On the Relaxed Total Variation of Singular Maps

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Abstract

We consider the *total variation* functional $TV(u) = \int |\det Du|$ defined on $W^{1,n}(\Omega, \mathbb{R}^n)$ for $\Omega \subset \mathbb{R}^n$. An extension TV^p is defined by relaxation in the weak topology of $W^{1,p}$ for $p < n$; so the relaxed functional is defined also on maps which may have singularities. In this paper we study the relaxed total variation and compute the functional on 0-homogeneous singular maps.

1 Introduction

See Section 3 if in doubt about notation. Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $u: \Omega \to \mathbb{R}^n$. We define the *total variation* of a map $u: \Omega \to \mathbb{R}^n$

$$TV(u) = \int_\Omega |\det Du(x)| \, dx$$

which is a continuous functional on $W^{1,n}(\Omega, \mathbb{R}^n) \to \mathbb{R}$. Altought $W^{1,n}$ is the natural ambient space for TV in many applications one is interested in extending TV on maps with singularities. The main example is $u: \mathbb{B}^n \to \mathbb{R}^n$, $u(x) = x/|x|$ which is in $W^{1,p}$ for all $p < n$ but not in $W^{1,n}$.

An extension of TV on $W^{1,p}$ for $p < n$ can be accomplished in two ways.

The first one is to define a distributional Jacobian determinant to replace $\det Du$. Since $\det Du$ is a sum of products of n terms which are derivatives of u, one can integrate by parts to obtain a distribution which involves the products of only $n-1$ derivatives. The distributional Jacobian has been widely studied, see [15] [1] [18] [4] [17] [6] [2].

The second possibility (which we consider in this paper) is to extend TV by relaxation either in the weak or strong topology of $W^{1,p}$ for $p < n$ as follows

$$\begin{aligned} TV^p(u) &= \inf\{\liminf_{k\to\infty} TV(u_k): u_k \stackrel{W^{1,p}}{\rightharpoonup} u,\ u_k \in W^{1,n}\}, \\ TV^p_s(u) &= \inf\{\liminf_{k\to\infty} TV(u_k): u_k \stackrel{W^{1,p}}{\to} u,\ u_k \in W^{1,n}\}. \end{aligned}$$

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See [14], [8], [10], [3], [13], [12], [7].

This paper bases itself on [7] where the relaxed total variation is introduced and widely studied.

2 Plan of the paper

Our aim is the computation of TV^p on singular maps. We will only consider 0-homogeneous maps. This class of maps is however relevant in the study of general maps with single point singularities. In fact when u has a singularity in one point x_0 we have

$$TV^p(u) = \int |\det Du| + TV^p(u_0)$$

where $u_0(y) = \lim_{\varepsilon\to 0^+} u(x_0 + \varepsilon y)$ is the 0-homogeneous blow-up of the singularity (see the proof of [7, Theorem 10]).

Given a Lipschitz continuous map $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{R}^n$, we will consider the 0-homogeneous extension $u_\varphi(x) = \varphi(x/|x|)$. This map is defined on $\mathbb{D}^n \setminus \{0\}$ and hence is defined almost everywhere on $\mathbb{B}^n$. It is easy to show that $u_\varphi \in W^{1,p}(\mathbb{B}^n, \mathbb{R}^n)$ for all $p < n$ but (apart from constants) $u_\varphi \notin W^{1,n}$.

In Theorem 4.3 we highlight the following identity

$$TV^p(u_\varphi) = TV^p_s(u_\varphi) = \inf\{TV(v)\colon v \llcorner \mathbb{S}^{n-1} = \varphi,\ v \text{ regular}\}.$$

Thought not explicitly stated, this fundamental result is proved in [7, Section 7]. We have rewritten the proof which is divided in the two estimates of Lemma 4.1 and 4.2.

In Theorem 4.7 we show that $TV^p(u_\varphi)$ only depends on the homotopy class of the map φ. Notice that the distributional Jacobian is instead tied to homological properties of φ (see [10]).

Applying Theorem 4.7 we easily find that the formula

$$TV^p(u_\varphi) = |\deg(\varphi)| \cdot |\mathbb{B}^n|$$

holds whenever $\varphi(\mathbb{S}^{n-1}) \subset \mathbb{S}^{n-1}$ (Theorem 5.1). This result extends [7, Corollary 16] and was conjectured in [7, Remark 11].

In Theorem 5.2 we compute $TV^p(u_8)$ where u_8 is the so called "eight curve". This example was first introduced in [9] where it is shown (thought in a slightly different context) that $TV^p(u_8) > 0$. This is the first example of a map on which the distributional Jacobian and the relaxed total variation differ. In [7, Example 47] it has been proved that $TV^p_s(u_8) \le 4\pi$ while in [12] the u_8 function is considered as an example and $TV^p_s(u_8) \ge 4\pi$ is claimed. Instead we show that $TV^p(u_8) = TV^p_s(u_8) = 2\pi$. This result was independently obtained in [16, Theorem 1.2].

Last, we also present a more complex example to show how to proceed to compute TV^p on a general 0-homogeneous map in $\mathbb{R}^2$.

A brief resume on the *topological degree* and on *homotopy* is given in Section 6.

3 Notation

Let $\mathbb{R}$ be the set of *real numbers* and $\mathbb{Z}$ the set of *whole numbers*. We will denote by $\mathbb{B}^n_\rho(x)$ the *open ball* of $\mathbb{R}^n$ with radius ρ centred in x. Default values $\rho = 1$, $x = 0$ and n are assumed. With $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ we denote the *sphere* $\partial\mathbb{B}^n$ and with $\mathbb{D}^n_\rho = \overline{\mathbb{B}^n_\rho}$ the *closed disc*. If $u: A \to B$ and $C \subset A$ we denote the *restriction* $u \llcorner C: C \to B$, $u \llcorner C(x) = u(x)$.

We denote with $\mathcal{C}^n$ the space of differentiable functions, with $W^{n,p}$ the Sobolev spaces and with Lip the space of Lipschitz continuous functions. If $A \subset \mathbb{R}^n$ we denote with χ_A the *characteristic function* of A which has value 1 in A and 0 outside. With $\deg(u, y)$ we denote the *topological degree* of u in the point y (see Section 6.1). With $\operatorname{sgn}(x)$ we denote the sign function: $x = \operatorname{sgn}(x) \cdot |x|$. If X is a set $\#X$ is the number of points or cardinality of X. With $\operatorname{mul}(u, y)$ we will denote the *multiplicity* of u in y that is $\#u^{-1}(y)$ (see Theorem 4.6).

Given two continuous maps $\varphi, \psi: \mathbb{S}^{n-1} \to X$ we say that φ and ψ are *homotopic* if there exists a continuous mapping (the homotopy) $\gamma: [0, 1] \times \mathbb{S}^{n-1} \to X$ such that $\gamma(0, \cdot) = \varphi$ and $\gamma(1, \cdot) = \psi$. With $\pi_k(X)$ we denote the k-th *homotopy group* of the topological space X (see Section 6.2). $G * H$ is the *free product* of groups, i.e. the group of words generated by the elements of G and H.

When $X \subset \mathbb{R}^n$ we denote with $|X|$ or $\mathcal{H}^n(X)$ the *Lebesgue measure* of X; we define $\operatorname{dist}(x, X) = \inf_{y \in X} |x - y|$ and we denote with $(X)_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, X) < \varepsilon\}$ the ε-*neighbourhood* of X.

In Definition 4.4 we introduce the notion of *Lipschitz set*.

4 Main Theorems

The following Lemma 4.1 and 4.2 can be found in [7, Section 7]. We prove them here for the convenience of the reader. In particular compare the following with [7, (108)].

Lemma 4.1 (upper bound). *Let $\varphi: \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a Lipschitz continuous map and let $v: \mathbb{B}^n \to \mathbb{R}^n$ be any Lipschitz continuous function with $v \llcorner \mathbb{S}^{n-1} = \varphi$. Let $u: \mathbb{B}^n \to \mathbb{R}^n$ be defined almost everywhere by $u(x) = \varphi(x/|x|)$. Then for all $p \in \,]1, n[$,*

$$TV^p_s(u) \le TV(v).$$

Proof. For $\varepsilon \in \,]0, 1[$ define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if} \quad |x| \ge \varepsilon \\ v(x/\varepsilon) & \text{if} \quad |x| \le \varepsilon. \end{cases}$$

Clearly u_ε is Lipschitz. Moreover $\det Du_\varepsilon(x) = 0$ when $|x| > \varepsilon$ while u_ε is a reparameterization of v in $\mathbb{B}_\varepsilon$. So $TV(u_\varepsilon) = TV(v)$.

Moreover

$$|Du(x)| = \frac{1}{|x|} |D_\tau v(x/|x|)| \le \frac{1}{|x|} \|Dv\|_{L^\infty}$$

and for $|x| < \varepsilon$

$$|Du_\varepsilon(x)| = \frac{1}{\varepsilon}|Dv(x/\varepsilon)| \leq \frac{1}{\varepsilon}\|Dv\|_{L^\infty}.$$

Hence

$$\begin{aligned} \|Du - Du_\varepsilon\|^p_{L^p} &= \int_{\mathbb{B}_\varepsilon} |Du - Du_\varepsilon|^p \\ &\leq c_1 \|Dv\|^p_{L^\infty} \int_{\mathbb{B}_\varepsilon} \left(\frac{1}{|x|} + \frac{1}{\varepsilon}\right)^p \,\mathrm{d}x = c_2 \varepsilon^{n-p} \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$ when $p < n$. Hence $u_\varepsilon \to u$ in $W^{1,p}$ and

$$TV^p_s(u) \leq \liminf_{\varepsilon \to 0^+} TV(u_\varepsilon) = TV(v). \qquad \square$$

$\square$

Compare the following with [7, Lemma 36].

Lemma 4.2 (lower bound). *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\Omega_0 \Subset \Omega$. Let $u \in \mathrm{Lip}(\overline{\Omega} \setminus \Omega_0, \mathbb{R}^n) \cap W^{1,p}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ for some $p \in]n-1, n[$. Then given any $\varepsilon > 0$ it is possible to find a map $v \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$, $v \llcorner \partial\Omega = u \llcorner \partial\Omega$ such that*

$$TV^p(u) \geq TV(v) - \varepsilon.$$

Proof. By the definition of TV^p we can find a sequence $u_k \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ such that

$$\liminf_{k\to\infty} TV(u_k) \leq TV^p(u) + \varepsilon/2, \tag{1}$$
$$u_k \overset{W^{1,p}}{\rightharpoonup} u.$$

Let $c_1 = \|u\|_{L^\infty}$. If we let u'_k be the truncation of u_k in the cube $[-c_1, c_1]^n$ we notice that $TV(u'_k) \leq TV(u_k)$ and also $u'_k \rightharpoonup u$. So by replacing u_k with u'_k we may also suppose that

$$\sup_k \|u_k\|_{L^\infty} \leq c_1.$$

Choose now $r > 0$ such that the set $A_r = \{x \in \Omega \colon \mathrm{dist}(x, \partial\Omega) < r\}$ is contained in $\Omega \setminus \Omega_0$ and consider the *cut-off* function $\varphi \colon \overline{\Omega} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} \mathrm{dist}(x, \partial\Omega)/r & \text{if } x \in A_r \cup \partial\Omega, \\ 1 & \text{if } x \in \Omega \setminus A_r. \end{cases}$$

Notice that φ is $1/r$-Lipschitz and $|\varphi| \leq 1$. So the following sequence $v_k \colon \overline{\Omega} \to \mathbb{R}^n$

$$v_k(x) = \varphi(x) u_k(x) + [1 - \varphi(x)] u(x)$$

is in $\mathrm{Lip}(\overline{\Omega}, \mathbb{R}^n)$, $v_k \llcorner \partial\Omega = u \llcorner \partial\Omega$ and we have

$$\begin{aligned} TV(v_k) &= \int_\Omega |\det Dv_k| = \int_{A_r} |\det Dv_k| + \int_{\Omega \setminus A_r} |\det Du_k| \\ &\leq \int_{A_r} |\det Dv_k - \det Du_k| + \int_\Omega |\det u_k|. \end{aligned} \tag{2}$$

We recall the inequality

$$|\det \xi - \det \eta| \le c_2(1 + |\xi|^{n-1} + |\eta|^{n-1})|\xi - \eta|$$

which holds for all matrices ξ and η with some constant c_2, in particular

$$|\det Dv_k - \det Du_k| \le c_2(1 + |Dv_k|^{n-1} + |Du_k|^{n-1})|Dv_k - Du_k|$$

and by Hölder inequality with $q = (p - n + 1)/p$ we get

$$\begin{aligned} &\int_{A_r} |\det Dv_k - \det Du_k| \\ &\le c_2 \left\|1 + |Dv_k|^{n-1} + |Du_k|^{n-1}\right\|_{L^{\frac{p}{n-1}}(A_r)} \cdot \|Dv_k - Du_k\|_{L^q(A_r)} \end{aligned} \tag{3}$$

From the definition of v_k we obtain

$$\begin{aligned} |Dv_k - Du_k| &= |D\varphi \otimes (u_k - u) + (1 - \varphi)(Du - Du_k)| \\ &\le \frac{|u - u_k|}{r} + |Du - Du_k|. \end{aligned}$$

Since $u_k \rightharpoonup u$ weakly in $W^{1,p}$ we know that $u - u_k$ is bounded in $W^{1,p}$ that is $\sup_k \|Du - Du_k\|_{L^p} \le c_3$. Moreover there exists a subsequence u_{k_j} such that $|u(x) - u_{k_j}(x)|/r \to 0$ for almost every x. So by Lebesgue convergence Theorem we obtain

$$\liminf_{k\to\infty} \|Dv_k - Du_k\|^q_{L^q(A_r)} \le \liminf_{k\to\infty} \int_{A_r} \left[\frac{|u - u_k|}{r} + c_3\right]^q = c_3^q |A_r|. \tag{4}$$

Being Du_k and Du bounded in L^p and being also

$$|Dv_k| \le 2|Du_k| + |Du| + \frac{|u - u_k|}{r}$$

we find that

$$\begin{aligned} &\left\|1 + |Dv_k|^{n-1} + |Du_k|^{n-1}\right\|_{L^{\frac{p}{n-1}}(A_r)} \\ &\le c_4 \left[1 + \|Dv_k\|^{n-1}_{L^p(A_r)} + \|Du_k\|^{n-1}_{L^p(A_r)}\right] \\ &\le c_5 \left[1 + \left\|\frac{u - u_k}{r}\right\|^{n-1}_{L^p(A_r)}\right] \end{aligned}$$

again, passing to the limit we obtain

$$\liminf_{k\to\infty} \left\|1 + |Dv_k|^{n-1} + |Du_k|^{n-1}\right\|_{L^{\frac{p}{n-1}}(A_r)} \le c_5. \tag{5}$$

From (3), (4) and (5) we have

$$\liminf_{k\to\infty} \int_{A_r} |\det Dv_k - \det Du_k| \le c_2 c_3 c_5 |A_r|^{\frac{1}{q}}$$

and from (2) and (1) we conclude

$$\liminf_{k\to\infty} TV(v_k) \le c_2 c_3 c_5 |A_r|^{\frac{1}{q}} + TV^p(u) + \frac{\varepsilon}{2}.$$

Since $|A_r| \to 0$ as $r \to 0$ we can choose r small enough so that for some k

$$TV(v_k) \le TV^p(u) + \varepsilon. \qquad \square$$

$\square$

Theorem 4.3 (characterization). *Let $\varphi: \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a Lipschitz map and let $u: \mathbb{B}^n \to \mathbb{R}^n$ be defined almost everywhere by $u(x) = \varphi(x/|x|)$. Then for all $p \in]n-1, n[$ we have $u \in W^{1,p}(\mathbb{B}^n, \mathbb{R}^n)$ and*

$$TV^p(u) = TV^p_s(u) = \inf\{TV(v): v \in \operatorname{Lip}(\mathbb{D}^n, \mathbb{R}^n),\ v\llcorner\mathbb{S}^{n-1} = \varphi\}.$$

Proof. Clearly $TV^p \le TV^p_2$ so the result comes at once from Lemma 4.1 and 4.2. $\square$ $\square$

Definition 4.4. *We say that a set $X \subset \mathbb{R}^n$ is* Lipschitz *if there exist $\varepsilon > 0$ and a Lipschitz mapping $\pi: (X)_\varepsilon \to X$ such that $\pi(x) = x$ for all $x \in X$.*

Notice that the projection π need not to be the minimum distance projection. In particular the set $X = \{(x,y) \in \mathbb{R}^2 : |y| = |x|\}$ is Lipschitz in our definition. An example of non Lipschitz set is instead $X = \{(x,y) \in \mathbb{R}^2 : |y| = x^2\}$.

The following theorem assures that when $X \subset \mathbb{R}^n$ is regular then homotopic maps can be joined with a Lipschitz continuous homotopy. This is true when X is Lipschitz. However we believe that when $n = 2$ and X is the image of a Lipschitz curve (as in Theorem 5.2) again this theorem should be valid even if X is not Lipschitz in our definition.

Lemma 4.5. *Let X be a bounded, Lipschitz subset of $\mathbb{R}^n$. Let $\varphi, \psi: \mathbb{S}^{n-1} \to X$ be two Lipschitz mappings. If φ and ψ are homotopic then there exists a Lipschitz homotopy $\bar\gamma: [0,1] \times \mathbb{S}^{n-1} \to X$ such that $\bar\gamma(0,\cdot) = \varphi$ and $\bar\gamma(1,\cdot) = \psi$.*

Proof. Being φ and ψ homotopic there exists a continuous mapping $\gamma: [0,1] \times \mathbb{S}^{n-1} \to X$. First of all we consider an extension $\tilde\gamma: \mathbb{R} \times \mathbb{S}^{n-1} \to X$

$$\tilde\gamma(t,\sigma) = \begin{cases} \gamma(0,\sigma) & \text{if} \quad t < 0 \\ \gamma(t,\sigma) & \text{if} \quad 0 \le t \le 1 \\ \gamma(1,\sigma) & \text{if} \quad t > 1 \end{cases}$$

Clearly $\tilde\gamma$ is uniformly continuous since γ is.

Then, given $\delta > 0$ we define, by regularization, $\gamma_\delta: [0,1] \times \mathbb{S}^{n-1} \to \mathbb{R}^n$

$$\gamma_\delta(t,\sigma) = \frac{1}{2\delta \mathcal{H}^{n-1}(B_\delta)} \int_{t-\delta}^{t+\delta} \int_{B_\delta(\sigma)} \tilde\gamma(s,\tau)\, \mathrm{d}\mathcal{H}^{n-1}(\tau)\, \mathrm{d}s$$

with $B_\delta(\sigma) = \mathbb{B}^n_\delta(\sigma) \cap \mathbb{S}^{n-1}$.

One can easily check that $\gamma_\delta \overset{L^\infty}{\to} \gamma$ and γ_δ is Lipschitz continuous.

Since X is Lipschitz we can choose $\varepsilon > 0$ such that there exists a Lipschitz projection $\pi: (X)_\varepsilon \to X$. Also fix $\bar\delta$ such that $\gamma_{\bar\delta}([0,1]\times\mathbb{S}^{n-1}) \subset (X)_\varepsilon$ (this is possible since $\gamma_\delta \to \gamma$ uniformly). Finally define $\bar\gamma: [0,1]\times\mathbb{S}^{n-1} \to X$ by

$$\bar\gamma(t,\sigma) = \begin{cases} \pi((1-3t)\gamma(0,\sigma) + 3t\gamma_{\bar\delta}(0,\sigma)) & \text{when} \quad t \in [0,1/3], \\ \pi(\gamma_{\bar\delta}(3t-1,\sigma)) & \text{when} \quad t \in [1/3,2/3], \\ \pi((3-3t)\gamma_{\bar\delta}(1,\sigma) + (3t-2)\gamma(1,\sigma)) & \text{when} \quad t \in [2/3,1]. \end{cases}$$

For $t \in [0,1/3]$ we are joining the curve φ with the curve $\gamma_{\bar\delta}(0,\cdot)$. Between $1/3$ and $2/3$ we consider the regularized homotopy $\gamma_{\bar\delta}$ and between $2/3$ and 1 we join the curve ψ. The resulting mapping $\bar\gamma$ is a Lipschitz homotopy between φ and ψ. □ □

We recall the following theorem (see for example [10]) which will be useful in the computation of TV

Theorem 4.6 (area formula). *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u \in \mathrm{Lip}(\Omega,\mathbb{R}^n)$. Then*

$$TV(u) = \int_\Omega |\det Du(x)|\,dx = \int_{u(\Omega)} \mathrm{mul}(u,y)\,dy$$

Theorem 4.7 (homotopic invariance). *Let $X \subset \mathbb{R}^n$ be a Lipschitz set with $|X| = 0$. Let $\varphi,\psi: \mathbb{S}^{n-1} \to X$ be two Lipschitz maps and let $u, v: \mathbb{B}^n \to \mathbb{R}^n$ be defined almost everywhere by $u(x) = \varphi(x/|x|)$, $v(x) = \psi(x/|x|)$.*

If φ and ψ are homotopic in X then $TV^p(u) = TV^p(v)$ for every $p \in]n-1,n[$.

Proof. Since X is a Lipschitz set and φ, ψ are homotopic, by Lemma 4.5 we know that there exists a Lipschitz mapping $\gamma: [0,1]\times\mathbb{S}^{n-1} \to X$ such that

$$\gamma(0,\sigma) = \varphi(\sigma), \quad \gamma(1,\sigma) = \psi(\sigma) \qquad \forall\sigma \in \mathbb{S}^{n-1}.$$

By Lemma 4.2 there exists a sequence $u_k \in \mathrm{Lip}(\mathbb{D}^n,\mathbb{R}^n)$ such that $\liminf_k TV(u_k) \le TV^p(u)$ and $u_k \llcorner \mathbb{S}^{n-1} = \varphi$. Define $v_k: \mathbb{D}^n \to \mathbb{R}^n$ as follows:

$$v_k(x) = \begin{cases} u_k(2x) & \text{when} \quad |x| \le \frac{1}{2} \\ \gamma(2|x|-1, x/|x|) & \text{when} \quad |x| \ge \frac{1}{2}. \end{cases}$$

Clearly $v_k \in \mathrm{Lip}(\mathbb{D}^n,\mathbb{R}^n)$ and $v_k \llcorner \mathbb{S}^{n-1} = \psi$ so that by Lemma 4.1 $TV^p(v) \le TV(v_k)$. We have

$$TV(v_k) = \int_{\mathbb{B}^n_{\frac{1}{2}}} |\det Dv_k| + \int_{\mathbb{B}^n \setminus \mathbb{B}^n_{\frac{1}{2}}} |\det Dv_k|.$$

The first integral on the right hand side of the previous equality is nothing else that $TV(u_k)$ since the map $u_k \llcorner \mathbb{B}^n_{\frac{1}{2}}$ is a reparameterization of u. On the other hand in view of Theorem 4.6 we have

$$TV(v_k \llcorner \mathbb{D}^n \setminus \mathbb{B}^n_{\frac{1}{2}}) = TV(\gamma) = \int_X \mathrm{mul}(\gamma,y)\,\mathrm{d}y = 0$$

since $|X| = 0$.

So we get $TV(v_k) = TV(u_k)$ and hence

$$TV^p(v) \le \liminf_{k\to\infty} TV(v_k) = \liminf_{k\to\infty} TV(u_k) \le TV^p(u).$$

Repeating the proof with φ and ψ exchanged we obtain the opposite inequality and hence the complete proof. □ □

5 Applications

Compare the following with [7, Theorem 1 and Corollary 16].

Theorem 5.1 (the sphere). *Consider any Lipschitz map $\varphi: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ and let $u: \mathbb{B}^n \to \mathbb{R}^n$ be defined almost everywhere by $u(x) = \varphi(x/|x|)$. Then*

$$TV^p(u) = |\deg(\varphi)| \cdot |\mathbb{B}^n|$$

for all $p \in]n-1, n[$.

Proof. Let $k = \deg(\varphi)$ and let $\bar\varphi: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be defined as follows

$$\bar\varphi(\rho\cos\theta, \rho\sin\theta, x_3, x_4, \ldots, x_n) = (\rho\cos(k\theta), \rho\sin(k\theta), x_3, x_4, \ldots, x_n),$$

where $\rho = \sqrt{1 - (x_3^2 + x_4^2 + \ldots + x_n^2)}$ and let $\bar u: \mathbb{D}^n \to \mathbb{R}^n$ be defined by $\bar u(x) = \bar\varphi(x/|x|)$.

Clearly $\deg(\bar\varphi) = k$ and hence $\bar\varphi$ is homotopic to φ (see Section 6.2) so that by Theorem 4.7 we have $TV^p(u) = TV^p(\bar u)$.

To apply Theorem 4.3 we consider any Lipschitz map $v: \mathbb{D}^n \to \mathbb{R}^n$ such that $v \llcorner \mathbb{S}^{n-1} = \bar\varphi$. One of such maps is $\bar v(x) = |x|\bar\varphi(x/|x|)$. Since for this map we have $\deg(\bar v, y) = k$ for all $y \in \mathbb{B}^n$ we also have $\deg(v, y) = k$ for all $y \in \mathbb{B}^n$ because v and $\bar v$ agree on $\mathbb{S}^{n-1}$ (see Section 6.1). So we have (see Theorem 4.6 and Section 6.1)

$$TV(v) = \int_{\mathbb{R}^n} \mathrm{mul}(v, y)\, dy \ge \int_{\mathbb{B}^n} |\deg(v, y)|\, dy = |k| \cdot |\mathbb{B}^n|$$

while

$$TV(\bar v) = \int_{\mathbb{B}^n} \mathrm{mul}(\bar v, y)\, dy = \int_{\mathbb{B}^n} |k| dy = |k| \cdot |\mathbb{B}^n|.$$

Hence

$$TV(u) = TV(\bar u) = \inf_v TV(v) = |k| \cdot |\mathbb{B}^n|. \qquad \square$$

□

Compare the following with [9, Theorem 1], [12, Example 3.8], and [7, Section 6 and Example 47].

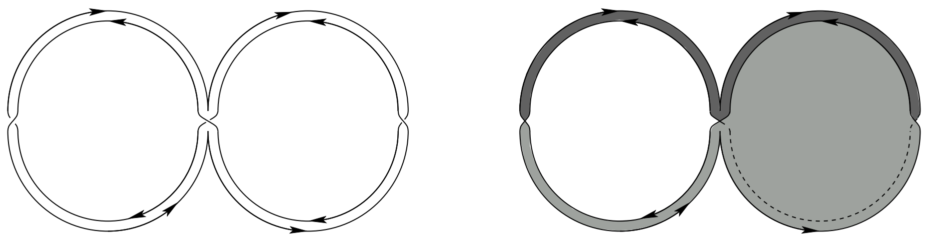

Figure 1: On the left the "eight curve". On the right a representation of the image of the map u_2. The right disc is spanned two times while the left remains empty.

Theorem 5.2 (the eight curve). *Consider the Lipschitz curve* $\varphi\colon \mathbb{S}^1 \to \mathbb{R}^2$ *defined by*

$$\varphi(\cos\theta, \sin\theta) = \begin{cases} (-1,0) + (\cos 4\theta, \sin 4\theta) & \text{when} \quad \theta \in [0, \pi/2] \\ (1,0) + (-\cos 4\theta, \sin 4\theta) & \text{when} \quad \theta \in [\pi/2, \pi] \\ (-1,0) + (\cos 4\theta, -\sin 4\theta) & \text{when} \quad \theta \in [\pi, 3\pi/2] \\ (1,0) + (-\cos 4\theta, -\sin 4\theta) & \text{when} \quad \theta \in [3\pi/2, 2\pi] \end{cases}$$

and consider the map $u_8\colon \mathbb{B}^2 \to \mathbb{R}^2$ *defined almost everywhere by* $u_8(x) = \varphi(x/|x|)$*. Then* $TV^p(u_8) = 2\pi$ *for all* $p \in]1, 2[$.

Proof. The image of φ is the "eight curve" $X = X_1 \cup X_2$ that is the union of the two circles $X_1 = \mathbb{S}^1(-1,0)$ and $X_2 = \mathbb{S}^1(1,0)$. Let D_1 and D_2 be respectively the two discs $D_1 = \mathbb{D}^2(-1,1)$ and $D_2 = \mathbb{D}^2(1,0)$. If a represent the curve around the left circle and b the curve around the right circle, then φ can be represented by $aba^{-1}b^{-1}$.

To compute $TV^p(u_8)$, by Theorem 4.3, it is enough to consider all Lipschitz continuous maps u which agree with φ on $\mathbb{S}^1$.

One such map is $u_1(x) = |x|\varphi(x/|x|)$. Clearly it holds $TV(u_1) = 4\pi$ since both circles D_1 and D_2 are spanned twice by this map. This gives at once $TV^p(u_8) \leq 4\pi$.

However a better result can be achieved. In fact it is possible to find a Lipschitz continuous function u_2 whose image spans one of the two circles (say D_2) two times while leaves empty the other circle (as suggested in Figure 1).

We decompose $\mathbb{D}^2$ into three pieces: $T = \mathbb{D}^2 \cap \{x - 1 \leq y \leq x + 1\}$, $A^+ = \mathbb{D}^2 \cap \{y \geq x + 1\}$ and $A^- = \mathbb{D}^2 \cap \{y \leq x - 1\}$. Also consider the two points $p^+ = (1/2, 1/2)$ and $p^- = (-1/2, -1/2)$ which respectively lie in $T \cap A^\pm$ (see Figure 2).

We define u_2 by the following properties:

1. $u_2 \llcorner \mathbb{S}^1 = \varphi$;

2. u_2 is constant on the lines $x + y = q$ for every q in $[-1; 1]$;

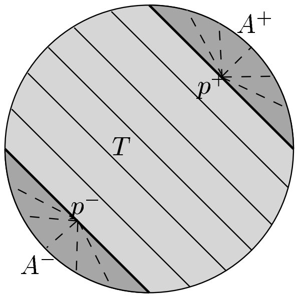

Figure 2: Here are the pieces in which $\mathbb{D}^2$ is broken in the definition of u_2. Along the circle, u_2 is equal to φ. Along continuous lines the function u_2 is constant; along dashed lines u_2 is a linear interpolation. Along the two strong lines u_2 is zero.

3. u_2 is linear on the intersection of A^+ (respectively A^-) with the lines through the point p^+ (respectively p^-).

As suggested by Figure 1 the image $u_2(T)$ of the strip T is the left circle X_1 while the image of both A^+ and A^- is the full disc D_2.

So applying Theorem 4.6

$$TV(u_2) = \int_{u_2(\mathbb{B}^2)} \operatorname{mul}(u_2, y)\,\mathrm{d}y = \int_{D_2} 2 = 2\pi.$$

And in view of Theorem 4.3 we obtain

$$TV^p(u_8) \leq 2\pi.$$

Indeed we claim that $TV^p(u_8) = 2\pi$. To prove this we have to show that given any regular map $v\colon \mathbb{D}^2 \to \mathbb{R}^2$ with $v \llcorner \mathbb{S}^1 = \varphi$ we have $TV(v) \geq 2\pi$. So let v be such a map. Since v and u_1 agree on the boundary of $\mathbb{D}^2$ they have the same topological degree. So we have (recall that $X = \varphi(\mathbb{S}^1)$)

$$\deg(v, y) = \deg(u_1, y) = 0 \quad \forall y \in \mathbb{R}^2 \setminus X.$$

We now claim that one of the two circles D_1 or D_2 is completely contained in the image of v. In fact suppose by contradiction that there exists two points y_1 and y_2 such that $y_1 \in D_1$, $y_2 \in D_2$ and $\operatorname{mul}(v, y_1) = \operatorname{mul}(v, y_2) = 0$. Let $Y = \mathbb{R}^2 \setminus \{y_1, y_2\}$. We know that the homotopy group $\pi_1(Y)$ is $\mathbb{Z} * \mathbb{Z}$ and, denoting with a and b the two generators of $\pi_1(Y)$, the homotopy class of φ is $[\varphi] = aba^{-1}b^{-1} \neq 0$ (see Section 6.2). On the other hand we are supposing that $v(\mathbb{D}^2) \subset Y$ so that v is an explicit homotopy in Y between φ and the constant map $v(0)$. This is a contradiction.

So at least one of the two circles (say D_1) must be completely spanned by v. On the other hand for almost every $y \in \mathbb{R}^n \setminus X$ we have $\deg(v, y) =$

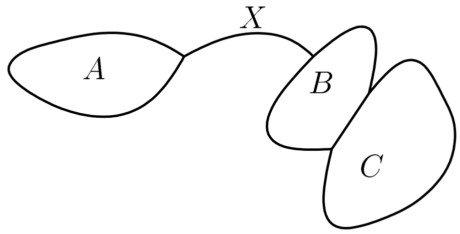

Figure 3: The curve $X = \varphi(\mathbb{S}^1)$ with three holes A, B and C.

$\sum_{v(x)=y} \operatorname{sgn} \det Dv(x) = 0$. Since $\operatorname{mul}(v, y) = \#\{v(x) = y\}$ we get that $\operatorname{mul}(v, y)$ is even for almost every $y \in \mathbb{R}^n$. Being different from 0 we obtain that $\operatorname{mul}(v, y) \geq 2$ for almost every $y \in D_1$ and hence

$$TV(v) \geq \int_{D_1} \operatorname{mul}(v, y)\, \mathrm{d}y \geq 2\pi. \qquad \square$$

$\square$

The technique used in the previous example can be extended to many 0-homogeneous maps $u \colon \mathbb{D}^2 \to \mathbb{R}^2$.

Let $\varphi \colon \mathbb{S}^1 \to \mathbb{R}^2$ be a Lipschitz continuous curve. Let $X = \varphi(\mathbb{S}^1)$. Clearly $|X| = 0$. Suppose, for example, that X is a set enclosing three "holes" A, B and C as in Figure 3.

So A, B, C are open, simply connected, disjoint subsets of $\mathbb{R}^2$. Since X is Lipschitz we know that any curve φ can be replaced with the canonical representant of its homology class $[\varphi]$. If a, b and c are three closed curves enclosing the sets A, B and C we know that $\pi_1(X) = \mathbb{Z}_a * \mathbb{Z}_b * \mathbb{Z}_c$ that is the group of all words composed with the letters a, b, c and their inverses a^{-1}, b^{-1}, c^{-1}.

Suppose, for example, that our curve φ is represented by the word

$$\varphi = aca^{-1}bc^{-1}ab^{-1}.$$

We now construct three different maps $v_1, v_2, v_3 \colon \mathbb{D}^2 \to \mathbb{R}^2$ with $v_i \llcorner \mathbb{S}^1 = \varphi$ such that

$$\inf\{TV^p(v) \colon v \in \operatorname{Lip}(\mathbb{D}^2, \mathbb{R}^2),\ v \llcorner \mathbb{S}^1 = \varphi\} = \min\{TV^p(v) \colon v \in \{v_1, v_2, v_3\}\}. \quad (6)$$

As in Theorem 5.2 the idea is to "collapse" some pieces of φ with strips joining two inverse paths.

For example we can collapse in $aca^{-1}bc^{-1}ab^{-1}$ the first a with a^{-1} joining them with a strip. The two remaining closed curves are then c and $bc^{-1}ab^{-1}$. The first curve cannot be simplified further and we put a disc on it. In the second curve we can collapse b with b^{-1} leaving the curve $c^{-1}a$ on which we put two discs. The resulting map $v_1 \colon \mathbb{D}^2 \to \mathbb{R}^2$ can be represented as in Figure 4.

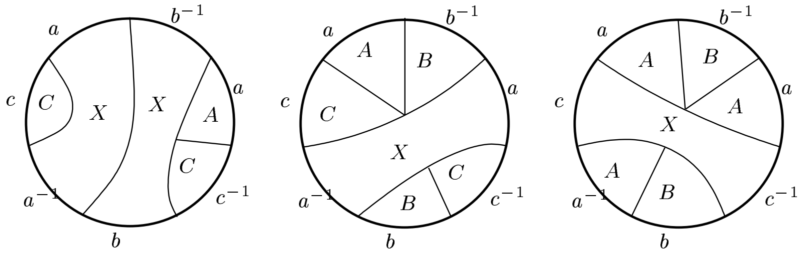

Figure 4: Here are represented the domains of the three maps $v_1, v_2, v_3 \colon \mathbb{D}^2 \to \mathbb{R}^2$. The letters in the regions indicate where the point of that regions are sent by v. In particular the regions marked with an X are mapped into a thin curve contained in X while the regions A, B and C are mapped, with a disc type surface, onto the corresponding region.

The strips have 0 total variation while the discs have a total variation which equals the measure of the corresponding hole. So $TV(v_1) = |A| + 2|C|$.

If we instead join the second a of $aca^{-1}bc^{-1}ab^{-1}$ with its inverse a^{-1} we are left with the two curves $b^{-1}ac$ and bc^{-1} that cannot be collapsed further. The resulting map v_2 (see Figure 4) has total variation $TV(v_2) = |A| + 2|B| + 2|C|$.

The last possibility is to collapse c with c^{-1} to obtain the two curves $a^{-1}b$ and $ab^{-1}a$. The resulting map v_3 has total variation $TV(v_3) = 3|A| + 2|B|$.

We have to prove that any other regular map $v \colon \mathbb{D}^2 \to \mathbb{R}^2$ is worse (i.e. has greater total variation) than one of v_1, v_2, v_3.

To achieve this we use the topological degree. Let $v \colon \mathbb{D}^2 \to \mathbb{R}^2$ be any $\mathcal{C}^1$ map with $v \llcorner \mathbb{S}^1 = \varphi$.

Since all maps with the same boundary have the same topological degree, we have[1] $\deg(v, A) = 1$, $\deg(v, B) = 0$ and $\deg(v, C) = 0$. This can be quickly obtained also by letting a, b and c commute in $aca^{-1}bc^{-1}ab^{-1}$ to obtain $aa^{-1}abb^{-1}cc^{-1} = a$. Hence $\mathrm{mul}(v, y) \geq \chi_A(y)$.

Then we claim that either B or C must be completely spanned by v. In fact suppose that there exist $y_1 \in B$ and $y_2 \in C$ such that $\mathrm{mul}(v, y_1) = \mathrm{mul}(v, y_2) = 0$. Then let $Y = \mathbb{R}^2 \setminus \{y_1, y_2\}$. The inclusion $i \colon X \to Y$ gives an homomorphism $i^* \colon \pi_1(X) \to \pi_1(Y)$ which can be computed by simply letting $a = 0$ since a is homotopic to 0 in Y. So $i^*(\varphi) = cbc^{-1}b^{-1}$ which, for sure, is different from 0 in $\pi_1(Y)$. However v is an explicit homotopy in Y between φ and 0. So either B or C is completely spanned by v.

Suppose that C is spanned by v. Since $\deg(v, C) = 0$ we get that $\mathrm{mul}(y, C) \geq 2$ as $\mathrm{mul}(y, C)$ is even. Hence $\mathrm{mul}(v, y) \geq 2\chi_C$ and recalling that $\mathrm{mul}(v, y) \geq \chi_A$ we obtain $TV(v) \geq |A| + 2|C| = TV(v_1)$.

Suppose instead that B is spanned by v. In this case we obtain $\mathrm{mul}(v, y) \geq 2\chi_B$. We now claim that either $\mathrm{mul}(v, C) \geq 1$ or $\mathrm{mul}(v, A) \geq 2$. Suppose by contradiction that there exist $y_1 \in A$ and $y_2 \in C$ such that $\mathrm{mul}(v, y_1) \leq 1$ and

[1] With $\deg(v, E) = d$ we mean $\deg(v, y) = d$ for all $y \in E$.

$\mathrm{mul}(v, y_2) = 0$. We can also suppose that y_1 is not a critical value of v. If $\mathrm{mul}(v, y_1) = 0$ we obtain a contradiction reasoning as before. So suppose that $\mathrm{mul}(v, y_1) = 1$. In this case there exists one unique point x_1 such that $v(x_1) = y_1$. Since x_1 is not critical and $\deg(v, A) = 1$ we know that $\det Dv(x_1) > 0$. So v is locally invertible, i.e. there exist $\varepsilon > 0$ such that $v \llcorner \mathbb{B}_\varepsilon(x_1)$ is a diffeomorphism with its image. Hence the mapping $v: \mathbb{D}^2 \setminus \mathbb{B}^2_\varepsilon(x_1) \to Y = \mathbb{R}^2 \setminus \{y_1, y_2\}$ is an explicit homotopy between φ and the curve $\psi = v \llcorner \partial \mathbb{B}_\varepsilon(x_1)$. Notice that, in Y, we have $[\varphi] = aca^{-1}c^{-1}a$ and $[\psi] = a$. But this is a contradiction since $aca^{-1}c^{-1}a$ and a are different in $\mathbb{Z}_a * \mathbb{Z}_c$. We conclude that either C is spanned by v or A is spanned at least twice.

In the first case we get $\mathrm{mul}(y, C) \geq 2$ and hence $TV(v) \geq |A| + 2|B| + 2|C| = TV(v_2)$.

In the second case we get $\mathrm{mul}(y, A) \geq 3$ (since $\deg(v, A) = 1$) and hence $TV(v) \geq 3|A| + 2|B| = TV(v_3)$.

We have so proved (6) and hence $TV^p(u) = |A| + 2|C|$ if $|C| \leq |A| + |B|$ and $TV^p(u) = 3|A| + 2|B|$ otherwise.

6 Appendix

6.1 Topological degree

Let Ω be an open and bounded subset of $\mathbb{R}^n$ and let $u: \Omega \to \mathbb{R}^n$ be a continuous function. Given $y \in \mathbb{R}^n \setminus u(\partial\Omega)$ it is defined the *topological degree* of u in y which is denoted by $\deg(u, y) \in \mathbb{Z}$ (see for example [5]).

Among other properties we recall the following:

1. if $u \in \mathcal{C}^1$ and y is not critical (i.e. $\det Du(x) \neq 0$ whenever $u(x) = y$) it holds
$$\deg(u, y) = \sum_{u(x)=y} \mathrm{sgn} \det Du(x);$$
2. $|\deg(u, y)| \leq \mathrm{mul}(u, y)$ for all $y \in \mathbb{R}^n \setminus u(\partial\Omega)$;
3. $\deg(u, \cdot)$ is constant on each connected component of $\mathbb{R}^n \setminus u(\partial\Omega)$;
4. if $u \llcorner \partial\Omega = v \llcorner \partial\Omega$ then $\deg(u, y) = \deg(v, y)$ for all $y \in \mathbb{R}^n \setminus u(\partial\Omega)$.

Given a map $\varphi: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ and given any continuous extension $u: \mathbb{D}^n \to \mathbb{R}^n$ (with $u \llcorner \mathbb{S}^{n-1} = \varphi$) the topological degree $\deg(u, y)$ is the same for any $y \in \mathbb{B}^n$ and for any choice of u. So it only depends on φ and we call it $\deg(\varphi)$ (see Theorem 5.1). For example one can always choose $\bar{u}(x) = |x|\varphi(x/|x|)$ and define $\deg(\varphi) = \deg(\bar{u}, 0)$. One could also define the topological degree for maps φ between manifolds. In this case the manifold $\mathbb{S}^{n-1}$ has no boundary so that $\deg(\varphi, y)$ is constant for all $y \in \mathbb{S}^{n-1}$.

6.2 Homotopy

Given $k \in \mathbb{N}$ the *homotopy group* $\pi_k(X)$ is a group which is associated to the pathwise connected topological space X (see for example [11]). We can associate to each continuous map $\varphi: \mathbb{S}^k \to X$ an element $[\varphi]$ of the homotopy group $\pi_k(X)$. If $[\varphi] = [\psi]$ there exists a continuous mapping $\gamma: [0,1] \times \mathbb{S}^k \to X$ such that $\gamma(0,\cdot) = \psi$, $\gamma(1,\cdot) = \psi$ and we say that φ and ψ are *homotopic.* On the other hand if φ is homotopic to a constant map then $[\varphi] = 0$ (the identity element of the group $\pi_k(X)$).

When $X = \mathbb{S}^k$ we have $\pi_k(\mathbb{S}^k) = \mathbb{Z}$ and each map φ is represented by means of the topological degree, i.e. $[\varphi] = \deg(\varphi)$.

When $X \subset \mathbb{R}^2$ is a set with h holes (for example $X = \mathbb{R}^2 \setminus \{x_1, \ldots, x_h\}$) we have $\pi_1(X) = \mathbb{Z} * \ldots * \mathbb{Z}$ that is the *free product* of h copies of $\mathbb{Z}$. The loops $\varphi_1, \ldots, \varphi_h$ around the holes are the generators of the homotopy group. Every curve φ is hence represented by a sequence of these generators. This group is not commutative, i.e. $[\varphi_i][\varphi_j] \neq [\varphi_j][\varphi_i]$ if $i \neq j$.

Notice instead that the groups $\pi_k(X)$ with $k > 1$ are always commutative.

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