholonomic constraints, the natural requirement is that of Hörmander condition for the vector fields V, $X_1, \dots, X_m$. In general, this is not sufficient for controllability. However, this requirement is sufficient in the case when the flow, generated by V, is recurrent [3].

1.0.2. *Our contribution.* In this paper, we prove a general sufficient condition on nonholonomic controllability of (1.1). In a sense, this condition is almost necessary once the drift V and vector fields X_j are sufficiently smooth. Namely, in this case, controllability is guaranteed if vector fields V, $X_1, \dots, X_m$ satisfy the Hörmander condition, and (1.1) is controllable by arbitrarily small controls (without any nonholonomic constraints). Moreover, in this case, the controls u_j can be chosen so that $u = u(t, x)$ given by (1.3) be as small as desired. This gives an equivalence of controllability by arbitrarily small controls in holonomic and nonholonomic settings once V, $X_1, \dots, X_m$ satisfy Hörmander condition.

The main obstacle for our proofs is the fact, that one cannot a priori direct the system infinitesimally in the direction opposite the drift. Therefore, in the classical results cited above, some conditions on the drift are used, mainly the recurrence property. By contrast, in our paper, we only assume the controllability of the system with arbitrarily small controls without any nonholonomic constraints, which is, of course, necessary for nonholonomic controllability. Finally, mention that all our results are constructive. Namely, we provide an algorithm to convert an arbitrary control to a nonholonomic one. Similar results may also be obtained for flows on non-compact (including Euclidean spaces) in some assumptions on the considered vector fields. This work, requiring much technical efforts, is postponed to further papers.

2. Notation and preliminaries

With a slight abuse of notation, we write a vector field X over M as $X\colon M \to TM$ with TM standing for the tangent bundle of M, meaning that $X(z) \in T_zM$ for every $z \in M$. By $C^k(M, TM)$ we denote the set of all $X\colon M \to TM$ with all derivatives up to the order k continuous (resp. all derivatives continuous, if $k = \infty$). Denote $\varphi_t^V(y) := x(t)$, where $x(\cdot)$ is a solution of the system (1.1) satisfying $x(0) = y$. Similarly, we define flows for other systems.

For any metric space M, we denote by $B_\rho(z)$ the open ball centered at $z \in M$ with radius $\rho > 0$, and for $E \subset M$ its closure is denoted by $\bar{E}$. Along with the notation presented in Introduction, we also denote by $|\cdot|$ the Euclidean norm in a finite-dimensional Euclidean space $\mathbb{R}^n$, and by $\|\cdot\|_\infty$ the supremum norm for real-valued functions defined over an interval. Given real numbers a and b, we denote their minimum by $a \wedge b$.

The matrix norm $\|\cdot\|$ for real $n \times n$ matrices will always be chosen to correspond to the Euclidean norm $|\cdot|$ in $\mathbb{R}^n$. By e_j, we denote the j-th unit coordinate vector in $\mathbb{R}^n$. Given a Lipschitz continuous function/vector field F we denote by $\operatorname{Lip} F$ its Lipschitz constant. Finally, we use the notion id_n for the unit matrix $n \times n$.

3. Hörmander conditions

Let $X_j\colon M \to TM$, $j = 0, \dots, m$ be vector fields over a smooth manifold M. We consider the set $\mathfrak{X}_p$ of all X_j and all their Lie brackets of orders $\le p$ (if the vector fields are smooth enough so that the respective brackets be defined), i.e.

$$\mathfrak{X}_p = \{X_0, \dots, X_m\} \cup \bigcup_{i,j=0}^{m} [X_i, X_j] \cup \bigcup_{i,j,k=0}^{m} [[X_i, X_j], X_k] \cup \dots$$

For an $Y \in \mathfrak{X}_p$, we inductively define the *maximum order of Lie brackets forming* Y (frequently abbreviated as the order of Lie brackets, or just order, if there is no possibility of confusion) as follows:

(i) it is zero when $Y = X_j$ for some j,

(ii) if $Y = [U, V]$ for some $\{U, V\} \subset \mathcal{X}_p$, then the maximum order of Lie bracket of this particular representation of Y is maximum between the orders of U and V plus one, and the maximum order of Lie brackets of Y is minimum among all the orders of possible representations.

We denote then by $\mathcal{X}_k$ the set of all X_j and all their Lie brackets up to the maximum order $k \in \mathbb{N}$, so that $\mathcal{X}_p = \cup_k \mathcal{X}_k$. For instance, if $\dim M = 3$, and X_1, X_2 is a pair of vector fields over M generating noninvolutive distribution of planes, then the orders of X_1 and X_2 are zero, and the order of $[X_1, X_2]$ is one, and hence $\mathcal{X}_0 = \{X_1, X_2\}$, $\mathcal{X}_1 = \{[X_1, X_2]\}$.

Definition 3.1. *The vector fields $X_j \colon M \to TM$, $j = 0, \ldots, m$ satisfy Hörmander condition, if $X_j \in C^p(M, TM)$ for all $j = 0, \ldots, m$ and*

$$\operatorname{span}\{Y(x) \colon Y \in \mathcal{X}_p\} = T_x M.$$

Remark 3.2. Observe that for a compact manifold, the Hörmander condition implies that even for $p = \infty$ a finite number of vector fields Y is enough to span spaces $T_x M$ for all $x \in M$.

Example 3.3. Let $M := (\mathbb{R}/\mathbb{Z})^3$ be the three-dimensional torus. Assume $0 < R_1 < R_2 < R_3 < R_4 < R_5 < 0.5$, and let η_{45} stand for a smooth function such that $\eta_{45}(x) = 1$ for $x \in B_{R_4}(0)$, $\eta_{45}(x) = 0$ for $x \notin B_{R_5}(0)$ and $|\eta_{45}(x)| \leq 1$ for all $x \in M$. Consider the vector fields $X_1(x) := (e_1 + x_2 e_3)\eta_{45}(x)$, $X_2(x) := (e_2 - x_1 e_3)\eta_{45}(x)$. Let η_{12} be a C^∞ smooth function such that $\eta_{12}(x) = 1$ if $x \in \bar{B}_{R_1}(0)$, $\eta_{12}(x) = 0$ for $x \notin B_{R_2}(0)$ and $|\eta_{12}(x)| \leq 1$ for all $x \in M$, and consider the vector fields

$$X_3(x) := e_1(1 - \eta_{12}(x)), \quad X_4(x) := e_2(1 - \eta_{12}(x)), \quad X_5(x) := e_3(1 - \eta_{12}(x)).$$

We have then that the vector fields X_1, X_2, X_3, X_4, X_5 satisfy Hörmander condition. In fact, let $R := \min\{R_4 - R_3, R_3 - R_2\}$. Then for every $z \in B_{R_3}(0)$ one has $B_R(z) \subset B_{R_4}(0)$ and the 3×3 matrix $\mathcal{Y}_1(z) := (X_1(z), X_2(z), [X_1, X_2](z))$ is non-degenerate for any z. On the other hand, for every $y \notin B_{R_3}(0)$ one has $B_R(y) \subset \Omega_2 := M \setminus \bar{B}_{R_2}(0)$ and the 3×3 matrix $\mathcal{Y}_2(z) := (X_3(z), X_4(z), X_5(z))$ is an identity matrix for all $z \in B_{R_4}(0)$.

4. Technical tools

Our principal tool is the following statement proven in Appendix A.

Lemma 4.1. *There is an $\varepsilon_0 > 0$ such that for every $Y \in \mathcal{X}_k$ there exist an $\alpha = \alpha(Y) \in (0, 1]$, $l = l(Y) \in \mathbb{N}$, $c_j = c_j(Y) > 0$, $j = 1, \ldots, l$, and vector fields $\{Z_1, \ldots, Z_l, Z_1', \ldots, Z_l'\} \subset \{0\} \bigcup \{V + X_j, V - X_j \colon j = 1, \ldots, m\}$ for which the function $\Delta \colon (-\bar{\varepsilon}_0, \bar{\varepsilon}_0) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\Delta(\varepsilon, x) := \begin{cases} (\varphi^{Z_1}_{c_1 \varepsilon^\alpha} \circ \ldots \circ \varphi^{Z_l}_{c_l \varepsilon^\alpha})(x) - \varphi^V_{\sum_j c_j \varepsilon^\alpha}(x), & \varepsilon \geq 0, \\ (\varphi^{Z_1'}_{-c_1 |\varepsilon|^\alpha} \circ \ldots \circ \varphi^{Z_l'}_{-c_l |\varepsilon|^\alpha})(x) - \varphi^V_{-\sum_j c_j |\varepsilon|^\alpha}(x), & \varepsilon < 0, \end{cases} \tag{4.1}$$

satisfies

$$\Delta'_\varepsilon(\varepsilon,x)|_{\varepsilon=0} = Y(x).$$

Under conditions of Lemma 4.1 we denote

$$\tau_Y(\varepsilon) := \begin{cases} \sum_{j=1}^k c_j\varepsilon^\alpha, & \varepsilon \geq 0, \\ -\sum_{j=1}^k c_j|\varepsilon|^\alpha, & \varepsilon < 0, \end{cases} \qquad \text{and}$$

$$\psi(Y,\varepsilon)(x) := \begin{cases} (\varphi^{Z_1}_{c_1\varepsilon^\alpha} \circ \ldots \circ \varphi^{Z_l}_{c_l\varepsilon^\alpha})(x), & \varepsilon \geq 0, \\ (\varphi^{Z'_1}_{-c_1|\varepsilon|^\alpha} \circ \ldots \circ \varphi^{Z'_l}_{-c_l|\varepsilon|^\alpha})(x), & \varepsilon < 0. \end{cases}$$

With these notations, we prove the key statement of this section.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. The following assertions hold.*

(i) *If $V(x) \neq 0$, and for some $t > 0$ and $x \in \Omega$ the vector fields*

$$\{Y_1,\ldots,Y_{n-1}\} \subset \mathfrak{X}_k$$

are such that the vectors $Y_1(z),\ldots,Y_{n-1}(z),Y_n(z)$ with $Y_n(z) := V(z)$ form a basis of $T_z\Omega = \mathbb{R}^n$ for $z := \varphi^V_t(x)$, then denoting

$$\Phi(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1},z) := \tag{4.2}$$
$$(\psi(Y_{n-1},\varepsilon_{n-1}) \circ \ldots \circ \psi(Y_1,\varepsilon_1) \circ \varphi^V_{-\sum_{i=1}^{n-1}\tau_{Y_i}(\varepsilon_i)} \circ \varphi^V_{\varepsilon_0})(z) - z,$$

one has that Φ is continuous in the neighborhood of zero

$$\{(\varepsilon_0,\ldots,\varepsilon_{n-1}) \in \mathbb{R}^n \colon |(\varepsilon_0,\ldots,\varepsilon_{n-1})| < t, \Phi(\varepsilon_0,\ldots,\varepsilon_{n-1},z) \in \Omega\},$$

and $\det \Phi'_{(\varepsilon_0,\ldots,\varepsilon_{n-1})}(0,\ldots,0,z) \neq 0$.

(ii) *If $V(x) = 0$ and the vector fields $\{Y_1,\ldots,Y_n\} \subset \mathfrak{X}_k$ are such that the vectors $Y_1(x),\ldots,Y_n(x)$ form a basis of $T_x\Omega$, then denoting*

$$\Phi(\varepsilon_1,\ldots,\varepsilon_n,x) := \quad (\psi(Y_n,\varepsilon_n) \circ \ldots \circ \psi(Y_1,\varepsilon_1) \circ \varphi^V_{-\sum_{i=1}^n \tau_{Y_i}(\varepsilon_i)})(x) - x, \tag{4.3}$$

one has that Φ is a continuous map in a neighborhood of zero,

$$\{(\varepsilon_1,\ldots,\varepsilon_n) \in \mathbb{R}^n \colon |(\varepsilon_0,\ldots,\varepsilon_{n-1})| < t, \Phi(\varepsilon_1,\ldots,\varepsilon_n,z) \in \Omega\},$$

and $\det \Phi'_{(\varepsilon_0,\ldots,\varepsilon_{n-1})}(0,\ldots,0,x) \neq 0$.

Proof. Given the set of vector fields $\{Y_j\}$, the continuity of the map Φ is obvious.

Clearly, $\psi(Y_j,0)$ is the identity map. So, if all $\varepsilon_j = 0$ except for $j = i \neq 0$, one has $\Phi(0,\ldots,0,\varepsilon_i,0,\ldots,0,z) = \psi(Y_i,\varepsilon_i) \circ \varphi^V_{-\tau_{Y_i}(\varepsilon_i)}(z) - z$. But from Lemma 4.1 with Y_i in place of Y, ε_i in place of ε and $\varphi^V_{-\tau_{Y_i}(\varepsilon_i)}(z)$ in place of x we get

$$\begin{aligned}\psi(Y_i,\varepsilon_i) \circ \varphi^V_{-\tau_{Y_i}(\varepsilon_i)}(z) &= \varphi^V_{-\tau_{Y_i}(\varepsilon_i)}(z) + \varepsilon_i Y_i\left(\varphi^V_{-\tau_{Y_i}(\varepsilon_i)}(z)\right) + o(\varepsilon_i) \\ &= z + \varepsilon_i(Y_i(z) + o(1)) + o(\varepsilon_i) = z + \varepsilon_i Y_i(z) + o(\varepsilon_i)\end{aligned}$$

as $\varepsilon_i \to 0$ so that

$$\Phi'_{\varepsilon_j}(0,\ldots,0,z) = Y_j(z). \tag{4.4}$$

On the other hand, $\Phi(\varepsilon_0, 0, \dots, 0, z) = \varphi^V_{\varepsilon_0}(z) - z$, so that

$$\Phi'_{\varepsilon_0}(0, \dots, 0, z) = V(z). \tag{4.5}$$

It is known that the existence of derivatives in the direction of the basis vectors does not imply the existence of the full derivative at zero (see the map $(x, y) \mapsto x + y + \sqrt[3]{xy}$ as an appropriate counterexample). This is why we need Lemma 4.3, formulated several lines below and proven in Appendix B.

Then relationships (4.4) and (4.5) imply that the corresponding Jacobi matrices are non-degenerate. □

Lemma 4.3. *Both mappings, defined by the formulae* (4.2) *and* (4.3) *have a derivative at zero with respect to n-tuples of variables* $(\varepsilon_0, \dots, \varepsilon_{n-1})$ *and* $(\varepsilon_1, \dots, \varepsilon_n)$ *respectively.*

To finish our proofs we need a technical statement, proven in Appendix C.

Lemma 4.4. *Let U be a neighbourhood of zero in $\mathbb{R}^n$, N be a compact metric space, and a function $\Phi : U \times N \mapsto \mathbb{R}^n$ be continuous and let there exist an invertible Jacobi matrix $A(x) := \Phi'_q(0, x)$ that is a continuous function of x. Then for any $\sigma > 0$ there exists a $\delta > 0$ such that for any $x \in M$ the following inclusion holds:*

$$\overline{B}_\delta(0) \subset \overline{\Phi(B_\sigma(0), x)}.$$

Now we go back to our problem of local controllability for the flow on the manifold M. We let $\mathcal{R}(x_0, L)$ stand for the set of points attainable from the given point $x_0 \in M$ by using controls bounded in the uniform norm by $L > 0$ and prove the following statement.

Lemma 4.5. *Let M be a compact Riemannian manifold and vector fields*

$$V, X_1, \dots, X_m$$

satisfy the Hörmander condition. Then for any $\delta > 0$ there is a $\rho > 0$ such that for any point x and z of M satisfying $z = \varphi^V_t(x)$ with $t \geq \delta$ all the points in the ρ-neighborhood of z are attainable from the initial point x with the help of piecewise constant controls $u_j \colon [0, T] \to \{-1, 0, 1\}$, $j = 1, \dots, m$. In particular, $B_\rho(z) \subset \mathcal{R}(x, 1)$ (see Fig. 1). In the case $V(x) = 0$ one has $B_\rho(x) \subset \mathcal{R}(x, 1)$.

Proof. Taking a chart (ψ, U) on M with $U \subset M$ being an open subset and $\psi : U \to \Omega \subset \mathbb{R}^n$, we can transfer the vector fields and the respective flows to Ω. Now we can prove the existence of ρ as claimed in Lemma 4.5 but possibly depending on z. Here we apply Lemma 4.4 to the function $\Phi(\cdot, \cdot)$ acting from $B_t(0) \times M$ to Ω. The existence of ρ independent on z follows from the compactness of M and the fact that the radius of the ball attainable from a point z is a continuous function of z. □

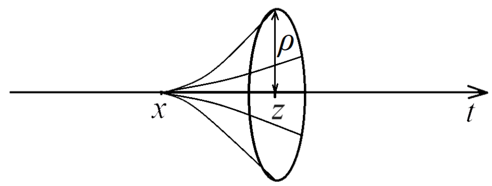


FIGURE 1. Local controllability.

5. Controllability

Theorem 5.1. *Let M be a compact Riemannian manifold with* $\dim M = n$. *Suppose that*

(i) *the vector fields V, $X_1, \dots, X_m$ satisfy Hörmander condition and*
(ii) *the problem* (1.1) *be controllable by arbitrary small controls.*

Then for any given $\{x_0, x_1\} \subset M$ and $\sigma > 0$ there is a $T > 0$ and $u_j \colon [0,T] \to \{0, \sigma\}$, $j = 1, \dots, m$ with the above properties, so that in particular $\mathcal{R}(x_0, \sigma) = M$ for arbitrarily small $\sigma > 0$ and every $x_0 \in M$, and hence, for every $\varepsilon > 0$ one can take σ to be so small that $\|u\|_\infty \le \varepsilon$, where u is defined by (1.3).

Proof. First of all, we formulate a simple statement that follows from Picard's approximations method [5, Chapter II,§1].

Lemma 5.2. *Let $x(\cdot) \in \mathbb{R}^n$, $y(\cdot) \in \mathbb{R}^n$ satisfy $\dot{x} = V(x)$, $\dot{y} = V(y) + v(t)$, $x(0) = y(0) = z$ for some $z \in \mathbb{R}^n$. Then $|x(t) - y(t)| \le \|v\|_\infty (e^{\mathrm{Lip}V t} - 1)/\mathrm{Lip}V$.*

Now fix a $\delta \in (0, \mathrm{diam}\, M/4\|V\|_\infty \wedge r/2\|V\|_\infty)$. By Lemma 4.5 there is a $\rho > 0$ such that for every $z \in M$ and for every $x \in M$ satisfying $z = \varphi_t^V(x)$ with $t \ge \delta$ one has that all the points in the ρ-neighborhood of z are attainable from the initial point x with the help of piecewise constant controls $u_j \colon [0,T] \to \{-1, 0, 1\}$, $j = 1, \dots, m$, so that in particular $B_\rho(z) \subset \mathcal{R}(x, 1)$,with $\rho > 0$ independent on z.

Let $\varepsilon \in \left(0, \|V\|_\infty \wedge \rho e^{-CL \mathrm{diam}\, M/2\|V\|_\infty}\right)$. By the chain transitivity of system (1.1), there is a continuous piecewise smooth curve $\tilde{x} \colon [0,T] \to M$ satisfying

$$\dot{\tilde{x}} = V(\tilde{x}(t)) + u(t, \tilde{x}(t)), \quad \tilde{x}(0) = x_0, \quad \tilde{x}(T) = x_1,$$

with $T \ge 1$ for some $u \in [0,T] \times M \to TM$ with $|u(t,x)|_{\tilde{x}(t)} \le \varepsilon$.

Let $N \in \mathbb{N} \cap (T/2\delta, T/\delta]$ (the latter intersection is nonempty because $T > 2\delta$), i.e. $\delta < T/N \le 2\delta$. Now, for each $j = 0, \dots, N-1$ setting $z_j := \varphi^V_{T/N}(\tilde{x}(jT/N))$, we get $B_\rho(z_j) \subset \mathcal{R}(\tilde{x}(jT/N), 1)$ with piecewise constant controls taking only values 0 or ± 1. On the other hand, the inequalities

$$\|V\|_\infty T/N \le 2\delta \|V\|_\infty \le r, \tag{5.1}$$

$$(\|V\|_\infty + \varepsilon) T/N \le 2\delta(\|V\|_\infty + \varepsilon) \le 4\delta \|V\|_\infty \le r, \tag{5.2}$$

imply both $z_j \in B_r(\tilde{x}(jT/N))$ and $\tilde{x}((j+1)T/N) \in B_r(\tilde{x}(jT/N))$. Therefore, by Lemma 5.2 applied with $\psi_* V$ instead of V, $\psi_* u$ instead of v, $z := \psi(\tilde{x}(jT/N))$ and $t := T/N$, we get

$$|\psi(z_j) - \psi(\tilde{x}((j+1)T/N))| \leq C\varepsilon e^{CLT/N} \leq C\varepsilon e^{2CL\delta} \leq C\rho,$$

and hence $\tilde{x}((j+1)T/N) \in B_\rho(z_j)$. Lemma 4.5 implies $\tilde{x}((j+1)T/N) \in \mathcal{R}(\tilde{x}(jT/N), 1)$, and by induction on j we get therefore $\tilde{x}(T) \in \mathcal{R}(\tilde{x}(0), 1)$, that is, $x_1 \in \mathcal{R}(x_0, 1)$ with piecewise constant controls taking only values 0 or ± 1.

Furthermore, if $D := \max_{j=1,\dots,m} \|X_j\|_\infty \leq \infty$, then given $\varepsilon > 0$ let $\tilde{X}_j := \varepsilon X_j / mD$ for all $j = 1, \dots, m$. The vector fields $\tilde{X}_j$ still satisfy Hörmander condition. Therefore there exist a $T > 0$ and piecewise constant control functions $\tilde{u}_j : [0, T] \to \{-1, 0, 1\}$ such that the boundary value problem

$$\dot{x} = V(x) + \tilde{u}_1(t)\tilde{X}_1(x(t)) + \ldots + \tilde{u}_m(t)\tilde{X}_m(x(t)), \qquad x(0) = x_0, \quad x(T) = x_1$$

be solvable. It suffices to set then $u_j := \varepsilon \tilde{u}_j / mD$, $j = 1, \dots, m$, to see that (1.2) is solvable with u is defined by (1.3) satisfying $\|u\|_\infty \leq \varepsilon$. □

The following corollary is valid.

Corollary 5.3. *Let M be a C^∞ smooth compact closed manifold, and C^∞ smooth vector fields V and X_j, $j = 1, \dots, m$ are such that $V, X_1, \dots, X_m$ satisfy Hörmander condition. The following conditions are sufficient for the ODE* (1.1) *to be controllable by arbitrarily small controls belonging to the distribution of planes defined by the vector fields $X_1 \dots, X_m$:*

(i) *V has a finite invariant measure μ (that is $\operatorname{div} \mu V = 0$ in the sense of distributions) with $\operatorname{supp} \mu = M$. In particular, this is true if $\operatorname{div} V = 0$ or if periodic points of* (1.1) *are dense in M.*
(ii) *All points of M are nonwandering for* (1.1).

Proof. If V has a finite invariant measure μ, then by Poincaré recurrence theorem μ-a.e. and hence all $x \in \operatorname{supp} \mu$ is a nonwandering point for the flow φ_t^V, and hence if $\operatorname{supp} \mu = M$, then every $X \in M$ is nonwandering for (1.1), which means that (i) follows from (ii). To prove (ii) note that if $\{x_0, x_1\} \subset M$ belongs to the set of nonwandering points for (1.1), then by proposition 3.3 of [7] for every $\varepsilon > 0$ there is a $T > 0$ and a control function $u \in L^\infty((0, T); TM)$ with $|u(t, x)|_{x(t)} \leq \varepsilon$ for all $t \in (0, T)$ such that the problem (1.2) admits a solution. Thus, under the conditions of (ii) he problem (1.1) is controllable by arbitrary small controls, and it suffices to refer now to Theorem 5.1 recalling that all he vector fields V and X_j, $j = 1, \dots, m$ and their derivatives are continuous. □

Remark 5.4. The above corollary is clearly valid if the manifold M, and the vector fields V and X_j, $j = 1, \dots, m$ be a C^k rather than C^∞ smooth with $k \geq 2$ such that $\operatorname{span} \{Y(x) \colon Y \in \mathcal{X}_k\} = T_x M$ for all $x \in M$.

6. Acknowledgements

Sergey Kryzhevich was supported by Gdańsk University of Technology by the DEC 14/2021/IDUB/I.1 grant under the Nobelium - 'Excellence Initiative - Research University' program. Eugene Stepanov acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001.

Appendix A. Proof of Lemma 4.1

In this section, we demonstrate how one can shift solutions of system (1.1) in prescribed directions of the set $\mathcal{X}$ by adding appropriate perturbations to system (1.1). To this aim, fix the vector field $Y(x)$ from the statement of the Lemma.

Given a Lipschitz continuous concerning the second argument vector field $U(t,\cdot)$, consider the system

$$\dot{x} = V(x) + U(t,x), \qquad x \in \mathbb{R}^n.$$

and compare its solution $\varphi_t^{V+U}(x_0)$ satisfying initial condition $x(0) = x_0$ with the solution $\varphi_t^V(x_0)$ of system (1.1) satisfying the same initial condition. Recall that $\Delta_t^U(x_0) = \varphi_t^{V+U}(x_0) - \varphi_t^V(x_0)$.

Proof of Lemma 4.1. First of all, recall a simple estimate:

$$\int_0^\varepsilon f(t)\,dt = f(\varepsilon/2)\varepsilon + O(\varepsilon^3) \tag{A.1}$$

for any C^2 – smooth function $f:\mathbb{R}\to\mathbb{R}^n$. One can obtain this estimate by writing down the Taylor series for f in $\varepsilon/2$.

We prove Lemma 4.1 by induction by the maximum order of Y (see the beginning of Section 3).

For any $k\in\mathbb{N}$ and any $Y\in\mathcal{X}_k$ we set $\tau_Y(\varepsilon) = \tau_k = 2^{5(1-2^{-k})}\varepsilon^{2^{-k}}$. In particular, $\tau_0 = \varepsilon$ and $\tau_{k+1} = 4\sqrt{2\tau_k}$ for any $k\in\mathbb{Z}^+$.

Step 1 (the simplest case). For any $Y\in\mathcal{X}_0$. For $Y = V$, the statement is trivial. Let $Y\in\{X_1,\dots,X_k\}$, consider the system $\dot{x} = V(x) + Y(x)$. Here we set $Z_1 = Y$, $c_1 = 1$, $\alpha = 1$ so that $\Delta(\varepsilon, x_0) = \Delta_\varepsilon^Y(x_0)$. By Lemma 5.2,

$$|\Delta_\varepsilon^Y(x_0)| \le \|Y\|_\infty(\exp(\operatorname{Lip} V\varepsilon) - 1)/\operatorname{Lip} V$$

where $\operatorname{Lip} V$ is the Lipschitz constant for the vector field V. Therefore, there exists a positive constant C such that

$$|\Delta_\varepsilon^Y(x_0)| \le C\varepsilon\|Y\|_\infty \tag{A.2}$$

for any ε sufficiently small. Now, proceed to more precise estimates. Clearly,

$$\begin{aligned}\varphi_\varepsilon^V(x_0) &= x_0 + \textstyle\int_0^\varepsilon V(\varphi_t^V(x_0))\,dt \qquad \text{and}\\ \varphi_\varepsilon^{V+Y}(x_0) &= x_0 + \textstyle\int_0^\varepsilon \left(V(\varphi_t^{V+Y}(x_0)) + Y(\varphi_t^{V+Y}(x_0))\right)\,dt.\end{aligned}$$

Consequently, using the estimate (A.2), we obtain

$$\Delta_\varepsilon^Y(x_0) = \int_0^\varepsilon Y(\varphi_t^{V+Y}(x_0))\,dt + \int_0^\varepsilon \big(V(\varphi_t^{V+Y}(x_0) - V(\varphi_t^V(x_0))\big)\,dt =$$
$$\int_0^\varepsilon Y(\varphi_t^{V+Y}(x_0))\,dt + \int_0^\varepsilon \big(DV(\varphi_t^{V+Y}(x_0))\Delta_t^Y(x_0) + O(\Delta_t^Y(x_0)^2)\big)\,dt =$$
$$\varepsilon Y(x_0) + O(\varepsilon^2) \qquad \text{as} \quad \varepsilon \to 0.$$

Step 2 (base of induction). In order to shift a solution in the directions of Lie brackets like $[X_i, V]$ or $[X_i, X_j]$, we derive the analog of the famous Trotter formula. Consider two vector fields $Y_1, Y_2 \in \mathfrak{X}_0$. We introduce a scheme that provides a shift of the solution in the direction close to $[Y_1, Y_2]$. Consider the vector field which depends on t (here $\delta > 0$ will be specified later):

$$U(t,x,\delta) := \begin{cases} Y_1(x), & \text{if} \quad t \in [0,\delta), \\ Y_2(x), & \text{if} \quad t \in [\delta, 2\delta), \\ -Y_1(x), & \text{if} \quad t \in [2\delta, 3\delta), \\ -Y_2(x), & \text{if} \quad t \in [3\delta, 4\delta). \end{cases}$$

This situation corresponds to the upper (solid lines) contour of Figure 2 below.

Now consider the respective ODE

$$\dot{x} = V(x) + U(t,x,\delta)$$

on the time segment $[0, 4\delta]$ and the corresponding solution $\varphi_t^{V+U}(x_0)$ with initial condition $x(0) = x_0$. Similarly to (A.2), we obtain

$$\sup_{t\in[0,4\delta]} |\Delta_t^U(x_0)| = O(\delta), \qquad \sup_{t\in[0,4\delta]} |\varphi_t^{V+U}(x_0) - x_0| = O(\delta). \tag{A.3}$$

Using formulas

$$\begin{aligned} \varphi_{4\delta}^{V+U}(x_0) &= x_0 + \int_0^{4\delta} \big(V(\varphi_t^{V+U}(x_0)) + U(t, \varphi_t^{V+U}(x_0)), \delta\big)\,dt, \\ \varphi_{4\delta}^{V}(x_0) &= x_0 + \int_0^{4\delta} V(\varphi_t^{V}(x_0))\,dt, \end{aligned} \tag{A.4}$$

we deduce that

$$\Delta_t^U(x_0) = \begin{cases} tY_1(x_0) + O(\delta^2), & \text{if } t \in [0,\delta], \\ \delta Y_1(x_0) + (t-\delta)Y_2(x_0) + O(\delta^2), & \text{if } t \in [\delta, 2\delta], \\ (3\delta - t)Y_1(x_0) + \delta Y_2(x_0) + O(\delta^2), & \text{if } t \in [2\delta, 3\delta], \\ (4\delta - t)Y_2(x_0) + O(\delta^2), & \text{if } t \in [3\delta, 4\delta]. \end{cases} \tag{A.5}$$

Now, we proceed to more precise estimates. Equation (A.4) implies

$$\begin{aligned} \Delta_{4\delta}^U(x_0) = \ & \varphi_{4\delta}^{V+U}(x_0) - \varphi_{4\delta}^{V}(x_0) = \\ & \int_0^{4\delta} \big(V(\varphi_t^{V+U}(x_0)) - V(\varphi_t^{V}(x_0))\big)\,dt + \int_0^{4\delta} U(t, \varphi_t^{V+U}(x_0), \delta)\,dt \end{aligned} \tag{A.6}$$

Let us write the asymptotic estimates for both integrals in the right-hand sides of (A.6) using (A.5). Obviously, for any $t \in [0, 4\delta)$

$$\begin{aligned} V(\varphi_t^{V+U}(x_0)) - V(\varphi_t^V(x_0)) &= DV(\varphi_t^V(x_0))\Delta_t^U(x_0) + O(\Delta_t^U(x_0)^2) = \\ \text{[due to (A.3)]} &= DV(\varphi_t^V(x_0))\Delta_t^U(x_0) + O(\delta^2). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^{4\delta} \left(V(\varphi_t^{V+U}(x_0)) - V(\varphi_t^V(x_0))\right) dt = \\ &\left(\int_0^{\delta} + \int_{\delta}^{2\delta} + \int_{2\delta}^{3\delta} + \int_{3\delta}^{4\delta}\right) \left(V(\varphi_t^{V+U}(x_0)) - V(\varphi_t^V(x_0))\right) dt = \\ &\delta \sum_{l=0}^{3} DV\left(\varphi^V_{\frac{(2l+1)\delta}{2}}(x_0)\right) \Delta^U_{\frac{(2l+1)\delta}{2}}(x_0) + O(\delta^3). \end{aligned} \tag{A.7}$$

Here we used (A.1) to obtain the last line of the latter equality. Applying estimates (A.5) we can rewrite (A.7) as follows:

$$\begin{aligned} &\int_0^{4\delta} \left(V(\varphi_t^{V+U}(x_0)) - V(\varphi_t^V(x_0))\right) dt = \\ &\frac{\delta^2}{2} DV\left(\varphi^V_{\frac{\delta}{2}}(x_0)\right) Y_1(x_0) + \delta^2 DV\left(\varphi^V_{\frac{3\delta}{2}}(x_0)\right)\left(Y_1(x_0) + \frac{Y_2(x_0)}{2}\right) + \\ &\delta^2 DV\left(\varphi^V_{\frac{5\delta}{2}}(x_0)\right)\left(\frac{Y_1(x_0)}{2} + Y_2(x_0)\right) + \\ &\frac{\delta^2}{2} DV\left(\varphi^V_{\frac{7\delta}{2}}(x_0)\right) Y_2(x_0) + O(\delta^3) = \\ &2\delta^2 DV(x_0)(Y_1(x_0) + Y_2(x_0)) + O(\delta^3). \end{aligned} \tag{A.8}$$

Now we estimate the second term of (A.6):

$$\begin{aligned} \int_0^{4\delta} U(t, x_0, \delta)\, dt &= \text{[by (A.1)]} = \\ &\varepsilon\Big(Y_1\left(\varphi^{V+U}_{\frac{\delta}{2}}(x_0)\right) + Y_2\left(\varphi^{V+U}_{\frac{3\delta}{2}}(x_0)\right) - \\ &Y_1\left(\varphi^{V+U}_{\frac{5\delta}{2}}(x_0)\right) - Y_2\left(\varphi^{V+U}_{\frac{7\delta}{2}}(x_0)\right)\Big) = \\ &\delta\Big(Y_1\left(\varphi^{V+U}_{\frac{\delta}{2}}(x_0)\right) - Y_1\left(\varphi^{V+U}_{\frac{5\delta}{2}}(x_0)\right) + \\ &Y_2\left(\varphi^{V+U}_{\frac{3\delta}{2}}(x_0)\right) - Y_2\left(\varphi^{V+U}_{\frac{7\delta}{2}}(x_0)\right)\Big) + O(\delta^3). \end{aligned} \tag{A.9}$$

Here we applied (A.1), (A.5) and (A.3) once again. Besides,

$$\begin{aligned}\varphi^{V+U}_{\frac{\delta}{2}}(x_0)-\varphi^{V+U}_{\frac{5\delta}{2}}(x_0) &= \int\limits_{\delta/2}^{5\delta/2}\left(V(\varphi^{V+U}_t(x_0))+U(t,\varphi^{V+U}_t(x_0),\delta)\right)\,dt = \\ & \int\limits_{\delta/2}^{5\delta/2} V(\varphi^{V+U}_t(x_0))\,dt+\int\limits_{\delta}^{2\delta} U(t,\varphi^{V+U}_t(x_0),\delta)\,dt+O(\delta^2)= \\ & -\delta(2V(x_0)+Y_2(x_0))+O(\delta^2).\end{aligned}$$

Similarly, $\varphi^{V+U}_{\frac{3\delta}{2}}(x_0)-\varphi^{V+U}_{\frac{7\delta}{2}}(x_0)=-\delta(2V(x_0)+Y_1(x_0))+O(\delta^2)$.

All in all, we obtain from (A.9) that

$$\int\limits_0^{4\delta} U(t,x_0,\delta)\,dt = 2\delta^2\left(DY_1(x_0)+DY_2(x_0)\right)V(x_0)+\delta^2[Y_1,Y_2](x_0).$$

The last formula, taken together with (A.8) implies that

$$\begin{aligned}\Delta^U_{4\delta}(x_0) &= 2\delta^2 DV(x_0)(Y_1(x_0)+Y_2(x_0))- \\ & 2\delta^2(DY_1(x_0)+DY_2(x_0))V(x_0)+ \\ & \delta^2 DY_2(x_0)Y_1(x_0)-DY_1(x_0)Y_2(x_0)+O(\delta^3)= \\ & 2\delta^2[Y_1+Y_2,V](x_0)+\delta^2[Y_1,Y_2](x_0)+O(\delta^3).\end{aligned}\tag{A.10}$$

This asymptotic estimate is the most important technical result of this section.

Remark A.1. Observe that, taking the zero vector field instead of Y_2, we obtain the expression $2\delta^2[Y_1,V](x_0)+O(\delta^3)$ in the right-hand side of (A.10).

Let Y_1 and Y_2 be not collinear to V. In order to 'model' the Lie bracket $[Y_1,Y_2]$ we have to eliminate the 'extra term' $2\delta^2[Y_1+Y_2,V](x_0)$ at the right-hand side of (A.10). To do this, we use the simple formula $[Y_1,Y_2]=[-Y_1,-Y_2]$. Besides, we 'rescale' the perturbation, selecting $\delta=\sqrt{\varepsilon/2}$. Let $\tilde{Y}_1=Y_1$ for $\varepsilon\ge 0$ and $\tilde{Y}_1=-Y_1$ for $\varepsilon\le 0$. All in all, we apply the perturbation

$$W(t,x,\varepsilon,[Y_1,Y_2]):=\begin{cases}\tilde{Y}_1(x), & \text{if} \quad t\in[0,\sqrt{|\varepsilon|/2}),\\ Y_2(x), & \text{if} \quad t\in[\sqrt{|\varepsilon|/2},2\sqrt{|\varepsilon|/2}),\\ -\tilde{Y}_1(x), & \text{if} \quad t\in[2\sqrt{|\varepsilon|/2},3\sqrt{|\varepsilon|/2}),\\ -Y_2(x), & \text{if} \quad t\in[3\sqrt{|\varepsilon|/2},4\sqrt{|\varepsilon|/2}),\\ -\tilde{Y}_1(x), & \text{if} \quad t\in[4\sqrt{|\varepsilon|/2},5\sqrt{|\varepsilon|/2}),\\ -Y_2(x), & \text{if} \quad t\in[5\sqrt{|\varepsilon|/2},6\sqrt{|\varepsilon|/2}),\\ \tilde{Y}_1(x), & \text{if} \quad t\in[6\sqrt{|\varepsilon|/2},7\sqrt{|\varepsilon|/2}),\\ Y_2(x), & \text{if} \quad t\in[7\sqrt{|\varepsilon|/2},8\sqrt{|\varepsilon|/2}).\end{cases}\tag{A.11}$$

Roughly speaking, we add the 'dashed contour' of Figure 2 to cancel brackets $[Y_1+Y_2,V]$, see (A.12) below. Figure 3a illustrates what the trajectories of the perturbed system look like in the considered case.

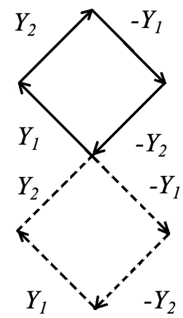


FIGURE 2. The order of applying controls while modeling the Lie bracket $[Y_1, Y_2]$.

If $\varphi_t^{V+W}(x_0)$ is the solution of the corresponding system with the initial condition $x(t_0) = x_0$, we have, due to (A.10)

$$\begin{aligned} \Delta^W_{8\sqrt{|\varepsilon|/2}}(x_0) &= \varphi^{V+W}_{8\sqrt{|\varepsilon|/2}}(x_0) - \varphi^V_{8\sqrt{|\varepsilon|/2}}(x_0) = \\ &\varepsilon[Y_1 + Y_2, V](x_0) + \frac{\varepsilon}{2}[Y_1, Y_2](x_0) - \\ &\varepsilon[Y_1 + Y_2, V](x_0) + \frac{\varepsilon}{2}[Y_1, Y_2](x_0) + O(|\varepsilon|^{3/2}) = \\ &\varepsilon[Y_1, Y_2](x_0) + O(\varepsilon^{3/2}), \qquad \text{see Fig. 4b.} \end{aligned} \tag{A.12}$$

Remark A.2. In order to obtain Lie brackets of the form $[Y_1, V]$, we consider the perturbation

$$W(\varepsilon, t, x) = \begin{cases} \tilde{Y}_1(x) & \text{if} \quad t \in [0, \sqrt{|\varepsilon|/2}), \\ 0 & \text{if} \quad t \in [\sqrt{|\varepsilon|/2}, 2\sqrt{|\varepsilon|/2}), \\ -\tilde{Y}_1(x) & \text{if} \quad t \in [2\sqrt{|\varepsilon|/2}, 3\sqrt{|\varepsilon|/2}), \\ 0 & \text{if} \quad t \in [3\sqrt{|\varepsilon|/2}, 8\sqrt{|\varepsilon|/2}). \end{cases}$$

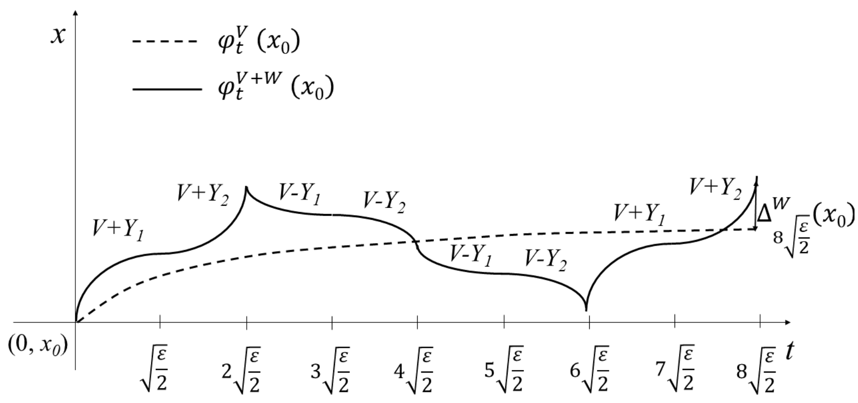
Thus, we proved the lemma for vector fields of orders 0 and 1.

Step 3. Before proceeding to the vector fields of higher orders, we prove the following technical statement. It will allow us to apply all the perturbations on intervals of time of the same length.

Lemma A.3. *Let θ_1 and θ_2 be strictly increasing continuous functions and $Y \in \mathcal{X}_p$ be a vector field such that*

(1) *$\theta_1(0) = \theta_2(0) = 0$ and $\theta_1(\varepsilon) < \theta_2(\varepsilon)$ for any $\varepsilon > 0$;*
(2) *there exists a control $W_Y = W(t, x, \varepsilon, Y)$, defined for $t \in [0, \theta_1]$ such that the solution $\varphi_t^{V+W_Y}(x_0)$ of system $\dot{x} = V(x) + W(t, x, \varepsilon, Y)$ is such that*

$$\varphi^{V+W_Y}_{\theta_1(\varepsilon)}(x_0) = \varphi^V_{\theta_1(\varepsilon)}(x_0) + \varepsilon Y(x_0) + o(\varepsilon).$$



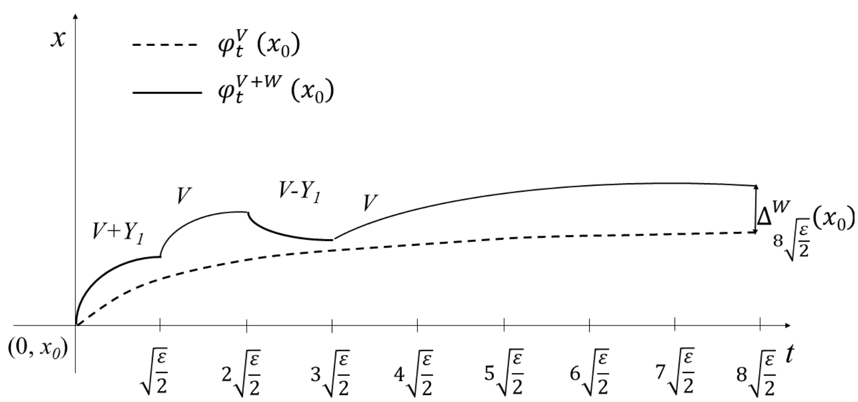


FIGURE 3. Trajectories of perturbed and unperturbed systems when we model a) $[Y_1, Y_2]$ with $Y_1, Y_2 \neq V$; b) $[Y_1, V]$.

Then, if we set $W(t, x, \varepsilon, Y) = 0$ *for* $t > \theta_1(\varepsilon)$, *we have*

$$\varphi^{V+W_Y}_{\theta_2(\varepsilon)}(x_0) = \varphi^V_{\theta_2(\varepsilon)}(x_0) + \varepsilon Y(x_0) + o(\varepsilon).$$

Proof of Lemma A.3. Let $\Delta^{W_Y}_\theta(x_0) = \varphi^{V+W_Y}_\theta(x_0) - \varphi^V_\theta(x_0)$. Obviously,

$$\Delta^{W_Y}_{\theta_1(\varepsilon)}(x_0) = \varepsilon Y(x_0) + o(\varepsilon).$$

Consider the linear system

$$\dot{u} = DV(\varphi^V_t(x_0))u. \tag{A.13}$$

Let $\Psi(t, s)$ be the fundamental matrix of (A.13) with $\Psi(s) = \mathrm{id}_n$. Observe that $\Psi(\theta_2(\varepsilon), \theta_1(\varepsilon)) = \mathrm{id}_n + o(1)$ as $\varepsilon \to 0$ since both the values $\theta_i(\varepsilon) = o(1)$ (they are continuous functions that vanish at zero). Then, by theorem on differentiation of solutions with respect to initial conditions, we have

$$\begin{aligned}\Delta^{W_Y}_{\theta_2(\varepsilon)}(x_0) = \ & \Psi(\theta_2(\varepsilon),\theta_1(\varepsilon))\Delta^{W_Y}_{\theta_1(\varepsilon)}(x_0) + o\left(\Delta^{W_Y}_{\theta_1(\varepsilon)}(x_0)\right) = \\ & \Delta^{W_Y}_{\theta_1(\varepsilon)(x_0)} + o(\Delta^{W_Y}_{\theta_1(\varepsilon)}(x_0)) = \varepsilon Y(x_0) + o(\varepsilon)\end{aligned}$$

as desired. □

Step 4 (the induction step). Now, we study the vector fields of order 2 and higher. Since the manifold M is compact, there exists a number $k \in \mathbb{N}$ such that for any point $x_0 \in M$ there exist vector fields $Y_1(x), \dots, Y_n(x)$ of $\mathfrak{X}_k$ that span the space T_xM for any point x of a neighborhood of x_0.

Now, we suppose that for given $k \in \mathbb{N}$, the desired controls $W(t,x,\varepsilon,Y_0)$ can be constructed for any $Y_0 \in \mathfrak{X}_k$. Let $X \in \mathfrak{X}_{k+1}$. Then there exist two vector fields Y and Z of $\mathfrak{X}_k$ such that $X = [Y,Z]$. By Lemma A.3, we may assume, without loss of generality, that $\tau_Y = \tau_Z = \tau_k$. Then there exist controls $W(t,x,\varepsilon,Y)$ and $W(t,x,\varepsilon,Z)$, such that the solutions $\varphi_t^{V+W_Y}(x_0)$ and $\varphi_t^{V+U_Z}(x_0)$ of systems

$$\dot{x} = V(x) + W(t,x,\varepsilon,Y) \qquad \text{and} \qquad \dot{x} = V(x) + W(t,x,\varepsilon,Z)$$

respectively with initial condition $x(0) = x_0$ satisfy asymptotic estimates

$$\begin{aligned}\varphi^{V+W_Y}_{\tau_k}(x_0) &= \varphi^{V}_{\tau_k}(x_0) + \varepsilon Y(x_0) + o(\varepsilon),\\ \varphi^{V+W_Z}_{\tau_k}(x_0) &= \varphi^{V}_{\tau_k}(x_0) + \varepsilon Z(x_0) + o(\varepsilon).\end{aligned}$$

Let $\sigma = \sqrt{\tau_k/2} = \tau_{k+1}/8$.

Observe that the perturbations $-W_Y$ and $-W_Z$ approximate shifts in directions $-Y$ and $-Z$ respectively. We can select $\varepsilon_{-Y} = \varepsilon_Y$ and $\varepsilon_{-Z} = \varepsilon_Z$.

Let us first assume that none of the fields Y and Z are collinear to V. Similarly to (A.11), we define the perturbation corresponding to X as follows:

$$W(t,x,\varepsilon,X) := \begin{cases} W(t,x,\varepsilon,Y), & \text{if} \quad t \in [0,\sigma),\\ W(t-\sigma,x,\varepsilon,Z), & \text{if} \quad t \in [\sigma,2\sigma),\\ -W(t-2\sigma,x,\varepsilon,Y), & \text{if} \quad t \in [2\sigma,3\sigma),\\ -W(t-3\sigma,x,\varepsilon,Z), & \text{if} \quad t \in [3\sigma,4\sigma),\\ -W(t-4\sigma,x,\varepsilon,Y), & \text{if} \quad t \in [4\sigma,5\sigma),\\ -W(t-5\sigma,x,\varepsilon,Z), & \text{if} \quad t \in [5\sigma,6\sigma),\\ W(t-6\sigma,x,\varepsilon,Y), & \text{if} \quad t \in [6\sigma,7\sigma),\\ W(t-7\sigma,x,\varepsilon,Z), & \text{if} \quad t \in [7\sigma,8\sigma).\end{cases}$$

If one of the vector fields Y or Z is collinear to V, the perturbation W_X can be defined similarly to the second formula of Remark A.2.

Now we consider the solution $\varphi_t^{V+W_X}(x_0)$ of the Cauchy problem

$$\dot{x} = V(x) + W(t, x, \varepsilon, X), \qquad x(0) = x_0.$$

The equality $\varphi_{\tau_{k+1}}^{V+W_X}(x_0) = \varphi_{\tau_{k+1}}^{V}(x_0) + \varepsilon X(x_0) + o(\varepsilon)$ is similar to (A.12).

In the case $X = [Y, V]$ (see Remark A.1) we set $\varepsilon_X = 4\sigma_Y$ and

$$W(t, x, \varepsilon, X) := \begin{cases} W(t, x, \varepsilon, Y), & \text{if} \quad t \in [0, \sigma), \\ 0, & \text{if} \quad t \in [\sigma, 2\sigma), \\ -W(t - 2\sigma, x, \varepsilon, Y), & \text{if} \quad t \in [2\sigma, 3\sigma), \\ 0, & \text{if} \quad t \in [3\sigma, 8\sigma]. \end{cases}$$

□

Observe that we need to construct perturbations for finitely many vector fields (those from $\mathfrak{X}_p$ where p is taken from the Hörmander condition). This fact, together with Lemma A.3, implies that we can construct all the necessary perturbations on time segments of the same length.

Appendix B. Proof of Lemma 4.3

We only give the proof for the map, defined by (4.2); the proof for the map defined by (4.3) is similar. For any $k \in \mathbb{Z}^+$ and any set of smooth vector fields $Y_1, \ldots, Y_k$ of $\mathfrak{X}$, we introduce the map

$$\Phi^{Y_1,\ldots,Y_k}(\varepsilon_0, \ldots, \varepsilon_k, z) := (\psi(Y_k, \varepsilon_k) \circ \ldots \circ \psi(Y_1, \varepsilon_1) \circ \varphi^V_{-\sum_{i=1}^k \tau_{Y_i}(\varepsilon_i)} \circ \varphi^V_{\varepsilon_0})(z) - z$$

and prove that it has the derivative at zero with respect to the variables $\varepsilon_0, \ldots, \varepsilon_k$ applying induction by k.

Step 1 (base of induction). For $k = 0$ (when the set $\{Y_1, \ldots, Y_k\}$ is empty) the statement is evident. Consider $k = 1$ and the map

$$\Phi^{Y_1}(\varepsilon_0, \varepsilon_1, z) = \psi(Y_1, \varepsilon_1) \circ \varphi^V_{\tau_{Y_1}(\varepsilon_1)} \circ \varphi^V_{\varepsilon_0}(z) - z.$$

By Lemma 4.1, the map $\Delta^{Y_1}_{\tau_{Y_1}(\varepsilon_1)}(z) = \psi(Y_1, \varepsilon_1)(z) - \varphi^V_{\tau_{Y_1}(\varepsilon_1)}(z)$ has a derivative with respect to ε_1 at zero. Then, the same is true for the function

$$\widetilde{\Delta}^{Y_1}(\varepsilon_1, z) := \psi(Y_1, \varepsilon_1)(z) \circ \varphi^V_{-\tau_{Y_1}(\varepsilon_1)}(z) - z = \Delta^{Y_1}_{\tau_{Y_1}(\varepsilon_1)}(z) + o(1)$$

does also have a derivative at zero. And now, we notice that

$$\Phi^{Y_1}(\varepsilon_0, \varepsilon_1)(z) = \widetilde{\Delta}^{Y_1}_{\tau_{Y_1}(\varepsilon_1)} \circ \varphi^V_{\varepsilon_0}(z) + \varphi^V_{\varepsilon_0}(z) - z$$

and, therefore this map has a derivative at zero with respect to the vector $(\varepsilon_0, \varepsilon_1)$.

Step 2 (the induction step). Consider the map $\widetilde{\Delta}^{Y_1,\ldots,Y_k}$ defined by the formula

$$\widetilde{\Delta}^{Y_1,\ldots,Y_k}(\varepsilon_0, \ldots, \varepsilon_k, z) = \psi(Y_k, \varepsilon_k) \circ \psi(Y_1, \varepsilon_1)(z) - \varphi^V_{\sum_{i=1}^k \tau_{Y_i}(\varepsilon_i)}(z)$$

and include the statement of differentiability at zero of the map $\widetilde{\Delta}^{Y_1,\dots,Y_{k-1}}$ to the induction hypothesis. For $\widetilde{\Delta}^{Y_1}$ this statement was proven in Step 1.

Observe that

$$\widetilde{\Delta}^{Y_1,\dots,Y_k}(\varepsilon_0,\dots,\varepsilon_k,z) =$$
$$= \psi(Y_k,\varepsilon_k)\circ\dots\circ\psi(Y_1,\varepsilon_1)(z) - \varphi^V_{\tau_{Y_k}(\varepsilon_k)}(z)\circ\psi(Y_{k-1},\varepsilon_{k-1})\circ\dots\circ\psi(Y_1,\varepsilon_1)(z) +$$
$$\varphi^V_{\tau_{Y_k}(\varepsilon_k)}(z)\circ\psi(Y_{k-1},\varepsilon_{k-1})\circ\dots\circ\psi(Y_1,\varepsilon_1)(z) - \varphi^V_{\sum_{i=1}^k \tau_{Y_i}(\varepsilon_i)}(z) = \Delta_1 + \Delta_2$$

where Δ_1 and Δ_2 are the first and the second difference, respectively.

Further,

$$\begin{aligned} \Delta_1 = \; & \Delta^{Y_k}_{\tau_{Y_k}(\varepsilon_k)}(\psi(Y_{k-1},\varepsilon_{k-1})\circ\dots\circ\psi(Y_1,\varepsilon_1)(z)) = \\ & Y_k(\psi(Y_{k-1},\varepsilon_{k-1})\circ\dots\circ\psi(Y_1,\varepsilon_1)(z))\varepsilon_k + o(\varepsilon_k) = \\ & Y_k(z)\varepsilon_k + o(\max(|\varepsilon_1|,\dots,|\varepsilon_k|)), \end{aligned}$$

so Δ_1 is differentiable at zero.

Meanwhile,

$$\begin{aligned} \Delta_2 = \; & \varphi^V_{\tau_{Y_k}(\varepsilon_k)}(z)\circ\psi(Y_{k-1},\varepsilon_{k-1})\circ\dots\circ\psi(Y_1,\varepsilon_1)(z) - \varphi^V_{\sum_{i=1}^k \tau_{Y_i}(\varepsilon_i)}(z) = \\ & \widetilde{\Delta}^{Y_1,\dots,Y_{k-1}}(\varepsilon_0,\dots,\varepsilon_{k-1},z) + o(\max(|\varepsilon_1|,\dots,|\varepsilon_k|)) \end{aligned}$$

which proves the existence of the Jacobi matrix at zero for Δ_2.

The trivial asymptotic estimate

$$\Phi^{Y_1,\dots,Y_k}(\varepsilon_0,\dots,\varepsilon_k,z) = \widetilde{\Delta}^{Y_1,\dots,Y_k}(\varepsilon_0,\dots,\varepsilon_k,z) + o(\max(|\varepsilon_0|,\dots,|\varepsilon_k|))$$

finishes the proof of the lemma. □

Appendix C. Proof of Lemma 4.4

Since the matrix $A(x) = \Phi'_q(0,x)$ is continuous and invertible for any x, the matrix $A^{-1}(x)$ is continuous and, therefore, bounded, so we select $C > 0$ so that $\|A(x)\| \le C$ for any $x \in N$. We need to prove that for any $\sigma > 0$ there exists a $\delta > 0$ such that for any $p \in B_\delta(0) \subset \mathbb{R}^n$ the equation

$$\Phi(q,x) = p, \qquad q \in B_\sigma(0) \tag{C.1}$$

is solvable. We represent the function $\Phi(q,x) = A(x)q + R(q,x)$.

Let $\rho > 0$ be such that $B_{r_0}(0) \subset U$ (recall that the function Φ is defined and continuous at $U \times N$). Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Observe that the function $\gamma : [0,r_0] \times S^{n-1} \times N \to \mathbb{R}^n$ defined by the formula

$$\gamma(r,\varphi,x) := \begin{cases} R(q,r\varphi)/r = (\Phi(r\varphi,x) - A(x)r\varphi)/r, & \text{if} \quad r > 0; \\ 0, & \text{if} \quad r = 0 \end{cases}$$

is continuous and, hence, uniformly continuous. This is why we can select a $\sigma > 0$ such that $|R(q,x)| \le |q|/2C$ for any $x \in N$ and $q \in B_\sigma(0)$.

Now, the equation (C.1) can be rewritten in the form $A(x)q + R(q, x) = p$ that is equivalent to

$$q = T(q, x) := A^{-1}(x)p - A^{-1}(x)R(q, x).$$

If $|q| \leq \sigma$ and $p \leq \sigma/8C$ then $|T(q, x)| \leq 2C|p| + \sigma/2 \leq 3\sigma/4$. This means that $T(\cdot, x)$ maps the ball $\overline{B}_\sigma(0)$ into itself. So, by Brouwer's fixed point theorem, the problem (C.1) is solvable. The lemma is proven. □

References

[1] Bloch, A. M., *Nonholonomic mechanics and control*, Interdisciplinary Applied Mathematics, Vol. 24, Second Edition, Springer, New York, 2015.

[2] Bonatti, C. and Crovisier, S. *Récurrence et généricité*, Invent. Math., **158:1** (2004), 33–104.

[3] Boscain, U. and Sigalotti, M., *Introduction to controllability of non-linear systems*, Contemporary Research in Elliptic PDEs and Related Topics, Springer, 2019.

[4] Chow, W.L. *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Mathematische Annalen, **117** (1939), 98–105, doi:10.1007/bf01450011.

[5] Hartman, Ph., *Ordinary Differential Equations*, John Wiley, New York, 1964.

[6] Jurdjevic, V. *Geometric control theory.* Cambridge University Press, 1997.

[7] Kryzhevich, S. and Stepanov, E., *The saga of a fish: from a survival guide to closing lemmas*, J. Differential Equations, **267:6**, (2019), 3442–3474.

[8] Rashevskii, P. K., *About connecting two points of complete non-holonomic space by admissible curve (in Russian)*, Uch. Zapiski ped. inst. Libknexta **2**, (1938): 83–94.

[9] Sussmann, H. J. *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. American Mathematical Society. **180** (1973), 171–188.

[10] Sussmann H. J. *Nonlinear Controllability and Optimal Control*, New York, Routledge, 1990.

(Marek Izydorek, Joanna Janczewska, Sergey Kryzhevich) Institute of Applied Mathematics, Faculty of Applied Physics and Mathematics and BioTechMed Center, Gdańsk University of Technology, ul. Narutowicza 11/12, 80-233 Gdańsk, Poland

Email address, Marek Izydorek: marek.izydorek@pg.edu.pl

Email address, Joanna Janczewska: joanna.janczewska@pg.edu.pl

Email address, Sergey Kryzhevich: serkryzh@pg.edu.pl

(Eugene Stepanov) Universitá di Pisa, Dipartimento di Matematica, Largo B. Pontecorvo 5, 56126 Pisa, Italy

Email address, Eugene Stepanov: stepanov.eugene@gmail.com