The regularity theory for the Mumford-Shah functional on the plane

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Contents

CHAPTER 1

Introduction

The aim of these notes is to give a complete self-contained account of the current state of the art in the regularity for planar minimizers and critical points of the Mumford-Shah functional. The relevant energies are computed on pairs (K, u) where K is a closed 1-rectifiable subset of some planar open set Ω and u is an element of $W^{1,2}_{loc}(\Omega \setminus K)$. For every bounded open set $U \subset \Omega$, every $g \in L^\infty(\Omega)$, and every $\lambda \geq 0$ we define

$$E_\lambda(K, u, U, g) := \int_{U \setminus K} |\nabla u|^2 + \mathcal{H}^1(K \cap U) + \lambda \int_U |u - g|^2 . \tag{1.0.1}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. The open set U will be omitted if it coincides with Ω or it is obvious from the context and likewise g will be omitted when $\lambda = 0$ or when it is obvious from the context.

The term

$$\mathscr{F}(u, U, g) = \int_U (u - g)^2$$

will be often called "fidelity term", while the remaining portion of the functional will be consistently denoted by E_0. In particular, we have the obvious identity $E_\lambda = E_0 + \lambda \mathscr{F}$. Under our assumption (i.e. $g \in L^\infty$) E_0 dominates the fidelity term at small scales. In particular, the case $\lambda > 0$ adds mainly technical complications and the reader who wishes to avoid these technicalities could just focus on the case $\lambda = 0$. Quite a few constants in the statements need to be adjusted if $\|g\|_\infty$ becomes high. In order to avoid discussing the size of these constants we will assume once and for all that

$$\|g\|_\infty \leq M_0\text{[1]} \qquad \text{and} \qquad 0 \leq \lambda \leq 1 \,, \tag{1.0.2}$$

where M_0 is any given positive number. In particular in the rest of the notes we will omit the dependence of the constants from M_0.

The energies can be naturally extended to the class of functions of special bounded variation (shortly $SBV(\Omega)$) and in that case we can regard K as the jump set of the SBV function u (cf. the monograph [4] for the definitions). This is not really relevant for our discussions as long as the reader is willing to accept some compactness statements which pertain to the *existence theory* (of minimizers and critical points) of the functionals. Such statements will play anyway an important role and for the sake of completeness we will include short proofs in the appendix: since we believe that it makes them conceptually easier and cleaner, those arguments take advantage of the SBV theory.

[1]Throughout the notes we will omit the reference domain in the L^∞ norm if it will be clear from the context.

1.1. Minimizers

We will consider three different notions of minimizers associated with the above energy functionals. We start defining the first two.

DEFINITION 1.1.1. A pair (K, u) in an open set $\Omega \subset \mathbb{R}^2$ will be called:

- an *(absolute) minimizer* of the functional E_λ if for every bounded open $U \subset \Omega$ and for every pair (J, v) such that $\overline{J \Delta K} \cup \operatorname{spt}(u - v) \subset\subset U$ we have
$$E_\lambda(K, u, U, g) \leq E_\lambda(J, v, U, g)\,; \tag{1.1.1}$$
- a *restricted minimizer* if (1.1.1) holds for those (J, v) with the additional property that the number of connected components of J does not exceed that of K.

Obviously a minimizer is also a restricted minimizer, while the converse is instead false in general. Note that the value of the functional is invariant under the addition of $\mathcal{H}^1$-null sets, which creates an annoying technicality when dealing with regulary issues. Assume for instance that $(\emptyset, u)$ is an absolute minimizer of E_0. Then u is a classical harmonic function and so our pair is as smooth as desirable. However, if K is any $\mathcal{H}^1$ null set, then (K, u) is as well an absolute minimizer, so that we have several other minimizers with sets K which can be quite irregular. In order to avoid this issue we wish to normalize minimizers in an appropriate way, throwing away "unnecessary pieces" of K.

DEFINITION 1.1.2. A pair (K, u) will be called normalized[2] if, for every open set U, $\mathcal{H}^1(K \cap U) = 0$ implies $K \cap U = \emptyset$.

It is easy to see (cf. Corollary 2.1.4 below) that if (K, u) is a minimizer or a restricted minimizer in Ω, then it can "be normalized" in the following sense: if we set $\mathcal{N} := \{U \text{ open } : \mathcal{H}^1(U \cap K) = 0\}$ and let
$$K' := K \setminus \bigcup_{U \in \mathcal{N}} U\,,$$
then u can be extended to a C^1 function u' on $\Omega \setminus K'$ (in fact an harmonic function when $\lambda = 0$). Clearly the energy of (K', u') coincides with that of (K, u). For this reason, from now on we will always assume that a minimizing pair (or restricted minimizing pair) is normalized, unless we explicitly say that it might not be.

1.2. Epsilon regularity

The celebrated Mumford-Shah conjecture, which is still unsolved, states the following.

CONJECTURE 1.2.1 (Mumford-Shah conjecture). *If (K, u) is a minimizer of E_λ then K can be described as the union of finitely many closed C^1 arcs γ_i which do not cross but can meet at their endpoints at 120 degrees in "triple junctions". In particular, if we fix a point $x \in K$, then in any sufficiently small disk $B_r(x)$ the set $K \cap B_r(x)$ is diffeomorphic to one of the following special types of closed sets:*

(a) *a radius of $B_r(x)$;*

[2]In the literature the term "reduced minimizer" is sometimes used, instead, cf. [14].

(b) *a diameter of* $B_r(x)$*;*
(c) *the union of three radii of* $B_r(x)$ *forming angles of* 120 *degrees.*

The main conclusion of the regularity theory developed thus far is that if K is close in the Hausdorff distance[3] to one of the three examples (a), (b) and (c) at some given scale, then it is indeed diffeomorphic to the corresponding model at a slightly smaller scale: a precise statement is given in Theorem 1.2.3 below. These results are typically called "ε-regularity theorems", the ε referring to the fact that the key assumption is the closeness (at a certain scale and in an appropriate sense) to a given model. Before coming to the precise statements we introduce the following terminology

DEFINITION 1.2.2. Let $x \in K$. The point will be called, respectively, a *regular loose end*, a *pure jump*, or a *triple junction* if for some $r > 0$ there is a C^1 diffeomorphism Φ of $B_r(x)$ with $\Phi(x) = x$, $D\Phi(x) = \mathrm{Id}$, and such that:
(i) $\Phi(B_r(x) \cap K)$ is a radius of $B_r(x)$;
(ii) $\Phi(B_r(x) \cap K)$ is a diameter of $B_r(x)$;
(iii) $\Phi(B_r(x) \cap K)$ is the union of three radii of $B_r(x)$ meeting at equal angles.

A maximal C^1 arc of K is a closed (relatively to the domain Ω) C^1 arc contained in K which is not contained in any larger C^1 arc. An extremum of a maximal C^1 arc is an endpoint belonging to the domain Ω.

If $K \cap B_r(x)$ is a single continuous arc with endpoints x and $y \in \partial B_r(x)$ which is C^1 away from $\{x\}$ but not necessarily up to x, we then call the point x a *loose end*. In fact a substantial portion of these notes will be dedicated to show that, if (K, u) is a minimizer (no matter whether absolute or restricted), then every loose end is regular. Likewise, an outcome of the regularity theory is also that, if $K \cap B_r(x)$ consists of three continuous arcs which meet at x, then these arcs are necessarily C^1 up to their common endpoint x.

The focus of this book is then proving the following theorem.

THEOREM 1.2.3. *There are* $\alpha > 0$ *and* $\varepsilon > 0$ *with the following property. Assume:*
(1) (K, u) *is an absolute minimizer of* E_λ *in* $B_r(x) \subset \mathbb{R}^2$*;*
(2) (1.0.2) *holds and* $r \leq 1$*;*
(3) *the Hausdorff distance of* $K \cap B_{2r}(x)$ *to a set* K_0 *as in Conjecture 1.2.1(a), (b) or (c) is smaller than* εr*.*

Then:
(i) *there is a* $C^{1,\alpha}$ *diffeomorphism* Φ *of* $B_r(x)$ *such that* $\Phi(K \cap B_r(x)) = K_0$*;*
(ii) *if* $\lambda = 0$ *or* $g \in C^0$*, then* K *is* C^2 *in some neighborhood of any pure jump;*
(iii) *if* $\lambda = 0$ *each maximal* C^1 *arc is* C^2 *up to its extrema, where its curvature vanishes.*

Moreover:
(iv) *the same conclusions hold in cases (b) and (c) if* (K, u) *is a restricted minimizer (in particular* ε *is independent of the number of connected components of* K*);*

[3]Due to suitable compactness properties of minimizers, it actually suffices that K is close in a weaker sense.

(v) *in case (a) the same conclusions hold if* (K, u) *is a restricted minimizer and* $K \cap B_{2r}(x)$ *is connected;*
(vi) *in case (c) the three maximal* C^1 *arcs forming* $K \cap B_r(x)$ *meet at equal angles.*

Note that in case (c) the meeting point y of the three maximal C^1 arcs forming $K \cap B_r(x)$ is not necessarily x. Likewise, in case (a) the interior regular loose end is not necessarily located at x.

As it is known from the work of Bonnet [8], if (K, u) is a restricted minimizer of E_λ and K has a finite number of connected components, then for every $x \in K$ the assumptions of Theorem 1.2.3 are satisfied in a sufficiently small disk $B_r(x)$. We thus get a quite satisfactory description of the regularity of the set K, namely the conclusions of Conjecture 1.2.1 hold.

Corollary 1.2.4. *Conjecture 1.2.1 hold for restricted minimizers* (K, u) *such that* K *has a finite number of connected components.*

If g is more regular, the regularity of K can be bootstrapped to higher regularity at regular jump points: loosely speaking the regularity of K at jump points is 2 derivatives more than the regularity of g. In fact, even the real analiticity of K holds if g is real analytic, cf. [26]. However, we will not pursue this issue in these notes.

We finally observe that the second part of conclusion (ii) in Theorem 1.2.3 has a curious outcome: for restricted minimizers of E_0 with finitely many connected components or anyway for any minimizer for which the Mumford-Shah conjecture holds, the points of maximal curvature of any maximal arc forming K must be pure jump points, unless the arc is a straight segment.

1.3. Global generalized minimizers and classification

Absolute minimizers enjoy a natural elementary energy upper bound on any disk $B_r(x) \subset \Omega$, more precisely

$$E_\lambda(K, u, B_r(x), g) \leq 2\pi r + \lambda\pi \|g\|_\infty^2 r^2 \,, \tag{1.3.1}$$

cf. Lemma 2.1.2 below. Restricted minimizers satisfy the same bound under the additional assumption that $\overline{B}_r(x) \cap K \neq \emptyset$. If we therefore consider the following rescalings

$$\begin{aligned} u_{x,r}(y) &:= r^{-1/2} u(x + ry) \\ K_{x,r} &:= \left\{ \tfrac{y-x}{r} : y \in K \right\}, \end{aligned}$$

then $E_0(u_{x,r}, K_{x,r}, B_1)$ enjoy a uniform bound. Since under the scaling detailed above the "fidelity term" becomes negligible as $r \downarrow 0$, the latter bound allows for a suitable compactness statement for $r \downarrow 0$, provided we introduce an appropriate concept of "generalized minimizer of E_0", cf. Definition 2.2.4 (this is done elegantly in [8] for the case of restricted minimizers). Note that in general, we do not have uniform bounds on the size of u, but only on the size of ∇u, hence the space of generalized minimizers of E_0 must allow for a suitable normalization of infinities. An important point is that the regularity theory detailed so far remains valid for them.

THEOREM 1.3.1. *The conclusions of Theorem 1.2.3 remain valid for generalized and generalized restricted minimizers of E.*

Coming back to the scaling procedure outlined above, as $r \downarrow 0$ the domain of definition of $(K_{x,r}, u_{x,r})$ becomes larger and in the limit we attain "global generalized minimizers", namely generalized minimizers on the entire $\mathbb{R}^2$. If for any such global minimizer (K, u) of E_0 the set K is empty, then u must be an harmonic function and the upper bound (1.3.1) implies, by Liouville's theorem, that u is constant. This is the easiest "elementary" global minimizer. Two elementary types of global generalized minimizers correspond to the cases (b) and (c) explained above: in one case the set K consists of an infinite line and in the other case it consists of three halflines meeting at a common origin at equal (120 degrees) angles. In both cases the function is piecewise constant (in fact, in an appropriate sense, the difference between all constant values must be ∞). This list exhausts the case of elementary global minimizers, which we define as generalized global minimizers whose Dirichlet energy vanishes identically on all compact sets, cf. Theorem 2.4.1.

If the Mumford-Shah Conjecture 1.2.1 holds, there is then only another possible type of generalized global minimizer, namely K can only be a half line. It can be proved that the corresponding u must then take a rather special form, cf. Chapter 4.

DEFINITION 1.3.2. We will call *cracktip* any pair (K, u) in $\mathbb{R}^2$ which, up to translations, rotations, subtraction of a constant and change of sign can be described as follows:

- K is the set $\mathbb{R}^+ := \{(t, 0) : t \in \mathbb{R}\}$;
- $u(x_1, x_2) = \sqrt{\frac{2}{\pi}}\mathrm{Re}\sqrt{x_1 + ix_2}$, where we use the convention that $\sqrt{z}$ is the branch of the square root yielding i on -1, with branch cut on $\mathbb{R}^+$, namely,
$$u(x_1, x_2) = \frac{1}{\sqrt{\pi}}\frac{x_2}{|x_2|}\sqrt{x_1 + \sqrt{x_1^2 + x_2^2}} \qquad \text{if } x_2 \neq 0,$$
and $u(x_1, 0) = 0$ if $x_1 \leq 0$.

Theorem 1.2.3 implies that if the list of generalized global minimizers is exhausted by the elementary ones and by cracktips, then the Mumford-Shah conjecture holds. Except for the determination of what happens at loose ends, the latter conclusion was in fact proved as a combination of the two remarkable works [13] and [9] (see also [14]). The only point left by those references was the possibility that the set K forms a rather slow spiral at loose ends: Theorem 1.2.3 excludes the latter possibility. This was first proved by Andersson and Mikayelyan for minimizers of E_0 under the assumption of connectedness of K in [7]. A different argument for the same theorem was then given by the three authors in [21], while the present reference is the first to handle the case of E_λ for $\lambda > 0$.

The Mumford-Shah conjecture is indeed fully equivalent to the classification of generalized global minimizers, a fact which we record in the following theorem.

THEOREM 1.3.3. *Conjecture 1.2.1 holds for absolute minimizers if and only if any global generalized minimizer is either one of the elementary minimizers listed in Theorem 2.4.1 or a cracktip.*

Observe that Theorem 1.2.3 does not guarantee that cracktips are indeed global minimizers: Theorem 1.2.3 would imply the latter conclusion only if one could find an absolute minimum (K, u) such that K is close, in some disk $B_r(x)$, to a single straight segment which terminates at x.

The global minimality of cracktip was however proved in the book [9]. While these notes will not cover the proof of the latter fact, we will however cover the proof of one fundamental conclusion of [9], which will be instrumental in showing case (c) of Theorem 1.2.3, and which is interesting in its own right. The relevant statement is recorded in the first half of the following theorem, while the second half is another remarkable result for global generalized minimizers proved in [15], of which we will report as well the proof.

THEOREM 1.3.4. *Let (K, u) be a generalized global minimizer in the sense of Definition 2.2.4. If all but at most one connected component of K are contained in a compact set of $\mathbb{R}^2$, then (K, u) is either one of the elementary global minimizers described in Theorem 2.4.1 or it is a cracktip.*

Let (K, u) be a generalized global minimizer for which K disconnects $\mathbb{R}^2$. Then (K, u) is an elementary global minimizer.

1.4. Structural results

Even though the Mumford-Shah conjecture is widely open for absolute minimizers with an infinite number of connected components, Theorem 1.2.3 and Theorem 1.3.4 tell us a lot about the structure of the set K for minimizers (K, u). We first introduce the following terminology.

DEFINITION 1.4.1. Consider a closed set $K \subset \mathbb{R}^2$. A point $p \in K$ is *nonterminal* if there is a continuous injective map $\gamma : (-1, 1) \to K$ such that $\gamma(0) = p$. Otherwise $p \in K$ is called *terminal*.

Given a pair (K, u) which is an absolute, restricted, generalized, or generalized restricted minimizer, we define:

- the regular part K^r of K, which consists of those points which are either regular loose ends, or pure jump points, or triple junctions;
- the irregular part K^i of K, which is the complement of K^r in K;
- the points $K^\sharp$ of high energy, which is the union of K^i and regular loose ends.

The following theorem summarizes all the structural results available in the literature (at least to our knowledge).

THEOREM 1.4.2. *Assume (K, u) is an absolute or a generalized minimizer of E_λ. Then*

(i) *K^i is a relatively closed subset, and its Hausdorff dimension is at most $1-\varepsilon$, where ε is a positive geometric constant;*
(ii) *Every nonterminal point of K is either a triple junction or a pure jump point;*
(iii) *If U is an open set such that $U \cap K$ consists of finitely many connected components, then $K^i \cap U = \emptyset$;*

(iv) *A point $p \in K$ belongs to $K^\sharp$ if and only if*

$$\limsup_{r\downarrow 0} \frac{1}{r}\int_{B_r(x)} |\nabla u|^2 > 0\,, \tag{1.4.1}$$

which occurs if and only if

$$\liminf_{r\downarrow 0} \frac{1}{r}\int_{B_r(x)} |\nabla u|^2 > 0\,; \tag{1.4.2}$$

(v) *Triple junctions and regular loose ends form a discrete set;*

(vi) *A terminal point which is not the accumulation point of infinitely many connected components of K is necessarily a regular loose end.*

Note that K^i could be further subdivided in the union of those points $\{p\}$ of K which are connected components of K (“singletons”) and of the irregular terminal points of the connected components of K with positive length. The current state of the art in the regularity theory does not allow to conclude that singletons are absent, or that terminal points of connected components of K with positive length are *always* regular loose ends. In both cases the scenario that cannot be excluded is that of points p which are the accumulation of an infinite sequence of connected components of K with positive length. It seems simpler to exclude that this might happen for terminal points of connected components with positive length: the latter would be implied by the following strengthening of the first part of Theorem 1.3.4, which is interesting in its own right.

Conjecture 1.4.3. *Let (K, u) be a global generalized minimizer with the property that K has one unbounded connected component. Then (K, u) is either an elementary global minimizer or a cracktip.*

We finally record here that the structure theorem yields another equivalent formulation of the Mumford-Shah conjecture in terms of the optimal summability of ∇u (cf. [20]).

Theorem 1.4.4. *Let (K, u) be an absolute or generalized minimizer of E_λ in Ω. The set K^i is empty if and only if $\nabla u \in L^{4,\infty}_{loc}$, namely if and only if for every compact set $U \subset \Omega$ there is a constant $C = C(U)$ such that*

$$|\{x \in U : |\nabla u(x)|^4 \leq M\}| \leq \frac{C}{M} \qquad \forall M \geq 1\,. \tag{1.4.3}$$

1.5. Critical points

Restricted, absolute, generalized, and generalized restricted minimizers are naturally critical points in the following sense (cf. Proposition 2.5.1). We start by defining two classes of appropriate variations of the relevant functionals.

Definition 1.5.1. Let (K, u) be an admissible pair in an open set U.

(Out) If $\varphi \in W^{1,2}(U \setminus K)$ with $\operatorname{spt}(\varphi) \subset\subset U$, then the 1-parameter family of competitors $(K_t, u_t) = (K, u + t\varphi)$ will be called an *outer variation* in U.

(In) Consider any one-parameter family of diffeomorphisms $(-\varepsilon, \varepsilon) \times \mathbb{R}^2 \ni (t, x) \mapsto \Phi_t(x) = \Phi(t, x)$ of U, which is C^1 in both variables and such that $\Phi_t(x) = x$ for all x outside a compact subset of U. Then $(K_t, u_t) = (\Phi_t(K), u \circ \Phi_t^{-1})$ will be called an *inner variation*.

As it can be naturally expected, for minimizers we will show that the condition

$$\frac{d}{dt}\Big|_{t=0} E_\lambda(K_t, u_t, U, g) = 0 \tag{1.5.1}$$

holds for every inner and outer variation in U. Equation (1.5.1) gives corresponding Euler-Lagrange conditions, which in turn play a crucial role in the regularity theory. In fact several conclusions can be inferred from such conditions alone and in these notes we will make an effort to keep track of them.

The first Euler-Lagrange condition, derived from outer variations is given by

$$-\int_{\Omega\setminus K} \nabla u \cdot \nabla \varphi = \lambda \int_\Omega \varphi(u-g) \qquad \forall \varphi \in W^{1,2}(\Omega \setminus K) \text{ with } \operatorname{spt}(\varphi) \subset\subset \Omega. \tag{1.5.2}$$

The second, derived from inner variations, takes a particularly simple form when $g \in C^1$:

$$\begin{aligned}&\int_{\Omega\setminus K} \Big(|\nabla u|^2 \operatorname{div}\psi - 2\nabla u^T \cdot D\psi\, \nabla u\Big) + \int_K e^T \cdot D\psi\, e\, d\mathcal{H}^1\\ &= \lambda \int_{\Omega\setminus K} (2(u-g)\nabla g \cdot \psi - |u-g|^2 \operatorname{div}\psi) \qquad \forall \psi \in C^1_c(\Omega, \mathbb{R}^2)\,,\end{aligned} \tag{1.5.3}$$

where $e(x)$ is a unit tangent vector field to K and the notations $a \cdot b$ and Ab refer to, respectively, the scalar product between two vectors a and b and the usual product of a matrix A and a (column) vector b (this convention will be followed through the rest of these notes). The test φ in the outer variation relates to u_t through the relation $u_t = u + t\varphi$ when it is C^1, while the test ψ relates to the one parameter family of diffeomorphisms in Definition 1.5.1(In) via $\psi(x) = \partial_t \Phi(0, x)$.

Note that (1.5.3) does not make sense when $g \in L^\infty$. In fact it is not at all obvious that $t \mapsto E_\lambda(K_t, u_t, \Phi_t(U), g \circ \Phi_t^{-1})$ is differentiable at $t = 0$ in that case. This can however be shown for minimizers and corresponding Euler-Lagrange conditions can be derived. In order to state it, we first introduce the normal ν, which is the counterclockwise rotation of e by an angle of 90 degrees. For minimizers we can then show the existence of suitable traces u^+, u^- and g_K on K, which are locally bounded Borel functions and such that

$$\begin{aligned}&\int_{\Omega\setminus K} \Big(|\nabla u|^2 \operatorname{div}\psi - 2\nabla u^T \cdot D\psi\, \nabla u\Big) + \int_K e^T \cdot D\psi\, e\, d\mathcal{H}^1\\ &= 2\lambda \int_{\Omega\setminus K} (u-g)\nabla u \cdot \psi + \lambda \int_K (|u^+ - g_K|^2 - |u^- - g_K|^2)\psi \cdot \nu d\mathcal{H}^1 \quad \forall \psi \in C^1_c(\Omega, \mathbb{R}^2)\,.\end{aligned} \tag{1.5.4}$$

The latter identity is in fact equivalent to (1.5.3) when $g \in C^1$.

Motivated by the above discussion we introduce the following notions.

Definition 1.5.2. Consider $g \in C^1$. A pair (K, u) in Ω is a critical point of E_λ if (1.5.2)-(1.5.3) hold for every $\varphi \in W^{1,2}(\Omega \setminus K)$ with $\mathrm{spt}\,(\varphi) \subset\subset \Omega$ and every $\psi \in C^1_c(\Omega, \mathbb{R}^2)$.

Consider $g \in L^\infty$. A pair (K, u) in Ω is a critical point of E_λ if (1.5.2) holds for every $\varphi \in W^{1,2}(\Omega \setminus K)$ with $\mathrm{spt}\,(\varphi) \subset\subset \Omega$ and if there are bounded Borel functions u^+, u^-, and g_K such that (1.5.4) holds for every $\psi \in C^1_c(\Omega, \mathbb{R}^2)$.

The variational identities (1.5.2)-(1.5.4) are proved for minimizers in Proposition 2.5.1. Both (1.5.2) and (1.5.3) are well-known and their proof are reported, for instance, in [4]. However we have not been able to find a reference in the literature for (1.5.4) and we thus give its derivation in the appendix.

Remark 1.5.3. In these notes we will take advantage of the identity (1.5.4) to prove several monotonicity formulae which we will freely use in the proofs of the regularity results of Theorem 1.2.3 in cases (b) and (c). On the other hand, those ε-regularity statements hold more generally for quasi-minimizers of the functional E_0, of which minimizers of E_λ are a distinguished example. In particular our use of the monotonicity formulae can be avoided.

More precisely, the conclusions of Theorem 1.2.3 in cases (b) and (c) of Conjecture 1.2.1 are proved in [13, 5, 3] (cf. also [4, 14]) for pairs (K, u) satisfying the following weakened version of (1.1.1): there are $\omega, \delta > 0$ such that

$$E_0(K, u, B_r(y)) \leq E_0(J, v, B_r(y)) + \omega r^{1+\delta}$$

(where x, r and (J, v) are arbitrary and satisfy the conditions for competitors given in Definition 1.1.1, with $U = B_r(x)$). Note that minimizers of E_λ are indeed quasi-minimizers of E with $\delta = 1$ and $\omega = 2\pi$ according to a simple comparison estimate.

However, the conclusion of Theorem 1.2.3 for Conjecture 1.2.1(a) seems to require crucially (1.5.4) or anyway a suitable approximate version. At present we do not know how to prove the same conclusion for quasi-minimizers or even whether to expect it to be true for them.

We also believe that the use of the variational identities in the proofs of the cases (b) and (c) has its own interest: besides providing a different point of view which covers the most important examples of quasi-minimizers of E_0 (i.e. minimizers of E_λ), it simplifies considerably some of the arguments.

1.6. Plan of the notes

We start the notes with Chapter 2, which will collect a number of preliminary results which have their own independent interest but will also be instrumental in the proofs of the main theorems. We will in particular:

- state and prove some elementary bounds and maximum principles;
- state a fundamental density lower bound on the discontinuity set K, due to De Giorgi, Carriero, and Leaci, whose proof is deferred to the appendix;
- introduce the “blow-up procedure” and state some relevant compactness properties of minimizers, with most of the proofs deferred to the appendix;

- detail the variational identities stated in Section 1.5 (whose proofs are given in the appendix) and derive a few important corollaries, in particular most of the monotonicity formulae.

Chapter 3 will prove the cases (b) and (c) of Theorem 1.2.3 (cf. Subsection 3.8, see also Theorems 3.1.1 and 3.1.2 for slightly different statements). The proof in case (b) (e.g. for "pure jumps") essentially follows the approach of Ambrosio, Fusco, and Pallara, cf. [4], but it must be noted that in our setting we can exploit the 1-dimensionality of the set K to take several shortcuts. For the case (c) of triple junctions we propose an approach which is different and new compared to that of David in [13]. In particular we take advantage of the monotonicity formulae of the first chapter and of a "blow-in" argument which considerably simplifies the overall proof.

Chapter 4 is a self-contained proof of Theorem 1.3.4. The arguments for the first part of the theorem are borrowed from the book [9] while the arguments for the second part are due to [15]. Here and there we propose alternative derivations and we streamline and simplify a few steps of the original arguments. Moreover, we prove first the second part of the theorem and hence take advantage of it whenever possible to prove the first part (a possibility which was not at disposal of the authors in [9], given that the paper [15] appeared afterwards).

The first part of Theorem 1.3.4 is in fact a stepping stone for case (c) of Theorem 1.2.3, more precisely it implies the intermediate result of Corollary 4.1.2, namely that when for an absolute minimizer (K, u) the set K is sufficiently close to a radius in the disk $B_r(x)$, then it consists of a single connected component in $B_{r/2}(x)$. Chapter 5 starts from this conclusion and leads to a complete proof of case (c) of Theorem 1.2.3 (cf. Theorem 5.0.1). We follow the arguments of our previous work [21] with some appropriate modifications to account for the fidelity term in the energy E_λ (the work [21] addressed the case of E_0).

Chapter 6 examines a few important consequences of the ε-regularity theory. It first studies which structural conclusions can be derived from it. More precisely it proves all the structural results of Theorem 1.4.2, apart from item (ii), which is exactly the content of Corollary 4.4.2: a quantitative version of item (i) is given in Corollary 6.1.3, and items (iii) to (vi) are implied by Theorem 6.1.1. Theorem 1.4.4 is the content of Theorem 6.1.5. Moreover the chapter addresses some other consequences of the ε-regularity theory, like the porosity of the singular portion of K, and the higher integrability of ∇u, conjectured by De Giorgi and first proved in [20] in the setting of this book (i.e. in 2-dimensions) and hence settled in all dimensions by De Philippis and Figalli in [22]. In fact we present both arguments.

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CHAPTER 2

Density bounds, compactness, variations, and monotonicity

In this chapter, we will collect some preliminary important considerations, before delving into the ε-regularity theory in the next four chapters. For some aspects which are not our main focus, but which are needed to understand the rest of the notes, we will nonetheless include the arguments for the reader's convenience (and because in some cases it has been difficult to track in the literature appropriate arguments that apply to our precise statements): however, when the proofs are long and technical we will move them to the appendix.

After collecting some preliminary lower and upper bounds on the energy of minimizers and on the length of K, we will use them to introduce a pivotal procedure in the regularity theory, that of "blow-up". This procedure will allow us to zoom around a point $x \in K$ and extract meaningful limits (global minimizers) which in turn will give us a first rough picture of the local behavior around the point x. The second part of the chapter is devoted to important variational identities that can be derived by plugging suitable tests in (1.5.4). We will review several consequences of these identities (the determination of the cracktip factor, Léger's magic formula, the David-Léger-Maddalena-Solimini identity, and the truncated test identities). Finally, in the last section, we will introduce two important monotonicity formulae due, respectively, to Bonnet, and to David and Léger. While the first has a straightforward proof, which will be given in this chapter, the latter requires a more involved argument, which will be detailed later in Chapter 4.

Though we have not yet introduced the concept of "generalized minimizer" for E_0 (which will be given in Definition 2.2.4), some preliminary statements hold for the latter as well and we point this out when needed. The proof will either become obvious once the concept is defined, or it is postponed to the appendix, where we will treat all cases.

2.1. Preliminaries

In this section, we collect some preliminary considerations.

2.1.1. A maximum principle. We first give the following elementary maximum principle (see Figure 1).

LEMMA 2.1.1. *Let (K, u) be an absolute or restricted minimizer of E_λ in Ω, or a generalized (resp. generalized restricted) minimizer of E_0 (cf. Definition 2.2.4) and let V be a connected component of $U \setminus K$ for some open $U \subset\subset \Omega$.*

(a) *If $V \cap \partial U = \emptyset$, then $\|u\|_{L^\infty(V)} \leq \|g\|_\infty$ when $\lambda > 0$ and $\nabla u|_V = 0$ when $\lambda = 0$;*

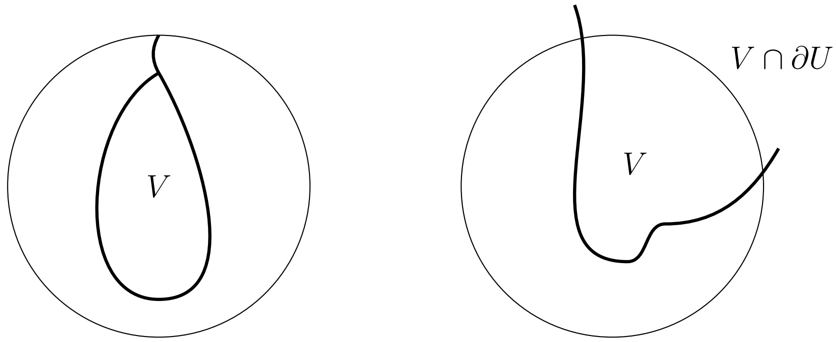

FIGURE 1. On the left case (a) and on the right case (b) of Lemma 2.1.1. In these examples U is a disk.

(b) *If V intersects ∂U, then if $\lambda > 0$*

$$\min\left\{\inf_V g,\ \inf_{V\cap\partial U} u\right\} \le \inf_V u \le \sup_V u \le \max\left\{\sup_V g,\ \sup_{V\cap\partial U} u\right\},$$

while if $\lambda = 0$

$$\inf_{V\cap\partial U} u \le \inf_V u \le \sup_V u \le \sup_{V\cap\partial U} u\,.$$

PROOF. In case statement (a) were false, we could define $\bar K = K$, $\bar u = u$ on $\Omega\setminus V$ and $\bar u = 0$ on V when (K,u) minimizes E_0: the pair $(\bar K, \bar u)$ would then have a lower energy. Similarly, when the pair minimizes E_λ with λ positive, u would be the absolute minimizer of $\int_V(|\nabla u|^2 + \lambda(u-g)^2)$ in V and hence its L^∞ norm is bounded by $\|g\|_\infty$ by an obvious truncation argument. As for (b), if $\lambda > 0$ set

$$m := \min\left\{\inf_V g,\ \inf_{V\cap\partial U} u\right\}, \qquad M := \max\left\{\sup_V g,\ \sup_{V\cap\partial U} u\right\}.$$

If (b) were false, we could just define $\bar K = K$, $\bar u = u$ on $\Omega\setminus V$ and $\bar u = \min\{M, \max\{m,u\}\}$ on V: the pair $(\bar K, \bar u)$ would contradict the minimality. Similarly, if $\lambda = 0$ we conclude by comparison with $\bar u = \min\{\sup_{V\cap\partial U} u, \max\{\inf_{V\cap\partial U} u, u\}\}$. □

2.1.2. Lower and upper bounds. Next, we point out that it is rather trivial to come up with upper bounds for the full energy in a disk $B_r(x)$ using a simple comparison argument .

LEMMA 2.1.2. *Assume (K,u) is an absolute or restricted minimizer of E_λ in $\Omega\subset\mathbb{R}^2$ or a generalized (resp generalized restricted) minimizer of E_0 (cf. Definition 2.2.4). Let $B_r(x)\subset\subset\Omega$ and in case (K,u) is a restricted minimizer assume in addition $\overline{B}_r(x)\cap K\neq\emptyset$. Then the estimate (1.3.1) holds.*

PROOF. Compare the energy of (K,u) to the competitor (J,v) defined by setting $J = \partial B_r(x)\cup(K\setminus B_r(x))$, $v = 0$ on $B_r(x)$ and $v = u$ on $\Omega\setminus(\overline{B_r(x)}\cup K)$. See Figure 2. □

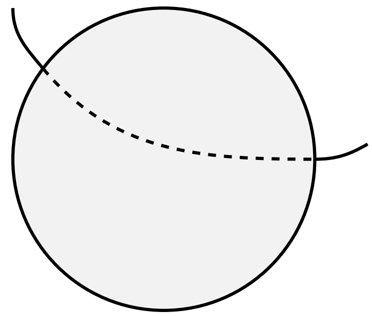

FIGURE 2. The competitor (J, v) in Lemma 2.1.2: we remove the dashed part of the set K, we add the circle $\partial B_r(x)$, we set $v = 0$ in the shaded disk and we keep $v = u$ outside of it.

A much more interesting and nontrivial fact is that it is possible to bound the length of K uniformly from below on any disk which is centered at a point of K. This fundamental discovery, which is at the foundation of all the regularity theory, is due to De Giorgi, Carriero and Leaci [18] (and an appropriate generalization appropriately holds in any dimension). Since then different proofs have appeared in the literature (see [11, 12, 13, 19, 10]). However, the statement below, which is valid across different formulations and for restricted minimizers independently of the number of the connected components of K, seems to be new and we include its proof in the appendix.

THEOREM 2.1.3. *There exists a geometric constant $\epsilon > 0$ with the following property.*

(a) *If (K, u) is a (absolute, restricted, generalized, or generalized restricted) minimizer of E_0 on Ω, then*

$$\mathcal{H}^1(K \cap B_\rho(x)) \geq \epsilon\rho \qquad \forall x \in K, \forall \rho \in (0, \operatorname{dist}(x, \partial\Omega))\,. \tag{2.1.1}$$

(b) *If (K, u) is a (absolute or restricted) minimizer of E_λ on Ω, then*

$$\mathcal{H}^1(K \cap B_\rho(x)) \geq \epsilon\rho \qquad \forall x \in K, \forall \rho \in (0, \min\{1, \operatorname{dist}(x, \partial\Omega)\})\,. \tag{2.1.2}$$

We remark that the constant ϵ in (2.1.2) depends indeed on λ and $\|g\|_\infty$, a dependence that we can ignore thanks to (1.0.2). Note that the two bounds (1.3.1)-(2.1.1) (resp. (2.1.2)) taken together describe a property of K which is often termed *Ahlfors regularity* in the literature.

2.1.3. Normalization and equivalence between classical and SBV formulations. The density lower bound in Theorem 2.1.3 and Lemma 2.1.1, together with an elementary fact about SBV functions observed by De Giorgi, Carriero and Leaci, gives the following corollary, whose proof is as well postponed to the appendix. We do not dwell here on the definition of BV and SBV functions and on the definition of the set S_u, but we refer the reader to the textbook [4].

COROLLARY 2.1.4. *Let (K, u) be a (absolute, restricted, generalized, or generalized restricted) minimizer of E_λ. Then $u \in SBV(\Omega)$ with $\overline{S}_u \subseteq K$. If u is an absolute minimizer, then we have in addition $\mathcal{H}^1(K \setminus \overline{S}_u) = 0$.*

Moreover, if $\mathcal{H}^1(K\cap U) = 0$ for some open domain $U \subset\subset \Omega$, then u extends to a function $u \in C^{1,\alpha}_{loc} \cap W^{2,p}_{loc}(U)$ for every $p < \infty$ and every $\alpha \in (0,1)$ which solves $\Delta u = \lambda(u-g)$.

REMARK 2.1.5. Corollary 2.1.4 implies that the function u and its gradient ∇u are pointwise defined on any open subset U in which $\mathcal{H}^1(K \cap U) = 0$, since u is continuous and continuously differentiable. This easily implies that any minimizer can be "normalized" in the sense of Definition 1.1.2, as explained in the introduction. In the rest of these notes we will make a similar assumption whenever dealing with general critical points, i.e. we will assume that they are normalized and that they belong to $C^{1,\alpha}_{loc} \cap W^{2,p}_{loc}(\Omega \setminus K)$.

2.2. Blow-up of minimizers

In this section we assume that (K, u) is an *absolute* or a *restricted* minimizer of the energy functional E_λ (and we recall that g is bounded). Observe that (1.3.1) gives locally an apriori estimate on the energy of u. Fix a point $x \in K$ and a sequence of radii $r_j \downarrow 0$. The bound suggests to consider the rescaled functions

$$K_j := \left\{ \frac{y-x}{r_j} : y \in K \right\} \tag{2.2.1}$$

$$u_j(y) := r_j^{-\frac{1}{2}} u(x + r_j y)\,. \tag{2.2.2}$$

When $\lambda = 0$ the pair (K_j, u_j) is then still a minimizer of the functional E_0 if (K, u) is. In case of $\lambda > 0$, the density lower bound (cf. Theorem 2.1.3) ensures that

$$\epsilon\, r \leq \int_{B_r(x)} |\nabla u|^2 + \mathcal{H}^1(K \cap B_r(x))$$

for every sufficiently small radius r, while the maximum principle gives

$$\mathscr{F}(u, B_r(x), g) = \int_{B_r(x)} |u-g|^2 \leq 4\pi \|g\|^2_{L^\infty(B_r)} r^2\,,$$

in particular the fidelity term $\mathscr{F}$ becomes negligible compared to E_0 at small scales.

It would be desirable to use now the upper bound on E_λ in order to provide a suitable compactness result. Note however that E_0 controls the values of the function only up to an additive constant in each connected component of $\Omega \setminus K$, while when $\lambda > 0$ the fidelity term does not help because it is a lower order perturbation. Thus the space of rescalings of minimizers is in general not (pre)compact in a classical sense: roughly speaking to find a meaningful compactification we must allow the limits of the rescalings to take infinite values, and in fact we need to distinguish between infinities of "different size".

We next deal with a slightly more general situation in which we make the following assumptions.

ASSUMPTION 2.2.1. We assume that

(i) λ_j is a sequence of numbers in $[0, 1]$;
(ii) g_j is a sequence of bounded functions with $\|g_j\|_\infty \leq M_0$;

(iii) (K^j, u^j) is a sequence of absolute or restricted minimizers of

$$E_{\lambda_j}(K, u, U_j, g_j) = E_0(K, u, U_j) + \lambda_j \mathscr{F}(u, U_j, g_j)$$

on a sequence of domains U_j;

(iv) $\lim_{j\uparrow\infty} \lambda_j r_j \|g_j\|^2_\infty = 0$;

(v) a certain open domain U satisfies $U' \subset \frac{U_j - x_j}{r_j}$ for every $U' \subset U$ compact and for j large enough (depending on U').

We define the rescaled sets and functions

$$K_j = K^j_{x_j, r_j} := \frac{K^j - x_j}{r_j}$$

$$u_j(y) = u^j_{x_j, r_j}(y) := \frac{u^j(r_j y + x_j)}{r_j^{1/2}}\,.$$

Up to subsequences (and using standard arguments) we can further assume the following.

(a) $K_j \cap U$ converges locally in the Hausdorff distance to a closed set $K \cap U$.

(b) We enumerate the connected components $\{\Omega_k\}_{k\in\mathscr{I}}$ of $U \setminus K$ and for each we select a point $z_k \in \Omega_k$. The real numbers

$$\{u_j(z_k) - u_j(z_l)\}_{j\in\mathbb{N}} \tag{2.2.3}$$

converge to some elements p_{kl} of the extended real line $[-\infty, \infty]$.

(c) The functions

$$v^k_j := u_j - u_j(z_k) \tag{2.2.4}$$

converge locally in $W^{1,2}_{loc}(\Omega_k)$ to an harmonic function v^k (this requires classical estimates for solution of the Laplace equation, for the details we refer the reader to the appendix).

We introduce further the function v on $U \setminus K$ by setting $v = v^k$ on each Ω_k.

When $x_j = x$, $r_j \downarrow 0$, $K^j \equiv K^0$ and $u^j \equiv u^0$, the above sequence will be called a *blow-up sequence*. Observe in addition that, because of (i) and (ii), (iv) is always satisfied for a blow-up sequence. It is elementary to see that Theorem 2.1.3 and Lemma 2.1.2 imply that K has locally finite $\mathcal{H}^1$ measure (cf. the proof in the appendix). The rectifiability of K is instead more delicate. Leaving that aside for the moment, we introduce the following terminology.

Definition 2.2.2. The triple (K, v, p_{kl}) as above will be called a *limit point* of (an appropriate subsequence of) (K_j, u_j) in the set U. In the case of a blow-up sequence, such triple will be called a *blow-up of* (K^0, u^0) *at* x. A pair (J, w) will be called an "admissible competitor" for the pair (K, v) if it satisfies the following assumptions:

(i) (J, w) coincides with (K, v) outside of an open set $O \subset\subset U$;

(ii) if $x, y \in U \setminus (O \cup K)$ belong to distinct connected components of $U \setminus K$, then they belong to distinct connected components of $U \setminus J$ (see Figure 3 for a simple illustration of the latter condition).

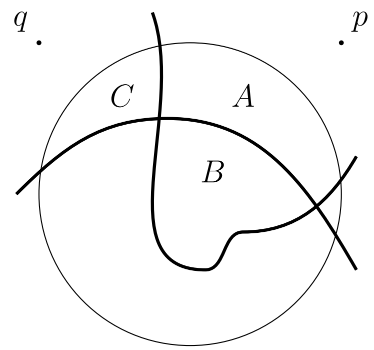

FIGURE 3. A visual explanation of the admissibility in point (ii) of Definition 2.2.2. The set K is given by the thick lines, while the open set O is the interior of the circle. In O we are allowed to change K to a new set J under the condition that any two points outside O which belong to distinct components of $U \setminus K$ will still belong to distinct connected components of $U \setminus J$. For instance, we cannot remove from K any arc which lies between the regions A and C, since such operation would "connect" the points p and q. However we are allowed to remove from K an arc which lies between A and B. The picture is also a good illustration of Lemma 2.2.5. As long as $J \cap O$ "separates" the four arcs which are the connected components of $\partial O \setminus K$ in O, the pair (J, w) is certainly an admissible competitor for (K, v).

It turns out that this is indeed a good candidate for a suitable variational problem in the limit. The proof of the following theorem will be given in the appendix for the reader's convenience.

THEOREM 2.2.3. *Let (K_j, u_j) be a sequence as in Assumption 2.2.1 and let $(K, v, \{p_{kl}\})$ be as in Definition 2.2.2. Then K is rectifiable and has locally finite $\mathcal{H}^1$ measure, while $v \in W^{1,2}(U' \setminus K)$ for every $U' \subset\subset U$. Moreover the following holds.*

(i) *For every $O \subset\subset U$ such that $\mathcal{H}^1(\partial O \cap K) = 0$, we have*

$$\lim_{j\to\infty} \mathcal{H}^1(K_j \cap \overline{O}) = \mathcal{H}^1(K \cap \overline{O})$$
$$\lim_{j\to\infty} \int_{O\setminus K_j} |\nabla u_j|^2 = \int_{O\setminus K} |\nabla v|^2 \, .$$

For every continuous compactly supported function $\varphi : \mathbb{P}^1\mathbb{R} \times U \to \mathbb{R}$ and for every O bounded open set with $\mathcal{H}^1(O \cap K) > 0$ we have

$$\lim_{j\to\infty} \int_{K_j \cap O} \varphi(T_x K_j, x) d\mathcal{H}^1(x) = \int_{K\cap O} \varphi(T_x K, x)\, d\mathcal{H}^1(x)\, . \tag{2.2.5}$$

(ii) *In the case of absolute minimizers, if (J, w) is an admissible competitor as in Definition 2.2.2, then $E_0(J, w) \geq E_0(K, v)$. In the case of restricted minimizers the same holds if J does not increase the number of connected components of K.*

(iii) *Let $\mathscr{A} \subseteq \mathscr{I}$ be a set of indices with the property that $-\infty < p_{kl} < \infty$ for every $k, l \in \mathscr{A}$. Define*
- $k_0 := \min \mathscr{A}$;
- $U_{\mathscr{A}} := \cup_{k \in \mathscr{A}} \Omega_k$ *and* $\Omega_{\mathscr{A}} := \mathrm{int}\big(\overline{U_{\mathscr{A}}}\big)$;
- $u_{\mathscr{A}} := v^{k_0}$ *on* Ω_{k_0} *and* $u_{\mathscr{A}} := v^k + p_{kk_0}$ *on* Ω_k *for any* $k \in \mathscr{A}$.

Then $(K, u_{\mathscr{A}})$ is an absolute (resp. restricted) minimizer of E_0 on the set $\Omega_{\mathscr{A}}$.

Definition 2.2.4. A triple $(K, v, \{p_{kl}\})$ on U as in the theorem above will be called a "generalized minimizer" resp. "generalized restricted minimizer" if it is the limit of absolute resp. restricted minimizers . If $U = \mathbb{R}^2$, then the triple will be called "global generalized minimizer", resp. "global generalized restricted minimizer". Finally, an "elementary global generalized minimizer" is a global generalized minimizer whose Dirichlet energy vanishes identically on compact sets.

In a variety of situations, it will be useful to have a quick way to identify whether a competitor (J, w) satisfies the requirement (ii) of Definition 2.2.2. The following simple remark will be widely used in that sense.

Lemma 2.2.5. *Let (K, v) and (J, w) be pairs in Ω which coincide outside of an open disk $B_r(x) \subset\subset \Omega$. Let $\{\gamma_i\}_i$ be the connected components of $\partial B_r(x) \cap K = \partial B_r(x) \cap J$. If no distinct pairs γ_i and γ_j are contained in the closure of the same connected component of $B_r(x) \setminus J$, then the pair (J, w) is admissible in the sense of Definition 2.2.2.*

The proof is left as a simple exercise to the reader, see again Figure 3.

2.3. Compactness for generalized minimizers

The theorem above can be extended to generalized minimizers of E_0 (absolute or restricted). This remark will be especially used in "blow-down" procedures for global generalized minimizers $(K, u, \{p_{ij}\})$, i.e. any limit of a subsequence of the rescalings $(K_{0,R}, u_{0.R}, \{p_{ij}\})$ as the radius diverges, where recall that $K_{0,R} = \frac{K}{R}$ and $u_{0,R}(y) = R^{-\frac{1}{2}} u(Ry)$.

Assumption 2.3.1. We assume that
(i) U_j is a sequence of monotonically increasing domains which converge to some domain Ω;
(ii) (K_j, u_j) is part of a triple $(K_j, u_j, \{p_{kl,j}\}_j)$ of generalized (restricted) minimizers.

Up to subsequences we further assume the following.
(a) K_j converges locally in the Hausdorff distance to a closed set K;
(b) We enumerate the connected components $\{\Omega_k\}_{k \in \mathscr{I}}$ of $\Omega \setminus K$, for each we select a point $x_k \in \Omega_k$. For each fixed x_k, x_l will belong to appropriate connected components $\Omega_{k,j}$ of $U_j \setminus K_j$ for j large enough and we will assume that
$$\{u_j(x_k) - u_j(x_l) - p_{kl,j}\}_{j \in \mathbb{N}} \tag{2.3.1}$$
converge to some elements p_{kl} of the extended real line;

(c) The functions
$$v_j^k := u_j - u_j(x_k) \tag{2.3.2}$$
converge locally in $W^{1,2}_{loc}(\Omega_k)$ to an harmonic function v^k.

We introduce further the function v on $\Omega \setminus K$ by setting $v = v^k$ on each Ω_k.

THEOREM 2.3.2. *Under the Assumption 2.3.1 all the conclusions of Theorem 2.2.3 apply to the triple $(K, v, \{p_{kl}\})$ and the corresponding sequence $\{(K_j, u_j)\}$.*

2.4. Elementary global generalized minimizers

We next turn to the simplest type of global generalized minimizers, i.e. those for which the Dirichlet energy vanishes identically.

THEOREM 2.4.1 (Classification of elementary global generalized minimizers). *Let $(K, v, \{p_{kl}\})$ be a global generalized minimizer of E_0 and assume that $\int |\nabla v|^2 = 0$. Then $(K, v, \{p_{kl}\})$ is either*

(a) *A* constant, *namely $K = \emptyset$ and v is a constant.*
(b) *A* global pure jump, *namely*
  (b1) *K is a straight line,*
  (b2) *v is constant on each connected component Ω_1 and Ω_2 of $\mathbb{R}^2 \setminus K$,*
  (b3) *and $|p_{12}| = \infty$.*
(c) *A* global triple junction, *namely:*
  (c1) *K is the union of three half lines originating at a common point where they form equal angles,*
  (c2) *v is constant on each of the three connected components Ω_1, Ω_2 and Ω_3 of $\mathbb{R}^2 \setminus K$,*
  (c3) *and $|p_{12}| = |p_{13}| = |p_{23}| = \infty$.*

For global generalized restricted minimizers such that $K \neq \emptyset$, then K is as in (b1) or as in (c1). The conclusions (b2) and (c2) hold as well. The conclusions (b3) and (c3) do not hold for all generalized restricted minimizers, but do hold for those satisfying the following stronger variational property:

- *$E_0(K, v, U) \leq E_0(J, w, U)$ for any bounded open set U and any pair (J, w) such that $\{v \neq w\} \subset\subset U$ and for which J consists of at most two connected components.*

Finally, if $(K, v, \{p_{kl}\})$ is a global generalized minimizer, then:

(i) *If K is empty and $(K, v, \{p_{kl}\})$ is an* absolute *minimizer, then it is necessarily a constant;*
(ii) *If K is a straight line, then it is necessarily a pure jump;*
(iii) *If K is the union of three half lines originating at a common point, then it is necessarily a triple junction.*

PROOF. We focus on the case of generalized minimizers, leaving the analogous one of generalized restricted minimizers to the reader.

We start off proving items (i)-(iii) in the second part of the statement.

Assume first $K = \emptyset$, then v is harmonic on $\mathbb{R}^2$. In view of the density upper bound (cf. (1.3.1)) and the mean value property for harmonic functions we conclude that $|\nabla v| = 0$ on $\mathbb{R}^2$.

If K is a line, w.l.o.g. we may assume $K = \{x \in \mathbb{R}^2 : x_2 = 0\}$ and set $H^\pm := \{x \in \mathbb{R}^2 : \pm x_2 > 0\}$. Then v is harmonic on $H^+ \cup H^-$ and $\frac{\partial v}{\partial \nu} = 0$ on K. By the Schwartz reflection principle, the even extensions $v_\pm$ of $v|_{H^\pm}$ across K are harmonic on the whole of $\mathbb{R}^2$ with

$$\int_{B_\rho} |\nabla v_\pm|^2 dx = 2 \int_{H^\pm \cap B_\rho} |\nabla v|^2 dx \le 4\pi\rho.$$

Arguing as above, we infer that $|\nabla v| = 0$ on $H^+ \cup H^-$, therefore v is locally constant on $\mathbb{R}^2 \setminus K$. To conclude the proof of item (ii) we have to show that $|p_{12}| = \infty$. Otherwise setting $\mathscr{A} = \{1, 2\}$ and $w = u_{\mathscr{A}}$, we can apply Theorem 2.3.2 to the rescalings $u_{0,R}(x) := R^{-\frac{1}{2}} v(Rx)$ and $K_{0,R}$. Obviously they converge to $(K, 0, \{0\})$ as $R \uparrow \infty$, and so $(\tilde{K}, \tilde{u}) = (K, 0)$ would have to be an absolute minimizer on any bounded open subset of $\mathbb{R}^2$. The latter assertion is false, as we can remove any compact subset of K and extend the function w to 0 on it.

In case (iii), $\mathbb{R}^2 \setminus K = \Omega_1 \cup \Omega_2 \cup \Omega_3$, each Ω_i being a convex cone with vertex p and opening $\alpha_i \in (0, 2\pi)$. Recalling that $\triangle v = 0$ on Ω_i and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega_i \cap K$, we may expand $w_i := v|_{\Omega_i}$ in Fourier series. In particular, given a point $x \in \Omega_i$, we set $r = \mathrm{dist}(x, p)$ and we let $\vartheta \in [0, \alpha_i]$ be the angle formed be the segment $[p, x]$ and one of the two half-lines delimiting Ω_i (the choice is not important). Hence, if we denote by $a_{i,k}$ the Fourier coefficients of $v|_{\partial B_1 \cap \Omega_i}$ in the angle ϑ, we can write

$$w_i(r, \vartheta) = \sum_{k=0}^{\infty} a_{i,k} r^{\frac{k\pi}{\alpha_i}} \cos\left(\frac{k\pi}{\alpha_i}\vartheta\right),$$

on Ω_i, while we can compute

$$\int_{\Omega_i \cap B_r(p)} |\nabla w_i|^2 = \sum_{k=1}^{\infty} \frac{k\pi}{2} a_{i,k}^2 r^{2\frac{k\pi}{\alpha_i}}. \tag{2.4.1}$$

Since $2k\pi \ge 2\pi > \alpha_i$, $k \ge 1$, we conclude that $a_{i,k} = 0$ for all $k \ge 1$ and $i \in \{1, 2, 3\}$, thanks again to inequality (1.3.1). Thus, v is locally constant on $\mathbb{R}^2 \setminus K$. In turn, being v a generalized global minimizer, $K \cap B_1(p)$ is a set with minimal length connecting three points on $\partial B_1(p)$, in particular the angles in p must be all equal to $\frac{2}{3}\pi$.

Assume now p_{ij} is finite for some $i \ne j$. Let $\mathscr{A} = \{i, j\}$ and consider $w = u_{\mathscr{A}}$ on $\Omega_{\mathscr{A}} = \mathrm{int}(\overline{\Omega_i \cup \Omega_j})$. Without loss of generality we can assume that $\partial\Omega_i \cap \partial\Omega_j = \{x_2 = 0, x_1 \ge 0\}$. We can then choose the points $y_k := (k^2, 0)$ and the radii $R_k := k$ and consider the pairs $(K_{y_k,R_k}, w_{y_k,R_k})$ on the domains $B_{\sqrt{k}}(0)$ obtained by rescaling and translating $\Omega_{\mathscr{A}}$. We can again apply Theorem 2.3.2 and we would conclude as above that $(K, 0)$ is a generalized minimizer, which is a contradiction.

Finally, we prove the classification of generalized global minimizers with null gradient energy as stated in (a)-(c). In this case K is a minimal Caccioppoli partition and then a minimal connection of $\partial B_R \cap K$ for all R (cf. [20, Lemma 12]). In particular, $R \mapsto \frac{\mathcal{H}^1(\partial B_R \cap K)}{R}$ is nondecreasing and $\frac{K}{R}$ is converging to a minimal cone K_∞ as $R \uparrow \infty$. Therefore, K_∞ is either a line or a propeller, the union of three half-lines meeting at a common point with equal angles. In turn, this implies that $\mathcal{H}^0(\partial B_R \cap K) \leq 3$ at least for some sequence of R's converging to infinity by the coarea formula [4, Theorem 2.93] (because $R^{-1}\mathcal{H}^1(B_R \cap K) \leq 3$ for every $R > 0$). In particular, either $\mathcal{H}^0(\partial B_\rho \cap K) = 3$ for some ρ, and in this case $B_\rho \cap K$ is a propeller, or $\mathcal{H}^0(\partial B_\rho \cap K) = 2$ for some ρ and in this case $B_\rho \cap K$ is a segment. Moreover, in the first case $\mathcal{H}^0(\partial B_R \cap K) = 3$ for all $R \geq \rho$, and thus K is a propeller, while in the second case $\mathcal{H}^0(\partial B_R \cap K) = 2$ for all $R \geq \rho$, and K is a line (cf. [20, Sections 2.3 and 3]).

We conclude the proof of (a)-(c) by appealing to the classification result contained in (i)-(iii). □

2.5. Variational identities

As we have already remarked, restricted and absolute minimizers (K, u) of E_λ are critical points, namely they satisfy the identities (1.5.2) and (1.5.4). The same applies to generalized minimizers of E_0. We can thus summarize our conclusions in the following statement (for the proof see Appendix A).

Proposition 2.5.1. *Assume (K, u) is a (restricted or absolute) minimizer of E_λ or a generalized (restriced or absolute) minimizer of E_0. Then (K, u) is a critical point, namely there exists suitable traces u^+, u^-, and $g_K \in L^\infty(\Omega, \mathcal{H}^1 \llcorner K)$ such that the identities (1.5.2) and (1.5.4) hold, with the properties that $\|g_K\|_\infty \leq \|g\|_\infty$ and that $u^\pm$ are the classical one-sided traces of K in the sense of Sobolev-space theory. If g is, in addition, C^1, (1.5.4) is indeed equivalent to (1.5.3).*

A more explicit form of the Euler-Lagrange conditions can be devised in case K is a smooth graph. To this aim we denote by ν the counterclockwise rotation by 90 degrees of a C^0 unit tangent vector e locally orienting K, while we denote by κ the curvature of a local compatible parametrization, namely

$$\kappa = \ddot{\gamma} \cdot \nu\,, \tag{2.5.1}$$

for an arclength parametrization γ such that $\dot{\gamma} = e$. Such classical definition of the curvature κ assumes in general C^2 regularity. We will use it under the assumption that K is $C^{1,1}$: the reader can check that local arc-length parametrizations belong to $W^{2,\infty}$ and thus $\ddot{\gamma}$ is interpreted as an L^∞ function of t. Under such assumptions the curvature κ is then a bounded Borel function defined $\mathcal{H}^1$-a.e. on K. Finally, w^+ and w^- are the one-sided traces of the relevant function w on K (following the obvious convention that w^+ is the trace on the side which ν is pointing to, cf. Figure 4).

Proposition 2.5.2. *Let (K, u) be a critical point of E_λ in U and assume that $K \cap U$ consists only of regular jump points (i.e. $K \cap U$ is a C^1 submanifold[1]). Then*

(a) *u has $C^{1,\alpha}$ extensions on each side of $K \cap A$, for every $\alpha < 1$,*
(b) *$K \cap A$ is locally $C^{1,1}$,*

[1] In particular we are assuming that $U \cap K$ does not contain any loose end of K.

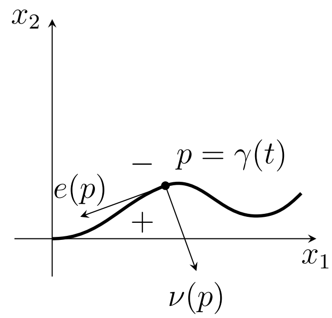

FIGURE 4. The tangent vector $e(p) = \dot\gamma(t)$ (for an-arc length parametrization) and the normal vector $\nu(p)$. The picture illustrates the convention for the symbols $\pm$ on traces of functions over γ.

(c) *the variational identities* (1.5.2)-(1.5.4) *are equivalent to the following three conditions*

$$\Delta u = \lambda(u-g) \qquad on\ \Omega\setminus K \tag{2.5.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad on\ K \tag{2.5.3}$$

$$\kappa = -(|\nabla u^+|^2 - |\nabla u^-|^2) - \lambda(|u^+ - g_K|^2 - |u^- - g_K|^2) \qquad \mathcal{H}^1\ a.e.\ on\ K\,, \tag{2.5.4}$$

where κ is the curvature of K and g_K is the function in Proposition 2.5.1.

The proof is given in the appendix for the reader's convenience. Therefore, in view of Remark 2.1.5 and item (a) above, in the rest of the notes we can simply assume that u is continuously differentiable in $\Omega\setminus K$.

We record here an important elementary consequence of (1.5.2) which will be used throughout the notes

COROLLARY 2.5.3. *Assume (K,u) is a critical point of E_λ in some domain Ω and fix $x\in\Omega$. For a.e. $r\in(0,\mathrm{dist}(x,\partial\Omega))$ the following holds. First of all we have*

$$\int_U |\nabla u|^2 + \lambda\int_U (u-g)u = \int_{\partial B_r(x)\cap\overline{U}} u\frac{\partial u}{\partial n} \tag{2.5.5}$$

for every connected component U of $B_r(x)\setminus K$ (where n is the unit normal to $\partial B_r(x)$).

Moreover if $\lambda = 0$ and γ is a connected component of $\partial B_r(x)\setminus K$ such that the endpoints of γ belong to the same connected component of K, then

$$\int_\gamma \frac{\partial u}{\partial n} = 0 \tag{2.5.6}$$

(see Figure 5 for an illustration of the two conclusions).

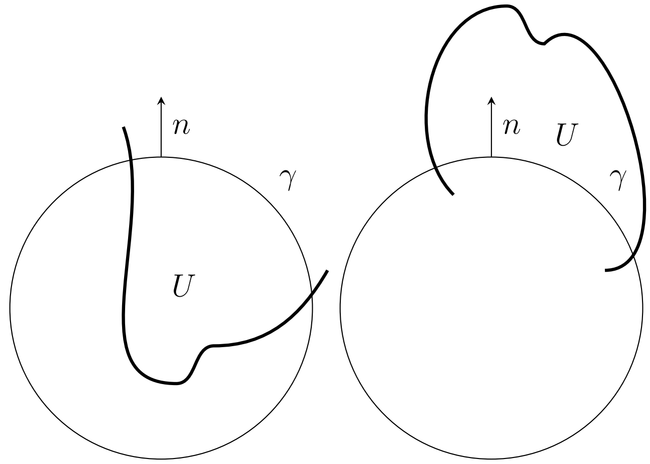

FIGURE 5. The arc γ is the set $\overline{U} \cap \partial B_r(x)$ and since its two endpoints belong to the same connected component of K, both conclusions of Corollary 2.5.3 apply. On the left the case $U \subset B_r(x)$, on the right the case $U \subset \Omega \setminus \overline{B}_r(x)$.

PROOF. Without loss of generality we assume $x = 0$. We will prove the claims for any radius r with the following property. First of all define the domain

$$\hat{K} := K \cup \{y \in \Omega : \tfrac{ry}{|y|} \in K\}$$

and we require that it is a Lebesgue null set. This is certainly the case for all r such that $\partial B_r \cap K$ is finite. Additionally we require that $u|_{\partial B_r(x) \setminus K} \in W^{1,2}(\partial B_r(x) \setminus K)$ and that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{B_{r+\delta} \setminus (B_{r-\delta} \cup \hat{K})} \left(|\nabla u(x) - \nabla u(\tfrac{rx}{|x|})|^2 + |u(x) - u(\tfrac{rx}{|x|})|^2\right) dx = 0\,. \tag{2.5.7}$$

This is again a property which certainly holds for a.e. r.

Let us first deal with the second statement and fix a corresponding γ. Then there is one connected component U of $\Omega \setminus (\gamma \cup K)$ with the property that $U \subset\subset \Omega$ and ∂U contains γ.

There are now two possibilities, either $U \subset B_r(x)$, or $U \subset \Omega \setminus \overline{B}_r(x)$. In the first we case we define

$$\psi_\delta(x) := \begin{cases} 1 & \text{for } |x| \leq 1 - \delta \\ 1 - \frac{1}{\delta}(|x| - 1 + \delta) & \text{for } 1 - \delta \leq |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.5.8}$$

while in the second we define

$$\psi_\delta(x) := \begin{cases} 0 & \text{for } |x| \leq 1 \\ \frac{1}{\delta}(|x| - 1) & \text{for } 1 \leq |x| \leq 1 + \delta \\ 1 & \text{otherwise.} \end{cases}$$

We then define the function χ on $\Omega \setminus (K \cup \gamma)$ as constantly equal to 1 on U and constantly equal to 0 otherwise, and we set $\varphi = \chi \psi_\delta(\frac{\cdot}{r})$. We then test (1.5.2) with φ and let $\delta \downarrow 0$. Using (2.5.7) we easily get (2.5.6) in the limit.

As for the first statement, we test (1.5.2) with $\chi u\psi_\delta(\frac{\cdot}{r})$ and let $\delta \downarrow 0$, with ψ_δ as in (2.5.8). The proof is entirely analogous and we leave the details to the reader. $\square$

2.5.1. Truncated tests. While obviously we cannot plug in (1.5.4) a test field which is not compactly supported, a standard approximation argument allows to derive an equivalent identity in that case too. This fact will play a pivotal role in the proof of several theorems described in these notes: specific choices of the test field will in fact deliver some remarkable identities. Analogous results can be inferred from (1.5.2), as done for instance in Corollary 2.5.3. We do not work out the details since we do not need those results in these notes.

PROPOSITION 2.5.4. *Let (K,u) be a critical point of E_λ in Ω and $y \in \Omega$. For a.e. $r \in (0, \operatorname{dist}(y,\partial\Omega))$ the following identity holds for every $\eta \in C^1(\overline{B}_r(y),\mathbb{R}^2)$*

$$
\begin{aligned}
&\int_{B_r(y)\setminus K} \Big(|\nabla u|^2 \operatorname{div}\eta - 2\nabla u^T\cdot D\eta\,\nabla u\Big) + \int_{B_r(y)\cap K} e^T\cdot D\eta\, e\, d\mathcal{H}^1\\
&= \int_{\partial B_r(y)\setminus K} \left(|\nabla u|^2 \eta\cdot n - 2\frac{\partial u}{\partial n}\nabla u\cdot\eta\right) d\mathcal{H}^1 + \sum_{p\in K\cap\partial B_r(y)} e(p)\cdot\eta(p)\\
&\quad + 2\lambda\int_{B_r(y)\setminus K}(u-g)\nabla u\cdot\eta + \lambda\int_{B_r(y)\cap K}\Big(|u^+-g_K|^2 - |u^- - g_K|^2\Big)\eta\cdot\nu\, d\mathcal{H}^1\,, \qquad (2.5.9)
\end{aligned}
$$

where $n(x) = \frac{x-y}{|x-y|}$ is the exterior unit normal to the circle, $e(p)$ is the tangent unit vector to K at p such that $e(p)\cdot n(p) > 0$, and $\nu(p) = e(p)^\perp$.

PROOF. We follow essentially the proof of [21]. In that reference we take advantage of the regularity theory to avoid technicalities. In principle we could still consistently follow the same approach: the regularity needed to carry over the proof of [21] relies only on case (b) of Theorem 1.2.3, while Proposition 2.5.4 will be used only in the proof of the cases (a) and (c). However, in order to keep our notes streamlined, we give here a proof which does not rely on any regularity. Moreover, since the proof in the case of $\lambda > 0$ is just a minor adjustment, we will focus on the case of E_0.

Fix $r > 0$ and $\eta \in C_c^\infty(\Omega,\mathbb{R}^2)$ and without loss of generality assume $y = 0$. Consider ψ_δ as in (2.5.8). We would like to plug $\psi_\delta(\frac{\cdot}{r})\eta$ in (1.5.4), however the latter is only Lipschitz continuous. We therefore first plug in (1.5.4) a suitable smoothed version $\psi_{\delta,\varepsilon}(\frac{\cdot}{r})\eta$, where $\psi_{\delta,\varepsilon}$ could for instance be the convolution of ψ_δ with a standard smooth kernel φ_ϵ. We then wish to pass to the limit as $\varepsilon \downarrow 0$. Observe first of all that the first summand in both the left and the right hand sides of (1.5.4) would carry to the obvious limits, respectively. Next recall that, by the coarea formula [4, Theorem 2.93], for $\mathcal{H}^1$-a.e. r the intersection of K with ∂B_r is finite. In particular, for $\mathcal{H}^1$-a.e. δ the function $\psi_\delta(\frac{\cdot}{r})\eta$ is differentiable $\mathcal{H}^1$-a.e. on K and the second term on the left hand side of (1.5.4) would then make sense. It is also a simple measure theoretic exercise to see that the identity (1.5.4) in fact holds under that assumption. We then can explicitly compute

$$
\begin{aligned}
&\int_{B_r\setminus K} \psi_\delta(\tfrac{x}{r})(|\nabla u|^2 \operatorname{div}\eta - 2\nabla u^T\cdot D\eta\,\nabla u) + \int_{B_r\cap K} \psi_\delta(\tfrac{x}{r}) e^T\cdot D\eta\, e\, d\mathcal{H}^1\\
=&\frac{1}{r\delta}\int_{B_r\setminus B_{r(1-\delta)}} \Big(|\nabla u|^2\eta\cdot \tfrac{x}{|x|} - 2(\eta\cdot\nabla u)(\nabla u\cdot \tfrac{x}{|x|})\Big) + \frac{1}{r\delta}\int_{K\cap B_r\setminus B_{r(1-\delta)}} (e\cdot\eta)(e\cdot\tfrac{x}{|x|})\, d\mathcal{H}^1\\
&+2\lambda\int_{B_r\setminus K} \psi_\delta(\tfrac{x}{r})(u-g)\nabla u\cdot\eta + \lambda\int_{B_r\cap K}\psi_\delta(\tfrac{x}{r})\Big(|u^+-g_K|^2-|u^--g_K|^2\Big)\eta\cdot\nu\, d\mathcal{H}^1\,.
\end{aligned}
$$

The terms on the left hand side in the equation above and the third and fourth integrals on the right hand side converge as $\delta\downarrow 0$ to the obvious limits by dominated convergence. The first term on the right hand side converge to the obvious limit for every r which satisfies (2.5.7) as in the proof of Corollary 2.5.3. The remaining term on the right hand side can be written in the following way using the coarea formula [4, Theorem 2.93]:

$$
\frac{1}{r\delta}\int_{r(1-\delta)}^{r}\Big(\sum_{p\in K\cap\partial B_s} e(p)\cdot\eta(p)\Big)\, ds\,.
$$

We then observe that, by standard measure theory, for a.e. r the integral converges, as $\delta\downarrow 0$, to the corresponding one in (2.5.9). We thus conclude that (2.5.9) holds for a.e. r.

The argument above makes, however, the choice of the radii dependent on the fixed smooth vector field η. In order to gain a set of full measure for which (2.5.9) is valid for every test η we use the following standard argument. We first select a countable family $\{\eta_j\}\subset C^\infty(\Omega,\mathbb{R}^2)$ which is dense in the C^1 topology in the space $C^1_c(\Omega,\mathbb{R}^2)$. Hence we observe that the above argument gives a set of full measure of radii r in $(0,\operatorname{dist}(y,\partial\Omega))$ for which (2.5.9) is valid for every η_j. Next we fix an r in this set and an $\eta\in C^1(\overline{B}_r(y),\mathbb{R}^2)$. We extend the latter to a vector field in $C^1_c(\Omega,\mathbb{R}^2)$, hence select a sequence η_{j_n} such that $\|\eta_{j_n}-\eta\|_{C^1}\to 0$ and derive (2.5.9) as limit of the corresponding identities for η_{j_n}. □

We collect next specific choices of the test field which will be crucial in several instances (cf. Lemma 2.6.5, Proposition 2.6.2, Proposition 5.2.1, Proposition 5.4.4).

Corollary 2.5.5. *Let (K,u), y, and r be a as in Proposition 2.5.4. If $\eta\in C^1(\overline{B}_r,\mathbb{R}^2)$ is conformal, then*

$$
\begin{aligned}
\int_{B_r(y)\cap K} e^T\cdot D\eta\, e\, d\mathcal{H}^1 &= \int_{\partial B_r(y)\setminus K}\left(|\nabla u|^2\eta\cdot n - 2\frac{\partial u}{\partial n}\nabla u\cdot\eta\right) d\mathcal{H}^1 + \sum_{p\in K\cap\partial B_r(y)} e(p)\cdot\eta(p)\\
&+2\lambda\int_{B_r(y)\setminus K}(u-g)\nabla u\cdot\eta + \lambda\int_{B_r(y)\cap K}\Big(|u^+-g_K|^2-|u^--g_K|^2\Big)\eta\cdot\nu\, d\mathcal{H}^1\,. \qquad (2.5.10)
\end{aligned}
$$

In particular, for every constant vector $v \in \mathbb{R}^2$ we have

$$0 = \int_{\partial B_r(y)\setminus K} \left(|\nabla u|^2 v \cdot n - 2\frac{\partial u}{\partial n}\frac{\partial u}{\partial v} \right) d\mathcal{H}^1 + \sum_{p \in K \cap \partial B_r(y)} e(p) \cdot v$$
$$+ 2\lambda \int_{B_r(y)\setminus K} (u-g)\nabla u \cdot v + \lambda \int_{\partial B_r(y)\cap K} \left(|u^+ - g_K|^2 - |u^- - g_K|^2 \right) u \cdot \nu \, d\mathcal{H}^1 \,, \quad (2.5.11)$$

and if $\tau(x) = \frac{(x-y)^\perp}{|x-y|}$, then

$$0 = -2 \int_{\partial B_r(y)\setminus K} \frac{\partial u}{\partial n}\frac{\partial u}{\partial \tau} d\mathcal{H}^1 + \sum_{p \in K \cap \partial B_r(y)} e(p) \cdot \tau(p) + 2\lambda \int_{B_r(y)\setminus K} (u-g)\nabla u \cdot \tau \,. \quad (2.5.12)$$

PROOF. Recall that, if $\eta \in C^1(\overline{B}_r, \mathbb{R}^2)$ is conformal, then $D\eta(x) = \vartheta(x)\mathbb{O}(x)$, with $\vartheta(x) > 0$ and $\mathbb{O}(x) \in O(2)$ for all $x \in \overline{B}_r$: in particular $|\xi|^2 \mathrm{div}\, \eta - 2\xi^T D\eta\, \xi$ vanishes identically for every vector ξ, leading immediately to formula (2.5.10). The identities in (2.5.11) and (2.5.12) are then simple consequences of (2.5.10) after we choose $\eta(x) \equiv v$ and $\eta(x) = (x-y)^\perp$, respectively. □

2.5.2. The factor of the cracktip. Consider the pair (K, u) given by $K = \mathbb{R}^+$ and, in polar coordinates, the function on $\mathbb{R}^2 \setminus K$ given by

$$u(\theta, r) = b r^{\frac{1}{2}} \cos\frac{\theta}{2} \,. \quad (2.5.13)$$

It is straightforward to see that (K, u) is a critical point of E_0 on any set $\Omega = B_r \setminus B_\delta$ for positive r and δ by taking advantage of Proposition 2.5.2. Note however that K is not any more smooth at the tip: in a suitable variational sense the curvature of K is singular at the origin. This singularity must be somewhat balanced by the variation of the Dirichlet energy and, remarkably, one outcome is that the constant b is then determined up to sign. Proposition 2.5.4 gives indeed a very short proof.

PROPOSITION 2.5.6. *Assume $K = \mathbb{R}^+$ and u is given by (2.5.13). If u is a critical point of E_0, then $b^2 = \frac{2}{\pi}$.*

PROOF. We use identity (2.5.11) in Corollary 2.5.5 on $B_r(x) = B_1(0)$ with vector $v = (1, 0)$. On the other hand, using polar coordinates, $|\nabla u|^2(\theta, 1) = \frac{b^2}{4}$, while $\eta \cdot n = \cos\theta$ and hence (2.5.9) becomes

$$1 = 2\int_{\partial B_1 \setminus \{(1,0)\}} \frac{\partial u}{\partial n}\frac{\partial u}{\partial \eta} \,. \quad (2.5.14)$$

We then compute

$$\frac{\partial u}{\partial n}(1, \theta) = \frac{b}{2}\cos\frac{\theta}{2} \quad (2.5.15)$$
$$\frac{\partial u}{\partial \theta}(1, \theta) = -\frac{b}{2}\sin\frac{\theta}{2} \quad (2.5.16)$$
$$\frac{\partial u}{\partial \eta}(1, \theta) = \cos\theta \frac{\partial u}{\partial n}(1, \theta) - \sin\theta \frac{\partial u}{\partial \theta}(1, \theta) = \frac{b}{2}\left(\cos\theta \cos\frac{\theta}{2} + \sin\theta \sin\frac{\theta}{2} \right) . \quad (2.5.17)$$

So the right hand side of (2.5.14) equals

$$2\int_0^{2\pi} \frac{b^2}{4}\left(\cos^2\frac{\theta}{2}\cos\theta + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\theta\right)d\theta = \frac{b^2}{2}\int_0^{2\pi}\left(\cos\theta\frac{\cos\theta+1}{2} + \frac{\sin^2\theta}{2}\right)d\theta$$
$$=\frac{b^2}{4}\int_0^{2\pi}(1+\cos\theta)\,d\theta = \frac{\pi b^2}{2}\,,$$

and inserting it in (2.5.14) we achieve $b^2 = \frac{2}{\pi}$. □

2.5.3. Léger's "magic formula". A remarkable discovery of Léger in [27] is a closed singular integral formula for

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)^2$$

when u is a global critical point of the Mumford-Shah functional. This formula will not be used in our notes, but since it can be seen as a simple and direct consequence of the inner variation identity, we include a very short proof in this section.

PROPOSITION 2.5.7. *Assume (K,u) is a global generalized restricted minimizer. Then the following formula holds for every $(x_0,y_0)\notin K$:*

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)^2(x_0,y_0) = -\frac{1}{2\pi}\int_K \frac{d\mathcal{H}^1(x,y)}{((x-x_0)+i(y-y_0))^2}\,. \tag{2.5.18}$$

REMARK 2.5.8. Introducing the complex coordinate $z = x+iy$ the formula (2.5.18) can be elegantly rewritten as:

$$\left(\frac{\partial u}{\partial z}\right)^2(z_0) = -\frac{1}{8\pi}\int_K \frac{d\mathcal{H}^1(w)}{(w-z_0)^2}\,. \tag{2.5.19}$$

REMARK 2.5.9. The assumptions on the global minimality of (K,u) can be considerably relaxed. The proof only uses the facts that the pair (K,u) is a critical point and that the growth estimate

$$\mathcal{H}^1(K\cap B_r) + \int_{B_r\setminus K}|\nabla u|^2 \le Cr \tag{2.5.20}$$

holds for all sufficiently large disks.

PROOF. First of all, by translation invariance it suffices to prove the formula when $(x_0,y_0)=0$. Next observe that the real part of (2.5.18) reads

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right](0) = \frac{1}{2\pi}\int_K \frac{y^2-x^2}{(x^2+y^2)^2}d\mathcal{H}^1\,, \tag{2.5.21}$$

while the imaginary part is in fact equivalent to (2.5.21) in the system of coordinates which results from a 45 degrees counterclockwise rotation of the standard one. We focus therefore on the proof of (2.5.21). Since (K,u) is a critical point of the Mumford-Shah functional without fidelity term, the inner variations (1.5.3) reads

$$\int_{\mathbb{R}^2\setminus K}(2\nabla u^T\cdot D\psi\nabla u-|\nabla u|^2\mathrm{div}\,\psi)=\int_K e^T\cdot D\psi\, e\,d\mathcal{H}^1\,. \tag{2.5.22}$$

We fix positive radii $\rho<R$ and consider the vector field $\psi(x,y)=\varphi(|(x,y)|)(x,-y)$ where

$$\varphi(t)=\begin{cases}\rho^{-2}-R^{-2} & \text{if } t\leq\rho\\ t^{-2}-R^{-2} & \text{if } \rho\leq t\leq R\\ 0 & \text{otherwise}\,.\end{cases}$$

Strictly speaking the latter is not a valid test in (2.5.22) because it is not continuously differentiable. However, if we assume that $\mathcal{H}^1(K\cap(\partial B_\rho\cup\partial B_R))=0$, it is easily seen that the right hand side (2.5.22) makes sense because ψ is $\mathcal{H}^1$-a.e. differentiable on K, while a standard regularization argument, analogous to the ones already used in the previous sections, shows the validity of the formula. Next we compute $D\psi$ in the two relevant domains where it does not vanish:

$$D\psi=\left(\frac{1}{\rho^2}-\frac{1}{R^2}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad\text{on } B_\rho$$

$$D\psi=-\frac{1}{R^2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{1}{(x^2+y^2)^2}\begin{pmatrix}y^2-x^2&2xy\\-2xy&y^2-x^2\end{pmatrix}\qquad\text{on } B_R\setminus\bar{B}_\rho\,.$$

Choose then ρ sufficiently small to have $B_\rho\cap K=\emptyset$ and obtain

$$\begin{aligned}&\frac{2}{\rho^2}\int_{B_\rho}\left(\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2\right)-\frac{2}{R^2}\int_{B_R}\left(\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2\right)\\ =&\frac{1}{R^2}\int_{K\cap B_R}(e_2^2-e_1^2)d\mathcal{H}^1+\int_{K\cap B_R}\frac{y^2-x^2}{(x^2+y^2)^2}\,d\mathcal{H}^1\,.\end{aligned}$$

Then, we first let $R\uparrow\infty$ and use (2.5.20) to obtain

$$\frac{2}{\rho^2}\int_{B_\rho}\left(\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2\right)=\int_K\frac{y^2-x^2}{(x^2+y^2)^2}\,d\mathcal{H}^1\,,$$

and hence we let $\rho\downarrow 0$ and use the regularity of u at 0 to infer (2.5.21). $\square$

2.6. Monotonicity formulae

An important role in our arguments will be played by three monotonicity statements, valid for minimizers of E_λ (irrespectively whether they are absolute, restricetd, generalized, or generalized restricted). The first statement was discovered by Bonnet in [8], while the other two were discovered and proved by David and Léger in [15]. The first and the second are considerably easier to prove and we will show them in this section.

PROPOSITION 2.6.1. *Assume (K,u) is a critical point of E_0. Fix $x \in \Omega$ and consider*

$$d(x,r) := \frac{D(x,r)}{r} := \frac{1}{r}\int_{B_r(x)\setminus K} |\nabla u|^2 \,. \tag{2.6.1}$$

Then $r \mapsto D(x,r)$ is an absolutely continuous function and $\frac{d}{dr}d(x,r) \geq 0$ for a.e. $r \in (0, \operatorname{dist}(x,\partial\Omega))$ which satisfies the following property:

(i) *The set $K \cap \partial B_r(x)$ belongs to the same connected component of K.*

Moreover, if (i) holds, $d(x,r)$ is constant on the interval $(0,r_0)$ and in addition (K,u) is a restricted, an absolute, or a generalized minimizer, then either $\nabla u = 0$ $\mathcal{L}^2$-a.e. on $B_{r_0}(x)$ or (K,u) is a cracktip with loose end located at x.

PROPOSITION 2.6.2. *Let (K,u) be an absolute, restricted, generalized, or generalized restricted minimizer of E_λ in Ω. Fix $x \in \Omega$ and consider*

$$F(x,r) := \frac{2}{r}\int_{B_r(x)\setminus K} |\nabla u|^2 + \frac{1}{r}\mathcal{H}^1(B_r(x)\cap K) =: 2d(x,r) + \frac{\ell(x,r)}{r}\,. \tag{2.6.2}$$

Then $r \mapsto F(x,r)$ is a function of bounded variation. The singular part of its derivative is a nonnegative measure, while there is a constant C such that the absolutely continuous part $F'(r)$ satisfies

$$F'(x,r) \geq \min\left(1, 3 - \frac{2\alpha}{\pi}\right)\frac{D'(x,r)}{r} - C\lambda \tag{2.6.3}$$

at a.e. radius $r \in (0, \operatorname{dist}(x,\partial\Omega))$ such that

(i) *each connected component of $\partial B_r(x)\setminus K$ has length less or equal to αr, with $\alpha \leq \frac{3}{2}\pi$.*

Moreover, if $\lambda = 0$, $r \mapsto F(x,r)$ is constant on $(0,r_0)$ and (i) holds for a.e. $r \in (0,r_0)$, then (K,u) coincides in $B_{r_0}(x)$ with an elementary global minimizer as in Theorem 2.4.1(b) and (c). In both cases necessarily $x \in K$ and in case (c) it must in fact be a triple junction.

The third is more laborious and will only be used in the second part of these notes. We therefore postpone its proof to Section 4.4.

PROPOSITION 2.6.3. *Let (K,u), Ω, x, and F be as in Proposition 2.6.2 with $\lambda = 0$. Then $F'(x,r) \geq 0$ at a.e. point r such that*

(i) *$N(r) := \sharp(K \cap \partial B_r(x)) \in \{0\} \cup [3,\infty)$*

(ii) *or $N(r) = 2$ and the two points of $K \cap \partial B_r(x)$ belong to the same connected component of K.*

Moreover, if $r \mapsto F(x,r)$ is constant on $(0,r_0)$ and (i) or (ii) hold for a.e. $r \in (0,r_0)$, then (K,u) coincides in $B_{r_0}(x)$ with an elementary global minimizer as in Theorem 2.4.1(a), (b), or (c). In both cases (b) and (c) necessarily $x \in K$ and in case (c) it must in fact be a triple junction.

REMARK 2.6.4. While it is plausible that (ii) in Proposition 2.6.3 could be weakened to the single assumption $N(r) = 2$, the monotonicity of F is certainly false when $N(r) = 1$. We indeed show here that it fails for any functional of type $F_c(x,r) := c\,d(x,r) + \frac{\ell(x,r)}{r}$.

Consider the cracktip pair (K,u) of Definition 1.3.2 and change it to the pair (K_a,u_a) where $K_a = K+(a,0)$ and $u_a(x_1,x_2) = u(x_1-a,x_2)$. We then introduce the function

$$f(a,r) := F_c((-a,0),r) = \frac{c}{r}\int_{B_r} |\nabla u_a|^2 + \frac{\mathcal{H}^1(K_a\cap B_r)}{r} =: D(a,r)+L(a,r)$$

and we study it on the domain $\Lambda := \{|a|<\frac{1}{2}, |r-1|<\frac{1}{2}\}$. First of all observe that $L(a,r) = \frac{r-a}{r}$ and it is thus a smooth function on Λ. Next we integrate by parts and write

$$D(a,r) = \frac{c}{r}\int_{\partial B_r\setminus(1,0)} u_a\frac{\partial u_a}{\partial n}\,.$$

The latter formula shows immediately that D is smooth as well on Λ. Next observe that $|\nabla u_a|^2(x) = \frac{1}{2\pi|x-(a,0)|}$ and in particular by symmetry we conclude $D(a,r) = D(-a,r)$. We thus infer $\frac{\partial^2 D}{\partial a\partial r}(0,r) = 0$. On the other hand we can explicitly compute $\frac{\partial^2 L}{\partial a\partial r} = \frac{1}{r^2}$ and thus we get $\frac{\partial^2 f}{\partial a\partial r}(0,r) = \frac{1}{r^2}$. Since it is obvious that $\frac{\partial f}{\partial r}(0,r)=0$, there is $\delta>0$ such that

$$\frac{\partial f}{\partial r}(a,r)<0 \qquad \forall r\in[\tfrac{3}{4},\tfrac{5}{4}], \forall a\in[-\delta,0)\,.$$

This shows that the function $r\to F_c((-a,0),r)$ is certainly not monotone for $a<0$ sufficiently small. In fact a simple scaling argument shows that the monotonicity fails on some interval for every a negative.

2.6.1. The David-Léger-Maddalena-Solimini identity. A first important ingredient in the proofs of both Proposition 2.6.2 and 2.6.3 is given by an interesting identity discovered independently by David and Léger in [15] and Maddalena and Solimini in [30] for critical points of E_0. In the following we state its version for critical points of E_λ

LEMMA 2.6.5. *Let F be as in Proposition 2.6.2. If (K,u) is an absolute, restricted, generalized, or a generalized restricted minimizer of E_λ in Ω, then for every $y\in\Omega$ and for a.e. $r\in(0,\operatorname{dist}(y,\partial\Omega))$ we have*

$$\begin{aligned}\int_{\partial B_r(y)\setminus K}\left(\frac{\partial u}{\partial n}\right)^2 &= \int_{\partial B_r(y)\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2 - \frac{\mathcal{H}^1(K\cap B_r(y))}{r} + \sum_{p\in\partial B_r(y)\cap K} e(p)\cdot n(p)\\ &+\frac{2\lambda}{r}\int_{B_r(y)\setminus K}(u-g)\nabla u\cdot(x-y) + \frac{\lambda}{r}\int_{B_r(y)\cap K}\Big(|u^+-g_K|^2-|u^--g_K|^2\Big)(x-y)\cdot\nu\,d\mathcal{H}^1\end{aligned} \tag{2.6.4}$$

where the vectors n and e are as in Proposition 2.5.4, in particular $n(p)\cdot e(p)>0$ for all $p\in\partial B_r(y)\cap K$.

PROOF. Test with $\eta(x) = \frac{x-y}{r}$ the equation (2.5.9). □

2.6.2. Elementary estimate on harmonic extensions. A second important ingredient is an elementary estimate on harmonic extensions.

LEMMA 2.6.6. *Let $U\subset B_r(x)$ be an open set such that*

- *$\partial B_r(x) \cap \partial U$ is a connected arc γ;*
- *U is contained in a circular sector of angle $\alpha < 2\pi$.*

Then for every $g \in W^{1,2}(\gamma)$ there is an harmonic extension $v \in W^{1,2}(U)$ such that

$$\int_U |\nabla v|^2 \le \frac{\alpha r}{\pi} \int_\gamma \left(\frac{\partial g}{\partial \tau}\right)^2 . \tag{2.6.5}$$

Proof. Since the statement is invariant under rotations, translations, and dilations, W.l.o.g. we assume $x = 0$, $r = 1$, and, using polar coordinates (θ, ρ), $U \subset V := \{\rho < 1, 0 < \theta < \alpha\}$. Observe that $\gamma = \{\rho = 1, a \le \theta \le b\}$ with $0 \le a < b \le \alpha$. Extend now g to $\{\rho = 1, 0 \le \theta \le \alpha\}$ by setting it to be constant on the arcs $\{\rho = 1, 0 \le \theta \le a\}$ and $\{\rho = 1, b \le \theta \le \alpha\}$: recalling that g is continuous by Morrey's embedding, it is obvious that the latter extension can be achieved in $W^{1,2}$. It then suffices to find a $W^{1,2}$ extension to the whole sector V which enjoys the desired bound. In other words, we can assume without loss of generality that U is itself the sector $V := \{\rho < 1, 0 < \theta < \alpha\}$. Consider now g as a function on the interval $[0, \alpha]$ and extend it to an even $W^{1,2}$ function on $[-\alpha, \alpha]$, which can be thought as a periodic function on $\mathbb{R}$ with period 2α. In particular we can write its Fourier series as

$$g(\theta) = a_0 + \sum_{k \ge 1} a_k \cos k \frac{\pi}{\alpha} \theta .$$

We then consider the harmonic extension

$$v(\theta, \rho) = a_0 + \sum_{k \ge 1} a_k \rho^{k\pi/\alpha} \cos k \frac{\pi}{\alpha} \theta$$

and standard computations yield

$$\int_V |\nabla v|^2 = \sum_{k \ge 1} \frac{k\pi}{2} a_k^2 \le \sum_{k \ge 1} \frac{k^2 \pi}{2} a_k^2 = \frac{\alpha}{\pi} \int_\gamma \left(\frac{\partial g}{\partial \tau}\right)^2 . \qquad \square$$

2.6.3. Proof of Proposition 2.6.2. We assume w.l.o.g. that $x = 0$, thus we drop the dependence on the base point in all the relevant quantities, i.e. for instance $D(r) = \int_{B_r \setminus K} |\nabla u|^2$ and $\ell(r) = \mathcal{H}^1(B_r \cap K)$. Observe that $r \mapsto D(r)$ is an absolutely continuous function with

$$D'(r) = \int_{\partial B_r \setminus K} |\nabla u|^2 = \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 + \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 . \tag{2.6.6}$$

$r \mapsto \ell(r)$ is a monotone nondecreasing function, and hence a function of bounded variation. Moreover, the absolutely continuous part of is derivative equals, by the coarea formula [4, Theorem 2.93],

$$\ell'(r) = \sum_{p \in \partial B_r \cap K} \frac{1}{e(p) \cdot n(p)} \tag{2.6.7}$$

(where we follow the notation of Proposition 2.5.4). The first claim of the proposition is thus obvious, while for a.e. r we have

$$r^2 F'(r) = 2r \int_{\partial B_r \setminus K} |\nabla u|^2 + r \sum_{p \in \partial B_r \cap K} \frac{1}{e(p)\cdot n(p)} - 2D(r) - \ell(r)\,. \tag{2.6.8}$$

We first prove the conclusion for $\lambda = 0$. Using (2.6.4) we get

$$\begin{aligned} r^2 F'(r) =& 3r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 \\ &+ r \sum_{p \in \partial B_r \cap K} \left(\frac{1}{e(p)\cdot n(p)} + e(p)\cdot n(p)\right) - 2(D(r) + \ell(r)) \\ \geq& 3r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 + 2rN(r) - 2E_0(K, u, B_r)\,, \end{aligned} \tag{2.6.9}$$

where $N(r) := \sharp(K \cap \partial B_r)$. Consider next a competitor (J, w) for (K, u) in the following fashion:

- $(J, w) = (K, u)$ outside B_r;
- $J \cap B_r$ consists of $N(r)$ straight segments joining each point of $\partial B_r \cap K$ with the origin;
- on each connected component of $B_r \setminus J$, which according to assumption (i) is a circular sector with angle $\alpha \leq \frac{3\pi}{2}$, we let w be the extension of the trace of u on the corresponding circular arc given by Lemma 2.6.6.

Observe that J does not increase the number of connected components and that we can apply Lemma 2.2.5. In particular (J, w) is an admissible competitor for restricted and generalized restricted minimizers as well. We can use estimate (2.6.5) in Lemma 2.6.6 to infer

$$E_0(K, u, B_r) \leq E_0(J, w, B_r) = \int_{B_r \setminus J} |\nabla w|^2 + rN(r) \leq \frac{\alpha}{\pi} r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + rN(r)\,.$$

Combined with (2.6.9) we then conclude

$$r^2 F'(r) \geq \left(3 - \frac{2\alpha}{\pi}\right) r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 .$$

In particular, we deduce from (2.6.6) that

$$F'(r) \geq \min\left(1, 3 - \frac{2\alpha}{\pi}\right) \frac{D'(r)}{r}.$$

Finally, if F is constant on $(0, r_0)$ and (i) holds for a.e. $r \in (0, r_0)$, we would necessarily conclude from the last but one inequality that, for a.e. $r \in (0, r_0)$,

$$\int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 = 0\,.$$

Equality (2.5.5) easily implies that u must be locally constant on $B_{r_0} \setminus K$, and thus K in ∂B_r is a minimizing network. More precisely, if r is a radius such that $N(r) < \infty$, $K \cap \overline{B}_r$ consists of finitely many connected components $K_1, \ldots, K_j$ and each K_j is a connected set which minimizes the length among all closed connected sets $\hat{K} \subset \overline{B}_r$ with $\hat{K} \cap \partial B_r = K_j \cap B_r$. In particular we conclude that $K_j \cap B_r$ is either a single segment or it consists of a network of a finite number of segments joining at triple junction.

Observe next that from (2.6.9), being u locally constant on $B_{r_0} \setminus K$, the constancy of F implies

$$2N(r) = \sum_{p \in \partial B_r \cap K} \left(\frac{1}{e(p) \cdot n(p)} + e(p) \cdot n(p) \right)$$

for a.e. $r \in (0, r_0)$. For every such r we must then have $e(p) = n(p)$ for every $p \in K \cap \partial B_r$. In particular we conclude that $K \cap B_{r_0}$ is a cone centered at the origin. But then it is either a straight segment or it is a collection of three radii meeting at the origin, or it is the empty set. The latter is excluded by assumption (i).

We now come to the monotonicity statement for $\lambda > 0$. We plug (2.6.4) in (2.6.8) to infer the following inequality from Cauchy-Schwartz and the energy upper bound in (1.3.1):

$$\begin{aligned}
r^2 F'(r) =& 3r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 \\
&+ r \sum_{p \in \partial B_r \cap K} \left(\frac{1}{e(p) \cdot n(p)} + e(p) \cdot n(p) \right) - 2(D(r) + \ell(r)) \\
&+ 2\lambda \int_{B_r \setminus K} (u - g) \nabla u \cdot y + \lambda \int_{B_r \cap K} \left(|u^+ - g_K|^2 - |u^- - g_K|^2 \right) y \cdot \nu \, d\mathcal{H}^1 \\
\geq & 3r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 + 2rN(r) - 2E_\lambda(K, u, B_r) - C\lambda r^2 \,,
\end{aligned} \tag{2.6.10}$$

where C depends on $\|g\|_\infty$ thanks to Lemma 2.1.1. We use the competitor (J, w) defined above to get

$$\begin{aligned}
E_\lambda(K, u, B_r) \leq & E_\lambda(J, w, B_r) = \int_{B_r \setminus J} |\nabla w|^2 + rN(r) + \lambda \int_{B_r} (w - g)^2 \\
& \leq \frac{\alpha}{\pi} r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + rN(r) + 4\pi\lambda \|g\|_\infty^2 r^2
\end{aligned}$$

(note that w consists of harmonic extensions in $B_r \setminus J$ and since the trace of w on ∂B_r is bounded by $\|g\|_\infty$, we see immediately that $\|w\|_\infty \leq \|g\|_\infty$). The latter estimate combined with (2.6.10) gives

$$r^2 F'(r) \geq \left(3 - \frac{2\alpha}{\pi}\right) r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 + r \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2 - C\lambda r^2 \,,$$

and (2.6.3) follows at once.

2.6.4. Proof of Proposition 2.6.1. First of all we start by assuming, without loss of generality, that ∇u does not vanish identically.

We follow the same approach of the previous section and will just carry on our computations assuming that we have selected a good radius. We thus have

$$\frac{d}{dr}d(r) = \frac{d}{dr}\frac{D(r)}{r} = \frac{1}{r}\int_{\partial B_r(x)\setminus K} |\nabla u|^2 - \frac{D(r)}{r^2} \tag{2.6.11}$$

Observe that we are in a position to apply both statements in Corollary 2.5.3. In particular, if we let c_i be the average of u on any connected component γ_i of $\partial B_r(x) \setminus K$ we can write

$$\begin{aligned}
D(r) &\overset{(2.5.5)}{=} \int_{\partial B_r(x)\setminus K} u\frac{\partial u}{\partial n} \overset{(2.5.6)}{=} \sum_i \int_{\gamma_i} (u-c_i)\frac{\partial u}{\partial n} \le \sum_i \left(\int_{\gamma_i} (u-c_i)^2\right)^{\frac{1}{2}} \left(\int_{\gamma_i}\left(\frac{\partial u}{\partial n}\right)^2\right)^{\frac{1}{2}} \\
&\le \sum_i \frac{\mathcal{H}^1(\gamma_i)}{\pi} \left(\int_{\gamma_i}\left(\frac{\partial u}{\partial \tau}\right)^2\right)^{\frac{1}{2}} \left(\int_{\gamma_i}\left(\frac{\partial u}{\partial n}\right)^2\right)^{\frac{1}{2}} \\
&\le 2r \left(\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2\right)^{\frac{1}{2}} \left(\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial n}\right)^2\right)^{\frac{1}{2}} \le r\int_{\partial B_r\setminus K}\left(\left(\frac{\partial u}{\partial \tau}\right)^2 + \left(\frac{\partial u}{\partial n}\right)^2\right) \\
&= r\int_{\partial B_r\setminus K} |\nabla u|^2\,,
\end{aligned} \tag{2.6.12}$$

where he have used that the sharp constant in the 1-dimensional Poincaré-Wirtinger inequality on an interval of length L is $\frac{L^2}{\pi^2}$.

Next notice that, if $\frac{D(r)}{r}$ is constant on $(0, r_0)$ then the equality holds for a.e. $r \in (0, r_0)$ in all the inequalities above. First of all observe that, if $\partial B_r \cap K$ consists of more than one point, then all the connected components of $\partial B_r \setminus K$ have length strictly less than 2π. This would give a strict inequality sign at the beginning of the third line, unless one of the two functions $\frac{\partial u}{\partial n}$ and $\frac{\partial u}{\partial \tau}$ vanish identically on $\partial B_r \setminus K$. If however only one of them vanishes identically on $\partial B_r \setminus K$, then the second inequality in the third line is strict. So they would have to vanish both identically on $\partial B_r \setminus K$. Then necessarily from (2.6.11) we would get

$$D(r) = r\int_{\partial B_r\setminus K} |\nabla u|^2 = 0.$$

So, if there is a set of radii of positive measure such that $\partial B_r \cap K$ consists of more than one point, then ∇u vanishes identically on B_r for any such radius r.

Observe that the same argument implies that there cannot be a set of radii of positive measure for which $\partial B_r \cap K$ is empty, because the constant in the sharp Poincaré-Wirtinger inequality on the unit circle equals the constant of the sharp Poincaré-Wirtinger inequality on an interval of length 2π, and we know that on the interval $[0, 2\pi[$ the latter is achieved by the functions of type

$$a + b\cos\frac{\theta}{2} \tag{2.6.13}$$

where a and b are constants.

In particular, we conclude that there exists a subset $\mathcal{R}$ of $(0, r_0)$ of full measure, functions $a, b : \mathcal{R} \to \mathbb{R}$, and a function $c : \mathcal{R} \to \mathbb{S}^1$ with the following properties

(a) $K \cap \partial B_r = \{(r\cos c(r), r\sin c(r))\} =: \{p(r)\}$ for all $r \in \mathcal{R}$;
(b) $u(\theta, r) = a(r) + b(r)\cos\left(\frac{\theta - c(r)}{2}\right)$ in polar coordinates[2];
(c) b never vanishes on $\mathcal{R}$.

Note in particular that $K \cap \partial B_r$ consists of exactly one point for a.e. r. Fix next $r \in \mathcal{R}$ and next define the function

$$\bar{u}(\theta, \rho) = a(r) + b(r)(r^{-1}\rho)^{\frac{1}{2}}\cos\frac{\theta - c(r)}{2}$$

and let $\bar{K}$ be the segment with endpoints the origin and $p(r)$.

This is obviously a competitor for (K, u). Moreover, a direct computation gives immediately

$$\int_{B_r\setminus\bar{K}} |\nabla \bar{u}|^2 = 2r\int_{\partial B_r\setminus\bar{K}}\left(\frac{\partial \bar{u}}{\partial \tau}\right)^2 = 2r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2 = \int_{B_r\setminus K}|\nabla u|^2\,,$$

where in the last equality we use the optimality conditions which can be derived from the constancy of $\frac{D(r)}{r}$ (cf. (2.6.12)). So, by minimality of (K, u), $\mathcal{H}^1(K) \le \mathcal{H}^1(\bar{K})$. In particular, since we already know that $\mathcal{H}^1(K \cap B_r) \ge r$ we actually conclude

$$\mathcal{H}^1(K \cap B_r) = r\,. \tag{2.6.14}$$

This holds for a.e. r and thus implies that the approximate tangent to the rectifiable set K must in fact be orthogonal to the circle ∂B_r at the point $(r\cos c(r), r\sin c(r))$ for a.e. $r \in (0, r_0)$. It also implies that, if we define

$$R := \{(r\cos c(r), r\sin c(r)) : r \in \mathcal{R}\}\,,$$

then $\mathcal{H}^1(K \setminus R) = 0$. On the other hand, by the density lower bound, this also means that R is dense in K.

Consider again a radius $r \in \mathcal{R}$ and, after applying a rotation, assume without loss of generality that $c(r) = \frac{\pi}{2}$. Notice that, since K is closed, there is a positive δ with the property that, if $\rho \in (r-\delta, r+\delta)$, the open set $U := \{(\rho\cos\phi, \rho\sin\phi) : r-\delta < \rho < r+\delta, 0 < \phi < \frac{\pi}{4}\}$ does not intersect K. We use the addition formula for the cosine to write

$$u(\theta, r) = a(r) + b(r)\cos\frac{c(r)}{2}\cos\frac{\theta}{2} + b(r)\sin\frac{c(r)}{2}\sin\frac{\theta}{2}$$

on the set U. Note that u is smooth over U and in particular the map

$$r \mapsto u(\theta, r)$$

must be smooth on the interval $I = (r-\delta, r+\delta)$ for every fixed $\theta \in (\frac{2\pi}{4}, \pi)$. This immediately implies that the three functions a, $b\cos\frac{c}{2}$ and $b\sin\frac{c}{2}$ have smooth extensions from $\mathcal{R} \cap I$ to I, because varying θ it is easy to find three linearly independent linear combinations of

[2] In the latter formula we understand that the angle of the polar coordinates is taken to vary in $[c(r), c(r) + 2\pi[$, so that the $\theta \mapsto u(\theta, r)$ has at most a jump discontinuity at $\theta = c(r)$ and is smooth everywhere else

these functions which are smooth. Using that $\cos^2 + \sin^2 = 1$, we then conclude that also b^2 has a smooth extension to I. Now, if such smooth extension were to vanish at some point ρ, we then would have that the trace of u is constant on ∂B_ρ. But this would immediately imply the constancy of the function u in B_ρ and then, as already argued, that $\nabla u \equiv 0$ on B_{r_0}.

Being that b^2 has a smooth extension which is bounded away from zero, b itself has a smooth extension if it does not change sign over I. Now, again because K is closed, for any fixed δ we can assume that $c(\rho) \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ for all $\rho \in I$, so that in particular $\cos \frac{c}{2}$ is positive and bounded away from zero on I. But then the existence of a smooth extension of $b \cos \frac{c}{2}$ over I would preclude b from changing sign. We thus conclude that b has a smooth extension. Moreover, over the interval $(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ the function $\cos \frac{\cdot}{2}$ has a smooth inverse, which allows us to conclude that c has a smooth extension to I as well.

Now, we conclude therefore that K is then a smooth curve in $B_{r+\delta} \setminus B_{r-\delta}$. But from the discussion above we also know that K intersects ∂B_ρ transversally for a.e. $\rho \in (r-\delta, r+\delta)$. This means that K is a straight segment, or in other words c is constant. We can now integrate by parts to conclude that

$$a'(r) = \frac{d}{dr} \frac{1}{2\pi r} \int_{\partial B_r} u = \frac{1}{2\pi r} \int_{\partial B_r} \frac{\partial u}{\partial n} = 0\,,$$

so that a is constant over I. Moreover we can use the fact that

$$\frac{1}{r} \int_{B_r} |\nabla u|^2 = \int_{\partial B_r} \left(\frac{\partial u}{\partial \tau}\right)^2$$

is constant in r to conclude that $b(r) = \beta r^{1/2}$ for some nonzero β.

Summarizing, for every $r \in \mathcal{R}$ we have concluded that there is an interval I containing it over which a and c are constant and b takes the form $\beta r^{1/2}$ for some constant β. Without loss of generality assume $a = c = 0$. Let I be a maximal such interval, which we denote by (s,t), and assume that $s > 0$. Then the trace of u on ∂B_s on the exterior of the disk B_s is of the form $\beta s^{1/2} \cos \frac{\theta}{2}$. We now claim that $K \cap \partial B_s$ must consist of the single point $(s, 0)$ (in cartesian coordinates). Indeed, if $K \cap \partial B_s$ contains some other point, then there is a sequence of radii $\{r_k\} \subset \mathcal{R}$ with $r_k \to s$ and $c(r_k) \to \phi \neq 0$, because we know that R is dense in K. But then it would follow from our formulas that $u|_{\partial B_{r_k}}$ converges to a function of the form $a' + b' \cos \frac{\theta - \phi}{2}$, which disagrees with $\beta s^{1/2} \cos \frac{\theta}{2}$ on the whole circle ∂B_s minus a discrete set of points. This would only be possible if $\partial B_s \subset K$, which however is excluded from the fact that $\mathcal{H}^1(K \setminus R) = 0$.

Now, if $K \cap \partial B_s$ consists only of the point $(s, 0)$, then it turns out that u is smooth in a neighborhood of $\partial B_s \setminus K$. We already know that the trace from the exterior of the disk B_s must be $\beta s^{1/2} \cos \frac{\theta}{2}$, in particular we can conclude that $s \in \mathcal{R}$. But then we can iterate the argument above and show that in fact there is an interval $(s - \delta, s + \delta)$ over which a and c are constant and b takes the form $\beta s^{1/2}$. This then shows that the same is true on the interval $(s - \delta, t)$, thereby contradicting the maximality of (s, t).

The conclusion is that in fact s must be 0. Likewise, an entirely analogous argument shows $t = r_0$. We thus have conclude that over B_{r_0} the discontinuity set K is a radius and u is given by the formula of the statement of the proposition.

CHAPTER 3

Pure jumps and triple junctions

This chapter is devoted to prove the cases (b) and (c) of Theorem 1.2.3.

3.1. Epsilon-regularity statements

We first state the two key ε-regularity theorems proved by David in his pioneering work [13] (see also [14]). The only difference with the statements in Theorem 1.2.3 is that we will make the additional assumption that the Dirichlet energy is also small, while the statements in Theorem 1.2.3 assume only the smallness of the Hausdorff distance to the model cases. We will show that it is rather straightforward to remove the smallness of the Dirichlet energy with a "blow-up" argument, thanks to Theorem 2.4.1.

We start off stating the case in which K is close in Hausdorff distance to a line. First we introduce some useful notation. We let:

- $\mathcal{R}_\theta$ be the counterclockwise rotation of angle $\theta \in [0, 2\pi]$ in $\mathbb{R}^2$.
- $\mathscr{V}_0$ be the infinite line $\{(t,0) : t \in \mathbb{R}\}$.

Hence we combine in a single quantity the measure of how distant is K from a diameter and how small is the Dirichlet energy

$$\Omega^j(\theta, x, r) := r^{-1}\mathrm{dist}_H(K \cap \overline{B}_{2r}(x), (x + \mathcal{R}_\theta(\mathscr{V}_0)) \cap \overline{B}_{2r}(x)) + r^{-1}\int_{B_{2r}(x)\setminus K} |\nabla u|^2 . \tag{3.1.1}$$

Moreover we will use the notation $\mathrm{gr}\,(f)$ for the graph of a given function f. Finally, in order to make our statements less cumbersome, we agree to use "minimizer" whenever the statement holds for absolute, restricted, generalized, and generalized restricted minimizers (i.e. for all type of minimizers considered in these notes).

THEOREM 3.1.1. *There are constants $\varepsilon, \alpha, C > 0$ with the following property. Assume*

(i) *(K,u) is a minimizer of E_λ on $B_{2r}(x)$ with $r \leq 1$;*
(ii) *There is $\theta \in [0, 2\pi]$ such that*

$$\Omega^j(\theta, x, r) + \lambda \|g\|_\infty^2 r^{\frac{1}{2}} < \varepsilon .$$

Then $K \cap B_r(x)$ is the graph of a $C^{1,\alpha}$ function f. More precisely there is $f : [-r, r] \to \mathbb{R}$ such that $K \cap B_r(x) = (x + \mathcal{R}_\theta(\mathrm{gr}\,(f))) \cap B_r(x)$ and

$$\|f\|_0 + r\|f'\|_0 + r^{1+\alpha}[f']_\alpha \leq Cr\Big(\Omega^j(\theta, x, r) + \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\Big)^{\frac{1}{2}} . \tag{3.1.2}$$

Actually, the proof we will give shows that a weaker assumption suffices for Theorem 3.1.1. Indeed, the Hausdorff distance in the definition of Ω^j can be substituted by an L^2, one-sided analogue (cf. the definition of the mean flatness β in (3.2.1) afterwards).

A higher dimensional version of the previous result has been established contemporarily and independently by Ambrosio, Fusco and Pallara [5, 3] (see also [4, Chapter 8]).

Next, we state the case in which K is close to a propeller. To that end we introduce further the notation

- $\mathscr{V}_0^+$ for the halfline $\{(t,0) : t \geq 0\}$;
- $\mathcal{T}_0$ for the global triple junction
$$\mathcal{T}_0 := \mathscr{V}_0^+ \cup \mathcal{R}_{\frac{2\pi}{3}}(\mathscr{V}_0^+) \cup \mathcal{R}_{\frac{4\pi}{3}}(\mathscr{V}_0^+)\,; \tag{3.1.3}$$
- $\Omega^t(\theta, x, r)$ for the analog of $\Omega^j(\theta, x, r)$:

$$\Omega^t(\theta, x, r) := r^{-1}\mathrm{dist}_H(K \cap \overline{B}_{2r}(x), (x + \mathcal{R}_\theta(\mathcal{T}_0)) \cap \overline{B}_{2r}(x)) + r^{-1}\int_{B_{2r}(x)\setminus K} |\nabla u|^2\,. \tag{3.1.4}$$

THEOREM 3.1.2. *There are constants $\varepsilon, \alpha, C > 0$ with the following property. Assume:*
(i) *(K, u) is a minimizer of E_λ in $B_{2r}(x)$ with $r \leq 1$;*
(ii) *There is $\theta \in [0, 2\pi]$ such that*
$$\Omega^t(\theta, x, r) + \lambda\|g\|_\infty^2 r^{\frac{1}{2}} < \varepsilon\,.$$
Then there is a $C^{1,\alpha}$ diffeomorphism $\Phi : B_r \to B_r(x)$ such that $K \cap B_r(x) = \Phi(\mathcal{T}_0 \cap B_r)$ and
$$\begin{aligned}|\Phi(0) - x| &+ r\left(\|D\Phi - \mathcal{R}_\theta\|_0 + \|D\Phi^{-1} - \mathcal{R}_{-\theta}\|_0\right)\\ &+ r^{1+\alpha}\left([D\Phi]_\alpha + [D\Phi^{-1}]_\alpha\right) \leq Cr\Big(\Omega^t(\theta, x, r) + \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\Big)^{\frac{1}{2}}\,.\end{aligned} \tag{3.1.5}$$

In 3d Lemenant [28] has proven an analogous statement provided that K in $B_{2r}(x)$ is close either to the union of three half-planes meeting along their edges by 120 degree angles (a $\mathbb{Y}$-cone), or to a cone over the union of the edges of a regular tetrahedron (a $\mathbb{T}$-cone).

We finally record an interesting corollary, which implies that, under the assumptions (i)-(ii) above, if $\lambda = 0$, then the three arcs forming K are in fact C^2 up to the triple junction and their respective curvatures vanish there.

COROLLARY 3.1.3. *Let (K, u) be a minimizer of E_0 satisfying the assumptions of Theorem 3.1.2. Then the three arcs forming $K \cap B_r(x)$ are C^2 up to the junction point $\bar{x} = \Phi(0)$ included and moreover $\kappa_i(\bar{x}) = 0$ for every $i \in \{1, 2, 3\}$.*

3.1.1. Useful corollaries of the epsilon-regularity at pure jumps and triple junctions. We observe here that, as it is standard for ε-regularity statements as Theorem 3.1.1 in geometric analysis, we can infer from these theorems a number of consequences which will be useful in the sequel, namely:

- an improved convergence result of the jump sets when the limit is itself smooth;
- a partial regularity result, which will be extensively used in the rest of the notes;
- a rigidity at “infinity” of certain global minimizers.

We start with the improved convergence.

COROLLARY 3.1.4. *Let $V \subset\subset U$ be two open planar domains and let:*

(a) *$K \subset U$ be a set which is the union of finitely many nonintersecting C^1 arcs (with endpoints in ∂U) and finitely many C^1 simple closed curves, all pairwise disjoint;*
(b) *$u : U \setminus K \to \mathbb{R}$ be a C^1 function with the property that, for every connected component A of $U \setminus K$, $u|_A$ has a C^1 extension to $\overline{A}$.*

Then, for every $\delta > 0$ there is a $\varepsilon(K, u, V, U, \delta) > 0$ with the following property. If (J, v) is a minimizer of E_λ and

$$\mathrm{dist}_H(J \cap U, K \cap U) + \int_{U \setminus (J \cup K)} |\nabla v - \nabla u|^2 + \lambda \mathrm{diam}\,(U) \|g\|^2_\infty < \varepsilon\,, \tag{3.1.6}$$

then $J \cap V$ is $C^{1,\alpha}$ close to $K \cap V$, where α is the constant of Theorem 3.1.1.

The proof follows from a simple application of Theorem 1.2.3 (and Theorem 1.3.1) for the case of a pure jump and a standard covering argument. An analogous statement, which we leave to the reader, holds if K is allowed to have a finite number of triple junctions.

We next state the partial regularity result.

COROLLARY 3.1.5. *Assume (K, u) is a minimizer of E_λ in some open domain U. Then the subset of K of pure jump points is relatively open and has full $\mathcal{H}^1$ measure.*

PROOF. The openness follows immediately from Corollary 3.1.4. Observe moreover that, since K is rectifiable, at $\mathcal{H}^1$-a.e. point $p \in K$ the sets

$$K_{p,r} := \frac{K - p}{r}$$

converge to the approximate tangent line ℓ_p to K at p. Although such convergence is just in a measure theoretic sense, the compactness Theorem 2.2.3 upgrades it to local Hausdorff convergence. By the case (b) of Theorem 1.2.3 every such p is a pure jump point and a point of regularity for K. $\square$

We close this section with the following rigidity theorem.

COROLLARY 3.1.6. *Assume that $(K, u, \{p_{kl}\})$ is a global generalized (or generalized restricted) minimizer and that, for some sequence of radii $r_j \uparrow \infty$, a subsequence of rescalings K_{0,r_j} converge to a pure jump or to a triple junction. Then (K, u) itself is, respectively, a pure jump or a triple junction.*

PROOF. By the ε-regularity theory, we conclude that in each disk B_{r_j} with r_j sufficiently large, $K \cap B_{r_j}$ is diffeomorphic either to a straight line or to a triple junction, and in the latter case the point of junction must be contained in $B_{r_j/2}$. It is then simple to see that K is connected. In particular, by the Bonnet's monotonicity formula Proposition 2.6.1, $\frac{1}{r} \int_{B_r} |\nabla u|^2$ is monotone nondecreasing in r. Since the limit of $\frac{1}{r_j} \int_{B_{r_j}} |\nabla u|^2$ is 0, we conclude that ∇u vanishes identically. But then by Theorem 2.4.1, (K, u) itself is either a pure jump or a triple junction. $\square$

3.2. Regularity at pure jumps: preliminaries

The proof of Theorem 3.1.1 is based on a suitable decay proposition for which we need some notation and terminology. Let

$$\mathcal{L} := \{\mathscr{V} \mid \mathscr{V} \subset \mathbb{R}^2 \text{ is a linear 1-dimensional subspace}\};$$
$$\mathcal{A} := \left\{z + \mathscr{V} \mid z \in \mathbb{R}^2 \text{ and } \mathscr{V} \in \mathcal{L}\right\}.$$

Let (K, u) be an admissible pair in B_1. For all $x \in B_1$ and $0 < r < 1 - |x|$ recall that

$$d(x, r) = \frac{D(x, r)}{r} = \frac{1}{r} \int_{B_r(x)\setminus K} |\nabla u|^2,$$

and define

$$\beta(x, r) := \min_{\mathscr{V} \in \mathcal{A}} \int_{B_r(x)\cap K} \frac{\operatorname{dist}^2(y, \mathscr{V})}{r^2} \frac{d\mathcal{H}^1(y)}{r} \tag{3.2.1}$$

$$m(x, r) := \max\{d(x, r), \beta(x, r), \lambda \|g\|_\infty^2 r^{\frac{1}{2}}\}. \tag{3.2.2}$$

In what follows in case $x = 0$ we shall drop the dependence on the base point from the notation introduced above. Observe that the following crude estimates are a simple consequence of our definitions:

$$\beta(x, \tau r) \leq \tau^{-3} \beta(x, r) \tag{3.2.3}$$

$$d(x, \tau r) \leq \tau^{-1} d(x, r). \tag{3.2.4}$$

The starting point to prove ε-regularity for K is the decay of the energy under a smallness condition at a certain radius.

PROPOSITION 3.2.1. *There are geometric constants $\varepsilon, \tau > 0$ such that, if (K, u) is a minimizer of E_λ on B_1 and $0 \in K$, then*

$$m(r) \leq \varepsilon \Rightarrow m(\tau r) \leq \tau^{\frac{1}{2}} m(r)\,. \tag{3.2.5}$$

The proof is based on two lemmas.

LEMMA 3.2.2. *There exists $\tau_1 \in (0, 1)$ such that for any $\tau \leq \tau_1$ and for any $\overline{\delta} > 0$ we can choose $\eta_1 = \eta_1(\overline{\delta}, \tau) > 0$ with the following property. If (K, u) is a minimizer of E_λ on B_1 and $0 \in K$, then for all $r \in (0, 1)$ such that*

$$\overline{\delta} \max\{d(r), \lambda \|g\|_\infty^2 r^{\frac{1}{2}}\} \leq \beta(r) \leq \eta_1,$$

we have

$$\beta(\tau r) \leq \tau \beta(r). \tag{3.2.6}$$

LEMMA 3.2.3. *There exists $\tau_2 \in (0, 1)$ such that for any $\tau \leq \tau_2$ and for any $\overline{\delta} > 0$ we can choose $\eta_2 = \eta_2(\overline{\delta}, \tau) > 0$ with the following property. If (K, u) is a minimizer of E_λ on B_1 and $0 \in K$, then for all $r \in (0, 1)$ such that*

$$\overline{\delta} \max\{\beta(r), \lambda \|g\|_\infty^2 r^{\frac{1}{2}}\} \leq d(r) \leq \eta_2,$$

we have

$$d(\tau r) \leq \tau^{\frac{1}{2}} d(r). \tag{3.2.7}$$

Before coming to the proofs of the two Lemmas we show how the decay proposition can be easily concluded from them.

Proof of Proposition 3.2.1. First of all fix $\tau = \min\{\tau_1, \tau_2\}$, denote by $\bar{\delta}$ a positive parameter smaller than 1 to be chosen later in terms of τ and let $\varepsilon = \min\{\eta_1(\tau, \bar{\delta}), \eta_2(\tau, \bar{\delta})\}$. We next claim that, for an appropriate choice of $\bar{\delta} \in (0, 1)$, the conclusion of the proposition holds (in fact we will see below that $\bar{\delta} \leq \tau^{\frac{7}{2}}$ suffices).

We distinguish four cases (actually, Case 4 below is not possible if $\lambda = 0$).

Case 1: $\bar{\delta}\max\{d(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\} \leq \beta(r)$ and $\bar{\delta}\max\{\beta(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\} \leq d(r)$. In this case $\beta(\tau r) \leq \tau\beta(r)$ by (3.2.6) and $d(\tau r) \leq \tau^{\frac{1}{2}} d(r)$, by (3.2.7), and thus $m(\tau r) \leq \tau^{\frac{1}{2}} m(r)$.

Case 2: $\bar{\delta}\max\{d(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\} \leq \beta(r)$ and $d(r) < \bar{\delta}\max\{\beta(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\}$. Observe that we have

$$\begin{aligned} d(\tau r) &\overset{(3.2.4)}{\leq} \tau^{-1} d(r) < \tau^{-1}\bar{\delta}\max\{\beta(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\} \\ \beta(\tau r) &\overset{(3.2.6)}{\leq} \tau\beta(r)\,. \end{aligned}$$

Hence, provided $\bar{\delta} \leq \tau^{\frac{3}{2}}$, we have $m(\tau r) \leq \tau^{\frac{1}{2}} m(r)$.

Case 3: $\beta(r) < \bar{\delta}\max\{d(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\}$ and $\bar{\delta}\max\{\beta(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\} \leq d(r)$. In this case

$$\begin{aligned} \beta(\tau r) &\overset{(3.2.3)}{\leq} \tau^{-3}\beta(r) < \tau^{-3}\bar{\delta}\max\{d(r), r^{\frac{1}{2}}\} \\ d(\tau r) &\overset{(3.2.7)}{\leq} \tau^{\frac{1}{2}} d(r)\,, \end{aligned}$$

Hence, by choosing $\bar{\delta} \leq \tau^{\frac{7}{2}}$ we conclude.

Case 4: $\beta(r) < \bar{\delta}\max\{d(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\}$ and $d(r) < \bar{\delta}\max\{\beta(r), \lambda\|g\|_\infty^2 r^{\frac{1}{2}}\}$. In this last case $m(r) = \lambda\|g\|_\infty^2 r^{\frac{1}{2}}$, being $\bar{\delta} < 1$, and $\max\{\beta(r), d(r)\} \leq \bar{\delta}\lambda\|g\|_\infty^2 r^{\frac{1}{2}}$. Hence, we conclude that

$$\begin{aligned} \beta(\tau r) &\overset{(3.2.3)}{\leq} \tau^{-3}\beta(r) \leq \tau^{-3}\bar{\delta}\lambda\|g\|_\infty^2 r^{\frac{1}{2}} \\ d(\tau r) &\overset{(3.2.4)}{\leq} \tau^{-1} d(r) \leq \tau^{-1}\bar{\delta}\lambda\|g\|_\infty^2 r^{\frac{1}{2}} \end{aligned}$$

It suffices to choose $\bar{\delta} \leq \tau^{\frac{7}{2}}$ to infer the desired decay. □

3.3. Lipschitz approximation

A key ingredient in the proof of both decay lemmas is the Lipschitz approximation for the set K. Before stating it we introduce a suitable notion of "excess" which, like the quantity β, measures the flatness of the set K. Given $\mathscr{V} \in \mathcal{L}$, we denote by $\pi_{\mathscr{V}} : \mathbb{R}^2 \to \mathscr{V}$ and $\pi_{\mathscr{V}}^{\perp} : \mathbb{R}^2 \to \mathscr{V}^{\perp}$ the orthogonal projections onto $\mathscr{V}$ and onto its orthogonal complement

$\mathscr{V}^\perp$, respectively. Given $\mathscr{V}, \mathscr{V}' \in \mathcal{L}$, consider the linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L := \pi_{\mathscr{V}} - \pi_{\mathscr{V}'}$, and denote by $|L|$ its Hilbert-Schmidt norm. In particular,

$$|L|^2 = 2 - 2\pi_{\mathscr{V}} : \pi_{\mathscr{V}'} ,$$

where : denotes the scalar product between 2×2 matrices. The excess is then given for $x \in B_1$ and $0 < r < 1 - |x|$ by

$$\mathrm{exc}(x, r) := \min_{\mathscr{V} \in \mathcal{L}} \mathrm{exc}_{\mathscr{V}}(x, r) \tag{3.3.1}$$

where for every $\mathscr{V} \in \mathcal{L}$

$$\mathrm{exc}_{\mathscr{V}}(x, r) := \frac{1}{r} \int_{B_r(x)\cap K} |\mathscr{V} - T_y K|^2 \, d\mathcal{H}^1(y). \tag{3.3.2}$$

It is also convenient to introduce a variant of the β-number: for all $\mathscr{A} \in \mathcal{A}$, we set

$$\beta_{\mathscr{A}}(x, r) := \int_{B_r(x)\cap K} \frac{\mathrm{dist}^2(y, \mathscr{A})}{r^2} \frac{d\mathcal{H}^1(y)}{r}$$

PROPOSITION 3.3.1. *There exist $C, \delta, \sigma, \alpha > 0$ geometric constants with the following properties. Assume that*

(a) *(K, u) is a minimizer of E_λ in B_r;*
(b) *$0 \in K$ and $\mathscr{V}_0$ is the horizontal axis;*
(c) *there exists $c \in \mathbb{R}$ such that setting $\bar\beta(r) := \beta_{(0,c)+\mathscr{V}_0}(r)$,*

$$d(r) + \bar\beta(r) + \lambda \|g\|^2_{L^\infty(B_r)} r < \delta .$$

Then,

(i) *$K \cap B_{\frac{r}{2}} \subset \{|x_2| \le Cr(\bar\beta(r))^\alpha\}$;*

and there exists $f : [-\sigma r, \sigma r] \to \mathbb{R}$ Lipschitz such that

(ii) *$\|f\|_{C^0} \le Cr\bar\beta(r)^\alpha$ and $\mathrm{Lip}(f) \le 1$;*
(iii) *the following estimates hold*

$$\mathcal{H}^1\left((\mathrm{gr}\,(f) \triangle K) \cap [-\sigma r, \sigma r]^2\right) \le Cr(d(r) + \bar\beta(r) + \lambda \|g\|^2_{L^\infty(B_r)} r)$$

and

$$\begin{aligned}\int |f'|^2 &\le \mathcal{H}^1(\mathrm{gr}\,(f) \setminus K) + C \int_{\mathrm{gr}\,(f)\cap K} |\mathscr{V}_0 - T_x K|^2 \, d\mathcal{H}^1(x) \\ &\le Cr(d(r) + \bar\beta(r) + \lambda \|g\|^2_{L^\infty(B_r)} r);\end{aligned}$$

(iv) *for any $\Lambda > 0$, δ can be chosen so that for any $\varepsilon > Cr(\bar\beta(r))^\alpha$ (cf. item (i) above),*

$$\mathcal{H}^1([-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(K \cap \{|x_2| < \varepsilon r\})) \le \Lambda r d(r) .$$

Finally, there are constants $\bar\tau > 0, \bar\varepsilon > 0$ with the property that

$$\tilde K := \left\{ x \in K \cap B_{\bar\tau r} \mid \sup_{0<\rho<\frac{r}{4}} (d(x, \rho) + \mathrm{exc}_{\mathscr{V}_0}(x, \rho)) < \bar\varepsilon \right\} \subseteq \mathrm{gr}\,(f). \tag{3.3.3}$$

3.3.1. Tilt Lemma. One basic tool to prove Proposition 3.3.1 is the following "tilt lemma", heavily inspired by a similar result in the theory of minimal surfaces due to Allard, cf. the seminal paper [1].

LEMMA 3.3.2. *There exists a universal constant $C > 0$ such that, if (K, u) is a minimizer of E_λ in B_1 and $0 \in K$,*

$$\operatorname{exc}\left(\tfrac{r}{4}\right) \leq C\left(d(r) + \beta(r) + \lambda\|g\|^2_{L^\infty(B_r)} r\right) \qquad \textit{for all } r \leq 1. \tag{3.3.4}$$

More precisely, if $\mathscr{A} = (0, c) + \mathscr{V}_0$ for some $c \in \mathbb{R}$, we have the more accurate estimate

$$\operatorname{exc}_{\mathscr{V}_0}\left(\tfrac{r}{4}\right) \leq C\left(d(r) + \beta_{\mathscr{A}}(r) + \lambda\|g\|^2_{L^\infty(B_r)} r\right). \tag{3.3.5}$$

PROOF. We start off noting that (3.3.5) implies (3.3.4). By rotating we can assume that $\mathscr{A} = (0, c) + \mathscr{V}_0$, for some constant $c \in \mathbb{R}$. By the density upper bound (1.3.1) we have $\operatorname{exc}_{\mathscr{V}_0}(\frac{r}{4}) \leq \frac{16}{r}\mathcal{H}^1(K \cap B_{\frac{r}{4}}) \leq 8\pi + \lambda\pi\|g\|^2_\infty r$. Let $\gamma > 0$ be a fixed a parameter which will be chosen appropriately later. If $\beta_{\mathscr{A}}(r) \geq \gamma$, then $\operatorname{exc}_{\mathscr{V}_0}(\frac{r}{4}) \leq \frac{8\pi}{\gamma}\beta_{\mathscr{A}}(r) + \lambda\pi\|g\|^2_\infty r$, and thus (3.3.5) follows in this case. Therefore, we may additionally suppose that

$$\beta_{\mathscr{A}}(r) \leq \gamma. \tag{3.3.6}$$

Next, we claim that

$$|c| \leq 2\left(\frac{\beta_{\mathscr{A}}(r)}{\epsilon}\right)^{\frac{1}{3}} r. \tag{3.3.7}$$

Indeed, first observe that $|c| < \frac{r}{2}$ for γ small enough. Otherwise, being $0 \in K$, the density lower bound inequality in (2.1.2) would imply

$$\frac{\epsilon}{64} \leq \frac{\epsilon}{4}\left(\frac{|c|}{2r}\right)^2 \leq \frac{1}{r^3}\int_{B_{\frac{r}{4}} \cap K} |x_2 - c|^2 \, d\mathcal{H}^1(x) \leq \beta_{\mathscr{A}}(r),$$

where in the second inequality we have used that $|x_2 - c| \geq \frac{|c|}{2}$ for all $x \in B_{\frac{r}{4}}$. By (3.3.6) this would give a contradiction for γ sufficiently small. Then, as $|c| < \frac{r}{2}$, arguing similarly we deduce that

$$\epsilon\left(\frac{|c|}{2r}\right)^3 \leq \frac{1}{r^3}\int_{B_{\frac{|c|}{2}} \cap K} |x_2 - c|^2 \, d\mathcal{H}^1(x) \leq \beta_{\mathscr{A}}(r),$$

as $|x_2 - c| \geq \frac{|c|}{2}$ for all $x \in B_{\frac{|c|}{2}}$.

We next use (3.3.6) and (3.3.7) to choose $\gamma > 0$ small enough to have $B_{\frac{r}{4}} \subset B_{\frac{r}{3}}((0, c)) \subset B_{\frac{2r}{3}}((0, c)) \subset B_r$. Hence, we conclude that it is enough to prove

$$\begin{aligned}
&\frac{1}{r}\int_{B_{\frac{r}{3}}((0,c)) \cap K} |\mathscr{V}_0 - T_x K|^2 \, d\mathcal{H}^1 \\
\leq &C\Big(d\left(\tfrac{2r}{3}\right) + \frac{1}{r^3}\int_{B_{\frac{2r}{3}}((0,c)) \cap K} |x_2 - c|^2 d\mathcal{H}^1 + \lambda\|g\|^2_{L^\infty(B_{\frac{2r}{3}}((0,c)))} r\Big),
\end{aligned}$$

which by translating is implied by

$$\frac{1}{r}\int_{B_{\frac{r}{3}}\cap K} |\mathscr{V}_0 - T_x K|^2 \, d\mathcal{H}^1 \leq C\Big(d(r) + \frac{1}{r^3}\int_{B_{\frac{2r}{3}}\cap K} |x_2|^2 \, d\mathcal{H}^1 + \lambda \|g\|^2_{L^\infty(B_{\frac{2r}{3}})} r\Big). \tag{3.3.8}$$

Let $e\colon K \to \mathbb{S}^1$ be a tangent vector field to K with $e(x) = (e_1(x), e_2(x))$. Then

$$|\mathscr{V}_0 - T_x K|^2 = \left| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} e_1^2 & e_1 e_2 \\ e_1 e_2 & e_2^2 \end{pmatrix} \right|^2 = 2 - 2e_1^2 = 2e_2^2 \, .$$

In particular, (3.3.8) is equivalent to

$$\frac{1}{r}\int_{K\cap B_{\frac{r}{3}}} e_2^2(x) \, d\mathcal{H}^1 \leq C\left(d(r) + \frac{1}{r^3}\int_{K\cap B_{\frac{2r}{3}}} |x_2|^2 \, d\mathcal{H}^1 + \lambda r \|g\|^2_{L^\infty(B_{\frac{2r}{3}})}\right). \tag{3.3.9}$$

Let $\eta(x) = \varphi^2(x)(0, x_2)$, $\varphi \in C_c^\infty(B_{\frac{2r}{3}}, [0,1])$ with $\varphi \equiv 1$ on $B_{\frac{r}{3}}$ and $\|\nabla\varphi\|_{L^\infty(B_{\frac{2r}{3}})} \leq \frac{6}{r}$. We have

$$D\eta(x) = 2\varphi(x)(0, x_2) \otimes \nabla\varphi(x) + \varphi^2(x) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus

$$\varphi^2 e_2^2 \leq e^T \, D\eta \, e + 2\varphi|\nabla\varphi||x_2||e_2|.$$

The internal variation formula (1.5.4) for critical points of E_λ, namely

$$\begin{aligned}\int_{B_r\cap K} e^T \cdot D\eta \, e \, d\mathcal{H}^1 &= -\int_{B_r\setminus K} \Big(|\nabla u|^2 \operatorname{div}\eta + 2\nabla u^T \cdot D\eta \, \nabla u\Big) \\ &\quad + 2\lambda \int_{B_r\setminus K} (u-g)\eta\cdot\nabla u + \lambda\int_{B_r\cap K}\Big(|u^+ - g_K|^2 - |u^- - g_K|^2\Big)\eta\cdot\nu \, d\mathcal{H}^1\end{aligned}$$

yields

$$\int_{K\cap B_r} \varphi^2 e_2^2 \, d\mathcal{H}^1 \leq 2\int_{K\cap B_r} \varphi|\nabla\varphi||x_2||e_2| \, d\mathcal{H}^1 + C\int_{B_{\frac{2r}{3}}\setminus K} |\nabla u|^2 + C\lambda\|g\|^2_{L^\infty(B_{\frac{2r}{3}})} r^2 \, ,$$

where we have used that $|D\eta| \leq C$ and that $\|u\|_{L^\infty(B_{\frac{2}{3}r})} \leq \|g\|_{L^\infty(B_{\frac{2}{3}r})}$. So we conclude

$$\int_{K\cap B_r} \varphi^2 e_2^2 \, d\mathcal{H}^1 \leq C\int_{K\cap B_r} |\nabla\varphi|^2 |x_2|^2 \, d\mathcal{H}^1 + C\left(rd\left(\frac{2r}{3}\right) + \lambda r^2 \|g\|^2_{L^\infty(B_{\frac{2r}{3}})}\right).$$

Since $\varphi \in C_c^\infty(B_{\frac{2r}{3}}, [0,1])$ with $\varphi \equiv 1$ on $B_{\frac{r}{3}}$ and $\|\nabla\varphi\|_{L^\infty(B_{\frac{2r}{3}})} \leq \frac{6}{r}$, (3.3.9) follows at once. $\square$

3.3.2. Vertical separation. The second main ingredient to prove Proposition 3.3.1 is the following "vertical separation lemma".

LEMMA 3.3.3. *There exist* $\varepsilon, \tau > 0$ *such that:*

(a) *if* (K, u) *is a minimizer of* E_λ *in* B_1*;*
(b) $z^1, z^2 \in K$ *and* $z^1 \in B_{\frac{1}{2}}$*;*
(c) $|z^1 - z^2| < \frac{\tau}{2}$ *and*

$$\sup_{\frac{|z^1-z^2|}{2}<\rho<\frac{|z^1-z^2|}{\tau}} \Big(d(z^1,\rho) + \mathrm{exc}_{\mathscr{V}_0}(z^1,\rho)\Big) < \varepsilon\,. \tag{3.3.10}$$

Then $|z^1_2 - z^2_2| \le |z^1_1 - z^2_1|$.

PROOF. We first consider the case of absolute and generalized absolute minimizers. For the sake of contradiction, assume the conclusion does not hold: in particular set $\varepsilon = \frac{1}{j}$ and $\tau = \frac{1}{j}$ and let $\{(K_j, u_j)\}$ and $z^1_j, z^2_j \in K_j$ be a sequence of minimizers and pairs of points which contradict the statement. Rescale the minimizers and translate so that $|z^1_j - z^2_j| = 1$ and $z^1_j = 0$. Up to subsequences assume that (K_j, u_j) converges to a generalized global minimizer $(\bar K, \bar u)$ and that z^i_j converges to some z^i, with $z^1 = 0$ and $|z^2| = 1$. From the hypothesis we know that $\int_{B_r} |\nabla \bar u|^2 = 0$ for every r. By Theorem 2.4.1, the pair $(\bar K, \bar u)$ can only be a constant, a pure jump, or a triple junction. We know $z^1, z^2 \in K$, so K is non-empty and u cannot be a constant. Consider now that, by (2.2.5) in Theorem 2.2.3, the set $\bar K$ has tangent $\mathscr{V}_0$ at $\mathcal{H}^1$-a.e. point. In particular, as $\mathrm{exc}_{\mathscr{V}_0}(0,\rho) = 0$ for all $\rho \in (\frac{1}{2}, +\infty)$ (and then for all $\rho > 0$), $\bar K$ cannot be the triple junction. So (K, u) must be a pure jump with a horizontal discontinuity. But from the contradiction assumption we also deduce that $|z^2_2| \ge |z^2_1|$. Since $z^1 = 0 \ne z^2$ and $|z^2| = 1$, the two points cannot belong to the same horizontal line. This gives a contradiction.

In the case of restricted and generalized restricted minimizers, note that we can again make a blow-up argument where we know that K_j is converging to an unbounded closed connected set $\bar K$ of locally finite length in $\mathbb{R}^2$. As above we conclude that the tangent to $\bar K$ is $\mathscr{V}_0$ $\mathcal{H}^1$-a.e., in particular we conclude that $\bar K$ is either a halfline or a line, both contained in $\mathscr{V}_0$. At any rate this implies that z^1 and z^2 both belong to the horizontal line $\mathscr{V}_0$, which is the same contradiction reached above. $\square$

3.3.3. Proof of Proposition 3.3.1. We may argue as in Lemma 3.3.2 (cf. (3.3.7)) to get $|c| \le 2\left(\frac{\bar\beta(r)}{\epsilon}\right)^{\frac{1}{3}} r$. In fact the same argument implies that

$$K \cap B_{\frac{r}{2}} \subset \{x : |x_2 - c| \le 2\left(\frac{\bar\beta(r)}{\epsilon}\right)^{\frac{1}{3}} r\}\,.$$

To this aim, assume that on the contrary there is $z \in K \cap B_{\frac{r}{2}}$ such that $|z_2 - c| > 2\left(\frac{\bar\beta(r)}{\epsilon}\right)^{\frac{1}{3}} r$. Set $\rho := \left(\frac{\bar\beta(r)}{\epsilon}\right)^{\frac{1}{3}} \frac{r}{2}$, if $\delta < \epsilon$ then $B_\rho(z) \subset B_r$ and we reach a contradiction as:

$$\frac{9}{8}\bar\beta(r) = \frac{9}{r^3}\epsilon\rho^3 \leq \frac{1}{r^3}\int_{B_\rho(z)\cap K} |x_2 - c|^2 \, d\mathcal{H}^1(x) \leq \bar\beta(r),$$

since $|x_2 - c| \geq 3\rho$ for every $x \in B_\rho(z)$.

In particular, we deduce that

$$K \cap B_{\frac{r}{2}} \subset \{|x_2| \leq Cr(\bar\beta(r))^{\frac{1}{3}}\}. \tag{3.3.11}$$

Fix ε and τ as in Lemma 3.3.3 and let

$$\tilde K := \left\{ x \in K \cap B_{\frac{\tau}{4}r} \mid \sup_{0<\rho<\frac{r}{2}} \left(d(x,\rho) + \mathrm{exc}_{\mathscr{V}_0}(x,\rho)\right) < \varepsilon \right\}.$$

On setting $\sigma := \frac{\tau}{4}$, using Lemma 3.3.3 we can define a 1-Lipschitz function $f : [-\sigma r, \sigma r] \to \mathbb{R}$ such that

$$\tilde K \subset \{(t, f(t)) \mid |t| \leq \sigma r\} = \mathrm{gr}\,(f).$$

In particular, by (3.3.11) we get conclusions (i) and (ii) with $\alpha = \frac{1}{3}$, as well as (3.3.3).

In addition, for what conclusion (iii) is concerned, being $\mathrm{Lip}(f) \leq 1$, the second estimate there follows immediately from Lemma 3.3.2 provided that the first estimate in conclusion (iii) itself is established. To this aim, by using Besicovitch covering theorem (see for instance [4, Theorem 2.18]), one can cover $(K \setminus \tilde K) \cap B_{\sigma r}$ with a countable family of balls with controlled overlapping such that in each ball the defining condition of $\tilde K$ does not hold, so that the ensuing estimate follows easily

$$\mathcal{H}^1((K\setminus\tilde K)\cap B_{\sigma r}) \leq \frac{C}{\varepsilon} r\left(d\left(\tfrac{3}{4}r\right) + \mathrm{exc}_{\mathscr{V}_0}\left(\tfrac{3}{4}r\right)\right) \leq \frac{C}{\varepsilon} r\Big(d(r) + \bar\beta(r) + \lambda r\|g\|^2_{L^\infty(B_r)}\Big), \tag{3.3.12}$$

where in the last inequality we have used (3.3.5) in the Tilt Lemma. Therefore, to conclude item (iii) we need to estimate $\mathcal{H}^1\left((\mathrm{gr}\,(f) \setminus K) \cap [-\sigma r, \sigma r]^2\right)$. To this aim, note that

$$\pi_{\mathscr{V}_0}(\mathrm{gr}\,(f) \setminus K) \cap [-\sigma r, \sigma r] \subset [-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(\tilde K),$$

so that (recalling $\mathrm{Lip}(f) \leq 1$) we get

$$\begin{aligned}
&\mathcal{H}^1\left((\mathrm{gr}\,(f) \setminus K) \cap [-\sigma r, \sigma r]^2\right) \\
&\leq \sqrt{2}\mathcal{H}^1\left(\pi_{\mathscr{V}_0}(\mathrm{gr}\,(f) \setminus K) \cap [-\sigma r, \sigma r]\right) \leq \sqrt{2}\mathcal{H}^1\left([-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(\tilde K)\right) \\
&\leq \sqrt{2}\mathcal{H}^1\left([-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(K)\right) + \sqrt{2}\mathcal{H}^1([-\sigma r, \sigma r] \cap \pi_{\mathscr{V}_0}(K \setminus \tilde K)) \\
&\leq \sqrt{2}\mathcal{H}^1\left([-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(K)\right) + \sqrt{2}\mathcal{H}^1((K \setminus \tilde K) \cap B_{\sigma r}),
\end{aligned}$$

In view of (3.3.12) we are left with estimating the measure of the set $A := [-\sigma r, \sigma r] \setminus \pi_{\mathscr{V}_0}(K)$. In this respect, consider the vertical segments $\mathscr{W}_t := \{t\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$\mathscr{W}_t \cap K = \emptyset \qquad \forall t \in A. \tag{3.3.13}$$

Theorem 2.4.1 and a compactness argument show that, choosing δ small enough in the assumption (c) of Proposition 3.3.1, (K,u) must be close to a pure jump (J,v), where $J=\{x_2=0\}$ and $v=v^+\chi_{\{x_2>0\}}+v^-\chi_{\{x_2<0\}}$, with $|v^+-v^-|\geq 2C_0>0$, for some universal constant C_0. In particular, $|u(t,\frac12)-u(t,-\frac12)|\geq C_0$ and then from Jensen inequality we get

$$\int_{\mathscr{W}_t}|\nabla u|^2dx_2\geq\left(\int_{\mathscr{W}_t}|\nabla u|dx_2\right)^2\geq C_0^2\qquad\forall t\in A.$$

From the latter we infer

$$\mathcal{H}^1(A)\leq\frac{1}{C_0^2}\int_{B_r}|\nabla u|^2,$$

that concludes the proof of (iii).

To prove (iv), we argue as above with

$$\tilde{A}:=[-\sigma r,\sigma r]\setminus\pi_{\mathscr{V}_0}(K\cap\{|x_2|<\varepsilon\})=[-\sigma r,\sigma r]\setminus\pi_{\mathscr{V}_0}(K).$$

Set

$$\mathscr{W}_t=\{t\}\times[-\gamma,\gamma]\,,\qquad\forall t\in\tilde{A}\,,$$

for a fixed $\gamma=O(\lambda^{\frac12})$. As before, by Jensen inequality we get

$$\frac{1}{2\gamma}\int_{\mathscr{W}_t}|\nabla u|^2dx_2\geq\left(\frac{1}{2\gamma}\int_{\mathscr{W}_t}|\nabla u|dx_2\right)^2\geq\left(\frac{C_0}{2\gamma}\right)^2,$$

and hence

$$\mathcal{H}^1(\tilde{A})\leq\frac{2\gamma}{C_0^2}\int_{B_r}|\nabla u|^2,$$

which implies (iv).

3.4. Regularity at pure jumps: decay lemmas

We are now ready to prove the two main decay lemmas.

3.4.1. Proof of Lemma 3.2.2. For the sake of contradiction, let $\tau_1>0$ to be chosen appropriately in what follows:then there are $\tau\in(0,\tau_1)$ and $\overline{\delta}>0$ such that there exist sequences (K_j,u_j) of minimizers of E_λ, radii $r_j\in(0,1)$, and real numbers c_j such that

$$\begin{aligned}\overline{\delta}\max\{d_j,\lambda_j\|g_j\|_\infty^2r_j^{\frac12}\}:=\overline{\delta}\max\left\{\frac{1}{r_j}\int_{B_{r_j}\setminus K_j}|\nabla u_j|^2,\lambda_j\|g_j\|_\infty^2r_j^{\frac12}\right\}&\\ \leq\frac{1}{r_j^3}\int_{B_{r_j}\cap K_j}|x_2-c_j|^2\,d\mathcal{H}^1=:\beta_j\to0,&\qquad(3.4.1)\end{aligned}$$

$$\min_{\mathcal{A}}\int_{B_{r_j}\cap K_j}\mathrm{dist}^2(x,\mathscr{A})d\mathcal{H}^1=\int_{B_{r_j}\cap K_j}|x_2-c_j|^2\,d\mathcal{H}^1,\qquad(3.4.2)$$

and that

$$\int_{B_{\tau r_j}\cap K_j}\mathrm{dist}^2(x,\mathscr{A})\,d\mathcal{H}^1\geq\tau^4r_j^3\beta_j\qquad\forall j,\ \forall\mathscr{A}\in\mathcal{A}\qquad(3.4.3)$$

(we can assume (3.4.2) thanks the the fact that our statement is invariant under rotations). To apply Proposition 3.3.1 let $\tau_1 \le \frac{\sigma}{2}$, σ being defined there. For j sufficiently large, let $f_j : [-\sigma r_j, \sigma r_j] \to \mathbb{R}$ be the 1-Lipschitz function provided by Proposition 3.3.1, and denote by Γ_j its graph. In light of conclusion (ii) in Proposition 3.3.1, we can assume Γ_j to be contained inside the rectangle $[-\sigma r_j, \sigma r_j] \times [-\frac{\sigma}{2} r_j, \frac{\sigma}{2} r_j]$. For any $\eta \in C_c^\infty((-\sigma r_j, \sigma r_j)^2; \mathbb{R}^2)$ consider the corresponding inner variation (1.5.4) to infer from the density upper bound in (1.3.1) and (3.4.1)

$$
\begin{aligned}
|\delta\Gamma_j(\eta)| &:= \left| \int_{\Gamma_j} e_j(x)^T \cdot D\eta \, e_j(x) \, d\mathcal{H}^1 \right| \\
&\le C\|\eta\|_{C^1} \int_{[-\sigma r_j, \sigma r_j]^2 \setminus K_j} |\nabla u_j|^2 + C\|\eta\|_{C^0} \lambda_j \|g_j\|_\infty^2 r_j \\
&\le C\bar{\delta}^{-1} \|\eta\|_{C^1} \beta_j r_j + C\bar{\delta}^{-1} \|\eta\|_{C^0} \beta_j r_j^{\frac{1}{2}} \,,
\end{aligned}
$$

where e_j is the unitary tangent vector field to Γ_j, and $C > 0$ is a constant.

Note that by choosing $\eta(x_1, x_2) = (0, \varphi(x_1)\psi(x_2))$, where $\varphi, \psi \in C_c^\infty((-\sigma r_j, \sigma r_j))$ and $\psi \equiv 1$ on $\left(-\frac{\sigma}{2} r_j, \frac{\sigma}{2} r_j\right)$, we can use the classical first variation formula for the length of the graph of a function to compute

$$
\delta\Gamma_j(\eta) = \int \frac{f_j' \varphi'}{\sqrt{1 + |f_j'|^2}}.
$$

In addition, by Proposition 3.3.1 (iii) and (3.4.1), we have that

$$
\int |f_j'|^2 \le C r_j (d_j + \beta_j + \lambda_j \|g_j\|_\infty^2 r_j) \le C r_j (\beta_j + \beta_j r_j^{\frac{1}{2}}) \le C\bar{\delta}^{-1} r_j \beta_j \,,
$$

where $C > 0$ is a constant. Thus, we set

$$
h_j(x_1) := \frac{f_j(r_j x_1) - f_j(0)}{r_j \beta_j^{\frac{1}{2}}},
$$

and conclude that, passing possibly to a subsequence, the h_j's converge weakly in the Sobolev space $W^{1,2}((-\sigma, \sigma))$ to a function h. Note that, because of the embedding $W^{1,2}((-\sigma, \sigma)) \subset C^{1/2}((-\sigma, \sigma))$ the convergence is uniform, and in particular we conclude that $h(0) = 0$. Moreover

$$
\int h' \zeta' = 0 \qquad \forall \zeta \in C_c^\infty((-\sigma, \sigma)).
$$

Hence, $h(x) = a x_1$ for some constant $a \in \mathbb{R}$. Consider $\mathscr{V}_j := \{(x_1, f_j(0) + \beta_j^{\frac{1}{2}} a x_1) \mid x_1 \in \mathbb{R}\}$, then from Proposition 3.3.1 (ii), recalling that $\tau < \tau_1 \le \frac{\sigma}{2}$, we conclude

$$
\int_{B_{\tau r_j} \cap \Gamma_j} \operatorname{dist}^2(x, \mathscr{V}_j) \, d\mathcal{H}^1 = o(r_j^3 \beta_j). \tag{3.4.4}
$$

On the other hand, from Proposition 3.3.1 (i), (iii) and (3.4.1) we have (assuming α there to be such that $2\alpha \le 1$)

$$\int_{B_{\tau r_j}\cap(K_j\setminus\Gamma_j)} \operatorname{dist}^2(x,\mathscr{V}_j)\,d\mathcal{H}^1 \le C r_j^2\beta_j^{2\alpha}\mathcal{H}^1(K_j\setminus\Gamma_j) \le C\bar\delta^{-1} r_j^3 \beta_j^{2\alpha+1}\,, \tag{3.4.5}$$

where $C>0$ is a constant. Hence, putting together (3.4.4) and (3.4.5),

$$\int_{B_{\tau r_j}\cap K_j} \operatorname{dist}^2(x,\mathscr{V}_j)\,d\mathcal{H}^1 = o(r_j^3\beta_j)$$

contradicting (3.4.3), and so we are done with the proof of (3.2.6).

3.4.2. Proof of Lemma 3.2.3. We argue by contradiction. Let $\tau_2>0$, to be suitably chosen afterwards: then there are $\tau\in(0,\tau_2)$, $\bar\delta>0$, and sequences (K_j,u_j) of minimizers of E_{λ_j}, real numbers c_j and radii r_j such that

$$d_j := \frac{1}{r_j}\int_{B_{r_j}\setminus K_j} |\nabla u_j|^2 \to 0\,,$$

$$\bar\delta \max\{\beta_j, \lambda_j\|g_j\|_\infty^2 r_j^{\frac12}\} := \bar\delta\max\left\{\frac{1}{r_j^3}\int_{B_{r_j}\cap K_j}|x_2-c_j|^2, \lambda_j\|g_j\|_\infty^2 r_j^{\frac12}\right\} \le d_j\,, \tag{3.4.6}$$

$$\min_{\mathcal{A}} \int_{B_{r_j}\cap K_j} \operatorname{dist}^2(x,\mathscr{A})d\mathcal{H}^1 = \int_{B_{r_j}\cap K_j} |x_2-c_j|^2\,d\mathcal{H}^1\,,$$

and

$$\int_{B_{\tau r_j}\setminus K_j} |\nabla u_j|^2 \ge \tau^{\frac32} r_j d_j. \tag{3.4.7}$$

Let $(\bar K_j, v_j)$ be defined by $\bar K_j := r_j^{-1}K_j$, and $v_j(y) := (d_jr_j)^{-\frac12}u_j(r_jy)$. In view of Proposition 3.3.1, the assumptions above and the density lower bound yield that $\bar K_j\cap\left[-\frac12,\frac12\right]^2\to\{x_2=0, |x_1|\le\frac12\}$ in Hausdorff convergence. In addition,

$$\int_{B_1\setminus\bar K_j} |\nabla v_j|^2 = 1, \tag{3.4.8}$$

thus by compactness for harmonic functions there exist harmonic functions $v^\pm$ and constants κ_j such that, up to subsequences,

$$v_j-\kappa_j \to v^\pm \qquad \text{in } W^{1,2}_{\mathrm{loc}}(B_1^\pm), \tag{3.4.9}$$

where $B_\rho^\pm := B_\rho\cap\{\pm x_2>0\}$, $\rho>0$. The heart of the proof is to show that

$$\lim_{j\to\infty}\int_{B_{\frac{\sigma}{2}}\setminus\bar K_j}|\nabla v_j|^2 = \int_{B^+_{\frac{\sigma}{2}}}|\nabla v^+|^2 + \int_{B^-_{\frac{\sigma}{2}}}|\nabla v^-|^2 \tag{3.4.10}$$

(σ the constant introduced in Proposition 3.3.1). In fact, once we have (3.4.10) we can easily conclude from (3.4.7) and (3.4.8) that

$$\tau^{\frac{3}{2}}\Big(\int_{B^+_{\frac{\sigma}{2}}}|\nabla v^+|^2+\int_{B^-_{\frac{\sigma}{2}}}|\nabla v^-|^2\Big)\le\tau^{\frac{3}{2}}\le\int_{B^+_\tau}|\nabla v^+|^2+\int_{B^-_\tau}|\nabla v^-|^2.$$

On the other hand, from the harmonicity of $v^\pm$ we have for $\tau<\frac{\sigma}{2}$

$$\int_{B^\pm_\tau}|\nabla v^\pm|^2\le\frac{4\tau^2}{\sigma^2}\int_{B^\pm_{\frac{\sigma}{2}}}|\nabla v^\pm|^2,$$

and we get a contradiction as long as $\frac{4\tau^2}{\sigma^2}<\tau^{\frac{3}{2}}$, for instance by choosing $\tau_2=\frac{\sigma^4}{32}$.

We are now left to establish (3.4.10). By (3.4.9), it is enough to prove that the Dirichlet energy of v_j on $B^\pm_{\frac{\sigma}{2}}$ does not concentrate on $\left[-\frac{\sigma}{2},\frac{\sigma}{2}\right]\times\{0\}$. For the sake of contradiction, if the energy concentrates, there would exist a constant $\theta>0$ and a sequence $\varepsilon_j\to0$ such that

$$\int_{(-\frac{\sigma}{2},\frac{\sigma}{2})\times(-\frac{\varepsilon_j}{2},\frac{\varepsilon_j}{2})\setminus\bar K_j}|\nabla v_j|^2\ge\theta. \tag{3.4.11}$$

Up to replacing ε_j with $\max\{\varepsilon_j,C(\bar\delta^{-1}d_j)^\alpha\}$, where C is the constant in (iv) of Proposition 3.3.1, we may assume $\varepsilon_j\ge C(\bar\delta^{-1}d_j)^\alpha$. Thus, using (3.4.6), we infer that $\varepsilon_j\ge C\beta_j^\alpha$.

To get a contradiction, we first note that being (K_j,u_j) a minimizer of E_{λ_j} on B_{r_j}, $(\bar K_j,v_j)$ minimizes on B_1 the functional

$$F_j(v,\bar K):=\int_{B_1}|\nabla v|^2+\frac{1}{d_j}\mathcal{H}^1(\bar K\cap B_1)+\lambda_jr_j^2\int_{B_1}|v-\bar g_j|^2 \tag{3.4.12}$$

where $\bar g_j(x):=(d_jr_j)^{-\frac12}g_j(r_jx)$. We next use (3.4.11) to build a competitor (K'_j,w_j) for $(\bar K_j,v_j)$ with $F_j(w_j,K'_j)<F_j(v_j,\bar K_j)$ for j large, which contradicts the minimality of $(\bar K_j,v_j)$. For the sake of simplicity we assume that the minimizers are absolute minimizers, the reader can anyway easily check that the competitor exhibited below is allowed also in the case of restricted and generalized minimizers.

To this aim consider the 1-Lipschitz function $f_j:[-\sigma r_j,\sigma r_j]\to\mathbb{R}$ given by Proposition 3.3.1 applied to (K_j,u_j). Note that by item (ii) there $\|f_j\|_{C^0}\le Cr_j(\bar\delta^{-1}d_j)^\alpha$, and by item (iii) there and by (3.4.6) we have

$$\int_{-\sigma r_j}^{\sigma r_j}|f_j'|^2\le C\bar\delta^{-1}r_jd_j,\qquad\mathcal{H}^1((\mathrm{gr}\,(f_j)\triangle K_j)\triangle[-\sigma r_j,\sigma r_j]^2)\le C\bar\delta^{-1}r_jd_j,$$

where $C>0$ is a constant, and by item (iv) for any $\Lambda,\varepsilon>0$ for j sufficiently large

$$\mathcal{H}^1([-\sigma r_j,\sigma r_j]\setminus\pi_{\mathscr{V}_0}(K_j\cap\{|x_2|<\varepsilon r_j\}))\le\Lambda r_jd_j.$$

In particular, the function $\bar f_j(t):=r_j^{-1}f_j(r_jt)$, $t\in[-\sigma,\sigma]$, is 1-Lipschitz with $\|\bar f_j\|_{C^0}\le C(\bar\delta^{-1}d_j)^\alpha<\varepsilon_j$, and

$$\int_{-\sigma}^{\sigma}|\bar f_j'|^2\le C\bar\delta^{-1}d_j,\qquad\mathcal{H}^1((\mathrm{gr}\,(\bar f_j)\triangle\bar K_j)\triangle[-\sigma,\sigma]^2)\le C\bar\delta^{-1}d_j, \tag{3.4.13}$$

and

$$\mathcal{H}^1([-\sigma,\sigma]\setminus\pi_{\mathscr{V}_0}(\bar{K}_j\cap\{|x_2|<\varepsilon\}))\le\Lambda d_j. \tag{3.4.14}$$

In view of (3.4.8) and (3.4.14), an elementary averaging argument implies that we can find two points, $a_j\in[-\sigma,-\frac{3}{4}\sigma]$ and $b_j\in[\frac{3}{4}\sigma,\sigma]$, with the property that on the region

$$R_j:=([a_j-\varepsilon_j,a_j+2\varepsilon_j]\cup[b_j-\varepsilon_j,b_j+2\varepsilon_j])\times\left[-\frac{3}{4}\sigma,\frac{3}{4}\sigma\right]$$

we have

$$\int_{R_j\setminus\bar{K}_j}|\nabla v_j|^2\le C\varepsilon_j,\qquad \mathcal{H}^1((\bar{K}_j\triangle\operatorname{gr}(\bar{f}_j))\cap R_j)\le C\varepsilon_j d_j, \tag{3.4.15}$$

for some positive constant C depending on σ and $\bar{\delta}$. Let h_j be the affine function such that $h_j(a_j)=\bar{f}_j(a_j)$ and $h_j(b_j)=\bar{f}_j(b_j)$. Note that $\|h_j\|_{C^0}\le C\bar{\delta}_j^{-\alpha}d_j^{\alpha}$, and $|h_j'|\le Cd_j^{\frac{1}{2}}$ by (3.4.13). Next, for simplicity of exposition we assume that the graph of h_j is horizontal (this is clearly always true up to a rotation). Let $h_j\equiv\bar{h}_j\in\mathbb{R}$ under the new choice of coordinates. In particular, $\bar{h}_j\to 0$ as $j\uparrow\infty$.

Set $Q_j:=[a_j,b_j]\times\left[-\frac{\sigma}{2},\frac{\sigma}{2}\right]$, assuming σ sufficiently small we construct a map $\Phi_j\colon B_1\to B_1$ as follows (see Figure 1 for a visual description of Φ_j):

- $\Phi_j(x_1,x_2)=(x_1,x_2)$, if $(x_1,x_2)\in B_1\setminus Q_j$;
- $\Phi_j(x_1,x_2)=(x_1,\varphi_j(x_1,x_2))$ if $(x_1,x_2)\in Q_j$;

where $\varphi_j:Q_j\to Q_j$ is defined by

- If $a_j+\varepsilon_j\le x_1\le b_j-\varepsilon_j$, the map $x_2\mapsto\varphi_j(x_1,x_2)$
  - i) is identically $\bar{h}_j$ for $|x_2-\bar{h}_j|\le 2\varepsilon_j$;
  - ii) maps linearly the segments $\{x_1\}\times[\bar{h}_j+2\varepsilon_j,\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j-2\varepsilon_j]$ onto $\{x_1\}\times[\bar{h}_j,\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j]$, respectively.
- If $a_j\le x_1\le a_j+\varepsilon_j$, the map $x_2\mapsto\varphi_j(x_1,x_2)$
  - i) is identically $\bar{h}_j$ if $|x_2-\bar{h}_j|\le 2(x_1-a_j)$;
  - ii) maps linearly the segments $\{x_1\}\times[\bar{h}_j+2(x_1-a_j),\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j-2(x_1-a_j)]$ onto $\{x_1\}\times[\bar{h}_j,\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j]$, respectively.
- If $b_j-\varepsilon_j\le x_1\le b_j$, the map $x_2\mapsto\varphi_j(x_1,x_2)$
  - i) is identically $\bar{h}_j$ if $|x_2-\bar{h}_j|\le 2(b_j-x_1)$;
  - ii) maps linearly the segments $\{x_1\}\times[\bar{h}_j+2(b_j-x_1),\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j-2(b_j-x_1)]$ onto $\{x_1\}\times[\bar{h}_j,\frac{\sigma}{2}]$ and $\{x_1\}\times[-\frac{\sigma}{2},\bar{h}_j]$, respectively.

Let $S_j:=[a_j,b_j]\times\{\bar{h}_j\}$, $\Sigma_j:=\Phi_j^{-1}(S_j)$, $T_j:=\{(x_1,x_2)\in Q_j:\ a_j+\varepsilon_j\le x_1\le b_j-\varepsilon_j,\ 2\varepsilon_j\le|x_2-\bar{h}_j|\le\frac{\sigma}{2}\}$, and $U_j:=[a_j+\varepsilon_j,b_j-\varepsilon_j]\times[-\frac{\sigma}{2},\frac{\sigma}{2}]$. Then

(a) $\Phi_j|_{\Sigma_j}$ is the projection onto $\mathscr{V}_j:=\{(x_1,\bar{h}_j):x_1\in\mathbb{R}\}$ with $\Phi_j(\Sigma_j)=S_j$, $\Phi_j\colon B_1\to B_1$ is Lipschitz, and $\Phi_j\colon B_1\setminus\Sigma_j\to B_1\setminus S_j$ is bi-Lipschitz;

(b) $\|D\Phi_j\|_{L^\infty((B_1\setminus Q_j)\cup\Sigma_j,B_1)}=1$, $\|D\Phi_j\|_{L^\infty(Q_j\setminus\Sigma_j,B_1)}\le 1+C\varepsilon_j$, and $\|D(\Phi_j^{-1})\|_{L^\infty(B_1\setminus S_j,B_1\setminus\Sigma_j)}\le 1+C\varepsilon_j$, for some universal $C>0$, and where we are using the operator norm on $D\Phi_j$;

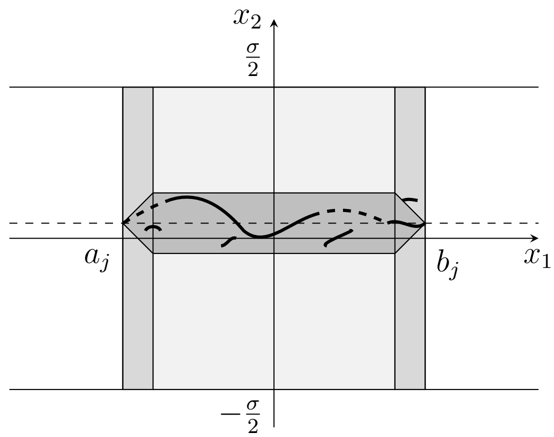

FIGURE 1. A visual description of the map Φ_j in the rotated coordinates (x_1, x_2). The map is the identity outside the gray zones. The diamond shaped dark gray zone is the region Σ_j: Φ_j "squeezes" it on the horizontal central segment S_j (which lies on the dashed horizontal line). The 6 lighter gray zones are consequently stretched by Φ_j: in T_j, the two central lightest ones, and in the four remaining lateral zones the Lipschitz constant is controlled by $1 + C\varepsilon_j$. The set $\bar{K}_j$ is depicted by the thick arcs, while the graph of $\bar{f}_j$ is thick and dashed (and has a large overlap with $\bar{K}_j$). While the graph of $\bar{f}_j$ must lie in the region Σ_j, there might well be portions of $\bar{K}_j$ which lie outside.

(c) $\operatorname{gr}(\bar{f}_j) \cap Q_j \subset \Sigma_j$.

The competitor (K'_j, w_j) to test the minimality of $(\bar{K}_j, v_j)$ for F_j is then given by

$$K'_j := (\bar{K}_j \setminus Q_j) \cup S_j \cup \Phi_j(\bar{K}_j \setminus \Sigma_j), \qquad w_j := v_j \circ \Phi_j^{-1} \quad \text{on } B_1 \setminus K'_j$$

In particular, note that the number of connected components of K'_j is less than or equal to that of $\bar{K}_j$.

We begin by estimating the length of K'_j. First note that $K'_j \setminus Q_j = \bar{K}_j \setminus Q_j$. Furthermore, we have

$$\begin{aligned}\mathcal{H}^1(S_j) &= \mathcal{H}^1(\Phi_j(\bar{K}_j \cap \Sigma_j)) + \mathcal{H}^1(S_j \setminus \Phi_j(\bar{K}_j \cap \Sigma_j)) \\ &\overset{(a),(b)}{\leq} \mathcal{H}^1(\bar{K}_j \cap \Sigma_j) + \mathcal{H}^1([a_j, b_j] \times \{\bar{h}_j\} \setminus \pi_{\mathscr{V}_j}(\bar{K}_j)) \overset{(3.4.14)}{\leq} \mathcal{H}^1(\bar{K}_j \cap \Sigma_j) + \Lambda d_j\end{aligned}$$

where we recall that $\mathscr{V}_j = \{(x_1, \bar{h}_j) : x_1 \in \mathbb{R}\}$. On the other hand, the very definition of K'_j gives

$$\begin{aligned}\mathcal{H}^1((K'_j \cap Q_j) \setminus S_j) &= \mathcal{H}^1(\Phi_j((\bar{K}_j \cap Q_j) \setminus \Sigma_j)) \overset{(b)}{\leq} (1 + C\varepsilon_j)\mathcal{H}^1((\bar{K}_j \cap Q_j) \setminus \Sigma_j) \\ &\overset{(c)}{\leq} \mathcal{H}^1((\bar{K}_j \cap Q_j) \setminus \Sigma_j) + C\varepsilon_j \mathcal{H}^1((\bar{K}_j \cap Q_j) \setminus \mathrm{gr}\,(\bar{f}_j)) \\ &\overset{(3.4.13)}{\leq} \mathcal{H}^1((\bar{K}_j \cap Q_j) \setminus \Sigma_j) + C\varepsilon_j d_j\,.\end{aligned}$$

Hence, we conclude that

$$\mathcal{H}^1(K'_j \cap B_1) \leq \mathcal{H}^1(\bar{K}_j \cap B_1) + \Lambda d_j + o(d_j). \tag{3.4.16}$$

Now, we estimate the Dirichlet energy. Note that $w_j = v_j$ on $B_1 \setminus Q_j$, and in addition that

$$\int_{U_j \setminus K'_j} |\nabla w_j|^2 \overset{(a),(b)}{\leq} (1 + C\varepsilon_j)^2 \int_{T_j \setminus \bar{K}_j} |\nabla v_j|^2 \overset{(3.4.11)}{\leq} (1 + C\varepsilon_j)^2 \left(\int_{Q_j \setminus \bar{K}_j} |\nabla v_j|^2 - \theta \right).$$

and that

$$\int_{Q_j \setminus (U_j \cup K'_j)} |\nabla w_j|^2 \leq \|D(\Phi_j^{-1})\|^2_{L^\infty(B_1 \setminus S_j, B_1 \setminus \Sigma_j)} \int_{Q_j \setminus (U_j \cup \bar{K}_j)} |\nabla v_j|^2 \overset{(3.4.15),\,(b)}{\leq} C\varepsilon_j\,.$$

Therefore, we conclude by (3.4.8) that

$$\int_{B_1 \setminus K'_j} |\nabla w_j|^2 \leq \int_{B_1 \setminus \bar{K}_j} |\nabla v_j|^2 - \theta + o(1). \tag{3.4.17}$$

By taking into account that $\|w_j\|_{L^\infty(B_1)} = \|v_j\|_{L^\infty(B_1)} \leq \|\bar{g}_j\|_{L^\infty(B_1)} = (d_j r_j)^{-\frac{1}{2}} \|g_j\|_{L^\infty(B_{r_j})}$, and that $\lambda_j \|g_j\|^2_{L^\infty(B_1)} r_j \leq \bar{\delta}^{-1} r_j^{1/2} d_j$ in view of (3.4.6), adding up (3.4.16) and (3.4.17) we get from the very definition of F_j in (3.4.12)

$$F_j(w_j, K_j) \leq F_j(v_j, \bar{K}_j) + \Lambda - \theta + o(1),$$

which, for j large enough and $\Lambda < \theta$, contradicts the fact that $(\bar{K}_j, v_j)$ minimizes F_j.

3.5. Regularity at pure jumps: conclusion

We are now ready to show how the ε-regularity Theorem 3.1.1 follows from the decay Proposition 3.2.1.

Proof of Theorem 3.1.1. Without loss of generality assume that $x = 0$ and that $\theta = 0$. First note that the very definition of β in (3.2.1) gives

$$\begin{aligned}\beta(0, 2r) &\leq (2r)^{-3} \mathrm{dist}^2_H(K \cap \bar{B}_{2r}, \mathscr{V}_0 \cap \bar{B}_{2r}) \mathcal{H}^1(K \cap \bar{B}_{2r}) \\ &\leq C(\Omega^j(0, 0, r))^2 (1 + \lambda \|g\|^2_\infty r^2) \leq C\Omega^j(0, 0, r)\end{aligned}$$

where $C > 0$ is a universal constant, assuming $\varepsilon \in (0,1)$, and using assumption (ii) and the energy upper bound in (1.3.1) for the last inequality. In turn, from this we deduce that

$$\beta(0,2r) + d(0,2r) + \lambda \|g\|_\infty^2 (2r)^{\frac{1}{2}} \le C(\Omega^j(0,0,r) + \lambda \|g\|_\infty^2 r^{\frac{1}{2}}) =: \varepsilon(r) < C\varepsilon\,.$$

Then from assumption (ii), (3.2.3) and (3.2.4) we get

$$\beta(z,r) + d(z,r) + \lambda \|g\|_\infty^2 r^{\frac{1}{2}} < C\varepsilon(r) \qquad \forall z \in B_r\,.$$

Using Proposition 3.2.1, if ε is chosen sufficiently small, we get for all $k \in \mathbb{N}$

$$\beta(z,\tau^k r) + d(z,\tau^k r) + \lambda \|g\|_\infty^2 (\tau^k r)^{\frac{1}{2}} < C\varepsilon(r)\tau^{\frac{k}{2}} \qquad \forall z \in K \cap B_r\,.$$

In particular, we can easily conclude

$$\beta(z,\rho) + d(z,\rho) + \lambda \|g\|_\infty^2 \rho^{\frac{1}{2}} \le C\varepsilon(r)\rho^{\frac{1}{2}} \qquad \forall \rho < r, \forall z \in K \cap B_r \tag{3.5.1}$$

(where from now on we stop keeping track of geometric constants). From the Tilt Lemma 3.3.2 we infer that

$$\mathrm{exc}(z,\rho) \le C\varepsilon(r)\rho^{\frac{1}{2}} \qquad \forall \rho < \frac{r}{4}\,, \forall z \in K \cap B_r. \tag{3.5.2}$$

For each z and ρ as above, let now $\mathscr{V}(z,\rho) \in \mathcal{L}$ be such that

$$\mathrm{exc}(z,\rho) = \mathrm{exc}_{\mathscr{V}(z,\rho)}(z,\rho),$$

(cf. (3.3.1) and (3.3.2) for the definition of excess), then observe that for all $\rho < t < \frac{r}{8}$ and $t \le 2\rho$, using the density lower bound in Theorem 2.1.3, we have

$$\begin{aligned} |\mathscr{V}(z,\rho) - \mathscr{V}(z,t)|^2 &\le \frac{C}{\rho} \int_{K \cap B_\rho(z)} |\mathscr{V}(z,\rho) - \mathscr{V}(z,t)|^2 d\mathcal{H}^1(y) \\ &\le C(\mathrm{exc}(z,\rho) + \mathrm{exc}(z,t)) \overset{(3.5.2)}{\le} C\varepsilon(r)\rho^{\frac{1}{2}}\,. \end{aligned} \tag{3.5.3}$$

Additionally, with a similar argument we infer that

$$|\mathscr{V}(z,\rho) - \mathscr{V}(y,\rho)|^2 \le C\varepsilon(r)\rho^{\frac{1}{2}} \qquad \forall z, y \in K \cap B_r \text{ with } |z-y| \le \tfrac{\rho}{4},\ \rho < \tfrac{r}{8}\,.$$

Combining both estimates with (3.5.2), an elementary summation argument on dyadic scales yields that

$$|\mathscr{V}(z,\rho) - \mathscr{V}(0,\tfrac{r}{16})|^2 \le C\varepsilon(r) r^{\frac{1}{2}} \qquad \forall z \in K \cap B_{\frac{r}{16}}\,, \forall \rho < \tfrac{r}{8}\,.$$

If we rotate the coordinates so that $\mathscr{V}(0,\frac{r}{16})$ is the horizontal line $\mathscr{V}_0$, the density lower bound in Theorem 2.1.3, (3.5.2) and the latter inequality imply that

$$\mathrm{exc}_{\mathscr{V}_0}(z,\rho) = \frac{1}{\rho}\int_{K \cap B_\rho(z)} |T_y K - \mathscr{V}_0|^2 \, d\mathcal{H}^1(y) \le C\varepsilon(r) r^{\frac{1}{2}} \qquad \forall z \in K \cap B_{\frac{r}{16}}\,, \forall \rho < \tfrac{r}{8}\,.$$

Therefore, thanks to (3.5.1) and (3.3.3) in Proposition 3.3.1, we know that for a choice of ε sufficiently small there is a 1-Lipschitz function $f : [-\sigma r, \sigma r] \to \mathbb{R}$ such that $K \cap B_{\sigma r} \subseteq \mathrm{gr}\,(f)$, where σ is the geometric constant in Proposition 3.3.1.

In addition, estimate (3.5.1) yields that

$$\lim_{\rho\downarrow 0}\frac{1}{\rho}\int_{B_\rho(z)}|\nabla u|^2=0$$

for every $z\in K\cap B_{\frac{r}{16}}$. Hence, by Theorem 2.4.1 any blow-up at every such z is either a pure jump or a triple junction. On the other hand, since $\beta(z,\rho)\downarrow 0$ as $\rho\downarrow 0$ (cf. (3.5.1)), we infer that any blow-up is in fact a pure jump. In particular, we conclude that

$$\lim_{\rho\downarrow 0}\frac{\mathcal{H}^1(K\cap B_\rho(z))}{2\rho}=1\,. \tag{3.5.4}$$

Note that, choosing ε sufficiently small, we can assume that $K\cap B_{\frac{\sigma r}{4}}$ is not empty (cf. item (i) in Proposition 3.3.1). In particular, $\|f\|_{C^0([-\frac{\sigma r}{4},\frac{\sigma r}{4}])}\le\frac{3}{4}\sigma r$. Denote by R the open rectangle

$$R=\Big(-\frac{\sigma r}{4},\frac{\sigma r}{4}\Big)\times\Big(-\frac{3}{4}\sigma r,\frac{3}{4}\sigma r\Big)\,.$$

We show next that $K\cap R=\mathrm{gr}\,(f)\cap R$. We know that $\pi_{\mathscr{V}_0}(K\cap R)$ is a relatively closed set inside $I=(-\frac{\sigma r}{4},\frac{\sigma r}{4})$ and it is not empty. If $I\setminus\pi_{\mathscr{V}_0}(K)$ is not empty, then it contains at least an open interval (a,b) with one extremum, say b, which belongs to $\pi_{\mathscr{V}_0}(K)$. Namely $y=(b,f(b))\in K\cap R$, but the density of the set K at y would be strictly less than 1, a contradiction to (3.5.4). Hence $K\cap R=\mathrm{gr}\,(f)\cap R$, in particular $K\cap B_{\frac{\sigma r}{4}}=\mathrm{gr}\,(f)\cap B_{\frac{\sigma r}{4}}$.

Finally, observe that the limit of $\mathscr{V}(z,\rho)$ for $\rho\downarrow 0$ exists at every $z=(s,f(s))\in K\cap\mathrm{gr}\,(f)$ by (3.5.3), that together with (3.5.2) yield that any $s\in I$ is a Lebesgue point of f'. In turn, this fact implies the differentiability of f at every $s\in I$, with the limit being the graph of the linear map $t\mapsto f'(s)t$ (cf., for instance, the proof of Rademacher's theorem in [4, Theorem 2.14]). In particular, the decay of $\mathrm{exc}(z,\cdot)$ in (3.5.2) can be translated into

$$\fint_{s-\delta}^{s+\delta}|f'(t)-f'(s)|^2\,dt\le C\varepsilon(r)\delta^{\frac{1}{2}}\qquad\forall\delta\le\sigma\rho_0-|s|,$$

and so the classical Campanato's theorem implies that $f\in C^{1,\frac{1}{4}}(I)$ with $[f']_{\frac{1}{4}}\le(C\varepsilon(r))^{\frac{1}{2}}$ (cf. [4, Theorem 7.51]). Estimate (3.1.2) follows from this and Proposition 3.3.1. This proves the claim of the theorem except that the graphicality of K has been shown in $B_{\frac{\sigma r}{4}}$ rather than in B_r. Note however that we can proceed with a simple covering argument to achieve graphicality in B_r (provided we choose ε even smaller). □

3.6. Triple junctions, closeness at all scales

We turn to the proof of Theorem 3.1.2. We start off proving the following fact: if around a point x a minimizer is close to a triple junction at some scale r, then there is a nearby point y such that the minimizer is close to a triple junction at *every* scale $\rho<r$ around y. In order to formulate our conclusion more precisely, we recall the notation introduced in Theorem 3.1.2

$$\Omega^t(\theta,x,r)=r^{-1}\mathrm{dist}_H\Big(K\cap\overline{B}_{2r}(x),(x+\mathcal{R}_\theta(\mathscr{T}_0))\cap\overline{B}_{2r}(x)\Big)+r^{-1}\int_{B_{2r}(x)\setminus K}|\nabla u|^2\,,$$

where $\theta \in [0, 2\pi]$, $\mathcal{R}_\theta$ is the corresponding rotation, and $\mathscr{T}_0$ is defined in (3.1.3). Furthermore we define

$$\Omega^t(x, r) := \inf_\theta \Omega^t(\theta, x, r) \tag{3.6.1}$$

LEMMA 3.6.1. *For every $\delta > 0$ there is $\eta > 0$ such that the following holds. Assume that (K, u) is a minimizer of E_λ in $B_{2r}(x)$ such that*

$$\Omega^t(x, r) + \lambda \|g\|_\infty^2 r^{\frac{1}{2}} < \eta .$$

Then there is a $y \in B_{\delta r}(x)$ such that

$$\Omega^t(y, \rho) + \lambda \|g\|_\infty^2 \rho^{\frac{1}{2}} \le \delta \qquad \forall \rho < r .$$

Lemma 3.6.1 follows in fact easily from the following more technical statement.

LEMMA 3.6.2. *There exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$ there is $\varepsilon_0(\gamma) > 0$ with the following property. For every $\varepsilon \in (0, \varepsilon_0)$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that for all $r \in (0, 1]$, for all (K, u) minimizer of E_λ in $B_{2r}(x)$, and for all $(N+1)$-ple of points $x_0 = x, x_1, \ldots, x_N$ in $B_{2r}(x)$ such that*

$$\begin{aligned} &\Omega^t(x_k, 2^{-k} r) + \lambda \|g\|_\infty^2 (2^{-k} r)^{\frac{1}{2}} \le \varepsilon \qquad \forall k \in \{0, \ldots, N\} \\ &|x_{k+1} - x_k| \le \gamma 2^{-k} r \qquad \forall k \in \{0, \ldots, N-1\} , \end{aligned}$$

then there is a point $x_{N+1} \in B_{2r}(x)$ such that

$$\begin{aligned} &\Omega^t(x_{N+1}, 2^{-N-1} r) + \lambda \|g\|_\infty^2 (2^{-N-1} r)^{\frac{1}{2}} \le \varepsilon \\ &|x_{N+1} - x_N| \le \gamma 2^{-N} r . \end{aligned}$$

3.6.1. Proof of Lemma 3.6.1. Without loss of generality $x = 0$. Fix $\delta > 0$, and additionally choose ε and γ sufficiently small, whose choice will be specified later, so that Lemma 3.6.2 is applicable. Let N be given by Lemma 3.6.2 and notice that, if η is chosen sufficiently small, the assumption of that Lemma holds with $x = x_0 = x_1 = \ldots = x_N = 0$. We thus find x_{N+1} as in the conclusion there. Observe therefore that we can apply the lemma again in $B_{\frac{r}{2}}(x_1)$, but this time the points $x_0, \ldots, x_N$ substituted by $x_1, \ldots, x_{N+1}$. Proceeding inductively we find a sequence of points $\{x_k\}$ with:

- $x_0 = 0$;
- $|x_{k+1} - x_k| \le \gamma 2^{-k} r$;
- $\Omega^t(x_k, 2^{-k} r) + \lambda \|g\|_\infty^2 (2^{-k} r)^{\frac{1}{2}} \le \varepsilon$.

Since $\{x_k\}$ is a Cauchy sequence, it has a limit y. Observe that $|y - x_k| \le \gamma 2^{-k+1} r$. Fix $\rho \le r$ and choose k such that $2^{-k-1} r < \rho \le 2^{-k} r$. In particular $B_\rho(y) \subset B_{2^{-k+1} r}(x_k)$. Let θ_k be such that

$$\Omega^t(\theta_k, x_k, 2^{-k} r) + \lambda \|g\|_\infty^2 (2^{-k} r)^{\frac{1}{2}} \le \varepsilon .$$

Observe that

$$\begin{aligned}
&\rho^{-1}\mathrm{dist}_H\Big(K\cap\overline{B}_{2\rho}(y),(y+\mathcal{R}_{\theta_k}(\mathscr{T}_0))\cap\overline{B}_{2\rho}(y)\Big)+\rho^{-1}\int_{B_{2\rho}(y)}|\nabla u|^2+\lambda\|g\|_\infty^2\rho^{\frac12}\\
&\le C\gamma+C(2^{-k}r)^{-1}\mathrm{dist}_H\Big(K\cap\overline{B}_{2^{-k+1}r},(x_k+\mathcal{R}_{\theta_k}(\mathscr{T}_0))\cap\overline{B}_{2^{-k+1}r}(x_k)\Big)\\
&\quad+C(2^{-k}r)^{-1}\int_{B_{2^{-k+1}r}(x_k)}|\nabla u|^2+C\lambda\|g\|_\infty^2(2^{-k}r)^{\frac12}\le C(\gamma+\varepsilon)\,,
\end{aligned}$$

where $C\ge 1$ is a geometric constant. We choose first $C\gamma\le\frac{\delta}{4}$. Having fixed γ, we can take $\varepsilon_0(\gamma)$ as in Lemma 3.6.2 and hence impose that $C\varepsilon<\min\{\varepsilon_0(\gamma),\frac{\delta}{4}\}$. This ensures the applicability of Lemma 3.6.2 in the argument above, and the inequality $C(\gamma+\varepsilon)<\delta$. We thus conclude

$$\Omega^t(y,\rho)+\lambda\|g\|_\infty^2\rho^{\frac12}\le\Omega^t(\theta,y,\rho)+\lambda\|g\|_\infty^2\rho^{\frac12}<\delta\,.$$

3.6.2. Proof of Lemma 3.6.2 for absolute and generalized minimizers. The argument for the two cases is entirely analogous and for simplicity we focus on absolute minimizers.

Without loss of generality $x=0$. We argue by contradiction. We assume that, for some $\gamma>0$ and $\varepsilon>0$ sufficiently small (the smallness will be specified later) and for every $N\in\mathbb{N}$ there are

(a) A family of numbers $\lambda_N\in[0,1]$;
(b) A family of fidelity functions g_N with $\|g_N\|_\infty\le M_0$;
(c) A family of radii $r_N\in(0,1]$;
(d) A family of points $x_{k,N}$, with $k\in\{0,\dots,N\}$, and

$$x_{0,N}=0 \tag{3.6.2}$$

$$|x_{k+1,N}-x_{k,N}|\le\gamma 2^{-k}r_N\qquad\forall k\in\{0,\dots,N-1\}\,; \tag{3.6.3}$$

(e) An absolute minimizing pair (K_N,u_N) of $E_{\lambda_N}(\cdot,\cdot,B_{2r_N},g_N)$ for which

$$\Omega^t(x_{k,N},2^{-k}r_N)+\lambda_N\|g_N\|_\infty^2(2^{-k}r_N)^{\frac12}\le\varepsilon\,, \tag{3.6.4}$$

for all $k\in\{0,\dots,N\}$;
(f) For every $y\in B_{\gamma 2^{-N}r_N}(x_{N,N})$

$$\Omega^t(y,2^{-N-1}r_N)+\lambda_N\|g_N\|_\infty^2(2^{-N-1}r_N)^{\frac12}>\varepsilon\,. \tag{3.6.5}$$

For each N we consider the rescaled pairs

$$v_N(x):=(2^{-N}r_N)^{-\frac12}u_N\Big(x_{N,N}+2^{-N}r_Nx\Big)\,, \tag{3.6.6}$$

$$J_N:=(2^{-N}r_N)^{-1}(K_N-x_{N,N})\,. \tag{3.6.7}$$

Next, observe that from items (a)-(c) above we achieve

$$\lim_{N\to\infty}\lambda_N\|g_N\|_\infty^2(2^{-N}r_N)^{\frac12}=0\,. \tag{3.6.8}$$

We can therefore apply Theorem 2.2.3 to conclude the convergence, up to subsequences, of (J_N,v_N) to a generalized minimizer $(J,v,\{p_{kl})\})$ of E_0.

Note that, the points $x_{k,N}$ are mapped to points $y_{k,N} := (2^{-N} r_N)^{-1}(x_{k,N} - x_{N,N})$. From (3.6.3) we immediately get

$$|y_{N-k,N}| \leq \gamma \sum_{j=1}^{k} 2^j \leq \gamma 2^{k+1}\,.$$

In particular, for every fixed $k \geq 1$, up to extraction of an appropriate subsequence, we can assume that $y_{N-k,N}$ converges to a point y_k with $|y_k| \leq \gamma 2^{k+1}$. In particular for (J, v), from (3.6.4) we immediately conclude

$$\Omega^t(y_k, 2^k) \leq \varepsilon \qquad \forall k \in \mathbb{N}\,. \tag{3.6.9}$$

But from the convergence in Theorem 2.2.3, from (3.6.5) and (3.6.8), we also conclude that

$$\Omega^t(z, 1/2) \geq \varepsilon \qquad \forall z \in B_\gamma\,. \tag{3.6.10}$$

For each k choose θ_k such that

$$\mathrm{dist}_H(J \cap B_{2^{k+1}}(y_k), y_k + \mathcal{R}_{\theta_k}(\mathscr{T}_0)) = \min_\theta \mathrm{dist}_H(J \cap B_{k+1}(y_k), y_k + \mathcal{R}_\theta(\mathscr{T}_0))$$

We fix $\delta > 0$ and we wish now apply Corollary 3.1.4 to the pair $(2^{-k}(J - y_k), 2^{-\frac{k}{2}} v(y_k + 2^k \cdot))$ choosing (K, u) equal to $(\mathcal{R}_{\theta_k}(\mathscr{T}_0)), 0)$, while $U = B_1 \setminus \bar{B}_\gamma$ and $V = B_{1-\gamma} \setminus B_{2\gamma}$. This just requires ε to be chosen sufficiently small depending on γ and δ. In particular we conclude that $2^{-k}(J - y_k) \cap (B_{1-\gamma} \setminus B_{2\gamma})$ is the union of three C^1 arcs, δ-close to the three segments $\mathcal{R}_{\theta_k}(\mathscr{T}_0)) \cap (B_{1-\gamma} \setminus B_{2\gamma})$. Scaling back, we conclude that in $A_k := J \cap (B_{2^k(1-\gamma)} \setminus B_{2^{k+1}\gamma})$ the set J consists of three C^1 arcs with Hausdorff distance less than $\delta 2^k$ from the three segments $y_k + \mathcal{R}_{\theta_k}(\mathscr{T}_0)$. Choosing γ smaller than a geometric constant, it is easy to see that each of the three C^1 curves $A_k \cap J$ coincide with one of the three C^1 curves $A_{k+1} \cap J$ in $A_k \cap A_{k+1}$, and viceversa, see Figure 2.

Thus $J \cap (\mathbb{R}^2 \setminus B_{\frac{1}{2}})$ consists of three distinct nonselfintersecting C^1 infinite curves Γ_i, which subdivide $\mathbb{R}^2 \setminus B_{\frac{1}{2}}$ in three infinite connected components. Note that, moreover, for each $r \geq \frac{1}{2}$, each curve Γ_i intersects each ∂B_r in exactly one point $p_i(r)$ and that, the three points $p_1(r), p_2(r), p_3(r)$ subdivide ∂B_r in three arcs of length $\frac{2\pi}{3} r(1-\delta) \leq l_i(r) \leq \frac{2\pi}{3} r(1+\delta)$.

Therefore, we are in a position to apply Proposition 2.6.2 and deduce from (2.6.3) that

$$\int_{2^k}^{2^{k+1}} \frac{1}{\rho} \int_{\partial B_\rho} |\nabla v|^2 d\rho \leq C\left(F(2^{k+1}) - F(2^k)\right) \qquad \forall k \geq -1\,,$$

where we recall that $F(r) := 2d(r) + \frac{\ell(r)}{r}$. Summing for $k = -1$ to $k_0 - 1$, using that $F(r)$ is positive and thanks to the density upper bound (1.3.1), we conclude

$$\int_{\frac{1}{2}}^{2^{k_0}} \frac{1}{\rho} \int_{\partial B_\rho} |\nabla v|^2 d\rho \leq C F(2^{k_0}) \leq \frac{C}{2^{k_0}} E_0(J, v, B_{2^{k_0}}) \leq C\,. \tag{3.6.11}$$

Letting k_0 to infinity we achieve

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\rho} \int_{\partial B_r} |\nabla v|^2 d\rho \leq C\,,$$

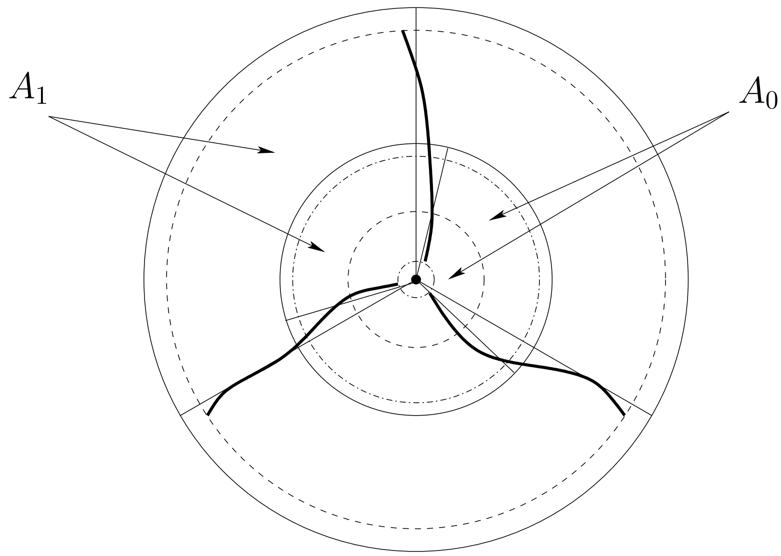

FIGURE 2. The region A_1 is delimited by the two dashed circles, while the region A_0 is delimited by the two dashed-and-dotted circles. In each of this regions the set J consists of three C^1 arcs which are close to three straight lines meeting at 120 degrees. The arcs must coincide where the regions A_0 and A_1 overlap, hence $J \cap (A_0 \cup A_1)$ consists of three C^1 arcs.

or equivalently by integration on dyadic intervals,

$$\sum_{k \geq 0} 2^{-k} \int_{B_{2^k} \setminus B_{2^{k-1}}} |\nabla v|^2 \leq C\,. \tag{3.6.12}$$

We claim that $(J, v, \{p_{ij}\})$ is indeed a triple junction, with z_0 being the point where the three half-lines meet together. This obviously would give a contradiction to (3.6.10), because $\Omega^t(0,2) \leq \varepsilon$, with $\varepsilon \leq \varepsilon_0(\gamma)$, would imply, for an appropriately chosen $\varepsilon_0(\gamma)$, that z_0 belongs to B_γ.

To this aim we start off showing that for all $R \geq 1$ the connected components of $\partial B_R \setminus J$ belong to three distinct connected components of $\mathbb{R}^2 \setminus J$. As a first step we show that each "blow-down" of $(J, v, \{p_{ij}\})$, i.e. any limit as $R \to \infty$ of a subsequence of the rescalings $(J_{0,R}, v_{0,R}, \{p_{ij}\})$, is a triple junction. For the sake of notational simplicity we drop 0 in the previous subscripts.

First of all, since v is harmonic in $\mathbb{R}^2 \setminus J$ and being $J \setminus B_{\frac{1}{2}}$ smooth, standard estimates for harmonic functions give for every $k \geq -1$

$$\|\nabla v\|^2_{L^\infty(B_{2^{k+1}} \setminus (B_{2^k} \cup J))} \leq C\Big(2^{-2k} \int_{B_{2^{k+2}} \setminus (B_{2^{k-1}} \cup J)} |\nabla v|\Big)^2 \leq C 2^{-2k} \int_{B_{2^{k+2}} \setminus (B_{2^{k-1}} \cup J)} |\nabla v|^2 \tag{3.6.13}$$

$C > 0$ independent of k. Recall that Γ_i is locally $C^{1,1}$ by item (b) in Proposition 2.5.2, and that by (2.5.4)

$$\kappa_i = -(|\nabla v^+|^2 - |\nabla v^-|^2) \qquad \mathcal{H}^1\text{-a.e. on } \Gamma_i$$

where κ_i denotes the curvature of Γ_i. Thus we conclude from the energy upper bound (1.3.1), and estimates (3.6.12) and (3.6.13)

$$\begin{aligned}\int_{J\setminus B_{\frac{1}{2}}} |\kappa_i| d\mathcal{H}^1 &\leq 2\sum_{k\geq 0} \|\nabla v\|^2_{L^\infty(B_{2^k}\setminus(B_{2^{k-1}}\cup J))}\mathcal{H}^1\Big(\Gamma_i\cap(B_{2^k}\setminus(B_{2^{k-1}}\cup J))\Big)\\ &\leq C\sum_{k\geq 0} 2^{-k}\int_{B_{2^{k+2}}\setminus(B_{2^{k-1}}\cup J)} |\nabla v|^2 \leq C\,.\end{aligned}$$

In particular, each Γ_i is asymptotic at ∞ to some straight line ℓ_i, and in particular as $R\uparrow\infty$, J_R converges, locally in the sense of Hausdorff, to a set J_∞ which is the union of three half-lines meeting at the origin. Apply Theorem 2.3.2 and let $(J_\infty, v_\infty, \{q_{kl}\})$ be any limit of a subsequence $R_j\uparrow\infty$ of $(J_{R_j}, v_{R_j}, \{p_{kl}\})$. It follows that v_∞ is harmonic in each of the three sectors which form the connected components of $\mathbb{R}\setminus J_\infty$ and that $\frac{\partial v_\infty}{\partial\nu}=0$ on J_∞. We also know that the angles formed by the three half-lines ℓ_i are all close to $\frac{2\pi}{3}$. Using Theorem 2.4.1(iii) we conclude that $(J_\infty, v_\infty, \{q_{kl}\})$ is a triple junction.

We next argue that the three connected components of $\mathbb{R}^2\setminus(B_{\frac{1}{2}}\cup J)$ belong to different connected components of $\mathbb{R}^2\setminus J$. In turn, this implies the claim that $\partial B_R\setminus J$ belongs to three different connected components of $\mathbb{R}^2\setminus J$ for all $R\geq 1$. Indeed, by contradiction fix two connected components Ω_1 and Ω_2 of $\mathbb{R}^2\setminus(B_{\frac{1}{2}}\cup J)$ and two arbitrary points $q_1\in\Omega_1$ and $q_2\in\Omega_2$ that are connected by a C^1 arc in $\mathbb{R}^2\setminus J$. The boundaries of Ω_1 and Ω_2 have one of the Γ_i in common and to fix ideas let us assume it is Γ_1. The latter is asymptotic to the half ℓ_1, which is common to the boundaries of two of the sectors of $\mathbb{R}^2\setminus J_\infty$. We denote Λ_1 and Λ_2 these two sectors and we fix two points $\hat{q}_1\in\Lambda_1$ and $\hat{q}_2\in\Lambda_2$ with $|\hat{q}_1|=|\hat{q}_2|=1$. It is then easy to see that there are two C^1 curves σ_1,σ_2 contained in $\mathbb{R}^2\setminus(B_{\frac{1}{2}}\cup J)$ and containing, respectively, q_1 and q_2, and such that for R sufficiently large, $\sigma_i\setminus B_R=\{\rho\hat{q}_i:\rho\geq R\}$. Thus by (3.6.12) and (3.6.13) we can write, for $2^{k-1}\geq R$,

$$\begin{aligned}|v(2^k\hat{q}_i)-v(2^{k-1}\hat{q}_i)| &\leq \int_{\sigma_i\cap(B_{2^k}\setminus B_{2^{k-1}})} |\nabla v|\, d\mathcal{H}^1\\ &\leq \mathcal{H}^1\Big(\sigma_i\cap(B_{2^k}\setminus B_{2^{k-1}})\Big)\|\nabla v\|_{L^\infty(B_{2^k}\setminus B_{2^{k-1}})}\leq C\,2^{\frac{k}{2}}\end{aligned}$$

Hence we can estimate

$$|v(2^k\hat{q}_1)-v(2^k\hat{q}_2)|\leq C2^{\frac{k}{2}}\,,$$

given that there is a curve connecting $R\hat{q}_1$ and $R\hat{q}_2$ which does not intersect J.

On the other hand, since all blow-downs of $(J, v, \{p_{kl}\})$ are triple junctions, from Theorem 2.3.2 we necessarily conclude that

$$\lim_{k\to\infty}\frac{|v(2^k\hat{q}_1)-v(2^k\hat{q}_2)|}{2^{\frac{k}{2}}}=\infty\,,$$

which would be a contradiction.

In particular, having shown that each connected component of $\partial B_R\setminus J$ belongs to distinct connected components of $\mathbb{R}^2\setminus J$, we can apply Proposition 2.6.1, and thus we conclude that

$$[1,\infty) \ni R \mapsto \frac{1}{R}\int_{B_R\setminus J} |\nabla v|^2$$

is nondecreasing in R. Since however we know that the blow-downs of $(J, v, \{p_{kl}\})$ are triple junctions, we have

$$\lim_{R\uparrow\infty} \frac{1}{R}\int_{B_R\setminus J} |\nabla v|^2 = 0\,.$$

We thus conclude that $\int_{B_R\setminus J} |\nabla v|^2 = 0$ for every $R \geq 1$. In particular by Theorem 2.4.1 $(J, v, \{p_{kl}\})$ is itself a triple junction.

3.6.3. Proof of Lemma 3.6.2 for restricted minimizers. In the case of restricted (and generalized restricted) minimizers, we apply the same procedure above. Note, however, that we cannot immediately conclude that the blow-downs are triple junctions, because the proof that $(J_\infty, v_\infty, \{q_{kl}\})$ are triple junctions, i.e. $|q_{kl}| = \infty$, relies on a comparison with a competitor which increases the number of connected components. Now, assume that for every fixed $R > 1$ the number of connected components of the sets J_{R_j} in the blow-down sequence is larger than 1, namely at least two, for an infinite number of j. Then, for the limiting blow-down we can use a competitor which has two connected components, because the cut-and-paste argument which from this competitor yields the competitors for the approximating sequence keeps the same number of connected components. On the other hand the argument $|q_{kl}| = \infty$ relies indeed in using a better competitor which has two connected components. Hence in this case the conclusion is valid and in particular the set J is connected. If instead there is an R for which the number of connected components of the J_{R_j} is 1 for an infinite set of j's, then we conclude directly that J is connected. At any rate, in both cases J must be connected. Hence we can apply the monotonicity formula in Proposition 2.6.1 and conclude that J is the union of three half lines, while v is locally constant. Note that the proof that the three half lines must meet at 120 degrees holds for restricted minimizers as well, since it is based on exhibiting competitors that consist of a single connected component.

3.7. Proof of Theorem 3.1.2

Denote by ε_0 the constant in assumption (ii) of Theorem 3.1.1 and fix a $\delta > 0$, whose choice will be specified later. Let then the constant ε in the statement of Theorem 3.1.2 be smaller than the constant η provided by the conclusion of Lemma 3.6.1. Let y be the point in $B_{\delta r}(x)$ provided by Lemma 3.6.1 itself. Arguing as in Lemma 3.6.2, if δ is chosen sufficiently small with respect to ε_0, then $K \cap B_r(y) \setminus \{y\}$ consists of three arcs $\Gamma_1, \Gamma_2, \Gamma_3$ with the following properties:

(i) Each Γ_i is a $C^{1,\frac{1}{2}}$ arc in $B_{r(1-s)}(y) \setminus B_{rs}(y)$ for every $s \in (0, \frac{1}{2})$.
(ii) For every $k \in \mathbb{N}$, $k \geq 1$, there is an angle θ_k such that $K \cap (B_{2^{-k}r}(y) \setminus B_{2^{-k-1}r}(y))$ is $\delta 2^{-k} r$ close in the Hausdorff distance to $(y + \mathcal{R}_{\theta_k}(\mathscr{T}_0))) \cap (B_{2^{-k}r}(y) \setminus B_{2^{-k-1}r}(y))$.
(iii) Each Γ_i is, in $B_{2^{-k}r}(y) \setminus B_{2^{-k-1}r}(y)$, a $C^{1,\frac{1}{2}}$ graph over the corresponding straight line $\ell_{k,i}$ of $y + \mathcal{R}_{\theta_k}(\mathscr{T}_0)$ of a function $\psi_{k,i}$ with $\|\psi'_{k,i}\|_\infty \leq C\delta$.

From now on, in order to simplify our notation, we assume without loss of generality that $y = 0$.

We are now in the position to apply Proposition 2.6.2 for all radii $\rho \in (0, r)$. Thus, from (2.6.3) we infer that

$$\frac{D'(\rho)}{\rho} \leq C\, F'(\rho) + C$$

for $\mathcal{L}^1$ a.e. $\rho \in (0, r)$. Therefore, by direct integration and the density upper bound (1.3.1) we deduce that

$$\int_0^r \frac{D'(\rho)}{\rho} d\rho \leq CF(r) + Cr \leq \frac{C}{r} E_\lambda(K, u, B_r, g) + Cr \leq C\,,$$

for a dimensional constant $C > 0$, or equivalently by passing to dyadic intervals,

$$\sum_{k \geq 2} \frac{1}{2^{-k} r} \int_{B_{2^{-k+2}r} \setminus B_{2^{-k+1}r}} |\nabla u|^2 \leq C\,. \tag{3.7.1}$$

Since $\Delta u = \lambda(u - g)$ on $B_{2r} \setminus K$, $K \cap (B_{2^{-k+2}r} \setminus B_{2^{-k+1}r})$ is $C^{1,\frac{1}{2}}$ and u satisfies the Neumann boundary conditions on K, by elliptic regularity we have

$$\|\nabla u\|^2_{L^\infty(B_{2^{-k+1}r} \setminus (B_{2^{-k}r} \cup K))} \leq \frac{C}{2^{-2k} r^2} \int_{B_{2^{-k+2}} \setminus (B_{2^{-k-1}r} \cup K)} |\nabla u|^2 + C\lambda 2^{-2k} r^2 (\|u\|^2_\infty + \|g\|^2_\infty) \tag{3.7.2}$$

for some constant $C > 0$.

Denote by κ_i the curvature of Γ_i and recall that

$$\kappa_i = -(|\nabla u^+|^2 - |\nabla u^-|^2) - \lambda(|u^+ - g_K|^2 - |u^- - g_K|^2)$$

$\mathcal{H}^1$-a.e. on Γ_i by (2.5.4) for an appropriate trace g_K of g (which enjoys the same upper bound on the L^∞ norm). From (3.7.2) we thus conclude

$$\int_{\Gamma_i \cap (B_{2^{-k+1}r} \setminus B_{2^{-k-1}r})} |\kappa_i| d\mathcal{H}^1 \leq \frac{C}{2^{-k} r} \int_{B_{2^{-k+2}r} \setminus B_{2^{-k-1}r}} |\nabla u|^2 + C 2^{-k} r \tag{3.7.3}$$

for some constant $C > 0$. We therefore conclude from (3.7.1)

$$\int_{\Gamma_i \cap B_r \setminus \{0\}} |\kappa_i| d\mathcal{H}^1 \leq C\,.$$

In turn, from this we deduce that $\overline{\Gamma_i}$ is a C^1 graph up to the origin. Moreover, a blow-up argument shows that the tangents to Γ_i in the origin form equal angles and thus, up to rotations, we can assume that they are given by $\{\theta = 0\}$, $\{\theta = \frac{2\pi}{3}\}$ and $\{\theta = \frac{4\pi}{3}\}$.

We shall prove next that $\overline{\Gamma_i}$ is a $C^{1,\gamma}$ graph up to the origin, $\gamma \in (0, 1)$, by means of a suitable monotonicity formula. First notice that for $\rho \in (0, r)$ we have $\mathcal{H}^0(\partial B_\rho \cap K) = 3$. Then, if we consider the set K' which is the union of the three segments obtained by joining each point in $\partial B_\rho \cap K$ with the origin we have

$$\mathcal{H}^1(B_\rho \cap K) \leq 3\rho = \mathcal{H}^1(B_\rho \cap K')\,.$$

In addition, $B_\rho \setminus K' = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where each Ω_i is a convex cone with vertex the origin and opening α_i, with $|\alpha_i - \frac{2}{3}\pi| \le C\delta(\rho)$, with $\delta(\rho) \to 0$ if $\rho \to 0$. Therefore if $C\delta(\rho) < \delta_0$ for $\rho \le \rho_0$ (δ_0 can be chosen as small as we want up to reducing ρ_0) and if w_i is the (harmonic) function provided by Lemma 2.6.6, we have

$$\int_{\Omega_i} |\nabla w_i|^2 \le \frac{\alpha_i}{\pi}\rho \int_{\partial\Omega_i \cap \partial B_\rho} \left(\frac{\partial u}{\partial \tau}\right)^2 . \tag{3.7.4}$$

If w is defined as $w|_{\Omega_i} = w_i$, $w = u$ on $B_{2r} \setminus B_\rho$, then testing the minimality of (K, u) with the competitor (K', w) we get for $\alpha_0 := \frac{2}{3}\pi + \delta_0$ (note that $\max\{\alpha_1, \alpha_2, \alpha_3\} \le \alpha_0$)

$$\int_{B_\rho} |\nabla u|^2 + \mathcal{H}^1(B_\rho \cap K) + \lambda \int_{B_\rho} |u - g|^2 \le \frac{\alpha_0}{\pi}\rho \int_{\partial B_\rho} \left(\frac{\partial u}{\partial \tau}\right)^2 + 3\rho + \lambda \int_{B_\rho} |w - g|^2 ,$$

from which we deduce straightforwardly that for $\mathcal{L}^1$-a.e. $\rho \in (0, \rho_0)$

$$D(\rho) \le \frac{\alpha_0}{\pi}\rho D'(\rho) + 4\pi\lambda \|g\|_\infty^2 \rho^2.$$

In turn, the latter inequality and the energy upper bound in (1.3.1) imply that for all $\rho \in (0, \rho_0)$

$$D(\rho) \le C\rho_0^{-\frac{\pi}{\alpha_0}}(D(\rho_0) + \rho_0^2)\rho^{\frac{\pi}{\alpha_0}} \le C\rho_0^{1-\frac{\pi}{\alpha_0}}\rho^{\frac{\pi}{\alpha_0}} , \tag{3.7.5}$$

for some constant $C > 0$ depending on λ, $\|g\|_\infty$ and α_0. Finally, estimates (3.7.3) and (3.7.5) yield

$$\int_{\Gamma_i \cap B_\rho} |\kappa_i| d\mathcal{H}^1 \le C\rho^{\frac{\pi}{\alpha_0}-1} ,$$

and the claimed $C^{1,\gamma}$ regularity of $\overline{\Gamma_i}$ up to the origin then follows at once choosing δ_0 sufficiently sufficiently small so that $\alpha_0 < \pi$.

We now come to the construction of the diffeomorphism Φ which is given as the composition $\Phi_0 \circ \Phi_1$ of two other diffeomorphisms. We note that due to our assumption we have actually set $\theta = 0$. Without loss of generality we assume $r = 1$. Φ_0 maps y (which without loss of generality has been assumed to be 0) into the origin. If we let $\varphi \in C_c^\infty(B_1)$ be a function which is identically equal to 1 on $B_{\frac{1}{2}}$, Φ_0 is then given by the formula

$$\Phi_0(z) := z + (1 - \varphi(z))x .$$

(observe that $B_\delta(x)$ contains the origin, in particular $|x| \le \delta$: thus if δ is sufficiently small, the latter map can be seen to be C^1 close to the identity, and hence a diffeomorphism). Φ_1 is then the inverse of a map Ψ which maps $K' := \Phi_0^{-1}(K)$ onto three straight half-lines emanating from 0 and forming equal angles. We know indeed that K' consists of three $C^{1,\alpha}$ arcs γ_1, γ_2, and γ_3 meeting at 0, where they form equal angles. We use polar coordinates (θ, ρ) to define the map Φ_1. Upon a suitable rotation we can also assume the tangents to γ_1, γ_2, and γ_3 at the origin are given by $\{\theta = 0\}$, $\{\theta = \frac{2\pi}{3}\}$, and $\{\theta = \frac{4\pi}{3}\}$. In particular each γ_i is given in polar coordinates by $\{(\rho, \theta_i(\rho))\}$, where $\theta_i : (0, 2r) \to \mathbb{S}^1$ is a $C^{1,\alpha}$ function and satisfies $\|\theta_i - (i-1)\frac{2\pi}{3}\| \le \delta$ and $|\theta_i'(r)| \le \delta r^{-1}$. In particular we can assume that $\delta < \frac{1}{2}$.

Let now $\psi \in C_c^\infty(-1,1)$ be a function which is identically equal to 1 on $(-\frac{1}{2},\frac{1}{2})$. In polar coordinates the map Φ_1 is then given by

$$(\theta,\rho) \mapsto \left(\theta + \sum_i \psi\left(\theta - (i-1)\tfrac{2\pi}{3}\right)\left(\theta_i(\rho) - (i-1)\tfrac{2\pi}{3}\right), \rho\right).$$

The estimate (3.1.5) follows from the corresponding estimates for the functions θ_i and for $|x|$ and is left to the reader.

3.7.1. Proof of Corollary 3.1.3.

Proof. We first note that $K \setminus \{\bar{x}\}$ is the union of three $C^{1,1}$ arcs in view of Proposition 2.5.2. We use the notation and arguments in the proof of Theorem 3.1.2, in particular $\bar{x} = 0$ by translation. We build upon the conclusions of Theorem 3.1.2 and substitute the monotonicity formula employed there by a sharper one inspired by Proposition 2.6.1.

Let $\rho \in (0, 2r)$, then $\mathcal{H}^0(\partial B_\rho \cap K) = 3$. Let $B_\rho \setminus K = S_1 \cup S_2 \cup S_3$ with $\partial B_\rho \setminus K = \gamma_1 \cup \gamma_2 \cup \gamma_3$, $\gamma_i \subset \partial S_i$, and $|\mathcal{H}^1(\gamma_i) - \frac{2}{3}\pi\rho| \leq C\varepsilon\rho^{1+\gamma}$. Denote by u_i the mean value of u on γ_i and recall that $\int_{\gamma_i} \frac{\partial u}{\partial \nu} = 0$ by (2.5.6) (here we use $\lambda = 0$). For every $i \in \{1,2,3\}$, by the sharp Poincaré-Wirtinger inequality we then get that

$$\begin{aligned}\int_{S_i} |\nabla u|^2 &\overset{(2.5.5)}{=} \int_{\gamma_i} u\frac{\partial u}{\partial \nu} \overset{(2.5.6)}{=} \int_{\gamma_i} (u-u_i)\frac{\partial u}{\partial \nu} \leq \int_{\gamma_i} \left(\frac{\delta}{2}(u-u_i)^2 + \frac{1}{2\delta}\left(\frac{\partial u}{\partial \nu}\right)^2\right) \\ &\leq \int_{\gamma_i} \left(\frac{\delta}{2}\left(\frac{\mathcal{H}^1(\gamma_i)}{\pi}\right)^2\left(\frac{\partial u}{\partial \tau}\right)^2 + \frac{1}{2\delta}\left(\frac{\partial u}{\partial \nu}\right)^2\right) = \frac{\mathcal{H}^1(\gamma_i)}{2\pi}\int_{\gamma_i} |\nabla u|^2\end{aligned}$$

having chosen $\delta = \frac{\pi}{\mathcal{H}^1(\gamma_i)}$ in the last inequality. In turn, from this we conclude that for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_0)$, $\rho_0 < 2r$ sufficiently small, we have

$$\begin{aligned}D(\rho) &= \int_{B_\rho \setminus K} |\nabla u|^2 = \sum_{i=1}^3 \int_{S_i} |\nabla u|^2 \leq \max_{i\in\{1,2,3\}} \frac{\mathcal{H}^1(\gamma_i)}{2\pi} \int_{\partial B_\rho \setminus K} |\nabla u|^2 \\ &= \max_{i\in\{1,2,3\}} \frac{\mathcal{H}^1(\gamma_i)}{2\pi} D'(\rho) \leq \frac{5}{12}\rho D'(\rho).\end{aligned}$$

By direct integration and the energy upper bound in (1.3.1) we deduce that for all $\rho \in (0,\rho_0)$

$$D(\rho) \leq D(\rho_0)\left(\frac{\rho}{\rho_0}\right)^{\frac{5}{12}} \leq 2\pi\rho_0^{-\frac{7}{5}}\rho^{\frac{12}{5}},$$

and thus estimate (3.7.3) yields

$$\int_{\Gamma_i \cap B_\rho} |\kappa_i| d\mathcal{H}^1 \leq C\rho^{\frac{7}{5}}.$$

In particular, since Γ_i is $C^{1,\gamma}$ up to the origin, we deduce that it is actually C^2 in the origin itself with $\kappa_i(0) = 0$.

Since u has $C^{1,\alpha}$ extensions on each side of $\Gamma_i \cap A$ (cf. item (a) in Proposition 2.5.2) and Γ_i is $C^{1,\gamma}$, the conclusion follows at once using (2.5.4) for $\lambda = 0$. ☐

REMARK 3.7.1. Actually, the arcs composing $K \setminus \{\bar{x}\}$ are C^∞ (resp. analytic) if g is C^∞ (resp. analytic) in view of the higher regularity theory contained in [4, Theorem 7.42] (resp. [26]).

3.8. Proof of Theorem 1.2.3 for pure jumps and triple junctions

We start by proving conclusion (i). First of all we observe that, by a standard covering argument, it is enough to prove the conclusion in a ball $B_{\delta r}(x)$ rather than in $B_r(x)$, where δ is a fixed geometric constant. We focus on the case of a pure jump and of absolute minimizers. The various other possibilities are all treated with the same idea, with minor changes. Without loss of generality assume $x = 0$.

We wish to apply Theorem 3.1.1 and in order to do it we claim that

(Cl) For any $\varepsilon > 0$ there are constants $\delta > 0$ and $\eta > 0$ such that, if (K, u) is an absolute minimizer and, for some $r > 0$, $\mathrm{dist}_H(\mathscr{V}_0 \cap \overline{B}_{2r}, K \cap \overline{B}_{2r}) \le \eta r$, then

$$\int_{B_{2\delta r}\setminus K} |\nabla u|^2 + \lambda \|g\|_\infty^2 (2\delta r)^{\frac{3}{2}} < 2\varepsilon\delta r\,.$$

First of all, since $\|g\|_\infty \le M_0$, $\lambda \le 1$, and $r \le 1$, it is easy to see that, for δ sufficiently small,

$$\lambda \|g\|_\infty^2 (2\delta r)^{\frac{3}{2}} < \varepsilon\delta r\,.$$

We therefore focus on the Dirichlet energy and argue by contradiction. In particular, if our claim is false, we find a sequence of radii $r_j \le 1$ and of absolute minimizers (K_j, u_j) in B_{2r_j} of E_{λ_j} with fidelity terms g_j, satisfying the following properties:

(1) $\mathrm{dist}_H(\mathscr{V}_0 \cap \overline{B}_{2r_j}, K_j \cap \overline{B}_{2r_j}) \le 2^{-j} r_j$;
(2) $\int_{B_{2j^{-1}r_j}\setminus K_j} |\nabla u_j|^2 \ge \varepsilon j^{-1} r_j$ for some positive ε.

We now consider the rescalings (J_j, v_j) of (K_j, u_j) given by

$$J_j := \frac{K_j}{j^{-1} r_j}$$
$$v_j(x) := (j^{-1} r_j)^{-\frac{1}{2}} u_j(j^{-1} r_j x)\,.$$

Note that $\lambda_j \|g_j\|_\infty^2 j^{-1} r_j \to 0$ as $j \uparrow \infty$. In particular we can apply Theorem 2.2.3 and assume that, up to a subsequence not relabeled, the pairs (J_j, v_j) converge to a generalized global minimizer (J, v). Observe that on any disk B_R we have

$$\mathrm{dist}\,(\mathscr{V}_0 \cap \overline{B}_R, J_j \cap \overline{B}_R) \le j 2^{-j}\,.$$

Hence J must coincide with $\mathscr{V}_0$. But then from Theorem 2.4.1 it follows that $\nabla v \equiv 0$ on $\mathbb{R}^2 \setminus \mathscr{V}_0$. In particular, by the convergence proved in Theorem 2.2.3,

$$\lim_{j\to\infty} \int_{B_1} |\nabla v_j|^2 = 0\,.$$

On the other hand

$$\int_{B_1} |\nabla v_j|^2 = \frac{1}{j^{-1} r_j} \int_{B_{2j^{-1} r_j}} |\nabla u_j|^2 \, ,$$

and we thus reach a contradiction with (2).

We next come to point (ii). In the case of pure jumps observe that if $\lambda = 0$ or $g \in C^0$, Δu is continuous and from classical estimates for the Neumann problem we infer that ∇u has a $C^{0,\alpha}$ extensions up to K on each side of it. In particular, from the Euler-Lagrange conditions of Proposition 2.5.2 we conclude that the distributional curvature of K is continuous, which in turn implies its C^2 regularity.

As for point (iii), it follows from Corollary 3.1.3 if the extremum is a triple junction.

CHAPTER 4

The Bonnet-David rigidity theorem for cracktip

4.1. Main statement and consequences

This part of our notes is devoted to prove the following rigidity theorem of Bonnet and David, which in turn is the first step towards the proof of case (c) in Theorem 1.2.3, namely Corollary 4.1.2. We start by stating both these facts.

THEOREM 4.1.1. *Let $(K, u, \{p_{kl}\})$ be a global generalized minimizer and assume that, for a sufficiently large radius R:*

(a) *$K \setminus B_R$ consists of a single unbounded connected component;*
(b) *$K \cap \partial B_R$ consists of a single point.*

Then $(K, u, \{p_{kl}\})$ is a cracktip.

COROLLARY 4.1.2. *There is a $\delta > 0$ with the following property. Assume that*

- *(1.0.2) holds;*
- *(K, u) is an absolute minimizer of E_λ in $B_{4r}(x)$ for some $4r \leq 1$;*
- *$\mathrm{dist}_H\left(K \cap \overline{B}_{4r}(x), (x + \mathcal{R}_\theta)(\mathscr{V}_0^+) \cap \overline{B}_{4r}(x)\right) < \delta r$.*

Then $B_{2r}(x) \cap K$ consists of a single continuous nonselfintersecting arc with an endpoint $y \in B_r(x)$ (which according to our terminology is a terminal point of K) and an endpoint in $\partial B_{2r}(x)$. Moreover the arc is $C^{1,1}$ in $B_{2r}(x) \setminus \{y\}$.

In this chapter we will not only prove Theorem 4.1.1 but also give some general properties of global generalized minimizers $(K, u, \{p_{kl}\})$ which are not elementary. In order to simplify our terminology, we will call them *nonelementary* global minimizers. An important result of David and Léger, presented below, shows that for a *nonelementary* global minimizer the set $\mathbb{R}^2 \setminus K$ is in fact connected. Due to this we can (and will) omit to mention the "normalizations" $\{p_{kl}\}$ for nonelementary global minimizers.

We conclude this section by showing how Corollary 4.1.2 follows from Theorem 4.1.1.

4.1.1. Proof of Corollary 4.1.2. The argument is similar to that used in Section 3.6. Following a similar path we introduce the quantities

$$\Omega^c(\theta, x, r) := r^{-1}\mathrm{dist}_H(K \cap \overline{B}_{2r}(x), (x + \mathcal{R}_\theta(\mathscr{V}_0^+)) \cap \overline{B}_{2r}(x))$$

and

$$\Omega^c(x, r) := \min_\theta \Omega^c(\theta, x, r)\,.$$

The following lemma is then the analog of Lemma 3.6.2.

LEMMA 4.1.3. *For every $\gamma > 0$ sufficiently small there exists $\varepsilon_0(\gamma)$ with the following property. Assume $\varepsilon \in (0, \varepsilon_0)$ and let $N = N(\varepsilon) \in \mathbb{N}$ be sufficiently large. Let $r \in (0, 1]$ and assume that (K, u) is an absolute minimizer of E_λ in $B_{2r}(x)$, while $x = x_0, x_1, \ldots, x_N$ are points such that*

$$\Omega^c(x_k, 2^{-k}r) + \lambda \|g\|^2_\infty (2^{-k}r)^{\frac{1}{2}} \le \varepsilon \qquad \forall k \in \{0, 1, \ldots, N\} \tag{4.1.1}$$

$$|x_{k+1} - x_k| \le \gamma 2^{-k} r \qquad \forall k \in \{0, 1, \ldots, N-1\}\,. \tag{4.1.2}$$

Then there is a point $x_{N+1} \in B_{2r}(x)$ such that $|x_{N+1} - x_N| \le \gamma 2^{-N} r$ and

$$\Omega^c(x_{N+1}, 2^{-N-1}r) + \lambda \|g\|^2_\infty (2^{-N-1}r)^{\frac{1}{2}} \le \varepsilon\,.$$

PROOF. We argue by contradiction and assume that, for some $\gamma > 0$ and $\varepsilon > 0$ sufficiently small (the smallness will be specified later) there are

(a) A family of numbers $\lambda_N \in [0, 1]$;
(b) A family of fidelity functions g_N with $\|g_N\|_\infty \le M_0$;
(c) A family of radii $r_N \in (0, 1]$;
(d) A family of points $x_{k,N}$, for $k \in \{0, \ldots, N\}$, with

$$x_{0,N} = x \qquad |x_{k+1,N} - x_{k,N}| \le \gamma 2^{-k} r_N \qquad \forall k \in \{0, \ldots, N-1\}; \tag{4.1.3}$$

(e) An absolute minimizing pair (K_N, u_N) of $E_{\lambda_N}(\cdot, \cdot, B_{2r_N}, g_N)$ for which

$$\Omega^c(x_{k,N}, 2^{-k}r_N) + \lambda_N \|g_N\|^2_\infty (2^{-k}r_N)^{\frac{1}{2}} \le \varepsilon\,, \tag{4.1.4}$$

for all $k \in \{0, \ldots, N\}$;

(f) For every $y \in B_{\gamma 2^{-N} r_N}(x_{N,N})$

$$\Omega^c(y, 2^{-N-1}r_N) + \lambda_N \|g_N\|^2_\infty (2^{-N-1}r_N)^{\frac{1}{2}} > \varepsilon\,.$$

For each N we consider the rescaled pairs

$$v_N(x) := (2^{-N}r_N)^{-\frac{1}{2}} u_N \left(x_{N,N} + 2^{-N} r_N x\right)$$
$$J_N := (2^{-N}r_N)^{-1}(K_N - x_{N,N})\,.$$

Next observe that from (4.1.4) we get

$$\lambda_N \|g_N\|^2_\infty 2^{-N} r_N \le \varepsilon (2^{-N} r_N)^{1/2}\,.$$

and thus in particular $\lambda_N \|g_N\|^2_\infty 2^{-N} r_N \to 0$. We can therefore apply Theorem 2.2.3 to conclude the convergence, up to subsequences, of (J_N, v_N) to a generalized minimizer $(J, v, \{p_{kl}\})$. Note that the points $x_{k,N}$ are mapped to the points

$$y_{k,N} := (2^{-N}r_N)^{-1}(x_{k,N} - x_{N,N})$$

We thus infer $y_{N,N} = 0$ for all N, and for $k \in \{1, \ldots, N\}$

$$|y_{N-k,N}| \le \gamma \sum_{j=1}^{k} 2^j \le \gamma 2^{k+1}\,.$$

Up to extraction of a subsequence we can assume that, for each fixed $k \geq 1$, $y_{N-k,N}$ converges, as $N \uparrow \infty$, to some y_k with $|y_k| \leq \gamma 2^{k+1}$, $k \geq 1$. Set moreover $y_0 = 0$. In particular for (J, v) we have

$$\Omega^c(y_k, 2^k) \leq \varepsilon \qquad \forall k \in \mathbb{N}\,,$$

On the other hand our contradiction assumption implies as well

$$\inf_{z \in B_\gamma} \Omega^c(z, 1/2) \geq \varepsilon\,. \tag{4.1.5}$$

Next observe that, by taking ε smaller than a suitably chosen $\varepsilon_0(\gamma)$, the ε-regularity theory at pure jumps would imply that, in the corona $A_k = B_{(1-\gamma)2^{k+1}}(y_k) \setminus B_{\gamma 2^{k+1}}(y_k)$ the set K consists of a single arc with endpoints on the circles $\partial B_{(1-\gamma)2^{k+1}}$ and $\partial B_{\gamma 2^{k+1}}$. If γ is smaller than a geometric constant, the coronas A_{k+1} and A_k have a large overlap. It then follows that K consists of a single unbounded curve in $\cup_{k \geq 1} A_k = \mathbb{R}^2 \setminus B_{2\gamma}$. We can thus apply Theorem 4.1.1 to conclude that (J, v) is a cracktip. If we denote by z_0 the starting point of the half-line J, the condition $\Omega^c(0, 1) \leq \varepsilon$ will imply that $z_0 \in B_\gamma$, provided ε is smaller than a suitably chosen positive $\varepsilon_0(\gamma)$. This however contradicts (4.1.5) and hence completes the proof. □

Proof of Corollary 4.1.2. The proof is entirely similar to that of Lemma 3.6.1. Like in there we assume without loss of generality that $x = 0$. We fix then δ, ε, and γ whose choice will be specified later. First of all ε is assumed to be smaller than the $\varepsilon_0(\gamma)$ given by Lemma 4.1.3, so that the latter is applicable. We then let N be the natural number given by the conclusion of the latter lemma. If we let $r_0 \leq 1$ be such that

$$\lambda M_0^2 r_0^{1/2} = \frac{\varepsilon}{2}\,,$$

we observe that, by choosing ε sufficiently small, we can at the same time ensure that

(1) Lemma 4.1.3 is applicable in $B_{2\bar r}$ with $\bar r = \min\{r, r_0\}$;
(2) $K \cap (B_{2r} \setminus B_{\bar r/2})$ is a single C^1 arc γ with endpoints in $\partial B_{\bar r/2}$ and ∂B_{2r}.

Indeed, the first point is simply because, by choosing δ sufficiently small, the condition of the Lemma is satisfied with $x_{k,N}$ all equal to 0, $k \in \{0, \ldots, N\}$ (note that N and ε are fixed at this point). As for the second point, it is just a consequence of the ε-regularity theory at pure jumps.

We now note that, after having applied Lemma 4.1.3 to find x_{N+1}, we can actually apply it again to $B_{\bar r/2}(x_1)$, but this time the points $x_0, \ldots, x_N$ would be substituted by $x_1, \ldots, x_{N+1}$. We then proceed inductively to produce a sequence of points x_k with

$$\begin{aligned}
&x_0 = 0\\
&|x_{k+1} - x_k| \leq \gamma 2^{-k} \bar r\\
&\Omega^c(x_k, 2^{-k}\bar r) + \lambda \|g\|_\infty^2 (2^{-k}\bar r)^{\frac12} \leq \varepsilon\,.
\end{aligned}$$

Since $\{x_k\}$ is a Cauchy sequence, it has a limit y. Observe that by choosing γ smaller than a geometric constant we can then ensure $y \in B_{\bar r}$. Moreover, it is easy to see that

$$\Omega^c(y, \rho) \leq 4(\varepsilon + \gamma)$$

for every $\rho \le \bar{r}$ simply by choosing k so that $2^{-k-2}\bar{r} \le \rho \le 2^{-k-1}\bar{r}$ and comparing $\Omega^c(y,\rho)$ with $\Omega^c(x_k, 2^{1-k}\bar{r})$, if $\gamma \le 1$.

In particular, if ε and γ are chosen sufficiently small, $K \cap (B_{2\rho/3} \setminus B_{\rho/3})$ consists of a single C^1 arc with endpoints in the respective circles $\partial B_{2\rho/3}$ and $\partial B_{\rho/3}$. This shows that $B_{2r} \cap K$ consists of a single continuous arc joining y with a point in ∂B_{2r}, which moreover is $C^{1,\alpha}$ in $B_{2r} \setminus \{y\}$. But then by the Euler-Lagrange conditions in Proposition 2.5.2 we conclude that the arc is $C^{1,1}$ in $B_{2r} \setminus \{y\}$. □

4.2. An overview of the ideas in the proof of Theorem 4.1.1

Even though for some steps we give independent proofs, the overall strategy and the main ideas for proving Theorem 4.1.1 are all taken from the book [9] of Bonnet and David. In this section we give an overview of the whole argument, which is very ingenious. Before coming to it, we first notice that in Section 4.4 we prove another Liouville-type Theorem, namely Theorem 4.4.1, which is due to David and Léger in the work [15], posterior to [9]. However, one big advantage of having Theorem 4.4.1 at disposal is that it implies a series of useful corollaries (above all Corollary 4.4.2) which cut a lot of technicalities of [9]. The proof of Theorem 4.4.1 are based on the monotonicity formulas in Propositions 2.6.2 and 2.6.3.

One main player in the proof of Theorem 4.1.1 is the "harmonic conjugate" v, a function with the property that $\nabla v = \nabla u^\perp$ on $\mathbb{R}^2 \setminus K$, and which can be shown (for a nonelementary global minimizer) to have a unique continuous extension to K. v is obviously harmonic on $\mathbb{R}^2 \setminus K$. Its existence and some preliminary properties are given in Section 4.5: one pivotal property is that v is constant on each connected component of K.

Much of the technical work for the proof of Theorem 4.1.1 goes into describing the structure of the level sets of v. The sections 4.6 and 4.7 will prove facts which are valid for all nonelementary global minimizers. The most important are that:

- no level set of v contains a loop (a particular case of this statement will actually be shown in the next section: K itself has no loops);
- most of them do not have "terminal points".

Section 4.8 then uses the additional assumption of Theorem 4.1.1 (namely that outside a sufficiently large ball K consists of a single connected component) together with Bonnet's monotonicity formula (cf. Proposition 2.6.1) to infer that asymptotically at infinity (K, u) is a cracktip. In particular any level set of v has precisely "two infinite ends" outside of a sufficiently large disk. This information, combined with the previous analysis allows to conclude that:

- up to a change of sign in u the function v achieves its absolute minimum m_0 exactly on the unbounded connected component of K;
- Most of the level sets of v are just single (i.e. nonintersecting) unbounded curves.

The first information allows us to define the maximum $\bar{m}$ of v on K, while another simple argument shows that if $\bar{m} = m_0$ then K is connected and hence it is the cracktip.

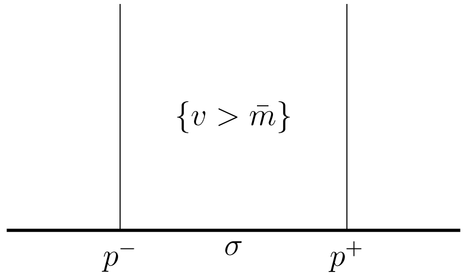

FIGURE 1. The picture is a visualization of the final argument for Theorem 4.1.1 under the simplifying assumption that G does not have triple junctions. The analysis in the Sections 4.6-4.9 implies that the global minimum and global maximum of the trace of u on G are assumed at two point p^- and p^+ "on the same side of G": the picture shows in particular the level set $\{v = \bar{m}\}$ departing at those points and delimiting the upper level set $\{v > \bar{m}\}$, whose closure cannot contain the terminal points of G. The range of the lower trace of u on the segment $\sigma = [p^+, p^-] \subset G$ is thus necessarily contained in the range of the upper trace over the same segment σ: the intermediate value theorem provides then a point $q \in \sigma$ where the two traces have the same value.

Arguing by contradiction that (K, u) is not a cracktip we can then introduce the pivotal object in the final argument for Theorem 4.1.1: a connected component G of K where the value of v equals $\bar{m}$. With a variant of the Bonnet's monotonicity formula we can show that K must have positive length and that no terminal point of G can actually be contained in the closure of $\{v > \bar{m}\}$, cf. Section 4.9.

The final argument comes now from looking at the trace of u as we go "around G" (which will be shown to be continuous) and is explained in Section 4.10 and it is particularly simple to explain when G does not have triple points and it is thus, topologically, a segment with two terminal points.

One outcome of the topological description of the level sets of v is that while going around G the trace of u has exactly two local extrema, which are clearly the global minimum point and the global maximum point. This happens because to each local extremum of the trace of u at G corresponds to a distinct end of the level set $\{v = \bar{m}\}$ at infinity: having proved that such ends are precisely 2, the trace must have precisely one local minimum and one local maximum. At the global maximum and the global minimum the level set $\{v = \bar{m}\}$ departs from G as two infinite half-lines, which delimit one connected component of $\mathbb{R}^2$ where v is above $\bar{m}$ and one connected component where v is below $\bar{m}$.

The other fundamental outcome of the previous analysis is that, since the terminal points of G are not in the closure of $\{v > \bar{m}\}$, the global minimum and maximum points of the trace of u on G (which for simplicity we denote by p^- and p^+) are on the "same side" of G: a schematic picture of what happens is give in Figure 1. $p^\pm$ delimit a segment σ in G. Referring to the Figure 1, the range of the trace of u on the "upper side" of σ is $[m, M]$,

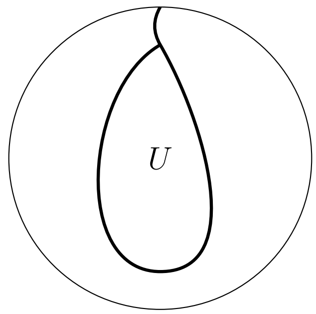

Figure 2. U is a "pocket", namely a bounded connected component of $\Omega \setminus K$ which does not intersect the boundary $\partial\Omega$ of the domain of (K, u).

where m denotes the global minimum and M the global maximum. So the range of the trace in the lower side must be a segment $[m', M']$ strictly contained in $[m, M]$. By the intermediate value theorem there must then be a point $q \in \sigma$ where the upper and lower traces coincide. However this point must be a jump point (cf. Corollary 4.4.2) and at those points upper and lower traces must necessarily differ. This gives the desired contradiction.

4.3. The absence of pockets

In this section we prove a simple fact valid for all concepts of minimizers of E_0, the absence of "internal pockets", see Figure 2.

Lemma 4.3.1. *Consider a minimizer (K, u) of E_0 in Ω. If U is a connected component of $\Omega \setminus K$, then either it is unbounded, or its closure has to intersect $\partial\Omega$.*

Proof. If the statement were false, then there would be a connected component U of $\Omega \setminus K$ with the property that $U \subset\subset \Omega$. In particular it turns out that u must necessarily be constant on U. Moreover, for any given constant c, if we define the function

$$u_c := \begin{cases} u & \text{on } \Omega \setminus \overline{U} \\ c & \text{on } U, \end{cases}$$

then (K, u_c) has the same energy has (K, u) and it is not difficult to check that it must have the same minimizing property of (K, u). Now, observe that $\mathcal{H}^1(\partial U) > 0$ and thus, by Corollary 3.1.5, there is at least one regular jump point $x \in K$, i.e. there is a neighborhood of x where K is a C^1 arc. Assume without loss of generality that $x = 0$ and that the tangent to K at 0 is $\{(x_1, x_2) : x_2 = 0\}$. In particular, for a sufficiently small $\delta > 0$ we have that $K \cap [-\delta, \delta]^2 = \mathrm{gr}\,(f)$ for some $C^{1,\alpha}$ function $f : [-\delta, \delta] \to [-\delta, \delta]$ with $f(0) = f'(0) = 0$. We next assume, w.l.o.g, that

$$U \cap [-\delta, \delta]^2 = \{(t, s) : s > f(t), s \in [-\delta, \delta]\}\,.$$

At the same time we know that the restriction of u to $[-\delta, \delta]^2 \setminus \mathrm{gr}\,(f)$ coincide with two $C^{1,\alpha}$ functions $u^\pm$ which can in fact be extended $C^{1,\alpha}$ up to the boundary $K \cap [-\delta, \delta]^2$. u^+ in fact is a constant while, by a classical extension theorem, we assume that u^- is extended C^1 to the full square $[-\delta, \delta]^2$.

Let now $c = u^-(0)$ and consider (K, u_c). We now consider a new function defined in the following way. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function which is identically equal to 1 on $[1,\infty)$ and identically equal to 0 on $(-\infty,-1]$ and let $\varphi \in C_c^\infty((-1,1)^2)$ be a function which is identically 1 on $(-1/2, 1/2)^2$. Then we first define

$$v(x) = c\chi\left(\frac{x_2}{\delta}\right) + u^-(x)\left(1 - \chi\left(\frac{x_2}{\delta}\right)\right),$$

which is a C^1 function on $[-\delta,\delta]^2$. Then on U we define

$$w(x) = c\left(1 - \varphi\left(\frac{x}{\delta}\right)\right) + v(x)\varphi\left(\frac{x}{\delta}\right),$$

while on the complement of $U \cup K$ we define

$$w(x) = u^-(x)\left(1 - \varphi\left(\frac{x}{\delta}\right)\right) + v(x)\varphi\left(\frac{x}{\delta}\right).$$

Observe that the function w coincides with a C^1 function on $Q := [-\frac{\delta}{2}, \frac{\delta}{2}] \times [-\delta, \delta]$ and we can thus consider $J := K \setminus Q$, so that (J, w) is a pair in the domain of E_0. Note that the pair is obviously a competitor for absolute minimizers. For restricted minimizers observe that we are not creating an additional connected component, while for generalized and generalized restricted minimizers it is easy to see that, if V is a bounded open set such that $V \supset\supset U$, there are no pair of points belonging to distinct connected components of $\Omega \setminus K$ and lying outside V which would belong to the same connected component of $\Omega \setminus J$. Finally, observe that $|\nabla w| \le C$ in Q, for some constant C independent of δ. So, while

$$\int_V |\nabla w|^2 \le \int_V |\nabla u|^2 + C\delta^2$$

we have

$$\mathcal{H}^1(J \cap V) \le \mathcal{H}^1(K \cap V) - \delta.$$

For a sufficiently small δ we then contradict the minimizing property of (K, u_c). □

4.4. The David-Léger rigidity theorem and consequences

In this section we prove the second part of Theorem 1.3.4, which we recall here for the reader's convenience.

THEOREM 4.4.1. *Assume $(K, u, \{p_{kl}\})$ is a global generalized minimizer of E_0 and K disconnects the plane. Then $(K, u, \{p_{kl}\})$ is either a pure jump or a triple junction.*

Before coming to the proof of Theorem 4.4.1, we register some very important corollaries.

COROLLARY 4.4.2. *Statement (ii) of Theorem 1.4.2 holds.*

COROLLARY 4.4.3. *If $(K, u, \{p_{kl}\})$ is a global generalized minimizer of E_0, then K cannot have two distinct unbounded connected components.*

COROLLARY 4.4.4. *If $(K, u, \{p_{kl}\})$ is a global generalized minimizer and ∇u vanishes on some open set, then $(K, u, \{p_{kl}\})$ is an elementary global minimizer.*

We start by giving a proof of the theorem, which relies on Proposition 2.6.3. Since the latter has not yet been proved, we will follow with its argument, and hence we will prove the three corollaries above.

Proof of Theorem 4.4.1. Let U_1 and U_2 be two connected components of $\mathbb{R}^2 \setminus K$. First of all observe that, by Lemma 4.3.1, both U_i are unbounded. In particular, for every r sufficiently large $B_r \cap U_i \neq \emptyset$ for both i's. Again by Lemma 4.3.1, $\partial B_r \cap U_i \neq \emptyset$ for both i's as well. It thus turns out that $K \cap \partial B_r$ has cardinality at least 2 for all r sufficiently large. If the cardinality is precisely 2, then $\partial B_r \setminus K$ consists of two arcs γ_1 and γ_2 and it turns out that one of them belongs to U_1 while the other to U_2. But then $K \cap \partial B_r$ must belong to the same connected component of $K \cap \overline{B}_r$.

Thus, in any case for all r sufficiently large we fall under the assumption of Proposition 2.6.3, which implies that the map $r \mapsto F(r)$ is monotone for r sufficiently large. Let F_0 be its limit as $r \uparrow \infty$, and consider the "blow-downs" of (K, u), namely any generalized global minimizer $(J, v, \{q_{kl}\})$ which is the limit of a sequence of rescalings (K_{0,r_j}, u_{0,r_j}) for some $r_j \uparrow \infty$. We then conclude that the function

$$r \mapsto \frac{2}{r} \int_{B_r \setminus J} |\nabla v|^2 + \frac{\mathcal{H}^1(B_r \cap J)}{r}$$

has the constant value F_0. We will show below that:

(Cl) J disconnects $\mathbb{R}^2$, and there is no $r > 0$ for which $B_r \setminus J$ belongs to the same connected component of $\mathbb{R}^2 \setminus J$.

Assuming the claim, we can argue as above to conclude that one of the two assumptions (i) and (ii) of Proposition 2.6.3 holds for a.e. $r > 0$. From the second part of the proposition we infer that (J, v) is an elementary global minimizer, with J that is either a straight line passing through the origin, or the union of three half lines meeting at the origin at equal angles. However, in one case $F_0 = 2$ and in the other $F_0 = 3$. We could now apply Corollary 3.1.6 to conclude that (K, u) itself is an elementary global minimizer.

In order to show (Cl), consider the set $V := \operatorname{Int} \overline{U}_1$ and the (local) Hausdorff limit $\tilde{J}$ of $\partial V_{0,r_j}$ (which exists up to subsequences). Clearly $\tilde{J} \subset J$. Fix moreover $\rho > 0$ and observe that for every sufficiently large j the set $B_{\rho/2} \setminus V_{0,r_j}$ is nonempty because $B_{\rho r_j/2} \cap U_2$ is not empty. So, for each $\rho > 0$ there is at least one point $p = p(\rho) \in \tilde{J} \cap B_r$. By Corollary 3.1.5, a.e. such p is a pure jump point. So there is a $\delta > 0$ such that $\tilde{J} \cap B_{2\delta}(p)$ divides $B_{2\delta}(p)$ in two regions W^+ and W^-. Again by Corollary 3.1.5, for j sufficiently large the same statement is correct for $K_{0,r_j} \cap B_\delta(p)$ and moreover $K_{0,r_j} \cap B_\delta(p)$ converges smoothly to $J \cap B_r$. Note that $K_{0,r_j} \cap B_\delta(p)$ must contain a point of $\partial V_{0,r_j}$ for all j large enough. But then the smoothness implies that $K_{0,r_j} \cap B_\delta(p) \subset \partial V_{0,r_j}$. One of the two connected components of K_{0,r_j} belongs then to V_{0,r_j}, while the other is contained in a different connected component of $\mathbb{R}^2 \setminus K_{0,j}$: this is a consequence of the definition of V as formed by the interior points of the closure of U_1. We thus infer that W^+ and W^- cannot belong to the same connected component of $\mathbb{R}^2 \setminus J$. Otherwise there would be an arc γ with endpoints $p^\pm \in W^\pm$, which is at positive distance from J. By the Hausdorff convergence, such path would be contained in $\mathbb{R}^2 \setminus K_{0,r_j}$ for all j large enough, implying then that $B_\delta(p)\setminus K_{0,r_j}$ is contained in the same connected component of $\mathbb{R}^2\setminus K_{0,r_j}$. This is a contradiction and hence completes the proof. $\square$

4.4.1. Proof of Proposition 2.6.3. Without loss of generality we assume $x=0$, and as usual we will drop the base point in the notation of all the relevant quantities.

First of all we argue as in the proof of Proposition 2.6.2 to conclude

$$r^2F'(r) = 2r\int_{\partial B_r\setminus K}|\nabla u|^2 + r\sum_{p\in\partial B_r\cap K}\frac{1}{e(p)\cdot n(p)} - 2D(r) - \ell(r) \tag{4.4.1}$$

and

$$r^2F'(r) \geq 3r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2 + r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial n}\right)^2 + 2rN(r) - 2E_0(K,u,B_r)\,, \tag{4.4.2}$$

Note however that (2.6.4) can be used to derive also

$$r^2F'(r) \geq r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2 + 3r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial n}\right)^2 - 2D(r)\,, \tag{4.4.3}$$

In the rest of the proof we consider several competitors for (K,u) in B_r, all constructed in the following fashion:

- (J,w) coincides with (K,u) on $\Omega\setminus\overline{B}_r$;
- $J\cap\overline{B}_r$ is connected and consists of finitely many segments;
- $J\cap\partial B_r\supset K\cap\partial B_r$;
- $J\cap B_r$ partitions B_r in finitely many connected components Ω_i;
- either $\mathcal{H}^1(\overline{\Omega_i}\cap\partial B_r)=0$, and w is defined to be constant on Ω_i, or $\overline{\Omega}_i\cap\partial B_r$ is a closed arc β_i which intersects only one connected component of $\partial B_r\setminus K$, and in that case we use Lemma 2.6.6 to extend $u|_{\beta_i}$ to Ω_i.

Since J does not increase the number of connected components of K and we can apply Lemma 2.2.5, the pair (J,w) is a valid competitor under all our assumptions and thus we can infer

$$E_0(K,u,B_r)\leq E_0(J,w,B_r)\leq \mathcal{H}^1(J\cap\overline{B}_r)+\frac{\alpha(J)}{\pi}r\int_{\partial B_r\setminus K}\left(\frac{\partial u}{\partial \tau}\right)^2\,, \tag{4.4.4}$$

where $\alpha(J)$ is the least real number such that each Ω_i is contained in an *open* circular sector of angle $\alpha(J)$ centred in the origin. Observe that in the extreme case $\alpha(J)=2\pi$, in order to apply Lemma 2.2.5, we need each Ω_i to be contained in a "slit domain". i.e. in $B_r\setminus[0,q_i]$ for some $q_i\in\partial B_r$.

In what follows, when invoking (4.4.4) we will then just need to specify $J\cap\overline{B}_r$ and estimate $\alpha(J)$ and $\mathcal{H}^1(J\cap\overline{B}_r)$. Moreover, since J coincides with K outside the closed disk $\overline{B}_r$, with a slight abuse of notation we will just refer to J for the piece inside the disk itself.

We next focus on proving $F'(r)\geq 0$, leaving the discussion of the implications of $F'\equiv 0$ to the very end. Observe that we can always resort to Proposition 2.6.3 to prove $F'(r)\geq 0$ when the largest arc of $\partial B_r\setminus K$ has length at most $\frac{3}{2}\pi r$. We therefore assume from now on that:

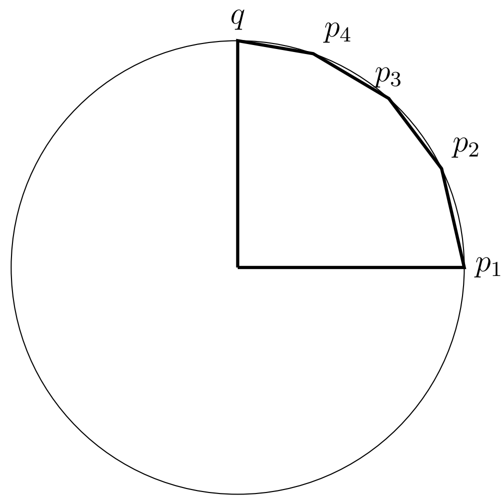

FIGURE 3. The set J in an example where $N(r) = 4$ and $K \cap \partial B_r$ lies in the first quadrant.

(A) $\partial B_r \cap K$ is contained in a subarc of ∂B_r which has length no larger than $\frac{\pi r}{2}$.

Case $N(r) \geq 4$**.** By applying a rotation (and because of (A)) we can assume that $K \cap \partial B_r = \{p_1, \ldots, p_N\}$ are contained in the quadrant $\{x_1 \geq 0, x_2 \geq 0\}$. Without loss of generality assume $p_1 = (r, 0)$ and order $p_2, \ldots, p_N$ counterclockwise, moreover denote by q the point $(0, r)$. We then use (4.4.4) with $J = [0, p_1] \cup [p_1, p_2] \cup \ldots \cup [p_{N-1}, p_N] \cup [p_N, q] \cup [q, 0]$ (cf. Figure 3). Observe that $\mathcal{H}^1(J) < 2r + \frac{\pi r}{2} < 4r \leq N(r)r$ and that $\alpha(J) = \frac{3\pi}{2}$, so that

$$E_0(K, u, B_r) < N(r)r + \frac{3}{2} \int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2$$

We then conclude from (4.4.2) that $r^2 F'(r) > 0$.

Case $N(r) = 3$**.** Let $\{p_+, p_-, q\} = K \cap \partial B_r$ and once again we invoke (A) and the possibility of rotating the domain to assume that $p_\pm = (c, \pm\sqrt{r^2 - c^2})$ for some $r\frac{\sqrt{2}}{2} \leq c \leq r$ and that $q = (d, \sqrt{r^2 - d^2})$ for some $0 < c \leq d$, cf. Figure 4. We now use (4.4.4) with two distinct (J, w): the corresponding sets J_1 and J_2 are specified in the picture Figure 4

Observe that

$$\begin{aligned}
\mathcal{H}^1(J_1) &\leq 2r + \sqrt{2}r \\
\mathcal{H}^1(J_2) &\leq r + \sqrt{2}r \\
\alpha(J_1) &= \pi \\
\alpha(J_2) &= 2\pi\,.
\end{aligned}$$

Apply now (4.4.4) with the two distinct competitors, average the inequalities and use (4.4.2) to conclude $F'(r) > 0$.

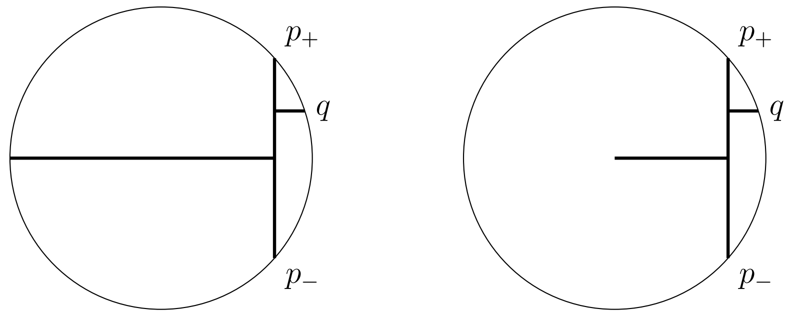

FIGURE 4. The sets J_1 and J_2 when $N(r) = 3$.

Case $N(r) = 2$**.** Let γ_1 and γ_2 be the two arcs delimited by $\partial B_r \cap K$ and again assume without loss of generality that $\omega := \frac{1}{\pi r}\max\{\ell(\gamma_i)\} \geq \frac{3}{2}$. Observe first that, by (1.5.2),

$$D(r) = \sum_i \int_{\gamma_i} u \frac{\partial u}{\partial n}\,.$$

Next, let c_i be the average of u on γ_i and recall Corollary 2.5.3 to estimate

$$\begin{aligned} D(r) =& \sum_i \int_{\gamma_i} (u - c_i)\frac{\partial u}{\partial n} \leq \left(\sum_i \int_{\gamma_i} (u - c_i)^2\right)^{\frac{1}{2}} \left(\int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2\right)^{\frac{1}{2}} \\ \leq& \omega r \underbrace{\left(\int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2\right)^{\frac{1}{2}}}_{=:a} \underbrace{\left(\int_{\partial B_r \setminus K} \left(\frac{\partial u}{\partial n}\right)^2\right)^{\frac{1}{2}}}_{=:b}\,. \end{aligned}$$

In particular, using (4.4.3) we conclude

$$rF'(r) \geq a^2 + 3b^2 - 2\omega ab =: P_1(\omega, a, b)\,. \tag{4.4.5}$$

Next we use (4.4.2) with two competitors J_1 and J_2 constructed in the following fashion. Without loss of generality we assume $\partial B_r = \{p_+, p_-\}$ where $p_\pm$ are, as in the previous step, symmetric with respect to the x_1-axis and in the quadrant $\{x_1 > 0, \pm x_2 > 0\}$, respectively. We then let J_1 be the connected set of minimal length joining the origin with the two points $p_\pm$ while, after setting $\bar{c} = (-r, 0)$, we let $J_2 = J_1 \cup [\bar{c}, 0]$, cf. Figure 5. Observe that J_1 is easily determined as the union of the three segments joining $0, p_+$ and p_- to the unique point c where such segments form equal angles.

Observe that $\alpha(J_1) = 2\pi$ and $\alpha(J_2) = \pi$, while $\mathcal{H}^1(J_2) = r + \mathcal{H}^1(J_1)$. In order to apply (4.4.4), we now wish to compute $\mathcal{H}^1(J_1)$, relating its length to the angle $2\pi - \omega\pi$, which is twice the angle formed by $p_\pm$ with the x_1 axis. Thus we can explicitly calculate

$$\begin{aligned} \mathcal{H}^1(J_1) =& \mathcal{H}^1([0, c]) + 2\mathcal{H}^1([c, p_+]) \\ =& r\Big(\cos(\pi(1 - \tfrac{\omega}{2}) - \tfrac{1}{\sqrt{3}}\sin(\pi(1 - \tfrac{\omega}{2}))\Big) + \tfrac{4r}{\sqrt{3}}\sin(\pi(1 - \tfrac{\omega}{2})) \\ =& 2r\Big(\tfrac{1}{2}\cos(\pi(1 - \tfrac{\omega}{2})) + \tfrac{\sqrt{3}}{2}\sin(\pi(1 - \tfrac{\omega}{2}))\Big) = 2r\sin(\tfrac{7\pi}{6} - \tfrac{\omega\pi}{2}) =: rg(\omega)\,. \end{aligned} \tag{4.4.6}$$

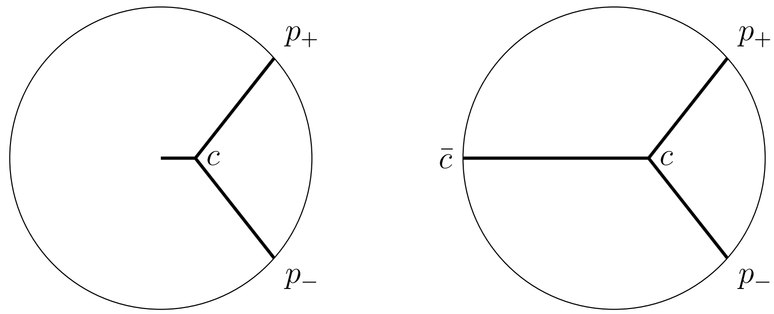

FIGURE 5. The sets J_1 and J_2 when $N(r) = 2$.

Applying now (4.4.2) we thus get

$$\begin{aligned} rF'(r) &\geq -a^2 + b^2 + 4 - 2g(\omega) =: P_2(\omega, a, b) \\ rF'(r) &\geq a^2 + b^2 + 2 - 2g(\omega)\,. \end{aligned}$$

Averaging between the two we then get

$$rF'(r) \geq b^2 + 3 - 2g(\omega) =: P_3(\omega, a, b)\,. \tag{4.4.7}$$

Introduce now the function $h(\omega, a, b) = \max\{P_1, P_2, P_3\}(a, b, \omega)$ (recall that P_1 is defined in (4.4.5)). We are thus reduced to show that $h \geq 0$ on the interval $[\frac{3}{2}, 2)$. First of all, $P_1 \geq 0$ if $\omega \leq \sqrt{3}$. Moreover, the function g is clearly a decreasing function of ω, which equals $\frac{3}{2}$ at the point ω_0 determined by

$$\omega_0 := \frac{7}{3} - \frac{2}{\pi} \arcsin \frac{3}{4}\,.$$

Hence $P_3 \geq 0$ for $\omega \geq \omega_0$. We thus have to show that h is positive when $\omega \in I := (\sqrt{3}, \omega_0)$. Observe that

- $h \geq P_3 \geq 0$ unless
$$b^2 < 2g(\omega) - 3\;; \tag{4.4.8}$$
- $h \geq P_2 \geq 0$ unless
$$b^2 + 4 - 2g(\omega) < a^2\,; \tag{4.4.9}$$
- $h \geq P_1 \geq 0$ unless
$$(\omega - \sqrt{\omega^2 - 3})^2 b^2 < a^2 < (\omega + \sqrt{\omega^2 - 3})^2 b^2\,. \tag{4.4.10}$$

Now, (4.4.9) and (4.4.10) would imply

$$4 - 2g(\omega) < ((\omega + \sqrt{\omega^2 - 3})^2 - 1)b^2\,,$$

which combined with (4.4.8) would imply

$$4 - 2g(\omega) < ((\omega + \sqrt{\omega^2 - 3})^2 - 1)(2g(\omega) - 3))\,,$$

which in turn becomes

$$1 < (\omega + \sqrt{\omega^2 - 3})^2 (2g(\omega) - 3)\,. \tag{4.4.11}$$

Recall that $I = (\sqrt{3}, \omega_0)$, and define

$$g_1(\omega) := (\omega + \sqrt{\omega^2 - 3})^2\,, \tag{4.4.12}$$

$$g_2(\omega) := 2g(\omega) - 3\,, \tag{4.4.13}$$

we then just need to show that $g_1 g_2 \le 1$ on I. Now, we already observed that g_2 is monotone decreasing, while it is easy to see that g_1 is monotone increasing. It can be explicitly computed that $g_2(\sqrt{3}) < \frac{1}{4}$, which in turn implies $\sup_I g_2 < \frac{1}{4}$. Since it can be readily checked that $g_1(\frac{7}{4}) = 4$, we conclude $\sup_{(\sqrt{3},7/4]} g_1 \le 4$, in turn implying $\sup_{(\sqrt{3},7/4]} g_1 g_2 < 1$. On the other hand it can be readily checked that $\omega_0 < 1.8$ and $g_1(1.8) < 5.3$, which in turn implies $\sup_I g_1 < 5.3$. Since $g_2(\frac{7}{4}) < 0.18$, we conclude $\max_{[7/4,\omega_0]} g_2 < 0.18$, from which we infer $\max_{[7/4,\omega_0]} g_1 g_2 < 0.18 \cdot 5.3 < 1$.

We also observe that in all these cases $F'(r)$ would actually result positive unless either a and b vanish.

Case $N(r) = 0$. In this case we compare (K, u) in B_r with the harmonic extension w of $u|_{\partial B_r}$. Recalling that the harmonic extension satisfies the estimate

$$\int_{B_r} |\nabla w|^2 \le r \int_{\partial B_r} \left(\frac{\partial u}{\partial \tau}\right)^2 ,$$

from (4.4.1) we immediately conclude

$$r^2 F'(r) \ge 2 \int_{\partial B_r} \left(\frac{\partial u}{\partial n}\right)^2 .$$

F **constant.** Observe that if $F'(r) = 0$ and we are in the assumptions of Proposition 2.6.2 then

$$\int_{\partial B_r} \left(\frac{\partial u}{\partial n}\right)^2 = 0\,.$$

However, the same conclusion can be drawn as well in all the cases examined above. But then the arguments of the final part of the proof of Proposition 2.6.2 apply here as well and we reach the conclusion that, if F is constant on $(0, r_0)$ and for a.a. $r \in (0, r_0)$ one of the assumptions (i) and (ii) holds, then $K \cap B_{r_0}$ coincides with one of three elementary global minimizers of Theorem 2.4.1, with the additional information that, if K is not empty, $0 \in K$ and that, if K is a triple junction, then 0 is the point of junction.

4.4.2. Proof of Corollary 4.4.2. We assume without loss of generality that the nonterminal point is the origin. By definition there is an injective continuous map $\gamma : [-1, 1] \to K$ such that $\gamma(0) = 0$. Set

$$2r := \min\{|\gamma(-1)|, |\gamma(1)|\} \tag{4.4.14}$$

and observe that there must be at least one negative s_- and one positive s_+ such that $|\gamma(s_\pm)| = r$. We let s_- be the largest negative number and s_+ the smallest positive number with the latter property. Using the Jordan curve theorem we then conclude that $B_r \setminus \gamma((s_-, s_+))$ consists of two connected components $\Omega^\pm$. Consider now the limit

$(K_\infty, u_\infty, \{p_{kl}\})$ of some sequence $\{(K_{0,r_k}, u_{0,r_k})\}$ with $r_k \downarrow 0$, as in Theorem 2.2.3. If we can show that K_∞ disconnects $\mathbb{R}^2$, then Theorem 4.4.1 would imply that (K_∞, u_∞) is an elementary global minimizer and thus, by the cases (b) and (c) of Theorem 1.2.3 (resp Theorem 1.3.1), 0 would be a regular point, i.e. a point for which there is a disk B_r in which K is diffeomorphic either to a diameter, or to three radii joining at the origin.

In order to show that K_∞ disconnects the plane, we consider the rescalings

$$\begin{aligned}\Omega^\pm_k &:= \{x : r_k x \in \Omega^\pm\} \cap B_1\\ \Gamma_k &:= \{x : r_k x \in \gamma(s_-, s_+)\} \cap B_1 .\end{aligned}$$

Up to extraction of a further subsequence, we can assume that $\Omega^\pm_k$ converge to some open sets $\Omega^\pm_\infty$ and Γ_k converges (locally in the Hausdorff distance) to some $\Gamma \subset K_\infty$. Since $\partial\Omega^\pm_k \subset \Gamma_k \cup \partial B_1$, the convergence of the open sets means that

(i) $|\Omega^+_\infty \Delta \Omega^+_k| + |\Omega^-_\infty \Delta \Omega^-_k| \to 0$;
(ii) $x \in \Omega^\pm_\infty$ if and only if there is a $\rho > 0$ and a $k_0 \in \mathbb{N}$ such that $B_\rho(x) \subset \Omega^\pm_k$ for $k \geq k_0$.

Notice also that each Γ_k contains at least one arc which connects the origin with ∂B_1. Hence Γ as well contains at least one arc which connects the origin with ∂B_1 and in particular Γ is not empty.

Pick now a point $q \in \Gamma$ which is a pure jump point (and whose existence is guaranteed by Corollary 3.1.5). Then, for sufficiently small ρ, $B_\rho(q) \cap \Gamma$ is a $C^{1,\alpha}$ arc which divides $B_\rho(q)$ in two regions. On the other hand, by the regularity theory developed in the first part of the notes, for a sufficiently large k we have that $K_{0,r_k} \cap B_\rho(q)$ consists also of a $C^{1,\alpha}$ arc which divides $B_\rho(q)$ into two regions. Since $\Gamma_k \to \Gamma$, for k sufficiently large we must have that $K_{0,r_k} \cap B_\rho(q) = \Gamma_k \cap B_\rho(q)$. In particular Γ_k divides $B_\rho(q)$ in two regions. By construction, one of the regions must be in Ω^+_k and the other must be in Ω^-_k. It then turns out that $\liminf_k \min\{|\Omega^+_k|, |\Omega^-_k|\} > 0$. Thus (i) above implies that both Ω^+_∞ and Ω^-_∞ are not empty.

We now use the latter fact to show that K_∞ disconnects $\mathbb{R}^2$. Consider indeed two points $q^\pm \in \Omega^\pm_\infty \setminus K_\infty$ whose existence is guaranteed by the fact that both $\Omega^\pm_\infty$ are not empty. If these points belonged to the same connected component of $\mathbb{R}^2 \setminus K_\infty$, then there would be a continuous curve $\eta : [0,1] \to \mathbb{R}^2 \setminus K_\infty$ with $\eta([0,1]) \cap K_\infty = \emptyset$, $\eta(0) = q^-$ and $\eta(1) = q^+$. For k sufficiently large we have

- $\eta([0,1])$ does not intersect K_{0,r_k} (by local Hausdorff convergence of K_{0,r_k} to K_∞);
- $q^\pm \in \Omega^\pm_k$ (by condition (ii) above);
- $\eta([0,1])] \subset B_{r/r_k}$, where r is the radius on (4.4.14).

Fix now a k sufficiently large so that all the conclusions above apply. Then:

- $p^- := r_k q^- \in \Omega^+$, $p^+ := r_k q^+ \in \Omega^-$;
- $t \mapsto r_k \eta(s)$ is a continuous curve contained in B_r which does not intersect K and hence does not intersect $\gamma((s^-, s^+))$.

However, since $\Omega^\pm$ are the two distinct connected components of $B_r \setminus \gamma((s^-, s^+))$, we have reached a contradiction.

4.4.3. Proof of Corollary 4.4.3. Assume by contradiction that (K, u) is a global generalized minimizer such that K has two unbounded connected components K_1 and K_2. Without loss of generality we can assume that both components intersect the unit disk B_1 and we fix two points $p_1 \in K_1 \cap B_1, p_2 \in K_2 \cap B_1$.

Fix now $k \in \mathbb{N} \setminus \{1, 2\}$ and consider the class of Lipschitz curves

$$\{\beta : [0, \ell(\beta)] \to K_1 : |\dot{\beta}| = 1, \beta(0) = p_1, |\beta(\ell(\beta))| \geq k^2\}\,.$$

By Lemma D.0.2 such class is nonempty. It is easy to see that there is a curve β in this class whose domain of definition has minimal length (the argument is the same as in Step 3 of the proof of Lemma D.0.2). Such curve must then be injective and $|\beta(\ell(\beta))| = k^2$, otherwise it would not have the minimizing property just described. We then consider the largest s such that $|\beta(s)| = 1$. The arc $\gamma_k : [s, \ell(\beta)] \ni \sigma \mapsto \beta(\sigma) \in K_1$ is thus an injective arc connecting a point in ∂B_1 to a point in ∂B_{k^2}. We find likewise an injective arc η_k in K_2 with the same properties. By the Jordan curve theorem the two paths $\gamma_k \cup \eta_k$ subdivide $B_{k^2} \setminus \overline{B}_1$ into two connected components $U_k^\pm$.

We consider now the rescaled pairs $(K_{0,k}, u_{0,k})$ and the corresponding domains $\Omega_k^\pm \subset B_k \setminus \overline{B}_{1/k}$, resulting from appropriately scaling the domains $U_k^\pm$. As in the proof of Corollary 4.4.2, up to subsequences we can assume that $(K_{0,k}, u_{0,k})$ converge to a global generalized minimizer (K_∞, u_∞), that the paths $\overline{\eta}_k \cup \overline{\gamma}_k$ converge locally in the Hausdorff topology to a connected closed set $\Gamma \subset K_\infty$ and that the sets $\Omega_k^\pm \cap B_1$ converge to open sets $\Omega_\infty^\pm$. As above $\Gamma \cap B_1$ is not empty and we can pick a point $q \in \Gamma \cap B_1$ which is a pure jump point for K_∞. Again the regularity theory will imply that there is a $B_\rho(q)$ and a k_0 such that for all $k \geq k_0$ $K_{0,k} \cap B_\rho(q)$ is a smooth arc subdividing $B_\rho(q)$ into two open subsets which have roughly the same area. Such arc is then either a subset of the rescaled η_k or a subset of the rescaled γ_k and in both cases we conclude that one the two regions in which $B_\rho(q)$ is subdivided by Γ is a subset of Ω_∞^+, while the other is a subset of Ω_∞^-.

Similarly to the proof of Corollary 4.4.2 we argue then that any pair of arbitry points $q^\pm \in \Omega_\infty^\pm \setminus K_\infty$ must belong to different connected components of $\mathbb{R}^2 \setminus K_\infty$. This in turn allows us to apply Theorem 4.4.1 and conclude that (K_∞, u_∞) is an elementary global generalized minimizer. But then Corollary 3.1.6 would imply that (K, u) was itself an elementary global generalized minimizer. In that case K must be connected, contradicting the initial assumption that K contains at least two distinct connected components.

4.4.4. Proof of Corollary 4.4.4. Assume $(K, u, \{p_{kl}\})$ is a global generalized minimizer and that ∇u vanishes on some open set. If K disconnects $\mathbb{R}^2$ then we know from Theorem 4.4.1 that the minimizer is elementary. We can thus assume that $\mathbb{R}^2 \setminus K$ is connected. But then, by the unique continuation for harmonic functions we conclude that $\nabla u = 0$ on the whole $\mathbb{R}^2 \setminus K$ and in particular that u is constant. This however is only possible if K is empty.

4.5. The harmonic conjugate

This and the next remaining four sections of the chapter will be dedicated to prove Theorem 4.1.1. A fundamental role in the proof will be played by the harmonic conjugate v

of the function u in a minimizing pair (K,u), which will be introduced in this section. v is, locally in $\mathbb{R}^2 \setminus K$, a classical harmonic conjugate of u, i.e. a function v whose gradient ∇v is the counterclockwise rotation of ∇u by 90 degrees, or alternatively such that the map $\zeta(x+iy) = u(x,y) + iv(x,y)$ is holomorphic. The main point is that v exists globally on $\mathbb{R}^2 \setminus K$, it is unique up to addition of a constant (if the minimizer is nonelementary), and it can be uniquely extended to a Hölder continuous function on the whole plane.

PROPOSITION 4.5.1. *Let (K,u) be a nonelementary global minimizer of E_0. Then there is a function $v : \mathbb{R}^2 \to \mathbb{R}$ with the following properties:*

(i) *v is harmonic and smooth on $\mathbb{R}^2 \setminus K$, where $\nabla v = \nabla u^\perp$;*

(ii) *$v \in W^{1,2}_{loc}(\mathbb{R}^2)$, it is Hölder continuous with exponent $\frac{1}{2}$ and the Hölder seminorm is globally bounded, i.e.*

$$\sup_{x \neq y} \frac{|v(x) - v(y)|}{|x-y|^{\frac{1}{2}}} < \infty\,;$$

(iii) *v is constant on each connected component of K;*

(iv) *v is unique up to addition of a constant.*

Before coming to the proof of the proposition we observe a simple consequence of Corollary 4.4.2, which will be useful in other occasions.

LEMMA 4.5.2. *Let (K,u) be a restricted, absolute, or generalized minimizer of E_λ. Let x, y be two nonterminal points in the same connected component of K. There is then an injective Lipschitz curve $\gamma : [0,1] \to K$ with $\gamma(0) = x$ and $\gamma(1) = y$ and a partition $0 = s_0 < s_1 < \ldots < s_{N+1} = 1$ such that:*

(i) *$\gamma|_{[s_i, s_{i+1}]} \in C^{1,\alpha}$;*

(ii) *Each $\gamma(s_i)$ with $i \in \{1, \ldots, N\}$ is a triple junction, while $\gamma(t)$ is a pure jump for every $t \neq s_i$.*[1]

In particular, if $K' \subset K$ is a connected component, the subset of nonterminal points of K' is still arc connected.

PROOF. Consider γ as in Lemma D.0.2. Since it is injective, $\gamma(s)$ is a nonterminal point for every $s \in (0,1)$. On the other hand $\gamma(0)$ and $\gamma(1)$ are nonterminal points by assumption. In particular, by the regularity theory developed so far, for each $s \in [0,1]$ there is a neighborhood U of s such that $\gamma(U \setminus \{s\})$ consist of pure-jump points. It thus turns out that there are at most finitely many s's such that $\gamma(s)$ is a triple junction. If we then parametrize γ with constant speed, assertion (i) follows from the regularity theory. □

PROOF OF PROPOSITION 4.5.1. Consider the L^2 vector field $\nabla u^\perp$. Observe that (1.5.2) implies that the distributional curl of $\nabla u^\perp$ vanishes. In particular $\nabla u^\perp$ must have a potential $v \in W^{1,2}_{loc}(\mathbb{R}^2)$. A simple direct proof of the latter claim can be found by convolving with a standard family of mollifiers φ_ε: clearly $\mathrm{curl}(\nabla u^\perp) * \varphi_\varepsilon$ is a smooth curl-free vector field by (1.5.2) and as such it has a potential v_ε which we can normalize to $v_\varepsilon(0) = 0$. We can

[1] Observe that 0 and 1 can be triple junctions or pure jumps.

then use the compact embedding of $W^{1,2}$ to extract a sequence $\varepsilon_k \downarrow 0$ such that v_{ε_k} has a local weak limit in $W^{1,2}$. Claim (i) is then an obvious consequence of the smoothness of v in $\mathbb{R}^2 \setminus K$, while the uniqueness up to constant is obvious because, by Theorem 4.4.1, $\mathbb{R}^2 \setminus K$ is connected. Observe next that

$$\int_{B_r(x)} |\nabla v|^2 = \int_{B_r(x)\setminus K} |\nabla u|^2 \leq 2\pi r$$

for every disk $B_r(x)$. Hence (ii) follows from the standard Morrey's estimate.

Finally, in order to prove (iii) fix a connected $K' \subset K$. Since by Corollary 3.1.5 pure jump points (which clearly are nonterminal) are dense, it suffices to prove that $v(x) = v(y)$ for every pair of pure jump points $x, y \in K'$, which from now on we assume to be fixed. Let γ be a curve as in Lemma 4.5.2 and observe that the corresponding partition must satisfy $0 < s_1 < s_2 < \ldots < s_N < 1$. Observe moreover that $v|_K$ must be constant in the neighborhood of every pure jump point by the regularity theory developed thus far: indeed at each jump point we have that u has C^1 extensions u^+ and u^- on both sides of K, while the Euler-Lagrange conditions for u imply $\frac{\partial u^\pm}{\partial \nu} = 0$. This in turn implies that v as well has C^1 extensions v^+ and v^- on both sides of K, and since ∇v is a counterclockwise 90 degree rotation of ∇u, we conclude that both ∇v^+ and ∇v^- are orthogonal to K. Hence $s \mapsto v(\gamma(s))$ is constant on the arcs $[0, s_1), (s_1, s_2), \ldots, (s_{N-1}, s_N), (s_N, 1]$. By continuity we conclude that $v \circ \gamma$ is constant over the whole interval $[0, 1]$, which in turn implies $v(x) = v(\gamma(0)) = v(\gamma(1)) = y$ as desired. □

4.6. The level sets of the harmonic conjugate: Part I

In this section we study the level sets of the harmonic conjugate introduced in Proposition 4.5.1. Our main conclusion is Proposition 4.6.2 below. Observe that for both v and u we have a quite good description of their behavior at every point of K which is either a pure jump point or a triple junction. We thus wish to first gain some more information on points which belong to the remaining set $K^\sharp$, i.e. points which either turn out to form a connected component of K by themselves or which are terminal points of a connected component with positive length (cf. Definition 1.4.1). At all such points we claim that u can be extended continuously.

PROPOSITION 4.6.1. *There is a universal constant C with the following property. Let (K, u) be a nonelementary global minimizer and assume that $x \in K^\sharp$ (i.e. that $x \in K$ is neither a triple junction nor a pure jump point). Then there is a $\bar{u} \in \mathbb{R}$ such that*

$$|\bar{u} - u(y)| \leq C|x - y|^{\frac{1}{2}} \qquad \forall y \notin K\,. \tag{4.6.1}$$

In particular there is a unique continuous extension of u to $K^\sharp$ and from now on we will just use the same notation u for such extension. We are now ready to state the following structural proposition.

PROPOSITION 4.6.2. *Let (K, u) be a nonelementary global minimizer and let v be as in Proposition 4.5.1. Then for a.e. $m \in \mathbb{R}$ we have the following properties:*

(i) *u is continuous at every point of $\{v = m\}$;*
(ii) *$\mathcal{H}^{\frac{1}{2}}(K \cap \{v = m\}) = 0$;*
(iii) *$\{v = m\} \setminus K$ is the union of countably many real analytic arcs;*
(iv) *$\{v = m\}$ has locally finite $\mathcal{H}^1$ measure;*
(v) *for every injective Lipschitz curve $\gamma : [0,1] \to \{v = m\}$ the function $u \circ \gamma$ is absolutely continuous, u is differentiable at $\gamma(s)$ for a.e. s such that $\dot\gamma(s) \neq 0$ and*

$$\frac{d}{dt}(u \circ \gamma) = ((\nabla u) \circ \gamma) \cdot \dot\gamma \tag{4.6.2}$$

where the right hand side is defined as 0 at every point s where $\dot\gamma(s) = 0$, irrespectively of whether u is differentiable or not at that point.

Remark 4.6.3. Note that, as a consequence of (v) we have the integral identity

$$u(\gamma(1)) - u(\gamma(0)) = \int_0^1 \nabla u(\gamma(t)) \cdot \dot\gamma(t)\, dt\,. \tag{4.6.3}$$

4.6.1. Proof of Proposition 4.6.1. Without loss of generality assume that $x = 0$. Given an open set U we define

$$\mathrm{osc}\,(u, U) := \sup\{|u(x) - u(y)| : x, y \in U \setminus K\}\,.$$

We just need to show that there is a universal constant C such that $\mathrm{osc}\,(u, B_r(0)) \leq C r^{\frac{1}{2}}$ and then we can simply define

$$\bar u := \lim_{y \notin K, y \to 0} u(x)$$

to conclude (4.6.1). Since we are claiming that the constant C in our estimate is independent of (K, u), we can assume by scaling that $r = 1$ and we are thus reduced to show

$$\sup\{|u(x) - u(y)| : x, y \in B_1 \setminus K\} \leq C\,. \tag{4.6.4}$$

We will accomplish the latter estimate in three steps.

Step 1 Firs of all consider $F := \{r \in (1,2) : \sharp(K \cap \partial B_r) < \infty\}$ and for each $r \in F$ we define

$$d(K, r) := \min\{|x - y| : x \neq y \text{ and } x, y \in K \cap \partial B_r\}\,.$$

We then claim that there is a $k \in \mathbb{N}$ such that

$$|\{r \in F : d(K, r) > k^{-1}\}| \geq 2k^{-1}\,. \tag{4.6.5}$$

Assume indeed that the statement is false for every k and let (K_k, u_k) be a counterexample for k. Hence a subsequence (K_{k_j}, u_{k_j}) converges to a global generalized minimizer (K_∞, u_∞). By the coarea formula [4, Theorem 2.93] there must be at least one radius $r \in (5/4, 7/4)$ such that $\partial B_r \cap K_\infty$ consists of finitely many pure jump points $x_1, \ldots, x_N$, intersecting ∂B_r transversally. But then there is $\delta > 0$ such that $(B_{r+2\delta} \setminus \overline{B}_{r-2\delta}) \cap K_\infty$ consists of N arcs, intersecting all circles at an angle no smaller than 2δ. In turn the regularity theory developed in the first part implies the existence of a positive η such that, for sufficiently large j, $d(K_{k_j}, \rho) > \eta$ for all $\rho \in (r - \eta, r + \eta)$, which is a contradiction.

Fix now a $k \in \mathbb{N}$ such that (4.6.5) holds and set

$$\mathcal{R} := \left\{ r \in F : d(K, r) > k^{-1} \quad \text{and} \quad \int_{\partial B_r} |\nabla u|^2 \leq 8\pi k \right\}.$$

Observe that $|\mathcal{R}| \geq k^{-1}$

For each $r \in \mathcal{R}$ we let $I_1(r), \ldots, I_{N(r)}(r)$ be the connected components of $\partial B_r \setminus K$. Observe that the length of each $I_j(r)$ is at least $\frac{1}{k}$. For each $I_j(r)$ let

$$\delta_j(r) := \max\{\text{dist}\,(x, K) : x \in I_j(r)\}$$

and

$$\delta(r) := \min_j \delta_j(r)\,.$$

We claim that there is some $M \in \mathbb{N}$ universal constant such that $\delta(r) > M^{-1}$ for at least one $r \in \mathcal{R}$. Assume the contrary and for each $M > 0$ let (K_M, u_M) be such that $\delta(r) \leq \frac{1}{M}$ for every r in the set

$$\mathcal{R}_M := \{r : d(K_M, r) > k^{-1}\}\,.$$

Let then (K_{M_j}, u_{M_j}) be a sequence which converges to (K_∞, u_∞). Let $\mathcal{R}_\infty$ be the set of all $\rho \in (1, 2)$ which are cluster points of a sequence $\{r_j\}$ with $r_j \in \mathcal{R}_{M_j}$. Then $\mathcal{R}_\infty$ cannot be finite. Fix now $\rho \in \mathcal{R}_\infty$. We then easily conclude that $\partial B_\rho \cap K_\infty$ contains at least one interval of length larger than $\frac{1}{k}$. However this would imply that $\mathcal{H}^1(K_\infty \cap B_2) = \infty$.

Step 2 Let now $\Omega(K, \delta, R) := \{x \in B_R : \text{dist}\,(x, K) > \delta\}$. We then claim that for every $\delta > 0$ there is $L \in \mathbb{N}$ such that $\Omega(K, \delta, 2)$ is contained in the same connected component of $\Omega(K, \frac{1}{L}, L)$. Indeed, assume the claim is false and for every L let (K_L, u_L) be a counterexample. Extract a subsequence (K_{L_j}, u_{L_j}) which is converging to a global generalized minimizer (K_∞, u_∞). Note that K_∞ must then disconnect $\mathbb{R}^2$ and thus, by Theorem 4.4.1, (K_∞, u_∞) is a generalized global minimizer. Recall that by assumption $0 \in K_L$ for every $L \in \mathbb{N}$ and hence $0 \in K_\infty$. In turn, by the regularity theory $0 \in K_{L_j}$ must be a triple junction or a pure jump point, while we are assuming that $0 \in K^\sharp$.

Step 3 Fix now M and k as in Step 1 and let $r \in \mathcal{R}$ be such that $\delta(r) > M^{-1}$. Let L be as in Step 2. Let I_i, $i \in \{1, \ldots, N\}$, be the connected components of $\partial B_r \setminus K$, and for each i select $x_i \in I_i$ such that $B_{M^{-1}}(x_i) \subset \mathbb{R}^2 \setminus K$. Fix now ℓ and consider that x_ℓ and $x_{\ell+1}$ are in the same connected component of $\Omega(K, \frac{1}{L}, L)$, which we denote by Ω'. Consider a maximal subset of points $\mathscr{S} = \{y_j\} \subset \Omega'$ with the property that $|y_i - y_j| \geq \frac{1}{8L}$ for each distinct pair y_i, y_j. We then have that $\{B_{(4L)^{-1}}(y_j)\}$ covers Ω'. At the same time the cardinality of $\mathscr{S}$ is bounded by a constant since $B_{(16L)^{-1}}(y_j)$ are pairwise disjoint.

A chain of $\mathscr{S}$ is given by a choice of balls $\{B_{(4L)^{-1}}(y_{i(j)})\}_{j\in\{1,\ldots N\}}$ where the $i(j)$ are all distinct and $B_{(4L)^{-1}}(y_{i(j)}) \cap B_{(4L)^{-1}}(y_{i(j+1)}) \neq \emptyset$. We say that $B_{(4L)^{-1}}(y_K)$ and $B_{(4L)^{-1}}(x_J)$ are chain-connected if there is a chain such that $I = i(1)$ and $J = i(N)$. Assume now, without loss of generality, that $B_{(4L)^{-1}}(y_1)$ contains x_ℓ and let $\mathscr{C} \subset \mathscr{S}$ be the subset of points such that $B_{(4L)^{-1}}(y_j)$ is chain-connected to $B_{(4L)^{-1}}(y_1)$. $\mathscr{C}$ must coincide with $\mathscr{S}$

otherwise the two open sets

$$U := \bigcup_{i\in\mathscr{C}} B_{(4L)^{-1}}(y_i) \tag{4.6.6}$$

$$V := \bigcup_{i\in\mathscr{S}\setminus\mathscr{C}} B_{(4L)^{-1}}(y_i) \tag{4.6.7}$$

would be disjoint and would disconnect Ω'. Upon reindexing our balls we can thus assume that $\{B_{(4L)^{-1}}(y_i)\}_{i\in\{1,\dots,N\}}$ is a chain such that $x_\ell \in B_{(4L)^{-1}}(y_1)$ and $x_{\ell+1} \in B_{(4L)^{-1}}(y_N)$. Set $z_0 = x_\ell$, $z_N = x_{\ell+1}$, and choose $z_i \in B_{(4L)^{-1}}(y_i)\cap B_{(4L)^{-1}}(y_{i+1})$ for every other $1 \leq i \leq N-1$. Consider then the piecewise linear curve γ consisting of joining the segments $[z_i, z_{i+1}]$. Since $|z_{i+1}-z_i| \leq 1/(2L)$, the length of the curve is bounded by a universal constant. Furthermore, each point z on the curve is at distance at least $1/2L$ from K. In turn this and the energy upper bound in (1.3.1) imply

$$\int_{B_{1/(2L)}(z)} |\nabla u|^2 \leq \frac{\pi}{L}\,.$$

By the mean-value property of harmonic functions, we conclude that $|\nabla u|$ is bounded by a universal constant on the curve γ. This then implies that $|u(x_\ell) - u(x_{\ell+1})|$ is bounded by a universal constant too.

Next recall that

$$\int_{\partial B_r} |\nabla u|^2 \leq 8\pi k\,.$$

Hence, by Morrey's embedding, we conclude that $\operatorname{osc}(u, \partial B_r \setminus K)$ is bounded by a universal constant. At this point the maximum principle of Lemma 2.1.1 implies that $\operatorname{osc}(u, B_r \setminus K)$ is bounded as well by the same constant, concluding the proof.

4.6.2. Proof of Proposition 4.6.2. Consider first the union $\tilde K$ of the connected components of K with positive length. Since $K\setminus\tilde K$ consists of irregular points, $\mathcal{H}^1(K\setminus\tilde K) = 0$ and thus Proposition F.0.1 implies that $\mathcal{H}^{\frac12}((K \setminus \tilde K)\cap\{v = t\}) = 0$ for a.e. t since $v \in C^{\frac12}$ by (ii) in Proposition 4.5.1. On the other hand, $v(\tilde K)$ is a countable set, because the connected components of K with positive length are countaly many, and on each such component the function v is constant. Thus, except for a countable values of t's we have $\tilde K \cap \{v = t\} = \emptyset$. In particular, (ii) follows. Moreover, Proposition 4.6.1 implies that u is continuous at every point $x \in \{v = t\}$ if $\{v = t\}$ does not intersect $\tilde K$, hence giving (i).

The fact that $\{v = t\} \setminus K$ consists of a locally finite union of real analytic arcs is a direct consequence of the harmonicity of u on $\mathbb{R}^2 \setminus K$. Next, for every $N \in \mathbb{N}$ consider that by the coarea formula [4, Theorem 2.93] it is true that

$$\int \mathcal{H}^1(\{v = t\} \cap (B_N \setminus K))\, dt = \int_{B_N\setminus K} |\nabla v| = \int_{B_N\setminus K} |\nabla u| < \infty\,.$$

Therefore for a.e. t we have that $\mathcal{H}^1(\{v = t\} \cap (B_N \setminus K)) < \infty$ for every $N \in \mathbb{N}$, thus implying (iv).

Again by the coarea formula [4, Theorem 2.93] we know that

$$\int\int_{B_N\cap\{v=t\}\setminus K} |\nabla u| d\mathcal{H}^1\, dt = \int_{B_N\setminus K} |\nabla v||\nabla u| = \int_{B_N\setminus K} |\nabla u|^2 < \infty\,.$$

We thus infer that for a.e. t

$$\int_{B_R\cap\{v=t\}\setminus K} |\nabla u| d\mathcal{H}^1 < \infty \qquad \forall R>0\,. \tag{4.6.8}$$

Fix now a t which satisfies (i), (ii), and (4.6.8) and let $\gamma : [0,1] \to \{v=t\}$ be an injective Lipschitz curve. We want to show that $u\circ\gamma \in W^{1,1}$. In order to do that we set

$$g(s) := \begin{cases} |\nabla u(\gamma(s))||\dot\gamma(s)| & \text{if } \gamma(s)\notin K \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, by injectivity of γ and the 1-dimensional area formula

$$\int_0^1 g(s)\, ds = \int_{\gamma([0,1])\setminus K} |\nabla u| < \infty\,.$$

The absolute continuity of $u\circ\gamma$ will then follow once we prove that, for all $\sigma, s\in[0,1]$,

$$|u(\gamma(s)) - u(\gamma(\sigma))| \le \int_\sigma^s g(\tau)\, d\tau\,. \tag{4.6.9}$$

First of all, (4.6.9) is obvious if $\gamma([\sigma,s])\subset \mathbb{R}^2\setminus K$, because in that case u is smooth on $[\sigma,s]$. On the other hand it is not possible that $\gamma([\sigma,s])\subset K$ for $\sigma<s$ because $\mathcal{H}^{\frac12}(K\cap\{v=t\})=0$, hence $K\cap\{v=t\}$ is totally disconnected and $\gamma([\sigma,s])\subset K$ would imply that γ is constant on $[\sigma,s]$, contradicting the injectivity of γ.

Fix now an arbitrary $\sigma<s$. By the above argument we know that there is a sequence $\sigma_k\downarrow\sigma$ and a sequence $s_k\uparrow s$ such that $\gamma(\sigma_k),\gamma(s_k)\notin K$, hence it suffices to prove (4.6.9) when $\gamma(s),\gamma(\sigma)\notin K$. However, if $\gamma([\sigma,s])\subset\mathbb{R}^2\setminus K$, then we already observed that the inequality is correct. Thus, we can assume the existence of at least one $\tau\in(\sigma,s)$ such that $\gamma(\tau)\in K$.

Fix now $\delta>0$ and using the fact that $\mathcal{H}^{\frac12}(K\cap\{v=t\})=0$ cover $\gamma([0,1])\cap K$ with a collection $\mathscr{C}$ of open disks $B_{r_i}(x_i)$ such that $x_i\in\gamma([0,1])\cap K$ and

$$\sum_i r_i^{\frac12} < \delta\,.$$

Observe that $\gamma([0,1])\cap K$ is compact and thus we can assume that the cover is finite. Moreover, by taking the disks suitably small, we can assume that $\gamma(\sigma)$ and $\gamma(s)$ do not belong to any of them. Finally, since $\mathcal{H}^{\frac12}(\gamma([0,1])\cap K)=0$ we can choose the disks so that $\partial B_{r_i}(x_i)\cap(\gamma([0,1])\cap K)=\emptyset$, which in turn implies that

$$\gamma([0,1])\cap K\cap B_{r_i}(x_i) \qquad \text{is compact for every } i. \tag{4.6.10}$$

Select now the smallest $s_1\in(\sigma,s)$ such that $\gamma(s_1)\in K$. Then clearly $\gamma(s_1)$ belongs to some disk of the cover $\mathscr{C}$ and by reindexing it we can assume it is $B_{r_1}(x_1)$. We then let σ_2 be the largest number such that $\gamma(\sigma_2)\in K\cap B_{r_1}(x_1)$, which exists by (4.6.10) and is smaller than s because $\gamma(s) \notin K$. If $\gamma((\sigma_2, s]) \cap K = \emptyset$ we stop the procedure, otherwise we select the smallest $s_2 > \sigma_2$ such that $\gamma(s_2) \in K$. Then $\gamma(s_2)$ must belong to a disk of $\mathscr{C}$ which however cannot be $B_{r_1}(x_1)$, and upon reindexing we can assume is $B_{r_2}(x_2)$. We then proceed as in the first step and since $\mathscr{C}$ is finite the procedure stops in finite time. We let σ_N be the last chosen number and set $\sigma_1 = \sigma$ and $s_N = s$. We can then partition $[\sigma, s]$ as

$$[\sigma_1, s_1) \cup [s_1, \sigma_2] \cup (\sigma_2, s_2) \cup \ldots \cup [s_{N-1}, \sigma_N] \cup (\sigma_N, s_N]\,.$$

Each (σ_i, s_i) is contained in $\mathbb{R}^2 \setminus K$. Thus u is smooth on the arc $\gamma((\sigma_i, s_i))$ and for every sufficiently small ε we can write

$$u(\gamma(s_i - \varepsilon)) - u(\gamma(\sigma_i + \varepsilon)) = \int_{\sigma_i+\varepsilon}^{s_i-\varepsilon} \frac{d}{d\tau}(u\circ\gamma)(\tau)\,d\tau\,.$$

In particular

$$|u(\gamma(s_i)) - u(\gamma(\sigma_i))| \le \int_{\sigma_i}^{s_i} g(\tau)\,d\tau\,.$$

On the other hand $u(\gamma(s_i)), u(\gamma(\sigma_{i+1})) \in B_{r_i}(x_i)$. Recall that $x_i \in \gamma([0,1]) \cap K$ and since $\gamma([0,1])$ does not intersect any connected component of K with positive length, we necessarily have $x_i \in K^\sharp$. Hence by Proposition 4.5.1

$$|u(\gamma(\sigma_{i+1})) - u(\gamma(s_i))| \le C r_i^{\frac{1}{2}}\,.$$

We can thus estimate

$$\begin{aligned}
|u(\gamma(s)) - u(\gamma(\sigma))| &\le \sum_{i=1}^{N} |u(\gamma(s_i)) - u(\gamma(\sigma_i))| + \sum_{i=1}^{N-1} |u(\gamma(\sigma_{i+1})) - u(\gamma(s_i))| \\
&\le \sum_{i=1}^{N} \int_{\sigma_i}^{s_i} g(\tau)\,d\tau + C\sum_{i=1}^{N-1} r_i^{\frac{1}{2}} \le \int_{\sigma}^{s} g(\tau)\,d\tau + C\delta\,.
\end{aligned}$$

Since δ is arbitrary, we then conclude (4.6.9).

As for (4.6.2) recall first that, since $u\circ\gamma$ is in $W^{1,1}$, it is a.e. differentiable. The derivative at a.e. $\tau \in [0,1] \setminus \gamma^{-1}(K)$ can be computed using the chain rule because u is smooth on $\mathbb{R}^2 \setminus K$. Nex observe that, by (4.6.9),

$$\left|\frac{d}{d\tau}(u\circ\gamma)\right| \le g \qquad \text{a.e.}$$

and since g vanishes on $\gamma^{-1}(K)$, we conclude that

$$\frac{d}{d\tau}(u\circ\gamma) = 0 \qquad \text{a.e. on } \gamma^{-1}(K).$$

However, as already noticed we have that $\dot\gamma = 0$ a.e. on $\gamma^{-1}(K)$. Hence we can claim (4.6.2) a.e. on $[0,1]$ if we interpret the expression as 0 at every point where $\dot\gamma$ exists and vanishes.

4.7. The level sets of the harmonic conjugate: Part II

In this section we deepen the analysis of the level sets of the harmonic conjugate. We will make use of the following terminology:

Definition 4.7.1. A set $J \subset \mathbb{R}^2$ contains no loops if any injective continuous map $\gamma : \mathbb{S}^1 \to J$ is necessarily constant. A point $p \in J$ is terminal if there is no injective continuous map: $\gamma : (-1,1) \to J$ such that $\gamma(0) = p$. Points $p \in J$ which are not terminal will be called nonterminal.

Note that the terminology is consistent with the terms terminal and nonterminal points introduced in Definition 1.4.1 and used thus far for points of K.

Proposition 4.7.2. *Let K, u, and v be as in Proposition 4.6.2. Then:*

(i) *For every m the level set $\{v = m\}$ does not disconnect $\mathbb{R}^2$ and in particular contains no loops.*

(ii) *v satisfies the maximum and minimum principle, i.e.*

$$\max_{\overline{U}} v = \max_{\partial U} v\,, \tag{4.7.1}$$

$$\min_{\overline{U}} v = \min_{\partial U} v\,, \tag{4.7.2}$$

for every bounded open set U. Moreover, if $x \in U$ is a local minimum (resp. maximum), then necessarily $x \in K$.

(iii) *For every m to which all the conclusions of Proposition 4.6.2 apply, the level set $\{v = m\}$ contains no terminal points.*

Proof. **Proof of (i)** If $\{v = m_0\}$ contains a loop, then it disconnects $\mathbb{R}^2$ and at least one connected component of $\{v \neq m_0\}$, which we denote by U, must be bounded. Clearly v on U must either be strictly larger than m_0 or strictly smaller than m_0. To fix ideas let us assume that it is strictly larger and let $\delta > 0$ be such that $U_m := \{v > m\} \cap U$ is not empty for every $m \in (m_0, m_0 + \delta)$. Consider that ∂U_m is rectifiable for a.e. m by Proposition 4.6.2 and that

$$\int_{m_0}^{m_0+\delta} \mathcal{H}^1(\partial U_m)\, dm < C\,.$$

We fix next a suitable regularization v_k of v, for instance by convolution, so that

$$\limsup_k \int_{m_0}^{m_0+\delta} \mathcal{H}^1(\partial\{v_k > m\} \cap U)\, dm < \infty\,.$$

By a standard diagonalization procedure we can find a sequence of values m_k converging to some $m \in (m_0, m_0 + \delta)$ with the properties that:

(a) All the conclusions of Proposition 4.6.2 apply.

(b) $\{v_k = m_k\} \cap U$ consist of non degenerate points (i.e. $\nabla v_k \neq 0$ everywhere on $\{v_k = m_k\} \cap U$).

(c) $\mathcal{H}^1(\{v_k = m_k\} \cap U)$ converge to $\mathcal{H}^1(\{v = m\} \cap U)$.

Since $\{v_k = m_k\} \cap U$ is (for sufficiently large k) compactly contained in U, it consists of finitely many loops. Each such loop bounds a corresponding disk. Fix a point p in $\{v > m\} \cap U$: for sufficiently large k's p must belong to one such disk. We denote it by D_k and we let γ_k be the corresponding loop. Note that the length of γ_k is uniformly bounded in k. We can thus parametrize γ_k by a constant multiple of the arc length over $\mathbb{S}^1$. Without loss of generality we keep denoting by γ_k such parametrization. By Ascoli-Arzelà we can assume that γ_k converges to some Lipschitz map $\gamma : \mathbb{S}^1 \to U$. Observe that $\gamma(\mathbb{S}^1) \subset \{v = m\} \cap U$. But observe also that the length of $\gamma(\mathbb{S}^1)$ must be equal to the limits of the lengths of $\gamma_k(\mathbb{S}^1)$ (which we might assume exist by extraction of a subsequence), otherwise item (c) above would be violated. In particular we obtain that

$$\int |\dot\gamma| = \mathcal{H}^1(\gamma(\mathbb{S}^1))\,.$$

Observe also that we can choose δ positive so that $\overline{B}_\delta(p) \subset \{v > m\} \cap U$. In particular, for all sufficiently large k the disk $\overline{B}_\delta(p)$ must be contained inside the disk D_k. We thus conclude from the isoperimetric inequality that

$$\mathcal{H}^1(\gamma(\mathbb{S}^1)) > 0\,.$$

Choose now the orientation of γ_k so that

$$\frac{\dot\gamma_k^\perp}{|\dot\gamma_k|} = \lambda_k \nabla v_k \circ \gamma_k$$

for some positive function λ_k. In particular

$$\frac{\dot\gamma_k^\perp}{|\dot\gamma_k|}$$

is the inward unit normal to D_k and v_k increases in its direction. It is then easy to see that

$$\frac{\dot\gamma^\perp}{|\dot\gamma|} \cdot \nabla v \circ \gamma \geq 0$$

holds a.e., since the convergence of v_k to v is in fact smooth on $\gamma(\mathbb{S}^1) \setminus K$, which has measure zero. But then we conclude that in fact

$$\frac{\dot\gamma^\perp}{|\dot\gamma|} \cdot \nabla v \circ \gamma > 0$$

a.e., since both vectors are collinear and nonzero a.e. on $\gamma(\mathbb{S}^1)$ (recall that v is harmonic outside K and as such the set of points not contained in K where its gradient vanishes must be countable).

In particular, we conclude

$$\dot\gamma \cdot \nabla u \circ \gamma > 0 \qquad \text{a.e.}.$$

However, by Proposition 4.6.2 the latter would imply

$$0 = u(\gamma(2\pi)) - u(\gamma(0)) = \int_0^{2\pi} \nabla u(\gamma(t)) \cdot \dot\gamma(t)\, dt > 0\,,$$

which is a contradiction.

Proof of (ii) The two cases are entirely analogous and we focus on the case of maxima for simplicity. Consider, by contradiction, a bounded open set V for which $\max_{\overline{V}} v > \max_{\partial V} v$. Then there is an m_0 such that $\{v = m_0\}$ does not intersect ∂V and $\{v > m_0\} \cap V$ is not empty. Setting $U := \{v > m_0\}$ we can argue as in the previous step to obtain a contradiction. Next, if $x \in U$ is a local maximum and $x \notin K$, by the harmonicity of v we conclude that ∇v must vanish in a neighborhood of x. But this is not possible because Corollary 4.4.4 would imply that (K, u) is an elementary minimizer.

Proof of (iii) We will prove the claim for any m such that all the conclusions of Proposition 4.6.2 apply. Fix then a point x_0 in $\{v = m\}$ and assume, without loss of generality, that $x_0 = 0$. The proof will be split in several step

Step 1. Choice of a good radius. Let $r > 0$, and G be any connected component of $\{v = m\} \cap \overline{B}_r$ and observe that it must necessarily intersect ∂B_r. Otherwise, using Lemma D.0.1 we find a Jordan curve γ which does not intersect $\{v = m\}$ and bounds a disk D which contains G. Clearly we must have either $v < m$ or $v > m$ on ∂D and in particular we would violate (ii).

We thus infer that $\{v = m\} \cap \partial B_r \neq \emptyset$ for every r, because some connected component of $\{v = m\} \cap \overline{B}_r$ must contain the origin (recall that $x_0 = 0 \in \{v = m\}$).

We next appeal to the coarea formula [4, Theorem 2.93] to choose an $r \in (1, 2)$ with the properties that

(1) $\{v = m\} \cap \partial B_r$ is finite;
(2) $\{v = m\} \cap \partial B_r \cap K = \emptyset$;
(3) $\nabla v(y) \neq 0$ for every $y \in \{v = m\} \cap \partial B_r$ (hence $\{v = m\}$ is a smooth arc in a sufficiently small neighborhood of any such y);
(4) $\{v = m\}$ intersects ∂B_r transversally at any $y \in \partial B_r \cap \{v = m\}$.

Fix now any point $y \in \partial B_r \cap \{v = m\}$. By (4) the normal to $\{v = m\}$ at y cannot be perpendicular to ∂B_r: recall that this normal is colinear with $\nabla v(y)$, which by (3) does not vanish. We thus conclude that $\nabla v(y)$ cannot be perpendicular to ∂B_r. Hence, if we consider the restriction of the function $v - m$ to ∂B_r, such function is changing sign at each point $y \in \partial B_r \cap \{v = m\}$. Therefore $\{v = m\} \cap \partial B_r$ consists of an even number of points which divide the cirle ∂B_r into an even number of arcs γ_i: on half of them $v - m$ is positive, while on the other half it is negative, cf. Figure 6.

Step 2. $0 \in \partial A$ **for some connected component** A **of** $B_r \setminus \{v = m\}$**.** Consider now the connected components $\{A_i\}$ of $B_r \setminus \{v = m\}$. $v - m$ does not change sign nor vanishes on each A_i. ∂A_i must then intersect ∂B_r: otherwise we would have $v \equiv m$ on ∂A_i, violating (ii). Since each A_i is connected, if ∂A_i intersects a given γ_j, the latter being one of the arcs introduced in the previous step, then $\gamma_j \subset A_i$. In particular each γ_j is contained in at most one ∂A_i, while each ∂A_i contains at least one γ_j. Hence the A_i's are finitely many.

Next, since $\{v = m\}$ contains no interior point, every $z \in \{v = m\} \cap B_r$ must be contained in ∂A_i for some i. This holds for 0 as well. We therefore assume that $0 \in \partial A_1$ and for simplicity we set $A = A_1$.

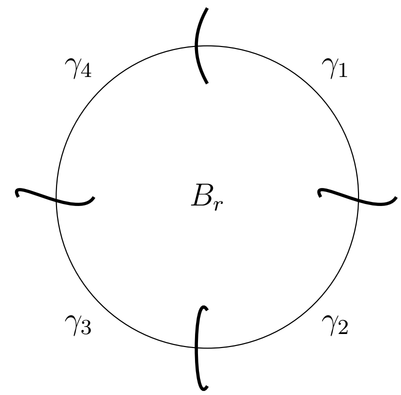

FIGURE 6. $\partial B_r \setminus \{v = m\}$ consists of an even number of arcs, on which the function $v - m$ takes alternating signs. The picture depicts $\{v = m\}$ only in a neighborhood of ∂B_r.

Step 3. A is simply connected. Furthermore, to simplify our discussion we assume that $v > m$ on A. Note next that A must be simply connected: if γ is a smooth simple Jordan curve in A, then it bounds a disk D and the latter must be contained in A otherwise D would contain a portion of ∂A which does not intersect γ, and from the latter we would get a connected component of $\{v = m\}$ which does not intersect ∂B_r. Since A contains D, γ is contractible in A.

Step 4. Finding a suitable Jordan curve in ∂A. Having established that A is simply connected, i.e. it is a topological disk, we infer that its boundary ∂A cannot be disconnected by a point $p \in \partial A$. We next devise a suitable algorithm to generate a suitable Jordan curve contained in ∂A.

Fix first a γ_i contained in ∂A, relabel it so that $i = 1$ and consider its two endpoints a_1 and b_1. If we remove the (open arc) γ_1 from ∂A, the remainder is a closed connected set with finite Hausdorff measure. We can thus apply Lemma D.0.2 to conclude that there is an injective curve η in $\partial A \setminus \gamma_1$ which connects b_1 and a_1. Consider it as a parametrized curve $\eta : [0, 1] \to \partial A$ with $\eta(0) = b_1$ and $\eta(1) = a_1$. Since in a neighborhood U of b_1 $\{v = m\} \cap U$ consists of a single smooth arc crossing $\{v = m\}$ precisely in b_1, for a sufficiently small $\delta > 0$ $\eta([0, \delta))$ must be contained in this arc. In particular $\eta((0, \delta))$ does not intersect ∂B_r. On the other hand $\eta(1) = a_1 \in \partial B_r$. Consider therefore the smallest positive s such that $\eta(s) \in \partial B_r$: the arc $\eta_1 = \eta([0, s])$ is then a simple arc joining b_1 with either a_1 or with one extremum of some other arc γ_j, and $\eta_1((0, s))$ is contained in $\partial A \setminus \partial B_r$. If $\eta_1(s) = a_1$ we stop and we conclude that $\gamma_1 \cup \eta_1$ is a Jordan curve.

Otherwise $\eta_1(s)$ must be an extremum of a second γ_j contained in ∂A. We relabel so that $j = 2$ and denote by a_2 and b_2 its extrema, with the convention that $a_2 = \eta_1(s)$. We now remove $\gamma_1 \cup \eta_1 \cap \gamma_2$ from ∂A (where we follow the convention that the arcs γ_i are open while the arcs η_i are closed). The remaining set is still connected, has finite $\mathcal{H}^1$ measure, and contains b_2 and a_1. We can thus repeat the procedure above to produce a second

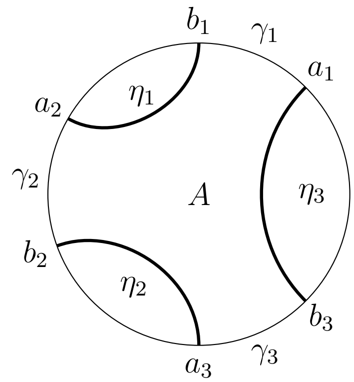

FIGURE 7. The picture is an illustration of the algorithm to find a Jordan curve inside ∂A. The algorithm finds inductively the pairs of Jordan arcs (γ_i, η_i) starting from γ_1.

simple arc $\eta_2 \subset \partial A \setminus (\gamma_1 \cup \eta_1 \cup \gamma_2)$ which connects b_2 with the endpoint of some other arc γ_k contained in ∂B_r, taking care that η_2 lies in B_r except for its two extrema. Recall that one extremum of η_2 is b_2: if the second is a_1 we stop the procedure. Otherwise note that it cannot be an extremum of either γ_1 or γ_2: it must be the extremum of some γ_k with $k \neq 1, 2$, cf. Figure 7. We then let $k = 3$ after relabeling and proceed as above.

Since the γ_j's are finitely many, the procedure must end. Assume it ends after N steps. If we string together $\gamma_1 \cup \eta_1 \cup \gamma_2 \cup \ldots \cup \eta_N$, we achieve a Jordan curve.

Step 5. Describing ∂A. The Jordan curve $\beta : \mathbb{S}^1 \to \partial A$ found by the previous algorithm bounds then a topological disk D which must be contained in B_r and which contains 0. Observe that A must be contained in D. Indeed D is a connected component of $B_r \setminus \beta(\mathbb{S}^1)$ and, since $\beta(\mathbb{S}^1) \subset \partial A \subset \{v = m\}$, any connected component of $B_r \setminus \{v = m\}$ either does not intersect D, or it is contained in D. On the other hand both A and D contains 0, so $A \cap D \neq \emptyset$. Next $v = m$ on all the η_i's, while $v > m$ on all the γ_i's ($v \geq m$ on each of them because they are in ∂A and the strict inequality is because none of them intersects $\{v = m\}$). But then we conclude that $v \geq m$ on D, by (ii). We now want to exclude that $v = m$ somewhere in D. Indeed, if $z \in \{v = m\} \cap D$ notice that there is a connected component G of $\{v = m\}$ which contains z and intersects ∂B_r. But then $\mathcal{H}^1(\{v = m\} \cap D) > 0$. Since $\mathcal{H}^1(\{v = m\} \cap K) = 0$, we must have some $z \in \{v = m\} \cap D \setminus K$. But then, by Proposition 4.6.2(ii) $v - m$ takes both signs in any neighborhood of z, contradicting $v \geq m$ in D. Having concluded that $v > m$ in D, we infer that $D \cap \{v = m\} = \emptyset$. In particular D is a connected component of $B_r \setminus \{v = m\}$. Since it contains A, it actually must be equal to A.

We are now ready to conclude: recalling that $0 \in \partial A$, it must be contained in some η_j. The latter is an injective curve, it is contained in $\{v = m\}$, and 0 is not one of its two extrema: in particular 0 is a nonterminal point of $\{v = m\}$. $\square$

4.8. The level sets of the harmonic conjugate: Part III

We are now ready to conclude our analysis of the level sets of v. While so far all the conclusions apply for a global generalized minimizer which is not elementary, in this section we use substantially the additional assumption that all but one connected component of K are contained in a fixed disk B_R. Before coming to the relevant statement, let us point out that, given any global generalized minimizer $(K, u, \{p_{kl}\})$ and any constant c, $(K, \pm(u-c), \{\pm p_{kl}\})$ is also a global generalized minimizer.

PROPOSITION 4.8.1. *Let (K, u) be a nonelementary global minimizer satisfying (a) and (b) in Theorem 4.1.1 and let v be as in Proposition 4.5.1. Then:*

(i) *every blow-down of (K, u) (i.e. any limit of (K_{0,r_j}, u_{0,r_j}) for $r_j \uparrow \infty$) is the cracktip;*
(ii) *$|\nabla v(x)| \geq c_0|x|^{-\frac{1}{2}}$ for all $x \notin K$ large enough and for a positive geometric c_0;*
(iii) *up to changing the sign of u, v achieves its global minimum $\min v$;*
(iv) *for every $m > \min v$ there is a radius $R = R(m)$ such that $\{v = m\} \setminus B_r$ is smooth and $\{v = m\}$ intersects each ∂B_r transversally in exactly two points for every $r \geq R(m)$;*
(v) *for a.e. $m > \min v$ the level set $\{v = m\}$ is a properly embedded unbounded line;*
(vi) *the unbounded connected component of K is $\{v = \min v\}$.*

PROOF. **Proof of (i).** Since $K \setminus B_{R_0}$ is a single connected component for a sufficiently large R_0, by Proposition 2.6.1 the quantity

$$\frac{1}{r}\int_{B_r} |\nabla u|^2$$

is monotone for $r > R_0$. Let (J, w) be any blow-down. It then turns out that J consists of a single unbounded connected component and that

$$\frac{1}{r}\int_{B_r} |\nabla w|^2$$

is a nonzero constant thanks to Corollary 4.4.4, being (K, u) nonelementary. But then Proposition 2.6.1 implies that it is a cracktip. Observe that, while we lack at this point any uniqueness statement and the half-line of the cracktip could change depending on the blow-down sequence, it is easy to verify that the sign of the constant b in (2.5.13) must always be the same, independently of the subsequence (cf. Proposition 2.5.6).

Proof of (ii), (iii), and (iv). Fix $r > 0$ sufficiently large and rotate the coordinates by an angle $\theta(r)$ so that $\mathcal{R}_{\theta(r)}(K) \cap \partial B_r = \{(r, 0)\}$. We consider the pair

$$K_r := \frac{1}{r}\mathcal{R}_{\theta(r)}(K) \tag{4.8.1}$$

$$u_r(x) := r^{-1/2} u(r\mathcal{R}_{-\theta(r)}(x)) \tag{4.8.2}$$

and the function

$$v_r(x) := r^{-1/2} v(r\mathcal{R}_{-\theta(r)}(x))\,.$$

Up to a change of the sign of u we can assume (K_r, u_r) is converging to the following specific cracktip (K_∞, w_∞):

$$K_\infty = \{(t,0) : t \geq 0\} \tag{4.8.3}$$

$$w_\infty(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \cos\frac{\theta}{2}\,. \tag{4.8.4}$$

On the other hand v_r is converging (locally uniformly, thanks to the Hölder estimate of Proposition 4.5.1) to a Hölder function v_∞ which is harmonic on $\mathbb{R}^2 \setminus K_\infty$ and satisfies $\nabla v_\infty = \nabla u_\infty^\perp$ and $v_\infty(0) = 0$. But then it turns out that

$$v_\infty(x) = \sqrt{\frac{2}{\pi}} r^{1/2} \sin\frac{\theta}{2}\,.$$

We next appeal to the ε-regularity theory at pure jumps to argue that the convergence of K_r to K_∞ is in $C^{1,\alpha}$ on $B_4 \setminus B_{\frac{1}{2}}$. Hence we can appeal to the regularity theory coming from the Euler-Lagrange conditions of Proposition 2.5.2 to conclude that the convergence of u_r and v_r to u_∞ and v_∞ is also in $C^{1,\alpha}$ "up to the discontinuity set K_r": more precisely, there are $C^{1,\alpha}$ diffeomorphisms Φ_r of $B_4 \setminus (B_{1/4} \cup K_r)$ onto $B_4 \setminus (B_{1/4} \cup K_\infty)$ such that

- Φ_r converges to the identity in $C^{1,\alpha}$ as $r \uparrow \infty$;
- $u_r \circ \Phi_r^{-1}$, $v_r \circ \Phi_r^{-1}$ converge in $C^{1,\alpha}$ to u_∞, v_∞ on $B_2 \setminus (B_{\frac{1}{2}} \cup K_\infty)$.

We can then easily draw the following conclusions:

(1) $|\nabla u_r| \geq c > 0$ on $B_2 \setminus (B_{\frac{1}{2}} \cup K_r)$ for some suitable positive constant c and every sufficiently large r. This clearly implies (ii).
(2) Since $|\nabla v_r| = |\nabla u_r|$ we infer that the level sets of v_r are smooth on $B_2 \setminus (B_{\frac{1}{2}} \cup K_r)$.
(3) Since $\pm\frac{\partial v_\infty}{\partial\theta}(1, \pm\theta) \geq \frac{1}{2\sqrt{2\pi}}$ for $\theta \in (0, \frac{\pi}{2})$, we conclude $\pm\frac{\partial v_r}{\partial\theta}(1, \pm\theta) \geq \frac{1}{4\sqrt{2\pi}}$ for $\theta \in (0, \frac{\pi}{2})$ and r large enough.
(4) Since $v_\infty(1,\theta) \geq \pi^{-1/2}$ for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, clearly $v_r(1,\theta) \geq \frac{1}{2}\pi^{-1/2}$ for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and r large enough.

The last two facts imply that, if r is large enough, then $v_r(1,\theta) > v_r(1,0)$ for all $\theta \in (0, 2\pi)$. Translating this information back to v, we conclude that, if we denote by $\bar{m}$ the constant value achieved by v on the unbounded connected component of K, then $v > \bar{m}$ on $\partial B_r \setminus K$. By the maximum principle of Proposition 4.7.2 the latter implies that $\bar{m}$ is the absolute minimum of v.

Fix now $m > \bar{m}$, then

$$\{v = m\} \setminus B_{\frac{r}{2}} = r(\{v_r = r^{-1/2} m\} \setminus B_{\frac{1}{2}})\,.$$

If r is sufficiently large, the set $\{v_r = r^{-1/2} m\}$ intersects ∂B_1 transversally in exactly two points, by (3) and (4). This shows (iv)

Proof of (v). Pick $m > \bar{m}$ to which all the conclusions of Proposition 4.7.2 apply and let r be large enough so that ∂B_r intersects $\{v = m\}$ transversally in exactly two points. Since $B_r \cap \{v = m\}$ has no terminal points and no loops, $\{v = m\} \cap B_r$ must be a Jordan arc. Indeed we could even notice that the very argument given in the previous section for Proposition 4.7.2(iii) shows this fact directly.

Proof of (vi). Let L be the unbounded connected component of K. Observe that we have already shown that $L \subset \{v = \bar{m}\}$, where $\bar{m} = \min v$. Next observe that, for a.e. $m > \bar{m}$, the level set $\{v \leq m\}$ is connected, because of (v). Since $\{v = \bar{m}\} = \bigcap_{m>\bar{m}} \{v \leq m\}$, we conclude that $\{v = \bar{m}\}$ is connected as well. Next notice that $\{v = \bar{m}\} \subset K$, because by Proposition 4.6.2(ii) any point of $\{v = \bar{m}\}$ is a global minimum for v. We have thus inferred that $L \subset \{v = \bar{m}\} \subset K$ and that $\{v = \bar{m}\}$ is connected. Recalling that L is a connected component of K, we must have $L = \{v = \bar{m}\}$. □

4.9. A special bounded connected component of K

From now on we will establish a set of conditions which we can assume for a couple (K, u) as in Theorem 4.1.1 without loss of generality, thanks to Proposition 4.8.1.

Assumption 4.9.1. (K, u) is a nonelementary global generalized minimizer (in particular $\mathbb{R}^2 \setminus K$ is connected) and v is an harmonic conjugate of u as in Proposition 4.5.1, which satisfy in addition the following properties.

(i) $v \geq 0$.
(ii) $L := \{v = 0\}$ is the only unbounded connected component of K.
(iii) There is $R > 0$ s.t. $K \setminus B_R \subset L$ and $L \cap \partial B_R$ has cardinality 1.

We then let m_0 be $\max_K v$.

Observe that m_0 is well defined because it is in fact $\max_{K \cap \overline{B}_R} v$ by (ii) and (iii) above, while v is continuous and $K \cap \overline{B}_R$ compact. We next point out a pivotal (yet simple) outcome of the Bonnet monotonicity formula.

Corollary 4.9.2. *Let (K, u) be as in Assumption 4.9.1. If $m_0 = 0$, then (K, u) is a cracktip.*

We postpone its simple proof, for the moment, and note that it gives us a route to the proof of Theorem 4.1.1: we assume that $m_0 > 0$ and wish to derive a contradiction. To that end it will be useful to make the following preliminary analysis.

Proposition 4.9.3. *Let (K, u) be as in Assumption 4.9.1 and assume that $m_0 > 0$. Then any $x \in K \cap \partial\{v > m_0\}$ is either a pure jump or a triple junction. In particular, if G is any connected component of K contained in $\{v = m_0\}$, then*

(i) *G is bounded and $\mathcal{H}^1(G) > 0$;*
(ii) *if $x \in G$ is a terminal point, then there is a neighborhood of x in which $v \leq m_0$.*

4.9.1. Proof of Corollary 4.9.2.

If $m_0 = 0$ it follows immediately from our assumptions that $K = \{v = 0\}$, it is connected, and it is unbounded. Since K does not disconnect $\mathbb{R}^2$, K must have at least one terminal point, which without loss of generality we can assume to be 0. By Proposition 2.6.1 we know that

$$\frac{D(r)}{r} = \frac{1}{r} \int_{B_r} |\nabla u|^2$$

is monotone nondecreasing. Recall that the blow-downs of (K, u) are cracktips by (i) Proposition 4.8.1. Observe that, if $\lim_{r\downarrow 0} r^{-1}D(r) = 0$, then the blow-ups of (K, u) at 0 are elementary minimizers. By the density lower bound none of them can be a constant, so they must be either pure jumps or triple junctions. But then the ε-regularity theory developed thus far would imply that $K \cap B_r$ is diffeomorphic either to a diameter of B_r or to three radii meeting at the origin, contradicting the hypothesis that 0 is a terminal point of K. We thus conclude that $c := \lim_{r\downarrow 0} r^{-1}D(r) > 0$. But then any blow-up (K_0, u_0) at 0 must have K_0 connected and must satisfy

$$\frac{1}{r}\int_{B_r} |\nabla u_0|^2 \equiv c > 0$$

Thus the second part of Proposition 2.6.1 implies that (K_0, u_0) is a cracktip. We thus conclude that $r^{-1}D(r)$ is as well constant and nonzero. Hence we can once again appeal to Proposition 2.6.1 to conclude that (K, u) itself is a cracktip.

4.9.2. Proof of Proposition 4.9.3. First of all we address the two consequences (i) and (ii) of the main statement. (ii) is indeed obvious because of the main part of the Proposition.

Next consider any connected component G of K in $\{v = m_0\}$, and note that G is necessarily bounded in view of Assumption 4.9.1. Let $x \in G$. Recall that the connected component J of $\{v = m_0\} \cap \overline{B}_r(x)$ which contains x must intersect $\partial B_r(x)$, by Proposition 4.7.2 (otherwise Lemma D.0.1 would provide an open disk D with smooth boundary γ containing J and with $\gamma \cap \{v = m_0\} = \emptyset$: such a D would then violate the maximum principle of Proposition 4.7.2). Therefore, if $\{x\}$ were a connected component of K, then there would be a sequence $x_k \in J \setminus K$ converging to x. v would be harmonic in a neighborhood of these points, and since it is nowhere constant because of Corollary 4.4.4, $v - m_0$ would need to change sign in these neighborhoods. In particular we could provide a sequence y_k such that $|x_k - y_k| \le \frac{1}{k}$ with $v(y_k) > m_0$, thus showing that $x \in K \cap \partial\{v > m_0\}$. But then by the first part of the Proposition x would be a pure jump point or a triple junction, showing that $K \cap B_\rho(x)$ is connected in some disk $B_\rho(x)$, a contradiction.

We next come to the main part of the Proposition, which will be split in several steps.

Step 1. Consider the set $V := \{v > m_0\}$ and fix any point x in $\mathbb{R}^2$. In this step we show that the functional

$$O(x, r) := \frac{1}{r}\int_{V\cap B_r(x)} |\nabla u|^2$$

is monotone in r. In this step for convenience of notation we assume that $x = 0$ and, arguing as in the proof of Proposition 2.6.1, we observe that $O(0, \cdot)$ is absolutely continuous and its derivative is given by

$$O'(0, r) = \frac{1}{r}\int_{V\cap\partial B_r} |\nabla u|^2 - \frac{1}{r^2}\int_{V\cap B_r} |\nabla u|^2 .$$

We next claim that

$$\int_{V\cap B_r} |\nabla u|^2 = \int_{V\cap\partial B_r\setminus K} u\frac{\partial u}{\partial n} . \tag{4.9.1}$$

In order to prove it we observe first that the identity can be justified if V is replaced by $\{v > m\}$ for some m bigger than m_0 for which the conclusion of (v) of Proposition 4.8.1 applies, because then we can let $m \downarrow m_0$. Next, for any such m the set $\{v = m\}$ is a nonintersecting infinite curve which does not intersect K (because by definition $v \leq m_0$ on K). It then turns out that ∇v cannot vanish on it: as it is well known, since v is harmonic, around a point p where ∇v vanishes the level set $\{v = v(p)\}$ cannot be a Jordan arc, and it is rather diffeomorphic to the union of $N \geq 4$ segments joining at p. This follows simply from the fact that, in a sufficiently small neighborhood of p, the level set $\{v = v(p)\}$ is diffeomorphic to the zero set of the first nontrivial harmonic polynomial in the Taylor expansion of $v - v(p)$ at p: up to rotations, the latter is necessarily of the form $\mathrm{Re}(z - z_0)^k$ for some $k \geq 2$, where $z_0 = x_0 + iy_0$ for $p = (x_0, y_0)$, and $z = x + iy$. It thus turns out that $\{v > m\}$ is a smooth set and we can use the relation $\frac{\partial u}{\partial n} = 0$ on $\partial\{v > m\} = \{v = m\}$.

Next observe that

$$\int_\gamma \frac{\partial u}{\partial n} = 0$$

for every connected component γ of $\{v > m_0\} \cap \partial B_r$. Again it suffices to show it first for every connected component of $\{v > m\} \cap \partial B_r$ for the $m > m_0$ to which we can apply Proposition 4.8.1 and then pass to the limit. In fact by varying m we can assume that the intersection of $\{v = m\}$ with ∂B_r is transversal. Then it suffices to observe that for each such γ we can find an arc η in $\{v = m\}$ with the same endpoints as γ such that $\eta \cup \gamma$ bounds a disk D. We then use

$$0 = \int_{\gamma \cup \eta} \frac{\partial u}{\partial n} = \int_\gamma \frac{\partial u}{\partial n},$$

where the second identity is due to $\frac{\partial u}{\partial n} = 0$ on η, and the first is implied by the harmonicity of u in D, if $D \cap K = \emptyset$. More generally we can use Proposition 2.5.1 if $K \cap D \neq \emptyset$, because (taking into account that $\lambda = 0$) (1.5.2) is the weak formulation of the Neumann condition $\frac{\partial u^\pm}{\partial n} = 0$ on K. However in this particular instance we can show that $D \cap K = \emptyset$. Indeed $\{v \geq m\} \subset \mathbb{R}^2 \setminus K$ as $m > m_0$ and $m_0 \geq v$ on K, therefore $K \cap D$ is contained in the endpoints of γ, which are actually points of η and $K \cap \eta = \emptyset$. At any rate we are now in the position of arguing as in Proposition 2.6.1 to show that

$$\frac{1}{r} \int_{V \cap \partial B_r} |\nabla u|^2 \geq \frac{1}{r^2} \int_{V \cap B_r} |\nabla u|^2 .$$

Step 2. Fix now any $x \in \mathbb{R}^2$ and observe that $V = \{v > m_0\} \subset \mathbb{R}^2 \setminus K$, and that by Proposition 4.8.1(i),

$$\lim_{r \uparrow \infty} O(x, r) = \lim_{r \uparrow \infty} \frac{1}{r} \int_{B_r(x) \setminus K} |\nabla u|^2 ,$$

while the latter value equals the constant value

$$C_\infty := \frac{1}{r} \int_{B_r \setminus K_c} |\nabla u_c|^2 = 1$$

for a cracktip (K_c, u_c) with terminal point at the origin. Indeed, observe first that

$$B_{r-|x|}(0) \subset B_r(x) \subset B_{r+|x|}(0)$$

and therefore it suffices to show that

$$\lim_{r\to\infty} \frac{1}{r}\int_{(\{v>m_0\}\cap B_r)\setminus K} |\nabla u|^2 = C_\infty\,.$$

Next, introduce $u_r(x) := r^{-1/2}u(rx)$, $K_r := r^{-1}K$, and $v_r(x) := r^{-1/2}v(rx)$ and consider that

$$\frac{1}{r}\int_{(\{v>m_0\}\cap B_r)\setminus K} |\nabla u|^2 = \int_{(\{v_r>r^{-1/2}m_0\}\cap B_1)\setminus K_r} |\nabla u_r|^2\,.$$

However, up to extraction of subsequences (which for the sake of keeping our notation simple we ignore) ∇u_r converges strongly in L^2 to ∇u_c, while v_r converges (uniformly, given that $\|v_r\|_{C^{1/2}} \leq C$) to the harmonic conjugate v_c of u_c normalized to be 0 at 0. Depending on the sign of the cracktip u_c, v_c is either nonnegative, or nonpositive. In the first case we have $\{v_c > 0\} = \mathbb{R}^2 \setminus K_c$, while in the second we have $\{v_c < 0\} = \mathbb{R}^2 \setminus K_c$. In particular $\{v_r > r^{-1/2}m_0\}$ converges to $\{v_c > 0\}$. The second case, namely $v_c \leq 0$, is then excluded because the limit of the integrals would be 0. We thus must have $\{v_c > 0\} = \mathbb{R}^2 \setminus K_c$, which in turn implies

$$\lim_{r\to\infty} \int_{(\{v_r>r^{-1/2}m_0\}\cap B_1)\setminus K_r} |\nabla u_r|^2 = \int_{B_1\setminus K_c} |\nabla u_c|^2\,.$$

In particular, if we set

$$C(x) := \lim_{r\downarrow 0} O(x,r)\,,$$

we then conclude that $C(x) \in [0,1]$. Observe that, arguing as in Proposition 2.6.1, if $C(x) = 1$, then $O'(x,\cdot) \equiv 0$, which in turn implies that:

- either $V \cap B_r(x)$ has Lebesgue measure zero for every r;
- or $B_r(x) \setminus V$ is a straight segment originating at x and u is $\frac{1}{2}$-homogeneous on V.

It is immediate to see that the first case is excluded as it would be $C(x) = 0$, while we are assuming that $C(x) = 1$. In the second case (K, u) is a cracktip with terminal point at x. But then m_0 could not be larger than 0, so the latter conclusion is excluded as well. Thus, $C(x) \in [0, 1)$.

Assume now that $C(x) \in (0, 1)$. Consider then any blow-up $(K_0, u_0, \{p_{kl}\})$ at x, limit of some sequence of rescalings (K_{x,r_j}, u_{x,r_j}) with $r_j \downarrow 0$, and consider the limit v_0 of the rescaled harmonic conjugate function

$$v_{x,r_j}(y) := r_j^{-1/2}(v(r_j y + x) - m_0)\,.$$

In view of Step 1, it then turns out that

$$\frac{1}{r}\int_{B_r\cap\{v_0>0\}} |\nabla u_0|^2 = C(x) > 0 \qquad \text{for all } r > 0.$$

We again argue as in Step 1 and conclude that the last identity yields that either $\{v_0 > 0\}\cap B_r$ has measure zero for all r (which is excluded from the fact that $C(x) > 0$), or that u_0 is the cracktip. In this case, however v_0 is the harmonic conjugate of the cracktip, normalized to be 0 at 0. Arguing as above, it follows again that either v_0 is nonnegative or it is nonpositive. But in the second case $\{v_0 > 0\}$ would be empty, which in turns gives $C(x) = 0$. It thus follows that v_0 is nonnegative, which in turn implies $\{v_0 > 0\} = \mathbb{R}^2 \setminus K_0$. In particular we have $C(x) = 1$, which is also a contradiction.

We thus have established that

$$\lim_{r\downarrow 0} O(x,r) = 0$$

for every $x \in \mathbb{R}^2$.

Step 3. Consider now $x \in K$ and let $(K_0, u_0, \{p_{kl}\})$ be a blow-up of (K, u) at x. Let moreover v_0 be as above. We distinguish two possibilities:

(a) $\{v_0 > 0\}$ is not empty;
(b) $\{v_0 > 0\}$ is empty.

In case (a), being $C(x) = 0$, we would have $|\nabla v_0| = |\nabla u_0| = 0$ on a nontrivial open set. It therefore follows that ∇u_0 vanishes identically on at least one connected component of $\mathbb{R}^2 \setminus K_0$. Since K_0 is nontrivial by the density lower bound, K_0 must disconnect $\mathbb{R}^2$ (cf. Corollary 4.4.4). In particular it follows that $(K_0, u_0, \{p_{kl}\})$ is either a pure jump or a triple junction. The ε-regularity theory for these two cases allows then to conclude the main claim of the proposition.

Step 4. We are now ready to conclude the proof of the Proposition: we just need to handle case (b). Consider $(K_0, u_0, \{p_{kl}\})$ and v_0 as above. Consider in addition the closure H of $\{v > m_0\}$ and observe that $H \cap B_r(x)$ is connected for every $r > 0$ by Proposition 4.8.1, while it also contains x. Consider now its rescalings $H_j := r_j^{-1}(H - x)$ and assume, up to subsequences, that it converges locally in the sense of Hausdorff to some closed set H_0. H_0 is connected and moreover it contains the origin. Observe that $v_0 \geq 0$ on H_0. Observe that, since we are in case (b) above, $v_0 \leq 0$. Therefore, $v_0 = 0$ on H_0. Now, if exists $z \in H_0 \setminus K_0$, then v_0 is harmonic in a neighborhood of z, and hence constant. We can thus argue as in the previous step to conclude that $(K_0, u_0, \{p_{kl}\})$ is either a pure jump or a triple junction. Otherwise we have $H_0 \subset K_0$. Since H_0 must be unbounded, we conclude that it contains some regular jump point z. Let $B_\rho(z)$ be such that $B_\rho(z) \cap H_0$ is a smooth arc γ and denote by $B^+_\rho(z)$ and $B^-_\rho(z)$ the connected components of $B_\rho(z) \setminus H_0$. By the unique continuation for harmonic functions at smooth boundaries, ∇u_0^+ cannot vanish identically on γ, unless u_0 is constant on $B^+_\rho(z)$. The latter situation however falls back in what already analyzed in Step 3 and we can ignore it. We thus conclude that ∇v_0^+ does not vanish indentically on γ and likewise ∇v_0^- does not vanish indentically on γ either, which means of course that $\frac{\partial v_0^\pm}{\partial n}$ do not vanish identically on γ. By possibly changing z and making ρ smaller we can assume that they do not vanish on γ at all (recall that $v_0 = 0$ on H_0).

Consider now that, by the ε-regularity theory, $K_{x,r_j} \cap B_\rho(z)$ is converging smoothly to $\gamma \cap B_\rho(z)$. So for a sufficiently large j it is an arc γ_j very close to γ. Now, the value of v_j

over γ_j is converging to 0. On the other hand we also know that $v_j \le 0$ on γ_j, because $v \le m_0$ on K.

Let now B_j^+ and B_j^- be the two connected components in which $B_\rho(z)$ is divided by γ_j. Consider that $|\frac{\partial v_j^\pm}{\partial n}| \ge c > 0$ on γ_j for some constant c. At the same time the second derivatives $D^2 v_j$ are uniformly bounded on the $B_j^\pm$. Observe that $\frac{\partial v_j^\pm}{\partial n}$ cannot change sign on γ_j, which is connected. We thus can examine the following cases, depending on their signs:

- $\frac{\partial v_j^+}{\partial n} \ge c > 0$ on γ_j or $\frac{\partial v_j^-}{\partial n} \ge c > 0$ on γ_j, for j sufficiently large. Thanks to the uniform bound on the second derivatives and to the smooth convergence towards γ, we can choose ρ sufficiently small but independent of j, so that $B_\rho^+(z)$ (or $B_\rho^-(z)$), is contained in H_0, which is a contradiction, because H_0 does not contain interior points.
- $\frac{\partial v_j^+}{\partial n} \le -c$ and $\frac{\partial v_j^-}{\partial n} \le -c < 0$ on γ_j for j sufficiently large. Again thanks to the bound on the second derivatives, if we then choose ρ sufficiently small and j sufficiently large, we must have $v_j < 0$ on B_j^+ and B_j^-. In particular, since H_j is the closure of $\{v_j > 0\}$, we conclude that $H_j \cap B_\rho(z) = \emptyset$. But that is also a contradiction because z is in H_0, which is the Hausdorff limit of H_j in $\overline{B}_\rho(z)$.

4.10. Proof of Theorem 4.1.1

In this section we complete the proof of Theorem 4.1.1. We thus fix a nonelementary global minimizer (K, u), an harmonic conjugate v as in Assumption 4.9.1, and set $m_0 := \max_K v$. Because of Corollary 4.9.2 we just need to show that $m_0 = 0$. Towards a contradiction we assume $m_0 > 0$ and we can now use Proposition 4.9.3 to fix a bounded connected component G of K with $\mathcal{H}^1(G) > 0$ and on which v takes the constant value m_0. We next recall that, because of Proposition 4.9.3 and Corollary 4.4.2, every point $p \in G$ can be assigned to one of the following three categories:

- Pure jump points p: for each of them there is a disk $B_\rho(p)$ such that $B_\rho(p) \cap K = B_\rho(p) \cap G$ is a smooth arc subdviding $B_\rho(p)$ into two topological disks. From Proposition 2.5.2 we gather immediately that this arc is smooth.
- Triple junctions p: for each of them there is a disk $B_\rho(p)$ such that $G \cap B_\rho(p) = K \cap B_\rho(p)$ is diffeomorphic to three radii of $B_\rho(p)$ joining at p at 120 degrees.
- Terminal points p (according to Corollary 4.4.2 then $p \notin \partial\{v > m_0\}$ in this case).

Obviously the triple junctions are countably many, because they form a discrete set: if x is a triple junction, by the ε-regularity theorem already established there is a punctured disk $B_r(x) \setminus \{x\}$ in which the set K is regular, in particular x is the only triple junction in $B_r(x)$. However, we caution the reader that for the terminal points the only information available is that they form a compact set of $\mathcal{H}^1$ measure zero.

4.10.1. Touring G counterclockwise. Without loss of generality we will assume that $\mathcal{H}^1(G) = \pi$ (this can be achieved by a simple rescaling argument). We now wish to find a surjective Lipschitz map $\alpha : \mathbb{S}^1 \to G$ with the following properties:

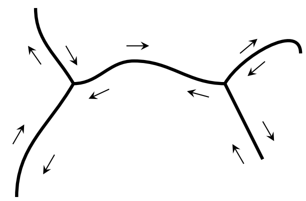

FIGURE 8. The picture is a visualization of the map α in case the number of maximal smooth open subarcs of G is finite. The map is particularly easy to define if the arcs are finitely many and very regular: in that case α is the limit as $t \downarrow 0$ of the clockwise arc-length parametrizations of the sets $\{x : \mathrm{dist}\,(x, G) = t\}$.

(i) Each jump point of G has exactly two counterimages;
(ii) Each triple junction point of G has exactly three counterimages;
(iii) Each terminal point has exactly one counterimage.

The idea is that such a map "goes around G". In fact it can be shown that, up to a change of phase in $\mathbb{S}^1$ and of orientation, the map is unique. We want then to select the one which "goes around G clockwise".

Before detailing the construction of this map, it is useful to break down G as the union of maximal smooth open subarcs: a C^∞ open subarc of G is the image of a smooth injective $\eta : (0, \ell) \to K$, parametrized by arc-length, and a maximal one is a C^∞ closed subarc of G which is not a strict subset of any other closed subarc. Observe that any maximal smooth open subarc consist only of jump points and has a unique arc-length parametrization up to orientation. If $\eta : (0, \ell) \to K$ is one such arc-length parametrization, arbitrarily chosen, then η can be extended continuously to both 0 and ℓ. The corresponding values, which we will call extrema of the subarc, cannot be equal (G contains no loops by Lemma 4.3.1) and cannot be jump points (otherwise the subarc would not be a maximal smooth one). Therefore, they are either terminal points, or triple junctions (and of course one of them could be a triple junction, while the other is a terminal point). From now on when treating a maximal smooth subarc of G we will always understand that its parametrization includes the two endpoints, even though the extension is not at all guaranteed to be smooth (the best we can say is that it is Lipschitz continuous at extrema which are terminal points, and C^2 at extrema which are triple junctions). The underlying idea about the map α is quite simple to define directly when the number of smooth maximal arcs is finite. We do not give a precise definition of the map in this case, since it can be easily inferred from the general construction below, we rather refer to reader to Figure 8 for a visual description of what α does.

We now describe in the detail the algorithm which produces the map α.

Construction of the map α. Fix a maximal smooth subarc (its choice is not important). We denote it by G_0 and let p_0 the point which divides it into two arcs of equal length. Without loss of generality we assume that $p_0 = 0$ and that the tangent to η_0 at 0 is horizontal. We then fix an arclength parametrization $\eta_0 : [-L_0, L_0] \to \mathbb{R}^2$ with $\eta_0(0) = 0$

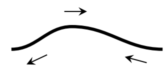

FIGURE 9. The map α_0 going around G_0.

and $\dot\eta_0(0) = (1,0)$. We start defining a map $\alpha_0 : [-L_0, 3L_0] \to \mathbb{R}^2$ as follows:

$$\alpha_0(\theta) = \begin{cases} \eta_0(\theta) & \text{if } -L_0 \leq \theta \leq L_0 \\ \eta_0(2L_0 - \theta) & \text{otherwise}. \end{cases}$$

α_0 is a map as in Figure 9.

If both extrema of G_0 are terminal points, then $G = \overline{G}_0$ and thus $L_0 = \frac{\pi}{2}$. In this case the map α is given by α_0, after we indentify $-\frac{\pi}{2}$ with $\frac{3\pi}{2}$ so that the domain of α is $\mathbb{S}^1$.

Otherwise, at least one of the extrema of η_0 is a triple junction. If only one of them is, we consider the two additional maximal subarcs which have that particular extremum in common, if both of them are, then we consider the four additional subarcs. Either way, the union of these subarcs and the initial one η_0 will be denoted by G_1. G_1 is, schematically, a tree with finitely many nodes, in the second alternative it will look like the set in Figure 8. We now define a new map $\alpha_1 : [-a_1, b_1] \to \mathbb{R}^2$ which is "going around G_1 clockwise" by putting together pairs of oppositely oriented parametrizations of each maximal subarc forming G_1. The parametrizations are chosen and joined in a unique way, once we prescribe that $[-L_0, L_0] \subset [-a_1, b_1]$ and that it must coincide with α_0 on $[-L_0, L_0]$. Note that G_1 has a finite number of terminal points. If all of them are terminal points of K we then stop the procedure. Otherwise we select all the ones which are triple junctions and add to G_1 all the maximal subarcs that have one of them as extrema. This forms G_2 and we can then construct the corresponding map α_2. We keep iterating this algorithm: if it stops after finitely many times N, we then set $\alpha = \alpha_N$. If the procedure never stops, then we can easily check that α_i converges to a unique map α (recall that $\mathcal{H}^1(G) < \infty$, and that the maximal subarcs are parametrized by arc-length).

Surjectivity of α. Once we show that α is surjective on G, it is a simple fact that it satisfies all the properties above.

In order to show its surjectivity, let $\tilde G$ be the image of α and observe at first the following property:

(R1) if $p, q \in G$ are two triple junctions extrema of the same maximal open subarc of G, then $p \in \tilde G$ if and only if $q \in \tilde G$.

In fact, assume $p \in \tilde G$. If $p \in G_i$ for some i, then certainly $q \in G_{i+1} \subset \tilde G$ by construction. On the other hand if p were not included in some G_i, then there would be a sequence of points $p_i \in G_i \setminus G_{i-1}$ converging to p. p_i would belong to an arc η_i added at the i-th stage. Since the total length of $\tilde G$ is finite, the length L_i must converge to 0 as $i \uparrow \infty$. Note however that each such L_i must have at least one extremum which is a triple junction. But then p would be a limit of triple junctions, which is not possible since the latter are isolated. We next observe the following further simple consequence of the same idea:

(R2) If $p \in \tilde G$ is a triple junction, then $\tilde G$ contains all three maximal smooth open subarcs of K which have p as one endpoint.

Fix now a point $q \in G$ and observe that, since G is a connected, there is an injective arc $\gamma : [0,L] \to G$ such that $\gamma(0) = 0$ and $\gamma(L) = q$, which we can assume to be parametrized by arc-length. If $\gamma([0,L])$ contains no triple junctions, then γ is contained in $\overline{G}_0$ and hence $q \in \overline{G}_0 \subset \tilde G$. Otherwise let $P \subset [0,L]$ be the subset of points s such that $\gamma(s)$ is a triple junction. This set must be discrete in $[0,L)$: we can order its elements as $p_1 < p_2 < \ldots$. Notice that p_1 is an extremum of G_0 and thus contained in $\tilde G$ by (R1). In particular by (R1) and (R2) we must have that $p_2 \in \tilde G$ as well. By induction all p_k belong to $\tilde G$. But then we have two possibilities:

- The number of p_k's is finite: if p_N is the largest of them, then $q = \gamma(L)$ is in the closure of a maximal smooth open subarc with endpoint p_N: by (R2) this subarc belongs to $\tilde G$ and since the latter is closed, $q \in \tilde G$;
- The number is infinite: in this case $p_k \to q$ and since $\tilde G$ is closed and $p_k \in \tilde G$ for every k, we conclude again $q \in \tilde G$.

4.10.2. Final contradiction. With the map α at hand, we define

$$J := \{t \in \mathbb{S}^1 : \alpha(t) \text{ is a jump point}\}\,.$$

For each $t \in J$ we then let $e(t) := \dot\alpha(t)$. Recall that the latter is well defined (as α is indeed smooth on J) and has unit norm. We then denote by $n(t)$ its counterclockwise rotation by 90 degrees. Recall that for every $t \in J$ there is one (and only one) other element $s \in J$ such that $\alpha(s) = \alpha(t)$. By construction we then have $e(s) = -e(t)$ and $n(s) = -n(t)$. Next recall that for every $t \in J$, if $p = \alpha(t)$, then there is a disk $B_\rho(p)$ in which G is a smooth arc dividing $B_\rho(p)$ in two topological disks. Observe that the restrictions of v and u to any of these two open sets have smooth extensions to its closure (in particular smooth extensions up to $G \cap B_\rho(p)$). We can therefore define

$$g(t) := \lim_{\delta\downarrow 0} n(t)\cdot\nabla v(\alpha(t)+\delta n(t))\,, \tag{4.10.1}$$

$$h(t) := \lim_{\delta\downarrow 0} u(\alpha(t)+\delta n(t))\,. \tag{4.10.2}$$

Clearly $g(t)$ tells us whether at the point p the function v is decreasing or increasing in the direction $n(t)$. But it is also easy to check that $h'(t) = g(t)$ (recall that $e(t) = \dot\alpha(t)$). Observe in addition that h has a continuous extension to $\mathbb{S}^1$. Indeed for a point $t \notin J$ we just have two possibilities.

- $\alpha(t)$ is a triple junction. In this case t is an isolated point of $\mathbb{S}^1 \setminus J$ and h can be continuously extended at t by the regularity theory at triple junctions.
- $\alpha(t)$ is a terminal point. But then by Proposition 4.6.1 u is continuous at $\alpha(t)$ and

$$|u(\alpha(t)) - h(s)| \le C|t-s|^{1/2}$$

for all $s \in J$ and a universal constant C.

The final contradiction argument then hinges on the following pivotal lemma.

LEMMA 4.10.1. *The continuous function $g : J \to \mathbb{R}$ defined above has the following properties:*

(i) $\{g = 0\}$ *has empty interior;*
(ii) *The closure S^+ of $\{g > 0\}$ in $\mathbb{S}^1$ is a connected arc.*

Before coming to the proof of Lemma 4.10.1 we show how to conclude the proof of Theorem 4.1.1 from it. First of all, since $v(\alpha(t)) = m_0$ for every t, when $t \in \{g > 0\}$ clearly there is a $\varepsilon > 0$ with the property that $v(\alpha(t) + \sigma n(t)) > m_0$ for every $\sigma \in (0, \varepsilon)$. In particular we conclude that $\alpha(t) \in \partial\{v > m_0\}$. Moreover $\{g > 0\}$ is dense in S^+ and therefore $\alpha(S^+) \subset \partial\{v > m_0\}$. Proposition 4.9.3 (ii) implies that $\alpha(S^+)$ consists all of jump points and triple junctions. Since G has at least two terminal points (cf. (i) Proposition 4.7.2), we conclude that the complement of S^+ is not empty, and we will denote it by S^-.

At this point it is convenient to change phase to the parametrization α so that $\alpha([0, M]) = S^+$ and $\alpha(M, 2\pi) = S^-$ (cf. (ii) Lemma 4.10.1). Clearly, as $h' = g$, the function h is increasing on S^+ and decreasing on S^-: $h(0)$ is then the minimum and $h(M)$ is the maximum. Observe also that in both intervals $(0, M)$ and $(M, 2\pi)$ the monotonicity of h is strict, because the zero set of h' has empty interior.

Since $[0, M]$ contains no terminal points, we can subdivide $[0, M]$ as $[0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_N, M]$, where each t_i is a triple junction and each open interval $(t_0, t_1), \ldots, (t_N, M)$ contains no triple junctions. Observe that α is injective on $[0, M]$ (because $\alpha([0, M])$ contains no terminal point).

The situation is particularly easy when $(0, M)$ itself contains no triple junctions. $\alpha((0, M))$ has a second, disjoint, counterimage $(s_0, s_0 + M)$. This counterimage must be contained in S^-. We therefore can draw the following conclusions:

- h is strictly decreasing on $[s_0, s_0 + M]$.
- $h(0) < h(s_0 + M) < h(s_0) < h(M)$ (the inequalities are justified as h is strictly decreasing on S^-, for the first we also use that $h(0) = h(2\pi)$);
- $\alpha(s) = \alpha(s_0 + M - s)$ (because recall that $\alpha|_{[0,M]}$ and $\alpha|_{[s_0,s_0+M]}$ are arclength parametrizations of the same smooth arc with opposite orientations.

Now, define the function

$$k(s) := h(s) - h(s_0 + M - s) \tag{4.10.3}$$

and note that k is strictly increasing on $[0, M]$ while $k(0) < 0 < k(M)$. In particular, $k(0) = [u](\alpha(0))$ and $k(M) = -[u](\alpha(M))$. Therefore k must have a zero z_0 in $(0, M)$. Such zero z_0 corresponds to a pure jump point $p = \alpha(z_0)$ with the property that the two traces $u^+(p)$ and $u^-(p)$ on the two sides of K at p are equal. But since u is an *absolute minimizer* this is not possible (note that here we are using the property that at a jump point the one-sided traces of u have to differ: this property does not hold for restricted minimizers).

We now analyze the more general case in which $(0, M)$ is not contained in J. We then let $t_1, \ldots, t_N$ be as above and set $t_0 = 0$ and $t_{N+1} = M$. We then find points $s_{N+1} < s_N^+ < s_N^- < s_{N-1}^+ < s_{N-1}^- < \ldots < s_1^- < s_0$ such that:

- α maps $[s_{N+1}, s_N^+]$ onto $\alpha([t_N, t_{N+1}])$, $[s_1^-, s_0]$ onto $\alpha([t_0, t_1])$, and $[s_j^-, s_{j-1}^+]$ onto $\alpha([t_{j-1}, t_j])$;

- $\alpha(s_{N+1}) = \alpha(t_{N+1})$, $\alpha(s_0) = \alpha(t_0)$, and $\alpha(s_j^-) = \alpha(s_j^+) = \alpha(t_j)$.

Moreover by (ii) Lemma 4.10.1:

- $h(t_0) < h(s_0)$;
- $h(s_{N+1}) < h(t_{N+1})$;
- $h(t_0) < h(t_1) < \ldots < h(t_{N+1})$;
- $h(s_0) < h(s_1^-) < h(s_1^+) < \ldots < h(s_N^-) < h(s_N^+) < h(s_{N+1})$.

In order to streamline the rest of the discussion we use the convention that $s_0^+ = s_0^- = s_0$.

Assume now there is $j \leq N$ such that $h(s_j^-) \leq h(t_j)$. In that case we let j be the smallest of them. Observe then that $h(s_{j-1}^+) \geq h(s_{j-1}^-) > h(t_{j-1})$. We then set $a := t_{j-1}$, $a + d := t_j$, $b := s_{j-1}^+$ and observe that $s_j^- = b + d$. Our function k is now defined on $[a, a + d]$ as

$$k(s) = h(s) - h(b + d - (s - a)) \tag{4.10.4}$$

Once again k is strictly increasing on $[a, a + d]$ and moreover $h(a) < 0 \leq h(a + d)$. Then h must have a zero z_0 in $(a, a + d]$. If this zero is smaller than $a + d$, then we are in the exact same situation as in the case analyzed when $\alpha((0, M_0))$ contains no triple junction. In case the zero is $a + d$, we then find a triple junction point at which two of the three traces of u coincide: this again (thanks to the regularity theory at triple junctions) contradicts the absolute minimality of u.

If there is no $j \leq N$ such that $h(s_j^-) \leq h(t_j)$, then obviously $h(s_N^+) > h(s_N^-) > h(t_N)$. But then the exact argument just given can be replicated defining $a = t_N$, $b + d = t_{N+1}$, $s_N^+ = b$. Then $s_{N+1} = b+d$ and defining k as in (4.10.4) we find this time $h(a) < 0 < h(a+d)$. Thus we can repeat the very same argument above and conclude that this case too leads to a contradiction.

4.10.3. Proof of Lemma 4.10.1. In order to conclude our proof of Theorem 4.1.1 we are thus left with giving an argument for Lemma 4.10.1. First of all, if $\{g = 0\}$ contains an interior point, then there is a jump point $p \in G$ and a neighborhood $B_\rho(p)$ with the properties that:

- $G \cap B_\rho(p)$ divides $B_\rho(p)$ into two topological disks, B^+ and B^-;
- the trace of ∇u on $G \cap B_\rho(p)$ from one of the two sides B^+ or B^- vanishes identically (as $\frac{\partial u}{\partial n}^\pm = 0$ on K).

Assume without loss of generality that the side is B^+. The unique continuation of harmonic functions then implies that ∇v vanishes identically on B^+. But then u would be constant on B^+ and we know this is not possible in view of Corollary 4.4.4 being (K, u) nonelementary.

In order to prove the second statement of the lemma, consider first $t \in J$ with $g(t) > 0$ and let s be the only other point on $\mathbb{S}^1$ such that $\alpha(s) = \alpha(t)$. We want to show that $g(s) \leq 0$. Assume indeed $g(s) > 0$. We can then select $\delta > 0$ such that v is strictly increasing on the two segments $[\alpha(s), \alpha(s) + \delta n(s)]$ and $[\alpha(t), \alpha(t) + \delta n(t)]$. For any $m > m_0 = v(\alpha(s)) = v(\alpha(t)$ sufficiently close to m_0 we find then $a, b \in (0, \delta)$ such that

$$v(\alpha(t) + an(t)) = v(\alpha(s) + bn(s)) = m\,.$$

For an appropriately chosen m we can apply the conclusion (v) of Proposition 4.8.1. So the two points $p = \alpha(t) + an(t)$ and $q = \alpha(s) + bn(s)$ lie in the properly embedded unbounded line $\{v = m\}$. In particular p and q determine a bounded Jordan arc β on $\{v = m\}$. The union of the arc β with the segment $[p, q]$ (which is directed along $n(t) = -n(s)$ and hence contains the point $\alpha(t) = \alpha(s)$) is a simple curve, which by the Jordan's Theorem bounds a disk D. Note that on ∂D we have $v \geq m_0$ and thus $v \geq m_0$ on D. In particular $D \cap K \subset \partial\{v > m_0\}$. But $K \cap \partial D$ consists only of the point $\alpha(s)$. In particular it would follow that $D \cap K$ contains a terminal point, contradicting Proposition 4.9.3.

Having established the claim above, we are now ready to prove the second conclusion of the lemma. Towards a contradiction we assume that there are points t_1, t_2, t_3, t_4 in $J \subset \mathbb{S}^1$ with the property that t_2 and t_4 belong to the two distinct arcs of $\mathbb{S}^1$ delimited by t_1 and t_3 and at the same time $g(t_3), g(t_1) > 0$ and $g(t_2), g(t_4) < 0$. Moreover, because of the first statement of the lemma, by perturbing t_1 and t_3 we can assume that, if s_1 and s_3 are the two other points such that $\alpha(s_1) = \alpha(t_1)$ and $\alpha(s_3) = \alpha(t_3)$, then $g(s_1), g(s_3) < 0$

Consider now as above a $\delta > 0$ such that v is strictly increasing on both segments $S_1 = [\alpha(t_1), \alpha(t_1) + \delta n(t_1)]$ and $S_3 = [\alpha(t_3), \alpha(t_3) + \delta n(t_3)]$. Fix as above an $m > m_0$ to which the conclusion (v) of Proposition 4.8.1 apply and let p_1, p_3 be the only intersections of S_1 and S_3 with $\{v = m\}$. Again as above let $\beta \subset \{v = m\}$ be the Jordan arc delimited by p_1 and p_3. Meanwhile let γ be a Jordan arc connecting $\alpha(t_1)$ and $\alpha(t_3)$ in G (which exists by Lemma D.0.2). The curve $\gamma \cup \beta \cup [\alpha(t_1), p_1)] \cup [\alpha(t_2), p_2]$ is simple and therefore it delimits a disk D. Arguing as above, $v \geq m_0$ in D and thus D cannot intersect K. In particular it follows that $\gamma = K \cap \partial D$ contains a finite number of triple junctions. So we can chop γ as the union of closed arcs

$$\gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_N \cup \gamma_{N+1}$$

such that:

- The right endpoint p_i of γ_i is the left endpoint q_{i+1} of γ_{i+1};
- For $i \in \{1, \ldots, N\}$ each γ_i is the closure of a maximal smooth open subarc γ_i joining the triple junctions q_i and p_i;
- γ_0 joins $q_0 = \alpha(t_1)$ to the triple junction $p_0 = q_1$ while γ_{N+1} joins the triple junction $p_N = q_{N+1}$ the $p_{N+1} = \alpha(t_3)$, but both γ_0 and γ_{N+1} contain no other jump points.

We parametrize γ by arclength so that, while we are following it from q_0 to p_{N+1}, the counterclockwise rotation of $\dot\gamma$ by 90 degrees points "inwards" with respect to D

It is now easy to see that there is an interval $I \subset \mathbb{S}^1$ over which α is injective and $\alpha(I) = \gamma$ and that we can choose it so that $n(s)$ always agrees with the counterclockwise rotation of $\dot\gamma(\sigma)$ for the only σ with $\gamma(\sigma) = \alpha(s)$. But then clearly the two extrema of the intervals must be t_1 and t_3, because $n(s_1)$ and $n(s_3)$ point outwards. We conclude that g is never negative over the interval I. So neither t_2 nor t_4 can belong to it, and rather we have $t_2, t_4 \in \mathbb{S}^1 \setminus I$. This is however precisely in contradiction to our initial assumption.

CHAPTER 5

Epsilon regularity at the cracktip

This chapter is devoted to the final part in the proof of Theorem 1.2.3. In view of the previous chapters, and in particular of Corollary 4.1.2 and of the ε-regularity of pure jumps, it will suffice to prove the following statement.

THEOREM 5.0.1. *There are positive constants ε_0, α_0 and r_0 with the following property. Assume that (K, u) is a critical point of E_λ in B_3 and that:*

(i) (1.0.2) *holds;*
(ii) *$K \cap B_2$ consists of a single Jordan arc γ with endpoints 0 and $p \in \partial B_2$;*
(iii) *γ is $C^{1,1}_{loc}$ in $B_2 \setminus \{0\}$;*
(iv) $\mathrm{dist}_H(K, [0,p]) \le \varepsilon_0$.

Then, up to a suitable rotation of coordinates, $K \cap [0, r_0]^2$ is given by $\{(t, \psi(t)) : t \in [0, r_0]\}$ for some C^{1,α_0} function $\psi : [0, r_0] \to \mathbb{R}$ with $\psi(0) = \psi'(0) = 0$. If $\lambda = 0$ then, in addition, $\psi \in C^{2,\alpha_0}$ and $\psi''(0) = 0$.

In order to prove the results in Theorem 5.0.1 we need some preparatory work. Thanks to Lemma 5.0.4 u is continuous at 0 and we may assume without loss of generality $u(0) = 0$. Thus, by Corollary 4.1.2 and the ε-regularity at pure jumps we infer (cf. again Lemma 5.0.4) that[1]

$$|u(x)| + |x||\nabla u(x)| \lesssim |x|^{1/2} \tag{5.0.1}$$

Moreover, in view of Proposition 2.5.2 u has $C^{1,1^-}$ extensions on each side of $K \cap B_2$[2] and the variational identities (1.5.2)-(1.5.4) imply the following three conditions

$$\Delta u = \lambda(u - g) \qquad \text{on } B_2 \setminus K \tag{5.0.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } K \tag{5.0.3}$$

$$\kappa = -(|\nabla u^+|^2 - |\nabla u^-|^2) - \lambda(|u^+ - g_K|^2 - |u^- - g_K|^2) \qquad \mathcal{H}^1 \text{ a.e. on } K \cap B_2. \tag{5.0.4}$$

We stress that the equivalence in Proposition 2.5.2 has been obtained assuming K to be locally a $C^{1,1}$ graph. We do not know that this property holds at the origin, the loose end, but we know it on $K \setminus \{0\}$. Indeed, Corollary 4.1.2 provides only a parametrization of K in polar coordinates smooth up to the tip excluded, i.e. $\gamma \in C^{1,1}((0,2))$ such that

$$K = \{\gamma : [0,2] \to B_2 : \gamma(r) = r(\cos\alpha(r), \sin\alpha(r))\}. \tag{5.0.5}$$

[1] Here and in what follows we adopt the notation $\varphi \lesssim \psi$ if there is $C > 0$ such that $\varphi \le C\psi$.
[2] namely, u has extensions on each side of $K \cap B_2$ that are $C^{1,\alpha}$ for every $\alpha < 1$.

On the other hand, it is reductive to focus only on (5.0.3)-(5.0.4) and it turns out that we can deduce more pieces of information from (1.5.2)-(1.5.4) using Proposition 2.5.4, which will play a key role in our analysis. In particular, we need the following simple consequence of Corollary 2.5.5 in which we consider the situation described in item (ii) above of Theorem 5.0.1, i.e. $K \cap \partial B_r$ consists of a single point. We can then take a suitable linear combination of (2.5.11) and (2.5.12) (in fact we subtract the second from the first) to derive a boundary integral identity, which does not involve the set K in case $\lambda = 0$. Recall the notation $n(x) = {}^{x}/_{|x|}$ and $\tau(x) = {}^{x^\perp}/_{|x|}$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

COROLLARY 5.0.2. *Let (K, u) be as in Theorem 5.0.1, and assume that $K \cap \partial B_r = \{p\}$. Then, for a.e. $r \in (0, \operatorname{dist}(0, \partial\Omega))$ it is true that*

$$
\begin{aligned}
0 = &\int_{\partial B_r \setminus \{p\}} \left(|\nabla u|^2 n \cdot \tau(p) + 2\frac{\partial u}{\partial n} \nabla u \cdot (\tau - \tau(p)) \right) d\mathcal{H}^1 \\
&- 2\lambda \int_{B_r \setminus K} (u - g)(\tau - \tau(p)) \cdot \nabla u \, dx + \lambda \int_{B_r \cap K} \left(|u^+ - g_K|^2 - |u^- - g_K|^2 \right) \tau(p) \cdot \nu \, d\mathcal{H}^1 .
\end{aligned}
\tag{5.0.6}
$$

It is also convenient to define a natural substitute of the harmonic conjugate in case $\lambda > 0$. We argue similarly to Proposition 4.5.1 and consider an auxiliary function u_a which satisfies

$$\Delta u_a = -\lambda(u - g) \qquad \text{on } B_2$$

and

$$u_a(0) = 0\,, \quad \nabla u_a(0) = 0. \tag{5.0.7}$$

In order to obtain a canonical choice, we fix a cut-off function φ supported in B_3 and identically 1 on B_2 and we let

$$\bar{u}_a = \lambda \Gamma * (\varphi(u - g))\,,$$

where $\Gamma(x) = -\frac{1}{2\pi} \log |x|$ is the fundamental solution of the Laplace equation. Clearly $\bar{u}_a \in H^2 \cap C^{1,1^-}(B_3)$ by elliptic regularity (namely $\bar{u}_a \in C^{1,\alpha}(B_2)$ for all $\alpha \in (0, 1)$). We then set $u_a(x) = \bar{u}_a(x) - \bar{u}_a(0) - \nabla \bar{u}_a(0) \cdot x$. Observe that, with this canonical choice, $u_a = 0$ if $\lambda = 0$.

By the very definition of u_a, the L^2 vector field $\nabla(u + u_a)^\perp$ is curl free on B_2 in the sense of distributions. By mollification we find a potential $w \in H^1_{loc}(B_2)$, i.e. $\nabla w = \nabla(u + u_a)^\perp$, which is harmonic and smooth on $B_2 \setminus K$. The following facts follow from simple modifications of arguments already used in the previous sections, which we anyway report for the reader's convenience.

LEMMA 5.0.3. *$w \in C^{0,1/2}(B_2)$ and we can normalize it so that $w(0) = 0$. Let $\gamma \in C^{1,1}((0, 2))$ be as in (5.0.5), and define*

$$h_1(r) := \int_0^r \nabla u_a(\gamma(\rho)) \cdot \dot{\gamma}^\perp(\rho) d\rho \tag{5.0.8}$$

and

$$
\begin{aligned}
h_2 &:= 2(\nabla u^+ - \nabla u^-)\cdot \nabla u_a - \lambda(|u^+ - g_K|^2 - |u^- - g_K|^2)\\
&= 2(\nabla u^+ - \nabla u^-)\cdot \nabla u_a - \lambda(|u^+|^2 - |u^-|^2 - 2(u^+ - u^-)g_K)\,.
\end{aligned}
\tag{5.0.9}
$$

Then

$$
\begin{cases}
\Delta w = 0 & \text{on } B_2\setminus K\\
w = h_1 & \text{on } K\\
\kappa = -(|\nabla w^+|^2 - |\nabla w^-|^2) + h_2 & \mathcal{H}^1 \text{ a.e. on } K\cap B_2,
\end{cases}
\tag{5.0.10}
$$

and the functions h_1 and h_2 satisfy the growth estimates[3]

$$
h_1 \in C^{1,1^-}_{\rm loc}([0,2)),\ \text{and} \qquad |h_1| + r|h_1'| \lesssim r^{2^-} \tag{5.0.11}
$$

$$
h_2 \in L^\infty_{\rm loc}(B_2),\ \text{and} \qquad |h_2(x)| \lesssim |x|^{1/2^-}. \tag{5.0.12}
$$

when $\lambda > 0$, while they vanish identically if $\lambda = 0$.

Finally, if $r\in(0,2)$ and $\partial B_r\cap K = \{p\}$, then the following identity holds

$$
\begin{aligned}
&2\int_{B_r} \nabla u_a^T\cdot D\tau\,\nabla u_a\\
&= \int_{\partial B_r}\Big(|\nabla u_a|^2 n\cdot\tau(p) + 2\frac{\partial u_a}{\partial n}\nabla u_a\cdot(\tau-\tau(p))\Big)d\mathcal{H}^1 + 2\lambda\int_{B_r}(u-g)(\tau-\tau(p))\cdot\nabla u_a\,,
\end{aligned}
\tag{5.0.13}
$$

where $\tau(x) = \frac{x^\perp}{|x|}$ (a function which belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$ for every $p\in[1,2)$).

Proof. Observe that, since $\|\nabla u_a\|_{C^0}\le C$, we have

$$
\int_{B_r(x)}|\nabla w|^2 = \int_{B_r(x)}|\nabla u + \nabla u_a|^2 \le 2\int_{B_r(x)}|\nabla u|^2 + Cr^2 \le Cr\,.
$$

In particular $w\in C^{0,1/2}$ follows from the usual Morrey's embedding. The equation $\Delta w = 0$ on $B_2\setminus K$ is obvious from the definition, while the Dirichlet condition $w|_K = h_1$ follows from $w(0)=0$ integrating ∇w along K. The last equation in (5.0.10) follows immediately from (5.0.4) using $|\nabla w^\pm|^2 = |\nabla u^\pm + \nabla u_a|^2$. The estimates (5.0.11)-(5.0.12) follow immediately from (5.0.1) and the formulas for h_1 and h_2. Finally, (5.0.13) is a simple integration by parts using $\Delta u_a = -\lambda(u-g)$. □

Lemma 5.0.4. *Under the assumptions of Theorem 5.0.1 u is continuous at 0 and moreover there are constants $C, r_1>0$ such that*

$$
|u(x)-u(0)| + |x||\nabla u(x)| \le C|x|^{1/2} \qquad \forall x\in B_{r_1}\setminus K\,. \tag{5.0.14}
$$

Proof. Consider a point $x\in(B_r\setminus B_{r/2})\setminus K$. We distinguish two situations:

[3] Here and in what follows we write $0\le\varphi(x)\lesssim(\psi(x))^{\beta^-}$ if $\varphi(x)\lesssim(\psi(x))^\alpha$ for all $\alpha<\beta$, with a constant depending on α in the last inequality.

- $B_{r/8}(x) \cap K = \emptyset$. Recalling that $\int_{B_{r/8}(x)} |\nabla u|^2 \leq Cr$ and using the mean value property for harmonic functions we immediately infer
$$|\nabla u(x)| \leq \frac{8^2}{\pi r^2} \int_{B_{r/8}(x)} |\nabla u| \leq \frac{8}{r} \left(\int_{B_{r/8}(x)} |\nabla u|^2 \right)^{1/2} \leq C r^{-1/2} \leq C|x|^{-1/2}.$$
- $B_{r/8}(x) \cap K \neq \emptyset$. We can then fix a point $\bar{x} \in B_{r/8}(x) \cap K$. We consider therefore $B_{r/4}(\bar{x}) \subseteq B_{3r/2} \setminus B_{r/8}$ and notice that we can apply the jump case of the ε-regularity theory to $B_{r/4}(\bar{x})$ because of Corollary 4.1.2, provided $|x| \leq r_1$ for a sufficiently small r_1. In particular the conclusion of the ε-regularity theory, classical estimates for the PDE (5.0.2) and (5.0.3), and the estimate
$$\int_{B_{r/4}(\bar{x})} |\nabla u|^2 \leq Cr,$$
we conclude again
$$|\nabla u(x)| \leq C r^{-1/2} \leq C|x|^{-1/2}.$$

We observe next that, again by the ε-regularity theory at pure jumps, for every $0 < r < r_1$ $\partial B_r \cap K$ consists of a single point. In particular, by the estimate on $|\nabla u|$, we conclude that $\operatorname{osc}(u, \partial B_r) \leq C r^{1/2}$. In particular, by the maximum principle Lemma 2.1.1, we conclude that
$$\operatorname{osc}(u, B_r) \leq C r^{1/2}.$$
In particular u is continuous at 0 and $|u(x) - u(0)| \leq C|x|^{1/2}$. □

5.1. Rescaling and reparametrization

Before starting our considerations, we introduce the model "tangent function" of a minimizer at a loose end. By Theorem 4.1.1, in polar coordinates the latter is given by the function
$$\operatorname{Rsq}(\phi, r) := \sqrt{\tfrac{2r}{\pi}} \cos \tfrac{\phi}{2} \tag{5.1.1}$$
with jump set K_{Rsq} equal to the open half line $\{(t,0) : t \in \mathbb{R}^+\}$ (in cartesian coordinates). Observe that Rsq is, up to the prefactor $\sqrt{\frac{2}{\pi}}$, the real part of a branch of the complex square root. We will likewise use the notation for its harmonic conjugate Isq, which is the imaginary part of the same branch, multiplied by the same prefactor, namely
$$\operatorname{Isq}(\phi, r) := \sqrt{\tfrac{2r}{\pi}} \sin \tfrac{\phi}{2}. \tag{5.1.2}$$

5.1.1. Rescalings. From now until the very last section, (K, u) will always denote a critical point of E_λ in B_2 satisfying the assumptions of Theorem 5.0.1. Keeping the notation introduced in (5.0.5), for $\rho > 0$ set
$$u^\rho(\phi, r) := \rho^{-1/2} u(\phi + \alpha(\rho r), \rho r), \tag{5.1.3}$$
$$\alpha^\rho(r) := \alpha(\rho r). \tag{5.1.4}$$

Lemma 5.1.1. *For every $\delta > 0$ the following holds.*

If $\lambda > 0$, then for every $\varepsilon > 0$ there is $\varepsilon_1 > 0$ such that, if (K, u) is as in Theorem 5.0.1 and α is as in (5.0.5) with $\varepsilon_0 \le \varepsilon_1$, then

$$\|u^\rho - \mathrm{Rsq}\|_{C^{1,1-\varepsilon}([0,2\pi]\times[1/2,2])} + \|\alpha^\rho\|_{C^{1,1-\varepsilon}([1/2,2])} \le \delta \qquad \forall\, \rho \le \frac{1}{4}\,. \tag{5.1.5}$$

If $\lambda = 0$, then for every $k \in \mathbb{N}$ there is $\varepsilon_1 > 0$ such that, under the very same assumptions,

$$\|u^\rho - \mathrm{Rsq}\|_{C^k([0,2\pi]\times[1/2,2])} + \|\alpha^\rho\|_{C^k([1/2,2])} \le \delta \qquad \forall\, \rho \le \frac{1}{4}\,. \tag{5.1.6}$$

Proof. First of all, we introduce the function $u^\rho(x) := \rho^{-1/2}u(\rho x)$ and the rescaled singular set $K_\rho := \frac{K}{\rho}$. Recall that (K_ρ, u^ρ) converges, up to subsequences, to a global minimizer of E_0, which we denote by (K_0, u_0). We also know that $|\nabla u(x)| \le C|x|^{-1/2}$ from Lemma 5.0.3. But then it follows easily that the rescaled set K_ρ satisfies uniform $C^{1,1}$ estimates on $B_1 \setminus B_{1/4}$. Hence the estimate of α^ρ in (5.1.5) simply follows from the fact that the curve parametrized by α^ρ is converging to the straight segment $[0,(1,0)]$, while α^ρ has a uniform $C^{1,1}$ bound. As for the other estimate in (5.1.5) it follows from classical regularity for the Neumann problem, using $\Delta u = \lambda(u-g)$ (and the uniform L^∞ bounds on u and g).

Next, observe that, for the case $\lambda = 0$, the rescaled pair (K_ρ, u^ρ) is still a minimizer of the Mumford-Shah functional, and the estimate (5.1.6) would follow (by compactness) once we show uniform $C^{k,1/2}$ estimates for both u^ρ and K_ρ in $B_1 \setminus B_{1/2}$. However, the latter is a simple bootstrapping process using the equation. Note for instance that Theorem 3.1.1 gives a uniform $C^{1,\alpha}$ bound on $K_\rho \cap (B_{2-\mu} \setminus B_\mu)$ for every $\mu \in (0,1)$. But then using the uniform L^2 bound for ∇u and classical estimates for the Neumann problem, we conclude uniform C^α estimates for ∇u in $B_{2-2\mu} \setminus (B_{2\mu} \cup K_\rho)$ for every $\mu \in (0, 1/2)$. We can now use the equation for the curvature κ to conclude uniform $C^{2,\alpha}$ estimates for $K_\rho \cap (B_{2-3\mu} \setminus B_{3\mu})$ for every $\mu \in (0, 1/3)$. In turn this implies uniform $C^{1,\alpha}$ estimates for ∇u in $B_{2-4\mu} \setminus (B_{4\mu} \cup K_\rho)$ for every $\mu \in (0, 1/4)$. This bootstrap argument can be repeated finitely many times, until we achieve $C^{k+1,\alpha}$ estimates. □

Corollary 5.1.2. *For every $\delta > 0$ the following holds.*

(a) *If $\lambda > 0$, then for every $\varepsilon > 0$ there is $\varepsilon_1 > 0$ such that, if (K, u) and α satisfy the assumptions of Theorem 5.0.1 with $\varepsilon_0 \le \varepsilon_1$, then*

$$\|r^{i-1/2}\partial_\phi^j\partial_r^i(u(\phi+\alpha(r), r) - \mathrm{Rsq}(\phi, r))\|_{C^{0,1-\varepsilon}([0,2\pi]\times(0,1/2))} \le \delta \quad \forall\, i+j \le 1, \tag{5.1.7}$$

$$\|r\alpha'(r)\|_{C^{0,1-\varepsilon}((0,1/2))} \le \delta\,, \tag{5.1.8}$$

(b) *If $\lambda = 0$, then for every $k \in \mathbb{N}$ there is $\varepsilon_1 > 0$ such that, if (K, u) and α satisfy the assumptions of Theorem 5.0.1 with $\varepsilon_0 \le \varepsilon_1$, then*

$$\sup_{[0,2\pi]\times(0,1/2)} r^{i-1/2}|\partial_\phi^j\partial_r^i(u(\phi+\alpha(r), r) - \mathrm{Rsq}(\phi, r))| \le \delta \quad \forall\, i+j \le k, \tag{5.1.9}$$

$$\sup_{(0,1/2)} r^i|\alpha^{(i)}(r)| \le \delta \qquad\qquad \forall\, i \le k\,. \tag{5.1.10}$$

Proof. We discuss only the case $\lambda = 0$, the other being analogous. Observe first that

$$(\alpha^\rho)^{(i)}(r) = \rho^i \alpha^{(i)}(\rho r)\,.$$

Taking the supremum in $r \in [1/2, 2]$ in the latter identity, we easily infer

$$\rho^i \|\alpha^{(i)}\|_{C^0([\rho/2, 2\rho])} = \|(\alpha^\rho)^{(i)}\|_{C^0([1/2,2])}\,,$$

and hence conclude (5.1.10) from Lemma 5.1.1.

Next, from (5.1.3) and the $1/2$-homogeneity of Rsq we conclude

$$u(\phi + \alpha(r), r) - \mathrm{Rsq}(\phi, r) = \rho^{1/2}\left(u^\rho\left(\phi, \tfrac{r}{\rho}\right) - \mathrm{Rsq}\left(\phi, \tfrac{r}{\rho}\right)\right)\,.$$

Differentiating the latter identity j times in θ and i times in r, we conclude

$$\partial_r^i \partial_\phi^j \left(u(\phi + \alpha(r), r) - \mathrm{Rsq}(\phi, r)\right) = \rho^{1/2 - i} \partial_r^i \partial_\phi^j \left(u^\rho - \mathrm{Rsq}\right)\left(\phi, \tfrac{r}{\rho}\right)$$

Substitute first $\rho = r$ and take then the supremum in ϕ and r to achieve (5.1.7), again from Lemma 5.1.1. □

5.1.2. Reparametrization. Following Simon's insight for studying the uniqueness of tangent cones to minimal surfaces [33] we next introduce the functions

$$\vartheta(t) := \alpha(e^{-t}) \tag{5.1.11}$$
$$\varrho(t) := e^{-t}(\cos\vartheta(t), \sin\vartheta(t)) \tag{5.1.12}$$
$$f(\phi, t) := e^{t/2}\, w(\phi + \vartheta(t), e^{-t}) = w^{e^{-t}}(\phi, 1)\,, \tag{5.1.13}$$
$$\mathrm{rsq}(\phi) := \mathrm{Rsq}(\phi, 1). \tag{5.1.14}$$
$$\mathrm{Isq}(\phi, r) := \sqrt{\tfrac{2r}{\pi}} \sin\left(\tfrac{\phi}{2}\right) \tag{5.1.15}$$
$$\mathrm{isq}(\phi) := \mathrm{Isq}(\phi, 1). \tag{5.1.16}$$

Note that the change of phase in the definition of f in (5.1.13) maps the set K onto the halfline $\{0\} \times [0, \infty)$.

In the next lemma we derive a system of partial differential equations for the functions f and ϑ, exploiting the Euler-Lagrange conditions satisfied by u and K (cf. (1.5.2) and (1.5.4)). The effect of the negative exponential reparametrization is that we will get an evolution equation for f. The claimed regularity for γ corresponds to exponential decay estimates in time t for $\dot{\vartheta}$, which will be the object of study in the next sections.

We also rewrite the estimates of Corollary 5.1.2 in terms of the new functions. It is more convenient to work with w rather than u. This is clear when $\lambda = 0$ because of the homogeneous Dirichlet boundary condition satisfied by w on K instead of its Neumann counterpart satisfied by u.

LEMMA 5.1.3. *If (K, u) satisfies the assumptions of Theorem 5.0.1 and ϑ, f are given by (5.1.11) and (5.1.13), then*

$$\begin{cases} f_t = \dfrac{f}{4} + f_{\phi\phi} + f_{tt} + \Big(\dot\vartheta f_\phi + \dot\vartheta^2 f_{\phi\phi} - 2\dot\vartheta f_{t\phi} - \ddot\vartheta f_\phi\Big) \\ f(0,t) = f(2\pi,t) = H_1(t) \\ \dfrac{\ddot\vartheta - \dot\vartheta - \dot\vartheta^3}{(1+\dot\vartheta^2)^{5/2}} = f_\phi^2(2\pi,t) - f_\phi^2(0,t) + H_2(t) \end{cases} \tag{5.1.17}$$

where (recalling the definition of h_1 in (5.0.8) and h_2 in (5.0.9)) the functions H_i are given by the following formulas:

$$H_1(t) := e^{\frac{t}{2}} h_1(e^{-t})\,, \tag{5.1.18}$$

$$H_2(t) := \frac{2\dot\vartheta(t)}{1+\dot\vartheta^2(t)}\Big(\frac{H_1(t)}{2} - \dot H_1(t)\Big)(f_\phi(2\pi,t) - f_\phi(0,t)) + \frac{e^{-t}}{1+\dot\vartheta^2(t)} h_2(\vartheta(t), e^{-t})\,. \tag{5.1.19}$$

Moreover, it is true that

$$H_1 \in H^{3/2} \cap C^{1,1^-}((1,+\infty)), \ \text{and} \qquad H_2 \in L^\infty((1,+\infty))\,, \tag{5.1.20}$$

and for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $t > 0$ and $s \geq t$

$$|H_1(t)| + |\dot H_1(t)| + |H_2(t)| \lesssim (e^{-t})^{3/2^-} \tag{5.1.21}$$

$$|\dot H_1(t) - \dot H_1(s)| \leq C_\varepsilon e^{-(\frac{3}{2}-\varepsilon)t}|s-t|^{1-\varepsilon}\,. \tag{5.1.22}$$

Finally, for every fixed σ, $\delta > 0$, $\varepsilon \in (0, \varepsilon_0)$ and $k \in \mathbb{N}$, the following estimates hold provided ε_0 in Theorem 5.0.1 is sufficiently small: if $\lambda > 0$

$$\|\dot\vartheta\|_{C^{0,1-\varepsilon}([\sigma,\infty))} \leq \delta\,, \tag{5.1.23}$$

$$\|\partial_\phi^i \partial_t^j (f - \mathrm{isq})\|_{C^{0,1-\varepsilon}([0,2\pi]\times[\sigma,\infty))} \leq \delta \qquad \text{for all } i+j \leq 1, \tag{5.1.24}$$

and if $\lambda = 0$

$$\|\vartheta^{(i)}\|_{C^0([\sigma,\infty))} \leq \delta \qquad \text{for all } i \leq k, \tag{5.1.25}$$

$$\|\partial_\phi^i \partial_t^j (f - \mathrm{isq})\|_{C^0([0,2\pi]\times[\sigma,\infty))} \leq \delta \qquad \text{for all } i+j \leq k. \tag{5.1.26}$$

PROOF. Let us first introduce the unit tangent and normal vector fields to K denoted by $e(t)$ and for the normal vector $\nu(t)$, the latter is obtained from $e(t)$ by a counterclockwise rotation of 90 degrees, that is:

$$e(t) := \frac{\dot\varrho(t)}{|\dot\varrho(t)|}, \quad \nu(t) := e^\perp(t).$$

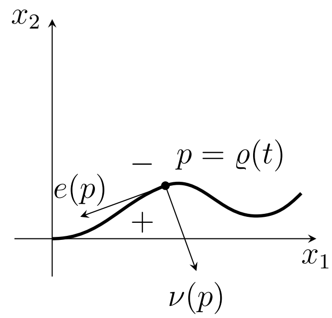

FIGURE 1. The tangent vector $e(p)$ and the normal vector $e(p)$ and a point $p \in K$. Since $t \mapsto |\varrho(t)|$ is a decreasing function, $e(p)$ points towards the origin. Consequently the convention for the symbols $\pm$ on traces of functions is as illustrated in the picture.

Moreover, we will denote by ∇u^+ and ∇u^- the traces of ∇u on K where $\pm$ is identified by the direction in which the vector ν is pointing. More precisely, if $p \in K$, then

$$\nabla u^+(p) = \lim_{s\downarrow 0} \nabla u(p + s\nu(p))\,,$$
$$\nabla u^-(p) = \lim_{s\downarrow 0} \nabla u(p - s\nu(p))\,.$$

Observe that, under the assumptions of Lemma 5.1.1, $e(t)$ is pointing "inward", i.e. towards the origin, and hence for $p = \varrho(t) = (e^{-t}(\cos(\vartheta(t)), \sin(\vartheta(t)))$ (cf. (5.0.5)) we have

$$\nabla u^+(p) = \lim_{\phi\uparrow 2\pi} \nabla u(e^{-t}(\cos(\vartheta(t)+\phi), \sin(\vartheta(t)+\phi))\,, \tag{5.1.27}$$
$$\nabla u^-(p) = \lim_{\phi\downarrow 0} \nabla u(e^{-t}(\cos(\vartheta(t)+\phi), \sin(\vartheta(t)+\phi))\,. \tag{5.1.28}$$

We refer to Figure 1 for a visual illustration.

Since (K, u) is a critical point of the E_λ energy on B_2, the identities (5.0.2)-(5.0.4) are true. Note that the curvature κ of K is given by

$$\kappa(t) = \frac{1}{|\dot\varrho(t)|}\dot e(t)\cdot\nu(t)\,.$$

In particular, the auxiliary function w defined in Lemma 5.0.3 satisfies (5.0.10), which we rewrite for the readers' convenience

$$\begin{cases} \triangle w = 0 & \text{on } B_1 \\ w = h_1 & \text{on } K \\ \kappa = -|\nabla w^+|^2 + |\nabla w^-|^2 + h_2 & \text{on } K\,, \end{cases} \tag{5.1.29}$$

where h_1 and h_2 are defined in (5.0.8) and (5.0.9), respectively. Recalling that

$$w(\phi, r) = r^{1/2} f(\phi - \vartheta(-\ln r), -\ln r), \tag{5.1.30}$$

we compute

$$w_r = r^{-1/2}\left(\frac{f}{2} - f_t + \dot\vartheta f_\phi\right), \quad w_\phi = r^{1/2} f_\phi. \tag{5.1.31}$$

Next we recall the formula for the Laplacian in polar coordinates:

$$\triangle w = 0 \quad \Longleftrightarrow \quad r^{-2} w_{\phi\phi} + r^{-1}(r w_r)_r = 0.$$

By means of (5.1.31) we get

$$r^{-2} w_{\phi\phi} = r^{-3/2} f_{\phi\phi}\,,$$

and

$$\begin{aligned} r^{-1}(r w_r)_r =& r^{-1}\left(r^{1/2}\left(\frac{f}{2} - f_t + \dot\vartheta f_\phi\right)\right)_r \\ =& r^{-3/2}\left(\frac{f}{4} - \frac{f_t}{2} + \frac{\dot\vartheta f_\phi}{2}\right) + r^{-1/2}\left(-r^{-1}\frac{f_t}{2} + r^{-1}\dot\vartheta\frac{f_\phi}{2}\right) \\ &+ r^{-1/2}\left(r^{-1} f_{tt} - 2r^{-1}\dot\vartheta f_{t\phi} - r^{-1}\ddot\vartheta f_\phi + r^{-1}\dot\vartheta^2 f_{\phi\phi}\right) \\ =& r^{-3/2}\left(\frac{f}{4} - f_t + \dot\vartheta f_\phi + f_{tt} - 2\dot\vartheta f_{t\phi} - \ddot\vartheta f_\phi + \dot\vartheta^2 f_{\phi\phi}\right). \end{aligned}$$

In conclusion, we get

$$f_t = \frac{f}{4} + f_{\phi\phi} + f_{tt} + \left(\dot\vartheta f_\phi + \dot\vartheta^2 f_{\phi\phi} - 2\dot\vartheta f_{t\phi} - \ddot\vartheta f_\phi\right). \tag{5.1.32}$$

Next, recalling equality (5.1.30), we may rewrite the Dirichlet condition in the new coordinates simply as

$$f(0,t) = f(2\pi, t) = e^{\frac{t}{2}} h_1(e^{-t}) = H_1(t)\,. \tag{5.1.33}$$

Finally, we derive the equation satisfied by the scalar curvature κ. To this end take into account that

$$\dot\varrho(t) = -\,\varrho(t) + \dot\vartheta(t)\varrho^\perp(t)\,, \tag{5.1.34}$$

and thus differentiating (5.1.34) we get

$$\ddot\varrho(t) = -\,\dot\varrho + \ddot\vartheta\varrho^\perp + \dot\vartheta\dot\varrho^\perp\,. \tag{5.1.35}$$

On the other hand, explicitly we have

$$\begin{aligned} \dot\varrho(t)^\perp &= -e^{-t}(-\sin\vartheta(t), \cos\vartheta(t)) - e^{-t}\dot\vartheta(t)(\cos\vartheta(t), \sin\vartheta(t)) \\ &= -\varrho^\perp(t) - \dot\vartheta(t)\varrho(t)\,. \end{aligned} \tag{5.1.36}$$

Hence, we conclude

$$\begin{aligned} \kappa(t) =& \frac{1}{|\dot\varrho(t)|}\left(\frac{d}{dt}\frac{\dot\varrho(t)}{|\dot\varrho(t)|}\right)\cdot\frac{\dot\varrho^\perp(t)}{|\dot\varrho(t)|} = \frac{\ddot\varrho(t)\cdot\dot\varrho^\perp(t)}{|\dot\varrho(t)|^3} \\ =& \frac{(\dot\vartheta + \dot\vartheta^3 - \ddot\vartheta)|\varrho(t)|^2}{(1+\dot\vartheta^2)^{3/2}|\varrho(t)|^3} = e^t\,\frac{\dot\vartheta + \dot\vartheta^3 - \ddot\vartheta}{(1+\dot\vartheta^2)^{3/2}}. \end{aligned} \tag{5.1.37}$$

As

$$|\nabla u|^2 = |\nabla w|^2 = (w_r)^2 + r^{-2}(w_\phi)^2 = r^{-1}\left(\frac{f}{2} + \dot\vartheta f_\phi - f_t\right)^2 + r^{-1} f_\phi^2$$

we get

$$\frac{\dot\vartheta + \dot\vartheta^3 - \ddot\vartheta}{(1+\dot\vartheta^2)^{3/2}} = -\left[\left(\frac{f}{2} + \dot\vartheta f_\phi - f_t\right)^2 + f_\phi^2\right]\Bigg|_0^{2\pi} + e^{-t} h_2(\vartheta(t), e^{-t})\,. \tag{5.1.38}$$

Thus, by taking into account (5.1.33) and (5.1.38) we conclude the third equation in (5.1.17).

If $\lambda = 0$ we note that in terms of ϑ the bound of α in (5.1.10) reads as

$$\sup_{t\in[\sigma,\infty)} |\vartheta^{(i)}(t)| \le C_i\,\delta \qquad \text{for every } i \le k.$$

Indeed, differentiating i times the identity $\vartheta(t) = \alpha(e^{-t})$ we get

$$\vartheta^{(i)}(t) = \sum_{j=1}^{i} b_{i,j} e^{-jt} \alpha^{(j)}(e^{-t})\,,$$

with $b_{i,j} \in \mathbb{R}$. Then, (5.1.25) follows at once.

Instead, the bound (5.1.26) is a consequence of the linearity and elementary arguments, together with the decay (5.1.9). Indeed, the latter translates into

$$\sup_\phi |\partial_\phi^i \partial_t^j (g - \mathrm{rsq})| \le C_j\,\delta \qquad \text{for every } t\in[\sigma,\infty) \text{ and } i+j\le k,$$

having set $g(\phi,t) := e^{t/2} u(\phi + \vartheta(t), e^{-t})$. To prove the latter estimate we argue as follows. Using the $1/2$-homogeneity of Rsq, we infer

$$\begin{aligned} g(\phi,t) - \mathrm{rsq}(\phi) = {}& e^{t/2}\Big(u(\phi + \alpha(e^{-t}), e^{-t}) - \mathrm{Rsq}(\phi, e^{-t})\Big)\\ =:{}& e^{t/2} h(\phi, e^{-t})\,. \end{aligned} \tag{5.1.39}$$

We conclude that (5.1.9) can be reformulated as

$$\sup_{r\in(0,1/2)} r^{i-1/2} \|\partial_\theta^j \partial_r^i h(\cdot, r)\|_{C^0} \le C_i\,\delta \qquad \text{for every } t\in[\sigma,\infty) \text{ and } i+j\le k.$$

On the other hand, differentiating (5.1.39) yields

$$\partial_\phi^j \partial_t^i (g(\phi,t) - \mathrm{rsq}(\phi)) = \sum_{\ell=0}^{i} b_{i,\ell}\, e^{t/2 - \ell t} [\partial_\phi^j \partial_t^\ell h](\phi, e^{-t})\,,$$

for some $b_{i,\ell} \in \mathbb{R}$. Setting $r = e^{-t}$, we then conclude

$$\|\partial_\phi^i \partial_t^j (g - \mathrm{rsq})\|_{C^0([0,2\pi]\times[\sigma,\infty))} \le \delta \qquad \text{for all } i+j\le k\,,$$

and thus (5.1.26) follows at once, using that the gradient of the harmonic conjugate is the counterclockwise rotation by 90 degrees of the gradient of u.

Instead, if $\lambda > 0$ we argue analogously to infer (5.1.23) and (5.1.24), using in addition classical estimates for the function u_a. Finally, being $u_a \in C^{1,1^-}_{\mathrm{loc}}(B_2)$, in view of the bounds in (5.0.11)-(5.0.12), we infer (5.1.20)-(5.1.22). $\square$

5.2. First linearization

In this section we consider a sequence (u_j, α_j) as in Theorem 5.0.1 where condition (iv) holds for a vanishing sequence $\varepsilon_0(j) \downarrow 0$. Without loss of generality we assume $\alpha_j(1) = 0$. With fixed $\hat{\varepsilon} \in (0, 3/2)$ we define δ_j, θ_j, and v_j thanks to (5.1.11)-(5.1.13) as follows:

$$\delta_j := \|f_j(\cdot, \cdot + j) - \mathrm{isq}\|_{H^2((0,2\pi)\times(0,3))} + \|\dot{\vartheta}_j(\cdot + j)\|_{H^1((0,3))} + \lambda e^{-(3/2-\hat{\varepsilon})j} \tag{5.2.1}$$

$$\theta_j(t) := \delta_j^{-1}\vartheta_j(t + j) \tag{5.2.2}$$

$$v_j(\phi, t) := \delta_j^{-1}(f_j(\phi, t + j) - \mathrm{isq}(\phi)) \tag{5.2.3}$$

Next, we show that the limit of (v_j, θ_j) solves a linearization of (5.1.17), and in addition it satisfies the linearization of (2.5.9) (actually it suffices to consider (5.0.6)). The latter remark is crucial for our purposes.

PROPOSITION 5.2.1. *Let (u_j, α_j) as in Theorem 5.0.1 where the smallness condition in item (iv) holds for a vanishing sequence $\varepsilon_0(j) \downarrow 0$. Assume $\alpha_j(1) = 0$ and define ϑ_j and f_j as in (5.1.11)-(5.1.13) and v_j and θ_j as above. Then, up to subsequences,*

(a) *v_j converge to some function v weakly in $H^2((0, 2\pi) \times (0, 3))$ and strongly in $W^{1,p}((0, 2\pi) \times (0, 3))$ for all $p \geq 1$;*

(b) *θ_j converge to some θ; weakly in $H^2((0, 2\pi))$ and in $C^{1,\frac{1}{2}-\varepsilon}([0, 3])$ for all $\varepsilon \in (0, 1)$.*

More precisely, the convergences are

(c) *either if $\lambda = 0$: in $C^{2,\alpha}([0, 2\pi] \times [\sigma, 3 - \sigma])$ for v_j, and in $C^{2,\alpha}([\sigma, 3 - \sigma])$ for θ_j, for all $\sigma \in (0, \frac{3}{2})$, $\alpha \in (0, 1)$, respectively;*

(d) *or if $\lambda > 0$: in $C^{1,1-\varepsilon}([0, 2\pi] \times [\sigma, 3-\sigma])$ and strongly in $H^2((0, 2\pi) \times (\sigma, 3-\sigma))$ for v_j, and strongly in $H^2((\sigma, 3 - \sigma))$ for θ_j, for all $\sigma \in (0, \frac{3}{2})$, $\varepsilon \in (0, 1)$, respectively.*

Moreover, the pair (v, θ) solves the following linear system of PDEs in $(0, 2\pi) \times (0, 3)$

$$\begin{cases} v_t - v_{tt} = \frac{v}{4} + v_{\phi\phi} + (\dot{\theta} - \ddot{\theta})\mathrm{isq}_\phi \\[2ex] v(0, t) = v(2\pi, t) = 0 \\[2ex] \dot{\theta}(t) - \ddot{\theta}(t) = \sqrt{\frac{2}{\pi}}(v_\phi(2\pi, t) + v_\phi(0, t)) \\[2ex] \theta(0) = 0\,, \end{cases} \tag{5.2.4}$$

and satisfies the following integral condition for every $t \in (0, 3)$:

$$\int_0^{2\pi} v_\phi(\phi, t) \sin\frac{\phi}{2}\, d\phi = 0\,. \tag{5.2.5}$$

PROOF. The statements in (a) and (b) are obvious consequences of the bounds on (v_j, θ_j) (and of the fact that $H^2((0, 2\pi) \times (0, 3))$, resp. $H^2((0, 3))$, embeds compactly in $W^{1,p}((0, 2\pi) \times (0, 3))$ for all $p \geq 1$, resp. $C^{1,\frac{1}{2}-\varepsilon}([0, 3])$ for all $\varepsilon \in (0, \frac{1}{2})$). Observe that, by assumption, $\theta_j(0) = 0$ and thus $\theta(0) = 0$ is a consequence of the uniform convergence.

We next observe that the PDE in (5.1.17) is linear in the unknown f. Hence, recalling that $(f_j(\phi, t+j), \vartheta_j(t+j)) = (\mathrm{isq}(\phi) + \delta_j v_j(\phi, t), \delta_j \theta_j(t))$, we infer

$$v_{j,tt} + v_{j,\phi\phi} = -\frac{v_j}{4} + v_{j,t} + (\ddot\theta_j - \dot\theta_j)(\mathrm{isq}_\phi + \delta_j v_{j,\phi}) + 2\delta_j \dot\theta_j v_{j,t\phi} - \delta_j \dot\theta_j^2 (\mathrm{isq}_{\phi\phi} + \delta_j v_{j,\phi\phi}) \tag{5.2.6}$$

and

$$v_j(0,t) = v_j(2\pi, t) = \delta_j^{-1} H_1(t+j)\,. \tag{5.2.7}$$

Passing into the limit we therefore conclude easily that v solves the PDE in the first line of (5.2.4). Likewise the boundary condition $v(0,\cdot) = v(2\pi, 0) = 0$ is also a consequence of uniform convergence, of the bounds (5.1.21) and (5.1.22), and of the very definition of δ_j in (5.2.1), in turn implying for all $\varepsilon \in (0, \hat\varepsilon)$

$$\|\delta_j^{-1} H_1(\cdot + j)\|_{C^{1,1-\varepsilon}([0,3])} + \|\delta_j^{-1} H_2(\cdot + j)\|_{L^\infty((0,3))} \to 0 \qquad j \to +\infty\,. \tag{5.2.8}$$

Moreover, note that again thanks to the bounds in (5.1.21) and (5.1.22) we can also deduce that for all $j \geq 1$

$$\begin{aligned}
[\dot H_1(\cdot + j)]^2_{H^{1/2}((0,\infty))} =& 2\int_j^\infty dt \int_t^\infty \frac{|\dot H_1(s) - \dot H_1(t)|^2}{|s-t|^2} ds \\
=& 2\int_j^\infty dt \sum_{k=0}^\infty \int_{t+k}^{t+k+1} \frac{|\dot H_1(s) - \dot H_1(t)|^2}{|s-t|^2} ds \\
\leq& 2C_\varepsilon^2 \int_j^\infty dt \sum_{k=0}^\infty \int_k^{k+1} \frac{e^{-(3-2\varepsilon)(t+k)}}{\tau^{2\varepsilon}} d\tau \leq \tilde C_\varepsilon e^{-(3-2\varepsilon)j}\,.
\end{aligned}$$

Therefore,

$$\|\delta_j^{-1} H_1(\cdot + j)\|_{H^{3/2}((0,\infty))} \to 0\,. \tag{5.2.9}$$

We next write the third equation in (5.1.17) in terms of θ_j and v_j:

$$\begin{aligned}
\ddot\theta_j - \dot\theta_j =& \delta_j^2 \dot\theta_j^3 + 4(1 + \delta_j^2 \dot\theta_j^2)^{5/2} \Big(\delta_j (v_{j,\phi}^2(2\pi, t) - v_{j,\phi}^2(0,t)) \\
& - \sqrt{\tfrac{2}{\pi}}(v_{j,\phi}(2\pi, t) + v_{j,\phi}(0,t)) + \delta_j^{-1} H_2(t+j)\Big)\,.
\end{aligned} \tag{5.2.10}$$

Observe that, by the trace theorems, $v_{j,\phi}(2\pi, \cdot)$ and $v_{j,\phi}(0,\cdot)$ enjoy uniform bounds in $H^{\frac{1}{2}}$. Clearly, by (5.2.8) we get that the third equation in (5.2.4) holds.

Moreover, by (5.2.8) and by the Sobolev embedding, we conclude that the right hand side of (5.2.10) has a uniform control in L^q for every $q < \infty$, in particular the same bound is enjoyed by $\ddot\theta_j - \dot\theta_j$ and, using that $\|\dot\theta_j\|_{C^0}$ is bounded, we conclude that $\dot\theta_j$ has a uniform $W^{1,q}$ bound for every $q < \infty$.

Next, we distinguish the two cases $\lambda = 0$ and $\lambda > 0$, the former being considerably simpler at least for what computations are concerned.

Case $\lambda = 0$. We start off rewriting the equation in (5.2.6) above in the following way:

$$(1 + \delta_j^2 \dot\theta_j^2) v_{j,\phi\phi} + v_{j,tt} - 2\delta_j \dot\theta_j v_{j,t\phi} = \underbrace{-\frac{v_j}{4} + v_{j,t} + (\ddot\theta_j - \dot\theta_j)(\mathrm{isq}_\phi + \delta_j v_{j,\phi}) - \delta_j \dot\theta_j^2 \mathrm{isq}_{\phi\phi}}_{=:F_j} \tag{5.2.11}$$

Observe that the left hand side is an elliptic operator with a uniform bound on the ellipticity constants and a uniform bound on the $C^{\frac{1}{2}-\varepsilon}$ norm of the coefficients, for all $\varepsilon \in (0,\hat{\varepsilon})$. Thanks to the uniform $W^{1,q}$ bound on θ_j and v_j for every $q<\infty$, we infer a uniform bound on $\|F_j\|_{L^q((0,2\pi)\times(0,3))}$ for every $q<\infty$.

In addition, as $v_j=0$ on $\{0,2\pi\}\times(0,3)$, using elliptic regularity we conclude a uniform bound for $\|v_j\|_{W^{2,q}((0,2\pi)\times(\sigma,3-\sigma))}$. We now can use Morrey's embedding to get a uniform estimate on $\|v_j\|_{C^{1,\alpha}([0,2\pi]\times[2\sigma,3-2\sigma])}$ for every $\alpha<1$. We now turn again to (5.2.10), to conclude that the right hand side has a uniform C^α bound in $[2\sigma,3-2\sigma]$ for every $\alpha>0$. This gives uniform $C^{1,\alpha}$ bounds on the coefficient of the elliptic operator in the left hand side of (5.2.11) and uniform C^α bounds on the right hand side of (5.2.11). We can thus infer a uniform $C^{2,\alpha}$ bound in $[0,2\pi]\times[3\sigma,3-3\sigma]$ on v_j from elliptic regularity.

It thus remains to prove (5.2.5). The latter will come from (5.0.6). First of all, we fix $t\in(0,3)$, set $t_j:=e^{-t-j}$ and observe that $\partial B_{t_j}\cap K_j$ consists of a single point p_j. We can thus apply Corollary 5.0.2. Hence using the relation between harmonic conjugates, we rewrite (5.0.6) as

$$0=\int_{\partial B_{t_j}\setminus\{p_j\}}\left(|\nabla w_j|^2 n\cdot\tau(p_j)-2\frac{\partial w_j}{\partial\tau}\nabla w_j\cdot(n-n(p_j))\right)\,d\mathcal{H}^1\,.$$

Next, we assume without loss of generality that $p_j=(t_j,0)$ and rewrite the latter equality using polar coordinates:

$$\underbrace{t_j\int_0^{2\pi}\left(t_j w_{j,r}^2-\frac{1}{t_j}w_{j,\phi}^2\right)(\phi,t_j)\sin\phi\,d\phi}_{=:A_{1,j}}-\underbrace{2\int_0^{2\pi}\left(w_{j,r}w_{j,\phi}\right)(\phi,t_j)(1-\cos\phi)\,d\phi}_{=:A_{2,j}}=0 \quad (5.2.12)$$

We next write w_j in terms of v_j, γ_j and θ_j as

$$w_j(\phi,r)=r^{1/2}\mathrm{isq}(\phi-\delta_j\theta_j(-\ln r-j))+\delta_j r^{1/2}v_j(\phi-\delta_j\theta_j(-\ln r-j),-\ln r-j)\ ,$$

Note that, having normalized so that $p_j=(t_j,0)$, we conclude that $\theta_j(-\ln t_j)=\theta_j(t+j)=0$. Using the latter we compute:

$$\frac{\partial w_j}{\partial r}(\phi,t_j)=\underbrace{t_j^{-1/2}\frac{\mathrm{isq}(\phi)}{2}}_{=:a_j(\phi)}+\delta_j\underbrace{t_j^{-1/2}\left(\frac{\dot\theta_j(t)}{\sqrt{2\pi}}\cos\frac{\phi}{2}+\frac{v_j(\phi,t)}{2}-v_{j,t}(\phi,t)\right)}_{=:b_j(\phi)}+o(\delta_j) \quad (5.2.13)$$

$$\frac{\partial w_j}{\partial\phi}(\phi,t_j)=\underbrace{t_j^{1/2}\frac{\mathrm{rsq}(\phi)}{2}}_{=:c_j(\phi)}+\delta_j\underbrace{t_j^{1/2}v_{j,\phi}(\phi,t)}_{=:d_j(\phi)}\,. \quad (5.2.14)$$

Note now that the function $a_j^2-\frac{1}{t_j}c_j^2$ is even. Since $\sin\phi$ is odd, we thus conclude

$$A_{1,j}=2\delta_j\int_0^{2\pi}\left(t_j a_j(\phi)b_j(\phi)-t_j^{-1}c_j(\phi)d_j(\phi)\right)\,\sin\phi\,d\phi\ +o(\delta_j)\,.$$

Letting $j \to \infty$ we obtain

$$\begin{aligned}\lim_{j\to\infty} \delta_j^{-1} A_{1,j} =& \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \left(\frac{\dot\theta(t)}{\sqrt{2\pi}} \cos\frac{\phi}{2} + \frac{v(\phi,t)}{2} - v_t(\phi,t) \right) \sin\frac{\phi}{2} \sin\phi \, d\phi \\ & - \sqrt{\frac{2}{\pi}} \int_0^{2\pi} v_\phi(\phi,t) \cos\frac{\phi}{2} \sin\phi \, d\phi \,. \end{aligned} \tag{5.2.15}$$

By direct computation it follows that

$$\int_0^{2\pi} a_j(\phi) c_j(\phi)(1-\cos\phi)\, d\phi = \frac{1}{8} \int_0^{2\pi} \sin\phi(1-\cos\phi)\, d\phi = 0 \,,$$

from which we conclude

$$A_{2,j} = 2\delta_j \int_0^{2\pi} (a_j(\phi) d_j(\phi) + b_j(\phi) c_j(\phi))(1-\cos\phi)\, d\phi + o(\delta_j) \,.$$

Hence

$$\begin{aligned}\lim_{j\to\infty} \delta_j^{-1} A_{2,j} =& \sqrt{\frac{2}{\pi}} \int_0^{2\pi} v_\phi(\phi,t) \sin\frac{\phi}{2}(1-\cos\phi)\, d\phi \\ & + \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \left(\frac{\dot\theta(t)}{\sqrt{2\pi}} \cos\frac{\phi}{2} + \frac{v(\phi,t)}{2} - v_t(\phi,t) \right) \cos\frac{\phi}{2}(1-\cos\phi)\, d\phi \,. \end{aligned} \tag{5.2.16}$$

Combining (5.2.15) and (5.2.16) with (5.2.12) we conclude

$$\begin{aligned} 0 =& \frac{\dot\theta(t)}{2\pi} \int_0^{2\pi} \left(\sin^2\phi - 2\cos^2\frac{\phi}{2}(1-\cos\phi) \right) \, d\phi \\ & + \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \left(\frac{v(\phi,t)}{2} - v_t(\phi,t) \right) \left(\sin\frac{\phi}{2}\sin\phi - \cos\frac{\phi}{2}(1-\cos\phi) \right) \, d\phi \\ & - \sqrt{\frac{2}{\pi}} \int_0^{2\pi} v_\phi(\phi,t) \left(\cos\frac{\phi}{2}\sin\phi + \sin\frac{\phi}{2}(1-\cos\phi) \right) \, d\phi \,. \end{aligned}$$

Using the identities

$$\begin{aligned} &\sin^2\phi - 2\cos^2\frac{\phi}{2}(1-\cos\phi) = \sin^2\phi - (1+\cos\phi)(1-\cos\phi) = 0 \,, \\ &\sin\frac{\phi}{2}\sin\phi - \cos\frac{\phi}{2}(1-\cos\phi) = \sin\frac{\phi}{2}\sin\phi + \cos\frac{\phi}{2}\cos\phi - \cos\frac{\phi}{2} = 0 \,, \\ &\cos\frac{\phi}{2}\sin\phi + \sin\frac{\phi}{2}(1-\cos\phi) = \cos\frac{\phi}{2}\sin\phi - \sin\frac{\phi}{2}\cos\phi + \sin\frac{\phi}{2} = 2\sin\frac{\phi}{2} \,, \end{aligned}$$

we conclude (5.2.5).

Case $\lambda > 0$. We rewrite the equation in (5.2.6) above in the following way:

$$v_{j,\phi\phi} + v_{j,tt} = \underbrace{-\frac{v_j}{4} + v_{j,t} + (\ddot\theta_j - \dot\theta_j)(\mathrm{isq}_\phi + \delta_j v_{j,\phi}) - \delta_j \dot\theta_j^2 \mathrm{isq}_{\phi\phi} - \delta_j^2 \dot\theta_j^2 v_{j,\phi\phi} + 2\delta_j \dot\theta_j v_{j,t\phi}}_{=:F_j}\,, \tag{5.2.17}$$

and observe that, as before in case $\lambda = 0$, we can now get a uniform bound on the norms $\|F_j\|_{L^q((0,2\pi)\times(0,3))}$ for every $q < \infty$.

With fixed $\sigma \in (0, \frac{3}{2})$ consider a domain Λ diffeomorphic to the circle such that $(0,2\pi)\times(0,3) \subset \Lambda \subset (0,2\pi)\times(-\sigma, 3+\sigma)$, and extend it to Λ with $C^{1,1-\varepsilon}(\partial\Lambda)$ and $H^{3/2}(\partial\Lambda)$ norms bounded by a multiple of the corresponding one for $\delta_j^{-1} H_1(\cdot + j)$. Denote by $\widetilde{H}_j$ the extended function, and define the auxiliary functions $v_j^{(1)}$ and $v_j^{(2)}$ to be the solutions respectively of

$$\begin{cases} v^{(1)}_{j,\phi\phi} + v^{(1)}_{j,tt} = 0 & \text{on } \Lambda \\ \\ v_j^{(1)} = \widetilde{H}_j & \text{on } \partial\Lambda\,, \end{cases}$$

and of

$$\begin{cases} v^{(2)}_{j,\phi\phi} + v^{(2)}_{j,tt} = F_j \chi_{(0,2\pi)\times(0,3)} & \text{on } \Lambda \\ \\ v_j^{(2)} = 0 & \text{on } \partial\Lambda\,. \end{cases}$$

[25, Corollary 8.36, Theorems 8.29 and 8.16] imply that $v_j^{(1)} \in C^{1,1-\varepsilon}(\overline{\Lambda})$ with

$$\|v_j^{(1)}\|_{C^{1,1-\varepsilon}(\overline{\Lambda})} \le c \|\delta_j^{-1} H_1(\cdot + j)\|_{C^{1,1-\varepsilon}([0,3])}\,,$$

where the constant c depends on Λ. In particular, from (5.2.8) we infer that $v_j^{(1)}$ converges strongly to zero in $C^{1,1-\varepsilon}(\overline{\Lambda})$. Moreover, [29, Theorem 5.1] imply that $v_j^{(1)} \in H^2(\Lambda)$ with

$$\|v_j^{(1)}\|_{H^2(\Lambda)} \le c \|\delta_j^{-1} H_1(\cdot + j)\|_{H^{3/2}([0,3])}\,,$$

where the constant c depends on Λ. Thus, from (5.2.9) we infer that $v_j^{(1)}$ converges to zero strongly in $H^2((0,2\pi)\times(0,3))$, as well. In addition, [25, Theorems 9.13 and 9.15] implies that $v_j^{(2)} \in W^{2,p}((0,2\pi)\times(\sigma, 3-\sigma))$ for all $p > 2$.

The function $v_j - v_j^{(1)} - v_j^{(2)}$ is then harmonic on $(0,2\pi)\times(0,3)$ with null Dirichlet boundary conditions on $\{0, 2\pi\}\times(0,3)$. By odd reflection with respect to those segments we find an harmonic function on $(-2\pi, 4\pi)\times(0,3)$. By interior estimates for harmonic functions (see for instance [25, Theorem 2.10]) we conclude that $v_j - v_j^{(1)} - v_j^{(2)} \in C^\infty([0,2\pi]\times[\sigma, 3-\sigma])$ with

$$\|D^\beta(v_j - v_j^{(1)} - v_j^{(2)})\|_{C^0([0,2\pi]\times[\sigma,3-\sigma])} \le (n|\beta|\sigma^{-1})^{|\beta|} \|v_j - v_j^{(1)} - v_j^{(2)}\|_{C^0([0,2\pi]\times[\sigma,3-\sigma])}\,,$$

for any multi-index β, where $|\beta|$ denotes the length of β. From this we conclude item (d).

Hence, v_j turns out to converge in $C^{1,1-\varepsilon}([0,2\pi]\times[\sigma, 3-\sigma])$ and strongly in $H^2((0,2\pi)\times(\sigma, 3-\sigma))$. Finally, from (5.2.10) we then conclude that θ_j converge strongly in $H^2((\sigma, 3-\sigma))$.

Let us now turn to the proof of (5.2.5). Assume $p_j = (r_j, 0))$ where $r_j := e^{-t-j}$ for some $t \in (0, 3)$. Then we rewrite (5.0.6) by taking advantage of (5.0.13), recalling that $\nabla^\perp(u_j + u_{j,a}) = \nabla w_j$ we conclude

$$\underbrace{\int_{\partial B_{r_j}\setminus\{p_j\}} \left(|\nabla w_j|^2 n\cdot\tau(p_j) - 2\frac{\partial w_j}{\partial\tau}\nabla w_j\cdot(n - n(p_j))\right)d\mathcal{H}^1}_{=:B_j} = C_{1,j} + C_{2,j} + C_{3,j}$$

where

$$\begin{aligned}
C_{1,j} &= 2\int_{\partial B_{r_j}\setminus\{p_j\}} \Big((\nabla w_j\cdot\nabla^\perp u_{j,a})n\cdot\tau(p_j) + \frac{\partial w_j}{\partial\tau}\nabla u_{j,a}\cdot(\tau - \tau(p_j)) \\
&\qquad\qquad - \frac{\partial u_{j,a}}{\partial n}\nabla w_j\cdot(n - n(p_j))\Big)d\mathcal{H}^1 \\
C_{2,j} &= 2\lambda\int_{B_{r_j}\setminus K_j}(u_j - g_j)\nabla w_j\cdot(n - n(p_j)) - 2\int_{B_{r_j}}\nabla^T u_{j,a}\cdot D\tau\nabla u_{j,a} \\
C_{3,j} &= -2\lambda\int_{B_{r_j}\cap K_j}\Big(|u_j^+ - g_{j,K_j}|^2 - |u_j^- - g_{j,K_j}|^2\Big)\tau(p_j)\cdot\nu\, d\mathcal{H}^1 .
\end{aligned}$$

Since the term B_j equals the sum of the terms $A_{1,j}$ and $A_{2,j}$ already discussed in the proof of the analogous identity when $\lambda = 0$, we immediately conclude that

$$\lim_{j\to+\infty}\delta_j^{-1}B_j = -2\sqrt{\frac{2}{\pi}}\int_0^{2\pi} v_\phi(\phi, t)\sin\frac{\phi}{2}\,d\phi\,. \tag{5.2.18}$$

Next, recall that $\nabla u_{j,a}(0) = 0$ while $u_{j,a} \in C^{1,1^-}$, so that in particular we can achieve $\|\nabla u_{j,a}\|_{L^\infty(B_r)} \le Cr^{1-\hat\varepsilon/2}$, where $\hat\varepsilon > 0$ has been chosen in the definition of δ_j (cf. (5.2.1)). Since $|\nabla w_j(x)| \le C|x|^{-1/2}$, we thus can estimate

$$|C_{1,j}| \le C\|\nabla w_j\|_{L^\infty(B_{r_j})}\|\nabla u_{j,a}\|_{L^\infty(B_{r_j})}r_j \le Cr_j^{3/2-\hat\varepsilon/2}$$

where C is a universal constant. On the other hand, $\delta_j \ge Cr_j^{3/2-\hat\varepsilon}$ for some positive constant C and so

$$\lim_{j\to+\infty}\delta_j^{-1}C_{1,j} = 0\,. \tag{5.2.19}$$

As for $C_{2,j}$ we write

$$\begin{aligned}
|C_{2,j}| &\le \left|\int_{B_{r_j}\setminus K}(u_j - g_j)\nabla w_j\cdot(n - n(p_j))\right| + \left|2\int_{B_{r_j}}\nabla^T u_{j,a}\cdot D\tau\nabla u_{j,a}\right| \\
&\le C\int_{B_{r_j}}|x|^{-1/2} \le Cr_j^{3/2}\,.
\end{aligned}$$

Likewise,

$$\left|\int_{B_{r_j}\cap K}\Big(|u_j^+ - g_{j,K_j}|^2 - |u_j^- - g_{j,K_j}|^2\Big)\tau(p_j)\cdot\nu\, d\mathcal{H}^1\right|$$
$$= \left|\int_{B_{r_j}\cap K}\Big(|u_j^+|^2 - |u_j^-|^2 - 2(u_j^+ - u_j^-)g_{j,K_j}\Big)\tau(p_j)\cdot\nu\, d\mathcal{H}^1\right| \le C\|g_j\|_\infty r_j^{3/2} \le C r_j^{3/2},$$

where we have used that $|u_j^\pm(x)| \le C|x|^{1/2}$ (cf. (5.0.1)), the estimate $\|g_{j,K_j}\|_{L^\infty(\Omega.\mathcal{H}^1\llcorner K_j)} \le \|g_j\|_\infty$, and the density upper bound in (1.3.1)). Therefore, thanks to the very definition of δ_j in (5.2.1) we conclude that

$$\lim_{j\to+\infty}\delta_j^{-1}(C_{2,j}+C_{3,j}) = 0\,. \tag{5.2.20}$$

Collecting (5.2.18)-(5.2.20) we deduce again (5.2.5). □

5.3. Spectral analysis

In this section we will find a suitable representation of solutions to (5.2.4) on $[0,2\pi]\times(0,3)$, based on the spectral analysis of a closely related linear PDE.

To this aim we introduce the following terminology: a function h on $(0,2\pi)\times(0,3)$ will be called even if $h(\phi,t) = h(2\pi-\phi,t)$ and odd if $h(\phi,t) = -h(2\pi-\phi,t)$. Moreover, a general h can be split into the sum of its odd part $h_j^o(\phi,t) := \frac{h(\phi,t)-h(2\pi-\phi,t)}{2}$ and its even part $h_j^e(\phi,t) = \frac{h(\phi,t)+h(2\pi-\phi,t)}{2}$. Note finally that, if h is even (resp. odd), then $\partial_\phi^j\partial_t^k h$ is odd (resp. even) for j odd, and even (resp. odd) for j even for every k.

In what follows it will be convenient to consider the change of variables

$$\zeta(\phi,t) := v^o(\phi,t) - \theta(t)\mathrm{isq}_\phi(t) = v^o(\phi,t) - \frac{\theta(t)}{\sqrt{2\pi}}\cos\frac{\phi}{2}\,. \tag{5.3.1}$$

Lemma 5.3.1. *The pair $(v,\theta)\in H^2((0,2\pi)\times(0,3))\times H^3((0,2\pi))$ solves (5.2.4) if and only if v^e, $\zeta\in H^2((0,2\pi)\times(0,3))$ are such that $v^e(\cdot,t)$ is even for every $t\in(0,3)$ and it solves the partial differential equation with homogeneous boundary conditions*

$$\begin{cases} v^e_{tt} + v^e_{\phi\phi} + \frac{v^e}{4} - v^e_t = 0 \\[2ex] v^e(0,t) = v^e(2\pi,t) = 0\,, \end{cases} \tag{5.3.2}$$

$\zeta(\cdot,t)$ is odd for every for $t\in(0,3)$, $\zeta(0,0)=0$, $\zeta(0,t) = -\frac{\theta(t)}{\sqrt{2\pi}}$, and ζ solves the partial differential equation with Ventsel boundary conditions

$$\begin{cases} \zeta_{tt} + \zeta_{\phi\phi} + \frac{\zeta}{4} - \zeta_t = 0 \\[2ex] \zeta_\phi(0,t) + \frac{\pi}{2}\left(\frac{\zeta}{4}(0,t) + \zeta_{\phi\phi}(0,t)\right) = 0\,, \end{cases} \tag{5.3.3}$$

and in addition $\theta \in H^3((0,2\pi))$ solves

$$\begin{cases} \dot\theta(t) - \ddot\theta(t) = 2\sqrt{\frac{2}{\pi}}\zeta_\phi(0,t) \\ \\ \theta(0) = 0\,. \end{cases} \tag{5.3.4}$$

Taking into account standard regularity theory, the lemma is reduced to elementary computations which are left to the reader.

From (5.3.4) it is evident that the decay properties of θ are related only to the odd part v^o of the solution v to (5.2.4) (cf. (5.3.1)). The analysis of the latter would suffice in case $\lambda = 0$. Instead, for $\lambda > 0$ it is necessary to discuss the spectral property of the even part v^e of the solutions to the linearized system, as well.

We aim at a representation for v^e and ζ, i.e. a representation as a series of functions in ϕ with coefficients depending on t, for which we can reduce (5.3.2), (5.3.3), respectively, to an independent system of ODEs for the coefficients. To that aim we introduce the spaces

$$\mathscr{E} := \{\psi \in H^1((0,2\pi)) : \psi(\phi) = \psi(2\pi - \phi)\}\,, \tag{5.3.5}$$

and

$$\mathscr{O} := \{\psi \in H^1((0,2\pi)) : \psi(\phi) = -\psi(2\pi - \phi)\}\,. \tag{5.3.6}$$

We start off discussing the spectral analysis for the even part which is standard. Instead, that for the odd part is more subtle in view of the Ventsel boundary conditions.

5.3.1. Spectral analysis of the even part. The representation for v^e is given in the next proposition.

PROPOSITION 5.3.2. *If $v^e \in H^2((0,2\pi)\times(0,3))$ is even and $v^e(0,t) = 0$ for every $t \in (0,3)$ then*

$$v^e(\phi,t) = \sum_{k=0}^{\infty} \frac{a_k(t)}{\sqrt{\pi}} \sin\left((k+\tfrac{1}{2})\phi\right) \tag{5.3.7}$$

where:

(a) $C^{-1}\sum_k k^2 a_k^2(t) \le \|v^e(\cdot,t)\|^2_{H^1} \le C\sum_k k^2 a_k^2(t)$ *for a universal constant C;*
(b) *For $k \geq 0$, the coefficients a_k satisfy $a_k(t) = \langle v^e(\cdot,t), \frac{1}{\sqrt{\pi}}\sin((k+\frac{1}{2})\cdot)\rangle_{L^2}$.*

If additionally v^e solves (5.3.2) then $v^e \in C^\infty([0,2\pi]\times(0,3))$ and for every $k \geq 0$ the coefficients $a_k(t)$ in the expansion satisfy

$$\ddot a_k(t) - \dot a_k(t) - (k^2+k)a_k(t) = 0\,. \tag{5.3.8}$$

From (5.3.8) it follows in particular that, for every $k \geq 0$

$$a_k(t) = C_k\, e^{(k+1)t} + D_k\, e^{-kt} \tag{5.3.9}$$

for some C_k, $D_k \in \mathbb{R}$.

REMARK 5.3.3. Observe that, as a consequence of the regularity of v_e, the coefficients C_k and D_k will satisfy appropriate decay estimates in k as $k \uparrow \infty$, even though this is not explicitly stated.

PROOF. Consider for $h \in \mathscr{E}$ the eigenvalue problem

$$\begin{cases} h_{\phi\phi} = \beta h \\ \\ h(0) = 0\,. \end{cases}$$

By solving the ODE and imposing the Dirichlet boundary condition it is easy to show that $\beta = -\nu^2 < 0$ and that 2ν is odd. From this one finds that necessarily $h(\phi) = A\sin\Big((k+\frac{1}{2})\phi\Big)$ for some $k \in \mathbb{N}$ and $A \in \mathbb{R} \setminus \{0\}$, and that a_k are as in item (b).

The rest of the statement follows easily from standard Fourier analysis, and explicit calculations. □

5.3.2. Spectral analysis of the odd part. The representation for ζ is detailed in the following

PROPOSITION 5.3.4. *If $\zeta \in H^2((0,2\pi)\times(0,3))$ is odd then*

$$\zeta(\phi,t) = \sum_{k=0}^{\infty} a_k(t)\zeta_k(\phi) \tag{5.3.10}$$

where:

(a) *$C^{-1}\sum_k \nu_k^2 a_k^2(t) \le \|\zeta(\cdot,t)\|^2_{H^1((0,2\pi))} \le C\sum_k \nu_k^2 a_k^2(t)$ for a universal constant C;*
(b) *The functions ζ_k are defined in Section 5.3.4 (cf. (5.3.20)-(5.3.22));*
(c) *For $k \ge 2$, the coefficients a_k satisfy $a_k(t) = \langle \zeta(\cdot,t), \zeta_k\rangle$ for the bilinear symmetric form $\langle \cdot,\cdot\rangle$ defined in Section 5.3.3 (cf. (5.3.15)),*
(d) *the coefficients $a_0(t)$ and $a_1(t)$ are given by $a_0(t) = \mathcal{L}_0(\zeta(\cdot,t))$ and $a_1(t) = \mathcal{L}_1(\zeta(\cdot,t))$ for appropriately defined linear bounded functionals $\mathcal{L}_0, \mathcal{L}_1 : \mathscr{O} \to \mathbb{R}$.*

Next, if additionally ζ solves (5.3.3) then $\zeta \in C^\infty([0,2\pi]\times(0,3))$ and for every $k \ge 2$ the coefficients $a_k(t)$ in the expansion satisfy

$$\ddot{a}_k(t) - \dot{a}_k(t) - (\nu_k^2 - \tfrac{1}{4})a_k(t) = 0 \tag{5.3.11}$$

where the number ν_k's are given in Lemma 5.3.7 (cf. (5.3.18)). In particular, for every $k \ge 2$

$$a_k(t) = C_k e^{\mu_{k,-}t} + D_k e^{\mu_{k,+}t} \tag{5.3.12}$$

where the C_k and D_k are constants and,

$$\mu_{k,\pm} = \frac{1}{2} \pm \nu_k\,.$$

The proof is an obvious consequence of Proposition 5.3.8 below, which will be the main focus of the rest of this section.

5.3.3. The Ventsel boundary condition. For every $\psi \in \mathscr{O}$ we look for solutions $h \in \mathscr{O}$ of the following equation:

$$\begin{cases} h_{\phi\phi} = \psi \\ \\ h_\phi(0) = -\dfrac{\pi}{2}\left(\dfrac{h(0)}{4} + h_{\phi\phi}(0)\right). \end{cases} \tag{5.3.13}$$

The following is an elementary fact of which we include the proof for the reader's convenience.

LEMMA 5.3.5. *For every $\psi \in \mathscr{O}$ there is a unique solution $h := \mathscr{A}(\psi) \in \mathscr{O}$ of (5.3.13). In fact the linear operator $\mathscr{A} : \mathscr{O} \to \mathscr{O}$ is compact.*

PROOF. $h \in \mathscr{O}$ solves the first equation in (5.3.13) if and only if

$$h(\phi) = h_\phi(\pi)(\phi - \pi) + \underbrace{\int_\pi^\phi \int_\pi^\tau \psi(s)\, ds\, d\tau}_{=:\Psi(\phi)}. \tag{5.3.14}$$

On the other hand the initial condition holds if and only if

$$h_\phi(\pi)\left(\frac{\pi^2}{8} - 1\right) = \Psi'(0) + \frac{\pi}{2}\left(\frac{\Psi(0)}{4} + \Psi''(0)\right).$$

Since Ψ is determined by ψ, the latter determines uniquely $h_\phi(\pi)$ and thus shows that there is one and only one solution $h = \mathscr{A}(\psi) \in \mathscr{O}$ of (5.3.13). Moreover, we obviously have

$$\|\mathscr{A}(\psi)\|_{H^3} \leq C\|\psi\|_{H^1},$$

which shows that the operator is compact. □

We next introduce in $\mathscr{O}$ a continuous bilinear map

$$\langle u, v\rangle := \int_0^{2\pi} u_\phi v_\phi - \frac{1}{4}\int_0^{2\pi} uv. \tag{5.3.15}$$

If $\langle\cdot,\cdot\rangle$ were a scalar product on $\mathscr{O}$, $\mathscr{A}$ would be a self-adjoint operator on $\mathscr{O}$ with respect to it and we would conclude that there is an orthonormal base made by eigenfunctions of $\mathscr{A}$. Unfortunately $\langle\cdot,\cdot\rangle$ is *not* positive definite, it has a one dimensional radical. This causes some technical complications.

LEMMA 5.3.6. *The bilinear map $\langle\cdot,\cdot\rangle$ satisfies the following properties:*
- (a) $\langle v, v\rangle \geq 0$ *for every* $v \in \mathscr{O}$*;*
- (b) $\langle v, v\rangle = 0$ *if and only if* $v(\phi) = \mu \cos\frac{\phi}{2}$ *for some constant* μ*;*
- (c) $\langle v, \cos\frac{\phi}{2}\rangle = 0$ *for every* $v \in \mathscr{O}$*;*
- (d) $\langle \mathscr{A}(v), w\rangle = \langle v, \mathscr{A}(w)\rangle$ *for every* $v, w \in \mathscr{O}$*.*

PROOF. **(a) & (b)** First observe that (a) is equivalent to

$$\frac{1}{4}\int_0^{2\pi} v^2 \leq \int_0^{2\pi} v_\phi^2. \tag{5.3.16}$$

If we write v using the Fourier series expansion $v(\phi) = \sum_{k=1}^{\infty} \alpha_k \cos \frac{k\phi}{2}$, the inequality becomes obvious and it is also clear that equality holds if and only if $\alpha_k = 0$ for every $k \geq 2$.

(c) Let $z(\phi) := \cos \frac{\phi}{2}$ and observe that $\frac{z}{4} + z_{\phi\phi} = 0$ and that $z_\phi(0) = z_\phi(2\pi) = 0$. We therefore compute

$$\langle w, z\rangle = \int_0^{2\pi} w_\phi z_\phi - \frac{1}{4}\int_0^{2\pi} zw = wz_\phi\Big|_0^{2\pi} - \int_0^{2\pi} w\left(z_{\phi\phi} + \frac{z}{4}\right) = 0\,.$$

(d) Consider $z = \mathscr{A}(v)$ and $u = \mathscr{A}(w)$. We then compute

$$\begin{aligned}
\langle \mathscr{A}(v), w\rangle =&\langle z, u_{\phi\phi}\rangle = \int_0^{2\pi} z_\phi u_{\phi\phi\phi} - \frac{1}{4}\int_0^{2\pi} z u_{\phi\phi}\\
=&z_\phi u_{\phi\phi}\Big|_0^{2\pi} - \int_0^{2\pi} z_{\phi\phi}u_{\phi\phi} - \frac{1}{4} z u_\phi\Big|_0^{2\pi} + \frac{1}{4}\int_0^{2\pi} z_\phi u_\phi\\
=&z_\phi u_{\phi\phi}\Big|_0^{2\pi} - z_{\phi\phi}u_\phi\Big|_0^{2\pi} + \int_0^{2\pi} z_{\phi\phi\phi}u_\phi - \frac{1}{4}zu_\phi\Big|_0^{2\pi} + \frac{1}{4}z_\phi u\Big|_0^{2\pi} - \frac{1}{4}\int_0^{2\pi} z_{\phi\phi}u\\
=&\; z_\phi\left(u_{\phi\phi} + \frac{u}{4}\right)\Big|_0^{2\pi} - u_\phi\left(z_{\phi\phi} + \frac{z}{4}\right)\Big|_0^{2\pi} + \langle z_{\phi\phi}, u\rangle\\
=&-\frac{2}{\pi}z_\phi u_\phi\Big|_0^{2\pi} + \frac{2}{\pi}z_\phi u_\phi\Big|_0^{2\pi} + \langle v, \mathscr{A}(w)\rangle = \langle v, \mathscr{A}(w)\rangle\,.
\end{aligned}$$

□

5.3.4. Spectral decomposition. We are now ready to prove the following spectral analysis. First of all we start with the following

LEMMA 5.3.7. *If β is a real number and $h \in \mathscr{O}$ a solution of the following eigenvalue problem*

$$\begin{cases} h_{\phi\phi} = \beta h \\[2ex] h_\phi(0) = -\frac{\pi}{2}\left(\frac{h(0)}{4} + h_{\phi\phi}(0)\right) \end{cases} \tag{5.3.17}$$

then

(a) *$\beta < 0$ and if we set $\beta = -\nu^2$ for $\nu > 0$, then ν is a positive solution of*

$$\nu\cos\nu\pi = \frac{\pi}{2}\left(\frac{1}{4} - \nu^2\right)\sin\nu\pi\,. \tag{5.3.18}$$

(b) *h is a constant multiple of $\sin(\nu(\phi - \pi))$.*

(c) *The positive solutions of (5.3.18) are given by an increasing sequence $\{\nu_k\}_k \in \mathbb{N}$ in which $\nu_1 = \frac{1}{2}$, $\nu_2 > \frac{3}{2}$ and*

$$\lim_{k\to\infty}\frac{\nu_k}{k} = 1 \tag{5.3.19}$$

We will postpone the proof of the lemma and introduce instead the following notation. For $k = 1$ we set

$$\zeta_1(\phi) = \cos\frac{\phi}{2}\,, \tag{5.3.20}$$

while for $k > 1$ we let

$$\zeta_k(\phi) := c_k \sin(\nu_k(\phi - \pi))\,, \tag{5.3.21}$$

where c_k is chosen so that $\langle \zeta_k, \zeta_k \rangle = 1$. Furthermore, we set

$$\zeta_0(\phi) := (\phi - \pi) \sin \frac{\phi}{2}\,, \tag{5.3.22}$$

the relevance of the latter function is that it solves

$$\begin{cases} \zeta_{\phi\phi} = -\frac{\zeta}{4} + \zeta_1 \\ \\ \zeta_\phi(0) = -\frac{\pi}{2}\left(\frac{\zeta(0)}{4} + \zeta_{\phi\phi}(0)\right)\,. \end{cases} \tag{5.3.23}$$

In particular, if we restrict the second derivative operator on the 2-dimensional vector space generated by ζ_1 and ζ_0, its matrix representation is given by

$$\begin{pmatrix} -1/4 & 0 \\ 1 & -1/4 \end{pmatrix}.$$

Consequently, the operator $\mathscr{A}$ is not diagonalizable in $\mathscr{O}$, which is the reason why its spectral analysis is somewhat complicated.

Proposition 5.3.8. *The set $\{\zeta_k\} \subset \mathscr{O}$ is an Hilbert basis for $\mathscr{O}$, namely for every $\zeta \in \mathscr{O}$ there is a unique choice of coefficients $\{a_k\}$ such that*

$$\zeta = \sum_{k=0}^{\infty} a_k \zeta_k\,, \tag{5.3.24}$$

where the series converges in H^1. The coefficients a_k in (5.3.24) are determined by

$$a_k = \langle \zeta, \zeta_k \rangle \qquad \text{for all } k \geq 2, \tag{5.3.25}$$

while a_0 and a_1 are continuous linear functionals on $\mathscr{O}$.

Proof of Lemma 5.3.7. First of all, consider $\beta = 0$. An odd solution of (5.3.17) must then take necessarily the form $c(\phi - \pi)$ and the boundary condition would imply $c = 0$. If $\beta > 0$ observe that a nontrivial function $\zeta \in \mathscr{O}$ solving (5.3.17) would also satisfy $\mathscr{A}(\zeta) = \frac{\zeta}{\beta}$. If $\beta = \nu^2 > 0$ for $\nu > 0$, then $h(\phi) = c(e^{\nu(\phi-\pi)} - e^{-\nu(\phi-\pi)})$ for some constant c. If $c \neq 0$ the boundary condition becomes

$$\nu\left(e^{-\nu\pi} + e^{\nu\pi}\right) = -\frac{\pi}{2}\left(\frac{1}{4} + \nu^2\right)\left(e^{-\nu\pi} - e^{\nu\pi}\right)\,. \tag{5.3.26}$$

The latter identity is equivalent to

$$e^{2\pi\nu}(\pi + 4\pi\nu^2 - 8\nu) = \pi + 4\pi\nu^2 + 8\nu\,. \tag{5.3.27}$$

If we make the substitution $x = 2\pi\nu$, we then are seeking for zeros of the function

$$\Phi(x) = e^x(\pi^2 + x^2 - 4x) - \pi^2 - x^2 - 4x = 0\,.$$

The derivative is given by

$$\Phi'(x) = e^x(x^2 - 2x + \pi^2 - 4) - 2(2 + x)\,,$$

the second derivative by

$$\Phi''(x) = e^x(x^2 + \pi^2 - 6) - 2 \geq 3e^x - 2 > 0\,.$$

In particular Φ is convex and $\Phi'(0) = \pi^2 - 8 > 0$. Thus Φ is strictly increasing and, since $\Phi(0) = 0$, it cannot have positive zeros.

Consider now $\mu = -\beta^2$ for $\nu > 0$. A solution of the PDE in (5.3.17) must then be a linear combination of $\sin \nu(\phi - \pi)$ and $\cos \nu(\phi - \pi)$: the requirement that $h \in \mathscr{O}$ excludes the multiples of $\cos \nu(\phi - \pi)$ in the linear combination.

For $h(\phi) = \sin \nu(\phi - \pi)$ the boundary condition becomes

$$\nu \cos(-\nu\pi) = -\frac{\pi}{2}\left(\frac{1}{4} - \nu^2\right) \sin(-\nu\pi)\,, \tag{5.3.28}$$

which is equivalent to (5.3.18). If we introduce the unknown $x = \pi\nu$, then the equation becomes

$$\Psi(x) := 8x \cos x - \left(\pi^2 - 4x^2\right) \sin x = 0\,.$$

Since $\Psi'(x) = (4x^2 + 8 - \pi^2) \cos x$, Ψ' has a single zero in the open interval $(0, \frac{\pi}{2})$. Since $\Psi(0) = \Psi(\frac{\pi}{2}) = 0$, we infer that there is no zero of Ψ in the open interval $(0, \frac{\pi}{2})$, i.e. any positive ν satisfying (5.3.28) cannot be smaller than $\frac{1}{2}$. Moreover, as Ψ' is strictly negative on $(\frac{\pi}{2}, \frac{3}{2}\pi)$ and $\Psi(\frac{3}{2}\pi) < 0 < \Psi(2\pi)$, the next solution ν lies in $(\frac{3}{2}, 2)$.

Next, there is a unique solution $\nu_k \in (k-1, k)$, for every $k \geq 3$. Indeed, $\Psi((k-1)\,\pi) \cdot \Psi(k\,\pi) < 0$ and Ψ' has a single zero in the open interval $((k-1)\,\pi, k\pi)$. Therefore $(\nu_k)_k$ satisfies (5.3.19). ☐

Proof of Proposition 5.3.8. Let Y be the closure in H^1 of the vector space V generated by $\{\zeta_k\}_{k\geq 2}$. First of all observe that, for some constant C independent of k,

$$1 = \langle \zeta_k, \zeta_k \rangle \geq C^{-1} \|\zeta_k\|^2_{H^1} \qquad \forall k \geq 2\,. \tag{5.3.29}$$

Indeed set $g_k := \sin \nu_k(\phi - \pi)$: (5.3.29) is then equivalent to say that the g_k's satisfy the same inequality. An explicit computation shows that this is equivalent to

$$\begin{aligned}&\int_0^{2\pi} \cos^2 \nu_k(\phi - \pi)\, d\phi - \frac{1}{4\nu_k^2} \int_0^{2\pi} \sin^2 \nu_k(\phi - \pi)\, d\phi\\ &\geq C^{-1} \left(\int_0^{2\pi} \cos^2 \nu_k(\phi - \pi)\, d\phi + \frac{1}{\nu_k^2} \int_0^{2\pi} \sin^2 \nu_k(\phi - \pi)\, d\phi\right)\,.\end{aligned}$$

For each fixed ν_k the fact that the inequality holds for a sufficiently large constant is an easy consequence of the fact that $\int \cos^2 \nu_k(\phi - \pi)$ is positive while $\int \sin^2 \nu_k(\phi - \pi)$ is finite, and $\nu_k \geq \nu_2 > \frac{3}{2}$ for $k \geq 2$. On the other hand by (5.3.19) both integrals converge to π as $k \uparrow \infty$ and thus for a sufficiently large k the inequality holds for $C \geq 2$. Now, for $k \neq j$ we have

$$\langle \zeta_k, \zeta_j \rangle = -\nu_k^2 \langle \mathscr{A}(\zeta_k), \zeta_j \rangle = -\nu_k^2 \langle \zeta_k, \mathscr{A}(\zeta_j) \rangle = \frac{\nu_k^2}{\nu_j^2} \langle \zeta_k, \zeta_j \rangle$$

implying that $\langle \zeta_k, \zeta_j \rangle = 0$.

We next claim that $\zeta_1(\phi) = \cos\frac{\phi}{2} \not\in Y$. Otherwise there is a sequence $\{v_n\} \subset V$ such that $v_n \to \zeta_1$ strongly in H^1. v_n takes therefore the form $v_n = \sum_{k=2}^{N(n)} a_{n,k}\zeta_k$. Using that $\langle v_n, v_n\rangle$ converges to $\langle \zeta_1, \zeta_1\rangle = 0$. Thus we have

$$\lim_{n\to\infty} \sum_{k=2}^{N(n)} a_{n,k}^2 = 0\,. \tag{5.3.30}$$

Now, given that the operator $\mathscr{A}$ is compact we also have that $z_n := \frac{\mathscr{A}(v_n)}{4}$ converges strongly in H^1 to $\frac{\mathscr{A}(\zeta_1)}{4} = -\cos\frac{\phi}{2}$. On the other hand

$$z_n = -\sum_{k=2}^{N(n)} \frac{1}{4\nu_k^2} a_{n,k}\zeta_k\,.$$

We then would have by item (c) of Lemma 5.3.7 and (5.3.29)

$$\begin{aligned}
0 <&\|\zeta_1\|^2_{H^1} = \lim_{n\to\infty} \|z_n\|^2_{H^1} \leq \lim_{n\to\infty} \sum_{k,j=2}^{N(n)} \frac{|a_{n,j}||a_{n,k}|}{16\nu_j^2\nu_k^2} \|\zeta_k\|_{H^1}\|\zeta_j\|_{H^1} \\
\leq& C \limsup_{n\to\infty} \left(\sum_{k=2}^{N(n)} \frac{|a_{n,j}|}{j^2}\right)^2 \leq C \limsup_{n\to\infty} \sum_{k=2}^{N(n)} \frac{1}{k^4} \sum_{j=2}^{N(n)} a_{n,j}^2 \leq C \limsup_{n\to\infty} \sum_{j=2}^{N(n)} a_{n,j}^2 \overset{(5.3.30)}{=} 0\,,
\end{aligned}$$

Consider now the standard H^1 scalar product $(\cdot,\cdot)$ on $\mathscr{O}$ and for every $\zeta \in Y$ let $\zeta = \zeta^\perp + \zeta^\parallel$ be the decomposition of ζ into a multiple of ζ_1 and an element $\zeta^\perp$ orthogonal in the scalar product $(\cdot,\cdot)$ to ζ_1. Since $\zeta_1 \not\in Y$ and Y is closed in H^1, there is a constant $\alpha > 0$ such that $\|\zeta^\perp\|^2_{H^1} \geq \alpha\|\zeta\|^2_{H^1}$. On the other hand using the Fourier expansion of ζ we easily see that $\langle \zeta, \zeta\rangle = \langle \zeta^\perp, \zeta^\perp\rangle \geq C^{-1}\|\zeta^\perp\|^2_{H^1}$ for some universal constant $C > 0$. In particular $\mathscr{A}$ is a compact self-adjoint operator on Y, which implies that $\{\zeta_k\}_{k\geq 2}$ is an orthonormal basis on the Hilbert space Y (endowed with the scalar product $\langle\cdot,\cdot\rangle$).

Consider now the 2-dimensional vector space $Z := \{a_0\zeta_0 + a_1\zeta_1 : a_i \in \mathbb{R}\}$. If $a_0\zeta_0 + a_1\zeta_1 = z \in Z \cap Y$, using Lemma 5.3.6 and the fact that $\langle y, \zeta_1\rangle = 0$ for every $y \in Y$, we can compute

$$\langle z, \zeta_j\rangle = a_0\langle \zeta_0, \zeta_j\rangle = -\nu_j^2\langle a_0\zeta_0, \mathscr{A}(\zeta_j)\rangle = -\nu_j^2\langle a_0\mathscr{A}(\zeta_0), \zeta_j\rangle = 4\nu_j^2\langle a_0\zeta_0, \zeta_j\rangle = 4\nu_j^2\langle z, \zeta_j\rangle$$

for every $j \geq 2$. Since $\nu_j > \frac{3}{2}$ we infer that $\langle z, \zeta_j\rangle = 0$, i.e. that $z = 0$, since $\{\zeta_j\}_{j\geq 2}$ is an orthonormal Hilbert basis of Y with respect to the scalar product $\langle\cdot,\cdot\rangle$. We have thus concluded that $Z \cap Y = \{0\}$. The proof of the proposition will be completed once we show that $Z + Y = \mathscr{O}$. Consider an element $\zeta \in \mathscr{O}$ and define

$$\bar\zeta := \frac{\langle \zeta_0, \zeta\rangle}{\langle \zeta_0, \zeta_0\rangle}\zeta_0 + \sum_{j\geq 2} \langle \zeta_j, \zeta\rangle \zeta_j\,.$$

It turns out that $\bar\zeta \in Z + Y$ and that $\hat\zeta := \zeta - \bar\zeta$ satisfies the condition $\langle \hat\zeta, z\rangle = 0$ for every element $z \in Z + Y =: X$. We claim that the latter condition implies that $\hat\zeta$ is a constant multiple of $\cos\frac{\phi}{2}$. Indeed set $X^\perp := \{v : \langle v, w\rangle = 0 \quad \forall w \in X\}$. Then clearly

$\mathscr{A}(X^\perp) \subset X^\perp$. Moreover $\mathscr{A}$ on $X^\perp$ has only one eigenvalue, namely -4. Consider now $X^\perp \ni v \mapsto Q(v,v) = \langle \mathscr{A}(v), \mathscr{A}(v)\rangle = \langle \mathscr{A}^2(v), v\rangle$ and set

$$m := \sup\{Q(v,v) : v \in X^\perp \quad \text{and} \quad \langle v, v\rangle = 1\}\,, \tag{5.3.31}$$

where at the moment m is allowed to be ∞ as well. If $m = 0$ we then have that $\mathscr{A}(v)$ is a multiple of ζ_1 for every v and this would imply that v itself is a multiple of ζ_1. We therefore assume that m is nonzero. Using the fact that $Q(v, \zeta_1) = 0$ for every v, we can find a maximizing sequence with Fourier expansion

$$v_k := \sum_{j\geq 1} c_{k,j} \cos\left((j + \tfrac{1}{2})\phi\right)$$

for which we easily see that $\langle v_k, v_k\rangle \geq C^{-1}\|v_k\|^2_{H^1}$. We can thus extract a subsequence converging weakly to some v. v clearly belongs to $X^\perp$ and, by the compactness of the operator $\mathscr{A}$ is actually a maximizer of (5.3.31). The Euler-Lagrange condition implies then that $\mathscr{A}^2(v) = mv + b\zeta_1$ for some real coefficients b. Consider now the vector space W generated by ζ_1, v and $\mathscr{A}(v)$. W is then either 2-dimensional or 3-dimensional and $\mathscr{A}$ maps it onto itself. If W were three-dimensional, then the matrix representation of $\mathscr{A}|_W$ in the basis ζ_1, v and $\mathscr{A}(v)$ would be

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & m \end{pmatrix}$$

Since the characteristic polynomial of the latter matrix is $(x-1)(x-m)(x+4)$, $\mathscr{A}$ would have an eigenvalue different from -4 on $W \subset X^\perp$, which is not possible. On the other hand if W were 2-dimensional, then v and $\cos\frac{\phi}{2}$ would be a basis and the matrix representation of $\mathscr{A}|_W$ in that basis would be

$$\begin{pmatrix} -4 & 0 \\ \alpha & \beta \end{pmatrix}$$

Since $\mathscr{A}|_W$ cannot have an eigenvalue different than -4 this would force $\beta = -4$. We then would have $\mathscr{A}(v) = -4v + \alpha\zeta_0$. This would imply that v is an odd solution of $v_{\phi\phi} + \frac{v}{4} = \alpha \cos\frac{\phi}{2}$. The general solution of the latter equation is given by $c_1 \cos\frac{\phi}{2} + c_2 \sin\frac{\phi}{2} + \alpha(\phi - \pi)\sin\frac{\phi}{2}$, for real coefficients c_1 and c_2. The fact that v is odd implies $c_2 = 0$, namely $c_1\zeta_1 + \alpha\zeta_0$. The fact that v is not colinear with ζ_1 implies that $\alpha \neq 0$, but on the other hand since $v \in X^\perp$, $\langle v, \zeta_1\rangle = 0$, which implies $\alpha = 0$. We have reached a contradiction: $X^\perp$ was thus the line generated by ζ_1, proving that indeed $X = \mathscr{O}$. $\square$

5.4. The linear three annuli property

We now define a functional which will be instrumental in proving a suitable decay property for the coefficients of solutions to (5.2.4) and hence to (5.1.17). The latter called three annuli property, is a way to encode the presence of positive exponentials among the coefficients a_k defined by (5.3.9), (5.3.11) in the representations of v^e, v^o, respectively.

We separate the behaviour of the even and odd parts. We start off with the former.

DEFINITION 5.4.1. Consider any $0 < \sigma < s < 3$ real numbers and functions v such that v is even, $v \in H^2((0, 2\pi) \times (\sigma, s))$ and $v(0, t) = v(2\pi, t) = 0$ for every t.

We then define the functional

$$\mathcal{G}_e(v, \sigma, s) := \int_\sigma^s \|v_{\phi\phi}(\cdot, t)\|^2_{L^2((0,2\pi))}\, dt \tag{5.4.1}$$

PROPOSITION 5.4.2. *There is a constant $C > 0$ such that for all $v \in H^2((0, 2\pi) \times (0, 3))$ even with $v(0, t) = 0$*

$$C^{-1} \int_\sigma^s \|v(\cdot, t)\|^2_{H^2((0,2\pi))}\, dt \le \mathcal{G}_e(v, \sigma, s) \le C \int_\sigma^s \|v(\cdot, t)\|^2_{H^2((0,2\pi))}\, dt\,. \tag{5.4.2}$$

Moreover, there is a constant $\eta \in (0, 1)$ such that the following property holds

$$\text{If } \mathcal{G}_e(v, 1, 2) \ge (1 - \eta)\mathcal{G}_e(v, 0, 1) \text{ then } \mathcal{G}_e(v, 2, 3) \ge (1 + \eta)\mathcal{G}_e(v, 1, 2), \tag{5.4.3}$$

for every even solution $v \in H^2((0, 2\pi) \times (0, 3))$ of (5.2.4) satisfying:

$$\int_0^{2\pi} v(\phi, t) \sin \frac{\phi}{2} d\phi = 0\,. \tag{5.4.4}$$

PROOF. Since $v(0, t) = v(2\pi, t) = 0$, we have by the Poincaré inequality

$$\|v(\cdot, t)\|_{L^2((0,2\pi))} \le C\|v_\phi(\cdot, t)\|_{L^2((0,2\pi))}\,.$$

Moreover, using that $v_\phi(\pi, t) = 0$, we conclude

$$\|v(\cdot, t)\|_{H^2((0,2\pi))} \le C\|v_{\phi\phi}(\cdot, t)\|_{L^2((0,2\pi))}\,,$$

which clearly implies (5.4.2).

We now establish (5.4.3). Recall that, since v is even and satisfies (5.4.4), the Fourier decomposition of v reads as (cf. (5.3.7))

$$v(\phi, t) = \sum_{k=1}^\infty \frac{a_k(t)}{\sqrt{\pi}} \sin\left((k + \tfrac{1}{2})\phi\right),$$

where by (5.3.9) the coefficients a_k's satisfy for all $k \in \mathbb{N}$ with $k \ge 1$

$$a_k(t) = C_k\, e^{(k+1)t} + D_k\, e^{-kt}$$

for some C_k, $D_k \in \mathbb{R}$. A simple calculation then gives for all $k \ge 1$

$$\frac{d^2}{dt^2}(a_k^2(t)) \ge a_k^2(t) \ge 0, \tag{5.4.5}$$

establishing the convexity of each a_k^2 for $k \ge 1$. Moreover, note that

$$\mathcal{G}_e(v, \sigma, s) = \int_\sigma^s \underbrace{\sum_{k=1}^\infty (k + \tfrac{1}{2})^4 a_k^2(t)}_{h(t):=}\, dt.$$

We now want to argue that, there is a constant $\eta > 0$ with the following property. If $h \geq 0$ is a nontrivial L^1 function such that $\ddot{h} \geq h$ on $(0,3)$, in particular h is convex, then

$$\int_1^2 h(t)\,dt \geq (1-\eta)\int_0^1 h(t)\,dt \quad \Longrightarrow \quad \int_2^3 h(t)\,dt \geq (1+\eta)\int_1^2 h(t)\,dt\,.$$

Indeed, assume by contradiction this were not true and let h_j be a sequence of nontrivial functions such that $\ddot{h}_j \geq h_j \geq 0$ and

$$\int_1^2 h_j(t)\,dt \geq \max\left\{(1-1/j)\int_0^1 h_j(t)\,dt, (1+1/j)^{-1}\int_2^3 h_j(t)\,dt\right\}\,.$$

After multiplying by a suitable constant we can then assume

$$\int_1^2 h_j(t)\,dt = 1\,.$$

The convexity of the h_j and the uniform bound on $\|h_j\|_{L^1((0,3))}$ implies easily a uniform bound on $\|h_j\|_{L^\infty((\sigma,s))}$ for any $0 < \sigma < s < 3$ and therefore (again by convexity) a uniform Lipschitz bound on any compact subset of $(0,3)$. This ensures the local uniform convergence of a subsequence of h_j (not relabeled) to a nonnegative convex function h, which is L^1 (and thus locally finite) on the open interval $(0,3)$. In particular, $\int_1^2 h(t)\,dt = 1$. On the other hand it is also easy to see that

$$\int_0^1 h(t)\,dt \leq 1, \quad \text{and} \quad \int_2^3 h(t)\,dt \leq 1\,.$$

By the mean-value theorem this implies the existence of three points $0 < t_1 < 1 < t_2 < 2 < t_3 < 3$ where $h(t_2) \geq 1 \geq \max\{h(t_1), h(t_3)\}$. But then the convexity of h implies that h must be constantly equal to 1 on $[t_1, t_3]$. Since the inequality $\ddot{h} \geq h$ is verified in the limit in the sense of distributions, this is a contradiction. □

Next we consider the odd part.

Definition 5.4.3. Fix a constant $c_0 > 0$ appropriately small (whose choice will be specified in Proposition 5.4.4 below). Consider now any couple of real numbers $0 \leq \sigma < s \leq 3$ and a pair of functions (v, θ) such that

(i) v is odd, $v \in H^2((0,2\pi)\times(\sigma,s))$ and $v(0,t) = v(2\pi,t) = 0$ for every t;
(ii) $\theta \in H^2([\sigma,s])$.

Define ζ as in (5.3.1) and let $a_k(t)$ be the coefficients in the representation (5.3.10) and ν_k the numbers in Lemma 5.3.7. We then define the functionals

$$\mathcal{E}(v,\theta,\sigma,s) := \sum_{k=2}^{\infty}\int_\sigma^s (\nu_k^4 a_k^2(t) + \ddot{a}_k^2(t))\,dt \tag{5.4.6}$$

$$\mathcal{F}(v,\theta,\sigma,s) := \int_\sigma^s (\dot{\theta}^2(t) + \ddot{\theta}^2(t) + a_0^2(t) + a_1^2(t) + \ddot{a}_0^2(t) + \ddot{a}_1^2(t))\,dt \tag{5.4.7}$$

$$\mathcal{G}_o(v,\theta,\sigma,s) := \max\{\mathcal{E}(v,\theta,\sigma,s), c_0\mathcal{F}(v,\theta,\sigma,s)\} \tag{5.4.8}$$

PROPOSITION 5.4.4. *There is a constant* $\eta \in (0,1)$ *such that the following property holds for every solution* $(v,\theta) \in H^2((0,2\pi)\times(0,3)) \times H^3((0,2\pi))$ *of* (5.2.4) *with* v *odd:*

(a) *If* $\mathcal{E}(v,\theta,1,2) \geq (1-\eta)\mathcal{E}(v,\theta,0,1)$ *then* $\mathcal{E}(v,\theta,2,3) \geq (1+\eta)\mathcal{E}(v,\theta,1,2)$.

Furthermore, there are positive constants C *and* c_0 *such that the following properties hold for every solution* (v,θ) *of* (5.2.4) *with* v *odd which satisfies in addition* (5.2.5)*:*

(b) *If* $\mathcal{G}_o(v,\theta,1,2) \geq (1-\eta)\mathcal{G}_o(v,\theta,0,1)$ *then* $\mathcal{G}_o(v,\theta,2,3) \geq (1+\eta)\mathcal{G}_o(v,\theta,1,2)$.

and

$$C^{-1}(\|v\|^2_{H^2((0,2\pi)\times(\sigma,s))} + \|\dot\theta\|^2_{H^1((\sigma,s))}) \leq \mathcal{G}_o(v,\theta,\sigma,s) \leq C(\|v\|^2_{H^2((0,2\pi)\times(\sigma,s))} + \|\dot\theta\|^2_{H^1((\sigma,s))})\,, \tag{5.4.9}$$

for all $0 \leq \sigma < s \leq 3$.

PROOF. In order to prove claim (a) consider any of the functions $a_k(t)$ and $\ddot a_k(t)$ and call it $\omega(t)$, and observe we know $k \geq 2$ by assumption. From Proposition 5.3.4 and Lemma 5.3.7 it follows that ω solves then the ODE

$$\ddot\omega(t) - \dot\omega(t) - c\,\omega(t) = 0\,,$$

where c is a constant which depends on k, but it satisfies the bound $c \geq \bar c > 0$ for some positive $\bar c$ independent of k. The polynomial $x^2 - x - c$ has then a positive and a negative solution α^+ and $-\alpha^-$ (also depending on k) with $\alpha^\pm \geq \alpha_0 > 0$. The function $\omega(t)$ is then given by $De^{\alpha^+ t} + Ce^{-\alpha^- t}$. A simple computations shows that

$$\frac{d^2}{dt^2}(\omega^2(t)) \geq \hat c\,\omega^2(t)\,,$$

where the positive constant $\hat c$ can be chosen to depend on α_0 and in particular independent of k. Summing the square of all the coefficients involved in the computation of $\mathcal{E}$ we find a non negative function $h(t)$ with the property that $\ddot h(t) \geq \hat c\, h(t)$ and $\mathcal{E}(v,\theta,s,\sigma) = \int_s^\sigma h(t)\,dt$. The rest of the argument follows the lines of that employed for proving the analogous property for $\mathcal{G}_e$ in Proposition 5.4.2, to which we refer.

Having shown (a) we now turn to (b). We claim that (b) holds for c_0 sufficiently small. Observe that if $\mathcal{E}(v,\theta,1,2) \geq c_0\mathcal{F}(v,\theta,1,2)$, then (b) is simply implied by (a). Thus we may assume $\mathcal{G}_o(v,\theta,1,2) = c_0\mathcal{F}(v,\theta,1,2)$. We argue by contradiction: for $c_0 = 1/j$ choose (v_j,θ_j) such that

$$\mathcal{G}_o(v_j,\theta_j,1,2) \geq \max\{(1-\eta)\mathcal{G}_o(v_j,\theta_j,0,1), (1+\eta)^{-1}\mathcal{G}_o(v_j,\theta_, 2,3)\}\,.$$

Using the linearity we can normalize it so that $\mathcal{F}(v_j,\theta_j,1,2) = 1$. Observe that we have the inequalities

$$1 = \mathcal{F}(v_j,\theta_j,1,2) \geq \max\{(1-\eta)\mathcal{F}(v_j,\theta_j,0,1), (1+\eta)^{-1}\mathcal{F}(v_j,\theta_j,2,3)\} \tag{5.4.10}$$

$$1 = \mathcal{F}(v_j,\theta_j,1,2) \geq j\max\{\mathcal{E}(v_j,\theta_j,1,2), (1-\eta)\mathcal{E}(v_j,\theta_j,0,1), (1+\eta)^{-1}\mathcal{E}(v_j,\theta_j,2,3)\}\,. \tag{5.4.11}$$

From Proposition 5.3.4 we gain a uniform bound on $\|v_j\|_{H^2([0,2\pi)\times(0,3))}$ and $\|\dot\theta_j\|_{H^1((0,3))}$ and consequently (since $\theta_j(0) = 0$) on $\|\theta_j\|_{H^2((0,3))}$. We then extract a sequence converging weakly to $(v,\theta) \in H^2$ which satisfies (5.2.4) and (5.2.5). Consider the functions v and ζ, which are the limit of the corresponding maps constructed from v_j. From (5.4.11) and (5.4.6) we conclude that $\zeta(\phi,t) = a_0(t)\zeta_0(\phi) + a_1(t)\zeta_1(\phi)$. Unraveling the definition of ζ we infer

$$v(\phi,t) = a_0(t)(\phi-\pi)\sin\frac{\phi}{2} + \bar{a}_1(t)\cos\frac{\phi}{2}\,,$$

where $\bar{a}_1(t) = a_1(t) + \frac{\theta(t)}{\sqrt{2\pi}}$. However the boundary conditions $v(0,t) = v(2\pi,t) = 0$ imply $\bar{a}_1 \equiv 0$. We are thus left with the formula $v(\phi,t) = a_0(t)(\phi-\pi)\sin\frac{\phi}{2}$. Inserting in (5.2.4) we get:

$$\begin{cases} \ddot{a}_0(t) - \dot{a}_0(t) = 0 \\ \dot{\theta}(t) - \ddot{\theta}(t) = -\sqrt{2\pi}a_0(t) \\ \theta(0) = 0\,. \end{cases} \tag{5.4.12}$$

From the first equation we find $a_0(t) = c_1 + c_2e^t$, while from the second we find $\theta(t) = d_1 - \sqrt{2\pi}c_1t + d_2e^t - c_2\sqrt{2\pi}te^t$, i.e. $\theta(t) = -\sqrt{2\pi}ta_0(t) + d_1 + d_2e^t$. Using $\theta(0) = 0$ we thus get $\theta(t) = -\sqrt{2\pi}ta_0(t) + d(e^t-1)$. We next use (5.2.5) to derive that a_0 is actually identically null. Indeed, the latter reads as

$$a_0(t)\underbrace{\int_0^{2\pi}\left(\sin\frac{\phi}{2} + \frac{\phi-\pi}{2}\cos\frac{\phi}{2}\right)\sin\frac{\phi}{2}\,d\phi}_{=:I} = 0\,.$$

We compute the integral I as

$$\begin{aligned} I &= \int_0^{2\pi}\left(\sin^2\frac{\phi}{2} + \frac{\phi-\pi}{2}\cos\frac{\phi}{2}\sin\frac{\phi}{2}\right)d\phi \\ &= \int_0^{2\pi}\left(\frac{1}{2}(1-\cos\phi) + \frac{\phi-\pi}{4}\sin\phi\right)d\phi \\ &= \pi - \frac{\phi-\pi}{4}\cos\phi\Big|_0^{2\pi} + \frac{1}{4}\int_0^{2\pi}\cos\phi\,d\phi = \frac{\pi}{2}\,. \end{aligned}$$

So we actually infer $a_0(t) = 0$, which in turn implies $\theta(t) = d(e^t-1)$.

Using the convergences established for (v_j,θ_j) and passing to the limit into (5.4.10) we find

$$1 = \mathcal{F}(v,\theta,1,2) \geq \max\{(1-\eta)\mathcal{F}(v,\theta,0,1), (1+\eta)^{-1}\mathcal{F}(v,\theta,2,3)\}\,,$$

to get by an explicit computation

$$1 = 2d^2\int_1^2 e^{2t}\,dt \geq 2d^2\max\left\{(1-\eta)\int_0^1 e^{2t}\,dt, (1+\eta)^{-1}\int_2^3 e^{2t}\,dt\right\}.$$

Thus, $d \neq 0$ and the latter inequality is equivalent to

$$e^4 - e^2 \geq \max\{(1-\eta)(e^2-1), (1+\eta)^{-1}(e^6-e^4)\}\,,$$

which in turn is equivalent to

$$e^2 \geq \max\{(1-\eta), (1+\eta)^{-1}e^4\}\,.$$

Since $0 < \eta < 1$, the latter would imply $e^2 \geq \frac{e^4}{2}$, which is clearly a contradiction.

The growth conditions in (5.4.9) easily follow from Proposition 5.3.4 by taking into account that (v, θ) solves (5.2.4). $\square$

5.5. Second linearization and proof of Theorem 5.0.1

The three annuli property of the previous section allows us to improve upon Proposition 5.2.1.

PROPOSITION 5.5.1. *Let v_j and θ_j be as in Proposition 5.2.1. Then, there is a pair $(v, \theta) \in C^2_{loc}([0, 2\pi] \times [0, \infty))$ and a subsequence, not relabeled, such that (v_j, θ_j) converges in $C^2([0, 2\pi] \times [0, T))$ to (v, θ) for every $T > 0$. Moreover, (v, θ) solves (5.2.4), satisfies (5.2.5) and there are positive constants ϖ and C such that:*

$$\|v^e - \mathrm{isq}\|_{C^2((0,2\pi)\times[k,k+1])} + \|v^o\|_{C^2((0,2\pi)\times[k,k+1])} + \|\dot\theta\|_{C^1([k,k+1])} \le Ce^{-\varpi k}$$

for all $k \in \mathbb{N} \setminus \{0\}$ if $\lambda = 0$, while

$$\|v^e - \mathrm{isq}\|_{H^2((0,2\pi)\times(k,k+1))} + \|v^o\|_{H^2((0,2\pi)\times(k,k+1))} + \|\dot\theta\|_{H^1((k,k+1))} \le Ce^{-\varpi k}$$

for all $k \in \mathbb{N} \setminus \{0\}$ if $\lambda > 0$.

PROOF. We prove the estimates claimed above separately for the odd and even parts by showing in both cases a nonlinear three annuli property.

We start with the case $\lambda > 0$ by observing that by Proposition 5.2.1 and by Proposition 5.4.4 we can prove: given $\beta > \ln 2$ (to be chosen suitably in what follows), if ε_0 in Theorem 5.0.1 is sufficiently small and (K, u) satisfies the assumptions of Theorem 5.0.1, for every $k \in \mathbb{N}$ we have that if

$$\begin{aligned}
&\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 1, 2), \lambda e^{-\beta(1+k)}\}\\
&\qquad \ge (1 - \tfrac{\eta}{2}) \max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 0, 1), \lambda e^{-\beta k}\}\\
\Longrightarrow &\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 2, 3), \lambda e^{-\beta(2+k)}\}\\
&\qquad \ge (1 + \tfrac{\eta}{2}) \max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 1, 2), \lambda e^{-\beta(1+k)}\}\,.
\end{aligned} \tag{5.5.1}$$

where $\mathcal{G}_o$ is the functional defined in (5.4.8), ϑ, f are given by (5.1.11) and (5.1.13), respectively.

Indeed, assume the claim is false, no matter how small ε_0 in Theorem 5.0.1 is chosen, and let thus f^o_j, ϑ_j be a sequence which violates it for some $k_j \in \mathbb{N}$ when we choose $\varepsilon_0 = \frac{1}{j}$. By rescaling the time variable t (which just implies a rescaling of the variable r in the original problem), we can assume $k_j = j$. Furthermore, by adding a rotation we can assume that $\vartheta_j(0) = 0$, so that we can apply Proposition 5.2.1. Assume by contradiction that

$$\begin{aligned}
&\max\{\mathcal{G}_o(f^o_j(\cdot, \cdot + j), \vartheta_j(\cdot + j), 1, 2), \lambda e^{-\beta(1+j)}\}\\
&\quad \ge \max\Big\{\Big(1 - \tfrac{\eta}{2}\Big) \max\{\mathcal{G}_o(f^o_j(\cdot, \cdot + j), \vartheta_j(\cdot + j), 0, 1), \lambda e^{-\beta j}\},\\
&\qquad\qquad \Big(1 + \tfrac{\eta}{2}\Big)^{-1} \max\{\mathcal{G}_o(f^o_j(\cdot, \cdot + j), \vartheta_j(\cdot + j), 2, 3), \lambda e^{-\beta(2+j)}\}\Big\}\,.
\end{aligned} \tag{5.5.2}$$

Note that if $\lambda > 0$ and $\beta > \ln 2$, then necessarily

$$\mathcal{G}_o(f_j^o(\cdot, \cdot + j), \vartheta_j(\cdot + j), (1,2)) \geq \lambda e^{-\beta(1+j)} \tag{5.5.3}$$

for all $j \geq 0$, since otherwise from (5.5.2) we would conclude for some $i \geq 0$ that

$$e^{-\beta(1+i)} \geq \max\left\{\left(1 - \tfrac{\eta}{2}\right) e^{-\beta i}, \left(1 + \tfrac{\eta}{2}\right)^{-1} e^{-\beta(2+i)}\right\},$$

which is equivalent to

$$\left(1 + \tfrac{\eta}{2}\right)^{-1} \leq e^{\beta} \leq \left(1 - \tfrac{\eta}{2}\right)^{-1},$$

in turn implying $e^\beta \in (2/3, 2)$, as $\eta \in (0,1)$. Clearly, this is a contradiction in view of the choice $\beta > \ln 2$.

Therefore, from this observation, from the definition of v_j in (5.2.3), from (5.5.2), and since the functional $\mathcal{G}_o$ is quadratic, we immediately obtain that

$$\mathcal{G}_o(v_j^o, \theta_j, 1, 2) \geq \max\left\{\left(1 - \tfrac{\eta}{2}\right)\mathcal{G}_o(v_j^o, \theta_j, 0, 1), \left(1 + \tfrac{\eta}{2}\right)^{-1}\mathcal{G}_o(v_j^o, \theta_j, 2, 3)\right\}. \tag{5.5.4}$$

Upon choosing $\beta > 3/2 - \hat{\varepsilon} (> \ln 2)$, where $\hat{\varepsilon}$ has been fixed in the definition of δ_j (cf. (5.2.1)), from (5.5.3) we conclude that $\liminf_j \|v_j^o\|_{H^2((0,2\pi)\times[1,2])} > 0$, because of (5.4.9).

Thus, we can apply Proposition 5.2.1 (d) (recall that $\theta_j(0) = 0$) to extract a subsequence converging to some (v, θ) strongly in $H^2((0,2\pi) \times (\sigma, 3-\sigma)) \times H^2((\sigma, 3-\sigma))$ for all $\sigma \in (0, 3/2)$. Therefore, we have that

$$\mathcal{G}_o(v^o, \theta, 1, 2) = \lim_j \mathcal{G}_o(v_j^o, \theta_j, 1, 2),$$

and in particular we conclude that the pair (v^o, θ) is nontrivial. On the other hand, the functional $\mathcal{G}_o$ is lower semicontinuous with respect to the mentioned convergences, and we thus infer from (5.5.4)

$$\mathcal{G}_o(v^o, \theta, 1, 2) \geq \max\left\{\left(1 - \tfrac{\eta}{2}\right)\mathcal{G}_o(v^o, \theta, 0, 1), \left(1 + \tfrac{\eta}{2}\right)^{-1}\mathcal{G}_o(v^o, \theta, 2, 3)\right\},$$

contradicting Proposition 5.4.4 (b) being (v^o, θ) nontrivial.

Therefore, having completed the proof of (5.5.1), if for some $k_0 \in \mathbb{N}$ we were to have

$$\begin{aligned}&\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k_0), \vartheta(\cdot + k_0), 1, 2), \lambda e^{-\beta(1+k_0)}\} \\ &\qquad \geq \left(1 - \tfrac{\eta}{2}\right)\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k_0), \vartheta(\cdot + k_0), 0, 1), \lambda e^{-\beta k_0}\},\end{aligned}$$

then from (5.5.1) itself, and the choice $\beta > \ln 2$, we would infer that for all $j \geq k_0 + 1$

$$\mathcal{G}_o(f^o(\cdot, \cdot + j), \vartheta, 0, 1) \geq \left(1 + \tfrac{\eta}{2}\right)^{j-(k_0+1)}\mathcal{G}_o(f^o(\cdot, \cdot + k_0), \vartheta(\cdot + k_0), 0, 1).$$

However the latter contradicts the fact that $f^o(\cdot, \cdot + j)$ and $\dot{\vartheta}(\cdot + j)$ converge smoothly to 0 for $j \to \infty$.

We thus conclude that for every $k \in \mathbb{N}$

$$\begin{aligned}&\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 1, 2), \lambda e^{-\beta(k+1)}\} \\ &\qquad \leq \left(1 - \tfrac{\eta}{2}\right)\max\{\mathcal{G}_o(f^o(\cdot, \cdot + k), \vartheta(\cdot + k), 0, 1), \lambda e^{-\beta k}\},\end{aligned}$$

in turn implying, by iteration and by (5.4.9), the existence of constants $C > 0$ and $\varpi \in (0, \ln 2)$ such that

$$\|f^o\|^2_{H^2((0,2\pi)\times(k,k+1))} + \|\dot\vartheta\|^2_{H^1((k,k+1))} \le Ce^{-\varpi k}\left(\|f^o\|^2_{H^2((0,2\pi)\times(0,1))} + \|\dot\vartheta\|^2_{H^1((0,1))} + \lambda\right).$$

In turn, if (v_j, θ_j) are as in the statement of the proposition, we infer

$$\begin{aligned}&\|v^o_j\|^2_{H^2((0,2\pi)\times(k,k+1))} + \|\dot\theta_j\|^2_{H^1((k,k+1))} \\ &\qquad \le Ce^{-\varpi k}\left(\|v^o_j\|^2_{H^2((0,2\pi)\times(0,1))} + \|\dot\theta_j\|^2_{H^1((0,1))} + \lambda\right) = \quad Ce^{-\varpi k}. \end{aligned} \tag{5.5.5}$$

The estimate for the even part follows analogously. Indeed, one first shows the nonlinear three annuli property for

$$g^e(\phi, t+k) := f^e(\phi, t+k) - \langle f^e(\cdot, t+k), \tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2})\rangle_{L^2}\tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2}).$$

Note that $g^e(\cdot, \cdot + j)$ is still even and rescaled by δ_j is converging to an even solution to (5.2.4) satisfying (5.4.4). Hence, by using Proposition 5.3.2 and by arguing as above one deduces that if

$$\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), 1, 2), \lambda e^{-\beta(1+k)}\} \ge (1 - \tfrac{\eta}{2})\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), \lambda e^{-\beta k}\}$$

then

$$\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), 2, 3), \lambda e^{-\beta(2+k)}\} \ge (1 + \tfrac{\eta}{2})\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), 1, 2), \lambda e^{-\beta(1+k)}\}.$$

where $\mathcal{G}_e$ is the functional defined in (5.4.1).

By assumption $f^e(\cdot, \cdot + j)$ converges smoothly to isq for $j \to \infty$, so that for every $k \in \mathbb{N}$

$$\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), 1, 2), \lambda e^{-\beta(k+1)}\} \le \left(1 - \tfrac{\eta}{2}\right)\max\{\mathcal{G}_e(g^e(\cdot, \cdot + k), \vartheta(\cdot + k), 0, 1), \lambda e^{-\beta k}\}.$$

In turn implying, by iteration and by (5.4.2), the existence of constants $C > 0$ and $\varpi \in (0, \ln 2)$ such that

$$\|g^e\|^2_{H^2((0,2\pi)\times(k,k+1))} \le Ce^{-\varpi k}\left(\|g^e\|^2_{H^2((0,2\pi)\times(0,1))} + \lambda\right).$$

Therefore, we conclude that

$$\begin{aligned}&\|v^e_j - \langle v^e_j, \tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2})\rangle_{L^2}\tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2})\|^2_{H^2((0,2\pi)\times(k,k+1))} \\ &\le Ce^{-\varpi k}\left(\|v^e_j - \langle v^e_j, \tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2})\rangle_{L^2}\tfrac{1}{\sqrt\pi}\sin(\tfrac{\phi}{2})\|^2_{H^2((0,2\pi)\times(0,1))} + \lambda\right).\end{aligned} \tag{5.5.6}$$

Having fixed $k \in \mathbb{N}$, in view of (5.5.5) and (5.5.6), the conclusion of the proposition is then a simple application of Proposition 5.2.1 with $[0,1]$ replaced by $[k, k+1]$, that is equivalently by applying it to $f(\cdot, \cdot + k + j)$ and $\vartheta(\cdot + k + j)$ on $[0,1]$, together with a diagonal argument over k and j.

The proof of the case $\lambda = 0$ proceeds similarly: the inequality analogous to (5.5.1) in this setting is obtained by revisiting the argument outlined above, upon normalizing $\mathcal{G}_o(v^o_j, \theta_j, 1, 2) = 1$ for every j. Then, the conclusion follows by taking advantage of (5.5.5), of elliptic regularity and of the stronger convergences described in Proposition 5.2.1 (c). $\square$

Using the second linearization procedure in Proposition 5.5.1 and again the spectral analysis for solutions of (5.2.4) we will then conclude the decay for the curvature at the tip when $\lambda = 0$.

COROLLARY 5.5.2. *There is a constant δ_0 with the following property. Assume (K, u) is as in Theorem 5.0.1 and ϑ as in (5.1.11). Then there are constants C, $\delta_0 > 0$ and $\delta_1 \in (0,1)$ such that, if $\lambda = 0$,*

$$\|f^o(\cdot,t)\|_{C^2([0,2\pi])} + |\dot\vartheta(t)| + |\ddot\vartheta(t)| \le Ce^{-(1+\delta_0)t}\,, \tag{5.5.7}$$

$$\|f^e(\cdot,t) - \mathrm{isq}\|_{C^2([0,2\pi])} \le Ce^{-(1-\delta_1)t}\,, \tag{5.5.8}$$

while, if $\lambda > 0$ for every $\varepsilon \in (0,1)$ there is a constant $C_\varepsilon > 0$ such that

$$\|f^o(\cdot,t)\|_{C^{1,1-\varepsilon}([0,2\pi])} + |\dot\vartheta(t)| \le C_\varepsilon e^{-\delta_0 t}\,, \tag{5.5.9}$$

$$\|f^e(\cdot,t) - \mathrm{isq}\|_{C^{1,1-\varepsilon}([0,2\pi])} \le C_\varepsilon e^{-\delta_0 t}\,. \tag{5.5.10}$$

In particular, in case $\lambda = 0$ we also have

$$\left|\frac{\ddot\vartheta(t) - \dot\vartheta(t) - \dot\vartheta^3(t)}{(1+\dot\vartheta^2(t))^{5/2}}\right| \le Ce^{-(1+\delta_0)t}\,. \tag{5.5.11}$$

PROOF. First of all consider any limit (v,θ) as in Proposition 5.5.1. We discuss separately the behavior of the odd and even parts of v. Let ζ be as in (5.3.1). Recalling that v satisfies (5.2.5), Proposition 5.3.4 yields the expansion

$$\zeta(\phi,t) = \bar a_1(t)\cos\frac{\phi}{2} + \sum_{k=2}^{\infty}(\bar a_{k,-}e^{\mu_{k,-}t} + \bar a_{k,+}e^{\mu_{k,+}t})\zeta_k(\phi)\,,$$

where $\zeta_k(\phi) = c_k\sin(\nu_k(\phi-\pi))$, c_k is such that $\langle\zeta_k,\zeta_k\rangle = 1$ for every $k \ge 2$ (cf. (5.3.21)), where the $\bar a_{k,\pm}$'s are constants, and $\mu_{k,\pm}$ is the positive/negative zero of the quadratic polynomial $x^2 - x - (\nu_k^2 - \frac14)$, i.e. (cf. (5.3.11))

$$\mu_{k,\pm} = \frac12 \pm \nu_k\,.$$

Recalling item (c) in Lemma 5.3.7, we have $\nu_k \ge \nu_2 > \frac32$ when $k \ge 2$ and thus we conclude that $\mu_{k,+} \ge \mu_{2,+} > 2$ and $\mu_{k,-} \le \mu_{2,-} < -1$ for all $k \ge 2$. Therefore, by the decay properties of v^o and $\dot\theta$ in Proposition 5.5.1, we easily infer that $\bar a_{k,+} = 0$ for every $k \ge 2$, so that

$$v^o(\phi,t) = \zeta(\phi,t) + \frac{\theta(t)}{\sqrt{2\pi}}\cos\frac{\phi}{2} = a_1(t)\cos\frac{\phi}{2} + \sum_{k=2}^{\infty}\bar a_k e^{-\mu_k t}\zeta_k(\phi)\,,$$

where we have set $\mu_k = |\mu_k^-| > 1$, $\bar a_k := \bar a_{k,-}$, and $a_1(t) := \bar a_1(t) + \frac{\theta(t)}{\sqrt{2\pi}}$. In particular, note that $\mu_k^2 + \mu_k = \nu_k^2 - \frac14$. From the boundary conditions $v^o(0,t) = v^o(2\pi,t) = 0$ we conclude that

$$a_1(t) = -\sum_{k=2}^{\infty}\bar a_k\zeta_k(0)e^{-\mu_k t}\,. \tag{5.5.12}$$

An elementary computation together with the Ventsel boundary condition satisfied by the ζ_k's (cf. (5.3.17)) and the definition of ν_k (cf. Lemma 5.3.7 (a)) imply that

$$v^o_\phi(0,t) = \frac{\pi}{2}\sum_{k=2}^{\infty} \bar a_k(\nu_k^2 - \tfrac14)\zeta_k(0)e^{-\mu_k t}\,. \tag{5.5.13}$$

Thus, using the ODE in (5.3.4) and $\mu_k^2+\mu_k = \nu_k^2 - \frac14$ we get that

$$\theta(t) = A + Be^t - \sqrt{2\pi}\sum_{k=2}^{\infty}\bar a_k\zeta_k(0)e^{-\mu_k t}\,.$$

As $\lim_{t\to+\infty}\theta(t)\in\mathbb{R}$ we infer that $B=0$, in turn implying $A=\lim_{t\to+\infty}\theta(t)$. Finally, as $\theta(0)=0$ we deduce that

$$\theta(t) = \sqrt{2\pi}\sum_{k=2}^{\infty}\bar a_k\zeta_k(0)(1-e^{-\mu_k t})\,. \tag{5.5.14}$$

Note that the latter together with (5.5.12) easily imply that $\bar a_1(t) = -\frac{A}{\sqrt{2\pi}}$.

We argue similarly for the even part. Namely, Proposition 5.3.2 yields the expansion

$$v^e(\phi,t) = \sum_{k=0}^{\infty}(C_k e^{(k+1)t} + D_k e^{-kt})\tfrac{1}{\sqrt{\pi}}\sin((k+\tfrac12)\phi)\,,$$

and by the decay properties of v^e in Proposition 5.5.1, we infer that $C_k=0$ for all $k\geq 0$, and moreover that $D_0=\sqrt{2}$, namely

$$v^e(\phi,t) = \mathrm{isq}(\phi) + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}D_k e^{-kt}\sin((k+\tfrac12)\phi)\,. \tag{5.5.15}$$

Case $\lambda = 0$. From (5.5.12), (5.5.13), (5.5.14) and (5.5.15) it is then easy to check that for every $T>0$ we have the estimate

$$\|v^o\|_{C^2([0,2\pi]\times[T,2T])} + \|\dot\theta\|_{C^1([T,2T])} \leq Ce^{-\mu_2 T}\left(\|v^o\|_{H^2([0,2\pi]\times[0,1])} + \|\dot\theta\|_{H^1([0,1])}\right)\,,$$

and

$$\|v^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[T,2T])} \leq Ce^{-T}\|v^e - \mathrm{isq}\|_{H^2([0,2\pi]\times[0,1])}\,,$$

where C is a constant independent of T. Fix now T, whose choice will be specified a few paragraphs below. Using the conclusions of Proposition 5.5.1 if $\lambda=0$ we then conclude that, if u is as in Theorem 5.0.1 and ϑ and f as in Lemma 5.1.3 and ε_0 sufficiently small (depending on T), then

$$\begin{aligned}\|f^o\|_{C^2([0,2\pi]\times[T,2T])} + \|\dot\vartheta\|_{C^1([T,2T])} &\leq 2Ce^{-\mu_2 T}\left(\|f^o\|_{H^2([0,2\pi]\times[0,1])} + \|\dot\vartheta\|_{H^1([0,1])}\right)\\ &\leq \bar Ce^{-\mu_2 T}\left(\|f^o\|_{C^2([0,2\pi]\times[0,T])} + \|\dot\vartheta\|_{C^1([0,T])}\right)\,,\end{aligned}$$

and

$$\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[T,2T])} \leq 2Ce^{-T}\|f^e - \mathrm{isq}\|_{H^2([0,2\pi]\times[0,1])} \leq \bar Ce^{-T}\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[0,T])}\,,$$

where the constant $\bar{C}$ is independent of T. By a simple rescaling argument, this actually implies that for all $k \in \mathbb{N}$

$$\begin{aligned}\|f^o\|_{C^2([0,2\pi]\times[(k+1)T,(k+2)T]} &+ \|\dot\vartheta\|_{C^1([(k+1)T,(k+2)T])}\\ &\leq \bar{C}e^{-\mu_2 T}\left(\|f^o\|_{C^2([0,2\pi]\times[kT,(k+1)T])} + \|\dot\vartheta\|_{C^1([kT,(k+1)T])}\right).\end{aligned}$$

and

$$\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[(k+1)T,(k+2)T])} \leq \bar{C}e^{-T}\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[kT,(k+1)T])}.$$

We stress that the constant $\bar{C}$ is independent of T. On the other hand, recalling that $\mu_2 > 1$ (because $\mu_2 = \nu_2 - \frac{1}{2} > 1$, cf. (c) Lemma 5.3.7), while given $\delta_0 \in (0, \mu_2 - 1)$ and $\delta_1 \in (0,1)$, we can choose T itself large enough so that

$$\bar{C}e^{-\mu_2 T} \leq e^{-(1+\delta_0)T}, \qquad \bar{C}e^{-T} \leq e^{-(1-\delta_1)T}.$$

We then can iterate the latter inequalities to infer

$$\begin{aligned}\|f^o\|_{C^2([0,2\pi]\times[(k+1)T,(k+2)T])} &+ \|\dot\vartheta\|_{C^1([(k+1)T,(k+2)T])}\\ &\leq e^{-(1+\delta_0)kT}\left(\|f^o\|_{C^2([0,2\pi]\times[0,T])} + \|\dot\vartheta\|_{C^1([0,T])}\right),\end{aligned}$$

and

$$\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[(k+1)T,(k+2)T])} \leq e^{-(1-\delta_1)kT}\|f^e - \mathrm{isq}\|_{C^2([0,2\pi]\times[0,T])}.$$

This easily gives the conclusions (5.5.7), (5.5.8) and, in particular, (5.5.11).

Case $\lambda > 0$.

The proof of the estimates in (5.5.9) and (5.5.10) follows as in the previous case by using the conclusions of Proposition 5.5.1 for $\lambda > 0$ rather than those for $\lambda = 0$ there. ☐

The latter Corollary 5.5.2 implies easily Theorem 5.0.1.

5.5.1. Proof of Theorem 5.0.1. Corollary 5.5.2 gives a C^{2,δ_0} estimate for the parametrization of $K \cap B_2$ if $\lambda = 0$ and a C^{1,δ_0} estimate if $\lambda > 0$. More precisely, the unit tangent $\tau(r)$ to $K \cap B_2$ at the point $\gamma(r) = r(\cos\alpha(r), \sin\alpha(r))$ (cf. (5.0.5)) is given by the expression

$$\tau(r) = \frac{1}{\sqrt{1 + r^2\alpha'(r)^2}}\Big((\cos\alpha(r), \sin\alpha(r)) + r\,\alpha'(r)\,(-\sin\alpha(r), \cos\alpha(r))\Big).$$

using the relation $r = e^{-t}$ and $\alpha(r) = \vartheta(|\ln r|)$ (cf. (5.1.11)) and (5.5.7) if $\lambda = 0$, respectively (5.5.9) if $\lambda > 0$, we easily check that $|\tau''(r)| \leq Cr^{\delta_0 - 1}$, respectively $|\tau'(r)| \leq Cr^{\delta_0 - 1}$. Integrating the latter inequalities between r_1 and r_2 we reach the estimates

$$|\tau'(r_2) - \tau'(r_1)| \leq C(r_2 - r_1)^{\delta_0} \qquad \forall\, 0 < r_1 < r_2 < 1/2,$$

for $\lambda = 0$, and for $\lambda > 0$

$$|\tau(r_2) - \tau(r_1)| \leq C(r_2 - r_1)^{\delta_0} \qquad \forall\, 0 < r_1 < r_2 < 1/2.$$

In turn, the latters imply respectively C^{1,δ_0} and C^{0,δ_0} estimates on the tangent $\tau(r)$ to K at the point $\gamma(r)$, and moreover that $\tau(r)$ has a limit $\tau_0 \in \mathbb{S}^1$ as $r \to 0^+$ in both cases.

In addition, we get a decay estimate for the curvature when $\lambda = 0$. Indeed, using (5.1.37), namely

$$\kappa(r) = r^{-1}\,\frac{\dot\vartheta(|\ln r|) + \dot\vartheta^3(|\ln r|) - \ddot\vartheta(|\ln r|)}{(1+\dot\vartheta^2(|\ln r|))^{3/2}}\,,$$

from estimate (5.5.11) in Corollary 5.5.2 we conclude that

$$|\kappa(r)| \leq C r^{\delta_0}$$

when $\lambda = 0$. Note that the denimonator of the quantity estimated in (5.5.11) differs from the denominator appearing in the formula of the curvature by a multiplicative factor which is $1+\dot\theta^2(|\ln r|)$: the latter however converges to 1 as $r \to 0$, and in fact according to our estimates is bounded above by an absolute constant on the interval of interest.

Moreover, it follows easily that there is an $\eta > 0$, depending only upon C and ϖ, such that $B_\eta \cap K$ is a graph $\{t\,\tau_0 + \psi(t)\,\tau_0^\perp\} \cap B_\eta$ for some function $\psi : [0,\eta] \to \mathbb{R}$. If $\lambda = 0$ the latter is C^{2,δ_0} smooth, with $\psi(0) = \psi'(0) = \psi''(0) = 0$ and $\|\psi\|_{C^{2,\delta_0}} \leq \bar C$. When $\lambda > 0$, ψ in C^{1,δ_0} with $\psi(0) = \psi'(0) = 0$ and $\|\psi\|_{C^{1,\delta_0}} \leq \bar C$.

On the other hand, if ε_0 is sufficiently small, since $|\alpha'(r)| \leq \varepsilon_0/\eta$ for $r \in (\eta, 1/2)$ (recall that $\gamma \in C^{1,1}((0,2)))$, we conclude that $K \cap B_{1/4}$ is a graph in the coordinates induced by the orthonormal base $\{\tau_0, \tau_0^\perp\}$. Finally, since such graph will have to be sufficiently close to the line $\{(s,0) : s \geq 0\}$ by assumption (iv), we conclude that τ_0 must be close to $(1,0)$. Therefore, $K \cap B_{1/4}$ is in fact a graph in the standard coordinates, as claimed in Theorem 5.0.1.

CHAPTER 6

Some consequences of the epsilon-regularity theory

6.1. Main statements

In this chapter, we use the ε-regularity theory and some more ideas to prove several structural results about the set K and a more refined analysis of the behavior of K around some particular points.

6.1.1. Structure of K. We consider the set K^j of pure jump points and note that, according to Theorem 3.1.1, they can be characterized as those points where the scaled Dirichlet energy and the flatness of K both converge to 0. Its complement $\Sigma := K \setminus K^j$, usually called in the literature the *singular set*, can then be decomposed as follows: $\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \Sigma^{(3)}$, where

$$\begin{aligned}
\Sigma^{(1)} &:= \{x \in \Sigma : \liminf_{r\downarrow 0} d(x,r) = 0\},\\
\Sigma^{(2)} &:= \{x \in \Sigma : \liminf_{r\downarrow 0} \beta(x,r) = 0\},\\
\Sigma^{(3)} &:= \{x \in \Sigma : \limsup_{r\downarrow 0} d(x,r) > 0, \text{ and } \limsup_{r\downarrow 0} \beta(x,r) > 0\}.
\end{aligned}$$

According to the Mumford-Shah conjecture, we should have $\Sigma^{(3)} = \emptyset$. Note also that the set $K^\sharp$ of high energy points is in fact given by $K \setminus (K^j \cup \Sigma^{(1)}) = \Sigma^{(2)} \cup \Sigma^{(3)}$.

We will use this subdivision to prove Theorem 1.4.2. A starting point will be the following one.

THEOREM 6.1.1. *Let (K, u) be an absolute, or generalized global minimizer of E_λ in Ω. Then the following holds:*

(i) $\Sigma^{(1)}$ *consists of the triple junctions of K and is discrete.*
(ii) $\Sigma^{(2)}$ *consists of the regular loose ends of K and is discrete.*
(iii) $\Sigma^{(3)}$ *consists of those connected components of K which are singletons and of irregular terminal points of connected components with positive length. Every $p \in \Sigma^{(3)}$ can also be characterized as an accumulation point of infinitely many connected components of K.*

In particular, we conclude that $K^i = \Sigma^{(3)}$ and coincides with the set $K^\sharp \setminus \Sigma^{(2)}$. In view of the above theorem we will refer to $\Sigma^{(1)}$ as the set of triple junctions and to $\Sigma^{(2)}$ as the set of cracktips, or regular loose ends. Observe that the discreteness of the sets immediately implies their countability. Finally, the case of restricted minimizers is not covered in Theorem 6.1.1 for the following reason:

- If the number of connected components of a minimizer is infinite, then the "restrictedness" is in fact an empty condition, and the minimizer is actually an absolute minimizer;
- If the number of connected components is finite, it follows from the theory exposed so far that the conclusion of the Mumford-Shah conjecture holds since any blow-up at a point of K is either a global pure jump, or a global triple junction, or a cracktip.

Finally, we remark that the first conclusion of the structure theorem can be slightly strengthened, in the sense that $\Sigma^{(1)}$ can be also characterized as those points where the liminf of $d(x, r)$ falls below a certain positive threshold, cf. Proposition 6.3.2 below.

6.1.2. Higher integrability of the gradient and the size of the singular set. As already remarked, the Mumford-Shah conjecture is equivalent to proving that the set K^i is empty for an absolute minimizer. An obvious consequence of the identity $K^i = K^{(3)} = K^\sharp \setminus K^{(2)}$ is the following

COROLLARY 6.1.2. $\mathcal{H}^1(K^i) = \mathcal{H}^1(K^\sharp) = 0$.

Using a porosity argument it was first shown by David that indeed the Hausdorff dimension of $K^\sharp$ must be strictly smaller than 1. We will show the latter theorem in a different way. Following the approach of Ambrosio, Fusco, and Hutchinson in [2], we relate a higher integrability estimate of ∇u to a suitable dimension estimate for $K^\sharp$. More precisely, using again $K^i = K^\sharp \setminus K^{(2)}$ we can immediately conclude the following

COROLLARY 6.1.3. *If $\nabla u \in L^p$, then $\mathcal{H}^{1-\frac{p}{2}}(K^i) = 0$.*

It was indeed conjectured by De Giorgi (in all space dimensions) that $\nabla u \in L^p_{loc}$ for all $p < 4$ (cf. with [17, conjecture 1]). So far only a first step into this direction has been established.

THEOREM 6.1.4. *There is $p > 2$ such that $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$ for all (K, u) either absolute minimizers of E_λ in Ω, and for all open sets $\Omega \subseteq \mathbb{R}^2$. The same conclusion holds for generalized global minimizers in $\mathbb{R}^2$.*

Theorem 6.1.4 has been proven first by De Lellis and Focardi in [20] using a reverse Hölder inequality (cf. Section 6.4). Shortly after De Philippis and Figalli established the result without any dimensional limitation in [22], as a more direct application of the ε-regularity theory at pure jumps (cf. Section 6.5). In this chapter we will present both proofs.

6.1.3. An equivalent formulation of the Mumford-Shah conjecture. Note that, because cracktip is indeed a minimizer, we certainly cannot expect $\nabla u \in L^4$, while on the other hand the Mumford-Shah conjecture would easily imply $\nabla u \in L^p_{loc}$ for every $p < 4$. Introducing a finer scale of spaces, it is in fact possible to characterize the conjecture in terms of a sharp integrability result for ∇u. The relevant statement was given in the introduction in Theorem 1.4.4, which for the reader convenience we recall here.

THEOREM 6.1.5. *Let (K, u) be an absolute, or generalized global minimizer of E_λ in Ω. The set K^i is empty if and only if $\nabla u \in L^{4,\infty}_{loc}$, namely if and only if for every compact set $U \subset \Omega$ there is a constant $C = C(U)$ such that*

$$|\{x \in U : |\nabla u(x)|^4 \leq M\}| \leq \frac{C}{M} \qquad \forall M \geq 1\,. \tag{6.1.1}$$

6.2. Proof of Theorem 6.1.1

By the ε-regularity theory, in order to show (i) and (ii) it suffices to show, respectively, that:

(i') If $x \in \Sigma^{(1)}$, then there is one blow-up $(\Sigma_\infty, u_\infty)$ at x which is a triple junction centered at the origin.

(ii') If $x \in \Sigma^{(2)}$, then there is one blow-up $(\Sigma_\infty, u_\infty)$ at x which is a cracktip centered at the origin.

In fact, in both cases it would follow that K is regular in some punctured disk $B_\rho(x) \setminus \{x\}$, implying that both $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are discrete sets.

In both cases, we will assume, without loss of generality, that $x = 0$. We start with (i'). Hence, we fix a sequence of radii $r_j \downarrow 0$ with the property that

$$\lim_{j\to\infty} \frac{1}{r_j} \int_{B_{r_j}} |\nabla u|^2 = 0$$

and assume, upon extraction of a subsequence, that the corresponding rescaled functions and sets as in Theorem 2.2.3 converge to a generalized global minimizer (K_∞, u_∞). By Theorem 2.2.3(i) we know that the Dirichlet energy of u_∞ vanishes identically, and so by Theorem 2.4.1 we know that (K_∞, u_∞) is either a constant, or a global pure jump, or a global triple junction. On the other hand, by the Hausdorff convergence of the rescaled sets K_{0,r_j} to K_∞, K_∞ must contain the origin, hence the constant is excluded. If it were a global pure jump, then the ε-regularity theory would imply that $0 \in K^j$, a contradiction to the assumption $0 \in \Sigma$. Hence, (K_∞, u_∞) must be a global triple junction. Now, if 0 were not the meeting point of the three half lines forming K_∞, then it would be a pure jump point of K_∞, and a simple diagonal argument would imply the existence of a blow-up of (K, u) at 0 which is a pure jump, a possibility which has already been excluded.

In case (ii') we choose our sequence so that $\beta(r_k)$ converges to 0 and we see immediately that the corresponding blow-up (K_∞, u_∞) has the property that K_∞ is contained in a line. As above observe also that K_∞ contains the origin. A result by Léger, cf. [27], implies then that K_∞ is either the whole line or a half-line. In Chapter 4 (cf. Theorems 4.1.1 and 4.4.1) we already saw the arguments leading to the conclusion that in the first case, we have a pure jump and in the second we have a cracktip. On the other hand, the pure jump case is excluded because the ε-regularity theory would imply that 0 is a pure jump of (K, u).

For completeness, we give a simple self-contained argument that K_∞ is either a line or a half-line. Even though we follow Léger's ideas, rather than relying on his "magic formula" in Proposition 2.5.7, we give a direct elementary derivation from the inner variations.

Without loss of generality we assume that $K_\infty \subset \{x : x_2 = 0\}$. We will show below that

(Cl) any nontrivial connected component of K_∞ is infinite.

Taking (Cl) for granted, observe that, since the ε-regularity theory implies that regular pure jumps are dense in K_∞, the existence of a connected component of K_∞ which is a singleton implies the existence of infinitely many connected components of K_∞ which are closed intervals of bounded positive length. Therefore, (Cl) implies that K_∞ is either a line or a half-line or the union of two half-lines. We then just need to exclude the possibility that K_∞ contains two connected components which are half-lines. To exclude the latter, observe that we can use Theorem 2.3.2 to "blow-down" the pair (K_∞, u_∞) introducing

$$K'_R := \{R^{-1}x : x \in K_\infty\} \qquad u'_R(x) := R^{-1/2}u(Rx)$$

and letting $R \uparrow \infty$. For an appropriate sequence R_k the pair (K'_{R_k}, u'_{R_k}) converges then to a global minimizer (J_∞, v_∞) for which J_∞ is a full line, which implies that the global minimizer is a pure jump. But then the ε-regularity theory would imply that for all sufficiently large k the set $K'_{R_k} \cap B_1$ is connected, which is a contradiction.

We now come to the core of the argument for (ii'), which is the proof of (Cl). Consider a connected component K' of K_∞ with positive length: it is either a closed interval or a closed half-line or the full line. By symmetry, it suffices to show that, if K' has a left extremum, which we will denote by a, then it cannot have a right extremum. Observe that any point $p = (x_1, 0) \in K'$ which is not an extremum is necessarily a regular jump point (since any blow-up at such points would be a global pure jump) and we define

$$u^\pm_\infty(p) = \lim_{t \downarrow 0} u_\infty(x_1, \pm t)\,.$$

It is also the case that, since we are assuming that K' is not the full line, K does not disconnect $\mathbb{R}^2$. It then follows from Proposition 4.6.1 that u_∞ is continuous at a. Recall that $\frac{\partial u^\pm_\infty}{\partial x_2} = 0$ at all $p \in K'$ which are not extrema, while (since the curvature of K_∞ is 0),

$$\left|\frac{\partial u^+_\infty}{\partial x_1}(p)\right| = \left|\frac{\partial u^-_\infty}{\partial x_1}(p)\right|\,.$$

The above derivatives can only vanish at isolated points: if the zeros were to accumulate to a point q which is not an extremum, the classical theory of harmonic functions would imply that both $u^\pm$ would be constant in a neighborhood of q. But then unique continuation would imply that u_∞ is actually constant on $\mathbb{R}^2 \setminus K$. If we next fix a point p where

$$\frac{\partial u^+_\infty}{\partial x_1}(p) \neq 0\,,$$

then in a neighborhood of it, we either have

$$\frac{\partial u^+_\infty}{\partial x_1} = \frac{\partial u^-_\infty}{\partial x_1} \tag{6.2.1}$$

or we have

$$\frac{\partial u^+_\infty}{\partial x_1} = -\frac{\partial u^-_\infty}{\partial x_1}\,. \tag{6.2.2}$$

But then, again the unique continuation theory for harmonic function would tell that one of the two alternatives holds on K' with the exception of its extrema. However, if the first alternative were to hold, integrating it starting from the left extremum a we would conclude that $u_\infty^+ = u_\infty^-$ on K': we could then remove it and decrease the energy, violating the minimality. We must therefore have (6.2.2). We next claim that $\frac{\partial u_\infty^+}{\partial x_1}$ never vanishes in the interior of K': that, and (6.2.2) would imply that there cannot be a right extremum of K', because at such extremum u_∞ would be continuous, and hence u_∞^+ and u_∞^- would have to coincide, while integrating (6.2.2) from a to b we would conclude that $u_\infty^+(b) \neq u_\infty^-(b)$.

We thus are left to show that, if $(x_1, 0)$ is an interior point of K_∞, then $\frac{\partial u_\infty^+}{\partial x_1}(x_1, 0) \neq 0$. By translation we can assume that $(x_1, 0) = (0, 0)$ and we will then show that

$$2\pi \left(\frac{\partial u_\infty^+}{\partial x_1}\right)^2 (0) = \int_{\{(t,0)\in(\mathbb{R}\times\{0\})\setminus K_\infty\}} \frac{dt}{t^2}\,. \tag{6.2.3}$$

Léger in [27] derives (6.2.3) directly from his "magic formula" (2.5.18), while here we will show how it follows immediately from the inner variation formula (1.5.3), using the same test of the proof of Proposition 2.5.7.

We fix positive radii $\rho < R$ and consider the vector field $\psi(x, y) = \varphi(|(x, y)|)(x, -y)$ where

$$\varphi(t) = \begin{cases} \rho^{-2} - R^{-2} & \text{if } t \leq \rho \\ t^{-2} - R^{-2} & \text{if } \rho \leq t \leq R \\ 0 & \text{otherwise}\,. \end{cases}$$

Strictly speaking, the latter is not a valid test in the inner variation formula, which in our case would read

$$\int_{\mathbb{R}^2\setminus K} (2\nabla^T u_\infty \cdot D\psi\, \nabla u_\infty - |\nabla u_\infty|^2 \mathrm{div}\, \psi) = \int_{K_\infty} e^T \cdot D\psi\, e\, d\mathcal{H}^1\,, \tag{6.2.4}$$

because ψ is not C^1. However, if we assume that $\mathcal{H}^1(K_\infty \cap (\partial B_\rho \cup \partial B_R)) = 0$, it is easily seen that the left hand side (6.2.4) makes sense because ψ is $\mathcal{H}^1$-a.e. differentiable on K_∞, while a standard regularization argument shows the validity of the formula. Next we compute $D\psi$ in the two relevant domains where it does not vanish:

$$D\psi = \left(\frac{1}{\rho^2} - \frac{1}{R^2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{on } B_\rho \tag{6.2.5}$$

$$D\psi = -\frac{1}{R^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{(x^2+y^2)^2} \begin{pmatrix} y^2 - x^2 & 2xy \\ -2xy & y^2 - x^2 \end{pmatrix} \qquad \text{on } B_R \setminus \bar{B}_\rho\,. \tag{6.2.6}$$

Inserting on (6.2.4) we immediately get

$$\begin{aligned} &\frac{2}{\rho^2} \int_{B_\rho\setminus K_\infty} \left(\left(\frac{\partial u_\infty}{\partial x_1}\right)^2 - \left(\frac{\partial u_\infty}{\partial x_2}\right)^2\right) - \frac{2}{R^2} \int_{B_R\setminus K_\infty} \left(\left(\frac{\partial u_\infty}{\partial x_1}\right)^2 - \left(\frac{\partial u_\infty}{\partial x_2}\right)^2\right) \\ =& \frac{1}{\rho^2}\mathcal{H}^1(K_\infty \cap B_\rho) - \frac{1}{R^2}\mathcal{H}^1(K_\infty \cap B_R) - \int_{\{(t,0)\in(B_R\setminus B_\rho)\cap K_\infty\}} \frac{dt}{t^2}\,. \end{aligned}$$

We choose ρ sufficiently small so that $B_\rho \cap K_\infty = B_\rho \cap (\mathbb{R} \times \{0\})$. Then we let $R \uparrow \infty$ and use

$$\int_{B_R \setminus K_\infty} |\nabla u_\infty|^2 + \mathcal{H}^1(B_R \cap K_\infty) \leq 2\pi R\,,$$

to conclude

$$\frac{2}{\rho^2} \int_{B_\rho \setminus K_\infty} \left(\left(\frac{\partial u_\infty}{\partial x_1} \right)^2 - \left(\frac{\partial u_\infty}{\partial x_2} \right)^2 \right) = \frac{2}{\rho} - \int_{\{(t,0) \in K_\infty \setminus B_\rho\}} \frac{dt}{t^2}\,.$$

Since, however, the choice of ρ yields

$$\frac{2}{\rho} = \int_{\{(t,0) \in (\mathbb{R} \times \{0\}) \setminus B_\rho\}} \frac{dt}{t^2}\,,$$

we arrive to

$$\frac{2}{\rho^2} \int_{B_\rho \setminus K_\infty} \left(\left(\frac{\partial u_\infty}{\partial x_1} \right)^2 - \left(\frac{\partial u_\infty}{\partial x_2} \right)^2 \right) = \int_{\{(t,0) \in (\mathbb{R} \times \{0\}) \setminus K_\infty\}} \frac{dt}{t^2}\,. \tag{6.2.7}$$

We now observe that, because 0 is a regular jump point, the function $\left(\frac{\partial u_\infty}{\partial x_1}\right)^2 - \left(\frac{\partial u_\infty}{\partial x_2}\right)^2$ is actually continuous over B_ρ and its value at 0 is $\left(\frac{\partial u_\infty^+}{\partial x_1}\right)^2(0)$. So, letting $\rho \downarrow 0$ in (6.2.7), we obtain (6.2.3).

It finally remains to show (iii). It is however a straightforward consequence of the material in Chapter 4 that, if a point $p \in K$ is not the accumulation point of infinitely many connected components of K, then the blow-ups at K are either cracktips or global pure jumps or triple junctions, while it is a straightforward consequence of the ε-regularity theory that if one of the blow-ups belong to these subsets, then $K \cap B_\rho(p)$ is connected for all sufficiently small radii ρ.

6.3. Proofs of Corollary 6.1.3 and Theorem 6.1.5

Corollary 6.1.3 is a simple consequence of the structure theorem and standard results in measure theory.

Proof of Corollary 6.1.3. Suppose that $|\nabla u| \in L^p_{loc}(\Omega)$ for some $p > 2$. For every $s \in [0, 2]$ consider the set

$$\Lambda_s := \left\{ x \in \Omega : \limsup_{r \to 0} r^{-s} \int_{B_r(x)} |\nabla u|^p > 0 \right\}.$$

Clearly, Λ_s is a subset of K. Choose $s := 1 - \frac{p}{2}$, then Hölder's inequality, yields that

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{B_r(x) \setminus K} |\nabla u|^2 = 0$$

for every $x \in K \setminus \Lambda_s$. In particular, Theorem 6.1.1 implies that every point $x \in K \setminus \Lambda_s$ is either a pure jump point or a triple junction. We thus conclude that $K^i \subset \Lambda_s$. Introduce finally the Radon measure defined on any subset $E \subseteq \Omega$ by

$$\mu(E) := \int_E |\nabla u|^p \,.$$

As $\Lambda_s \subset K$, then $\mu(\Lambda_s) = 0$. We can employ [4, Theorem 2.56] to infer that $\mathcal{H}^s(\Lambda_s) = 0$, which concludes the proof. □

For the proof of Theorem 6.1.5 we first make the following preliminary observation.

LEMMA 6.3.1. *Let $f \in L^{4,\infty}_{loc}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, then for all $\varepsilon > 0$ the set*

$$D_\varepsilon := \left\{ x \in \Omega :\ \liminf_r \frac{1}{r} \int_{B_r(x)} f^2(y) \geq \varepsilon \right\} \tag{6.3.1}$$

is locally finite.

PROOF. We shall show in what follows that if $f \in L^{4,\infty}(\Omega)$ then D_ε is finite. An obvious localization argument then gives the general case.

Let $\varepsilon > 0$ and consider the set D_ε in (6.3.1) above. First note that for any $B_r(x) \subset \Omega$ and any $\lambda > 0$ we have the estimate

$$\begin{aligned} \int_{\{y \in B_r(x) :\, |f(y)| \geq \lambda\}} f^2(y) &\leq \int_{\{y \in \Omega :\, |f(y)| \geq \lambda\}} f^2(y) \\ &= 2 \int_\lambda^\infty t\ |\{y \in \Omega :\ |f(y)| \geq t\}|\ dt \leq \int_\lambda^\infty \frac{2C}{t^3} dt = \frac{C}{\lambda^2}\,, \end{aligned} \tag{6.3.2}$$

where $C > 0$ is the constant introduced in (6.1.1). If $x \in D_\varepsilon$ and $r > 0$ satisfy

$$\int_{B_r(x)} f^2(y) \geq \frac{\varepsilon}{2} r, \tag{6.3.3}$$

by choosing $\lambda = 2(C/r\varepsilon)^{1/2}$ in (6.3.2) we conclude

$$\int_{\{y \in B_r(x) :\, |f(y)| < 2(\frac{C}{r\varepsilon})^{1/2}\}} f^2(y) \geq \frac{\varepsilon}{4} r. \tag{6.3.4}$$

Furthermore, the trivial estimate

$$\int_{\{y \in B_r(x) :\, |f(y)| < \lambda\}} f^2(y) < \pi \lambda^2 r^2,$$

implies for $\lambda = (\varepsilon/8\pi r)^{1/2}$

$$\int_{\{y \in B_r(x) :\, |f(y)| < (\frac{\varepsilon}{8\pi r})^{1/2}\}} f^2(y) < \frac{\varepsilon}{8} r. \tag{6.3.5}$$

By collecting (6.3.4) and (6.3.5) we infer

$$\int_{\{y \in B_r(x) :\, (\frac{\varepsilon}{8\pi r})^{1/2} \leq |f(y)| < 2(\frac{C}{r\varepsilon})^{1/2}\}} f^2(y) \geq \frac{\varepsilon}{8} r,$$

that in turn implies

$$\Big|\{y \in B_r(x) : |f(y)| \geq (\frac{\varepsilon}{8\pi r})^{1/2}\}\Big| \geq \frac{\varepsilon^2 r^2}{32C}. \tag{6.3.6}$$

Let $\{x_1, \ldots, x_N\} \subseteq D_\varepsilon$ and $r > 0$ be a radius such that the balls $B_r(x_i) \subsetneq \Omega$ are disjoint and (6.3.3) holds for each x_i. Then, from (6.3.6) and the fact that $f \in L^{4,\infty}(\Omega)$, we infer

$$N\frac{\varepsilon^2 r^2}{32C} \leq \Big|\{y \in \Omega : |f(y)| \geq (\frac{\varepsilon}{8\pi r})^{1/2}\}\Big| \leq \frac{C(8\pi r)^2}{\varepsilon^2} \Longrightarrow N \leq \frac{2^{11}C^2\pi^2}{\varepsilon^4},$$

and the conclusion follows at once. □

Another ingredient in the proof of Theorem 6.1.5 is the following strengthened version of the first conclusion in Theorem 6.1.1.

PROPOSITION 6.3.2. *There is a universal constant $\varepsilon > 0$ with the following property. If (K, u) is an absolute or generalized minimizer of E_λ, then*

$$\Sigma^{(1)} \cup K^j = \left\{x \in K : \liminf_{r\downarrow 0} d(x, r) < \varepsilon\right\} .$$

PROOF. We first of all observe that there is an absolute constant $\bar{\varepsilon} > 0$ such that, if (K, u) is a generalized global minimizer with $0 \in K$ and $\int_{B_8\setminus K} |\nabla u|^2 < \bar{\varepsilon}$, then either $B_1 \cap K$ is diffeomorphic to a pure jump or $B_4 \cap K$ is diffeomorphic to a triple junction. This can be done using the compactness Theorem 2.3.2 and the ε-regularity theory. Indeed, assume by contradiction (K_j, u_j) is a sequence of generalized global minimizers which satisfy

$$\int_{B_8\setminus K_j} |\nabla u_j|^2 < \frac{1}{j}$$

but violate our claim. Then up to subsequences they converge to a global generalized minimizer (K_∞, u_∞) with $0 \in K_\infty$ and such that

$$\int_{B_8\setminus K_\infty} |\nabla u_\infty|^2 = 0 .$$

It follows from the theory of Chapter 5 that u_∞ is an elementary global minimizer. If it is a pure jump, then it would follow from the ε-regularity theory that $K_j \cap B_1$ is diffeomorphic to a line for a sufficiently large j. If it is a triple junction, but the meeting point of the three half-lines is not contained in B_2, then the very same conclusion can again be drawn. If on the other hand, it is a triple junction with a meeting point at $x \in B_2$, then we can use the ε-regularity theory to say that $B_6(x) \cap K_j$ is diffeomorphic to a triple junction for a sufficiently large j.

Given the first part, consider now $\varepsilon := \frac{\bar{\varepsilon}}{8}$ and fix a point where $\liminf_{r\downarrow 0} r^{-1} d(x, r) < \varepsilon$ for some absolute minimizer or generalized global minimizer (K, u). Then we draw from the above argument that there is at least one blow-up (K_∞, u_∞) at x with the property that $0 \in K_\infty$ is either a pure jump point or a triple junction. A further blow-up with a simple diagonal argument implies then that there is a blow-up of (K, u) at x which is either a pure jump or a triple junction. This concludes the proof. □

Proof of Theorem 6.1.5. First of all assume that (K, u) is an absolute minimizer of E_λ (or a generalized minimizer of E) in some Ω and that $\nabla u \in L^{4,\infty}_{loc}(\Omega)$. Without loss of generality we can assume that Ω is the unit ball B_1 and that $\nabla u \in L^{4,\infty}(B_1)$. Let ε be the positive number of Proposition 6.3.2 and define

$$D_\varepsilon := \left\{x : \liminf_{r\downarrow 0} \frac{1}{r}\int_{B_r(x)\setminus K} |\nabla u|^2 \geq \varepsilon\right\}.$$

By Lemma 6.3.1 D_ε is a finite set, which we enumerate as $\{p_1, \ldots, p_N\}$ and by Proposition 6.3.2 any point $p \in K \setminus \{p_1, \ldots, p_N\}$ is either a pure jump or a triple junction. It then turns out that, if $p \in K \setminus \{p_1, \ldots, p_N\}$ there are only two possible alternatives: either there is a Lipschitz curve $\gamma \subset K$ connecting p with a point $q \in \partial B_1$, or there is a Lipschitz curve $\gamma \subset K$ connecting p to one of the p_i's. So, we can partition K into a collection of those finitely many connected components which contain at least one of the p_i's and a countable number of connected components which do not, but whose closure must contain a point of ∂B_1. A point $q \in \partial B_1$ cannot be the accumulation point of an infinite number of the latter, because otherwise, we would have $\mathcal{H}^1(K) = \infty$. It follows therefore from Theorem 6.1.1 that each p_i is necessarily a regular loose end of K.

Consider on the contrary an absolute minimizer (K, u) in Ω for which the Mumford-Shah conjecture holds. Let $U \subset\subset \Omega$. Then there is a finite subset $\{p_1, \ldots, p_N\} \subset U \cap K$ of regular loose ends, while all the other points of $K \cap U$ are regular jump points or triple junctions. From the regularity theorem at regular loose ends, we infer the existence of balls $B_{r_i}(p_i)$ and of an absolute constant C with the properties that $|\nabla u(x)| \leq C|x - p_i|^{-1/2}$. On the other hand by the regularity theory at triple junctions and regular jump points, $|\nabla u|$ is bounded on $U \setminus \bigcup_i B_{r_i}(p_i)$. Hence, we conclude that there is a constant $C(U)$ such that

$$|\nabla u(x)| \leq C \max\left\{|x - p_i|^{-1/2}\right\}.$$

Since the function on the right-hand side belongs to $L^{4,\infty}(U)$, this completes the proof. $\square$

6.4. Reverse Hölder inequality for ∇u

Following a classical path, the key ingredient used in [20, Theorem 1.3] to establish Theorem 6.1.4 (for E_0) is a reverse Hölder inequality for the gradient, which we state independently.

Theorem 6.4.1. *For all $q \in (1, 2)$ there exist $C_0 > 0$, $r_0 \in (0, 1)$ such that if (K, u) is either a restricted, or an absolute, or a generalized global minimizer of $E_\lambda(\cdot, \cdot, B_1, g)$, with $\lambda \leq 1$, $\|g\|_\infty \leq M_0$ (we use the notation introduced in Assumption 2.2.1), then for every $x \in B_{1/2}$ and $r \in (0, r_0)$*

$$\left(\fint_{B_r(x)\setminus K} |\nabla u|^2\right)^{1/2} \leq C_0 \left(\fint_{B_{2r}(x)\setminus K} |\nabla u|^q\right)^{1/q} + C_0 \|g\|_\infty. \tag{6.4.1}$$

Theorem 6.1.4 is then a consequence of a by now classical result by Giaquinta and Modica [24].

THEOREM 6.4.2. *Let $v \in L^q_{loc}(\Omega)$, $q > 1$, be nonnegative such that for some constants $\beta > 0$, $t \geq 1$ and $R_0 > 0$*

$$\left(\fint_{B_r(z)} v^q\right)^{1/q} \leq \beta \fint_{B_{t\,r}(z)} v + \fint_{B_{t\,r}(z)} h$$

for all $z \in \Omega$, $r \in (0, R_0 \wedge \operatorname{dist}(z, \partial\Omega))$, and with $h \in L^s(\Omega)$ for some $s > 1$.

Then $v \in L^p_{loc}(\Omega)$ for some $p > q$ and there is $C = C(\beta, n, q, p, \lambda) > 0$ such that

$$\left(\fint_{B_r(z)} v^p\right)^{1/p} \leq C \left(\fint_{B_{2r}(z)} v^q\right)^{1/q} + C \left(\fint_{B_{2r}(z)} h^q\right)^{1/q}.$$

We provide here a new proof of Theorem 6.4.1 relying on the compactness properties established in Theorem 2.2.3 rather than on the theory of Caccioppoli partitions as originally done in [20]. Nevertheless, the overall strategy is the same.

PROOF OF THEOREM 6.4.1. Assume by contradiction that there is $q \in (1, 2)$ such that for every $j \in \mathbb{N}$ one can find (K_j, u_j) either restricted, or absolute, or generalized global minimizer of $E_{\lambda_j}(\cdot, \cdot, B_1, g_j)$ with $\lambda_j \leq 1$ and $\|g_j\|_\infty \leq M_0$, radii $r_j \downarrow 0$, and points $x_j \in B_{1/2}$ such that

$$j \left(\fint_{B_{2r_j}(x_j)\setminus K_j} |\nabla u_j|^q\right)^{1/q} + j\|g_j\|_\infty \leq \left(\fint_{B_{r_j}(x_j)\setminus K_j} |\nabla u_j|^2\right)^{1/2} \tag{6.4.2}$$

Consider the rescalings $(\widetilde{K}_j, \widetilde{u}_j) := ((K_j)_{x_j,r_j}, (u_j)_{x_j,r_j})$, the conditions in Assumption 2.2.1 are satisfied, then by Theorem 2.2.3 $(\widetilde{K}_j, \widetilde{u}_j)$ converge up to a subsequence to a blow-up limit (K_∞, u_∞) which is a global generalized (restricted) minimizer of E_0 (in case (K_j, u_j) are restricted minimizers). Moreover, by (6.4.2) and the density upper bound (1.3.1) we get

$$\left(\fint_{B_2\setminus\widetilde{K}_j} |\nabla\widetilde{u}_j|^q\right)^{1/q} \leq j^{-1} \left(\fint_{B_1\setminus\widetilde{K}_j} |\nabla\widetilde{u}_j|^2\right)^{1/2} \leq j^{-1}(2\pi + \lambda_j \pi M_0^2 r_j)^{1/2}.$$

The latter inequality and Theorem 2.2.3 yield that

$$\lim_j \int_{B_2\setminus\widetilde{K}_j} |\nabla\widetilde{u}_j|^2 = \int_{B_2\setminus K_\infty} |\nabla u_\infty|^2 = 0.$$

Therefore, Corollary 4.4.4 implies that (K_∞, u_∞) is an elementary global minimizer; in turn, this and Theorem 2.2.3 imply that for every $R > 0$

$$\lim_j \int_{B_R\setminus\widetilde{K}_j} |\nabla\widetilde{u}_j|^2 = 0\,. \tag{6.4.3}$$

In particular, (K_∞, u_∞) is either a constant, or a global pure jump or a global triple junction (cf. Theorem 2.4.1). In any case, for all $R > 0$

$$\mathcal{H}^0(K_\infty \cap \partial B_R) \leq 3, \qquad \mathcal{H}^1(K_\infty \cap B_R) \leq 3R\,.$$

With fixed $R > 0$ we shall now choose a radius $\rho \in [\frac{R}{4}, R]$ conveniently as outlined in what follows

(a$_1$) if $\mathcal{H}^0(K_\infty \cap \partial B_R) = 0$ choose $\rho = R$;
(a$_2$) if $\mathcal{H}^0(K_\infty \cap \partial B_R) = 2$ then $K_\infty \cap B_R$ is a segment and $\partial B_R \setminus K_\infty$ is the union of two arcs. Then, either each of them has a length less or equal to $\frac{4\pi}{3}R$ or $K \cap B_{\frac{R}{2}} = \emptyset$. In the first instance set $\rho = R$, in the second $\rho = \frac{R}{2}$;
(a$_3$) if $\mathcal{H}^0(K_\infty \cap \partial B_R) = 3$, then $K_\infty \cap B_R$ is a (possibly off-centered) propeller and $\partial B_R \setminus K_\infty$ is the union of three arcs. Then, either each of them has length less or equal to $(2\pi - \frac{1}{8})R$, in this case set $\rho = R$, or $\mathcal{H}^0(K_\infty \cap \partial B_{\frac{R}{2}}) = 2$. In the last event, we are back to the setting of item (a2) with the radius $\frac{R}{2}$ playing the role of R. Thus, $K_\infty \cap B_{\frac{R}{2}}$ is a segment, and $\partial B_{\frac{R}{2}} \setminus K_\infty$ is the union of two arcs, and either each of them has a length less or equal to $\frac{2\pi}{3}R$ or $K \cap B_{\frac{R}{4}} = \emptyset$. In the first instance set $\rho = \frac{R}{2}$, in the second $\rho = \frac{R}{4}$.

By (6.4.3) and Theorem 2.2.3 we may select $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$

$$\int_{B_\rho \setminus \widetilde{K}_j} |\nabla \widetilde{u}_j|^2 + \mathrm{dist}_H^2(\widetilde{K}_j \cap B_\rho, K_\infty \cap B_\rho) \leq \varepsilon\rho\,.$$

Therefore, thanks to Theorem 3.1.1 and Theorem 3.1.2 for all $j \geq j_0$ and for some $\beta \in (0, \frac{1}{3})$ one of the following alternatives is true

(b$_1$) $\widetilde{K}_j \cap B_\rho = \emptyset$;
(b$_2$) For each $t \in ((1-\beta)\rho, \rho)$, $\partial B_t \setminus \widetilde{K}_j$ is the union of two arcs γ_1^j and γ_2^j each with length $< (2\pi - \frac{1}{9})t$, whereas $\widetilde{K}_j \cap B_t$ is connected and divides B_t in two components B_1^j, B_2^j with $\partial B_i^j = \gamma_i^j \cup (\widetilde{K}_j \cap \overline{B_t})$;
(b$_3$) For each $t \in ((1-\beta)\rho, \rho)$, $\partial B_t \setminus \widetilde{K}_j$ is the union of three arcs γ_1^j, γ_2^j and γ_3^j each with length $< (2\pi - \frac{1}{9})t$, whereas $B_t \cap \widetilde{K}_j$ is connected and divides B_t in three connected components B_1^j, B_2^j and B_3^j with $\partial B_i^j \subset \gamma_i^j \cup (\widetilde{K}_j \cap \overline{B_t})$.

We finally choose $r \in ((1-\beta)\rho, \rho)$ and a subsequence, not relabeled, such that

(A) $h_j := \widetilde{u}_j|_{\partial B_r}$ belongs to $W^{1,2}(\gamma)$ for any connected component γ of $\partial B_r \setminus \widetilde{K}_j$;
(B) h_j satisfies

$$\int_{\partial B_r \setminus \widetilde{K}_j} |h_j'|^q d\mathcal{H}^1 \leq \frac{1}{\beta\rho} \int_{B_\rho} |\nabla \widetilde{u}_j|^q\,.$$

Let us conclude our argument by showing that (6.4.2) is violated for j sufficiently big. To this aim, we note first that the choice $\beta < \frac{1}{3}$ yields that $r > \frac{2}{3}\rho$.

In case (b$_1$) holds, $\partial B_r \cap \widetilde{K}_j = \emptyset$ and $\widetilde{u}_j$ is the harmonic extension of its boundary trace h_j. Hence, the embedding $W^{1,q}(\partial B_r) \to H^{1/2}(\partial B_r)$ implies for some constant $C > 0$ (independent of j)

$$\begin{aligned}\int_{B_{\frac{2}{3}\rho}} |\nabla \widetilde{u}_j|^2 &\leq \int_{B_r} |\nabla \widetilde{u}_j|^2 \leq C \min_{c \in \mathbb{R}} \|h_j - c\|^2_{H^{1/2}(\partial B_r)} \\ &\leq C \left(\int_{\partial B_r} |h_j'|^q \, d\mathcal{H}^1\right)^{2/q} \overset{(B)}{\leq} C \left(\frac{1}{\beta\rho} \int_{B_\rho} |\nabla \widetilde{u}_j|^q\right)^{2/q}.\end{aligned}$$

Choosing $R = \rho \in [\frac{3}{2}, 2]$ and scaling back, we get a contradiction to (6.4.2).

In case (b$_2$) or (b$_3$) hold the construction is similar. Denote by J_j a *minimal connection* relative to $\widetilde{K}_j \cap \partial B_r$, i.e. a closed and connected set in $\overline{B}_r$ containing $\widetilde{K}_j \cap \partial B_r$ with shortest length (cf. [6, Theorem 4.4.20]). Then J_j splits $\overline{B_r}$ into two (case (b$_2$)) or three (case (b$_3$)) regions denoted by B_r^i. Let γ^i be the arc of ∂B_r contained in the boundary of B_r^i. The ensuing Lemma 6.4.3 provides a function $w_j^i \in W^{1,2}(B_r)$ with boundary trace h_j and satisfying for some absolute constant $C > 0$

$$\int_{B_r} |\nabla w_j^i|^2 \leq C \left(\int_{\gamma^i} |h_j'|^q \, d\mathcal{H}^1 \right)^{2/q} . \tag{6.4.4}$$

Denote by w_j the function equal to w_j^i on B_j^i, equal to $\widetilde{u}_j$ on $B_{r_j^{-1}}(-x_j) \setminus \cup_i B_j^i$, and extend J_j as $J_j \setminus B_r = \widetilde{K}_j \setminus B_r$. Clearly, $w_j \in H^1(B_{r_j^{-1}}(-x_j) \setminus J_j)$. Scaling back (J_j, w_j) to B_1 we obtain a pair admissible to test the minimality of (K_j, u_j). On setting $\widetilde{g}_j := r_j^{-1/2} g_j(x_j + r_j \cdot)$, for some constant $C > 0$ we then deduce that

$$\begin{aligned}
\int_{B_{\frac{2}{3}\rho} \setminus \widetilde{K}_j} |\nabla \widetilde{u}_j|^2 &\leq \int_{B_r \setminus J_j} |\nabla \widetilde{u}_j|^2 \\
&\leq \int_{B_r \setminus J_j} |\nabla w_j|^2 + \underbrace{\mathcal{H}^1(J_j \cap B_r) - \mathcal{H}^1(\widetilde{K}_j \cap B_r)}_{\leq 0} \\
&\qquad + \lambda r_j^2 \int_{B_r \setminus J_j} |w_j - \widetilde{g}_j|^2 - \lambda r_j^2 \int_{B_r \setminus \widetilde{K}_j} |\widetilde{u}_j - \widetilde{g}_j|^2 \\
&\leq \int_{B_r \setminus J_j} |\nabla w_j|^2 + C \|\widetilde{g}_j\|_\infty^2 \rho^2 \, r_j^2 \overset{(6.4.4)}{\leq} C \left(\int_{\partial B_r \setminus \widetilde{K}_j} |h_j'|^q \, d\mathcal{H}^1 \right)^{2/q} + C \|g_j\|_\infty^2 \rho^2 r_j \\
&\overset{(B)}{\leq} C \left(\frac{1}{\beta \rho} \int_{B_\rho \setminus \widetilde{K}_j} |\nabla \widetilde{u}_j|^q \right)^{2/q} + C \|g_j\|_\infty^2 \rho^2 r_j \, .
\end{aligned}$$

Choosing $\rho \in [\frac{3}{2}, 2]$ and R accordingly, scaling back we find a contradiction to (6.4.2). $\square$

In the proof above we have used an elementary extension result that we prove in what follows.

Lemma 6.4.3. *For any $q \in (1, 2)$ there exists $C = C(q) > 0$ such that the following holds. For any arc $\gamma \subseteq \partial B_1$ and any $h \in W^{1,q}(\gamma)$, there exists $w \in W^{1,2}(B_1)$ with trace g on γ and*

$$\|\nabla u_\infty\|_{L^2(B_1)} \leq \frac{C}{\left(2\pi - \mathcal{H}^1(\gamma)\right)^{1-\frac{1}{q}}} \|h'\|_{L^q(\gamma)}. \tag{6.4.5}$$

In addition, if $h \in L^\infty(\gamma)$, then $\|w\|_{L^\infty(B_1)} \leq 2\|h\|_{L^\infty(\gamma)}$.

Proof. Let α, $\beta \in \partial B_1$ denote the extreme points of γ. By the Hölder inequality

$$|h(\alpha) - h(\beta)| = \left| \int_\gamma h' d\mathcal{H}^1 \right| \leq \left(\mathcal{H}^1(\gamma)\right)^{1-\frac{1}{q}} \|h'\|_{L^q(\gamma)} \, .$$

Linearly interpolating h on $\partial B_1 \setminus \gamma$, we get an extension $\widetilde{h} \in W^{1,q}(\partial B_1)$ of h satisfying the estimate

$$\|\widetilde{h}'\|^q_{L^q(\partial B_1\setminus\gamma)} = (2\pi - \mathcal{H}^1(\gamma))^{1-q}|h(\alpha)-h(\beta)|^q \leq \left(\frac{\mathcal{H}^1(\gamma)}{2\pi - \mathcal{H}^1(\gamma)}\right)^{q-1} \|h'\|^q_{L^q(\gamma)} . \tag{6.4.6}$$

In turn, if we set $\hat{h} := \widetilde{h} - \fint_{\partial B_1} \widetilde{h}$, the Poincaré inequality and (6.4.6) yield

$$\|\hat{h}\|^q_{L^q(\partial B_1)} \leq C\|\widetilde{h}'\|^q_{L^q(\partial B_1)} \leq C\left(\frac{2\pi}{2\pi - \mathcal{H}^1(\gamma)}\right)^{q-1} \|h'\|^q_{L^q(\gamma)} . \tag{6.4.7}$$

The embedding $W^{1,q}(\partial B_1) \to H^{1/2}(\partial B_1)$ provides us with a function $v \in W^{1,2}(B_1)$ with boundary trace $\hat{h}$ and such that

$$\|\nabla v\|_{L^2(B_1)} \leq C\|\hat{h}\|_{H^{1/2}(\partial B_1)} \leq C\|\hat{h}\|_{W^{1,q}(\partial B_1)} \overset{(6.4.7)}{\leq} \frac{C}{(2\pi - \mathcal{H}^1(\gamma))^{1-\frac{1}{q}}} \|h'\|_{L^q(\gamma)}.$$

By the latter inequality the function $w := v + \fint_{\partial B_1} \widetilde{h}$ fulfills the assertions of the Lemma. □

6.5. Higher integrability of ∇u via the porosity of K

A central role in the approach by De Philippis and Figalli [22] to establish the higher integrability of the gradient in any dimension is played by the ensuing improvement of Theorem 3.1.1 due to David [13], and in higher dimension to Rigot [32], and Maddalena and Solimini [30]. Here, we provide a proof in the two-dimensional setting using again Theorem 2.2.3.

THEOREM 6.5.1. *There are constants C_1, r_1, $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ there exists $t_\varepsilon \in (0, \frac{1}{2})$ with the following property. If (K, u) is either a restricted, or an absolute, or a generalized global minimizer of $E_\lambda(\cdot, \cdot, B_2, g)$, with $\lambda \leq 1$, $\|g\|_\infty \leq M_0$ (we use the notation introduced in Assumption 2.2.1), then for every $x \in K \cap B_{1/2}$ and for every $r \in (0, r_1)$ there exist $y \in K \cap B_r(x)$ and $\rho \in (t_\varepsilon r, r)$ such that $B_\rho(y) \subset B_r(x)$ and the assumptions of Theorem 3.1.1 are satisfied in $B_\rho(y)$, namely for some $\theta \in [0, 2\pi]$*

$$\Omega^j(\theta, y, \rho) + \lambda\|g\|^2_\infty \rho^{\frac{1}{2}} < \varepsilon_0$$

($\varepsilon_0 > 0$ being the constant in Theorem 3.1.1).

In addition, it is true that

(i) *$K \cap B_\rho(y)$ is a $C^{1,\alpha}$ graph;*
(ii) *and,*

$$\rho\|\nabla u\|^2_{L^\infty(B_\rho(y))} \leq C_1\, \varepsilon. \tag{6.5.1}$$

PROOF. Assume by contradiction that we can find sequences $\varepsilon_j \downarrow 0$, $r_j \downarrow 0$, (K_j, u_j) of either restricted, or absolute, or generalized global minimizers of $E_{\lambda_j}(\cdot, \cdot, B_2, g_j)$, with

$\lambda_j \leq 1$, $\|g_j\|_\infty \leq M_0$, points $x_j \in K_j \cap B_{1/2}$, and scalars $t_j \in (0, \frac{1}{2})$ such that for every $y \in K_j \cap B_{r_j}(x_j)$ and $\rho \in (t_j r_j, r_j)$ with $B_\rho(y) \subset B_{r_j}(x_j)$, then

$$\rho^{-2}\mathrm{dist}^2_H(K_j \cap B_{2\rho}(y), \mathcal{R}_\theta(\mathscr{V}_0)) \cap B_{2\rho}(y)) + \rho^{-1}\int_{B_{2\rho}(y)\setminus K_j} |\nabla u_j|^2 + \lambda_j\|g_j\|^2_\infty \rho^{\frac{1}{2}} \geq \varepsilon_0 \quad (6.5.2)$$

for every $\theta \in [0, 2\pi]$ (we are using the notation introduced at the beginning of Section 3.1).

Consider the rescalings $(\widetilde{K}_j, \widetilde{u}_j) := ((K_j)_{x_j,r_j}, (u_j)_{x_j,r_j})$, being the conditions in Assumption 2.2.1 satisfied, by Theorem 2.2.3 $(\widetilde{K}_j, \widetilde{u}_j)$ converge up to a subsequence to a blow-up limit (K_∞, u_∞) which is a global generalized (restricted) minimizer of E_0 (in case (K_j, u_j) are restricted minimizers). Moreover, Corollary 3.1.5 yields that the subset of K_∞ of pure jump points is relatively open and has full $\mathcal{H}^1$ measure. Let $z \in K_\infty$ be one such point, then necessarily Theorem 3.1.1 implies that

$$t^{-2}\mathrm{dist}^2_H(K_\infty \cap B_{2t}(z), \mathcal{R}_{\theta_z}(\mathscr{V}_0)) \cap B_{2t}(z)) + t^{-1}\int_{B_{2t}(z)\setminus K_\infty} |\nabla u_\infty|^2 < \varepsilon_0$$

for some $\theta_z \in [0, 2\pi]$ and for every $t > 0$ sufficiently small. In turn, Theorem 2.2.3 implies the existence of points $z_j \in K_j \cap B_{r_j}(x_j)$ such that

$$\begin{aligned}(tr_j)^{-2}&\mathrm{dist}^2_H(K_j \cap B_{2tr_j}(z_j), \mathcal{R}_{\theta_z}(\mathscr{V}_0)) \cap B_{2tr_j}(z_j)) \\ &+ (tr_j)^{-1}\int_{B_{2tr_j}(z_j)\setminus K_j} |\nabla u_j|^2 + \lambda_j\|g_j\|^2_\infty (tr_j)^{\frac{1}{2}} < \varepsilon_0\end{aligned}$$

contradicting (6.5.2).

Item (i) follows directly from Theorem 3.1.1; instead estimate (6.5.1) is a consequence of item (i) and classical elliptic regularity results for the Neumann problem since u satisfies (2.5.3) and (2.5.4) (cf. [4, Theorem 7.53]). □

Theorem 6.5.1 can be rephrased by asserting that the set $K \setminus K^j$ is $(t_\varepsilon, 1)$-*porous* in K (cf. for instance [31]). Hence, following the papers by David [13], Rigot [32] and Maddalena and Solimini [30], one can estimate the Minkowski dimension (and thus the Hausdorff dimension) of the set $K \setminus K^i$ using Theorem 6.5.1, a by now classical result by David and Semmes [16], and the Alfohrs regularity of K (cf. Lemma 2.1.2 and Theorem 2.1.3), which we restate for convenience: if (K, u) is either an absolute or a restricted, or a generalized minimizer of E_λ on B_2 then

$$\epsilon\, r \leq \mathcal{H}^1(K \cap B_r(x)) \leq \varepsilon_\lambda r \quad (6.5.3)$$

for some constant $\epsilon > 0$, for $\varepsilon_\lambda := (2 + \lambda\|g\|^2_\infty)\pi$, for every $x \in K$, and for every $r \in (0, 1 \wedge \mathrm{dist}(x, \partial\Omega))$.

Instead, in Proposition 6.5.4 below, an estimate of the Minkowski dimension will be obtained directly. The key technical results to prove the higher integrability of ∇u are contained in Proposition 6.5.4 for which we need two preparatory lemmas. The first one provides the selection of suitable families of radii obtained via the De Giorgi's slicing/averaging principle.

LEMMA 6.5.2. *There are positive constants M_1, C_1 such that if $M \geq M_1$ for every (K,u) (absolute, restricted, generalized, or generalized restricted) minimizer of E_λ on B_2 we can find three decreasing sequences of radii such that*

(i) $1 \geq R_h > S_h > T_h > R_{h+1}$;
(ii) $8M^{-(h+1)} \leq R_h - R_{h+1} \leq M^{-\frac{(h+1)}{2}}$, *and* $S_h - T_h = T_h - R_{h+1} = 4M^{-(h+1)}$;
(iii) $\mathcal{H}^1(K \cap (\overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}})) \leq C_1 M^{-\frac{(h+1)}{2}}$;
(iv) $R_\infty = S_\infty = T_\infty \geq 1/2$.

PROOF. Let $R_1 = 1$, given R_h we construct S_h, T_h and R_{h+1} as follows.

Set $N_h := \lfloor \frac{M^{\frac{h+1}{2}}}{8} \rfloor \in \mathbb{N}$ and fix $M_1 \in \mathbb{N}$ such that $N_h \geq \lfloor \frac{M^{\frac{h+1}{2}}}{16} \rfloor$ for $M \geq M_1$. Here, $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$.

The annulus $B_{R_h} \setminus \overline{B}_{R_h - 8M^{-\frac{h+1}{2}}}$ contains the N_h disjoint sub annuli $\overline{B}_{R_h - 8(i-1)M^{-(h+1)}} \setminus \overline{B}_{R_h - 8iM^{-(h+1)}}$, $i \in \{1, \ldots, N_h\}$, of equal width $8\,M^{-(h+1)}$. By averaging we can find an index $i_h \in \{1, \ldots, N_h\}$ such that

$$\begin{aligned}\mathcal{H}^1\Big(K \cap (\overline{B}_{R_h - 8(i_h-1)M^{-\frac{h+1}{2}}} \setminus \overline{B}_{R_h - 8i_h\, M^{-\frac{h+1}{2}}})\Big)& \\ \leq \frac{1}{N_h}\,\mathcal{H}^1\Big(K \cap (\overline{B}_{R_h} \setminus \overline{B}_{R_h - 8M^{-\frac{h+1}{2}}})\Big) &\overset{(6.5.3)}{\leq} \varepsilon_\lambda \frac{R_h}{N_h} \leq 8\varepsilon_\lambda\, M^{-\frac{h+1}{2}},\end{aligned}$$

so that (iii) is established with $C_1 := 8\varepsilon_\lambda$. Finally, set

$$S_h := R_h - 8(i_h - 1)M^{-(h+1)},\ R_{h+1} := R_h - 8i_h\, M^{-(h+1)},\ T_h := \frac{1}{2}(S_h + R_{h+1}),$$

then items (i) and (ii) follow by the very definition, and item (iv) from (ii) if M_1 is sufficiently big. □

The second lemma has a geometric flavor.

LEMMA 6.5.3. *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz with*

$$f(0) = 0, \quad \|\nabla f\|_{L^\infty(\mathbb{R}^{n-1},\mathbb{R}^{n-1})} \leq \eta. \tag{6.5.4}$$

If $G := \mathrm{graph}(f) \cap B_2$ and $\eta \in (0, 1/15]$, then for all $\delta \in (0, 1/2)$ and $x \in (\overline{B}_{1+\delta} \setminus B_1) \cap G$

$$\mathrm{dist}(x, (\overline{B}_{1+2\delta} \setminus B_{1+\delta}) \cap G) \leq \frac{3}{2}\delta.$$

PROOF. Clearly by (6.5.4) we get

$$\|f\|_{W^{1,\infty}(B_2)} \leq 3\eta.$$

Let $x = (y, f(y)) \in (\overline{B}_{1+\delta} \setminus B_1) \cap G$ and $\hat{x} := (\lambda\, y, f(\lambda\, y))$, with λ to be chosen suitably. Note that as $|x| \geq 1$ we have

$$\begin{aligned}|f(\lambda\, y) - \lambda\, f(y)| &\leq |f(\lambda\, y) - f(y)| + |\lambda - 1|\,|f(y)| \\ &\leq |\lambda - 1|\,(\|\nabla f\|_{L^\infty(\mathbb{R}^{n-1},\mathbb{R}^{n-1})}\,|y| + \|f\|_{L^\infty(\mathbb{R}^{n-1})}) \leq 3\eta|\lambda - 1|\,|x|.\end{aligned}$$

Hence,

$$|\hat{x} - x| \leq |\hat{x} - \lambda\, x| + |\lambda - 1||x| \leq (3\eta + 1)|\lambda - 1|\,|x|.$$

It is easy to check that the choice $\lambda = 1 + \frac{5}{4}\delta|x|^{-1}$ gives the conclusion. □

Following De Philippis and Figalli [22], we prove next a version of the mentioned porosity result by David and Semmes [16] that is suitable for our purposes. In what follows, $(E)_r$ denotes the open r-neighborhood of a given set E, i.e. $(E)_r = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, E) < r\}$.

PROPOSITION 6.5.4. *There exist constants C_2, $M_2 > 0$ and $\alpha, \beta \in (0, 1/4)$ such that for every $M \geq M_2$, for every (K, u) (absolute, restricted, generalized, or generalized restricted) minimizer of E_λ on B_2, we can find families $\mathcal{F}_j$ of disjoint balls*

$$\mathcal{F}_j = \{B_{\alpha M^{-j}}(y_i) : y_i \in K, 1 \leq i \leq N_j\}$$

such that for all $h \in \mathbb{N}$

(i) *if B, $B' \in \cup_{j=1}^h \mathcal{F}_j$ are distinct balls, then $(B)_{4M^{-(h+1)}} \cap (B')_{4M^{-(h+1)}} = \emptyset$;*

(ii) *if $B_{\alpha M^{-j}}(y_i) \in \mathcal{F}_j$, then $K \cap B_{2\alpha M^{-j}}(y_i)$ is a $C^{1,\gamma}$ graph, $\gamma \in (0, 1)$, containing y_i,*

$$\inf_\theta \Omega^j(\theta, y_i, 2\alpha M^{-j}) + \lambda \|g\|_\infty^2 (\alpha M^{-j})^{\frac{1}{2}} < \varepsilon_0 ,$$

$$\|\nabla u\|_{L^\infty(B_{2\alpha M^{-j}}(y_i))} < M^{j+1}; \tag{6.5.5}$$

(iii) *let $\{R_h\}_{h\in\mathbb{N}}$, $\{S_h\}_{h\in\mathbb{N}}$, $\{T_h\}_{h\in\mathbb{N}}$ be the sequences of radii in Lemma 6.5.2, and let*

$$K_h := (K \cap \overline{B}_{S_h}) \setminus \left(\cup_{j=1}^h \cup_{\mathcal{F}_j} B\right),$$

(by construction $K_{h+1} \subset K_h \setminus \cup_{\mathcal{F}_{h+1}} B$), and

$$\widetilde{K}_h := (K \cap \overline{B}_{T_h}) \setminus \left(\cup_{j=1}^h \cup_{\mathcal{F}_j} (B)_{2M^{-(h+1)}}\right) \subset K_h. \tag{6.5.6}$$

Then, there exists a finite set of points $\mathcal{C}_h := \{x_i\}_{i\in I_h} \subset \widetilde{K}_h$ such that

$$|x_j - x_k| \geq 3M^{-(h+1)} \quad \forall j, k \in I_h,\ j \neq k; \tag{6.5.7}$$

$$(K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+1)}} \subset \cup_{i\in I_h} B_{8M^{-(h+1)}}(x_i); \tag{6.5.8}$$

$$\mathcal{H}^1(K_h) \leq C_1\, h\, M^{-2h\beta}; \tag{6.5.9}$$

$$\left|(K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+1)}}\right| \leq C_2\, h\, M^{-h(1+2\beta)-1}. \tag{6.5.10}$$

(iv) *$\Sigma \cap B_{1/2} \subset K_h$ for all $h \in \mathbb{N}$ and*

$$\left|(\Sigma \cap B_{1/2})_r\right| \leq C_2\, r^{1+\beta} \qquad \forall r \in (0, 1/2]. \tag{6.5.11}$$

In particular, $\dim_{\mathcal{M}}(\Sigma \cap B_{1/2}) \leq 1 - \beta$.

PROOF. In what follows we shall repeatedly use Theorem 6.5.1 with $\varepsilon \in (0, 1)$ fixed and sufficiently small. We split the proof into several steps.

In addition, consider the constants ϵ, C_1, M_1 introduced in (6.5.3) and Lemma 6.5.2, respectively.

Step 1. *Inductive definition of the families $\mathcal{F}_h$.*

For $h = 1$ we define

$$\mathcal{F}_1 := \emptyset,\ K_1 = K \cap \overline{B}_{S_1},\ \widetilde{K}_1 = K \cap \overline{B}_{T_1},$$

and choose $\mathcal{C}_1$ to be a maximal family of points in $\widetilde{K}_1$ at distance $3M^{-2}$ from each other. Of course, properties in items (i) and (ii) and in (6.5.7) are satisfied. To check the others, one can argue as in the verification below.

Suppose that we have built the families $\{\mathcal{F}_j\}_{j=1}^h$ as in the statement, to construct $\mathcal{F}_{h+1}$ we argue as follows. Let $\mathcal{C}_h = \{x_i\}_{i\in I_h} \subset \widetilde{K}_h$ be a maximal family of points satisfying (6.5.7), i.e. $|x_i - x_k| \geq 3M^{-(h+1)}$ for all j, $k \in I_h$ with $j \neq k$. Consider

$$\mathcal{G}_{h+1} := \{B_{M^{-(h+1)}}(x_i)\}_{i\in I_h}.$$

For every ball $B_{M^{-(h+1)}}(x_i) \in \mathcal{G}_{h+1}$ we can find a sub-ball $B_{2\alpha\, M^{-(h+1)}}(y_i) \subset B_{M^{-(h+1)}}(x_i)$, $\alpha \in (0, 1/4)$ for which the theses of Theorem 6.5.1 are satisfied. Then, define for $M \geq M_1$

$$\mathcal{F}_{h+1} := \{B_{\alpha\, M^{-(h+1)}}(y_i)\}_{i\in I_h}.$$

By condition (6.5.7), the balls $B_{\frac{3}{2}M^{-(h+1)}}(x_i)$ are disjoint and do not intersect

$$\cup_{j=1}^h \cup_{\mathcal{F}_j} (B)_{\frac{1}{2}M^{-(h+1)}}$$

by the very definition of $\widetilde{K}_h$ in (6.5.6). Thus, item (i) and (ii) are satisfied for M_2 sufficiently large. Hence, we can define K_{h+1}, $\widetilde{K}_{h+1}$ and $\mathcal{C}_{h+1}$ as in the statement.

We verify next the rest of the conclusions.

Step 2. *Proof of* (6.5.8).

Let $x \in (K_{h+1}\cap\overline{B}_{R_{h+2}})_{M^{-(h+2)}}$ and let z be a point of minimal distance from $K_{h+1}\cap\overline{B}_{R_{h+2}}$. In case $z \in \widetilde{K}_{h+1}$, by maximality of $\mathcal{C}_{h+1}$ there is some point $x_i \in \mathcal{C}_{h+1}$ such that $|z - x_i| < 3M^{-(h+2)}$ and thus we conclude $x \in B_{4M^{-(h+2)}}(x_i)$. Instead, if $z \in (K_{h+1} \cap \overline{B}_{R_{h+2}}) \setminus \widetilde{K}_{h+1}$, the definitions of K_{h+1} and $\widetilde{K}_{h+1}$ yield the existence of a ball $\widetilde{B} \in \cup_{j=1}^{h+1}\mathcal{F}_j$ for which $z \in (K \cap (\widetilde{B})_{2M^{-(h+2)}}) \setminus \widetilde{B}$. Item (ii) and a scaled version of Lemma 6.5.3 (applied with $\delta = 2M^{-(h+2)}$, and radii ρ (that of $\widetilde{B}$) and $\rho + 2M^{-(h+2)}$ in place of 1 and $1 + \delta$) give a point y satisfying

$$y \in \Big(K \cap (\widetilde{B})_{4M^{-(h+2)}}\Big) \setminus (\widetilde{B})_{2M^{-(h+2)}}, \quad \text{and} \quad |z - y| \leq 3M^{-(h+2)}.$$

Therefore, as $z \in \overline{B}_{R_{h+2}}$ and $T_{h+1} = R_{h+2} + 4M^{-(h+2)}$ we get by property (i) and the definition of $\widetilde{K}_{h+1}$

$$y \in \Big(K \cap B_{T_{h+1}} \cap (\widetilde{B})_{3M^{-(h+2)}}\Big) \setminus (\widetilde{B})_{2M^{-(h+2)}} \subset \widetilde{K}_{h+1}.$$

Finally, by maximality of $\mathcal{C}_{h+1}$ we may find $x_i \in \mathcal{C}_{h+1}$ such that $|y - x_i| < 3M^{-(h+2)}$. In conclusion, we have

$$|x - x_i| \leq |x - z| + |z - y| + |y - x_i| \leq 7M^{-(h+2)},$$

so that (6.5.8) follows at once.

Step 3. *For every* $h \in \mathbb{N}$ *and for every* $x_i \in \mathcal{C}_h$

$$K_h \cap B_{M^{-(h+1)}}(x_i) = K \cap B_{M^{-(h+1)}}(x_i)\,. \tag{6.5.12}$$

Indeed, assume by contradiction that we can find $x_i \in \mathcal{C}_h$ and $x \in (K \setminus K_h) \cap B_{M^{-(h+1)}}(x_i)$. As $x_i \in \widetilde{K}_h$ then $x_i \in B_{T_h}$, in turn implying $x \in B_{S_h}$ since $S_h - T_h = 4M^{-(h+1)}$. Therefore

$x \in (K \setminus K_h) \cap \overline{B}_{S_h}$, and by definition of K_h we can find a ball $B \in \mathcal{F}_j$, $j \le h$, such that $x \in B$. We conclude that

$$\mathrm{dist}(x_i, B) \le |x - x_i| \le M^{-(h+1)},$$

contradicting the assumption that $x_i \in \widetilde{K}_h$.

Step 4. *Proof of* (6.5.9).

We get first a lower bound for $\mathcal{H}^0(I_h)$: use (6.5.8) and the density upper bound in (6.5.3) to get

$$\mathcal{H}^1(K_h \cap \overline{B}_{R_{h+1}}) = \mathcal{H}^1(K_h \cap \overline{B}_{R_{h+1}} \cap \cup_{i \in I_h} B_{8M^{-(h+1)}}(x_i)) \le \varepsilon_\lambda 8 M^{-(h+1)} \mathcal{H}^0(I_h)$$

where we recall that $\varepsilon_\lambda = (2 + \lambda \|g\|_\infty^2)\pi$. Equivalently, we have that

$$\mathcal{H}^0(I_h) M^{-(h+1)} \ge \frac{1}{8\varepsilon_\lambda} \mathcal{H}^1(K_h \cap \overline{B}_{R_{h+1}}). \tag{6.5.13}$$

Thus, we estimate as follows by taking into account that by item (i) the balls in the family $\mathcal{F}_{h+1}$ are disjoint

$$\begin{aligned}
\mathcal{H}^1(K_{h+1}) &\le \mathcal{H}^1\left(K_h \setminus \cup_{\mathcal{F}_{h+1}} B\right) = \mathcal{H}^1(K_h) - \sum_{\mathcal{F}_{h+1}} \mathcal{H}^1(K_h \cap B) \\
&\overset{(6.5.3),(6.5.12)}{\le} \mathcal{H}^1(K_h) - \epsilon\alpha M^{-(h+1)} \mathcal{H}^0(I_h) \overset{(6.5.13)}{\le} \mathcal{H}^1(K_h) - \frac{\epsilon}{8\varepsilon_\lambda}\alpha \mathcal{H}^1(K_h \cap \overline{B}_{R_{h+1}}) \\
&= (1-\eta)\mathcal{H}^1(K_h) + \eta\left(\mathcal{H}^1(K_h) - \mathcal{H}^1(K_h \cap \overline{B}_{R_{h+1}})\right) \\
&\le (1-\eta)\mathcal{H}^1(K_h) + \eta\,\mathcal{H}^1(K \cap (\overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}})) \le (1-\eta)\mathcal{H}^1(K_h) + C_1 M^{-\frac{h+1}{2}},
\end{aligned} \tag{6.5.14}$$

where we have set $\eta := \frac{\epsilon}{8\varepsilon_\lambda}\alpha \in (0,1)$, and we have used the very definition of K_h in the last but one inequality, and item of (iii) Lemma 6.5.2 in the last one. By taking into account (iii) in Lemma 6.5.2 and (6.5.13), an iteration of (6.5.14) implies

$$\mathcal{H}^1(K_h) \le C_1 \sum_{i=1}^{h} (1-\eta)^{h-i} M^{-i/2} \le C_1\, h\, (\max\{1-\eta, M^{-1/2}\})^h.$$

Choose $\beta \in (0, 1/4)$ such that $(1-\eta) \le M^{-2\beta}$, the previous estimate then yields (6.5.9),

$$\mathcal{H}^1(K_h) \le C_1\, h\, \max\{M^{-2h\beta}, M^{-h/2}\} = C_1\, h\, M^{-2h\beta}.$$

Step 5. *Proof of* (6.5.10).

We use (6.5.8) to get for some dimensional constant C

$$\begin{aligned}
\left|(K_{h+1} \cap \overline{B}_{R_{h+2}})_{M^{-(h+2)}}\right| &\le \left|\cup_{i \in I_{h+1}} B_{8M^{-(h+2)}}(x_i)\right| \le C M^{-2(h+2)} \mathcal{H}^0(I_{h+1}) \\
&\overset{(6.5.3),(6.5.12)}{\le} C\epsilon M^{-(h+2)} \sum_{i \in I_{h+1}} \mathcal{H}^1(K_{h+1} \cap B_{M^{-(h+2)}}(x_i)) \\
&\le C\epsilon M^{-(h+2)} \mathcal{H}^1(K_{h+1}) \overset{(6.5.9)}{\le} C\epsilon C_1\,(h+1)\, M^{-2(h+1)\beta - (h+2)}.
\end{aligned}$$

where in the last but one inequality we have used that the balls $B_{M^{-(h+2)}}(x_i)$ are disjoint by construction.

Step 6. *Proof of* (6.5.11)

By construction we have that $\Sigma \cap B_{1/2} \subseteq K_h$. Therefore, (6.5.10) gives as $R_h \geq R_\infty \geq 1/2$

$$\left|(\Sigma \cap B_{1/2})_{M^{-(h+1)}}\right| \leq \left|(K_h \cap \overline{B}_{R_{h+1}})_{M^{-(h+1)}}\right| \leq C_2\, h\, M^{-h(1+2\beta)-1}.$$

Hence, if $r \in (M^{-(h+2)}, M^{-(h+1)}]$ we get

$$\left|(\Sigma \cap B_{1/2})_r\right| \leq C_2\, h\, M^{-h(1+2\beta)-1} \leq C_2\, M^{-h(1+\beta)-1} \leq C_2\, r^{1+\beta}. \qquad \square$$

We are now ready to establish the higher integrability of the gradient.

THEOREM 6.5.5. *There is $p > 2$ such that for all open sets $\Omega \subseteq \mathbb{R}^2$ and for all (K, u) either a restricted, or an absolute, or a generalized global minimizer of $E_\lambda(\cdot,\cdot,\Omega,g)$ then $\nabla u \in L^p_{\mathrm{loc}}(\Omega \setminus K)$.*

PROOF. Clearly, it is sufficient for our purposes to prove a localized estimate. Hence, for the sake of simplicity, we suppose that $\Omega = B_2$.

We keep the notation of Proposition 6.5.4 and furthermore denote for all $h \in \mathbb{N}$

$$A_h := \{x \in B_2 \setminus K : |\nabla u(x)|^2 > M^{h+1}\}\,. \tag{6.5.15}$$

We claim that

$$A_{h+2} \cap B_{R_{h+2}} \subset (K_h \cap B_{R_{h+1}})_{M^{-(h+1)}}. \tag{6.5.16}$$

Given this, for granted we conclude as follows: we use (6.5.10) to deduce that

$$|A_{h+2} \cap B_{R_{h+2}}| \leq \left|(K_h \cap B_{R_{h+1}})_{M^{-(h+1)}}\right| \leq C_2\, h\, M^{-h(1+2\beta)-1}. \tag{6.5.17}$$

Therefore, recalling that $\frac{1}{2} \leq R_\infty \leq R_h$, in view of (6.5.17) and Cavalieri's formula for $q > 1$ we get that

$$\begin{aligned}
\int_{B_{\frac{1}{2}}} |\nabla u|^{2q} &= q \int_0^\infty t^{q-1} |\{x \in B_{1/2} \setminus K : |\nabla u(x)|^2 > t\}|\, dt \\
&\leq q \sum_{h \geq 3} \int_{M^h}^{M^{h+1}} t^{q-1} |\{x \in B_{1/2} \setminus K : |\nabla u(x)|^2 > t\}|\, dt + M^{3q}|B_{1/2}| \\
&\leq \sum_{h \geq 0} M^{(h+4)q} |A_{h+2} \cap B_{\frac{1}{2}}| + M^{3q}|B_{\frac{1}{2}}| \\
&\leq C_2 \sum_{h \geq 0} h\, M^{(h+4)q - h(1+2\beta)-1} + M^{3q}|B_{\frac{1}{2}}|.
\end{aligned}$$

The conclusion follows at once by taking $q \in (1, 1+2\beta)$ and $p = 2q$.

To conclude we prove formula (6.5.16) in two steps.

Step 1. *For every $M > 0$ large enough, for every $h \in \mathbb{N}$ and for every $R \in (0, 1]$ we have that*

$$A_h \cap B_{R - 2M^{-h}} \subset (K \cap B_R)_{M^{-h}}\,. \tag{6.5.18}$$

Indeed, for $x \in A_h \cap B_{R-2M^{-h}}$ let $z \in K$ be such that $\mathrm{dist}(x, K) = |x-z|$. If $|x-z| > M^{-h}$ then $B_{M^{-h}}(x) \cap K = \emptyset$ so that u solves (2.5.2), i.e. $\triangle u = \lambda(u-g)$, on $B_{M^{-h}}(x)$. Therefore, being the right-hand side of the PDE in L^∞, by standard Lipschitz bounds in elliptic regularity [25, estimate (4.45)] and the density upper bound in (6.5.3), as $x \in A_h$ we infer that

$$M^{h+1} \le |\nabla u(x)|^2 \le C\, M^h$$

with C depending on $\|g\|_\infty$. The latter estimate is clearly impossible for M large enough. Furthermore, as $x \in B_{R-2M^{-h}}$ and $|x-z| \le M^{-h}$ we conclude that $z \in B_R$.

Step 2. *Proof of* (6.5.16).

Since $R_{h+1} - R_{h+2} \ge 8M^{-(h+2)}$ (cf. (i) Lemma 6.5.2), we apply Step 1 to A_{h+2} and $R = R_{h+1}$ and then (6.5.18) implies that

$$A_{h+2} \cap B_{R_{h+2}} \subset \Big(K \cap B_{R_{h+1}}\Big)_{M^{-(h+1)}}.$$

Let $x \in A_{h+2} \cap B_{R_{h+2}}$, $z \in K \cap B_{R_{h+1}}$ be a point of minimal distance, and suppose that $z \in K \setminus K_h$. Being $R_{h+1} \le S_h$, by the very definition of K_h there is a ball $B \in \cup_{j=1}^h \mathcal{F}_j$ such that $z \in B$. In turn, since $B = B_\rho(y)$ for some y and radius $\rho \ge t\, M^{-h}$, then $x \in B_{2t}(y)$ as $|x-z| \le M^{-(h+1)}$ for M sufficiently large. Thus, estimate $|\nabla u(x)|^2 < M^{h+1}$ follows from (6.5.5) in item (ii) of Proposition 6.5.4. This is in contradiction with $x \in A_{h+2}$. $\square$

APPENDIX A

Variational identities

We give here the proofs of Proposition 2.5.1 and Proposition 2.5.2. We recall that ν denotes the counterclockwise rotation by 90 degrees of a C^0 unit tangent vector e locally orienting K, while κ is the curvature of the curve, namely $\ddot{\gamma} \cdot \nu$, for an arclength parametrization γ such that $\dot{\gamma} = e$. Finally, w^+ and w^- are the one-sided traces of the relevant function w on K (following the obvious convention that w^+ is the trace on the side which ν is pointing to).

Proof of Proposition 2.5.1. The proofs of the outer variations formula (1.5.2) is standard, and we leave it to the reader.

Let $\psi \in C^1_c(\Omega, \mathbb{R}^2)$, and consider the Cauchy problems (parametrized in terms of the initial condition $x \in \Omega$)

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi_t(x) = \psi(\Phi_t(x)) \\ \Phi_0(x) = x\,, \end{cases}$$

then the map $\Phi_t(x)$ is a diffeomorphism of Ω onto itself for $t \in (-t_0, t_0)$, $t_0 > 0$ sufficiently small. We recall the notation $(K_t, u_t(x)) = (\Phi_t(K), u(\Phi_t^{-1}(x)))$, and define $f(t) := E_0(K_t, u_t)$. We claim that f is differentiable on $(-t_0, t_0)$ with

$$f'(t) = \int_{\Omega\setminus\Phi_t(K)} \Big(|\nabla u_t|^2 \mathrm{div}\,\psi - 2\nabla u_t^T \cdot D\psi\,\nabla u_t\Big) + \int_{\Phi_t(K)} e_t^T \cdot D\psi\, e_t\, d\mathcal{H}^1\,, \qquad \text{(A.0.1)}$$

where $e_t : \Phi_t(K) \to \mathbb{S}^1$ is a Borel vector field tangent to the rectifiable set $\Phi_t(K)$. Given this, if (K, u) is either an absolute or a restricted or a generalized minimizer of E_0 we get (1.5.4) (for $\lambda = 0$) as necessarily $f'(0) = 0$.

First, note that for all ε small

$$\begin{aligned} \nabla\Phi_\varepsilon^{-1}(\Phi_\varepsilon(x)) &= [\mathrm{Id} + \varepsilon D\psi(x)]^{-1} = \mathrm{Id} - \varepsilon D\psi(x) + o(\varepsilon) \\ \det\nabla\Phi_\varepsilon(x) &= \det[\mathrm{Id} + \varepsilon D\psi(x)] = 1 + \varepsilon\,\mathrm{div}\psi(x) + o(\varepsilon), \end{aligned}$$

$o(\varepsilon)$ uniform with respect to $x \in \Omega$. Thus, by changing variables as $u_{t+\varepsilon} = u_t(\Phi_\varepsilon^{-1})$, for $t \in (-t_0, t_0)$ and $\varepsilon \in \mathbb{R}$ sufficiently small, we get

$$\begin{aligned}
&\int_{\Omega\setminus\Phi_{t+\varepsilon}(K)} |\nabla u_{t+\varepsilon}|^2 = \int_{\Omega\setminus\Phi_t(K)} |\nabla u_t \cdot D\Phi_\varepsilon^{-1}(\Phi_\varepsilon)|^2 |\det D\Phi_\varepsilon| \\
&= \int_{\Omega\setminus\Phi_t(K)} |\nabla u_t - \varepsilon \nabla u_t \cdot D\psi|^2 (1 + \varepsilon \operatorname{div}\psi) + o(\varepsilon) \\
&= \int_{\Omega\setminus\Phi_t(K)} |\nabla u_t|^2 + \varepsilon \int_{\Omega\setminus\Phi_t(K)} \Big(|\nabla u_t|^2 \operatorname{div}\psi - 2\nabla u_t^T \cdot D\psi\, \nabla u_t \Big) + o(\varepsilon)\,. \qquad \text{(A.0.2)}
\end{aligned}$$

In addition, as $\Phi_{t+\varepsilon}(K) = \Phi_t(\Phi_\varepsilon(K))$, from the coarea formula [4, Theorem 2.93] we infer that

$$\mathcal{H}^1(\Phi_{t+\varepsilon}(K)) = \mathcal{H}^1(\Phi_t(K)) + \varepsilon e_t^T \cdot D\psi\, e_t + o(\varepsilon) \qquad \text{(A.0.3)}$$

In conclusion, we deduce from (A.0.2) and (A.0.3) that f is differentiable in $t \in (-t_0, t_0)$ and that (A.0.1) holds.

To prove (1.5.4) in the general case we need to consider further the fidelity term. To this aim, for $t \in (-t_0, t_0)$ set

$$h_g(t) := \int_{\Omega\setminus\Phi_t(K)} |u_t - g|^2\,.$$

Let us first assume that $g \in C^1(\Omega)$, and prove that h_g is differentiable on $(-t_0, t_0)$ with

$$h_g'(t) = -2\int_\Omega (u_t - g)\psi \cdot \nabla u_t - \int_{\Phi_t(K)} (|u_t^+ - g|^2 - |u_t^- - g|^2)\psi \cdot \nu_{\Phi_t(K)} d\mathcal{H}^1\,, \qquad \text{(A.0.4)}$$

where $\nu_{\Phi_t(K)}$ denotes the counterclockwise rotation by 90 degrees of the Borel unit tangent vector e_t to $\Phi_t(K)$.

Indeed, by changing variables, the fact that $g \in C^1(\Omega)$ implies that, for some function $o(\varepsilon)$ that is uniform with respect to $x \in \Omega$,

$$\begin{aligned}
h_g(t+\varepsilon) &= \int_{\Omega\setminus\Phi_{t+\varepsilon}(K)} |u_{t+\varepsilon} - g|^2 = \int_{\Omega\setminus\Phi_t(K)} |u_t - g(\Phi_\varepsilon)|^2 |\det D\Phi_\varepsilon| \\
&= \int_{\Omega\setminus\Phi_t(K)} |u_t - g - \varepsilon\nabla g \cdot \psi + o(\varepsilon)|^2 (1 + \varepsilon \operatorname{div}\psi) + o(\varepsilon) \\
&= h_g(t) + \varepsilon \int_{\Omega\setminus\Phi_t(K)} \Big(|u_t - g|^2 \operatorname{div}\psi - 2(u_t - g)\nabla g \cdot \psi \Big) + o(\varepsilon).
\end{aligned}$$

Therefore,

$$h_g'(t) = \int_{\Omega\setminus\Phi_t(K)} \Big(|u_t - g|^2 \operatorname{div}\psi - 2(u_t - g)\nabla g \cdot \psi \Big)\,,$$

and the identity in (A.0.4) follows from an integration by parts by taking into account that, under the standing assumptions, $|u - g|^2 \in BV_{loc} \cap L^\infty_{loc}(\Omega)$.

If $g \in L^\infty(\Omega)$, let $g_\varepsilon \in C^1 \cap L^\infty(\Omega)$ be such that $g_\varepsilon \to g$ in $L^2(\Omega)$. Consider therefore the functions

$$f_\varepsilon(t) := f(t) + \lambda h_{g_\varepsilon}(t)$$

and observe that, by the above formulas $|f'_\varepsilon(t)|$ is uniformly bounded. Thus $f_\varepsilon(t)$ converge uniformly to the Lipschitz function $f_0(t) = f(t) + \lambda h_g(t)$, which has a minimum in 0. We rewrite formula (A.0.4) for the derivative of h_{g_ε} as follows

$$h'_{g_\varepsilon}(t) := \underbrace{-2\int_{\Omega\setminus\Phi_t(K)} (u_t - g_\varepsilon)\psi \cdot \nabla u_t}_{=:\Lambda_\varepsilon(t)} \underbrace{- \int_{\Phi_t(K)} ((u_t^+)^2 - (u_t^-)^2)\psi \cdot \nu_{\Phi_t(K)} d\mathcal{H}^1}_{=:\Gamma(t)}$$
$$+ \underbrace{2\int_{\Phi_t(K)} (u_t^+ - u_t^-) g_\varepsilon\, \psi \cdot \nu_{\Phi_t(K)} d\mathcal{H}^1}_{=:L_{t,\varepsilon}(\psi)} .$$

Note that $\Lambda_\varepsilon(t)$ converge uniformly as $\varepsilon \to 0$ to

$$\Lambda(t) = -2\int_{\Omega\setminus\Phi_t(K)} (u_t - g)\psi \cdot \nabla u_t \,,$$

while $\Gamma \in C^0((-t_0, t_0))$, because $u \in W^{1,2}(\Omega \setminus K)$ and we can use the Generalized Area Formula [4, Theorem 2.91]. We use the Generalized Area Formula also to rewrite

$$L_{t,\varepsilon}(\psi) = 2\int_K (u^+ - u^-) g_\varepsilon(\Phi_t)\psi(\Phi_t) \cdot \nu_{\Phi_t(K)}(\Phi_t)\, J_t\, d\mathcal{H}^1 \,,$$

where $J_t(x) = |D\Phi_t(x)(e(x))|$, and $e(x)$ is the unit tangent to K at x.

If we consider the vector-valued measures

$$\mu_t := 2(u^+ - u^-)\, \nu_{\Phi_t(K)} \circ \Phi_t\, J_t\, \mathcal{H}^1 \llcorner K \,,$$

and the pushforward $\alpha_t := (\Phi_t)_\sharp \mu_t$, we can then write

$$L_{t,\varepsilon}(\psi) = \int g_\varepsilon\, \psi \cdot d\alpha_t \,.$$

Next, consider the measures $g_\varepsilon d\alpha_t \otimes dt$ on $\Omega \times (-t_0, t_0)$ and, given the uniform boundedness of g_ε (in a pointwise sense), we can assume that a suitable subsequence, not relabeled, converges to a measure of the form

$$\bar{g} d\alpha_t \otimes dt \,,$$

for some Borel function $\bar{g}$ with $\|\bar{g}\|_{L^\infty(\Omega\times(-t_0,t_0), d\alpha_t\otimes dt)} \le \|g\|_\infty$. So we can rewrite

$$f_0(t) = f_0(0) + \int_0^t f'(s)\, ds + \lambda \int_0^t (\Lambda(s) + \Gamma(s))\, ds + \lambda \int_0^t \int \bar{g}\psi \cdot d\alpha_s\, ds \,.$$

Since f_0 is Lipschitz and has a minimum in 0, while f', Λ, and Γ are continuous, we conclude that, for every positive s_0

$$|f'(0) + \lambda(\Lambda(0) + \Gamma(0))| \le \lambda \|g\|_\infty \sup_{|s|\le s_0} \left| \int \psi \cdot d\alpha_s \right|$$

We now use the fact that ψ is continuous and that so are the maps $t \mapsto \nu_{\Phi_t(K)}(\Phi_t(x)) = \frac{D\Phi_t(\nu(x))}{|D\Phi_t(\nu(x))|}$ and $t \mapsto J_t(x)$ to conclude that

$$\liminf_{s_0 \downarrow 0} \sup_{|s| \le s_0} \left| \int \psi \cdot d\alpha_s \right| \le 2 \int_K |u^+ - u^-| |\psi \cdot \nu| d\mathcal{H}^1 .$$

Since the map $\psi \mapsto (f'(0) + \lambda(\Lambda(0) + \Gamma(0)))$ is linear in ψ, we conclude from the Riesz representation theorem that there exists a function $g_K \in L^\infty(\Omega, \mathcal{H}^1 \llcorner K)$ with $\|g_K\|_{L^\infty(\Omega,\mathcal{H}^1 \llcorner K)} \le \|g\|_\infty$ such that

$$f'(0) + \lambda(\Lambda(0) + \Gamma(0)) = 2\lambda \int_K (u^+ - u^-) g_K \psi \cdot \nu \, d\mathcal{H}^1 .$$

□

We show next the second form of the Euler-Lagrange conditions together with the higher regularity of K.

Proof Proposition 2.5.2. Assume that $K \cap A$ is a graph for some open subset $A \subseteq \Omega$. Up to a rotation, there are an open interval $I \subset \mathbb{R}$ and $\phi : I \to \mathbb{R}$ such that

$$K \cap A = \{(t, \phi(t)) : t \in I\} .$$

Let $A^\pm := \{(t,s) \in A : \pm s > \phi(t)\}$, and let $\varphi \in C^1(\bar{A})$ be such that $\varphi = 0$ in a neighbourhood of $\partial A^+ \setminus K$. Let $v = u$ on A^- and $v = u + \varepsilon\varphi$ on A^+, then from the (restricted) minimality of (K,u) for E_λ we infer

$$\int_{A^+} \Big(\nabla u \cdot \nabla \varphi + \lambda \varphi (u - g) \Big) = 0 .$$

Clearly, a similar identity can be obtained on A^-. Therefore, u is a weak solution to

$$\begin{cases} \triangle u = \lambda(u-g) & A^\pm \\ \frac{\partial u}{\partial \nu} = 0 & K \cap A . \end{cases} \tag{A.0.5}$$

From elliptic regularity theory we infer that if $\phi \in C^{1,\alpha}(I)$ then $u \in C^{1,\alpha}(A^\pm)$ (see [4, Theorem 7.49]). The conclusions in item (a), in (2.5.2) and in (2.5.3) then follow at once.

We assume next that $u \in W^{2,2}(A^+ \cup A^-)$, and moreover that ν is the interior normal vector to A^+. Let $\psi \in C^1_c(A;\mathbb{R}^2)$, then integrating twice by parts give

$$\begin{aligned} \int_{A^\pm} |\nabla u|^2 \mathrm{div}\psi &= -2 \int_{A^\pm} \nabla u \cdot \nabla^2 u \, \psi \mp \int_{K\cap A} |\nabla u^\pm|^2 \psi \cdot \nu d\mathcal{H}^1 \\ &= 2 \int_{A^\pm} \Big(\triangle u \, \nabla u \cdot \psi + \nabla u^T \cdot D\psi \, \nabla u \Big) \\ &\quad \mp \int_{K \cap A} \Big((\nabla u \cdot \psi) \frac{\partial u}{\partial \nu} + |\nabla u^\pm|^2 \psi \cdot \nu \Big) d\mathcal{H}^1 \\ &\overset{(A.0.5)}{=} 2 \int_{A^\pm} \Big(\lambda(u-g) \nabla u \cdot \psi + \nabla u^T \cdot D\psi \, \nabla u \Big) \mp \int_{K\cap A} |\nabla u^\pm|^2 \psi \cdot \nu d\mathcal{H}^1 . \end{aligned}$$

Therefore, from (1.5.4) we infer that

$$\int_{K\cap A} e^T \cdot D\psi\, e\, d\mathcal{H}^1$$
$$= \int_{K\cap A} (|\nabla u^+|^2 + \lambda|u^+ - g_K|^2 - |\nabla u^-|^2 - \lambda|u^- - g_K|^2)\psi \cdot \nu d\mathcal{H}^1\,. \tag{A.0.6}$$

The last formula still holds even if $u \in W^{2,2}_{loc}(A^+ \cup A^-)$, which is actually the known regularity for u. Indeed, it suffices to deform smoothly K inside $A^\pm$ and to take into account (A.0.5) and the fact that both u and ∇u are bounded to conclude.

Consider $\varphi \in C^1_c(A)$ and let $\psi(x) = (0, \varphi(x))$, as $K \cap A = \{(t, \phi(t)) :\, t \in I\}$, we infer that

$$e(t,\phi(t))^T \cdot D\psi(t,\phi(t))\, e(t,\phi(t)) = \frac{d}{dt}\left(\varphi(t,\phi(t))\right) \frac{\phi'(t)}{1+|\phi'(t)|^2}\,,$$

and thus

$$\int_{K\cap A} e^T \cdot D\psi\, e\, d\mathcal{H}^1 = \int_I \frac{d}{dt}\left(\varphi(t,\phi(t))\right) \frac{\phi'(t)}{\sqrt{1+|\phi'(t)|^2}} dt\,.$$

Moreover, if $\varphi(t,s) = \zeta(t)\eta(s)$, where $\zeta \in C^1_c(I)$ and $\eta \in C^1_c(\mathbb{R})$ such that $\eta = 1$ on $[-2\|\phi\|_{L^\infty(I)}, 2\|\phi\|_{L^\infty(I)}]$, on setting $H := |\nabla u^+|^2 + \lambda|u^+ - g_K|^2 - |\nabla u^-|^2 - \lambda|u^- - g_K|^2 \in L^\infty(K \cap A)$, (A.0.6) rewrites as

$$\int_I \zeta'(t) \frac{\phi'(t)}{\sqrt{1+|\phi'(t)|^2}} dt = \int_I H(t,\phi(t))\, \zeta(t) dt\,. \tag{A.0.7}$$

In particular, $\frac{\phi'(t)}{\sqrt{1+|\phi'(t)|^2}} \in W^{1,\infty}(I)$, and in turn this implies $\phi' \in W^{1,\infty}(I)$, as $\phi' \in L^\infty(I)$. Item (b) is then established.

Eventually, (A.0.7) yields that the distributional curvature of the graph of ϕ satisfies

$$\frac{d}{dt}\left(\frac{\phi'(t)}{\sqrt{1+|\phi'(t)|^2}}\right) = -H(t,\phi(t)) \qquad \mathcal{L}^1 \text{ a.e. on } I,$$

(2.5.4) then follows at once. $\square$

APPENDIX B

Density lower bound

The aim of this section is to give a direct simple proof of the density lower bound estimate valid for (restricted) minimizers of the functions E_λ, and also for minimizers of the corresponding SBV relaxed versions.

We start with proving the following decay result.

Lemma B.0.1. *Consider an admissible couple (K,u) on Ω, then the following properties hold. For every $\tau>0$ there are constants $\varepsilon(\tau)>0$ and $\vartheta(\tau)\in(0,1)$ such that if (K,u) is a (restricted) minimizer of E_λ on Ω with $\mathcal{H}^1(K\cap B_\rho(x))\le\varepsilon\rho$ for some $\rho\in(0,\operatorname{dist}(x,\partial\Omega))$ with $\lambda\in[0,1]$ and $\|g\|_\infty\le M_0$, then*

$$E_0(K,u,B_{\tau\rho}(x))\le\max\{\tau^{3/2}E_0(K,u,B_\rho(x)),\tfrac{8\pi\lambda}{\vartheta}\|g\|_\infty^2\rho^2\}\,. \tag{B.0.1}$$

Proof. We start off by proving the result for $\lambda=0$. Assume by contradiction that for some $\tau>0$ there is a sequence of (restricted) minimizers (K_j,u_j) on Ω, points $x_j\in\Omega$ and radii ρ_j such that $\rho_j^{-1}\mathcal{H}^1(K_j\cap B_{\rho_j}(x_j))$ is infinitesimal and, on setting $E_j:=E_0(K_j,u_j,B_{\rho_j}(x_j))$,

$$E_0(K_j,u_j,B_{\tau\rho_j}(x_j))>\tau^{3/2}E_j\,.$$

Define next $v_j:B_1\to\mathbb{R}$ by $v_j(x):=E_j^{-1/2}u_j(x_j+\rho_jx)$ and $H_j:=B_1\cap\frac{1}{\rho_j}(K_j-x_j)$. By scaling we get

$$\int_{B_1\setminus H_j}|\nabla v_j|^2=E_j^{-1}\int_{B_{\rho_j}(x_j)\setminus K_j}|\nabla u_j|^2\le1,\qquad\mathcal{H}^1(H_j)=\rho_j^{-1}\mathcal{H}^1(K_j\cap B_{\rho_j}(x_j))\,.$$

As $\mathcal{H}^1(H_j)$ is infinitesimal, for a subsequence not relabeled, by the corea formula [4, Theorem 2.93] and by the Mean-value theorem we may find a radius $r\in(\tau^{1/4},1)$ (independent from j) such that

$$H_j\cap\partial B_r=\emptyset\,,\qquad\int_{\partial B_r}|\nabla v_j|^2\le(1-\tau^{1/4})^{-1}\,. \tag{B.0.2}$$

In particular, $v_j\in W^{1,2}(\partial B_r)$ with $\operatorname{osc}_{\partial B_r}v_j\le C$ (independent from j). Therefore, up to subtracting the mean value of v_j over ∂B_r and relabeling the sequence v_j, we may conclude that $\|v_j\|_{L^\infty(\partial B_r)}\le C$. In turn, scaling back to u_j and using Lemma 2.1.1 yields that $\|v_j\|_{L^\infty(B_r)}\le C$. Define for an admissible couple (J,w) in B_1 and for all open subsets $A\subseteq B_1$

$$F_j(J,w,A):=\int_{A\setminus J}|\nabla w|^2+\frac{1}{E_j}\mathcal{H}^1(J\cap A)\,. \tag{B.0.3}$$

Then, the conditions above rewrite as

$$F_j(H_j, v_j, B_1) = 1, \qquad F_j(J, w, B_1) \geq F_j(H_j, v_j, B_1)\,, \tag{B.0.4}$$

for all admissible couple (J, w) such that $\{v_j \neq w\} \cup (H_j \triangle J) \subset\subset B_1$ (with the number of connected components less than that of K_j in case u_j is a restricted minimizer of E), and

$$F_j(H_j, v_j, B_\tau) > \tau^{3/2}\,. \tag{B.0.5}$$

Moreover, $v_j \in SBV(B_1)$ with $S_{v_j} \subseteq H_j$ being v_j harmonic on $B_1 \setminus H_j$. Hence, thanks to the first condition in (B.0.4), and the $L^\infty(B_r)$ bound on v_j, Ambrosio's compactness theorem [4, Theorem 4.8] implies that there exists a subsequence (not relabeled) and a function $v \in SBV(B_r)$ such that v_j converge to v in $L^2(B_r)$, and that for all open subsets $A \subseteq B_r$

$$\mathcal{H}^1(S_v \cap A) \leq \liminf_j \mathcal{H}^1(S_{v_j} \cap A) = 0, \quad \int_A |\nabla v|^2 \leq \liminf_j \int_A |\nabla v_j|^2 \leq 1\,.$$

In particular, $v \in W^{1,2}(B_r)$. Actually, we claim that v turns out to be harmonic on B_r and satisfying for all $s \in (0, r)$

$$\lim_j \int_{B_s} |\nabla v_j|^2 = \int_{B_s} |\nabla v|^2\,. \tag{B.0.6}$$

Given this for granted, we conclude as follows: on one hand from (B.0.5) and the energy convergence in (B.0.6) and being $\tau < r^4 < r < 1$, we infer that

$$\int_{B_\tau} |\nabla v|^2 \geq \tau^{3/2}\,;$$

but on the other hand being v harmonic on B_r we have that

$$\int_{B_\tau} |\nabla v|^2 \leq \frac{\tau^2}{r^2} \int_{B_r} |\nabla v|^2 \leq \frac{\tau^2}{r^2}\,,$$

therefore we get a contradiction recalling that $\tau < r^4 < r < 1$.

We finally establish (B.0.6) together with the harmonicity of v on B_r. To this aim let $w \in W^{1,2}(B_r)$ with $\{v \neq w\} \subset\subset B_s$, $s \in (0, r)$. Let $s < t \in (0, 1)$ and $\varphi \in C_c^\infty(B_t)$ be such that $\varphi = 1$ on B_s. Define functions $\zeta_j = \varphi w + (1 - \varphi) v_j$ and sets J_j by $J_j \cap B_s = \emptyset$, $J_j \cap (B_t \setminus \overline{B_s}) = H_j \cap (B_t \setminus \overline{B_s})$. Then, (J_j, ζ_j) is an admissible couple to test the (restricted) minimality of (H_j, v_j) for F_j (note that the number of connected components of J_j is less than that of H_j, i.e. of K_j). The locality of the energy leads to

$$\begin{aligned}
F_j(H_j, v_j, B_s) &\leq F_j(H_j, v_j, B_t) \leq F_j(J_j, \zeta_j, B_t) \\
&\leq F_j(\emptyset, w, B_s) + C\, F_j(H_j, w, B_t \setminus \overline{B_s}) + C\, F_j(H_j, v_j, B_t \setminus \overline{B_s}) + \frac{C}{(t-s)^2} \int_{B_t \setminus \overline{B_s}} |v_j - w|^2 \\
&\leq \int_{B_s} |\nabla w|^2 + C \int_{B_t \setminus \overline{B_s}} |\nabla v|^2 + C \int_{B_t \setminus \overline{B_s}} |\nabla v_j|^2 + C\mathcal{H}^1(H_j) + \frac{C}{(t-s)^2} \int_{B_t \setminus \overline{B_s}} |v_j - v|^2\,.
\end{aligned}$$

As the sequence of Radon measures $(F_j(H_j, v_j, \cdot))_{j \in \mathbb{N}}$ is equi-bounded in mass on B_1 in view of (B.0.4), it converges to some Radon measure μ on B_1 up to a subsequence not relabeled for convenience. Assume that $\mu(\partial B_s) = \mu(\partial B_t) = 0$, by passing to the limit as $j \uparrow \infty$ and by Ambrosio's lower semicontinuity result we find

$$\begin{aligned}\int_{B_s} |\nabla v|^2 &\le \liminf_j F_j(H_j, v_j, B_s) \le \limsup_j F_j(H_j, v_j, B_s) \\ &\le \int_{B_s} |\nabla w|^2 + C \int_{B_t \setminus \overline{B_s}} |\nabla w|^2 + C\,\mu(B_t \setminus \overline{B_s})\,.\end{aligned}$$

Thus, by letting $t \downarrow s^+$ along values satisfying $\mu(\partial B_t) = 0$ we conclude that for all but a countable set of radii in $(0, r)$ we have

$$\int_{B_s} |\nabla v|^2 \le \liminf_j F_j(H_j, v_j, B_s) \le \limsup_j F_j(H_j, v_j, B_s) \le \int_{B_s} |\nabla w|^2\,. \tag{B.0.7}$$

The convergence for all radii $s \in (0, r)$ is an easy consequence of the fact that the Dirichlet energy of v (and of w) as a set function is the trace of a Radon measure on open sets.

Eventually, equality (B.0.6) follows by taking $w = v$ in the construction above and noting that in this case the radius $s \in (0, r)$ can be taken to be arbitrary.

The proof of the general case needs a further argument in addition to those used for $\lambda = 0$. We fix $\lambda \in (0, 1]$ and we claim that

$$E_0(K, u, B_{\tau\rho}(x)) \le \tau^{3/2} E_0(K, u, B_\rho(x)), \tag{B.0.8}$$

if in addition

$$(1 - \vartheta) E_0(K, u, B_\rho(x)) \le E_0(J, w, B_\rho(x)) \tag{B.0.9}$$

for all (J, w) with $\{u \neq w\} \cup (K \triangle J) \subset\subset B_\rho(x)$ and $\|w\|_\infty \le \|g\|_\infty$ (where $\vartheta(\tau) \in (0, 1)$ is the parameter in the statement). In case the condition in (B.0.9) is not satisfied for some admissible couple (J, w) with $\{u \neq w\} \cup (K \triangle J) \subset\subset B_\rho$, then we have

$$E_0(K, u, B_\rho(x)) < \tfrac{1}{\vartheta}(E_0(K, u, B_\rho(x)) - E_0(J, w, B_\rho(x)))\,.$$

Using that (K, u) is a (restricted) minimizer of E_λ on Ω and that $\|w\|_\infty \le \|g\|_\infty$, we get from the latter inequality

$$\begin{aligned}E_0(K, u, B_{\tau\rho}(x)) \le E_0(K, u, B_\rho(x)) &\le \frac{1}{\vartheta}(E_0(K, u, B_\rho(x)) - E_0(J, w, B_\rho(x)) \\ &\le \frac{1}{\vartheta}(E_\lambda(K, u, B_\rho(x)) - E_\lambda(J, w, B_\rho(x))) + \frac{\lambda}{\vartheta}\int_{B_\rho(x)} |w - g|^2 \\ &\le \tfrac{8\pi\lambda}{\vartheta}\|g\|_\infty^2 \rho^2\,.\end{aligned}$$

Thus, to conclude (B.0.1) we are left with proving (B.0.8) under the further condition (B.0.9).

The proof in the general case is very similar to that already presented in the case $\lambda = 0$, so that we highlight only the necessary changes. We argue by contradiction and consider $\tau > 0$ for which we can find (restricted) minimizers (K_j, u_j) of E_λ on Ω and infinitesimal sequences ε_j and ϑ_j, points $x_j \in \Omega$ and radii ρ_j so that $\mathcal{H}^1(K_j \cap B_{\rho_j}(x_j)) \le \varepsilon_j \rho_j$ and, on setting $E_j := E_0(K_j, u_j, B_{\rho_j}(x_j))$, we have

$$(1-\vartheta_j)E_j \le E_0(J, w, B_{\rho_j}(x_j)), \qquad E_0(K_j, u_j, B_{\tau\rho_j}(x_j)) > \tau^{3/2} E_j \,. \tag{B.0.10}$$

for all (J, w) with $\{u \neq w\} \cup (K_j \triangle J) \subset\subset B_{\rho_j}(x_j)$ and $\|w\|_\infty \le \|g\|_\infty$. Consider next the functions $v_j(x) := E_j^{-1/2} u_j(x_j + \rho_j x)$, the sets $H_j := B_1 \cap \frac{1}{\rho_j}(K_j - x_j)$, and the functionals F_j as defined in (B.0.3). By rescaling (B.0.10)

$$F_j(H_j, v_j, B_1) = 1 \,, \qquad 1 - \vartheta_j \le F_j(J, w, B_1) \tag{B.0.11}$$

for all (J, w) with $\{v_j \neq w\} \cup (H_j \triangle J) \subset\subset B_1$

$$F_j(H_j, v_j, B_\tau) > \tau^{3/2} \,, \tag{B.0.12}$$

(cf. with (B.0.4) and (B.0.5)). Thus, we may find a radius $r \in (\tau^{1/4}, 1)$ such that (B.0.2) hold, $v_j \in W^{1,2}(\partial B_r)$ with $\mathrm{osc}_{\partial B_r} v_j \le C$, and, up to subtracting the mean value of v_j over ∂B_r and relabeling the sequence v_j, we infer that $\|v_j\|_{L^\infty(\partial B_r)} \le C$. Consider next the function

$$\tilde v_j := \begin{cases} \min\{\|v_j\|_{L^\infty(\partial B_r)}, \max\{v_j, -\|v_j\|_{L^\infty(\partial B_r)}\}\} & B_r \\ v_j & B_1 \setminus B_r \end{cases}$$

Thus, $\tilde v_j \in SBV(B_1)$ with $\mathcal{H}^1(S_{\tilde v_j} \setminus S_{v_j}) = 0$. In particular, $\mathcal{H}^1(S_{\tilde v_j}) \le \mathcal{H}^1(S_{v_j}) \le \mathcal{H}^1(H_j)$. Ambrosio's compactness theorem implies that there exists a subsequence (not relabeled) and a function $v \in SBV(B_r)$ such that $\tilde v_j$ converge to v in $L^2(B_r)$, and that for all open subsets $A \subseteq B_r$

$$\mathcal{H}^1(S_v \cap A) \le \liminf_j \mathcal{H}^1(S_{\tilde v_j} \cap A) = 0, \qquad \int_A |\nabla v|^2 \le \liminf_j \int_A |\nabla \tilde v_j|^2 \le 1 \,.$$

Therefore, $v \in W^{1,2}(B_r)$. Moreover, for every $t \in (0, r)$ we have

$$\lim_j \Big(\int_{B_t} |\nabla v_j|^2 - \int_{B_t} |\nabla \tilde v_j|^2 \Big) = 0 \,. \tag{B.0.13}$$

Indeed, on one hand being $\tilde v_j$ a truncation of v_j, we have for every Borel subset B of B_1

$$\int_B |\nabla \tilde v_j|^2 \le \int_B |\nabla v_j|^2 \,.$$

In turn, from this and (B.0.11) we deduce that for all $t \in (0, 1)$

$$\int_{B_t} |\nabla v_j|^2 \le \int_{B_t} |\nabla \tilde v_j|^2 + \vartheta_j \,.$$

Therefore, we conclude that

$$0 \le \int_{B_t} |\nabla v_j|^2 - \int_{B_t} |\nabla \tilde v_j|^2 \le \vartheta_j \,.$$

Arguing as for item (a), by means of (B.0.13) one can prove that for all $t \in (0, r)$

$$\lim_j \int_{B_t} |\nabla v_j|^2 = \int_{B_t} |\nabla v|^2$$

and furthermore that v is harmonic on B_r by exploiting the (restricted) almost minimality of v_j for F_j on B_1, namely the second condition in (B.0.11). Eventually, the final contradiction follows again as for item (a) from (B.0.12). □

An immediate consequence of Lemma B.0.1 are the density lower bound estimates for (restricted) minimizers of E_λ. We follow the argument in [4, Theorem 7.21].

Proof of Theorem 2.1.3. We establish the estimates separately in the two cases $\lambda = 0$ and $\lambda > 0$. In both cases we fix $\tau = 1/2$ in Lemma B.0.1 and set $\epsilon := \varepsilon(1/2)$.

Case $\lambda = 0$. Consider the set

$$\Omega_u := \{x \in \Omega : \mathcal{H}^1(K \cap B_\rho(x)) < \epsilon\rho \text{ for some } \rho \in (0, \operatorname{dist}(x, \partial\Omega))\}.$$

Note that if $x \in \Omega_u$ and $\rho \in (0, \operatorname{dist}(x, \partial\Omega))$ is the corresponding radius, then $\mathcal{H}^1(K \cap B_\rho(x)) < (1-\mu)\epsilon\rho$ for some $\mu \in (0,1)$. It is then easy to check that $B_{(1-\mu/2)\rho}(x) \subset \Omega_u$, so that Ω_u is an open subset of Ω.

Let $\sigma > 0$ be such that $2\pi\sigma^{1/2} < \epsilon$. We claim that if $\mathcal{H}^1(K \cap B_\rho(x)) < \varepsilon(\sigma)\rho$ for some $\rho \in (0, \operatorname{dist}(x, \partial\Omega))$, then for all $j \in \mathbb{N}$

$$E_0(K, u, B_{\sigma 2^{-j}\rho}(x)) \le \epsilon 2^{-j/2} (\sigma 2^{-j}\rho)\,. \tag{B.0.14}$$

In particular, given this for granted, we get

$$\lim_{j\to\infty} \frac{E_0(K, u, B_{\sigma 2^{-j}\rho}(x))}{\sigma 2^{1-j}\rho} = 0,$$

so that $x \notin K_1$, where $K_1 := \{x \in K : \lim_r \frac{1}{2r}\mathcal{H}^1(K \cap B_r(x)) = 1\}$.

As the rectifiability of K implies that K_1 is dense in K, we conclude that $K = \overline{K_1} \subseteq \Omega \setminus \Omega_u$, being Ω_u open. On the other hand, u is harmonic on $\Omega \setminus K$, so that $\Omega \setminus K \subseteq \Omega_u$, and (2.1.1) thus follows at once.

Let us now prove (B.0.14) by an induction argument. Indeed, the case $j = 0$ follows immediately by from Lemma B.0.1, the energy upper bound and the choice of σ

$$E_0(K, u, B_{\sigma\rho}(x)) \le \sigma^{3/2} E_0(K, u, B_\rho(x)) \le 2\pi\sigma^{3/2}\rho < \epsilon\sigma\rho\,.$$

Assume now that (B.0.14) is true for some $j \in \mathbb{N}$, we show then that it holds for $j+1$. Indeed, in such a case Lemma B.0.1 and the induction assumption imply that

$$E_0(K, u, B_{\sigma 2^{-j-1}\rho}(x)) \le 2^{-3/2} E_0(K, u, B_{\sigma 2^{-j}\rho}(x)) \le \epsilon 2^{-(j+1)/2} (\sigma 2^{-j-1}\rho)\,.$$

Case $\lambda > 0$. The proof is very similar to that in the case $\lambda = 0$, we highlight only the necessary changes. Moreover we will prove the estimate, without loss of generality, for radii smaller than some suitably chosen constant R.

Choose $\sigma \in (0,1)$ such that $2\pi\sigma^{1/2} < \frac{\epsilon}{2}$, and $R > 0$ such that $\frac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2 R < \epsilon\sigma$. Note that, since $\|g\|_\infty \le M_0$, $\lambda \le 1$ and the parameter ϑ is fixed from Lemma B.0.1, R can be indeed chosen to be an absolute constant.

We claim that if $\mathcal{H}^1(K\cap B_\rho(x))<\varepsilon(\sigma)\rho$ for some $\rho\in(0,\min\{R,\mathrm{dist}(x,\partial\Omega)\})$, then (B.0.14) hold true. Indeed, if $j=0$, Lemma B.0.1, the energy upper bound and the choices of σ and R give

$$\begin{aligned}E_0(K,u,B_{\sigma\rho}(x)) &\le \max\{\sigma^{3/2}E_0(K,u,B_\rho(x)),\tfrac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2\rho^2\}\\ &\le \max\{\sigma^{3/2}E_\lambda(K,u,B_\rho(x)),\tfrac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2R\rho\}\\ &\le \max\{\sigma^{3/2}2\pi(1+\lambda\|g\|_\infty^2R),\tfrac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2R\}\rho<\epsilon\sigma\rho\,.\end{aligned}$$

Assume now that (B.0.14) is true for some $j\in\mathbb{N}$, we show that then it holds for $j+1$. Indeed, in such a case the induction assumption, Lemma B.0.1 and the choices of σ and R imply that

$$\begin{aligned}E_0(K,u,B_{\sigma2^{-j-1}\rho}(x)) &\le \max\{2^{-3/2}E_0(K,u,B_{\sigma2^{-j}\rho}(x)),\tfrac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2(\sigma2^{-j}\rho)^2\}\\ &\le \max\{2^{-3/2}\epsilon2^{-j/2}(\sigma2^{-j}\rho),\tfrac{8\pi\lambda}{\vartheta(1/2)}\|g\|_\infty^2R(\sigma2^{-j})^2\rho\}\\ &\le \max\{\epsilon2^{-(j+1)/2}(\sigma2^{-j-1}\rho),\epsilon\sigma(\sigma2^{-j})^2\rho\}\le\epsilon2^{-(j+1)/2}(\sigma2^{-j-1}\rho)\,.\end{aligned}$$

Therefore, (B.0.14) holds true. Set now

$$\Omega_u:=\{x\in\Omega:\ \mathcal{H}^1(K\cap B_\rho(x))<\epsilon\rho \text{ for some } \rho\in(0,\min\{R,\mathrm{dist}(x,\partial\Omega))\}\}\,.$$

It is easy to check that Ω_u is an open subset of Ω, that together with (B.0.14) give $K\subseteq\Omega\setminus\Omega_u$.

Eventually, by taking into account that $u\in W^{2,p}_{loc}(\Omega\setminus K)$ we conclude (2.1.2). ☐

Similar results hold true for minimizers of the corresponding weak formulations of the problem. To this aim we define for every open subset A of Ω and for every $v\in SBV(\Omega)$ the (formal) extensions of the functionals E_λ as follows:

$$\tilde{E}_\lambda(v,A):=\int_A|\nabla v|^2+\mathcal{H}^1(S_v\cap A)+\lambda\int_A|v-g|^2\,. \tag{B.0.15}$$

The very same arguments used in Lemma B.0.1 and Theorem 2.1.3 can be used to infer corresponding density lower bound estimates.

THEOREM B.0.2. *Let $u\in SBV(\Omega)$.*

(a) *There exists a constant $\epsilon>0$ such that if u is a minimizer of $\tilde{E}_0$ on Ω, then for all $x\in\overline{S}_u$*

$$\mathcal{H}^1(\overline{S}_u\cap B_\rho(x))\ge\epsilon\rho\qquad \textit{for all } \rho\in(0,\mathrm{dist}(x,\partial\Omega)). \tag{B.0.16}$$

(b) *There exist constant $\epsilon>0$ such that if u is a minimizer of $\tilde{E}_\lambda$ on Ω, then for all $x\in\overline{S}_u$*

$$\mathcal{H}^1(\overline{S}_u\cap B_\rho(x))\ge\epsilon\rho\qquad \textit{for all } \rho\in(0,\min\{1,\mathrm{dist}(x,\partial\Omega)\}). \tag{B.0.17}$$

Standard density estimates imply the essential closure of the jump set in the SBV setting (cf. [4, Theorem 2.56]).

Corollary B.0.3. *Let $u \in SBV(\Omega)$ be a minimizer of $\tilde{E}_\lambda$ on Ω. Then*
$$\mathcal{H}^1((\overline{S}_u \setminus S_u) \cap \Omega) = 0\,.$$

Observe that, given for granted the lower semicontinuity of the functional in SBV and the compactness theorem of Ambrosio, cf. [3], from the latter corollary we infer the existence of the "classical" minima for E_λ when $\lambda > 0$.

APPENDIX C

Normalization

PROOF OF COROLLARY 2.1.4. We start proving the first assertion for (restricted) minimizers u of E_λ. The fact that $u \in W^{2,p}_{loc}(\Omega \setminus K)$ for every $p < \infty$ and solves $\Delta u = \lambda(u-g)$ on each open subset $U \subset\subset \Omega \setminus K$ easily follows from the outer variation equation (1.5.2) and elliptic regularity theory. Hence, $\overline{S}_u \subseteq K$.

Let us now consider an absolute minimizer (K, u) of E_λ on Ω. With fixed $x \in \Omega$, $\mathcal{H}^0(K \cap \partial B_r(x))$ is finite except for a denumerable set of radii $I \subseteq (0, \mathrm{dist}(x, \partial\Omega))$. Thus $u \in W^{1,2}(\partial B_r(x) \setminus K)$ for $r \notin I$, and in particular $u \in L^\infty(\partial B_r(x))$. Therefore, $u \in L^\infty(B_r(x))$ thanks to Lemma 2.1.1. For any such radius, let $\tilde{u} \in SBV(B_r(x))$ be a minimizer of the functional $\tilde{E}_\lambda$ on $B_r(x)$ (cf. (B.0.15)), namely

$$\tilde{E}_\lambda(v, B_r(x)) = \int_{B_r(x)} |\nabla v|^2 + \mathcal{H}^1(S_v \cap B_r(x)) + \lambda \int_{B_r(x)} |v-g|^2$$

among functions $v \in SBV(B_r(x))$ with $\mathrm{spt}\{u \neq v\} \subset\subset B_r(x)$ (its existence follows from Ambrosio's lower semicontinuity theorem and the L^∞ bound on u). In this respect we recall that [18, Proposition 4.4] of De Giorgi, Carriero and Leaci implies that $w \in SBV(B_r(x))$ and $\mathcal{H}^1((S_w \setminus J) \cap B_r(x)) = 0$, for every couple (J, w) with $w \in W^{1,2}(B_r(x) \setminus J) \cap L^\infty(B_r(x))$. Therefore, we infer that

$$\begin{aligned} \tilde{E}_\lambda(\tilde{u}, B_r(x)) &\le \tilde{E}_\lambda(u, B_r(x)) \le E_\lambda(K \cap B_r(x), u, B_r(x)) \\ &\le E_\lambda(\overline{S_{\tilde{u}}} \cap B_r(x), \tilde{u}, B_r(x)) = \tilde{E}_\lambda(\tilde{u}, B_r(x)), \end{aligned}$$

where we have used the minimality of $\tilde{u}$ for $\tilde{E}_\lambda(\cdot, B_r(x))$ in the first inequality, the inclusion $\overline{S_u} \subseteq K$ in the second, the minimality of (K, u) for E_λ in the third, and the equality $\mathcal{H}^1((\overline{S_{\tilde{u}}} \setminus S_{\tilde{u}}) \cap B_r(x)) = 0$ in the last equality, that is a consequence of Corollary B.0.3. In particular, $\tilde{E}_\lambda(u, B_r(x)) = E_\lambda(K \cap B_r(x), u, B_r(x))$, in turn implying $\mathcal{H}^1((K \setminus \overline{S_u}) \cap B_r(x)) = 0$. By a covering argument we get $\mathcal{H}^1(K \setminus \overline{S_u}) = 0$.

Eventually, if (K, u) is a restricted minimizer of E_λ and $U \subset\subset \Omega$ is an open set such that $\mathcal{H}^1(K \cap U) = 0$, then $\mathcal{H}^1(\overline{S_u} \cap U) = 0$, and [18, Proposition 4.4] yields that $u \in W^{1,2}_{loc}(U)$. Hence, u extends to a harmonic function when $\lambda = 0$, resp. to a function in $W^{2,p}_{loc}(U)$ when $\lambda > 0$, in view of (1.5.2) and standard elliptic regularity theory. $\square$

APPENDIX D

Useful results from elementary topology

This section collects some useful elementary results, mostly of topological nature, which have been used extensively in the notes

LEMMA D.0.1. *Consider a closed set $K \subset \mathbb{R}^2$ and let J be a bounded connected component of K. Then, for every $\delta > 0$ there is a smooth Jordan curve γ such that*

(a) $\operatorname{dist}(y, J) < \delta$ *for every* $y \in \gamma$*;*
(b) $\gamma \cap K = \emptyset$*;*
(c) *J is contained in the bounded connected component of* $\mathbb{R}^2 \setminus \gamma$.

LEMMA D.0.2. *Let $K \subset \mathbb{R}^2$ be a closed connected set with locally finite Hausdorff measure. Then K is arcwise connected. Moreover, for every $x, y \in K$ there is an injective Lipschitz path $\gamma : [0,1] \to K$ such that $\gamma(0) = x$, $\gamma(1) = y$, and its length is minimal among all paths in K connecting x and y.*

PROOF OF LEMMA D.0.1. Consider the open set $U_\eta := \{y : \operatorname{dist}(y, K) < \eta\}$ and let V_η be the connected component of U_η which contains J. Clearly, each point $y \in \partial V_\eta$ has distance η from K. We next claim that for η sufficiently small $\overline{V}_\eta$ is compact. Indeed first of all observe that there is an open set Z such that $J \subset Z$ and $\partial Z \cap K = \emptyset$. Next notice that, since J is bounded, it is contained in some open ball $B_R(0)$. If we take $Z' := Z \cap B_R(0)$, then $J \subset Z'$ and moreover $\partial Z' \subset (B_R(0) \cap \partial Z) \cup (Z \cap \partial B_R(0))$, hence it does not intersect K. Since $\partial Z'$ is compact, it means that there is some positive constant c_0 such that $\operatorname{dist}(y, K) \geq c_0$ for every $y \in Z'$. In particular, for η sufficiently small, $\partial U_\eta \cap \partial Z' = \emptyset$. Thus $Z' \cap U_\eta$ is an open subset of U_η which contains J, implying that $V_\eta \subset Z'$.

Next we claim that, if η is sufficiently small, then we have

$$V_\eta \subset \{z : \operatorname{dist}(z, J) < \frac{\delta}{2}\}\,. \tag{D.0.1}$$

Indeed, if the latter were not true, then $\{\overline{V}_\eta\}_{\eta>0}$ would be a nested family of connected closed sets, each of which contains a point $z_\delta \in K$ at distance at least $\frac{\delta}{2}$ from J. Moreover, for η sufficiently small the sets would be bounded and hence compact. Their intersection K_∞ would then be a compact connected subset of K and since it contains J, it must be J. On the other hand, any accumulation point of $\{z_\delta\}_{\delta>0}$ would be an element of K_∞ at a positive distance from J, which is a contradiction.

Consider next $\eta > 0$ for which (D.0.1) holds and a standard family of mollifiers φ_ε, the function $\psi_\varepsilon := \mathbf{1}_{V_\eta} * \varphi_\varepsilon$ and the open sets $V_{\varepsilon,t} := \{y : \psi_\varepsilon(y) > t\}$. For ε sufficiently small we have that:

- $J \subset V_{\varepsilon,t}$ for every $1 > t \geq 0$;
- $\partial V_{\varepsilon,t} \cap K = \emptyset$ for every $1 > t \geq \frac{1}{4}$;
- $V_{\varepsilon,t} \subset \{y : \operatorname{dist}(y, J) < \delta\}$ for every $t \geq \frac{1}{4}$.

Fix such a small ε use Sard's theorem to select a $t \in (\frac{1}{4}, 1)$ such that $V_{\varepsilon,t}$ has smooth boundary. Furthermore pick the connected component W of $V_{\varepsilon,t}$ which contains J. By smoothness ∂W consists of a finite number of disjoint smooth Jordan curves and by classical differential topology there is one which is uttermost, i.e. such that W is contained in the topological disk bounded by it. The latter is a curve γ which satisfies all the requirements of the lemma. $\square$

Proof of Lemma D.0.2. **Step 1** We prove the first claim of the lemma when K is compact. In that case $\mathcal{H}^1(K)$ is finite. Fix now $\delta > 0$ and let $\mathscr{C} := \{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ be a cover of K such that $\sup_i r_i < \delta$ and $\sum_i 2r_i < \mathcal{H}^1(K) + \delta$. By compactness, we can assume, without loss of generality that $\mathscr{C}$ is finite and we can also assume that $B_{r_i}(x_i) \cap K \neq \emptyset$ for all i. A chain of $\mathscr{C}$ is given by a choice of balls in $\mathscr{C}$ with radii $\{r_{i(j)}\}_{j\in\{1,\ldots N\}}$ where the $i(j)$ are all distinct and $B_{r_{i(j)}} \cap B_{r_{i(j+1)}} \neq \emptyset$. We say that $B_{r_K}(x_K)$ and $B_{r_J}(x_J)$ are chain-connected if there is a chain such that $K = i(1)$ and $J = i(N)$. Assume now, w.l.o.g., that $B_{r_1}(x_1)$ contains x and let $\mathscr{C}' \subset \mathscr{C}$ be the set of balls $B_{r_i}(x_i)$ which are chain connected to $B_{r_1}(x_1)$. $\mathscr{C}'$ must coincide with $\mathscr{C}$ otherwise the two open sets

$$U := \bigcup_{i\in\mathscr{C}'} B_{r_i}(x_i) \tag{D.0.2}$$

$$V := \bigcup_{i\in\mathscr{C}\setminus\mathscr{C}'} B_{r_i}(x_i) \tag{D.0.3}$$

would be disjoint and would disconnect K. Upon reindexing our balls we can thus assume that $\{B_{r_i}(x_i)\}_{i\in\{1,\ldots,N\}}$ is a chain such that $x \in B_{r_1}(x_1)$ and $y \in B_{r_N}(x_N)$. Set $z_0 = x$, $z_N = y$, and choose $z_i \in B_{r_i}(x_i) \cap K$ for every other i. Consider then the piecewise linear curve consisting of joining the segments $[z_i, z_{i+1}]$. Since $|z_{i+1} - z_i| \leq 2(r_{i+1} + r_i)$, such curve has length at most $2\mathcal{H}^1(K) + 2\delta$. Observe moreover that, since $r_i < \delta$ for all i, each point of the curve has distance at most 2δ from K.

Let now $\delta := 1/j$ and let $\gamma_j : [0,1] \to \mathbb{R}^2$ be a constant speed parametrization of the latter curve. It turns out that $\operatorname{Lip}(\gamma_j) \leq 2\mathcal{H}^1(K) + 2$ and thus by Ascoli-Arzelà we can extract a subsequence converging uniformly to a Lipschitz curve $\gamma : [0,1] \to K$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Step 2 We next prove the first claim for K closed. We fix $x, y \in K$ and we seek for an arc connecting x and y in K. By Step 1 it suffices to show that for a sufficiently large $R > 0$ the points x and y must be contained in the same connected component of $K \cap \overline{B}_R$. Now, for each $N \in \mathbb{N}$ with $N \geq \max\{|x|, |y|\}$ let K_N be the connected component of $K \cap \overline{B}_N$ which contains x and assume by contradiction that $y \notin K_N$ for every N. We set $\tilde{K} := \bigcup_N K_N$. The latter set does not contain y: if we can show that it is at the same time open and closed in K, we have contradicted the connectedness of K.

First of all observe that, by Step 1, $K_N \subset K_{N+1}$, which in turn implies that $\tilde{K}$ is open in K. We next claim that:

(S) for every fixed $R > |x| + 1$ there is $N(R) > R$ such that $K_{N'} \cap \overline{B}_R = K_{N(R)} \cap \overline{B}_R$ for all $N' > N(R)$.

This would imply that $\overline{B}_R \cap \tilde{K} = \overline{B}_R \cap K_{N(R)}$ is closed, in turn implying that $\tilde{K}$ is closed.

If (S) is false for some R, then we can find a monotone sequence $N_k \geq R$ such that $K_{N_{k+1}} \cap \overline{B}_R$ is strictly larger than $K_{N_k} \cap \overline{B}_R$. In particular, there is a point $x_{k+1} \in K_{N_{k+1}} \cap \overline{B}_R$ which is not contained in $X_{N_k} \cap \overline{B}_R$. Let $\gamma_k : [0,1] \to K_{N_{k+1}}$ be a curve which connects x to x_{k+1} in $K_{N_{k+1}}$. Such curve cannot be contained in $\overline{B}_R$ and cannot intersect K_{N_k}, otherwise x_{k+1} would belong to K_{N_k}. Let now s_k be the smallest positive number such that $|\gamma_k(s_k)| = R$. It then turns out that the arc $\gamma_{k+1}([0, s_k])$ is contained in $K_{N_{k+1}} \cap \overline{B}_R$, but it has empty intersection with $K_{N_k} \cap \overline{B}_R$. Since such arc has length at least 1 (recall that $|x| \leq R - 1$), we conclude

$$\mathcal{H}^1(N_{k+1} \cap \overline{B}_R) \geq \mathcal{H}^1(N_k \cap \overline{B}_R) + 1\,.$$

Letting $k \uparrow \infty$ we would conclude $\mathcal{H}^1(K \cap \overline{B}_R) = \infty$.

Step 3 To prove the last claim, fix $x, y \in K$. By Step 1 and 2 we know the existence of a Lipschitz curve $\gamma : [0,1] \to K$ joining x and y in K. Let $L(\gamma) := \int |\dot{\gamma}(t)|\, dt$ and assume without loss of generality that it is parametrized at constant speed. Consider now the set $\mathscr{S}$ of Lipschitz curves $\gamma : [0,1] \to K$ joining x and y and parametrized at constant speed $L(\gamma)$. If L_0 is the infimum of $L(\gamma)$ for $\gamma \in \mathscr{S}$, then there is indeed a curve $\bar{\gamma}$ which attains it: this follows easily from Ascoli-Arzelà and the fact that every Lipschitz curve can be reparametrized to constant speed. $\bar{\gamma}$ is easily seen to be injective, otherwise it could not be a minimizer of the functional L. □

APPENDIX E

Proof of Theorem 2.2.3

Step 1 We start showing that there is a constant $C > 0$ such that

$$\frac{1}{C} \limsup_{j\to\infty} \mathcal{H}^1(K_j \cap B_r(x)) \le \mathcal{H}^1(K \cap \overline{B}_r(x)) \le C \liminf_{j\to\infty} \mathcal{H}^1(K_j \cap B_{2r}(x)) \tag{E.0.1}$$

for every ball $B_r(x) \subset \mathbb{R}^2$. Fix a positive $\delta \in (0, r]$ and consider any covering of disks $\{B_{r_k}(x_k)\}$ of $K \cap \overline{B}_r(x)$ with $\sup_k r_k \le \delta$ and $x_k \in K \cap \overline{B}_r(x)$. By compactness extract a finite subcover and using Vitali's covering Lemma, let $\{B_{r_\ell}(x_\ell)\}$ be a subfamily of pairwise disjoint disks such that $\{B_{5r_\ell}(x_\ell)\}$ is still a cover. Since the family is finite, for every j large enough we can find $y^j_\ell \in K_j \cap B_{r_\ell/2}(x_\ell)$. In particular, using the density lower bound (which under our assumption holds with a uniform constant independent of j) we have

$$\begin{aligned}\mathcal{H}^1_\delta(K \cap B_r(x)) &\le 10\pi \sum_\ell r_\ell \le C \sum_\ell \mathcal{H}^1(K_j \cap B_{r_\ell/2}(y^j_\ell)) \\ &\le C\mathcal{H}^1(K_j \cap B_{2r}(x))\,.\end{aligned}$$

Since δ is arbitrary, this implies the right inequality in (E.0.1). Next, fix $\delta > 0$ and let $\{E_k\}$ be a family of closed sets covering $K \cap \overline{B}_r(x)$ such that

$$\sum_k \operatorname{diam}(E_k) \le \mathcal{H}^1(K \cap \overline{B}_r(x)) + \delta\,.$$

Let $r_k := \operatorname{diam}(E_k)$ and consider disks $B_{2r_k}(x_k)$ containing E_k. By Hausdorff convergence, for j large enough $K_j \cap B_r(x) \subset \bigcup_\ell B_{2r_\ell}(x_\ell)$. We can thus estimate

$$\mathcal{H}^1(K_j \cap B_r(x)) \le \sum_\ell \mathcal{H}^1(K_j \cap B_{r_k}(x_k)) \le 4\pi \sum_k r_k \le C\mathcal{H}^1(K \cap \overline{B}_r(x)) + C\delta\,.$$

The arbitrariness of δ completes the proof.

We observe that this step is valid also for a sequence of restricted minimizers.

Step 2 We first give the argument for (K_j, u_j) absolute minimizers. Assume that $W \subset\subset U$ is an open set whose closure is a closed topological disk with the property that $\partial W \cap K = \emptyset$. By a standard approximation theorem there is a second topological disk $W \subset V \subset\subset U$ with $\partial V \cap K = \emptyset$ and which has smooth boundary. Then for a sufficiently large j we have as well that $\partial V \cap K_j = \emptyset$. Set $m_j := \min_{\partial V} u_j$ and $M_j := \max_{\partial V} u_j$ and use the PDE $\Delta u_j = \lambda_j(u_j - g_j)$ in a neighborhood of ∂V (independent of j) to conclude that $M_j - m_j$ is bounded uniformly independently of j. Moreover, by the maximum principle in Lemma 2.1.1 (b) it turns out that $\min\{-\|g_j\|_\infty, m_j\} \le u_j|_V \le \max\{M_j, \|g_j\|_\infty\}$. In particular, $u_j - m_j$ has a uniform BV bound on V and, up to subsequences, converges to an SBV function $\tilde{u}$ by the SBV closure theorem [4, Theorem 4.7]. Denote by $S_{\tilde{u}}$ its set of approximate discontinuities. The lower semicontinuity of the Mumford-Shah functional shows that

$$\int_V |\nabla \tilde{u}|^2 + \mathcal{H}^1(S_{\tilde{u}} \cap V) \leq \liminf_j \left(\int_{V \setminus K_j} |\nabla u_j|^2 + \mathcal{H}^1(K_j \cap V) \right)$$

cf. [4, Theorems 4.8]. On the other hand, a simple comparison argument (recall that $u_j|_{\partial V} \to u|_{\partial V}$ smoothly) shows at the same time that the latter is instead an equality (and the liminf is an actual limit), while $\tilde{u}$ is in fact a minimizer. In particular, De Giorgi's density lower bound applies and shows that $\mathcal{H}^1(\overline{S}_{\tilde{u}} \setminus S_{\tilde{u}}) = 0$ (cf. Corollary B.0.3). Setting $\tilde{K} := \overline{S}_{\tilde{u}}$ we obtain that $\mathcal{H}^1((K \setminus \tilde{K}) \cap V) = 0$ by Corollary 2.1.4. In particular this proves the rectifiability of $K \cap V$.

We next argue that in fact all of K is rectifiable. Let $K^\sharp$ be the union of the connected components of K which are singletons. If $y \in K^\sharp$, then there is a closed topological disk D with smooth boundary and containing y in the interior and such that $\partial D \cap K = \emptyset$ (cf. Lemma D.0.1). We can therefore apply the argument above to conclude that $K^\sharp$ is rectifiable. Consider now $K \setminus K^\sharp$. Then every $y \in K \setminus K^\sharp$ is contained in a nontrivial connected component of K: since each such component has positive length, there are countably distinct ones. As it is well known each connected closed set with finite Hausdorff measure is rectifiable, which completes the proof of our claim (cf. [6, Theorem 4.4.7]).

In the case of restricted minimizers we consider again the situation in which $W \subset\subset U$ is an open set whose boundary is a closed topological disk with the property that $\partial W \cap K = \emptyset$. Choose V as above and denote by $N(j)$ the number of connected components of $V \cap K_j$. If $\bar{N} := \liminf_j N(j) < \infty$, then $K \cap V$ consists of at most $\bar{N}$ components and it is therefore rectifiable. If $\lim_j N(j) = \infty$, it is not difficult to see that we can apply the same argument above, i.e. the SBV function $\tilde{u}$ is a minimizer. The reason is that any SBV function $\tilde{u}$ can be approximated in the Mumford-Shah energy with an SBV function whose jump set is closed and has a finite number of connected components, see for instance [12].

Step 3 We next wish to show that

$$\liminf_{j \to \infty} \mathcal{H}^1(K_j \cap A) \geq \mathcal{H}^1(K \cap A)$$

for every open set A. Consider the measures $\mu_j(E) := \mathcal{H}^1(K_j \cap E)$, E Borel, and assume that, up to subsequences, $\mu_j \rightharpoonup^\star \mu$ for some measure μ. Observe that, by Step 1,

$$C^{-1}\mathcal{H}^1(K \cap E) \leq \mu(E) \leq C\mathcal{H}^1(K \cap E) \qquad \text{for every Borel } E.$$

We thus have

$$\mu(K \cap E) = \int_{K \cap E} \theta(x) d\mathcal{H}^1(x)$$

for some Borel function θ taking values in $[C^{-1}, C]$. Our claim is thus equivalent to $\theta(x) \geq 1$ for $\mathcal{H}^1$-a.e. $x \in K$. Assume not, since K is rectifiable we can choose a point $x \in K$ where the approximate tangent to K exists, the 1-dimensional density of K equals 1 and moreover

$$\lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{2\rho} = \theta(x) < 1\,.$$

Using the density lower bounds for K, it then follows that the rescaled sets $K_{x,\rho} := \{\frac{y-x}{\rho} : y \in K\}$ converge locally in the Hausdorff distance to a 1-dimensional subspace ℓ of $\mathbb{R}^2$. Without loss of generality we can assume that $\ell = \{(x_1, 0) : x_1 \in \mathbb{R}\}$. In addition, by taking a diagonal sequence, we can assume that $\tilde{K}_j := (K_j)_{x,\rho_j}$ has the following properties:

$$\lim_j \mathcal{H}^1(\tilde{K}_j \cap B_1) = 2\theta(x) < 2\,.$$

Introducing the appropriate rescalings of the functions u_j we then have the following situation:

- $(\tilde{K}_j, \tilde{u}_j)$ are minimizers of the Mumford-Shah functional in B_2 with appropriate fidelity functions $\tilde{g}_j$ and fidelity constants $\tilde{\lambda}_j$;
- $\tilde{K}_j$ converges in the Hausdorff sense to the line $\{x_2 = 0\}$;
- $\mathcal{H}^1(\tilde{K}_j \cap B_1) \leq 2\theta < 2$ for some θ and for all j;
- $\tilde{\lambda}_j \|\tilde{g}_j\|^2_\infty \to 0$.

Observe that the following holds:

(R) if we replace the sequence ρ_j with any sequence $\tilde{\rho}_j$ such that $0 < \liminf_j \frac{\rho_j}{\tilde{\rho}_j} \leq \limsup_j \frac{\rho_j}{\tilde{\rho}_j} < \infty$, then all the properties above remain true.

Now, for each γ sufficiently small compared to $2-2\theta$, it is easy to see that there is $\rho \in (\gamma, 1)$ and a subsequence such that

$$\lim_{\ell\to\infty} \int_{\partial B_\rho} |\nabla \tilde{u}_{j_\ell}|^2 < \infty$$
$$\lim_{\ell\to\infty} \mathcal{H}^0(\tilde{K}_{j_\ell} \cap \partial B_\rho) \leq 2\,.$$

On the other hand, by possibly changing the sequence ρ_j to a new sequence $\tilde{\rho}_j$ satisfying (R) above, and after extracting a further subsequence, we can assume that $\rho \in (1-\gamma, 1)$.

Moreover, again by a Fubini argument, if γ is sufficiently small compared to $2-2\theta$, we can find a $t \in (-(1-\gamma), 1-\gamma)$ such that

$$\lim_{\ell\to\infty} \int_{-1}^1 |\nabla \tilde{u}_j(t, x_2)|^2\, dx_2 < \infty$$
$$\tilde{K}_{j_\ell} \cap \{(t, x_2) : |x_2| \leq 1\} = \emptyset \quad \forall \ell\,.$$

In particular, from this, it is again simple to see that $M_\ell - m_\ell := \sup_{\partial B_\rho} \tilde{u}_{j_\ell} - \inf_{\partial B_\rho} \tilde{u}_{j_\ell}$ is uniformly bounded in ℓ. We thus conclude that $\tilde{u}_{j_\ell}$ converges, up to subsequences, to an SBV function $\tilde{u}$. Again the latter is a minimizer of the Mumford-Shah functional and $K \cap B_\rho$ must be the closure of $J_{\tilde{u}}$ and

$$2(1-\gamma) \leq 2\rho = \mathcal{H}^1(K \cap B_\rho) \leq \liminf_{\ell\to\infty} \mathcal{H}^1(\tilde{K}_{j_\ell} \cap B_\rho) \leq 2\theta\,.$$

Since γ is arbitrary, the latter is a contradiction.

In the case of restricted minimizers we again consider the number of connected components $N(j)$ of $\tilde{K}_j \cap B_\rho$. If the latter goes to infinity, we see that, as in the previous step, $\tilde{u}$

is a minimizer of the Mumford-Shah functional. Otherwise there is a uniform upper bound on the number of connected components and then the inequality

$$\mathcal{H}^1(K \cap B_\rho) \le \liminf_{\ell\to\infty} \mathcal{H}^1(\tilde{K}_{j,\ell} \cap B_\rho)$$

follows from the classical Golab's theorem (cf. [6, Theorem 4.4.17]).

Step 4 Next observe that

$$\lim_{j\to\infty} \int_O |\nabla u_j|^2 = \int_O |\nabla v|^2$$

for every open set $O \subset\subset U \setminus K$. The latter in fact is a consequence of the Hausdorff convergence of K_j to K and of standard regularity properties of harmonic functions.

Next, if we fix any open set $O \subset U$, choosing a sequence $O_\ell \uparrow O \setminus K$ of open sets with $\overline{O}_\ell \cap K = \emptyset$ we easily conclude

$$\int_{O\setminus K} |\nabla v|^2 = \lim_{\ell\to\infty} \int_{O_\ell\setminus K} |\nabla v|^2 \le \liminf_{j\to\infty} \int_O |\nabla u_j|^2 \,.$$

In particular, from the last inequality and Step 3, in order to conclude point (i) in the statement of the theorem we just need to show

$$\limsup_{j\to\infty} \left(\int_{O\setminus K_j} |\nabla u_j|^2 + \mathcal{H}^1(K_j \cap O) \right) \le \int_{O\setminus K} |\nabla v|^2 + \mathcal{H}^1(K \cap O)$$

under the assumption that $\mathcal{H}^1(\partial O \cap K) = 0$. Assume that the inequality fails and fix a subsequence, not relabeled, for which the limsup on the left is a limit. After possibly extracting a further subsequence, we can assume that the measures μ_j defined through

$$\mu_j(E) := \int_{E\setminus K_j} |\nabla u_j|^2 + \mathcal{H}^1(K_j \cap E)$$

converge weakly* to some measure μ and so far we can conclude that

$$\mu(E) = \int_{E\setminus K} |\nabla v|^2 \qquad \text{if } E \text{ is Borel and } K \cap E = \emptyset.$$

$$\mu(E) \ge \mathcal{H}^1(K \cap E) \qquad \text{if } E \text{ is Borel and } E \subset K$$

$$\mu(F) > \mathcal{H}^1(K \cap F) \qquad \text{for some } F \subset K \text{ Borel.}$$

Note however that from the upper bound (1.3.1) we immediately conclude $\mu(B_r(x)) \le 2\pi r$ for every disk $B_r(x) \subset U$. In particular

$$\mu(E) = \int_{E\setminus K} |\nabla v|^2 + \int_{K\cap E} \theta(x)\, d\mathcal{H}^1(x)\,,$$

for some density θ with $1 \le \theta \le \pi$, but which must be strictly larger than 1 on a set of positive $\mathcal{H}^1$ measure. Arguing as in the previous step, we pick a Lebesgue point $x \in K$ for θ with respect to the measure $\mathcal{H}^1$ and we can assume this is a point where the approximate tangent to the rectifiable set K exists. We can thus use the procedure in the previous step to produce a new sequence $(\tilde{K}_j, \tilde{u}_j)$ in B_3 with corresponding limits $\tilde{K}$ and $\tilde{v}$ and corresponding measures $\tilde{\mu}_j$ converging to $\tilde{\mu}$, with the additional properties that $\tilde{K}$ is a straight segment and the corresponding density $\tilde{\theta}$ is a constant strictly larger than 1. In order to simplify our notation we drop the $\tilde{}$, and we assume that the segment is $\sigma_0 := \{(x_1, 0) : |x_1| < 2\}$. Next we choose a vanishing sequence ε_j with the property that

$$K_j \cap B_2 \subset \{|x_2| \leq \varepsilon_j\} \tag{E.0.2}$$

$$\lim_{j\to\infty} \mu_j((a_j, b_j) \times (-\varepsilon_j, \varepsilon_j)) = \theta(b-a) \tag{E.0.3}$$

for sequences $a_j \to a$ and $b_j \to b$ in $[-2, 2]$ defined as follows: for each j choose two points $a_j \in [-2, -\frac{3}{2}]$ and $b_j \in [\frac{3}{2}, 2]$ with the property that, upon setting $L_j := ([a_j, a_j + \varepsilon_j) \cup (b_j - \varepsilon_j, b_j]) \times [-1, 1]$, then

$$\mu_j(L_j) \leq C\varepsilon_j \mu_j([-2, 2] \times [-1, 1]) \leq 6C\varepsilon_j\,, \tag{E.0.4}$$

where C is a geometric constant independent of j. Construct then the following Lipschitz deformation $\Phi_j : Q_j \to Q_j$, $Q_j := [a_j, b_j] \times [-1, 1]$, defined as $\Phi_j(x_1, x_2) := (x_1, \varphi_j(x_1, x_2))$ where, upon setting $f_j(x_1) := \min\{|x_1 - a_j|, |b_j - x_1|, \varepsilon_j\}$, the second component is given by

$$\varphi_j(x_1, x_2) = \begin{cases} 0 & \text{if } x \in \Sigma_j := \{|x_2| \leq f_j(x_1)\} \\ \frac{x_2 - f_j(x_1)}{1 - f_j(x_1)} & \text{if } x \in T_j^+ := \{f_j(x_1) < x_2 \leq 1\} \\ \frac{x_2 + f_j(x_1)}{1 - f_j(x_1)} & \text{if } x \in T_j^- := \{-1 \leq x_2 < -f_j(x_1)\} \end{cases} \tag{E.0.5}$$

(for a similar construction see the definition of the map Φ_j in the proof of Lemma 3.2.3). Note that Φ_j is the identity on ∂Q_j. Moreover, if we introduce the regions $Q_j^\pm := Q_j \cap \{\pm x_2 > 0\}$, then

- Φ_j is a bi-Lipschitz map of $T_j^\pm$ onto $Q_j^\pm$, with a uniform bound of the Lipschitz constant of both the map and its inverse;
- Φ_j maps the rectangles $R_j^\pm := [a_j + \varepsilon_j, b_j - \varepsilon_j] \times \{\varepsilon_j < \pm x_2 \leq 1\}$ onto the rectangles $[a_j + \varepsilon_j, b_j - \varepsilon_j] \times \{0 < \pm x_2 \leq 1\}$ and in these regions both $\mathrm{Lip}(\Phi_j)$ and $\mathrm{Lip}(\Phi_j^{-1})$ are bounded by $1 + C\varepsilon_j$ for a universal constant C;
- The restriction of Φ_j on Σ_j is the orthogonal projection onto the horizontal axis.

Let thus (J_j, v_j) be the pair:

- $J_j = ([a_j, b_j] \times \{0\}) \cup \Phi_j(K_j \setminus \Sigma_j)$;
- $v_j = u_j \circ \Phi_j^{-1}$ on $Q_j \setminus J_j$.

It easy to estimate the Mumford-Shah energy of (J_j, v_j) in the rectangle Q_j as

$$\begin{aligned} E_j &:= \int_{Q_j \setminus J_j} |\nabla v_j|^2 + \mathcal{H}^1(J_j \cap Q_j) \\ &\leq (1 + C\varepsilon_j)^3 \int_{R_j^+ \cup R_j^-} |\nabla u_j|^2 + (b_j - a_j - 2\varepsilon_j) + C\mu_j(L_j) \\ &\leq (1 + C\varepsilon_j)^3 \int_{Q_j \setminus K_j} |\nabla u_j|^2 + (b_j - a_j - 2\varepsilon_j) + C\mu_j(L_j)\,. \end{aligned}$$

Then from (E.0.3), (E.0.4) and the convergences $b_j \to b$ and $a_j \to a$, we easily conclude that

$$\lim_{j\to\infty}\left(\int_{Q_j\setminus K_j} |\nabla u_j|^2 + \mathcal{H}^1(K_j\cap Q_j) - E_j\right) \geq (\theta-1)(b-a)$$

contradicting the minimality of (K_j, u_j) for j large enough, recalling that $\theta > 1$ and considering that after rescaling the fidelity terms converge to 0 for our competitors.

Note that the competitor does not increase the number of connected components and therefore the argument remains valid for restricted minimizers.

Step 5 We now prove property (ii) (and come back later to (2.2.5) to complete the proof of (i)). Fix thus an open set O and a competitor (J, w) as in the statement and observe that the competitor keeps the same property if we replace O with a larger open set. In particular, without loss of generality, we can assume that O has a smooth boundary. Consider now a tubular neighborhood of ∂O where the distance function to ∂O gives a smooth foliation. By a Fubini-type argument, we can change slightly the set and assume that for a subsequence, not relabeled, we have

$$\sup_j \left(\int_{\partial O\setminus K_j} |\nabla u_j|^2 + \mathcal{H}^0(K_j\cap \partial O)\right) < \infty$$
$$\int_{\partial O\setminus K} |\nabla v|^2 + \mathcal{H}^0(K\cap\partial O) < \infty\,.$$

Denote by O_δ the set $O_\delta := \{x \in O : \mathrm{dist}(x, \partial O) > \delta\}$. We leave as an exercise to the reader to show that for each δ sufficiently small there is a diffeomorphism Φ_δ of O_δ onto O with the property that $\|D\Phi_\delta - \mathrm{Id}\|_{C^0} + \|D\Phi_\delta^{-1} - \mathrm{Id}\|_{C^0}$ is infinitesimal as $\delta \to 0$. Enumerate the connected components of $U \setminus K$ which intersect ∂O and denote them by $U_1, \ldots, U_N$. We leave to the reader to prove the simple fact that, by our definition of the v^i's, there is a choice of constants p_{ik}'s with the property that, if we define the map $\hat{v}_j$ on each U_i to be equal to $v^i + p_{ik}$, then there is a suitable sequence $\delta_j \downarrow 0$ and a suitable interpolating pair (J_j, z_j) on $O \setminus O_{\delta_j}$ such that

- $z_j = u_j$ on ∂O and $J_j \cap \partial O = K_j \cap \partial O$;
- $z_j = \hat{v}_j \circ \Phi_{\delta_j}$ on ∂O_{δ_j} and $J_j \cap \partial O_{\delta_j} = \Phi_{\delta_j}^{-1}(\partial O \cap J)$;
- $\int_{O\setminus O_{\delta_j}} |\nabla z_j|^2 + \mathcal{H}^1(J_j \cap (O \setminus O_{\delta_j})) \to 0$ as $j \uparrow \infty$.

Now, the technical condition in item (ii) of Definition 2.2.2 on the competitor makes sure that, if $x, y \in \partial O$ belong to two distinct connected components Ω_i and Ω_k of $U \setminus K$, then x, y are in distinct connected components of $U \setminus J$. Therefore, given $q \in U \setminus J$, we define $\hat{w}_j(q) := w(q) + p_{ik}$ if the connected component of $U \setminus J$ containing q intersects ∂O in Ω_i, while we define $\hat{w}_j(q) := w(q)$ if the connected component of $U \setminus J$ does not intersect ∂O. We are now ready to define (J_j, z_j) inside O_{δ_j} as well: we put $J_j \cap O_{\delta_j} := \Phi_{\delta_j}^{-1}(J \cap O)$ and we define $z_j := \hat{w}_j \circ \Phi_{\delta_j}$. Observe now that, if the energy of (J, w) were smaller than that of (K, v), then for sufficiently large j the energy of (J_j, z_j) in O would be smaller than that of (K_j, u_j), which is a contradiction.

In the case of restricted minimizers we split again into two situations, one in which the number of connected components in O converges to infinity (in which case it can be shown that the limit is an absolute minimizer) or the one in which the number of connected components stays bounded. Since the number of connected components of the limit is at most the limit of the number of connected components of the approximating sequence, the limit is a restricted minimizer.

Step 6 We next prove (iii). To this aim we first draw some conclusions from (ii). First of all consider an open set $O \subset\subset U$ and assume there is a connected component A of $O \setminus K$ whose closure does not intersect ∂O. Now observe that, if we change the value of the constant in A, any admissible competitor for the new modified function is an admissible competitor for the old function, and viceversa. We can thus assume without loss of generality that v is identically equal to 0 on any such connected component, and ultimately on any bounded connected component of $U \setminus K$ whose closure does not have some portion of the boundary in common with ∂U.

Assume now that O intersects K in a finite number of points, that its boundary ∂O is regular and that the restriction of v on each connected component of $\partial O \setminus K$ belongs to $W^{1,2}$. Observe that all these properties imply the boundedness of v on $\partial O \setminus K$. If O' is a connected component of $O \setminus K$ which is not compactly contained in O, $v|_{O'}$ minimizes the Dirichlet energy among all $W^{1,2}$ functions which agree with it on $O' \cap \partial O$. By the maximum principle we conclude that v is bounded on every such O'. Since we have normalized v to be 0 on the remaining ones (which do not "touch" ∂O), we conclude that v is bounded in O. Ultimately, since for every open $A \subset\subset U$ we can easily find a slightly larger O with all the properties above, we conclude that v is locally bounded.

Consider now $\Omega_{\mathscr{A}}$ as in the statement of point (iii). It then turns out that, for every $A \subset\subset \Omega_{\mathscr{A}}$, the function $u_{\mathscr{A}}$ is bounded and has bounded variation in A. Using analogous reasonings, it is not difficult to see that, upon subtraction of a suitable constant c_j, we conclude as well that $u_j - c_j$ is uniformly bounded and has uniform bound on its BV norm. Moreover $u_j - c_j$ converge to $v - c$ for some suitable constant c. In particular the conclusion of point (iii) that $u_{\mathscr{A}}$ is a minimizer in A follows from the SBV existence theory of minimizers, as already argued in Step 2.

In the case of restricted minimizers we argue similarly.

Step 7 In order to show (2.2.5), and hence complete the proof of (i), consider the measures α_j on $\mathbb{P}^1\mathbb{R} \times \mathbb{R}^2$ given by

$$\int \varphi(\pi, x) d\alpha_j(\pi, x) := \int_{K_j} \varphi(T_x K_j, x) d\mathcal{H}^1(x)\,,$$

for every $\varphi \in C_c(\mathbb{P}^1\mathbb{R} \times U)$. From now on we use the notation $\alpha_j = \delta_{T_x K_j} \otimes \mathcal{H}^1 \llcorner K_j$. The convergence in (2.2.5) is equivalent to say that α_j converges weakly* to the measure $\alpha = \delta_{T_x K} \otimes \mathcal{H}^1 \llcorner K$. First of all, up to subsequences we can assume that $\alpha_j \rightharpoonup^\star \beta$ for some measure β. Secondly, by what proved so far $\beta(\mathbb{P}^1\mathbb{R} \times E) = \mathcal{H}^1(K \cap E)$. In particular, by the classical theorem on disintegration of measures, we can write $\beta = \beta^x \otimes \mathcal{H}^1 \llcorner K$, where β^x is a weakly* measurable family of probability measures on $\mathbb{P}^1\mathbb{R}$. Thus, (2.2.5) is equivalent to $\beta^x = \delta_{T_x K}$ for $\mathcal{H}^1$-a.e. $x \in K$. If the latter conclusion were false, we could then find a point x_0 where $\beta^{x_0} \neq \delta_{T_{x_0} K}$, the approximate tangent $T_{x_0} K$ to K at x_0 exists, and where the rescaled measures $\beta^{x_0,r}$ defined by

$$\int \varphi(\pi, y) d\beta^{x_0,r}(\pi, y) = \frac{1}{r} \int \varphi\left(\pi, \frac{y - x_0}{r}\right) d\beta(\pi, y)$$

converges to the product measure $\beta^{x_0} \times \mathcal{H}^1 \llcorner T_{x_0} K$. Observe moreover that, since

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{B_r(x)} |\nabla u|^2 = 0 \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in K\,,$$

we can assume to have selected $x_0 \in K$ such that the last condition holds, as well.

We can now extract a diagonal sequence and find a new sequence of minimizers $(\tilde{K}_j, \tilde{u}_j)$, with the property that $\tilde{K}_j$ converges to the line $\ell = T_{x_0} K$ uniformly on compact subsets, but the corresponding measures $\tilde{\alpha}_j = \delta_{T_x \tilde{K}_j} \otimes \mathcal{H}^1 \llcorner \tilde{K}_j$ converge weakly* to $\beta^{x_0} \times \mathcal{H}^1 \llcorner \ell$, with $\beta^{x_0} \neq \delta_\ell$. Moreover, up to subsequences, $(\tilde{K}_j, \tilde{u}_j)$ converge to a global generalized minimizer $(\ell, \tilde{u}, p_{12})$. Note that necessarily $|\nabla \tilde{u}| = 0$ vanishes identically. Therefore $p_{12} = \pm\infty$ (as it follows from Theorem 2.4.1 that the global minimizer is a pure jump).

To fix ideas rotate the coordinates so that $\ell = \{(t, 0) : t \in \mathbb{R}\}$. For each $t \in [-2, 2]$ consider the segment $\sigma_t := \{(t, s) : -1 \leq s \leq 1\}$ and define

$$E_j := \{t \in [-2, 2] : \sigma_t \cap \tilde{K}_j \neq \emptyset\}\,.$$

Since for sufficiently large j the $\tilde{u}_j$ will be harmonic on $[-1, 1] \times ([-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2])$ and by Chebyshev there is always a $t \in [-2, 2] \setminus E_j$ such that

$$\int_{\sigma_t} |\nabla u_j|^2 \leq \frac{1}{4 - |E_j|} \int_{[-2,2]^2} |\nabla \tilde{u}_j|^2\,,$$

we must necessarily have $|E_j| \to 4$, otherwise we would conclude $p_{12} = 0$.

Next for each $x \in \tilde{K}_j$ let $\theta(x)$ be the angle between the lines ℓ and $T_x \tilde{K}_j$. Recall that, by the generalised coarea formula [4, Theorem 2.93],

$$\mathcal{H}^1(\tilde{K}_j \cap [-2, 2]^2) = \int_{E_j} \sum_{x \in \sigma_t \cap \tilde{K}_j} (\cos\theta(x))^{-1}\, dt\,, \tag{E.0.6}$$

while by the conclusions in item (i)

$$\lim_{j \to \infty} \mathcal{H}^1(\tilde{K}_j \cap [-2, 2]^2) = 4\,. \tag{E.0.7}$$

Define thus $F_j := \{t \in E_j : \sharp(\sigma_t \cap \tilde{K}_j) = 1\}$ and $G_j := \tilde{K}_j \cap (F_j \times [-2, 2])$. Then from (E.0.6) we have

$$\mathcal{H}^1(\tilde{K}_j \cap [-2, 2]^2) \geq |F_j| + \mathcal{H}^1(\tilde{K}_j \cap ([-2, 2]^2 \setminus G_j)) \geq |F_j| + 2|E_j \setminus F_j|\,.$$

In particular, from (E.0.7) and $|E_j| \to 4$, we conclude $|F_j| \to 4$ and $\mathcal{H}^1(\tilde{K}_j \cap ([-2, 2]^2 \setminus G_j)) \to 0$. Next, for each $\delta > 0$ consider the set

$$H_j := \{x \in G_j : \cos\theta(x) < 1 - \delta\}$$

and its projection $\pi_\ell(H_j)$ on ℓ. We then have from (E.0.6)

$$\mathcal{H}^1(\tilde{K}_j \cap [-2,2]^2) \geq |F_j \setminus \pi_\ell(H_j)| + \mathcal{H}^1(H_j) \geq |F_j \setminus \pi_\ell(H_j)| + \frac{1}{1-\delta}|\pi_\ell(H_j)|\,.$$

In particular, we again conclude from (E.0.7) and $|F_j| \to 4$, that $|\pi_\ell(H_j)| \to 0$ and $\mathcal{H}^1(H_j) \to 0$. Summarizing, for any positive $\eta > 0$ we have

$$\lim_{j\to\infty} \mathcal{H}^1(\{x \in \tilde{K}_j \cap [-2,2]^2 : |T_x\tilde{K}_j - \ell| > \eta\}) = 0\,.$$

This however easily implies that $\delta_{T_x\tilde{K}_j} \otimes \mathcal{H}^1 \llcorner \tilde{K}_j$ converges to $\delta_\ell \times \mathcal{H}^1 \llcorner \ell$ on the open set $(-2,2)^2$, contradicting our assumption that $\beta^{x_0} \neq \delta_\ell$.

APPENDIX F

Hirsch's coarea inequality for Hölder maps

In this section we include an elementary observation by Jonas Hirsch, which leads to a coarea inequality for Hölder maps. The argument is similar to that of [23, Theorem 2.10.25] and, in fact, what we need could be concluded directly from the very statement of [23, Theorem 2.10.25] by selecting an appropriate target metric space Y in there. However the interesting point is not so much in the argument, but rather in the realization that it is indeed possible to have a coarea inequality for Hölder maps, a fact which we have not seen anywhere in the literature.

PROPOSITION F.0.1. *Le $f \in C^\alpha(\mathbb{R}^m)$ and $A \subset \mathbb{R}^m$ closed. For every $\beta \geq 0$ there is then a constant $C(\alpha, \beta)$ such that*

$$\int \mathcal{H}^\beta(f^{-1}(\{t\}) \cap A)\, dt \leq C[f]_\alpha \mathcal{H}^{\beta+\alpha}(A)\,. \tag{F.0.1}$$

PROOF. W.l.o.g. we assume $\mathcal{H}^{\alpha+\beta}(A) < \infty$. Fix $i \in \mathbb{N} \setminus \{0\}$ an almost optimal $\frac{1}{i}$ cover A with compact sets $\{B^i_j\}$, i.e.

$$\operatorname{diam}(B^i_j)^{\alpha+\beta} \leq \frac{1}{i} \tag{F.0.2}$$

$$\sum_j (\operatorname{diam}(B^i_j)) \leq C(\alpha, \beta)\mathcal{H}^{\alpha+\beta}_{i^{-1}}(A) + \frac{1}{i}\,. \tag{F.0.3}$$

The functions

$$g^i_j(y) := (\operatorname{diam}(B^i_j))^\beta \mathbf{1}_{f(B^i_j)}(y)$$

are nonnegative and measurable and so is

$$g^i(y) := \sum_j g^i_j(y)\,.$$

So

$$\begin{aligned}\int g^i(y)\, dy &= \sum_j (\operatorname{diam}(B^i_j))^\beta |f(B^i_j)| \leq \sum_j [f]_\alpha (\operatorname{diam}(B^i_j))^{\alpha+\beta}\\ &\leq C[f_\alpha]\mathcal{H}^{\alpha+\beta}_{i^{-1}}(A) + \frac{1}{i}\,.\end{aligned}$$

Note however that

$$\mathcal{H}^\alpha_{i^{-1}}(A \cap f^{-1}(\{y\}) \leq g^i(y)\,.$$

Since $\mathcal{H}^\alpha_{i^{-1}}(A \cap f^{-1}(\{y\}) \uparrow \mathcal{H}^\alpha(A \cap f^{-1}(\{y\})$ and $\mathcal{H}^{\alpha+\beta}_{i^{-1}}(A) \uparrow \mathcal{H}^{\alpha+\beta}(A)$ monotonically, the desired inequality follows from letting $i \uparrow \infty$. $\square$

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Index