On the L^1-relaxed area of graphs of BV piecewise constant maps taking three values

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Abstract

Given a bounded open connected set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, we consider the class of piecewise constant maps u taking three fixed values $\alpha, \beta, \gamma \in \mathbb{R}^2$, vertices of an equilateral triangle; for any u in this class, using a weak notion of Jacobian determinant valid for BV functions, we give a precise description of $\mathrm{Det}\,(\nabla u)$ and show that the relaxed graph area of u is bounded from above by a quantity related to the flat norm of $\mathrm{Det}\,(\nabla u)$. The provided upper bound allows to show the validity of a De Giorgi conjecture regarding the relaxed area functional when one restricts to this class of piecewise constant functions.

Key words: Plateau problem, relaxation, minimal connections, area functional, minimal surfaces, $\mathbb{S}^1$-valued maps.

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1 Introduction

Problems of relaxation of non-convex functionals with non-standard growth arise in many contexts of calculus of variations. In the non-parametric approach to Plateau problem, as in capillarity problems and other settings related with minimal surfaces in higher codimension, the area functional is the canonical example of energy which, in codimension greater than 1, is non-convex (but policonvex). This leads to non-trivial questions when one tries to relax the functional, and a full understanding of the properties of the relaxed area functional is far from being reached; even basic questions regarding the characterization of the domain itself and the expression of the relaxed area functional are open.

In this paper we focus on a class of piecewise constant maps u from a planar domain to $\mathbb{R}^2$, which actually generalizes the classical triple junction map. The latter was introduced by De Giorgi in [19] (see also [1, 8, 30]) in order to prove a conjecture regarding the lack of integral representations for the area functional. Before explaining our main results, let us introduce the area functional and the related open questions.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded open set (that, without loss of generality, we assume connected). For a given function $v \in C^1(\Omega;\mathbb{R}^2)$ we indicate by

$$\mathbb{A}(v,\Omega) := \int_\Omega \sqrt{1+|\nabla v|^2 + |\,\mathrm{Det}\ (\nabla v)|^2}dx \tag{1.1}$$

the classical 2-dimensional area of the graph $G_v = \{(x,y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of v. For any $u \in L^1(\Omega;\mathbb{R}^2)$ the L^1-relaxed area of the graph of u is defined as

$$\overline{\mathcal{A}}(u,\Omega) := \inf\{\liminf_{k\to+\infty} \mathbb{A}(v_k,\Omega),\ v_k \in C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ in } L^1(\Omega;\mathbb{R}^2)\}. \tag{1.2}$$

It is well known that, when v is scalar valued, the study of the relaxed area is crucial in the study of the Cartesian Plateau problem [25]. In particular, for scalar valued maps, the graph area is obtained by relaxing the classical functional $\mathbb{A}(v,\Omega) = \int_\Omega \sqrt{1+|\nabla v|^2}dx$ for $v \in C^1(\Omega)$. It is also well known that in this case, for any $u \in L^1(\Omega)$, it holds

$$\overline{\mathcal{A}}(u,\Omega) = \begin{cases} \int_\Omega \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega) & \text{if } u \in BV(\Omega),\\ +\infty & \text{otherwise,}\end{cases} \tag{1.3}$$

where ∇u is the approximate gradient of u and $D^s u$ is the singular part of Du (see [25]). In the case of vector-valued maps, the characterization of the domain $\mathrm{Dom}(\overline{\mathcal{A}}(\cdot,\Omega))$ of $\overline{\mathcal{A}}(\cdot,\Omega)$, and the computation of its corresponding values seem at the moment out of reach, due to the presence of highly nonlocal phenomena. Specifically, for vector-valued maps it can be proved that

$$\overline{\mathcal{A}}(u,\Omega) \geq \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega), \tag{1.4}$$

and strict inequality might occurs (actually, equality holds only in very special cases [1]).

Starting from [19], the relaxed area of the triple junction map $u_T : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ has been studied. This map takes only three values $\alpha, \beta, \gamma \in \mathbb{R}^2$, which are the vertices of an equilateral triangle $T \subset \mathbb{R}^2$ inscribed in a circle of radius 1; in particular, $|\alpha| = |\beta| = |\gamma| = 1$. The map u_T assumes these values in three equal circular sectors of the circle $B_r(0)$ whose boundaries are three radii of $B_r(0)$ meeting at the origin with angles of 120°. In [1], the authors positively answered to a conjecture of De Giorgi [19], stating that the set function[1] $A \mapsto \overline{\mathcal{A}}(u_T;A)$ is not subadditive, and hence it cannot have an integral representation. These results have been obtained using suitable estimates on $\overline{\mathcal{A}}(u_T;A)$, for suitable subdomains $A \subset B_r(0)$, even if the precise expression of $\overline{\mathcal{A}}(u_T;B_r(0))$ is missing in [1]. Sharp estimates are instead contained in [8, 30], where the precise value of $\overline{\mathcal{A}}(u_T;B_r(0))$ has been proved to coincide with

$$\overline{\mathcal{A}}(u_T;B_r(0)) = |B_r(0)| + 3m_r, \tag{1.5}$$

where $|\cdot|$ denotes the Lebesgue measure, and m_r is the area of an area-minimizing minimal surface obtained as the solution of a Dirichlet-Neumann nonparametric Plateau problem in codimension 1. The techniques used to show (1.5) are based on the notion of integral currents, Cartesian currents [23, 24], together with a Steiner type symmetrization machinary adapted for integral currents which strongly relies on the symmetries of the domain $\Omega = B_r(0)$ and of the target triangle T (we refer to [8, 30] for details). It is here important to point out that m_r enjoys the following features: for all $r > 0$

(a) $m_r > rl$, where $l = \sqrt{3}$ is the side of the triangle T;

[1] Defined for any open set $A \subseteq B_r(0)$.

(b) $3m_r < 3rl + |T|$, with $|T|$ denoting the area of T.

As a consequence of (a) it turns out that

$$\overline{\mathcal{A}}(u_T; B_r(0)) > |B_r(0)| + |Du_T|(B_r(0)),$$

in particular showing that, in contrast with the scalar case (1.3), the presence of the Jacobian determinant in (1.2) plays a crucial role.

Before explaining the main consequence of (b) instead, we first point out that similar phenomena have been observed for the vortex map $u_V : B_r(0) \to \mathbb{R}^2$, $u_V(x) = \frac{x}{|x|}$, $x \neq 0$, and more in general for functions $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Specifically, if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, in [11] it has been proved that the distributional Jacobian determinant of u provides a nontrivial contribution in the computation of $\overline{\mathcal{A}}(u, \Omega)$. This contribution can always be estimated from above as the following formula shows:

$$\overline{\mathcal{A}}(u, \Omega) \leq \int_\Omega \sqrt{1 + |\nabla u|^2} dx + \| \operatorname{Det} (\nabla u)\|, \tag{1.6}$$

where the quantity $\| \cdot \|$ is a suitable norm on $C_c^{0,1}(\Omega)'$ equivalent to the standard flat norm (see [11]). We emphasize that this inequality may hold strict, and an explicit example is given in [6], where it is proved the following precise formula

$$\overline{\mathcal{A}}(u_V, B_r(0)) = \int_\Omega \sqrt{1 + |\nabla u_V|^2} dx + \mathcal{C}_r, \tag{1.7}$$

for the vortex map, where $\mathcal{C}_r$ is again, in a similar fashion as in (1.5), the area of an area-minimizing minimal surface obtained as a solution of a non-parametric Plateau problem with partial free boundary (see [6] for details, and [7] for the general approach to this Plateau-type problem). Here, for r small enough, it holds that $\mathcal{C}_r < 2\pi r$, where $2\pi r = \| \operatorname{Det} (\nabla u_V)\|$ turns out to be the $\| \cdot \|$-norm of the distributional determinant $\operatorname{Det} (\nabla u_V) = \pi \delta_0$ in $B_r(0)$.

In this paper we prove a formula similar to (1.6) for piecewise constant maps u taking only the three values α, β, γ. This requires to introduce a notion of distributional determinant for this kind of functions; in particular, using the notion of minimal lifting introduced in [27], in [28] the author proved that, for suitable maps $u \in BV(\Omega; \mathbb{R}^2)$, a component of a suitable Cartesian current[2] with underlying map u replaces the distributional Jacobian determinant of u. In [20], extending this result to every function $u \in BV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, a weak notion of Jacobian determinant for these maps is provided, and in particular it turns out that if $u \in BV(\Omega; \{\alpha, \beta, \gamma\})$, then $\operatorname{Det} (Du)$ is well-defined and can be written, simililarly to the case of $\mathbb{S}^1$-valued Sobolev maps, as a series of weighted Dirac deltas. As a consequence, it follows that $\operatorname{Det} (Du) \in C_c^{0,1}(\Omega)'$ and then $\operatorname{Det} (Du)$ has finite flat norm.

The first main result of the present paper is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz boundary and let $u \in BV(\Omega; \{\alpha, \beta, \gamma\})$. Then*

$$\overline{\mathcal{A}}(u, \Omega) \leq |\Omega| + |Du|(\Omega) + 4\| \operatorname{Det} (\nabla u)\|_{\text{flat}}. \tag{1.8}$$

Also in this case, we emphasize that the inequality in (1.8) can be strict. For instance, in the case of the triple junction map $u_T : B_r(0) \to \{\alpha, \beta, \gamma\}$, it holds that

$$\operatorname{Det} (\nabla u_T) = |T|\delta_0,$$

[2]This current, called completely vertical lifting of u, is unique.

where $|T|$ is the area of the triangle T with vertices α, β, γ. It then follows that $\| \mathrm{Det}\ (\nabla u_T)\|_{\mathrm{flat}} = |T|r$, and so for $r \geq 1$ we have

$$\overline{\mathcal{A}}(u_T, B_r(0)) = |B_r(0)| + 3m_r < |B_r(0)| + 3rl + |T| \leq |B_r(0)| + |Du|(B_r(0)) + \| \mathrm{Det}\ (\nabla u_T)\|_{\mathrm{flat}},$$

where we have used condition (b) above. We conjecture that the inequality in (1.8) is always strict (apart from the trivial case $\mathrm{Det}\ (\nabla u) = 0$, which essentially occurs only when there is no presence of multiple points (see Section 2.6)). The presence of the flat norm encodes, in (1.8), the aforementioned nonlocality of the relaxed area functional.

Following De Giorgi [19], it seems interesting to consider a further relaxation of $\overline{\mathcal{A}}$, this time looking at the functional $\overline{\mathcal{A}}(u, \cdot)$ as a function of the open set: for every $V \subseteq \Omega$, we set

$$\overline{\overline{\mathcal{A}}}(u, V) := \inf \left\{ \sum_{k=1}^{+\infty} \overline{\mathcal{A}}(u, A_k) : A_k \subseteq \Omega \text{ open }, \bigcup_{k=1}^{+\infty} A_k \supseteq V \right\}. \tag{1.9}$$

The advantage of this second relaxation is that, for all $u \in L^1(\Omega; \mathbb{R}^2)$, $\overline{\overline{\mathcal{A}}}(u, \cdot)$ is the trace of a regular Borel measure restricted to the open subsets of Ω. Moreover, $\overline{\overline{\mathcal{A}}}(u, \cdot)$ coincides with the greatest subaddivite functional which is less or equal to $\overline{\mathcal{A}}(u, \cdot)$; in some sense, $\overline{\overline{\mathcal{A}}}(u, \cdot)$ should encode the local part of the relaxed area functional, just excluding the singular contribution. Specifically, De Giorgi conjectured the following statement[3]:

Conjecture ([19, Conjecture 3]). For any $u \in L^1(\Omega; \mathbb{R}^2)$ with $\overline{\mathcal{A}}(u, \Omega) < +\infty$ it holds that

$$\overline{\overline{\mathcal{A}}}(u, \Omega) = \inf\{\overline{\mathcal{A}}(u, \Omega \setminus C) : C \text{ is closed with } \mathcal{H}^1(C \cap \Omega) = 0\}. \tag{1.10}$$

In [11] we partially answer to this conjecture, proving that it is true when $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. The second main result of this paper states that such conjecture still holds for maps $u \in BV(\Omega; \{\alpha, \beta, \gamma\})$. Precisely, we have the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ and u be as in Theorem 1.1. Then*

$$\overline{\overline{\mathcal{A}}}(u, \Omega) = |\Omega| + |Du|(\Omega), \tag{1.11}$$

and

$$\overline{\overline{\mathcal{A}}}(u, \Omega) = \inf\{\overline{\mathcal{A}}(u, \Omega \setminus C) : C \textit{ is closed with } \mathcal{H}^0(C \cap \Omega) < +\infty\}. \tag{1.12}$$

In particular (1.10) *holds.*

In order to prove Theorem 1.1 we approximate the map u with maps $u_k \in BV(\Omega; \{\alpha, \beta, \gamma\})$ which are polyhedral (namely, their jump set S_{u_k} is a finite union of segments) and moreover which enjoy the feature that for all $x \in S_{u_k}$ there is a neighborhood of x in which u_k takes only two values. For this kind of maps it is known that

$$\overline{\mathcal{A}}(u_k, \Omega) = |\Omega| + |Du_k|(\Omega),$$

and so Theorem 1.1 follows if one shows that $\liminf_{k \to \infty} |Du_k|(\Omega) \leq |Du|(\Omega) + 4\| \mathrm{Det}\ (\nabla u)\|_{\mathrm{flat}}$. This is provided by Proposition 3.1 in Section 3. In order to show this, we use suitable density theorems for polyhedral maps, which are proved in Section 2.5, whose starting point is the approximation result contained in [14]. The main point here is to show that we can suitably

[3]The conjecture is here presented in the case $\Omega \subset \mathbb{R}^2$, and $u \in L^1(\Omega; \mathbb{R}^2)$, even if it was stated in any dimension.

approximate the flat norm of the Jacobian determinant of $u \in BV(\Omega; \{\alpha, \beta, \gamma\})$. This requires a characterization of $\mathrm{Det}\,(\nabla u)$, which is given in Section 2.4.

To prove Theorem 1.2 we first prove (1.11), and then we show that, erasing a suitable finite set C of points in Ω, we can apply formula (1.8) with the domain $\Omega \setminus C$ and show that the flat norm contribution can be made arbitrarily small. Notice however that the domain $\Omega \setminus C$ is not Lipschitz, and so we cannot apply directly Theorem 1.1, but need a technical modification of it.

The paper is divided into two main parts. In Section 2 we set the notation and give some preliminary results. In particular we characterize the Jacobian determinant of piecewise constant maps and show the needed density of polyhedral maps. Then in Section 3 we prove Theorems 1.1 and 1.2.

2 Preliminaries

2.1 Notation

Let $U \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz boundary (a Lipschitz domain in the sequel). Let $\delta > 0$; we denote by

$$U_\delta := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \overline{U}) < \delta\}, \tag{2.1}$$

the δ-neighborhood of U, where $\mathrm{dist}(\cdot, \overline{U})$ is the distance from $\overline{U}$.

Given a vector field $\phi = (\phi_1, \phi_2) : U \to \mathbb{R}^2$ we define $\mathrm{Curl}\,\phi := \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2}$. We also denote by $\phi^\perp$ the vector $\phi^\perp = (-\phi_2, \phi_1)$, namely its counterclockwise rotated by a $\pi/2$-angle.

We introduce the following quantity, for all $x, y \in \overline{U}$,

$$d_U(x, y) := \min\{|x - y|, \mathrm{dist}(x, \partial U) + \mathrm{dist}(y, \partial U)\}.$$

This well-known pseudometric on $\overline{U}$ is useful to describe atomic distributions (see next section) arising as Jacobian distributional determinant of maps with values in $\mathbb{S}^1$ (see [13, 15, 16, 29] and references therein), and is related with its minimal connection when dealing with domains with boundary (see [15, Chapter 14]).

We denote by $\mathcal{M}_b(U)$ the space of Radon measures with bounded total variation in U. We denote by $e_1, e_2, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$, which is naturally identified with a basis of 1-vectors. The symbols $dx_1, dx_2, \ldots, dx_n$ denote a basis of 1-covectors. We denote by $\mathcal{D}^k(U)$ the space of k-forms on U, and by $\mathcal{D}_k(U)$ the space of k-currents on U. Any 0-current T in $\mathcal{D}_0(U)$ can be naturally identified with a distribution in $\mathcal{D}'(U)$.

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^2$. A $\mathcal{H}^1$-rectifiable subset S of $\mathbb{R}^2$ is said to be polyhedral if it is a finite union of segments.

Lipschitz maps and dual norms. For an open set $U \subset \mathbb{R}^2$ we denote by $C^{0,1}(U)$ the space of Lipschitz functions on U, and by $C_c^{0,1}(U)$ its subspace of compactly supported maps. We define $l(\psi)$ the Lipschitz constant of $\psi \in C^{0,1}(U)$, namely

$$l(\psi) := \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|\psi(x) - \psi(y)|}{|x - y|} \right\},$$

and we define the Lipschitz norm in $C^{0,1}(U)$ as

$$\|\psi\|_{C^{0,1}} := \max\{\|\psi\|_{L^\infty}, l(\psi)\}. \tag{2.2}$$

For bounded domains U, on the subspace $C_c^{0,1}(U)$ it turns out that $l(\cdot)$ is a norm equivalent to $\|\cdot\|_{C^{0,1}}$. For any $\Lambda \in C_c^{0,1}(U)'$ we introduce its flat norm as

$$\|\Lambda\|_{\mathrm{flat},U} := \sup_{\substack{\psi\in C_c^{0,1}(U)\\ l(\psi)\le 1}} \langle \Lambda, \psi\rangle. \tag{2.3}$$

Here, brackets $\langle\cdot,\cdot\rangle$ denote the duality paring between $C_c^{0,1}(U)'$ and $C_c^{0,1}(U)$. We also denote by $\langle\cdot,\cdot\rangle_A$ the duality paring between $C_c^{0,1}(A)'$ and $C_c^{0,1}(A)$, when the open set A is not clear from the context.

Diffeomorphisms of $\mathbb{R}^2$. We denote by $id : \mathbb{R}^2 \to \mathbb{R}^2$ the identity map, namely $id(x) = x$ for all $x \in \mathbb{R}^2$; let further I denote the 2×2 identity matrix. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism of class C^1; let $\delta > 0$, then if $\|G - id\|_{W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)} < \delta$ it happens that also $\|G^{-1} - id\|_{W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)} < O(\delta)$. Indeed, using the triangle inequality and the submultiplicativity of the Frobenius norm for 2×2-matrices, it is not hard to prove that $\|\nabla G^{-1}\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} < \frac{|I|}{1-\delta} = \frac{\sqrt{2}}{1-\delta}$. Next, for every $y \in \mathbb{R}^2$, $|G^{-1}(y)-y| = |G^{-1}(y)-G^{-1}(G(y))| \le 2\|\nabla G^{-1}\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}|G(y)-y|$, so $\|G^{-1} - id\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} < 2\frac{\sqrt{2}}{1-\delta}\|G - id\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}$. Furthermore, $|\nabla G^{-1}(y) - I| = |\nabla G^{-1}(y)(I - \nabla G(G^{-1}(y))| \le |\nabla G^{-1}(y)||(I-\nabla G(G^{-1}(y))|$, so $\|\nabla G^{-1} - I\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} < \frac{\sqrt{2}}{1-\delta}\|\nabla G - I\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}$. From these two estimates we finally get $\|G^{-1} - id\|_{W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)} < 2\frac{\sqrt{2}}{1-\delta}\delta$, so $O(\delta)$ can be chosen equal to $2\frac{\sqrt{2}}{1-\delta}\delta$.

In this case, redefining δ if necessary, we will often assume that G satisfies

$$\max\{\|G - id\|_{W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)}, \|G^{-1} - id\|_{W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)}\} < \delta. \tag{2.4}$$

We introduce the following definition:

Definition 2.1. We say that a diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ is regular if $G, G^{-1} \in C^1(\mathbb{R}^2;\mathbb{R}^2)$, and there exists $\delta > 0$ such that (2.4) holds.

If G is a regular diffeomorphism as above, we can estimate, for any vector $v \in \mathbb{R}^2$,

$$\left|v^T(\nabla G)\right| \le \left|v^T(\nabla G - I)\right| + \left|v^T\right| \le (1+\delta)|v|,$$

and hence, if φ is a Lipschitz map, $\varphi\circ G$ will be Lipschitz as well with $l(\varphi\circ G) \le (1+\delta)l(\varphi)$. In particular, the flat norm of the push-forward by G of any 0-current $T \in \mathcal{D}_0(\mathbb{R}^2)$ satisfies

$$\|G_\# T\|_{\mathrm{flat},\mathbb{R}^2} = \sup_{\substack{\varphi\in C_c^{0,1}(\mathbb{R}^2)\\ l(\varphi)\le 1}} T(\varphi\circ G) = (1+\delta) \sup_{\substack{\varphi\in C_c^{0,1}(\mathbb{R}^2)\\ l(\varphi)\le 1}} T\left(\frac{\varphi\circ G}{(1+\delta)}\right) \le (1+\delta)\|T\|_{\mathrm{flat},\mathbb{R}^2}. \tag{2.5}$$

Therefore, if $T \in \mathcal{D}_0(\mathbb{R}^2)\cap C_c^{0,1}(\mathbb{R}^2)'$ then also $G_\# T \in \mathcal{D}_0(\mathbb{R}^2)\cap C_c^{0,1}(\mathbb{R}^2)'$. Further, if T has support in the closure $\overline{U}$ of a Lipschitz domain U, one has

$$\|G_\# T\|_{\mathrm{flat},U} \le (1+\delta)\|T\|_{\mathrm{flat},G^{-1}(U)} \le (1+\delta)\|T\|_{\mathrm{flat},U_\delta}. \tag{2.6}$$

2.2 Atomic distributions

We introduce the following subclass of Radon measures on a Lipschitz domain U:

$$X_f(U) := \Big\{\Lambda \in \mathcal{M}_b(U) : \exists n\in\mathbb{N},\ \exists (x_i,y_i)\in \overline{U}\times\overline{U} \text{ for } i = 1,\dots,n : \Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})\Big\}. \tag{2.7}$$

Every $\Lambda \in X_f(U)$ can be identified with an integral 0-current in $\mathcal{D}_0(U)$. The points x_i, y_i are referred to as poles of Λ. Notice that if $x_i \in \partial U$, it does not contribute in $\mathcal{M}_b(U)$, and hence its presence in the representation $\Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$ is only taken for convenience. In particular, a single Dirac delta $\Lambda = \delta_x \in X_f(U)$, because for any $y \in \partial U$ it holds $\Lambda = \delta_x - \delta_y$.

The more general set of atomic distributions in U is defined as

$$X(U) := \Big\{ \Lambda \in C_c^{0,1}(U)' : \exists (x_i, y_i) \in \overline{U} \times \overline{U} \ \forall i \in \mathbb{N} : \Lambda = \sum_{i=1}^{+\infty} (\delta_{x_i} - \delta_{y_i}), \ \sum_{i=1}^{+\infty} d_U(x_i, y_i) < +\infty \Big\}. \tag{2.8}$$

For all $\Lambda \in X(U)$ it holds

$$\langle \Lambda, \varphi \rangle = \sum_{i=1}^{+\infty} (\varphi(x_i) - \varphi(y_i)) \qquad \forall \varphi \in C_c^{0,1}(U). \tag{2.9}$$

The fact that $\Lambda \in C_c^{0,1}(U)'$ (that is equivalent to require that $\|\Lambda\|_{\mathrm{flat},U} < +\infty$) implies that for any $\varphi \in C_c^{0,1}(U)$ the series in (2.9) is convergent.

Remark 2.2. Notice that $\Lambda = \sum_{i=1}^{+\infty} (\delta_{x_i} - \delta_{y_i})$ has not a unique representation as a series; two sequences $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{U} \times \overline{U}$ and $((\widehat{x}_i, \widehat{y}_i))_{i\in\mathbb{N}} \subset \overline{U} \times \overline{U}$ define the same linear functional on $C_c^{0,1}(U)$ if

$$\Big\langle \sum_{i=1}^{+\infty} (\delta_{x_i} - \delta_{y_i}), \varphi \Big\rangle = \Big\langle \sum_{i=1}^{+\infty} (\delta_{\widehat{x}_i} - \delta_{\widehat{y}_i}), \varphi \Big\rangle \qquad \forall \varphi \in C_c^{0,1}(U). \tag{2.10}$$

We point out that the hypothesis $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{U} \times \overline{U}$ is done for convenience, and it may happen that for some $i \in \mathbb{N}$, either $x_i \in \partial U$ or $y_i \in \partial U$, or even both. Notice that if $x_i \in \partial U$ then $\delta_{x_i} = 0$ in $C_c^{0,1}(U)'$; the presence of x_i only affects the representation of Λ, and not its action on $C_c^{0,1}(U)$.

Remark 2.3. Let $\Lambda \in X_f(U)$ be nonzero, and write $\Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$ for some points $(x_i, y_i) \in \overline{U} \times \overline{U}$. We define $I^+(\Lambda) := \{x_i : x_i \in U\}$, and $I^-(\Lambda) := \{y_i : y_i \in U\}$. Of course, the measure Λ depends only on the points in $I^+(\Lambda)$ and $I^-(\Lambda)$ and not on the points belonging to ∂U. Namely

$$\Lambda = \sum_{i: x_i \in I^+(\Lambda)} \delta_{x_i} - \sum_{i: y_i \in I^-(\Lambda)} \delta_{y_i}.$$

Up to erasing points x_i which coincide with some y_j, we can always suppose that there are no cancellation in the sum above, namely that $I^+(\Lambda) \cap I^-(\Lambda) = \varnothing$. We can also relabel the indeces of the points in $I^{\pm}(\Lambda)$ and suppose that

$$I^+(\Lambda) = \{x_1, \dots, x_m\} \qquad I^+(\Lambda) = \{y_{m+1}, \dots, y_{m+k}\},$$

where we do not exclude that there are repeated points in $I^+(\Lambda)$ (resp., in $I^-(\Lambda)$) or that some of these sets is empty. For all $i = 1, \dots, m$ we introduce a point $y_i \in \partial U$ so that $|x_i - y_i| = \mathrm{dist}(x_i, \partial U)$, and for all $i = m+1, \dots, m+k$ we introduce a point $x_i \in \partial U$ so that $|y_i - x_i| = \mathrm{dist}(y_i, \partial U)$. Since these new points belong to ∂U they do not contribute on $C_c^{0,1}(U)'$, and so, $\Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$, where we have renamed $n := m + k$.

With this procedure we show that, given $\Lambda \in X_f(U)$, we can always suppose that the representation $\Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$ satisfies the following property:

(P) for all $i = 1, \dots, n$ either $x_i \in U$ and $y_i \in \partial U$, or $x_i \in \partial U$ and $y_i \in U$. In the first case $|x_i - y_i| = \mathrm{dist}(x_i, \partial U)$, in the latter $|x_i - y_i| = \mathrm{dist}(y_i, \partial U)$. Moreover, the two families of points $I^+(\Lambda) := \{x_i : x_i \in U\}$ and $I^-(\Lambda) := \{y_i : y_i \in U\}$ are disjoint.

In the sequel we will consider the following class of rectifiable currents in $\mathbb{R}^2$:

$$\mathcal{S} := \Big\{ S \in \mathcal{D}_1(\mathbb{R}^2) : S = \sum_{k=1}^{m} [\![\overline{w_k z_k}]\!] \text{ for some sequence } ((w_k, z_k))_k \subset \mathbb{R}^2,\ m \in \mathbb{N} \Big\}, \tag{2.11}$$

and denote by $\mathcal{S}(U)$ the class in (2.11) when the currents are restricted to an open set $U \subset \mathbb{R}^2$.

Let U be a Lipschitz domain, and let $\Lambda \in X(U)$; then the supremum in the right-hand side of (2.3) can be extended to Lipschitz maps vanishing at the boundary, namely

$$\|\Lambda\|_{\mathrm{flat},U} := \sup_{\substack{\psi \in C_0^{0,1}(U) \\ l(\psi) \le 1}} \langle \Lambda, \psi \rangle, \tag{2.12}$$

where $C_0^{0,1}(U)$ denotes the space of Lipschitz maps ψ on U whose extension on $\overline{U}$ satisfies $\psi = 0$ on ∂U. According to [29] (see also [15] and references therin, and [11]), the supremum in (2.12) is achieved. Moreover, standard results (see [21, page 367] and [11, Lemma 8.1]) entail

$$\|\Lambda\|_{\mathrm{flat,U}} = \inf\{|S|_U : S \in \mathcal{D}_1(U),\ \Lambda = \partial S\} \qquad \forall \Lambda \in X(U), \tag{2.13}$$

and the infimum is attained. According to [11, Proposition 3.5] (which can be straightforwardly adapted to this case), the following statement can be proved:

Proposition 2.4. *Let $\Lambda \in X(U)$; then the infimum in the right-hand side of (2.13) is attained and there is a minimizer $S \in \mathcal{D}_1(\Omega)$ that is also an integer multiplicity current.*

In the case that $\Lambda \in X_f(U)$ something more can be proved for the minimizer S. Assume that $\Lambda = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$ is a representation satisfying property (P); we define

$$P(\Lambda) := \{i \in \{1, \dots, n\} : x_i \in U\}, \qquad \text{and} \qquad N(\Lambda) := \{i \in \{1, \dots, n\} : y_i \in U\}.$$

By (P) we obviously have $P(\Lambda) \cup N(\Lambda) = \{1, \dots, n\}$, and $P(\Lambda) \cap N(\Lambda) = \varnothing$. For all $I \subseteq P(\Lambda)$ we introduce $\mathcal{T}(I)$ the class of injective maps $\tau : I \to N(\Lambda)$ (this class might be empty, e.g., if $I = \varnothing$ or $\Lambda(N) = \varnothing$). Then we introduce the following minimum problem

$$m(\Lambda) := \min_{\substack{I \subseteq P(\Lambda) \\ \tau \in \mathcal{T}(I)}} \left(\sum_{i \in I} |x_i - y_{\tau(i)}| + \sum_{j \in P(\Lambda) \setminus I} |x_j - y_j| + \sum_{j \in N(\Lambda) \setminus \tau(I)} |x_j - y_j| \right). \tag{2.14}$$

If $I = \varnothing$ we have $\mathcal{T}(I) = \varnothing$ and the quantity between brackets is intended to be

$$\sum_{j \in P(\Lambda)} |x_j - y_j| + \sum_{j \in N(\Lambda)} |x_j - y_j|.$$

The minimum above is always attained as the number of competitors is finite. Correspondingly, if $I_{\min}$ and $\tau_{\min}$ are minimizers of (2.14), we denote by $S_{\min} \in \mathcal{S}$ the 1-current

$$S_{\min} := \sum_{i \in I_{\min}} [\![\overline{y_{\tau_{\min}(i)} x_i}]\!] + \sum_{j \in P(\Lambda) \setminus I_{\min}} [\![\overline{y_j x_j}]\!] + \sum_{j \in N(\Lambda) \setminus \tau_{\min}(I_{\min})} [\![\overline{y_j x_j}]\!]. \tag{2.15}$$

In the special case $I_{\min} = \varnothing$ we will have

$$S_{\min} := \sum_{j \in P(\Lambda)} [\![\overline{y_j x_j}]\!] + \sum_{j \in N(\Lambda)} [\![\overline{y_j x_j}]\!].$$

Notice that, in any case, it trivially holds $\partial S_{\min} = \Lambda$ in $\mathcal{D}_0(U)$. To shortcut the notation, for all couples (I, τ) admissible for the minimum problem (2.14), we introduce the couple $(\widehat{I}, \widehat{\tau})$ defined as follows: We set

$$\widehat{I} := P(\Lambda) \cup (N(\Lambda) \setminus \tau(I)) \tag{2.16}$$

(where we notice that the union is made through mutually disjoint sets) and define $\widehat{\tau} : \widehat{I} \to \{1, \dots, n\}$ as

$$\widehat{\tau}(i) := \begin{cases} \tau(i) & \text{if } i \in I, \\ i & \text{if } i \in (P(\Lambda) \setminus I) \cup (N(\Lambda) \setminus \tau(I)). \end{cases} \tag{2.17}$$

In the special case $I = \varnothing$ we define $\widehat{I} = P(\Lambda) \cup N(\Lambda)$ and $\widehat{\tau}(i) = i$ for all i. Using this notation, for any couple (I, τ) admissible for the problem (2.14), we introduce the corresponding 1-current $S_{(I,\tau)} \in \mathcal{S}$ defined as

$$S_{(I,\tau)} = \sum_{i \in \widehat{I}} [\![\overline{y_{\widehat{\tau}(i)} x_i}]\!]. \tag{2.18}$$

By (2.15) it follows that $S_{\min} = S_{(I_{\min}, \tau_{\min})}$.

Lemma 2.5. *Let $\Lambda \in X_f(U)$ with $U \subset \mathbb{R}^2$ a Lipschitz domain. Then*

$$\|\Lambda\|_{\text{flat}, U} = m(\Lambda), \tag{2.19}$$

and there is a minimizer S for (2.13) *that satisfies the following properties:*

(i) S belongs to $\mathcal{S}(U)$;

(ii) if (I, τ) is a minimizer for (2.14)*, than $S = S_{\min}$ defined in* (2.15)*. In particular there is $(\widehat{I}, \widehat{\tau})$ as in* (2.16) *and* (2.17) *such that $S_{\min} = S_{(I,\tau)}$ in* (2.18)*. Furthermore, for all $i \in \widehat{I}$ the interior of the segment $\overline{y_{\widehat{\tau}(i)} x_i}$ is contained in U, and*

$$|S|_U = \sum_{i \in \widehat{I}} |x_i - y_{\widehat{\tau}(i)}|;$$

(iii) if the points in the family $\{x_i, y_j : i \in P(\Lambda), j \in N(\Lambda)\}$ are three by three not collinear, then for every $i, j \in \widehat{I}$, $i \neq j$, if $\overline{y_{\widehat{\tau}(i)} x_i} \cap \overline{y_{\widehat{\tau}(j)} x_j} \cap U \neq \varnothing$, one of the following holds:

(a) $x_i = x_j \in \partial U$, and either $\overline{y_{\widehat{\tau}(i)} x_i} \subseteq \overline{y_{\widehat{\tau}(j)} x_j}$ or $\overline{y_{\widehat{\tau}(j)} x_j} \subseteq \overline{y_{\widehat{\tau}(i)} x_i}$;

(b) $y_{\widehat{\tau}(i)} = y_{\widehat{\tau}(j)} \in \partial U$, and either $\overline{y_{\widehat{\tau}(i)} x_i} \subseteq \overline{y_{\widehat{\tau}(j)} x_j}$ or $\overline{y_{\widehat{\tau}(j)} x_j} \subseteq \overline{y_{\widehat{\tau}(i)} x_i}$.

Proof. Let us prove (2.19). Let (I, τ) be a minimizer of (2.14) and let S be the current in (2.18). Since $S \in \mathcal{D}_1(\mathbb{R}^2)$ and $\partial S = \Lambda$, we obviously have, by (2.13), $\|\Lambda\|_{\text{flat},U} \leq m(\Lambda)$. Let us prove the opposite inequality.

Thanks to Proposition 2.4, as the minimizer S has integer multiplicity, by Federer decomposition theorem for integral 1-currents [21, 4.2.25], we can write $S := \sum_{i=1}^{\infty} S_i$, where S_i are the indecomposable components of S and are the push-forward of the integration on $[0,1]$ by Lipschitz maps. Hence, either $\partial S_i = \delta_{w_i} - \delta_{z_i}$ or $\partial S_i = 0$. We exclude the second case, since we could erase S_i, define $\widehat{S} := S - S_i$, and see that $\widehat{S}$ becomes a better competitor than S for (2.13), contradicting the minimality. Similarly, by minimality and indecomposability, we have that $|S_i|_U \leq |[\![z_i w_i]\!]|_U$, and one easily sees that it must be $S_i = [\![z_i w_i]\!]$.

Let now $\Lambda = \sum_{i=1}^{n}(\delta_{x_i} - \delta_{y_i})$ be a representation of Λ satisfying (P). As $\partial S = \sum_{i=1}^{m} \partial S_i$ and S_i are indecomposable, one sees[4] that $m \leq n$, and so $S \in \mathcal{S}$. Hence, for any $i = 1, \dots, m$ we have the following exclusive possibilities:

1. $w_i = x_{k(i)} \in U$, for some index $k(i) \in P(\Lambda)$, and $z_i \in \partial U$;
2. $z_i = y_{h(i)}$ for some index $h(i) \in N(\Lambda)$, and $w_i \in \partial U$;
3. both $w_i, z_i \in U$, and so there are two indeces $k(i) \in P(\Lambda)$ and $h(i) \in N(\Lambda)$ such that $w_i = x_{k(i)}$ and $z_i = y_{h(i)}$.

In all the cases, since $\partial S = \sum_{i=1}^{m}(\delta_{w_i} - \delta_{z_i}) = \Lambda$ we can choose the functions k and h injective, and see that for all $j \in P(\Lambda)$ there exists $i \in \{1, \dots, m\}$ such that $k(i) = j$, and for all $j \in N(\Lambda)$ there is i so that $h(i) = j$. Further, by minimality and property (P), we can suppose that in the first case $z_i = y_{k(i)}$, in the second one that $w_i = x_{h(i)}$, and in the latter case that $k(i) \neq h(i)$. We define

$$I := \{k(i) : i \in \{1, \dots, m\} \text{ and 3 holds}\},$$

and introduce the function $\tau : I \to N(\Lambda)$ as $\tau(j) = h(k^{-1}(j))$. Finally we set

$$\widehat{\tau}(j) := \begin{cases} j & \text{if } j \in P(\Lambda) \text{ and 1 holds, or } j \in N(\Lambda) \setminus \tau(I), \\ \tau(j) & \text{otherwise,} \end{cases}$$

for all $j \in P(\Lambda) \cup (N(\Lambda) \setminus \tau(I))$. Now we observe that S coincides with the one in formula (2.18). Further, so far we have shown that any indecomposable component S_i of S is one addendum in (2.18). Then, any segment $\overline{y_{\widehat{\tau}(i)} x_i}$ cannot intersect the boundary of U in its interior, thanks to indecomposability. Therefore, by (2.14), we conclude that $\|\Lambda\|_{\mathsf{flat},U} = |S|_U \geq m(\Lambda)$. This implies (2.19), (i), and (ii).

Let us now show (iii): let $i \neq j$ be such that $\overline{y_{\widehat{\tau}(i)} x_i} \cap \overline{y_{\widehat{\tau}(j)} x_j} \neq \varnothing$ and assume that the intersection consists of a unique point q in the interior of the two segments. Then we modify $\widehat{\tau}$ into a new function $\overline{\tau}$ which differs from $\widehat{\tau}$ only on $\{i, j\}$ and $\overline{\tau}(j) = \widehat{\tau}(i)$ and $\overline{\tau}(i) = \widehat{\tau}(j)$. In this way, since by the triangle inequality $|x_i - y_{\widehat{\tau}(j)}| + |x_j - y_{\widehat{\tau}(i)}| < |x_i - y_{\widehat{\tau}(i)}| + |x_j - y_{\widehat{\tau}(j)}|$, the corresponding current $\sum_{i \in \widehat{I}} [\![\overline{y_{\overline{\tau}(i)} x_i}]\!]$ is a better competitor for (2.14), leading to a contradiction. In particular we conclude that if $\overline{y_{\widehat{\tau}(i)} x_i} \cap \overline{y_{\widehat{\tau}(j)} x_j} \cap U \neq \varnothing$ then such intersection must contain an extremum of (at least) one of the two segments. Assume that such point is $x_i \in U$; x_j and $y_{\widehat{\tau}(j)}$ cannot both belong to U, by assumption of non-collinearity. Hence either $x_j \in \partial U$ or $y_{\widehat{\tau}(j)} \in \partial U$. Let us treat separately the two cases:

- $x_j \in \partial U$: in this case we find a contradiction with minimality, since the segments $\overline{x_i y_{\widehat{\tau}(j)}}$ and $\overline{x_j y_{\widehat{\tau}(i)}}$ have total length strictly smaller than $|x_i - y_{\widehat{\tau}(i)}| + |x_j - y_{\widehat{\tau}(j)}|$;
- $y_{\widehat{\tau}(j)} \in \partial U$: we find a contradiction by the triangle inequality, because, as $|x_i - y_{\widehat{\tau}(i)}| \leq |x_i - y_{\widehat{\tau}(j)}|$ we will have, unless $y_{\widehat{\tau}(j)} = y_{\widehat{\tau}(i)}$, that $|x_j - y_{\widehat{\tau}(i)}| < |x_j - y_{\widehat{\tau}(j)}|$ which is absurd[5] by property (P). If instead $y_{\widehat{\tau}(j)} = y_{\widehat{\tau}(i)}$, we are in case (b).

[4]This follows since for all $i = 1, \dots, m$, at least one among z_i and w_i must coincide with a pole x_j or y_j with index $j \in P(\Lambda) \cup N(\Lambda)$.

[5]As $y_{\widehat{\tau}(j)} \in \partial U$, necessarily $\widehat{\tau}(i) = i$ and y_i minimizes the distance from x_j to ∂U.

Similarly, if the extremum of the segment belonging to $\overline{y_{\hat{\tau}(i)}x_i} \cap \overline{y_{\hat{\tau}(j)}x_j} \cap U \neq \varnothing$ is $y_{\hat{\tau}(j)}$, we will end up with case (a). This concludes the proof. □

Let $\delta > 0$ and U be a Lipschitz domain. Let $\Lambda \in X_f(U_\delta)$, where U_δ is the δ-neighborhood of U (see (2.1)). The following theorem provides a property of continuity of $\|\Lambda\|_{\mathrm{flat},U_\delta}$ with respect to δ.

Lemma 2.6. *Let $\delta_0 > 0$, let U be a Lipschitz domain, and let $\Lambda \in X_f(U_{\delta_0})$. Then*

$$\lim_{\delta \to 0^+} \|\Lambda\|_{\mathrm{flat},U_\delta} = \|\Lambda\|_{\mathrm{flat},U}.$$

Proof. Since $U_\delta \subset U_{\delta'}$ for $\delta < \delta'$, the quantity $\|\Lambda\|_{\mathrm{flat},U_\delta}$ is nondecreasing in δ, and so the limit exists. Also, $\lim_{\delta\to 0^+} \|\Lambda\|_{\mathrm{flat},U_\delta} \geq \|\Lambda\|_{\mathrm{flat},U}$. Let us show the opposite inequality.

As the supremum in (2.12) is achieved, let $\psi_\delta \in C_0^{0,1}(U_\delta)$ be a maximizer in U_δ, for all $0 < \delta < \delta_0$. We can trivially extended ψ_δ to zero on $U_{\delta_0} \setminus U_\delta$. Up to subsequences, there is some $\psi \in C_0^{0,1}(U)$ such that $\psi_\delta \to \psi$ pointwisely, and then uniformly since $l(\psi_\delta) \leq 1$ for all $\delta \in (0, \delta_0)$. Therefore, using that Λ is a Radon measure on U_{δ_0} with finite total variation, we easily get

$$\|\Lambda\|_{\mathrm{flat},U_\delta} = \langle \Lambda, \psi_\delta \rangle_{\mathbb{R}^2} \to \langle \Lambda, \psi \rangle_{\mathbb{R}^2} \leq \|\Lambda\|_{\mathrm{flat},U},$$

as $\delta \to 0$. The thesis is achieved. □

We now extend the continuity property of the flat norm for general atomic distributions $\Lambda \in X(U_{\delta_0})$.

Lemma 2.7. *Let $\delta_0 > 0$, let U be a Lipschitz domain, and let $\Lambda \in X(U_{\delta_0})$. Then*

$$\lim_{\delta \to 0^+} \|\Lambda\|_{\mathrm{flat},U_\delta} = \|\Lambda\|_{\mathrm{flat},U}.$$

Proof. Since $\Lambda \in X(U_{\delta_0})$, we find a sequence of couples $(x_i, y_i) \in \overline{U_{\delta_0}} \times \overline{U_{\delta_0}}$ such that $\Lambda = \sum_{i=1}^{+\infty}(\delta_{x_i} - \delta_{y_i})$, and $\sum_{i=1}^{+\infty} d_{U_{\delta_0}}(x_i, y_i) < +\infty$. For $\epsilon > 0$ we find $N > 0$ so that

$$\sum_{i=N+1}^{+\infty} d_{U_{\delta_0}}(x_i, y_i) < \epsilon;$$

this in particular implies, setting $\Lambda_\epsilon := \sum_{i=N+1}^{+\infty}(\delta_{x_i} - \delta_{y_i})$ and $\Lambda_N := \sum_{i=1}^{N}(\delta_{x_i} - \delta_{y_i})$, that

$$\begin{aligned} \|\Lambda_N\|_{\mathrm{flat},V} &\leq \|\Lambda\|_{\mathrm{flat},V} + \|\Lambda_\epsilon\|_{\mathrm{flat},V} \leq \|\Lambda\|_{\mathrm{flat},V} + \epsilon, \\ \|\Lambda_N\|_{\mathrm{flat},V} &\geq \|\Lambda\|_{\mathrm{flat},V} - \|\Lambda_\epsilon\|_{\mathrm{flat},V} \geq \|\Lambda\|_{\mathrm{flat},V} - \epsilon, \end{aligned} \tag{2.20}$$

for any open set $V \subset U_{\delta_0}$. As a consequence, by Lemma 2.6, we infer

$$\lim_{\delta\to 0^+} \|\Lambda\|_{\mathrm{flat},U_\delta} \leq \epsilon + \lim_{\delta \to 0^+} \|\Lambda_N\|_{\mathrm{flat},U_\delta} = \epsilon + \|\Lambda_N\|_{\mathrm{flat},U} \leq 2\epsilon + \|\Lambda\|_{\mathrm{flat},U}.$$

This concludes the proof thanks to arbitrariness of $\epsilon > 0$, since on the other hand we always have $\lim_{\delta\to 0^+} \|\Lambda\|_{\mathrm{flat},U_\delta} \geq \|\Lambda\|_{\mathrm{flat},U}$. □

2.3 Functions of bounded variation

Let U be a fixed bounded Lipschitz domain. Let $u \in BV(U;\mathbb{R}^2)$; we recall that the distributional gradient of u is a measure $Du \in \mathcal{M}_b(U;\mathbb{R}^{2\times 2})$ which writes as

$$Du = \nabla u \mathcal{L}^2 + D^c u + D^J u,$$

where $\nabla u \in L^1(U;\mathbb{R}^2)$ is the approximate gradient, $D^c u$ is the Cantor part of Du and $D^J u$ is the jump part which is absolutely continuous with respect to the one dimensional Hausdorff measure $\mathcal{H}^1$. As in the scalar case, there exists a 1-rectifiable set S_u such that

$$\langle D^J_j u_i, \varphi\rangle = \int_{S_u} (u_i^+ - u_i^-)\nu_j \varphi \, d\mathcal{H}^1, \qquad \forall \varphi \in C_c(\Omega),$$

where ν is a unit normal vector to S_u chosen so that, for $\mathcal{H}^1$-a.e. $x \in S_u$, it holds

$$u^+(x) = \operatorname{aplim}_{\substack{y\to x \\ (y-x)\cdot\nu>0}} u(y), \qquad u^-(x) = \operatorname{aplim}_{\substack{y\to x \\ (y-x)\cdot\nu<0}} u(y). \tag{2.21}$$

Currents induced by scalar maps. Let $f \in BV(U)$ be a given real valued function. We introduce $T_f \in \mathcal{D}_2(\Omega)$ the following 2-dimensional current: for every 2-form $\alpha \in \mathcal{D}^2(U)$ we set

$$T_f(\alpha) := \int_U \langle f(x) e_1 \wedge e_2, \alpha(x)\rangle dx. \tag{2.22}$$

The boundary ∂T_f of T_f can be identified with the gradient of f; namely, for all $\omega \in \mathcal{D}^1(U)$, $\omega = \phi_1 dx_1 + \phi_2 dx_2$, one has

$$\partial T_f(\omega) = T_f(d\omega) = T_f(\operatorname{Curl} \phi \; dx_1 \wedge dx_2) = \int_U f(x) \operatorname{Curl} \phi(x) dx = \int_U \phi^\perp(x) dDf(x). \tag{2.23}$$

Push-forward and boundaries. Let $G \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ be a diffeomorphism which preserves orientation. If $\{e_1, e_2\}$ is a basis of 1-vectors in $\Lambda_1(\mathbb{R}^2)$, we denote by $\{\varepsilon_1, \varepsilon_2\}$ a basis for $\Lambda_1(G(\mathbb{R}^2)) \cong \Lambda_1(\mathbb{R}^2)$.

Given $g \in BV(G(U))$ we denote by $T_g^G \in \mathcal{D}_2(G(U))$ the current

$$T_g^G(\beta) := \int_{G(U)} \langle g(y)\varepsilon_1 \wedge \varepsilon_2, \beta(y)\rangle dy, \tag{2.24}$$

for all $\beta \in \mathcal{D}^2(G(U))$. Further, for $f \in BV(U)$ and any $\beta \in \mathcal{D}^2(G(U))$, writing $\beta = \varphi dy_1 \wedge dy_2$ (so that $G^\# \beta = (\varphi \circ G) dG_1 \wedge dG_2$), we have

$$\begin{aligned} G_\#(T_f)(\beta) &= \int_U \langle f(x) e_1 \wedge e_2, G^\#\beta(x)\rangle dx = \int_U f(x)\varphi(G(x)) \det\,(\nabla G(x)) dx \\ &= \int_{G(U)} f(G^{-1}(y))\varphi(y) dy = \int_{G(U)} \langle f\circ G^{-1}(y)\varepsilon_1 \wedge \varepsilon_2, \beta(y)\rangle dy = T^G_{f\circ G^{-1}}(\beta). \end{aligned}$$

Hence

$$G_\# T_f = T^G_{f\circ G^{-1}}. \tag{2.25}$$

As for the boundary of $G_\# T_f$, given $\varpi = \phi_1 dy_1 + \phi_2 dy_2 \in \mathcal{D}^1(G(U))$, one has $d\varpi = \text{ Curl } \phi dy_1 \wedge dy_2$, and

$$\begin{aligned}\partial G_\#(T_f)(\varpi) &= G_\#(T_f)(d\varpi) = \int_{G(U)} \langle f \circ G^{-1}(y)\varepsilon_1 \wedge \varepsilon_2, d\varpi(y)\rangle dy \\ &= \int_{G(U)} f \circ G^{-1}(y) \text{ Curl } \phi(y) dy = \int_{G(U)} \phi^\perp(y) \cdot dD(f \circ G^{-1})(y) = \partial T^G_{f\circ G^{-1}}(\varpi).\end{aligned} \tag{2.26}$$

Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a regular diffeomorphism as in Definition 2.1, and let $\delta > 0$ such that (2.4) holds. Observe that, as $G(U) \subset U_\delta$, if $f : U_\delta \to \mathbb{R}^2$, then $f \circ G : G^{-1}(U_\delta) \to \mathbb{R}^2$, and since $U \subset G^{-1}(U_\delta)$ both f and $f \circ G$ are defined on U.

Let $f \in BV(U_\delta)$; then the currents T_f and $T^G_{f\circ G^{-1}}$ are well-defined in $\mathcal{D}_2(U_\delta)$ and $\mathcal{D}_2(G(U_\delta))$, respectively, and as $G(U_\delta) \supset U$, both are well-defined in $\mathcal{D}_2(U)$. We have, from (2.22) and (2.24),

$$T^G_{f\circ G^{-1}} \llcorner U = T_{f\circ G^{-1}} \llcorner U,$$

so that, from (2.25), we have

$$T_{f\circ G^{-1}} \llcorner U = (G_\# T_f) \llcorner U. \tag{2.27}$$

BV **piecewise constant maps.** Let $U \subset \mathbb{R}^2$ be a Lipschitz domain and G and $\delta > 0$ be as above. We now discuss the special case in which the map $f \in BV(U_\delta)$ is piecewise constant. This implies that the approximate gradient ∇f of f (as well as the Cantor part of Df), is constantly null, and Df consists only of the jump part, namely

$$Df = (f^+ - f^-)\nu \cdot \mathcal{H}^1 \llcorner S_f,$$

where $S_f \subset U_\delta$ is the jump set. Let $\omega \in \mathcal{D}^1(U)$, $\omega = \phi_1 dx_1 + \phi_2 dx_2$, so (2.23) and (2.26) imply

$$\partial T_f(\omega) = \int_{S_f \cap U} (f^+ - f^-)\phi^\perp \cdot \nu d\mathcal{H}^1 = \int_{S_f} (f^+ - f^-)\phi \cdot \tau d\mathcal{H}^1, \tag{2.28}$$

$$\begin{aligned}\partial T_{f\circ G^{-1}}(\omega) &= \int_{S_{f\circ G^{-1}} \cap U} ((f \circ G^{-1})^+ - (f \circ G^{-1})^-)\phi^\perp \cdot \nu d\mathcal{H}^1 \\ &= \partial T_f(G^\# \omega) = \int_{S_f} (f^+ - f^-)\left((\phi \circ G)\nabla G\right)^\perp \cdot \nu d\mathcal{H}^1 \\ &= \int_{S_f} (f^+ - f^-)(\phi \circ G)\nabla G \cdot \tau d\mathcal{H}^1 = \int_{S_f} (f^+ - f^-)(\phi \circ G) \cdot \frac{\partial G}{\partial \tau} d\mathcal{H}^1,\end{aligned} \tag{2.29}$$

where we denote $-\nu^\perp = \tau$ a unit tangent vector to S_f. Here we have used also (2.27).

2.4 Weak Jacobian determinant of vector-valued functions of bounded variation

Let $U \subset \mathbb{R}^2$ be a Lipschitz domain. If $u \in BV(U; \mathbb{R}^2)$, we will have $u_i \in BV(U)$, $i = 1, 2$. According to formula (2.22), we have at our disposal two currents $T_{u_i} \in \mathcal{D}_2(U)$, $i = 1, 2$.

Let $u \in BV(U; \mathbb{R}^2) \cap L^\infty(U; \mathbb{R}^2)$ be given. We introduce the measure $\lambda_u \in \mathcal{M}_b(U; \mathbb{R}^2)$ defined as

$$\begin{aligned}\int_U \varphi(x) \cdot d\lambda_u(x) := &-\frac{1}{2}\int_{U\setminus S_u} \overline{u}_1(x)\varphi^\perp(x) \cdot dDu_2 + \frac{1}{2}\int_{U\setminus S_u} \overline{u}_2(x)\varphi^\perp(x) \cdot dDu_1 \\ &- \frac{1}{2}\int_{S_u} (u_2^+(x)u_1^-(x) - u_1^+(x)u_2^-(x))\varphi^\perp(x) \cdot \nu d\mathcal{H}^1(x),\end{aligned} \tag{2.30}$$

for all $\varphi \in C_c(U;\mathbb{R}^2)$. Here, $\overline{u}(x)$ denotes the Lebesgue value of u at x, defined $\mathcal{H}^1$-a.e. on $U \setminus S_u$. As a vector valued Borel measure with bounded total variation, λ_u can be identified with a 1-current in $\mathcal{D}_1(U)$.

Definition 2.8. *Let $u \in BV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$; then we denote by $Ju \in \mathcal{D}_0(U)$ the current*

$$Ju := \partial\lambda_u.$$

The definition of $Ju \in \mathcal{D}_0(U)$ provides a weak notion of Jacobian determinant of u, and extends the classical distributional determinant of ∇u for Sobolev maps (see [20], and also [28]).

The following theorem (see [20]) provides a property of continuity for Ju.

Theorem 2.9. *Let $u_j, u \in BV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$ be such that $u_j \rightharpoonup u$ strictly in $BV(U;\mathbb{R}^2)$ and $\sup_j \|u_j\|_{L^\infty} < C$. Then*

$$Ju_j \to Ju \text{ wealky in } \mathcal{D}_0(\Omega).$$

Moreover, $\|Ju\|_{\mathrm{flat},U} \leq |\lambda_u|(U) \leq C\|u\|_{BV(U;\mathbb{R}^2)}$.

As a consequence of the last assertion in the previous theorem we get:

Corollary 2.10. *Assume the hypotheses of Theorem 2.9. If $|\lambda_u - \lambda_{u_j}|(U) \to 0$ as $j \to \infty$, then $Ju_j \to Ju$ with respect to the flat norm.*

BV **functions taking three values.** From now on we suppose that $\alpha, \beta, \gamma \in \mathbb{S}^1$ are the vertices of an equilateral triangle centered at the origin, with edge of length $l := \sqrt{3}$ and

$$\frac{\sqrt{3}}{2} = \alpha \times \beta = \beta \times \gamma = \gamma \times \alpha. \tag{2.31}$$

Here we have noted $\alpha \times \beta := -\alpha \cdot \beta^\perp$. The last requirement implies that α, β, and γ, are in couterclockwise order on $\mathbb{S}^1$. Set

$$\sigma := \frac{\sqrt{3}}{4} = \frac{|\alpha \times \beta|}{2} = \frac{|\alpha \times \gamma|}{2} = \frac{|\beta \times \gamma|}{2}. \tag{2.32}$$

For a piecewise constant map $u \in BV(U;\{\alpha,\beta,\gamma\})$ the weak Jacobian determinant will read, using (2.30), for all $\varphi \in \mathcal{D}(U)$,

$$\begin{aligned} Ju(\varphi) &= -\frac{1}{2}\int_{S_u} (u_2^+(x)u_1^-(x) - u_1^+(x)u_2^-(x))\nabla^\perp\varphi(x) \cdot \nu d\mathcal{H}^1(x) \\ &= \frac{1}{2}\int_{S_u} (u^+ \times u^-)(x)\frac{\partial\varphi}{\partial\tau}(x) d\mathcal{H}^1(x). \end{aligned}$$

Here, since u takes values in $\{\alpha,\beta,\gamma\}$, it holds that $u^+ \times u^-$ takes only a finite number of possible values. In particular, thanks to (2.32), it happens that $\frac{u^+\times u^-}{2} \in \{\pm\sigma\}$ $\mathcal{H}^1$-a.e. on S_u. In particular

$$\lambda_u = \frac{1}{2}(u^+ \times u^-)\tau \cdot \mathcal{H}^1 \llcorner S_u \tag{2.33}$$

is the multiple of an integer multiplicity 1-current (namely, the multiplicity is in $\{\pm\sigma\}$).

We will now discuss more in details the structure of the Jacobian determinant Ju when u has polyhedral jump set. Specifically, we introduce the following definition:

Definition 2.11. A function $u \in BV_{\rm loc}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ is called polyhedral if $S_u = \cup_{i=1}^N \overline{n_i p_i}$ is the union of finitely many segments, with $\overline{n_i p_i} \subset \overline{U}$. We call n_i, p_i, $i = 1, \dots, N$ (the extrema of the open segment $\overline{n_i p_i}$), the vertices of S_u. Additionally, we suppose that if $i \neq j, 1 \leq i, j \leq N$ then either $\overline{n_i p_i} \cap \overline{n_j p_j}$ is empty or it is a vertex.

We say that a map $u \in BV(U; \{\alpha, \beta, \gamma\})$ is polyhedral if it is the restriction on U of a polyhedral function.

Notice that if $u \in BV_{\rm loc}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ is polyhedral then it must be constant outside some ball $B_R(0)$.

As a consequence of the definition, if u is polyhedral, then in the segment $\overline{n_i p_i}$ the jump $u^+ - u^-$ of u is constant, for all i. Therefore we easily obtain

$$Ju(\varphi) = \frac{1}{2} \sum_{i=1}^N \int_{\overline{n_i p_i}} (u^+ \times u^-) \frac{\partial \varphi}{\partial \tau} d\mathcal{H}^1 = \frac{1}{2} \sum_{i=1}^N \left(u^+ \times u^-\right) \llcorner \overline{n_i p_i} \left(\varphi(p_i) - \varphi(n_i)\right), \tag{2.34}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Using (2.32), the previous expression is equal to

$$Ju(\varphi) = \sigma \sum_{i=1}^N \gamma_i \left(\varphi(p_i) - \varphi(n_i)\right), \qquad \gamma_i := \frac{(u^+ \times u^-)}{|\,(u^+ \times u^-)\,|} \in \{-1, +1\}.$$

Namely

$$Ju := \sigma \sum_{i=1}^N \gamma_i (\delta_{p_i} - \delta_{n_i}), \tag{2.35}$$

is a finite Radon measure; notice that if $u \in BV(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ the points p_i, n_i are not in general distinct and can also lie on ∂U. It turns out that

$$\frac{1}{\sigma} Ju \in X_f(U).$$

Let us now consider a regular diffeomorphism $G \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ satisfying (2.4). Assume $u \in BV_{\rm loc}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ be constant outside a ball $B_R(0)$, and let us consider the currents $\lambda_u, \lambda_{u\circ G} \in \mathcal{D}_1(\mathbb{R}^2)$ in (2.33) related to u and $u \circ G$: namely

$$\begin{aligned} \lambda_u &= \frac{1}{2}(u^+ \times u^-)\tau \cdot \mathcal{H}^1 \llcorner S_u, \\ \lambda_{u\circ G} &= \frac{1}{2}((u\circ G)^+ \times (u \circ G)^-)\widehat{\tau} \cdot \mathcal{H}^1 \llcorner S_{u\circ G}, \end{aligned} \tag{2.36}$$

where $\widehat{\tau} = -\widehat{\nu}^\perp$, with $\widehat{\nu}$ the unit normal to $S_{u\circ G}$. We can now consider the push-forward of $\lambda_{u\circ G}$ by G. For all $\omega = \phi_1 dx_1 + \phi_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$ we have

$$\begin{aligned} G_\# \lambda_{u\circ G}(\omega) = \lambda_{u\circ G}(G^\# \omega) &= \frac{1}{2} \int_{S_{u\circ G}} ((u\circ G)^+ \times (u\circ G)^-)(\phi \circ G) \frac{\partial G}{\partial \widehat{\tau}} d\mathcal{H}^1 \\ &= \frac{1}{2} \int_{S_u} (u^+ \times u^-) \phi \cdot \tau d\mathcal{H}^1 = \lambda_u(\omega), \end{aligned}$$

that is,

$$G_\# \lambda_{u\circ G} = \lambda_u \qquad \text{in } \mathcal{D}_1(\mathbb{R}^2).$$

In particular, $\partial(G_{\#}\lambda_{u\circ G}) = \partial\lambda_u$, i.e.

$$G_{\#}J(u\circ G) = Ju \qquad \text{in } \mathcal{D}_0(\mathbb{R}^2). \tag{2.37}$$

Equivalently, we will also have

$$J(u\circ G) = (G^{-1})_{\#}Ju \qquad \text{in } \mathcal{D}_0(\mathbb{R}^2).$$

Now, if Ju has the form in (2.35), we see that

$$J(u\circ G) := \sigma\sum_{i=1}^{N}\gamma_i(\delta_{G^{-1}(p_i)} - \delta_{G^{-1}(n_i)}). \tag{2.38}$$

We have proved the following:

Lemma 2.12. *Let $G \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ be a regular diffeomorphism as in* (2.4)*. Assume $u \in BV_{\rm loc}(\mathbb{R}^2;\{\alpha,\beta,\gamma\})$ be polyhedral. Then $\frac{1}{\sigma}Ju$ and $\frac{1}{\sigma}J(u\circ G)$ are Radon measures in $X_f(\mathbb{R}^2)$ and write as in* (2.35) *and* (2.38) *respectively. In particular, for all $\varphi \in C_c^{0,1}(\mathbb{R}^2)$ one has*

$$Ju(\varphi) = \sigma\sum_{i=1}^{N}\gamma_i(\varphi(p_i) - \varphi(n_i)), \qquad J(u\circ G) = \sigma\sum_{i=1}^{N}\gamma_i(\varphi(G^{-1}(p_i)) - \varphi(G^{-1}(n_i))). \tag{2.39}$$

Observe that (2.5) and (2.6) apply to $J(u\circ G)$ and Ju as in Lemma 2.12.

Remark 2.13. To show Lemma 2.12 we have proved (2.37); notice that this formula holds for any piecewise constant $u \in BV_{\rm loc}(\mathbb{R}^2;\{\alpha,\beta,\gamma\})$. In particular, if $\frac{1}{\sigma}Ju \in X(\mathbb{R}^2)$, also $\frac{1}{\sigma}J(u\circ G) \in X(\mathbb{R}^2)$, and hence,

$$Ju(\varphi) = \sigma\sum_{i=1}^{\infty}(\varphi(x_i) - \varphi(y_i)), \qquad J(u\circ G) = \sigma\sum_{i=1}^{\infty}(\varphi(G^{-1}(x_i)) - \varphi(G^{-1}(y_i))). \tag{2.40}$$

2.5 Approximation of BV piecewise constant maps

Since we deal with functions taking only the 3 values $\alpha,\beta,\gamma \in \mathbb{R}^2$, we need the following approximation theorem contained in [14, Theorem 2.2].

Theorem 2.14. *Let $u \in BV_{\rm loc}(\mathbb{R}^2;\mathcal{Z})$, with $\mathcal{Z} := \{z_1,\dots,z_m\} \subset \mathbb{R}^2$ a finite set. Assume that u is constant outside a ball $B_R(0)$ (hence $|Du|(\mathbb{R}^2) < +\infty$). Then there exists a sequence $u_j \in BV_{\rm loc}(\mathbb{R}^2;\mathcal{Z})$ such that the jump set S_{u_j} of u_j is polyhedral and $u_j \to u$ in $L^1_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$. Furthermore, there are bijective functions $f_j \in C^1(\mathbb{R}^2;\mathbb{R}^2)$, with $f_j^{-1} \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ as well, such that $f_j \to id$ strongly in $W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$ and $|Du_j - D(u\circ f_j)|(\mathbb{R}^2) \to 0$ as $j \to \infty$.*

By [14, Lemma 2.7], the following extension result holds:

Theorem 2.15. *Let $U \subset \mathbb{R}^2$ be a Lipschitz domain and let $u \in BV(U;\mathcal{Z})$, with $\mathcal{Z} := \{z_1,\dots,z_m\} \subset \mathbb{R}^2$ a finite set. Then there exist $C > 0$ (depending only on U) and a function $\widetilde{u} \in BV_{\rm loc}(\mathbb{R}^2;\mathcal{Z})$ such that*

$$\begin{aligned} &\widetilde{u} = u && \text{on } U,\\ &|D\widetilde{u}|(\partial U) = 0, &&\\ &|D\widetilde{u}|(\mathbb{R}^2) \le C|Du|(U). && \end{aligned} \tag{2.41}$$

Furthermore, we can always assume that $\widetilde{u}$ is constant outside $B_R(0)$, for some $R > 0$ large enough such that $U \subset\subset B_R(0)$.

Remark 2.16. Let $u \in BV(U; \{\alpha, \beta, \gamma\})$, and let us denote by u itself an extension of u as in Theorem 2.15. Let $(u_j) \subset BV_{\mathrm{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ be the polyhedral approximations of $u \in BV_{\mathrm{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ provided by Theorem 2.14. Since u can be taken constant outside $B_R(0)$, we can easily see that u_j as well can be choosen constant outside $B_R(0)$, and $u_j = u$ on $\mathbb{R}^2 \setminus B_R(0)$.

The condition $|Du_j - D(u \circ f_j)|(\mathbb{R}^2) \to 0$ as $j \to \infty$ implies that for any U' with $U \subset U' \subset B_R(0)$ we have

$$|Du_j - D(u \circ f_j)|(U') \to 0, \qquad \text{as } j \to \infty.$$

Let us denote by $w_j := u_j - u \circ f_j$ and by w_j^1 and w_j^2 its components; according to (2.28), this means that for $i = 1, 2$, the current $\partial T_{w_j^i}$ tends to zero strongly in $\mathcal{D}_1(U')$ (i.e., its mass in U' tends to zero as $j \to \infty$). On the other hand, since $f_j \to id$ strongly in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$, taking into account (2.29), also

$$\partial T_{w_j^i \circ f_j^{-1}} \to 0 \qquad \text{strongly in } \mathcal{D}_1(U'), \qquad i = 1, 2. \tag{2.42}$$

Now $w_j \circ f_j^{-1} = u_j \circ f_j^{-1} - u$; hence, we easily infer[6]

$$u_j \circ f_j^{-1} \to u \qquad \text{strongly in } BV(U'; \mathbb{R}^2) \tag{2.43}$$

and

$$\partial T_{u_j^i \circ f_j^{-1}} \to \partial T_{u^i} \qquad \text{strongly in } \mathcal{D}_1(U'), \qquad i = 1, 2. \tag{2.44}$$

Using again (2.28) and (2.29), exploiting the fact that $|u^+ - u^-| = l > 0$, $\mathcal{H}^1$-a.e. on S_u, we conclude that

$$\mathcal{H}^1(S_u \Delta S_{u_j \circ f_j^{-1}}) \to 0 \qquad \text{as } j \to \infty. \tag{2.45}$$

With (2.43) and (2.45) at our disposal we are now able to prove the following:

Theorem 2.17. *Let $U \subset \mathbb{R}^2$ be a Lipschitz domain and let $u \in BV(U; \{\alpha, \beta, \gamma\})$. Then $\frac{1}{\sigma} Ju \in X(U)$.*

Proof. As in Remark 2.16, we extend u to all $\mathbb{R}^2$ constantly outside $B_R(0)$, and choose polyhedral approximations of u given by functions $u_j \in BV_{\mathrm{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ which are constant outside $B_R(0)$. Let $f_j : \mathbb{R}^2 \to \mathbb{R}^2$ be diffeomorphisms as in Theorem 2.14; by Lemma 2.12, we know that

$$Ju_j = \sigma \sum_{i=1}^{N_j} (\delta_{p_i} - \delta_{n_i}),$$

and

$$J(u_j \circ f_j^{-1}) = \sigma \sum_{i=1}^{N_j} (\delta_{f_j(p_i)} - \delta_{f_j(n_i)}),$$

for some points $p_i, n_i \in \mathbb{R}^2$. We claim that

$$J(u_j \circ f_j^{-1}) \to Ju \qquad \text{in the flat norm.} \tag{2.46}$$

[6] Fixing any subsequence, this must be true for a suitable sub-subsequence, and hence it holds for the full sequence.

Indeed, let λ_u and $\lambda_{u_j\circ f_j^{-1}}$ be the measures as in (2.36) associated with u and $u\circ f_j^{-1}$ respectively. Specifically,

$$\lambda_{u\circ f_j^{-1}} = \frac{1}{2}((u_j \circ f_j^{-1})^+ \times (u_j \circ f_j^{-1})^-)\widehat{\tau}_j \cdot \mathcal{H}^1 \llcorner S_{u_j\circ f_j^{-1}}.$$

To prove the claim it is sufficient to show that $|\lambda_{u_j\circ f_j^{-1}} - \lambda_u|(\mathbb{R}^2) \to 0$ as $j \to \infty$. This is straighforward, since

$$|\lambda_{u_j\circ f_j^{-1}} - \lambda_u|(\mathbb{R}^2) \leq \sigma\mathcal{H}^1(S_u \Delta S_{u_j\circ f_j^{-1}}),$$

which goes to zero thanks to (2.45).

We now choose a (not-relabelled) subsequence of (u_j), such that

$$\|J(u_j \circ f_j^{-1}) - J(u_{j-1} \circ f_{j-1}^{-1})\|_{\text{flat},U} \leq \frac{1}{2^j}, \tag{2.47}$$

for all $j \geq 1$. We write $(J(u_j \circ f_j^{-1}) - J(u_{j-1} \circ f_{j-1}^{-1})) = \sigma \sum_{i=M_{j-1}+1}^{M_j} (\delta_{x_i} - \delta_{y_i})$ and $J(u_0 \circ f_0^{-1}) = \sigma \sum_{i=1}^{M_0} (\delta_{x_i} - \delta_{y_i})$, for a suitable increasing sequence of natural numbers $M_j > 0$, and suitable points $x_i, y_i \in \overline{U}$. Notice that we can choose these representations in such a way that

$$\|(J(u_j \circ f_j^{-1}) - J(u_{j-1} \circ f_{j-1}^{-1})\|_{\text{flat},U} \leq \sum_{i=M_{j-1}+1}^{M_j} \sigma|x_i - y_i| \leq \frac{1}{2^{j-1}} \qquad \forall j \geq 1.$$

Hence

$$J(u_n \circ f_n^{-1}) = J(u_0 \circ f_0^{-1}) + \sum_{j=1}^{n} \Big(J(u_j \circ f_j^{-1}) - J(u_{j-1} \circ f_{j-1}^{-1}) \Big) = \sigma \sum_{i=1}^{M_n} (\delta_{x_i} - \delta_{y_i}),$$

and

$$\sum_{i=1}^{M_n} \sigma|x_i - y_i| \leq 2\|J(u_0 \circ f_0^{-1})\|_{\text{flat},U} + 2.$$

Letting $n \to \infty$ in the two previous expressions, we infer $\frac{1}{\sigma}Ju \in X(U)$. The thesis then follows by using that $\sum_{i=1}^{\infty} d_U(x_i, y_i) \leq \sum_{i=1}^{\infty} |x_i - y_i|$. □

As a consequence of the previous proof we have the following:

Corollary 2.18. *Let $u \in BV_{loc}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$. Then there exist a sequence of polyhedral maps $(u_j) \subset BV_{loc}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ and a sequence $(f_j) \subset C^1(\mathbb{R}^2; \mathbb{R}^2)$ of diffeomorphisms with $f_j \to id$ strongly in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ such that*

$$\|J(u_j \circ f_j) - Ju\|_{\text{flat},\mathbb{R}^2} \to 0.$$

The previous result implies in turn

$$\|J(u_j \circ f_j) - Ju\|_{\text{flat},V} \to 0,$$

for any Lipschitz domain $V \subset \mathbb{R}^2$.

A further consequence of the proof of Theorem 2.17 is the following:

Corollary 2.19. *The space $X(U)$ is sequentially closed with respect to the flat topology.*

Proof. Let $(\Lambda_k) \subset X(U)$ be a sequence converging to Λ in the flat topology. Starting from (2.47) and replacing $J(u_k \circ f_k^{-1})$ with Λ_k we can employ the same argument of the proof of Theorem 2.17 and conclude that $\Lambda \in X(U)$. □

The following approximation result will be crucial to prove our main theorem.

Theorem 2.20. *Let $U \subset \mathbb{R}^2$ be a Lipschitz domain and let $u \in BV(U; \{\alpha, \beta, \gamma\})$. Then there exists a sequence $(u_j) \subset BV(U; \{\alpha, \beta, \gamma\})$ of polyhedral maps such that*

$$\begin{aligned} &u_j \to u \qquad \textit{strictly in } BV(U; \{\alpha, \beta, \gamma\}), \\ &Ju_j \to Ju \qquad \textit{in } \mathcal{D}_0(U), \\ &\|Ju_j\|_{\text{flat},U} \to \|Ju\|_{\text{flat},U}. \end{aligned} \tag{2.48}$$

Proof. We extend u to $\mathbb{R}^2$ constantly outside $B_R(0)$ (with $R > 0$ large enough so that $U \subset\subset B_R(0)$). Let $u_j \in BV_{\text{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ be polyhedral approximations of u given by functions which are constant outside $B_R(0)$, and let $f_j \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ be diffeomorphisms as in Theorem 2.14; let $\delta > 0$ be arbitrary, and assume without loss of generality that (2.4) holds for all f_j's. By Theorem 2.14 we know that $\|u_j - (u \circ f_j)\|_{BV(B_R(0);\mathbb{R}^2)} \to 0$ as $j \to \infty$. Moreover, as a consequence of [14, Corollary 2.4] we also know that $u_j \to u$ strictly in $BV(U; \mathbb{R}^2)$. By Theorem 2.9, also $Ju_j \to Ju$ weakly in $\mathcal{D}_0(U)$; to conclude, we have hence to prove the last condition in (2.48).

We have

$$\|Ju_j\|_{\text{flat},U} \le \|Ju_j - J(u \circ f_j^{-1})\|_{\text{flat},U} + \|J(u \circ f_j^{-1})\|_{\text{flat},U}. \tag{2.49}$$

The first term in the right-hand side tends to zero as $j \to \infty$. Indeed, by (2.37), we can write $Ju_j - J(u \circ f_j^{-1}) = (f_j^{-1})_{\#}(J(u_j \circ f_j) - Ju)$; thus by (2.5), one has

$$\|Ju_j - J(u \circ f_j^{-1})\|_{\text{flat},U} \le \|Ju_j - J(u \circ f_j^{-1})\|_{\text{flat},\mathbb{R}^2} \le (1+\delta)\|J(u_j \circ f_j) - Ju\|_{\text{flat},\mathbb{R}^2},$$

the last term vanishing as $j \to 0$ thanks to Corollary 2.18. Let us now analyse the second term in the right-hand side of (2.49). Let $\delta > 0$ be arbitrary, and assume j is large enough so that f_j satisfies (2.4). Thanks to Theorem 2.17, we can write

$$Ju = \sigma \sum_{i=1}^{\infty} (\delta_{x_i} - \delta_{y_i}), \tag{2.50}$$

for suitable points $x_i, y_i \in \mathbb{R}^2$. Let $I := \{i : \text{ either } x_i \text{ or } y_i \in U_\delta\}$, so that

$$Ju \llcorner U = \sigma \sum_{i \in I} (\delta_{x_i} - \delta_{y_i}).$$

By (2.40) we also have

$$J(u \circ f_j^{-1}) \llcorner U = \sigma \sum_{i \in I} (\delta_{f_j(x_i)} - \delta_{f_j(y_i)}),$$

because $f_j(x_i) \in U$ implies $x_i \in U_\delta$ (and similarly for y_i). In particular, by (2.6)

$$\|J(u \circ f_j^{-1})\|_{\text{flat},U} \le (1+\delta)\|Ju\|_{\text{flat},f_j(U)} \le (1+\delta)\|Ju\|_{\text{flat},U_\delta}.$$

We can hence pass to the limsup and obtain

$$\limsup_{j\to\infty} \|J(u \circ f_j^{-1})\|_{\text{flat},U} \le (1+\delta)\|Ju\|_{\text{flat},U_\delta},$$

which holds for any $\delta > 0$ small enough, and thus, by Lemma 2.7, we conclude

$$\limsup_{j\to\infty} \|J(u\circ f_j^{-1})\|_{\mathrm{flat},U} \leq \|Ju\|_{\mathrm{flat},U}.$$

The opposite inequality follows by the second equation in (2.48) and by lower semicontinuity of the flat norm. □

2.6 Density of polyhedral maps

Let $u \in BV_{\mathrm{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ be polyhedral. According to Definition 2.11, there is $N > 0$ such that

$$S_u = \cup_{i=1}^{N} \overline{n_i p_i},$$

for suitable points $p_i, n_i \in \mathbb{R}^2$, vertices of the jump set of u. We recall that every segment $\overline{n_i p_i}$ does not partially overlap each other, and neither trasversally intersects any other, but they only can share an endpoint; i.e., for $i \neq j$, $\overline{n_i p_i} \cap \overline{n_j p_j}$ is either empty or a vertex. Moreover, if x is a vertex of S_u, we define its multiplicity as $m(x) := \#\{i \in \{1, \dots, N\} : x = n_i, \text{ or } x = p_i\}$.

Equivalently, a vertex x has multiplicity $m > 0$ (and in such a case is called m-vertex) if there exists $\delta > 0$ such that, for all $r \in (0, \delta)$, $B_r(x) \cap S_u$ consists exactly of m segments (which will be radii of $B_r(x)$). For this reason, the multiplicity of any vertex is at least 2.

Let x be a 3-vertex (also referred to as a triple vertex) and let $r > 0$ be small enough so that $B_r(x) \cap S_u$ consists of exactly three radii R_1, R_2, and R_3 of $B_r(x)$, chosen in counterclockwise order around x. Let $S_{i,i+1}$ be the circular sector enclosed by R_i and R_{i+1}, $i = 1, 2, 3$, with $i + 1$ intended mod(3). Finally, set $\theta_i := u \llcorner S_{i,i+1}$ the value of u on $S_{i,i+1}$. Then, since x is a 3-vertex, it follows that the triple $(\theta_1, \theta_2, \theta_3)$ is a permutation of (α, β, γ). We say that the 3-vertex is positively (negatively) oriented if the sign of the permutation is positive (negative, respectively).

Eventually, a couple (x, y) of distinct triple vertices is called a dipole if y is negatively and x is positively oriented.

Let x be a vertex of S_u and let $\overline{xy}$ be a segment in S_u. Choose a Cartesian coordinate system (x', y') with origin at x so that the halfline $\{x' > 0, y' = 0\}$ contains the segment $\overline{xy}$. If $\tau = -\nu^\perp = (1, 0)$ is a tangent vector to $\overline{xy}$, thanks to (2.34), it turns out that $\nu = (0, 1)$ and so $u^\pm$ corresponds to the value that u takes for $y' > 0$ (for $y' < 0$, respectively) just above (below) the segment $\overline{xy}$. In particular, if x is a 3-vertex positively oriented, the possible values of (u^+, u^-) are only the following: $(u^+, u^-) = (\alpha, \gamma)$, $(u^+, u^-) = (\beta, \alpha)$, $(u^+, u^-) = (\gamma, \beta)$; hence, in any case, $\frac{1}{2}(u^+ \times u^-) = \sigma$. Therefore, if x is a positively oriented 3-vertex, using again (2.34), by (2.31) and (2.32), we infer that

$$Ju \llcorner B_r(x) = 3\sigma\delta_x \qquad \text{in } \mathcal{D}_0(B_r(x)), \tag{2.51}$$

for $r > 0$ small enough. If instead x is negatively oriented, we will have a minus sign in the right-hand side of the previous expression.

We will now state and prove the following density result:

Theorem 2.21. *Let U be a Lipschitz domain and $u \in BV(U; \{\alpha, \beta, \gamma\})$. Then for all $\epsilon > 0$ there is a map $u_\epsilon \in BV(U; \{\alpha, \beta, \gamma\})$ such that*

(i) u_ϵ is polyhedral, and its jump set writes as $S_{u_\epsilon} = \cup_{i=1}^{N_\epsilon} \overline{n_i^\epsilon p_i^\epsilon} \cap U$, for suitable points $p_i^\epsilon, n_i^\epsilon \in \mathbb{R}^2$, vertices of the jump set;

(ii) for all vertices $x \in U$, the multiplicity of x is at most 3;

(iii) *the vertices* $p_i^\epsilon, n_i^\epsilon$ *of* S_{u_ϵ} *which are contained in* U *form a family of three by three not collinear points;*

(iv) *if* $\Lambda = \Lambda_\epsilon := \frac{1}{3\sigma} Ju_\epsilon$ *and* $S = S_\epsilon := \sum_{i\in\widehat{I}} [\![\overline{y_{\widehat{\tau}(i)} x_i}]\!]$ *is the minimizer of* (2.13) *provided by Lemma 2.5, then the segments* $\overline{y_{\widehat{\tau}(i)} x_i} \cap U$, $i \in \widehat{I}$, *are mutually disjoint;*

(v) *it holds that*

$$\|u - u_\epsilon\|_{L^1} + \big||Du|(U) - |Du_\epsilon|(U)\big| + \big|\|Ju\|_{\mathrm{flat,U}} - \|Ju_\epsilon\|_{\mathrm{flat,U}}\big| \le \epsilon. \tag{2.52}$$

To prove this, we will combine the two following lemmas. We start with:

Lemma 2.22. *Suppose* $u \in BV(U; \{\alpha,\beta,\gamma\})$ *is polyhedral. Then there exists a finite set of couples* $\{(x_i, y_i) : i = 1,\dots,N\} \subset \overline{U}\times\overline{U}$ *such that* $Ju = 3\sigma\sum_{i=1}^N(\delta_{x_i} - \delta_{y_i})$. *Furthermore, there is a sequence of polyhedral maps* u_j *with vertices of* S_{u_j} *in* U *of multiplicities at most* 3 *and such that*

$$\begin{aligned} &u_j \to u \qquad \textit{strongly in } BV(U;\mathbb{R}^2),\\ &\|Ju_j - Ju\|_{\mathrm{flat},U} \to 0, \end{aligned} \tag{2.53}$$

as $j \to \infty$.

Proof. We divide the proof into three steps.

Step 1. Let the jump set of u write as $S_u = \cup_{i=1}^m \overline{n_i p_i}$. By Lemma 2.12 the Jacobian determinant of Du has the form

$$Ju = \sigma\sum_{i=1}^m \gamma_i(\delta_{p_i} - \delta_{n_i}),$$

for suitable signs $\gamma_i \in \{\pm 1\}$. Up to switch the notation for p_i and n_i we assume that $\gamma_i = 1$ for all $i = 1,\dots,m$.

A vertex x of multiplicity 2 has null contribution, since in this case $x = p_i = n_j$ for some i, j. Instead, if x is a triple point then, if $x \in U$, by (2.51) we have that its contribution is $\pm 3\sigma\delta_x$. Adding, if necessary, points on ∂U, the lemma is proved if any vertex has multiplicity at most 3, as we can take $u_j := u$ for all $j > 0$.

Step 2: Let us prove the statement for a general u. Let $\{x_k : k = 1,\dots,K\}$, be the family of vertices in U of S_u with multiplicities $m_k \ge 4$, $k = 1,\dots,K$. Let $\epsilon > 0$ be small enough so that $B_\epsilon(x_k)\cap S_u$ consists of m_k radii for all $k = 1,\dots,K$ and let $v \in \{\alpha,\beta,\gamma\}$ be a fixed vector. Let $Q_{\epsilon,k}$ be a closed square with baricenter in x_k with vertices on $\partial B_\epsilon(x_k)$. We define the function

$$u_\epsilon(x) := \begin{cases} u(x) & \text{if } x \in U\setminus \cup_{k=1}^K Q_{\epsilon,k}\\ v & \text{if } x \in \cup_{k=1}^K Q_{\epsilon,k}. \end{cases}$$

The maps $u_\epsilon \in BV(\Omega;\{\alpha,\beta,\gamma\})$ are polyhedral, with vertices at most of multiplicity 3, and satisfy $u_\epsilon \to u$ strongly in $BV(U;\mathbb{R}^2)$ as $\epsilon \to 0$. Furthermore, as u_ϵ and u differ only on $\cup_{k=1}^K Q_{\epsilon,k}$, owing to (2.30), we easily see that $\lambda_u - \lambda_{u_\epsilon}$ is a measure concentrated only on $\bigcup_{k=1}^K (Q_{\epsilon,k}\cap S_u)\cup \bigcup_{k=1}^K \partial Q_{\epsilon,k}$, whose total variation goes to zero as $\epsilon \to 0$. In particular

$$|\lambda_u - \lambda_{u_\epsilon}|(U) \to 0 \qquad \text{as } \epsilon \to 0,$$

and then by Corollary 2.10, $Ju_\epsilon \to Ju$ in the flat norm. The same holds for $u_j := u_{\epsilon_j}$ as $j\to\infty$, with ϵ_j an infinitesimal sequence.

Step 3: We show that the weak Jacobian determinant of u takes the form $Ju = 3\sigma\sum_{i=1}^N(\delta_{x_i} - \delta_{y_i})$. But this is a consequence of Corollary 2.19, applied to the distributions $\frac{1}{3\sigma}Ju_j$. The proof is complete. ☐

Lemma 2.23. *Let $u \in BV(U;\{\alpha,\beta,\gamma\})$ be polyhedral and such that every vertices of S_u in U has multiplicity at most 3. Then for all $\epsilon > 0$ there is a polyhedral map $u_\epsilon \in BV(U;\{\alpha,\beta,\gamma\})$ such that*

$$\|u_\epsilon - u\|_{L^1} + \big||Du_\epsilon|(U) - |Du|(U)\big| + \big|\|Ju_\epsilon\|_{\mathrm{flat,U}} - \|Ju\|_{\mathrm{flat,U}}\big| \le \epsilon, \tag{2.54}$$

and all the vertices of S_{u_ϵ} which are contained in U have multiplicity at most 3 and are three by three not collinear. Moreover, we can find u_ϵ so that, if $\Lambda := \frac{1}{3\sigma} Ju_\epsilon$ and $S = \sum_{i\in\widehat{I}} [\![\overline{y_{\widehat{\tau}(i)}x_i}]\!]$ is the minimizer of (2.13) provided by Lemma 2.5, then the segments $\overline{y_{\widehat{\tau}(i)}x_i} \cap U$, $i \in \widehat{I}$, are mutually disjoint.

Proof. Let $S_u = \cup_{i=1}^m \overline{n_i p_i}$ and denote by $\{z_i,\ i = 1,\dots,n\}$ the points among p_i and n_i which are contained in U. Given $\eta > 0$ small enough, let $T_\eta := \{x \in U : \mathrm{dist}(x, S_u) < \eta\}$ be a tubular neighborhood of S_u; we assume also that the balls $B_\eta(z_i) \subset U$ are mutually disjoint. We will modify u in T_η in order to move the points z_i. For all $i = 1,\dots,n$, by assumption, z_i is either a 2-vertex or a 3-vertex. In the first case, let w_1 and w_2 be the points among the z_i's which are connected to z_i by a segment in S_u; we choose $\widehat{z}_i \in B_\eta(z_i)$ so that $\widehat{z}_i$ does not belong to any line ℓ_{jk} passing through z_j and z_k, with $j,k \ne i$. We can choose $\widehat{z}_i$ arbitrarily close to z_i. We now define $\widehat{u}$ in such a way that it coincides with u outside T_η and has $S_{\widehat{u}}$ which is given by

$$S_{\widehat{u}} = \big(S_u \setminus (\overline{w_1 z_i} \cup \overline{w_2 z_i})\big) \cup (\overline{w_1 \widehat{z}_i} \cup \overline{w_2 \widehat{z}_i}).$$

If η is small enough, it is easily seen that $\widehat{u}$ is uniquely determined, and it holds

$$\|\widehat{u} - u\|_{L^1} + ||D\widehat{u}|(U) - |Du|(U)| \le C\eta,$$

for some constant $C > 0$ independent of η. Furthermore, we also estimate

$$\big|\|J\widehat{u}\|_{\mathrm{flat,U}} - \|Ju\|_{\mathrm{flat,U}}\big| \le 6\sigma|z_i - \widehat{z}_i| \le C\eta.$$

In the case that z_i is a triple point, we proceed in the same way and define $\widehat{u}$ so that

$$S_{\widehat{u}} = \big(S_u \setminus (\overline{w_1 z_i} \cup \overline{w_2 z_i} \cup \overline{w_3 z_i})\big) \cup (\overline{w_1 \widehat{z}_i} \cup \overline{w_2 \widehat{z}_i} \cup \overline{w_3 \widehat{z}_i}),$$

where w_1, w_2, and w_3, are the vertices of S_u linked to z_i with a segment in S_u. Similar estimates lead to

$$\|\widehat{u} - u\|_{L^1} + \big||D\widehat{u}|(U) - |Du|(U)\big| + \big|\|J\widehat{u}\|_{\mathrm{flat,U}} - \|Ju\|_{\mathrm{flat,U}}\big| \le C\eta.$$

Then we iterate the construction moving every z_i, for $i = 1,\dots,n$. The thesis then follows by fixing $\epsilon > 0$, and choosing η small enough in order that (2.54) holds for $u_\epsilon = \widehat{u}$.

Eventually, let $\Lambda = \frac{1}{3\sigma} J\widehat{u} = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i})$ be a representation satisfying (P) and let $S = \sum_{i\in\widehat{I}} [\![\overline{y_{\widehat{\tau}(i)}x_i}]\!]$ be the minimizer of (2.13) provided by Lemma 2.5. If $\overline{y_{\widehat{\tau}(i)}x_i} \cap \overline{y_{\widehat{\tau}(j)}x_j} \cap U \ne \varnothing$ for some $i,j \in \widehat{I}$, $i \ne j$, then by Lemma 2.5 (iii) either (a) or (b) holds, and hence it means that the four points $y_{\widehat{\tau}(i)}, x_i, y_{\widehat{\tau}(j)}, x_j$ are on the same line. So, if for instance $x_i \in U$ (if not, necessarily $y_{\widehat{\tau}(i)} \in U$), it is sufficient to repeat the preceeding procedure to move x_i (respectively, $y_{\widehat{\tau}(i)}$) a bit in order that it is not aligned with any segment $\overline{y_k x_h}$, $k,h \in \{1,\dots,n\}$, $h \ne i$. □

We are now ready to prove Theorem 2.21.

Proof of Theorem 2.21. Given $u \in BV(U;\{\alpha,\beta,\gamma\})$ and $\epsilon > 0$, we use the approximation result given by Theorem 2.20, and combine this with Lemma 2.22 and Lemma 2.23. □

Corollary 2.24. *Let U be a Lipschitz domain and let $u \in BV(U;\{\alpha,\beta,\gamma\})$. Then $\frac{1}{3\sigma}Ju \in X(U)$.*

Proof. We denote by $u \in BV_{\rm loc}(\mathbb{R}^2;\{\alpha,\beta,\gamma\})$ itself an extension of u as given by Theorem 2.15, and let $u_j \in BV_{\rm loc}(\mathbb{R}^2;\{\alpha,\beta,\gamma\})$ be polyhedral maps as in Corollary 2.18. Thanks to Lemma 2.12 the restrictions to U of the maps $u_j \circ f_j$ enjoy $J(u_j\circ f_j) \in X(U)$; hence, since $J(u_j \circ f_j) \to Ju$ with respect to the flat distance, we conclude by Corollary 2.19. □

3 Proof of the main results

In this section we prove Theorems 1.1 and 1.2. Throughtout the section $\Omega \subset \mathbb{R}^2$ denotes a Lipschitz domain.

We start with the following:

Proposition 3.1. *Let $u \in BV(\Omega;\{\alpha,\beta,\gamma\})$ be a polyhedral map such that $S_u := \cup_{i=1}^N \overline{n_i p_i}$ satisfies:*

(1) the points z_i's in the family $\{z_i : i = 1,\dots,m\}$ of vertices n_i, p_i, which belong to Ω, are three by three not collinear;

(2) the multiplicity of each vertex of $S_u \cap \Omega$ is at most 3.

Let $\Lambda := \frac{1}{3\sigma}Ju = \sum_{i=1}^n(\delta_{x_i} - \delta_{y_i})$, where we assume that this representation satisfies hypothesis (P), let $S \in \mathcal{S}(U)$ be a minimizer of (2.13) provided by Lemma 2.5, $S = \sum_{i\in\widehat{I}} [\![\overline{y_{\widehat{\tau}(i)}x_i}]\!]$, and suppose that

(3) the segments $\overline{y_{\widehat{\tau}(i)}x_i} \cap \Omega$, $i \in \widehat{I}$, are mutually disjoint.

Set $l_i := |x_i - y_{\widehat{\tau}(i)}|$, $i \in \widehat{I}$; then there exists a sequence $(u_k) \subset BV(\Omega;\{\alpha,\beta,\gamma\})$ of polyhedral maps satisfying:

(i) the multiplicities of the vertices of $S_{u_k} \cap \Omega$ are at most 2;

(ii) $u_k \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$ as $k \to \infty$, and

$$\liminf_{k\to\infty} |Du_k|(\Omega) \le l\mathcal{H}^1(S_u) + 3l\sum_{i\in\widehat{I}} l_i = |Du|(\Omega) + 4\|Ju\|_{\mathrm{flat},\Omega}.$$

Proof. Let $\{w_i : i = 1,\dots,K\}$ be the points among $\{x_i, y_i : i = 1,\dots,n\}$ which are contained in Ω. By assumption, all such points must be triple points. Let $\eta > 0$ be small enough so that the balls $B_\eta(w_i)$, $i = 1,\dots,K$ are mutually disjoint, contained in Ω, and such that $B_\eta(w_i) \cap S_u$ consists of three radii.

To prove the thesis, for all $\epsilon > 0$ small enough we show that there exists a polyhedral map $u_\epsilon \in BV(\Omega;\{\alpha,\beta,\gamma\})$ satisfying (i) and such that

$$|Du_\epsilon|(\Omega) \le l\mathcal{H}^1(S_u) + 3l\sum_{i=1}^n l_i + O(\epsilon), \tag{3.1}$$

where $O(\epsilon) \to 0$ as $\epsilon \to 0$.

First we observe that, thanks to hypotheses (1), (2) and (3), by (iii) of Lemma 2.5, the segments $\overline{y_{\widehat{\tau}(i)}x_i} \cap \Omega$ are mutually disjoint and they can share only endpoints lying on $\partial\Omega$. Moreover, thanks again to (1), for all $i \in \widehat{I}$, the segment $\overline{y_{\widehat{\tau}(i)}x_i}$ enjoys one and only one of the following:

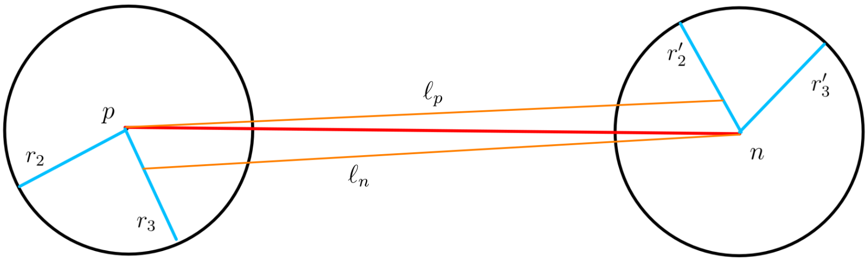

Figure 1: Construction in Step 1, a subcase of (a).

(a) $\overline{y_{\widehat{\tau}(i)}x_i}$ is contained in a segment $\overline{n_jp_j}$, for some $j = 1, \dots, N$;

(b) for any $j = 1, \dots, N$, the intersection $\overline{y_{\widehat{\tau}(i)}x_i} \cap \overline{n_jp_j}$ is either empty or is a single point.

To construct u_ϵ we will recursively modify u in a tubular neighborhood of the segment $\overline{y_{\widehat{\tau}(i)}x_i}$, for all $i \in \widehat{I}$.

Step 1. In this step we describe how to modify u around the segment $\overline{y_{\widehat{\tau}(i)}x_i}$ in case that (a) above holds. To simplify the notation, set $p := x_i$, $n := y_{\widehat{\tau}(i)}$. We first discuss the case in which $p, n \in \Omega$: Denote $T_\eta(\overline{np}) := \{x \in \mathbb{R}^2 : \text{dist}(x, \overline{np}) < \eta\}$. Up to taking η small enough, we can suppose that $T_\eta(\overline{np}) \subset \Omega$ and that u takes only two values in $(T_\eta(\overline{np}) \setminus B_\eta(p)) \setminus B_\eta(n)$, say α and β. Further, $\overline{B_\eta(p)} \cap S_u$ consists of three radii, r_1, r_2, and r_3 chosen in counterclockwise order around p, and with $r_1 \subset \overline{np}$. Similarly, we note by r'_1, r'_2, and r'_3 the three radii of $\overline{B_\eta(n)} \cap S_u$ chosen in clockwise order around n and $r'_1 \subset \overline{np}$. Let (x, y) be a Cartesian coordinate system so that $\overline{np} \subset \{y = 0\}$. Assume first that either for $\overline{y} > 0$ small enough the line $y = \overline{y}$ intersects both r_2 and r'_2, or for $\overline{y} < 0$ small enough the line $y = \overline{y}$ intersects both r_3 and r'_3. Suppose without loss of generality we are in the first case, and denote by p' and n' the corresponding intersections between $y = \overline{y}$ and r_2 and r'_2, respectively. Finally we set

$$u_\epsilon := \begin{cases} \gamma & \text{in } Q_{pnn'p'}, \\ u & \text{elsewhere in } \Omega, \end{cases} \tag{3.2}$$

where $Q_{pnn'p'}$ is the quadrilateral with vertices p, n, n', and p'. The new family of triple points of u_ϵ is $\{w_i : i = 1, \dots, K\} \setminus \{p, n\}$, and it is straightforward to check that

$$\begin{aligned} &\|u - u_\epsilon\|_{L^1} \leq \eta l_i + O(\eta), \\ &|Du_\epsilon|(\Omega) \leq |Du|(\Omega) + 2l l_i + O(\eta). \end{aligned} \tag{3.3}$$

Assume now that no lines $y = \overline{y}$ intersect both r_2 and r'_2, or both r_3 and r'_3. This means that r_2 and r_3 are contained in $\{y \leq 0\}$, and r'_2 and r'_3 in $\{y \geq 0\}$ (or viceversa). Let $v \in \{\alpha, \beta, \gamma\}$ be the value that u takes in the circular sector enclosed by r_2 and r_3 (which is the same value in the sector between r'_2 and r'_3). Let us assume that $v = \gamma$. Then we build two parallel segments ℓ_p and ℓ_n, originating from p and n respectively, and ending at points p' and n' on r'_2 and r_3, respectively (see Figure 1). We then set u_ϵ as in (3.2). Also in this case the triple points p and n disappear, and (3.3) holds true.

It remains to discuss the case in which one among x_i and $y_{\widehat{\tau}(i)}$ belongs to $\partial\Omega$. As before, set $p := x_i$, $n := y_{\widehat{\tau}(i)}$, assume that $n \in \partial\Omega$, and that u takes the values α and β in $(T_\eta(\overline{np}) \setminus B_\eta(p)) \setminus$

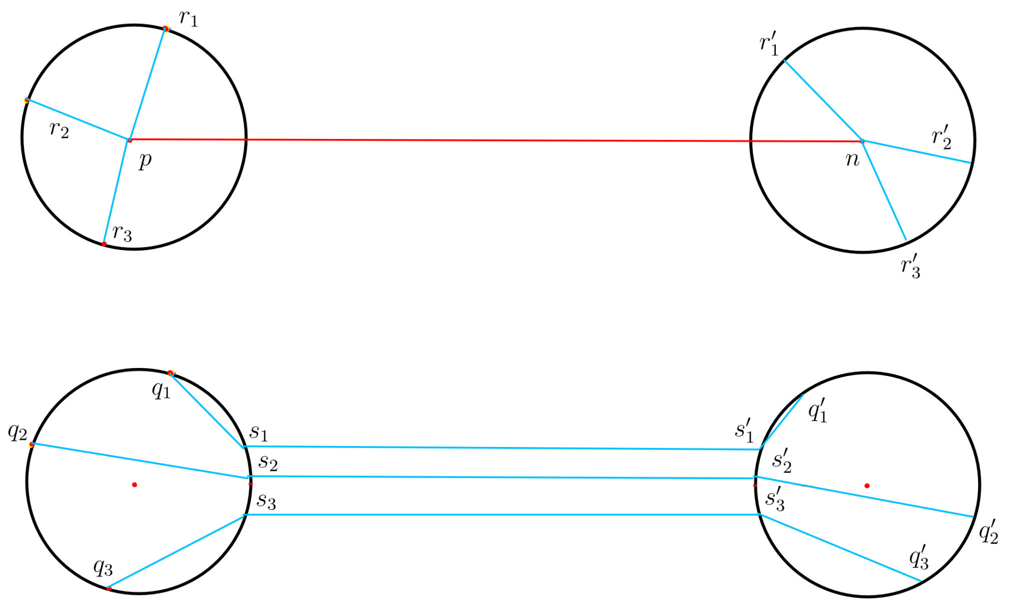

Figure 2: Construction in Step 2, case $M = 2$.

$B_\eta(n)$. Let $\overline{y}$ and p' be as above and choose a point $n' \in \{y = \overline{y}\} \cap \big(T_\eta(\overline{np}) \setminus B_\eta(p) \setminus B_\eta(n)\big)$; then we define u_ϵ as in (3.2). It is easy to see that the triple points of u_ϵ in Ω are in the family $\{w_i : i = 1, \ldots, K\} \setminus \{p\}$, and that (3.3) holds.

Step 2. Let us now treat case (b). We have to distinguish the cases in which both the points $p := x_i$, $n := y_{\widehat{\tau}(i)}$ belong to Ω and in which one of them is on $\partial\Omega$. Let us treat the first one: Thanks to (b), the segment $\overline{np}$ intersects S_u in a finite set of $M \geq 2$ points, containing p and n.

Suppose first that $M = 2$, i.e., $\overline{np}$ does not intersect S_u in its interior. Let, as in Step 1, r_1, r_2, r_3, and r'_1, r'_2, r'_3 be radii in $B_\eta(p)$ and $B_\eta(n)$ respectively, contained in S_u and with $\overline{np} \cap B_\eta(p)$, r_1, r_2, and r_3 chosen in counterclockwise order around p, and $\overline{np} \cap B_\eta(n)$, r'_1, r'_2, and r'_3 chosen in clockwise order around n. Let q_j be the endpoint of r_j on $\partial B_\eta(p)$ and let q'_j the endpoint of r'_j on $\partial B_\eta(n)$, for $j = 1, 2, 3$. We choose three points (in clockwise order) s_j, $j = 1, 2, 3$, between q_1 and q_3, on $\partial B_\eta(p)$, and we choose three points (in counterclockwise order) s'_j, $j = 1, 2, 3$, between q'_1 and q'_3, on $\partial B_\eta(n)$. We can choose the points s_j and s'_j close to the segment $\overline{np}$ in such a way that the three segments $\overline{s_j s'_j}$, $j = 1, 2, 3$, are parallel to $\overline{np}$ (see Figure 2).

Since p and n are two triple points, one positively and one negatively oriented, and since $\overline{np}$ does not intersect any other point in S_u except p and n, the map u takes the same values on the arcs[7] $\widehat{q_j q_{j+1}}$ and $\widehat{q'_j q'_{j+1}}$, for $j = 1, 2, 3$. Assume, without loss of generality, that $u = \alpha$ on $\widehat{q_3 q_1}$, $u = \beta$ on $\widehat{q_1 q_2}$, and $u = \gamma$ on $\widehat{q_2 q_3}$; then we define

$$u_\epsilon := \begin{cases} \beta & \text{on } T_{12} \cup S_{12} \cup S'_{12} \\ \gamma & \text{on } T_{23} \cup S_{23} \cup S'_{23} \\ u & \text{elsewhere,} \end{cases} \tag{3.4}$$

[7]These arcs are intended on $\partial B_\eta(p)$ and $\partial B_\eta(n)$ respectively, j is intended mod 3, and it is intended that $\widehat{q_j q_{j+1}}$ does not contain q_{j+2}.

where T_{12} is the region in $(\Omega \setminus B_\eta(p)) \setminus B_\eta(n)$ enclosed by $\overline{s_1 s_1'}$ and $\overline{s_2 s_2'}$, T_{23} is the region in $(\Omega \setminus B_\eta(p)) \setminus B_\eta(n)$ enclosed by $\overline{s_2 s_2'}$ and $\overline{s_3 s_3'}$, $S_{12} \subset B_\eta(p)$ is enclosed between $\overline{q_1 s_1}$ and $\overline{q_2 s_2}$, $S_{12}' \subset B_\eta(n)$ is enclosed between $\overline{q_1' s_1'}$ and $\overline{q_2' s_2'}$, and similarly S_{23} and S_{23}'. It is not difficult to see that the new map u_ϵ has not triple points in $B_\eta(p)$ and $B_\eta(n)$ (and neither we have added other triple points). Moreover, it is easily checked that

$$\begin{aligned} &\|u - u_\epsilon\|_{L^1} \leq \eta l_i + O(\eta), \\ &|Du_\epsilon|(\Omega) \leq |Du|(\Omega) + 3l l_i + O(\eta). \end{aligned} \tag{3.5}$$

Let us now discuss how to modify u in the case that $M > 2$. The procedure is the same: we choose the points s_1, s_2, and s_3 as before in $\partial B_\eta(p)$, and then, denoting by ℓ the first segment in S_u, starting from p, that $\overline{np}$ intersects, we choose s_1', s_2', and s_3' on ℓ in such a way that the segments $\overline{s_j s_j'}$, $j = 1, 2, 3$, are parallel to $\overline{np}$. Assuming, as before, that $u = \alpha$ on $\widehat{q_3 q_1}$, $u = \beta$ on $\widehat{q_1 q_2}$, and $u = \gamma$ on $\widehat{q_2 q_3}$, we define

$$u_\epsilon := \begin{cases} \beta & \text{on } T_{12} \cup S_{12} \\ \gamma & \text{on } T_{23} \cup S_{23} \\ u & \text{elsewhere,} \end{cases} \tag{3.6}$$

where T_{12} is the region in $\Omega \setminus B_\eta(p)$ enclosed by $\overline{s_1 s_1'}$, $\overline{s_2 s_2'}$, and ℓ, T_{23} is the region in $\Omega \setminus B_\eta(p)$ enclosed by $\overline{s_2 s_2'}$, $\overline{s_3 s_3'}$, and ℓ, S_{12} and S_{23} defined as before. With this definition, u_ϵ has no anymore triple points in $B_\eta(p)$, but has a new triple point p' either in s_1' or in s_3' (and positively oriented). However, we notice also that the segment $\overline{p'n}$, if η was chosen small enough, does intersect S_u in exactly $(M-1)$ points. Then we can repeat the argument above, inductively, starting from the segment $\overline{p'n}$, and redefining u_ϵ $(M-1)$ times, up to erase also the triple point in n. We easily check that, at the end of the procedure, also in this case we have the estimate (3.5).

To conclude Step 2, we have to describe how to modify u in the case that one point among p and n belongs to $\partial\Omega$. Assume without loss of generality that $n \in \partial\Omega$, then if $M > 2$ we proceed as in the previous case. So we have only to specify how to modify u in the case $M = 2$. In this case, we proceed as before defining u_ϵ as in (3.6), but with the difference that we choose the points $s_1' = s_2' = s_3' = n \in \partial\Omega$. Also in this case (3.5) still holds.

Step 3. We iterate the procedure described in Step 1 and Step 2 for all couples $(x_i, y_{\widehat{\tau}(i)})$. In the end, summing the estimates (3.3) and (3.5) for all $i \in \widehat{I}$, by the triangle inequality we can choose η small enough so that u_ϵ satisfies (3.1). The thesis follows. □

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1. Let $u \in BV(\Omega; \{\alpha, \beta, \gamma\})$ be given; we have to prove that

$$\overline{\mathcal{A}}(u, \Omega) \leq |\Omega| + |Du|(\Omega) + 4\|Ju\|_{\text{flat},\Omega}. \tag{3.7}$$

By Theorem 2.15 there exists an extension $\overline{u} \in BV_{\text{loc}}(\mathbb{R}^2; \{\alpha, \beta, \gamma\})$ of u such that $|D\overline{u}|(\partial\Omega) = 0$. For $\delta \in (0, 1)$, we consider the δ-neighborhood Ω_δ of Ω, defined as in (2.1). Since $|D\overline{u}|(\partial\Omega) = 0$, we have

$$|D\overline{u}|(\Omega_\delta \setminus \Omega) \to 0 \qquad \text{as } \delta \to 0. \tag{3.8}$$

Since δ is arbitrary, by Lemma 4.1 we can assume that Ω_δ is a Lipschitz domain (and we can take it as small as we want).

Furthermore, thanks to Theorem 2.17 and Lemma 2.7, we have

$$\|J\overline{u}\|_{\mathrm{flat},\Omega_\delta} \to \|J\overline{u}\|_{\mathrm{flat},\Omega} = \|Ju\|_{\mathrm{flat},\Omega} \qquad \text{as } \delta \to 0. \tag{3.9}$$

For all $\delta > 0$ small enough, and all $k > 0$, we use Theorem 2.21 to find a polyhedral map $u_{\delta,k} \in BV(\Omega_\delta; \{\alpha,\beta,\gamma\})$ satisfying (1), (2), and (3) of Proposition 3.1, and such that

$$\|\overline{u} - u_{\delta,k}\|_{L^1(\Omega_\delta)} + \big||D\overline{u}|(\Omega_\delta) - |Du_{\delta,k}|(\Omega_\delta)\big| + \big|\|J\overline{u}\|_{\mathrm{flat},\Omega_\delta} - \|Ju_{\delta,k}\|_{\mathrm{flat},\Omega_\delta}\big| \le \frac{1}{k}. \tag{3.10}$$

In turn, by Proposition 3.1, for all such δ and k there exists a polyhedral map $\widehat{u}_{\delta,k}$ with the vertices of $S_{\widehat{u}_{\delta,k}}$ in Ω having multiplicities at most 2, and such that

$$\|\widehat{u}_{\delta,k} - u_{\delta,k}\|_{L^1(\Omega_\delta)} \le \frac{1}{k}, \qquad |D\widehat{u}_{\delta,k}|(\Omega_\delta) \le |Du_{\delta,k}|(\Omega_\delta) + 4\|Ju_{\delta,k}\|_{\mathrm{flat},\Omega_\delta} + \frac{1}{k}. \tag{3.11}$$

Moreover, using the density result in Lemma 2.23 we can assume that $|D\widehat{u}_{\delta,k}|(\partial\Omega) = 0$ for all δ and $k > 0$. From (3.10) and (3.11) it follows that

$$\|\widehat{u}_{\delta,k} - u\|_{L^1(\Omega)} \le \frac{2}{k}, \qquad |D\widehat{u}_{\delta,k}|(\Omega) \le |D\overline{u}|(\Omega_\delta) + 4\|J\overline{u}\|_{\mathrm{flat},\Omega_\delta} + \frac{2}{k}. \tag{3.12}$$

We now invoke [1, Theorem 3.14], which implies that

$$\overline{\mathcal{A}}(\widehat{u}_{\delta,k},\Omega) = |\Omega| + |D\widehat{u}_{\delta,k}|(\Omega) \le |\Omega| + |D\overline{u}|(\Omega_\delta) + 4\|J\overline{u}\|_{\mathrm{flat},\Omega_\delta} + \frac{2}{k}.$$

and so, letting $k \to \infty$, by lower semicontinuity of $\overline{\mathcal{A}}$, we infer

$$\overline{\mathcal{A}}(u,\Omega) \le |\Omega| + |D\overline{u}|(\Omega_\delta) + 4\|J\overline{u}\|_{\mathrm{flat},\Omega_\delta}. \tag{3.13}$$

Finally, using (3.8) and (3.9), we can let $\delta \to 0$ and conclude (3.7). The thesis is achieved. □

Remark 3.2. Notice that for the functions $\widehat{u}_{\delta,k}$ we can extract a sequence $\delta_k \searrow 0$, such that, as $k \to \infty$, it holds

$$\begin{aligned}
&\widehat{u}_k := \widehat{u}_{\delta_k,k} \to u \qquad \text{in } L^1(\Omega),\\
&\limsup_{k\to\infty} |D\widehat{u}_k|(\Omega) \le |Du|(\Omega) + 4\|Ju\|_{\mathrm{flat},\Omega},\\
&\limsup_{k\to\infty} \overline{\mathcal{A}}(\widehat{u}_k,\Omega) \le |\Omega| + |Du|(\Omega) + 4\|Ju\|_{\mathrm{flat},\Omega}.
\end{aligned}$$

Moreover, any map $\widehat{u}_k$ admits an extension to Ω_δ which is polyhedral with, for all the vertices of the jump contained in Ω_δ, multiplicities at most 2. This observation will be useful in the sequel.

We now focus on the proof of Theorem 1.2. We recall that if $u \in BV(\Omega; \{\alpha,\beta,\gamma\})$, then by Corollary 2.24 we have

$$\frac{1}{3\sigma} Ju = \sum_{i=1}^{\infty} (\delta_{x_i} - \delta_{y_i}),$$

for suitable points $x_i, y_i \in \overline{\Omega}$, such that $\sum_{i=1}^{\infty} d_\Omega(x_i,y_i) < +\infty$.

Proof of Theorem 1.2. We divide the proof into two steps.

Step 1(Proof of (1.11)*).* For all $\epsilon > 0$ small enough we fix $N > 0$ so that

$$\sum_{i=N+1}^{\infty} d_\Omega(x_i, y_i) < \epsilon. \tag{3.14}$$

This in particular implies that, setting $\Lambda_\epsilon := 3\sigma \sum_{i=N+1}^{\infty}(\delta_{x_i} - \delta_{y_i})$, we have

$$\|\Lambda_\epsilon\|_{\mathrm{flat},\Omega} \le 3\sigma\epsilon. \tag{3.15}$$

Let us denote by $\{w_k : k = 1, \dots, m\} = \{x_i \in \Omega,\ y_j \in \Omega,\ i \le N, j \le N\}$ the family of points x_i, y_j (with $i, j \le N$) which are contained in Ω. We can choose $r > 0$ small enough so that the closed balls $\overline{B_{2r}}(w_k)$, $k = 1, \dots, m$ are contained in Ω and are mutually disjoint. It turns out that the domain $U_r := \Omega \setminus (\cup_{k=1}^m \overline{B_r}(w_k))$ is a Lipschitz domain, so that we can apply Theorem 1.1 and obtain

$$\overline{\mathcal{A}}(u, U_r) \le |U_r| + |Du|(U_r) + 4\|Ju\|_{\mathrm{flat},U_r}. \tag{3.16}$$

Notice that $Ju \llcorner U_r = \Lambda_\epsilon \llcorner U_r$ in $\mathcal{D}'(U_r)$, so from (3.15) we readily infer

$$\|Ju\|_{\mathrm{flat},U_r} \le 3\sigma\epsilon. \tag{3.17}$$

On the other hand, denoting $D^r := \cup_{k=1}^m B_{2r}(w_k)$, by Theorem 2.9 we also deduce that

$$\|Ju\|_{\mathrm{flat},D^r} \le C|Du|(D^r) \to 0 \text{ as } r \to 0^+,$$

so we choose $r > 0$ small enough in order that

$$\|Ju\|_{\mathrm{flat},D^r} \le \epsilon. \tag{3.18}$$

Therefore, again Theorem 1.1 implies that

$$\overline{\mathcal{A}}(u, D^r) \le |D^r| + |Du|(D^r) + 4\epsilon. \tag{3.19}$$

Eventually, by definition of $\overline{\overline{\mathcal{A}}}$, by (3.16), (3.17), and (3.19), we get

$$\overline{\overline{\mathcal{A}}}(u, \Omega) \le |U_r| + |Du|(U_r) + |D^r| + |Du|(D^r) + o_\epsilon(1), \tag{3.20}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0^+$. Since (3.20) holds for all $r > 0$ small enough, we conclude that $\overline{\overline{\mathcal{A}}}(u, \Omega) \le |\Omega| + |Du|(\Omega) + o_\epsilon(1)$, and thus

$$\overline{\overline{\mathcal{A}}}(u, \Omega) \le |\Omega| + |Du|(\Omega),$$

by arbitrariness of $\epsilon > 0$. The opposite inequality simply follows from the fact that

$$\overline{\mathcal{A}}(u; U) \ge |U| + |Du|(U),$$

for any open set U and any $u \in BV(U; \{\alpha, \beta, \gamma\})$, as a consequence of (1.4).

Step 2. As in Step 1, we fix $\epsilon > 0$ and $N > 0$ so that (3.14), (3.15), and (3.17) hold. Furthermore, in the Lipschitz domain $U_r = \Omega \setminus (\cup_{h=1}^m \overline{B_r}(w_h))$ we consider a sequence $\widehat{u}_k$ as in Remark 3.2 (applied with Ω replaced by U_r). To emphasize the dependence of this sequence on

$r > 0$ we denote such maps $u^r_k := \widehat{u}_k$. We recall that u^r_k admits an extension on a neighborhood of U_r which is polyhedral with vertices of $S_{u^r_k}$ in U_r of multiplicity at most 2, and

$$\limsup_{k\to\infty} \overline{\mathcal{A}}(u^r_k, U_r) \leq |U_r| + |Du|(U_r) + 4\|Ju\|_{\mathrm{flat},U_r} \leq |U_r| + |Du|(U_r) + 12\sigma\epsilon, \tag{3.21}$$

for all $r > 0$ sufficiently small. In particular

$$\overline{\mathcal{A}}(u^r_k, U_r) \leq |U_r| + |Du|(U_r) + 12\sigma\epsilon + o_k(1), \tag{3.22}$$

where $o_k(1) \to 0$ as $k \to \infty$. On the other hand

$$\liminf_{k\to\infty} \overline{\mathcal{A}}(u^r_k, U_{2r}) \geq |U_{2r}| + |Du|(U_{2r}), \tag{3.23}$$

where $U_{2r} = \Omega \setminus (\cup_{h=1}^m \overline{B_{2r}}(w_h))$, so

$$\overline{\mathcal{A}}(u^r_k, U_{2r}) \geq |U_{2r}| + |Du|(U_{2r}) + o'_k(1), \tag{3.24}$$

where $o'_k(1) \to 0$ as $k \to \infty$.

Fixing r and k, we can find a sequence $(v^{r,k}_j) \subset C^1(U_r;\mathbb{R}^2) \cap W^{1,\infty}(U_r;\mathbb{R}^2)$ such that

$$\begin{aligned} &v^{r,k}_j \to u^r_k \qquad \text{in } L^1(U_r;\mathbb{R}^2) \text{ as } j \to \infty, \\ &\lim_{j\to\infty} \mathbb{A}(v^{r,k}_j, U_r) = \overline{\mathcal{A}}(u^r_k, U_r) = |U_r| + |Du^r_k|(U_r), \end{aligned} \tag{3.25}$$

where the last equality follows from [1, Theorem 3.14]. Combining with (3.21) and (3.24), we have

$$\begin{aligned} |U_r \setminus U_{2r}| + |Du^r_k|(U_r \setminus U_{2r}) &\leq \lim_{j\to\infty} \mathbb{A}(v^{r,k}_j, U_r \setminus U_{2r}) = \lim_{j\to\infty} \Big(\mathbb{A}(v^{r,k}_j, U_r) - \mathbb{A}(v^{r,k}_j, U_{2r})\Big) \\ &\leq \overline{\mathcal{A}}(u^r_k, U_r) - \liminf_{j\to\infty} \mathbb{A}(v^{r,k}_j, U_{2r}) \leq \overline{\mathcal{A}}(u^r_k, U_r) - \overline{\mathcal{A}}(u^r_k, U_{2r}) \\ &\leq |U_r| + |Du|(U_r) + 12\sigma\epsilon - |U_{2r}| - |Du|(U_{2r}) + \eta_k \\ &= |U_r \setminus U_{2r}| + |Du|(U_r \setminus U_{2r}) + 12\sigma\epsilon + \eta_k, \end{aligned}$$

where $\eta_k \to 0$ as $k \to \infty$. From this we get

$$|Du^r_k|(U_r \setminus U_{2r}) \leq |Du|(U_r \setminus U_{2r}) + 12\sigma\epsilon + \eta_k. \tag{3.26}$$

Next we observe that, since $v^{r,k}_j$ is built by mollification (see [1, Theorem 3.14] for details), it also follows that

$$v^{r,k}_j \to u^r_k \qquad \text{strictly in } BV(U_r;\mathbb{R}^2) \text{ as } j \to \infty. \tag{3.27}$$

By Fubini and the mean value theorems we can find a set $I_h \subset (r, 2r)$ of positive measure such that for all $\rho \in I_h$ we have

$$|D\widetilde{u}^r_k|(\partial B_\rho(w_h)) \leq \frac{1}{r}|Du^r_k|(B_{2r}(w_h) \setminus B_r(w_h)), \tag{3.28}$$

where we have denoted by $\widetilde{u}^r_k \in BV(\partial B_\rho(w_h);\mathbb{R}^2)$ the trace[8] of u^r_k on $\partial B_\rho(w_h)$. By (3.27), applying [4, Lemma 2.5], for all $h = 1, \ldots, m$ we can find $r_h \in I_h$ such that (up to extracting a subsequence)

$$v^{r,k}_j \to \widetilde{u}^r_k \qquad \text{strictly in } BV(\partial B_{r_h}(w_h);\mathbb{R}^2) \qquad \text{as } j \to \infty, \qquad \forall h = 1, \ldots, m. \tag{3.29}$$

[8] This coincides with the restriction of u^r_k for a.e. $\rho > 0$, and we can assume this is a BV function.

For all j we now define

$$\overline{v}_j^{r,k}(x) = \begin{cases} v_j^{r,k}(x) & \text{if } x \in \Omega \setminus (\cup_{h=1}^m B_{r_h}(w_h)) \\ v_j^{r,k}\left(w_h + r_h \frac{x-w_h}{|x-w_h|}\right) & \text{if } x \in B_{r_h}(w_h) \setminus \{w_h\},\ h = 1, \dots, m. \end{cases} \tag{3.30}$$

Hence we have $\overline{v}_j^{r,k} \in W^{1,\infty}_{\rm loc}(\Omega \setminus \{w_h : h = 1, \dots, m\}; \mathbb{R}^2)$, it is piecewise C^1-regular, and furthermore, for $x \in B_{r_h}(w_h) \setminus \{w_h\}$,

$$\nabla \overline{v}_j^{r,k}(x) = \nabla v_j^{r,k}\left(w_h + r_h \frac{x-w_h}{|x-w_h|}\right) \nabla \left(r_h \frac{x-w_h}{|x-w_h|}\right),$$

whose Jacobian determinant is null as $\det\left(\nabla\left(\frac{x-w_h}{|x-w_h|}\right)\right) = 0$, and

$$|\nabla \overline{v}_j^{r,k}(x)| \le \left|\nabla v_j^{r,k}\left(w_h + \frac{x-w_h}{|x-w_h|}\right)\right| \frac{r_h}{|w_h - x|}.$$

As a consequence

$$\begin{aligned} \mathbb{A}(\overline{v}_j^{r,k}, \cup_{h=1}^m (B_{r_h}(w_h) \setminus \{w_h\})) &= \int_{\cup_{h=1}^m B_{r_h}(w_h)} \sqrt{1 + |\nabla \overline{v}_j^{r,k}|^2}\, dx \le |\cup_{h=1}^m B_{r_h}(w_h)| + \int_{\cup_{h=1}^m B_{r_h}(w_h)} |\nabla \overline{v}_j^{r,k}| dx \\ &\le 4\pi r^2 m + \sum_{h=1}^m \int_0^{2\pi} \int_0^{r_h} r_h \left|\nabla v_j^{r,k}(w_h + r_h(\cos\theta, \sin\theta))\right| d\rho d\theta \\ &= 4\pi r^2 m + \sum_{h=1}^m \int_0^{r_h} \int_{\partial B_{r_h}} |\nabla \overline{v}_j^{r,k}| d\mathcal{H}^1 d\rho \\ &= 4\pi r^2 m + \sum_{h=1}^m r_h |D\widetilde{u}_k^r|(\partial B_{r_h}(w_h)) + o_j(1), \end{aligned} \tag{3.31}$$

where, using (3.29), we have $o_j(1) \to 0$ as $j \to \infty$. In turn, from (3.26), (3.28), and since $r_h \in I_h$, we infer

$$\begin{aligned} \mathbb{A}(\overline{v}_j^{r,k}, \cup_{h=1}^m (B_{r_h}(w_h) \setminus \{w_h\})) &\le 4\pi r^2 m + |Du_k^r|(U_r \setminus U_{2r}) + o_j(1) \\ &\le 4\pi r^2 m + |Du|(U_r \setminus U_{2r}) + 12\sigma\epsilon + \eta_k + o_j(1). \end{aligned} \tag{3.32}$$

Eventually, by (3.25) and using (3.22), this implies

$$\begin{aligned} \mathbb{A}(\overline{v}_j^{r,k}, \Omega \setminus \{(w_h) : h = 1, \dots, m\}) &\le \mathbb{A}(\overline{v}_j^{r,k}, U_r) + \mathbb{A}(\overline{v}_j^{r,k}, \cup_{h=1}^m (B_{r_h}(w_h) \setminus \{w_h\})) \\ &\le \overline{\mathcal{A}}(u_k^r, U_r) + 4\pi r^2 m + |Du|(U_r \setminus U_{2r}) + 12\sigma\epsilon + \widehat{\eta}_k + o_j(1) \\ &\le |\Omega| + |Du|(\Omega) + 3\pi r^2 m + 12\sigma\epsilon + \widehat{\eta}_k + o_j(1), \end{aligned} \tag{3.33}$$

where $\widehat{\eta}_k \to 0$ as $k \to \infty$. Hence we conclude, by a diagonal argument, that there exists a sequence $v_k := \overline{v}_{j(k)}^{r,k} \in W^{1,\infty}_{\rm loc}(\Omega \setminus \{w_h : h = 1, \dots, m\}; \mathbb{R}^2)$, piecewise C^1-regular on $\Omega \setminus \{w_h : h = 1, \dots, m\}$, such that $v_k \to u$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$\liminf \mathbb{A}(\overline{v}_k, \Omega \setminus \{(w_h) : h = 1, \dots, m\}) \le |\Omega| + |Du|(\Omega) + 12\sigma\epsilon. \tag{3.34}$$

This is sufficient to ensure that

$$\overline{\mathcal{A}}(u, \Omega \setminus \{(w_h) : h = 1, \dots, m\}) \le |\Omega| + |Du|(\Omega) + 12\sigma\epsilon. \tag{3.35}$$

Therefore, for all $\epsilon > 0$ we have found a finite set of points $C_\epsilon := \{(w_h) : h = 1, \dots m\}$ such that (3.35) holds, and we conclude that the right-hand side of (1.12) coincides with $|\Omega| + |Du|(\Omega)$. This is exactly $\overline{\overline{\mathcal{A}}}(u, \Omega)$ as proved in Step 1, so the thesis is achieved. □

4 Appendix

We collect here two useful observations. The first one consists in the following lemma, whose content can be found in [22] (see also references therein):

Lemma 4.1. *Let $U \subset \mathbb{R}^2$ be a relatively compact set; then for a.e. $\delta > 0$ the δ-neighborhood U_δ of U has Lipschitz boundary.*

As a second remark, we see that we can equivalently relax the area functional using $W^{1,\infty}_{\mathrm{loc}}$ functions instead of C^1 maps:

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $u \in L^1(\Omega; \mathbb{R}^2)$. Then*

$$\overline{\mathcal{A}}(u, \Omega) = \inf\{\liminf_{k\to+\infty} \mathbb{A}(v_k, \Omega),\ v_k \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^2),\ v_k \to u \text{ in } L^1(\Omega; \mathbb{R}^2)\}. \tag{4.1}$$

This follows from the fact that for $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ it holds $\overline{\mathcal{A}}(v, \Omega) = \mathbb{A}(v, \Omega)$ (see [1]), which trivially implies the inequality $\leq$ in the formula above. The opposite inequality is obtained by simply observing that $C^1(\Omega; \mathbb{R}^2) \subseteq W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$.

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