Quasiconvex Bulk and Surface Energies: $C^{1,\alpha}$ Regularity

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July 27, 2023

Abstract

We establish regularity results for equilibrium configurations of vectorial multidimensional variational problems, involving bulk and surface energies. The bulk energy densities are uniformly strictly quasiconvex functions with p-growth, $p \geq 2$, without any further structure conditions. The anisotropic surface energy is defined by means of an elliptic integrand Φ not necessarily regular. For a minimal configuration (u, E), we prove partial Hölder continuity of the gradient ∇u of the deformation.

AMS Classifications. 49N15, 49N60, 49N99.
Key words. Regularity, nonlinear variational problem, free interfaces.

1 Introduction and statements

In this paper we study multidimensional vectorial variational problems involving bulk and surface energies, mainly related to problems issuing from material science and computer vision. Namely, we deal with regularity properties of solutions to such problems. The model problem

$$\int_\Omega \sigma_E(x)|\nabla u|^2\,dx + P(E,\Omega), \tag{1.1}$$

where $\sigma_E(x) := a\mathbb{1}_E + b\mathbb{1}_{\Omega\setminus E}$, $0 < a < b$, with $E \subset \Omega \subset \mathbb{R}^n$, and $P(E,\Omega)$ stands for the perimeter of the set E in Ω, dates back to the works of L. Ambrosio & G. Buttazzo and F.H. Lin. In [4, 35], the authors proved the existence and regularity for minimal configurations (u, E) of (1.1) in the scalar case. Furthermore, in [36] F.H. Lin & R.V. Kohn treated more general Dirichlet energies as the following

$$\mathcal{F}(u,E) := \int_\Omega \left(F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u)\right)\,dx + P(E,\Omega)\,, \tag{1.2}$$

with the constraints

$$u = \Psi \text{ on } \partial\Omega \text{ and } |E| = d,$$

$\Psi \in H^1(\Omega)$, $0 < d < |\Omega|$. They considered F and G convex functions growing quadratically on the gradient, and satisfying restrictive structure assumptions.
In the cited papers, it is proved $C^{0,\alpha}$ regularity for minimizers u in Ω and some estimates on the singular set of ∂E are given. More precisely, defined the set of regular points of ∂E as follows

$$\mathrm{Reg}(E) := \left\{x \in \partial E \cap \Omega : \partial E \text{ is a } C^{1,\gamma} \text{ hypersurface in } B_\varepsilon(x) \text{ for some } \varepsilon > 0 \text{ and } \gamma \in (0,1)\right\},$$

where $B_\varepsilon(x)$ denotes the ball of center x and radius ε, and, accordingly, the set of singular points of ∂E

$$\Sigma(E) := (\partial E \cap \Omega) \setminus \mathrm{Reg}(E),$$

then $\mathcal{H}^{n-1}(\Sigma(E)) = 0$, whereas (u, E) minimizes the functional (1.2).
More recently, G. De Philippis & A. Figalli in [17] and N. Fusco & V. Julin in [27], improved this result showing that minimal configurations of Dirichlet type functional (1.1) satisfy,

$$dim_{\mathcal{H}}(\Sigma(E)) \leq n - 1 - \varepsilon,$$

for some $\varepsilon > 0$ depending only on a, b. The same kind of estimate for the singular set $\Sigma(E)$ has been proved in [23, 24] for the more general Dirichlet functional (1.2).

The case of general functionals with p-growth on the gradient is still not completely understood especially regarding the regularity of the free interface ∂E and the dimension of the singular set $\Sigma(E)$. A first step in this direction has been done in [10] where the authors deal with constrained convex scalar problems, without structure assumptions on the bulk energies. They prove $C^{0,\alpha}$ regularity for minimizers u, but they do not give estimates for the singular set, which is an issue still unsolved, under the assumption of p-growth (see also contributions due to [21] and [34]). Nevertheless, as originally stressed in [22], for minimizers of Dirichlet functional (1.1), the exponent α, relative to the $C^{0,\alpha}$ regularity of minimizers u, can be affected by a closeness assumption on the coefficients a, b of $\sigma_E(x)$ appearing in (1.1). More precisely, in [22] it has been proved that the hypothesis $1 \leq \frac{a}{b} < \gamma_n$ ensures that $u \in C^{0,\frac{1}{2}+\varepsilon}$, for some $\varepsilon > 0$. Exploiting this information, the regularity of the boundary ∂E can be easily achieved by managing the bulk term as a perturbative term, since, by virtue of $C^{0,\frac{1}{2}+\varepsilon}$ regularity, the bulk term is asymptotically smaller than the perimeter term. Therefore, it can be invoked a well known regularity result for almost minimal perimeter minimizers due to Tamanini (see [40]), thus proving the regularity of the free boundary ∂E.

Differently from the scalar case, in the vectorial setting only a few regularity results for minimizers of integral functionals involving both bulk and interfacial energies are available in literature. According to our knowledge, the only papers dealing with the vectorial case are [6] and [11]. In [6] the regularity for vector valued free interface variational problems is treated within the context of k-th order homogeneous partial differential operators $\mathcal{A}$ (for a detailed study of $\mathcal{A}$-quasiconvexification see [28, 29]). In [11] the authors study minimal configurations of energy of the type

$$\mathcal{F}(v, A) := \int_\Omega (F(Dv) + \mathbb{1}_A G(Dv))\, dx + P(A, \Omega), \tag{1.3}$$

where $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ and $F, G : \mathbb{R}^{n\times N} \to \mathbb{R}$ are C^2 integrands, satisfying, for $p > 1$ and for positive constants ℓ_1, ℓ_2, L_1, $L_2 > 0$, the following growth and uniformly strict p-quasiconvexity conditions,

$$0 \leq F(\xi) \leq L_1(1 + |\xi|^2)^{\frac{p}{2}}, \tag{$F1$}$$

$$\int_\Omega F(\xi + D\varphi)\, dx \geq \int_\Omega \left(F(\xi) + \ell_1 |D\varphi|^2 (1 + |D\varphi|^2)^{\frac{p-2}{2}}\right) dx, \tag{$F2$}$$

$$0 \leq G(\xi) \leq L_2(1 + |\xi|^2)^{\frac{p}{2}}, \tag{$G1$}$$

$$\int_\Omega G(\xi + D\varphi)\,dx \geq \int_\Omega \Big(G(\xi) + \ell_2 |D\varphi|^2 (1+|D\varphi|^2)^{\frac{p-2}{2}}\Big)\,dx, \tag{G2}$$

for every $\xi \in \mathbb{R}^{n\times N}$ and $\varphi \in C^1_0(\Omega;\mathbb{R}^N)$.

Under these assumptions, the authors proved the existence of local minimizers for the functional (1.3), for any $p > 1$. Furthermore, they proved a partial $C^{1,\alpha}$ regularity result for minimal configurations in the quadratic case $p = 2$.

In this paper, we generalize the results given in [11] under two viewpoints. First we treat the more general case of p-growth with $p \geq 2$. Moreover, we deal with anisotropic surface energies.

In the rest of the paper we focus our attention on integral functionals defined as follows,

$$\mathcal{I}(v, A) := \int_\Omega \left(F(Dv) + \mathbb{1}_A G(Dv)\right)\,dx + \int_{\Omega\cap\partial^* A} \Phi(x, \nu_A(x))\,d\mathcal{H}^{n-1}(x)\,, \tag{1.4}$$

where $A \subset \Omega$ is a set of finite perimeter, $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $\mathbb{1}_A$ is the characteristic function of the set A. Here $\partial^* A$ denotes the reduced boundary of A in Ω and ν_A is the measure-theoretic outer unit normal to A, see Section 2.1.

We assume that Φ is an elliptic integrand on Ω (see Definition 2.6), i.e. $\Phi : \overline{\Omega}\times\mathbb{R}^n \to [0,\infty]$ is lower semicontinuous, $\Phi(x,\cdot)$ is convex and positively one-omogeneous, $\Phi(x, t\nu) = t\Phi(x,\nu)$ for every $t \geq 0$. Accordingly, we define the following anisotropic surface energy of a set A of finite perimeter in Ω:

$$\mathbf{\Phi}(A; G) := \int_{G\cap\partial^* A} \Phi(x, \nu_A(x))\,d\mathcal{H}^{n-1}(x), \tag{1.5}$$

for every Borel set $G \subset \Omega$. The assumption

$$\frac{1}{\Lambda} \leq \Phi(x,\nu) \leq \Lambda, \tag{1.6}$$

with $\Lambda > 1$, allows us to compare the surface energy introduced in (1.5) with the usual perimeter. Anisotropic surface energies arise in many physical areas such as the formation of crystals (see [7, 8]), liquid drops (see [16, 26]), capillary surfaces (see [18, 19]). F.J. Almgren was the first to study the regularity of surfaces that minimize anisotropic variational problems in his celebrated paper [3].

In the early stages the studies in this area had been done in the setting of varifolds and currents. These results can be applied to surfaces of arbitrary codimension, but with rather strong regularity assumptions on the integrands of the anisotropic energies, see [9, 39].

More recently, the regularity assumptions on the integrands Φ of the anisotropic energies have been weakened, see [20, 25], assuming that $\Phi(x,\cdot)$ is of class C^1 and $\Phi(\cdot,\xi)$ is Hölder continuous.

In the vectorial setting debated in this paper, where the bulk energy is of general type with p-growth, the regularity that we can expect for the gradient of the minimal deformation $u : \Omega \to \mathbb{R}^N$, $(N > 1)$, even in absence of a surface term, is a partial regularity result, that is outside a negligible set. As we observed above, the regularity of the free interface ∂E can be achieved by means of the regularity of u. On the other hand, knowing that the singular set S of the gradient ∇u has Lebesgue measure zero does not give informations on the singular set Σ of the free boundary that could also be totally contained in S.

We say that a pair (u, E) is a local minimizer of $\mathcal{I}$ in Ω, if for every open set $U \Subset \Omega$ and every pair (v, A), where $v - u \in W_0^{1,p}(U;\mathbb{R}^N)$ and A is a set of finite perimeter with $A\Delta E \Subset U$, we have

$$\int_U (F(\nabla u) + \mathbb{1}_E G(\nabla u))\,dx + \mathbf{\Phi}(E;U) \leq \int_U (F(\nabla v) + \mathbb{1}_A G(\nabla v))\,dx + \mathbf{\Phi}(A;U).$$

Existence and regularity of local minimizers of integral functionals of the type

$$\int_\Omega F(Du)\,dx,$$

with uniformly strict p-quasiconvex integrand F, and also in the non autonomous case, have been widely investigated (we refer to [1, 2, 12, 13, 14, 15, 31, 38] and for an exhaustive treatment to [30, 32]).

In order to prove the existence of local minimizers for functionals involving both bulk and surface energies of general type (1.4), we invoke a result stated in [11]. The only difference in our setting is the presence of the anisotropic term $\mathbf{\Phi}(A;U)$, and we give a semicontinuity result for the anisotropic energy (1.5), thus ensuring the existence got in Section 3. Therefore, we deduce the following theorem.

Theorem 1.1. *Let $p > 1$ and assume that $(F1)$, $(F2)$, $(G1)$, $(G2)$ hold. Then, if $v \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $A \subset \Omega$ is a set of finite perimeter in Ω, for every sequence $\{(v_k, A_k)\}_{k\in\mathbb{N}}$ such that v_k weakly converges to v in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $\mathbb{1}_{A_k}$ strongly converges to $\mathbb{1}_A$ in $L^1_{\mathrm{loc}}(\Omega)$, we have*

$$\mathcal{I}(v, A) \leq \liminf_{k\to\infty} \mathcal{I}(v_k, A_k).$$

In particular, $\mathcal{I}$ admits a minimal configuration $(u, \mathbb{1}_E) \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \times BV_{\mathrm{loc}}(\Omega;[0,1])$.

Afterwards, we get $C^{1,\beta}$ partial regularity for minimizers u guaranteed by Theorem 1.1, in the case of general interfacial energies given in (1.5) just assuming the comparability hypothesis (1.6). Moreover, if a closeness condition on F and G is assumed, i.e. the condition (H) is in order, then we can prove a sharp regularity for u, that is $u \in C^{1,\gamma}(\Omega_1)$ for every $\gamma \in (0, \frac{1}{p'})$ for a full measure set $\Omega_1 \subset \Omega$. It is worth pointing out that we do not need any regularity assumption on the integrand Φ to prove the regularity of u.

Theorem 1.2. *Let (u, E) be a local minimizer of $\mathcal{I}$. Let the bulk density energies satisfy $(F1)$, $(F2)$, $(G1)$, $(G2)$, and let the surface energy be of general type (1.5) with Φ satisfying (1.6). Then there exist an exponent $\beta \in (0, 1)$ and an open set $\Omega_0 \subset \Omega$ with full measure such that $u \in C^{1,\beta}(\Omega_0;\mathbb{R}^N)$. In addition, if we assume*

$$\frac{L_2}{\ell_1 + \ell_2} < 1, \tag{H}$$

then there exists an open set $\Omega_1 \subset \Omega$ with full measure such that $u \in C^{1,\gamma}(\Omega_1;\mathbb{R}^N)$ for every $\gamma \in (0, \frac{1}{p'})$.

The proof of the previous result is based on a comparison argument with solutions of a suitable linearized system. We establish decay estimates for the "hybrid" excess functions U_*

and U_{**} (see (5.2) and (5.48)). We look at the points in which the excess is small and we use, as usual for this kind of analysis, a blow-up argument reducing the problem to the study of convergence of the minimal configurations (u_h, E_h) of rescaled functionals in the unit ball. We need two Caccioppoli type inequalities for minimizers of perturbed rescaled functionals (see (5.17) and (5.58)) involving also the perimeter of the rescaled minimal set E_h.

2 Notation and Preliminary Results

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. We deal with vectorial functions $u : \Omega \to \mathbb{R}^N$, $N > 1$. The open ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ is defined as

$$B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}.$$

We denote by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$ and by c a generic constant that may vary in the same formula and between formulae. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. For $B_r(x_0) \subset \mathbb{R}^n$ and $u \in L^1(B_r(x_0); \mathbb{R}^N)$ we denote

$$(u)_{x_0,r} := \fint_{B_r(x_0)} u(x)\, dx.$$

We omit the dependence on the center when it is clear from the context.

$$\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta),$$

for the usual inner product of ξ and η, and accordingly $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$.
If $F \colon \mathbb{R}^{n\times N} \to \mathbb{R}$ is sufficiently differentiable, we write

$$DF(\xi)\eta := \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha}(\xi)\eta_i^\alpha \quad \text{and} \quad D^2F(\xi)\eta\eta := \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha \partial \xi_j^\beta}(\xi)\eta_i^\alpha \eta_j^\beta,$$

for ξ, $\eta \in \mathbb{R}^{n\times N}$.

It is well–known that for quasiconvex C^1 integrands the assumptions $(F1)$ and $(G1)$ yield the upper bounds

$$|D_\xi F(\xi)| \leq c_1 L_1 (1 + |\xi|^2)^{\frac{p-1}{2}} \quad \text{and} \quad |D_\xi G(\xi)| \leq c_2 L_2 (1 + |\xi|^2)^{\frac{p-1}{2}} \tag{2.1}$$

for all $\xi \in \mathbb{R}^{n\times N}$, with c_1 and c_2 constants depending only on p (see [32, Lemma 5.2] or [38]). Furthermore, if F and G are C^2, then $(F2)$ and $(G2)$ imply the following strong Legendre-Hadamard conditions

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial F}{\partial \xi_i^\alpha \partial \xi_j^\beta}(Q)\lambda_i\lambda_j\mu^\alpha\mu^\beta \geq c_3|\lambda|^2|\mu|^2 \quad \text{and} \quad \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \frac{\partial G}{\partial \xi_i^\alpha \partial \xi_j^\beta}(Q)\lambda_i\lambda_j\mu^\alpha\mu^\beta \geq c_4|\lambda|^2|\mu|^2,$$

for all $Q \in \mathbb{R}^{n\times N}$, $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$, where $c_3 = c_3(p, \ell_1)$ and $c_4 = c_4(p, \ell_2)$ are positive constants (see [32, Proposition 5.2]). We will need the following regularity result (see [30, 32]).

Proposition 2.1. *Let $v \in W^{1,2}(\Omega;\mathbb{R}^N)$ be such that*

$$\int_\Omega Q^{ij}_{\alpha\beta} D_i v^\alpha D_j \varphi^\beta \, dx = 0,$$

for every $\varphi \in C^\infty_c(\Omega;\mathbb{R}^N)$, where $Q = \{Q^{ij}_{\alpha\beta}\}$ is a constant matrix satisfying $|Q^{ij}_{\alpha\beta}| \le L$ and the strong Legendre-Hadamard condition

$$Q^{ij}_{\alpha\beta}\lambda_i\lambda_j\mu^\alpha\mu^\beta \ge \ell|\lambda|^2|\mu|^2,$$

for all $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$ and for some positive constants $\ell, L > 0$. Then $v \in C^\infty$ and, for any $B_R(x_0) \subset \Omega$, the following estimate holds

$$\fint_{B_{\frac{R}{2}}(x_0)} |Dv - (Dv)_{x_0,\frac{R}{2}}|^2 \, dx \le cR^2 \fint_{B_R(x_0)} |Dv - (Dv)_{x_0,R}|^2 \, dx,$$

where $c = c(n, N, \ell, L) > 0$.

The next iteration lemma has important applications in the regularity theory (for its proof we refer to [32, Lemma 6.1]).

Lemma 2.2. *Let $0 < \rho < R$ and let $\psi \colon [\rho, R] \to \mathbb{R}$ be a bounded nonnegative function. Assume that for all $\rho \le s < t \le R$ we have*

$$\psi(s) \le \vartheta\psi(t) + A + \frac{B}{(s-t)^\alpha} + \frac{C}{(s-t)^\beta}$$

where $\vartheta \in [0, 1)$, $\alpha > \beta > 0$ and $A, B, C \ge 0$ are constants. Then there exists a constant $c = c(\vartheta, \alpha) > 0$ such that

$$\psi(\rho) \le c\left(A + \frac{B}{(R-\rho)^\alpha} + \frac{C}{(R-\rho)^\beta}\right).$$

Given a C^1 function $f : \mathbb{R}^{n\times N} \to \mathbb{R}$, $Q \in \mathbb{R}^{n\times N}$ and $\lambda > 0$, we set

$$f_{Q,\lambda}(\xi) := \frac{f(Q + \lambda\xi) - f(Q) - Df(Q)\lambda\xi}{\lambda^2}, \quad \forall \xi \in \mathbb{R}^{n\times N}.$$

We state the following lemma about the growth of $f_{Q,\lambda}$ and $Df_{Q,\lambda}$, whose proof can be found in [2, Lemma II.3].

Lemma 2.3. *Let $p \ge 2$, and let f be a $C^2(\mathbb{R}^{n\times N})$ function such that*

$$|f(\xi)| \le C\big(1 + |\xi|^p\big) \quad \text{and} \quad |D_\xi f(\xi)| \le C\big(1 + |\xi|^{p-1}\big),$$

for any $\xi \in \mathbb{R}^{n\times N}$. Then for every $M > 0$ there exists a constant $c = c(M) > 0$ such that, for every $Q \in \mathbb{R}^{n\times N}$, $|Q| \le M$ and $\lambda > 0$, it holds that

$$|f_{Q,\lambda}(\xi)| \le c(|\xi|^2 + \lambda^{p-2}|\xi|^p) \qquad \text{and} \qquad |Df_{Q,\lambda}(\xi)| \le c(|\xi| + \lambda^{p-2}|\xi|^{p-1}), \tag{2.2}$$

for all $\xi \in \mathbb{R}^{n\times N}$.

2.1 Sets of finite perimeter

If $E \subset \mathbb{R}^n$ and $t \in [0,1]$, the set of points of E of density t is defined as

$$E^{(t)} = \left\{ x \in \mathbb{R}^n : \ |E \cap B_r(x)| = t|B_r(x)| + o(r^n) \text{ as } r \to 0^+ \right\}.$$

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$ and an open set $U \subset \mathbb{R}^n$, we say that E is of locally finite perimeter in U if there exists a $\mathbb{R}^n$-valued Radon measure μ_E (called the Gauss-Green measure of E) on U such that

$$\int_E \nabla \phi \, dx = \int_U \phi \, d\mu_E, \quad \forall \phi \in C_c^1(U).$$

Moreover, we denote the perimeter of E relative to $G \subset U$ by $P(E,G) = |\mu_E|(G)$.
The support of μ_E can be characterized by

$$\text{spt}\mu_E = \left\{ x \in U : 0 < |E \cap B_r(x)| < \omega_n r^n, \, \forall r > 0 \right\}, \tag{2.3}$$

(see [37, Proposition 12.19]). It holds that $\text{spt}\mu_E \subset U \cap \partial E$. The *essential boundary* of E is defined as $\partial^e E := \mathbb{R}^n \setminus (E^0 \cup E^1)$. If E is of finite perimeter in an open set U, then the *reduced boundary* $\partial^* E \subset U$ of E is the set of those $x \in U$ such that

$$\nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} \tag{2.4}$$

exists and belongs to $\mathbb{S}^{n-1}$. It is well known that

$$\partial^* E \subset U \cap \partial^e E \subset \text{spt}\mu_E \subset U \cap \partial E, \qquad U \cap \overline{\partial^* E} = \text{spt}\mu_E.$$

Federer's criterion, see for instance [37, Theorem 16.2], ensures that

$$\mathcal{H}^{n-1}((U \cap \partial^e E) \setminus \partial^* E) = 0.$$

Remark 2.4 (Minimal topological boundary)**.** *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in U and $F \subset \mathbb{R}^n$ is such that $|(E \Delta F) \cap U| = 0$, then F is a set of locally finite perimeter in U with $\mu_E = \mu_F$. In the rest of the paper, the topological boundary ∂E must be understood by considering the correct representative of E. We will choose $E^{(1)}$ as representative of E. With such a choice it can be easily verified that*

$$U \cap \partial E = \left\{ x \in U : 0 < |E \cap B_r(x)| < \omega_n r^n, \forall r > 0 \right\}.$$

Therefore, by (2.3)*,*

$$\overline{\partial^* E} = \text{spt}\mu_E = \partial E \cap U.$$

Finally by De Giorgi's rectifiability theorem (see [37, Theorem 15.5]) we get

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{2.5}$$

on Borel sets compactly contained in U where, given a Radon measure μ and a Borel set G, by $\mu \llcorner G$ we refer to the measure given by $\mu \llcorner G(F) = \mu(G \cap F)$.

It is well known that if E and F are of locally finite perimeter in U then $E \cap F$, $E \cup F$ and $E \setminus F$ are sets of locally finite perimeter in U. In this paper we use competitors obtained using set operations to test minimality inequalities. In fact, we just need properties involving the union of sets. A convenient way to handle these inequalities is to use the properties of Gauss–Green measures. The following result can be found in [37, Theorem 16.3].

Proposition 2.5 (Gauss-Green measure and set operations)**.** *If E and F are sets of finite perimeter it results that $\nu_E(x) = \pm\nu_F(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap \partial^* F$. Setting*

$$\{\nu_E = \nu_F\} = \{x \in \partial^* E \cap \partial^* F \,|\, \nu_E(x) = \nu_F(x)\},$$

then

$$\mu_{E\cup F} = \mu_E \llcorner F^{(0)} + \mu_F \llcorner E^{(0)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}\,. \tag{2.6}$$

2.2 Anisotropic surface energy

Definition 2.6 (Elliptic integrands)**.** *Given an open set Ω in $\mathbb{R}^n$, $\Phi : \overline{\Omega} \times \mathbb{R}^n \to [0, \infty]$ is an elliptic integrand on Ω if it is lower semicontinuous, with $\Phi(x, \cdot)$ convex and positively one-homogeneous for any $x \in \overline{\Omega}$, i.e. $\Phi(x, t\nu) = t\Phi(x, \nu)$ for every $t \geq 0$. Accordingly, the anisotropic surface energy of a set E of finite perimeter in Ω is defined as*

$$\mathbf{\Phi}(E; G) := \int_{G\cap\partial^* E} \Phi(x, \nu_E(x))\, d\mathcal{H}^{n-1}(x), \tag{2.7}$$

for every Borel set $G \subset \Omega$.

Remark 2.7 (Comparability to perimeter)**.** *In order to prove the regularity of minimizers of anisotropic surface energies, it is well known that (see the seminal paper [3]) a C^k-dependence of the integrand Φ on the variable ν, and a continuity condition with respect to the variable x, must be assumed. In fact, one more condition is essential, that is a non-degeneracy type condition for the integrand Φ. More precisely, we have to assume that there exists a constant $K > 1$ such that*

$$\frac{1}{K} \leq \Phi(x, \nu) \leq K, \tag{2.8}$$

for any $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$. We do not need any further hypotheses on the elliptic integrands. We observe that, if the elliptic integrand Φ satisfies condition (2.8)*, then the anisotropic surface energy* (2.7) *satisfies the following comparability condition*

$$\frac{1}{\Lambda}\mathcal{H}^{n-1}(G \cap \partial^* E) \leq \mathbf{\Phi}(E; G) \leq \Lambda \mathcal{H}^{n-1}(G \cap \partial^* E),$$

for any set E of finite perimeter in Ω and any Borel set $G \subset \Omega$.

Proposition 2.8. *Let $U \subset \mathbb{R}^n$ be an open set and let $E, F \subset U$ be two sets of finite perimeter in U. It holds that*

$$\mathbf{\Phi}(E \cup F; U) = \mathbf{\Phi}(E; F^{(0)}) + \mathbf{\Phi}(F; E^{(0)}) + \mathbf{\Phi}(E; \{\nu_E = \nu_F\}).$$

Proof. Let us observe that, since $\partial^* E \cap F^{(0)} \subset E^{(\frac{1}{2})} \cap F^{(0)}$ and $\partial^* F \cap E^{(0)} \subset F^{(\frac{1}{2})} \cap E^{(0)}$, we deduce that $(\partial^* E \cap F^{(0)}) \cap (\partial^* F \cap E^{(0)}) = \emptyset$. Similarly we have $(\partial^* E \cap F^{(0)}) \cap \{\nu_E = \nu_F\} = \emptyset$ and $(\partial^* F \cap E^{(0)}) \cap \{\nu_E = \nu_F\}) = \emptyset$. Thus, by [37, Theorem 16.3], it holds that

$$\begin{aligned}\mathbf{\Phi}(E \cup F; U) &= \int_{U\cap\partial^*(E\cup F)} \Phi(x, \nu_{E\cup F}(x))\, d\mathcal{H}^{n-1}(x)\\ &= \int_{F^{(0)}\cap\partial^* E} \Phi(x, \nu_{E\cup F}(x))\, d\mathcal{H}^{n-1}(x) + \int_{E^{(0)}\cap\partial^* F} \Phi(x, \nu_{E\cup F}(x))\, d\mathcal{H}^{n-1}(x)\end{aligned}$$

$$+ \int_{\{\nu_E=\nu_F\}} \Phi(x,\nu_{E\cup F}(x))\, d\mathcal{H}^{n-1}(x).$$

By using (2.6), we deduce $\nu_{E\cup F} = \nu_E$ $\mathcal{H}^{n-1}$–a.e. on $F^{(0)} \cap \partial^* E$ and $\nu_{E\cup F} = \nu_F$ $\mathcal{H}^{n-1}$–a.e. on $E^{(0)} \cap \partial^* F$, thus we reach the thesis. □

3 Lower Semicontinuity

In this section we prove Theorem 1.1. We base ourselves on the proof given in [11], focusing our attention on the only difference we have, the presence of the anisotropic surface energy. We get a semicontinuity result for the anisotropic perimeter, essential to handle this novelty.

Given a one-homogeneous Borel function $\Phi : \overline{\Omega} \times \mathbb{R}^n \to [0,\infty]$ as in Definition 2.6, for any $\mathbb{R}^n$-valued Radon measure μ on $\mathbb{R}^n$ and any Borel set $F \subset \mathbb{R}^n$, we define the Φ-anisotropic total variation of μ on F as

$$\mathbf{\Phi}(\mu, F) = \int_F \Phi\Big(x, \frac{\mu}{|\mu|}(x)\Big)\, d|\mu|(x),$$

where $\frac{\mu}{|\mu|}$ denotes the Radon-Nikodym derivative of μ with respect to its total variation. We also refer to the following theorem (see [5, Theorem 2.38]).

Theorem 3.1 (Reshetnyak's lower semicontinuity Theorem)**.** *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_h be two finite $\mathbb{R}^n$-valued Radon measures on Ω. If $\mu_h \overset{*}{\rightharpoonup} \mu$ in Ω then*

$$\int_\Omega \Phi\Big(x, \frac{\mu}{|\mu|}(x)\Big)\, |\mu|(x) \le \liminf_{h\to\infty} \int_\Omega \Phi\Big(x, \frac{\mu_h}{|\mu_h|}(x)\Big)\, d|\mu_h|(x),$$

for every lower semicontinuous function $\Phi : \Omega \times \mathbb{R}^n \to [0,\infty]$, positively 1-homogeneous and convex in the second variable.

We are now able to prove the following result.

Proposition 3.2. *Let Φ be an elliptic integrand as in Definition 2.6 and let $\mathbf{\Phi}$ be defined as in (2.7). Then*

$$\mathbf{\Phi}(E;U) \le \liminf_{h\to\infty} \mathbf{\Phi}(E_h;U),$$

whenever $U \subset \mathbb{R}^n$ is open, $\{E_h\}_{h\in\mathbb{N}}$ and E are sets of locally finite perimeter such that $\mathbb{1}_{E_h} \to \mathbb{1}_E$ in $L^1_{loc}(U)$ and $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$.

Proof. We recall that if E is a set of locally finite perimeter, then, by (2.5), $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$ and, by the definition of reduced boundary (2.4), $\mu_E/|\mu_E| = \nu_E$ on $\partial^* E$. Therefore, the Φ-anisotropic total variation of μ_E is equal to the Φ-surface energy of E, that is

$$\mathbf{\Phi}(\mu_E; F) = \mathbf{\Phi}(E; F).$$

Thus, the claim follows by virtue of Reshetnyak's lower semicontinuity Theorem. □

Proof of Theorem 1.1. We omit the proof as it can be obtained following verbatim the arguments used in [11, Section 3]. The only difference with the proof given in [11] concerns the presence of the anisotropic perimeter. In this regard we can use the lower semicontinuity result given above. We mention that the convergence $\mathbb{1}_{E_h} \to \mathbb{1}_E$ in $L^1_{loc}(U)$ assumed in Theorem 1.1 and the condition $\limsup_{k\to\infty} P(E_h, K) < \infty$ for every K compact set in Ω ensure that $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$. $\square$

4 Higher integrability result

This section is devoted to the proof of a higher integrability result for the gradient of the function u of the minimal configuration (u, E).

Theorem 4.1. *Assume that $(F1)$, $(F2)$, $(G1)$, $(G2)$ hold and let (u, E) be a local minimizer of $\mathcal{I}$. Then there exists $\delta = \delta(n, p, \ell_1, L_1, L_2) > 0$ such that for every $B_{2r}(x_0) \Subset \Omega$ it holds*

$$\left(\fint_{B_r(x_0)} |Du|^{p(1+\delta)}\,dx \right)^{\frac{1}{1+\delta}} \leq C \left[\fint_{B_{2r}(x_0)} |Du|^p\,dx + 1 \right],$$

where $C = C(n, p, \ell_1, L_1, L_2)$ is a positive constant.

Proof. We consider $0 < r < s < t < 2r$ and let $\eta \in C^\infty_0(B_t)$ be a cut-off function between B_s and B_t, i.e. $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B_s and $|\nabla\eta| \leq \frac{c}{t-s}$.
Setting

$$\psi_1 := \eta(u - (u)_{x_0,2r}) \quad \text{and} \quad \psi_2 := (1-\eta)(u - (u)_{x_0,2r}),$$

by the uniformly strict quasiconvexity of F in $(F2)$, we have

$$\ell_1 \int_{B_t} |D\psi_1(x)|^p\,dx \leq \int_{B_t} F(D\psi_1)\,dx = \int_{B_t} F(Du - D\psi_2)\,dx. \tag{4.1}$$

We write

$$\begin{aligned}
&\int_{B_t} F(Du - D\psi_2)\,dx = \int_{B_t} F(Du)\,dx + \int_{B_t} F(Du - D\psi_2)\,dx - \int_{B_t} F(Du)\,dx \\
&= \int_{B_t} F(Du)\,dx - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx \\
&\leq \int_{B_t} \big[F(Du) + \mathbb{1}_E G(Du)\big]\,dx - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx \\
&\leq \int_{B_t} \big[F(Du - D\psi_1) + \mathbb{1}_E G(Du - D\psi_1)\big]\,dx - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx,
\end{aligned} \tag{4.2}$$

where we used the fact that $G(\xi) \geq 0$ and the minimality of (u, E) with respect to $(u - \psi_1, E)$. Combining (4.2) in (4.1) and using the upper bound on DF given by (2.1), we obtain

$$\ell_1 \int_{B_s} |Du|^p\,dx = \ell_1 \int_{B_s} |D\psi_1|^p\,dx \leq \int_{B_t} F(D\psi_2)\,dx + \int_{B_t} \mathbb{1}_E G\big(D\psi_2\big)\,dx$$

$$
\begin{aligned}
&+ c(p,L_1)\int_{B_t\setminus B_s}(1+|Du|^2+|D\psi_2|^2)^{\frac{p-1}{2}}|D\psi_2|\,dx\\
&\le c(p,L_1,L_2)\left[\int_{B_t\setminus B_s}|D\psi_2|^p\,dy+\int_{B_t\setminus B_s}|Du|^p\,dx+|B_t|\right]\\
&\le c(p,L_1,L_2)\left[\int_{B_t\setminus B_s}|Du|^p\,dx\ +\ c\int_{B_t\setminus B_s}\frac{|(u-(u)_{x_0,2r})|^p}{(t-s)^p}\,dx+|B_t|\right],
\end{aligned}
$$

where we used assumptions $(F1)$ and $(G1)$, Young's inequality and the properties of η. Adding $c(p,L_1,L_2)\int_{B_s}|Du|^p\,dx$ to both sides of the previous estimate we get

$$
\begin{aligned}
(\ell_1+c(p,L_1,L_2))\int_{B_s}|Du|^p\,dx &\le c(p,L_1,L_2)\left[\int_{B_t}|Du|^p\,dx+\int_{B_t\setminus B_s}\frac{|u-(u)_{x_0,2r}|^p}{(t-s)^p}\,dx+|B_t|\right]\\
&\le c(p,L_1,L_2)\left[\int_{B_t}|Du|^p\,dx+c\int_{B_{2r}}\left(1+\frac{|u-(u)_{x_0,2r}|^p}{(t-s)^p}\right)\,dx\right],
\end{aligned}
$$

and, by Lemma 2.2, we deduce that

$$
\fint_{B_r}|Du|^p\,dx\le c(p,\ell_1,L_1,L_2)\fint_{B_{2r}}\left[1+\frac{|u-u_{2r}|}{r}^p\right]\,dx.
$$

The Sobolev-Poincaré inequality (see [32, p.102]) implies that

$$
\fint_{B_r}|Du|^p\,dx\le c(n,p,\ell_1,L_1,L_2)\left[\left(\fint_{B_{2r}}|Du|^{\frac{np}{n+p}}\,dx\right)^{\frac{n+p}{n}}+1\right],
$$

and the conclusion follows by virtue of Giaquinta-Modica Theorem (see [32, p.203]). □

5 Decay Estimates

In this section we collect some energy estimates for minimizers of the functional (1.4) that will be crucial in the proof of Theorem 1.2. In order to get them we will employ a well-known blow-up technique involving a quantity called excess, which includes all the energy terms of the functional. We have to use different type of excess depending on whether the assumption (H) is in force or not. We consider the bulk excess function defined as

$$
U(x_0,r):=\fint_{B_r(x_0)}\left[|Du(x)-(Du)_{x_0,r}|^2+|Du(x)-(Du)_{x_0,r}|^p\right]dx, \tag{5.1}
$$

for $B_r(x_0)\subset\Omega$. In the case that the assumption (H) is in force we will use the following "hybrid" excess

$$
U_*(x_0,r):=U(x_0,r)+\frac{P(E,B_r(x_0))}{r^{n-1}}+r. \tag{5.2}
$$

Proposition 5.1. *Let* (u, E) *be a local minimizer of the functional* $\mathcal{I}$ *introduced in* (1.4)*, and the assumptions* $(F1)$*,* $(F2)$*,* $(G1)$*,* $(G2)$ *and* (H) *hold. For every* $M > 0$ *and every* $0 < \tau < \frac{1}{4}$*, there exist* $\varepsilon_0 = \varepsilon_0(\tau, M) > 0$ *and* $c_* = c_*(n, p, \ell_1, \ell_2, L_1, L_2, \Lambda, M) > 0$ *such that, whenever* $B_r(x_0) \Subset \Omega$ *verifies*

$$|(Du)_{x_0,r}| \leq M \qquad \text{and} \qquad U_*(x_0, r) \leq \varepsilon_0 \,,$$

then

$$U_*(x_0, \tau r) \leq c_* \, \tau \, U_*(x_0, r) \,. \tag{5.3}$$

Proof. In order to prove (5.3), we argue by contradiction. Let $M > 0$ and $\tau \in (0, 1/4)$ be such that for every $h \in \mathbb{N}$, $C_* > 0$, there exists a ball $B_{r_h}(x_h) \Subset \Omega$ such that

$$|(Du)_{x_h,r_h}| \leq M, \qquad U_*(x_h, r_h) \to 0 \tag{5.4}$$

and

$$U_*(x_h, \tau r_h) \geq C_* \tau U_*(x_h, r_h). \tag{5.5}$$

The constant C_* will be determined later. Remark that we can confine ourselves to the case in which $E \cap B_{r_h}(x_h) \neq \emptyset$, since the case in which $B_{r_h}(x_h) \subset \Omega \setminus E$ is easier because $U = U_*$.

Step 1. *Blow-up.*

Set $\lambda_h^2 := U_*(x_h, r_h)$, $A_h := (Du)_{x_h,r_h}$, $a_h := (u)_{x_h,r_h}$, and define

$$v_h(y) := \frac{u(x_h + r_h y) - a_h - r_h A_h y}{\lambda_h r_h}, \quad \forall y \in B_1.$$

One can easily check that $(Dv_h)_{0,1} = 0$ and $(v_h)_{0,1} = 0$.
Set

$$E_h := \frac{E - x_h}{r_h}, \qquad E_h^* := \frac{E - x_h}{r_h} \cap B_1.$$

Note that

$$\begin{aligned} \lambda_h^2 = U_*(x_h, r_h) &= \fint_{B_1} \left[|Du(x_h + r_h y) - A_h|^2 + |Du(x_h + r_h y) - A_h|^p \right] dy + \frac{P(E, B_{r_h}(x_h))}{r_h^{n-1}} + r_h \\ &= \fint_{B_1} |\left[\lambda_h Dv_h|^2 + |\lambda_h Dv_h|^p \right] dy + P(E_h, B_1) + r_h. \end{aligned} \tag{5.6}$$

It follows that $r_h \to 0$, $P(E_h, B_1) \to 0$, and

$$\frac{r_h}{\lambda_h^2} \leq 1, \qquad \fint_{B_1} \left[|Dv_h|^2 + \lambda_h^{p-2} |Dv_h|^p \right] dy \leq 1, \qquad \frac{P(E_h, B_1)}{\lambda_h^2} \leq 1. \tag{5.7}$$

Therefore, by (5.4) and (5.7), there exist a (not relabeled) subsequence of $\{v_h\}_{h \in \mathbb{N}}$, $A \in \mathbb{R}^{n \times N}$ and $v \in W^{1,2}(B_1; \mathbb{R}^N)$, such that

$$\begin{aligned} &v_h \rightharpoonup v \quad \text{weakly in } W^{1,2}(B_1; \mathbb{R}^N), \quad v_h \to v \quad \text{strongly in } L^2(B_1; \mathbb{R}^N), \\ &A_h \to A, \quad \lambda_h Dv_h \to 0 \quad \text{in } L^p(B_1; \mathbb{R}^{n \times N}) \quad \text{and pointwise a.e. in } B_1, \end{aligned} \tag{5.8}$$

where we used the fact that $(v_h)_{0,1}=0$. Moreover, by (5.7) and (5.4), we also deduce that

$$\lim_{h\to\infty}\frac{(P(E_h,B_1))^{\frac{n}{n-1}}}{\lambda_h^2}=\lim_{h\to\infty}(P(E_h,B_1))^{\frac{1}{n-1}}\limsup_{h\to\infty}\frac{P(E_h,B_1)}{\lambda_h^2}=0. \tag{5.9}$$

Therefore, by the relative isoperimetric inequality,

$$\lim_{h\to\infty}\min\left\{\frac{|E_h^*|}{\lambda_h^2},\frac{|B_1\setminus E_h|}{\lambda_h^2}\right\}\le c(n)\lim_{h\to\infty}\frac{(P(E_h,B_1))^{\frac{n}{n-1}}}{\lambda_h^2}=0. \tag{5.10}$$

In the sequel the proof will proceed differently depending on whether

$$\min\{|E_h^*|,|B_1\setminus E_h|\}=|E_h^*| \qquad \text{or} \qquad \min\{|E_h^*|,|B_1\setminus E_h|\}=|B_1\setminus E_h|.$$

The first case is easier to handle. To understand the reason let us introduce the expansion of F and G around A_h as follows:

$$F_h(\xi):=\frac{F(A_h+\lambda_h\xi)-F(A_h)-DF(A_h)\lambda_h\xi}{\lambda_h^2}, \tag{5.11}$$
$$G_h(\xi):=\frac{G(A_h+\lambda_h\xi)-G(A_h)-DG(A_h)\lambda_h\xi}{\lambda_h^2},$$

for any $\xi\in\mathbb{R}^{n\times N}$. In the first case the suitable rescaled functional to consider in the blow-up procedure is the following

$$\mathcal{I}_h(w):=\int_{B_1}\big[F_h(Dw)dy+\mathbb{1}_{E_h^*}G_h(Dw)\big]\,dy\,. \tag{5.12}$$

We claim that v_h satisfies the minimality inequality

$$\mathcal{I}_h(v_h)\le\mathcal{I}_h(v_h+\psi)+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h)D\psi(y)\,dy, \tag{5.13}$$

for any $\psi\in W_0^{1,p}(B_1;\mathbb{R}^N)$. Indeed, using the change of variable $x=x_h+r_hy$, the minimality of (u,E) with respect to $(u+\varphi,E)$, for $\varphi\in W_0^{1,p}(B_{r_h}(x_h);\mathbb{R}^N)$, setting $\psi(y):=\frac{\varphi(x_h+r_hy)}{r_h}$, it holds that

$$\begin{aligned}&\int_{B_1}\big[(F_h(Dv_h(y))+\mathbb{1}_{E_h^*}G_h(Dv_h(y))\big]\,dy\\ &\le\int_{B_1}\big[F_h(Dv_h(y)+D\psi(y))+\mathbb{1}_{E_h^*}G_h(Dv_h(y)+D\psi(y))\big]\,dy+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h)D\psi(y)\,dy,\end{aligned}$$

and (5.13) follows by the definition of $\mathcal{I}_h$ in (5.12).

In the second case the suitable rescaled functional to consider in the blow-up procedure is

$$\mathcal{H}_h(w):=\int_{B_1}\big[F_h(Dw)+G_h(Dw)\big]\,dy.$$

Then we claim that

$$\mathcal{H}_h(v_h)\le\mathcal{H}_h(v_h+\psi)+\frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap\operatorname{supp}\psi}(\mu^2+|A_h+\lambda_hDv_h|^2)^{\frac{p}{2}}\,dy, \tag{5.14}$$

for all $\psi \in W_0^{1,p}(B_1;\mathbb{R}^N)$. Indeed, the minimality of (u,E) with respect to $(u+\varphi,E)$, for $\varphi \in W_0^{1,p}(B_{r_h}(x_h);\mathbb{R}^N)$, implies that

$$\begin{aligned}
\int_{B_{r_h}(x_h)}(F+G)(Du)\,dx &= \int_{B_{r_h}(x_h)}\big[F(Du)+\mathbb{1}_E G(Du)\big]dx+\int_{B_{r_h}(x_h)\setminus E}G(Du)dx\\
&\le \int_{B_{r_h}(x_h)}\big[F(Du+D\varphi)+\mathbb{1}_E G(Du+D\varphi)\big]dx+\int_{B_{r_h}(x_h)\setminus E}G(Du)dx\\
&= \int_{B_{r_h}(x_h)}(F+G)(Du+D\varphi)dx+\int_{B_{r_h}(x_h)\setminus E}\big[G(Du)-G(Du+D\varphi)\big]dx\\
&\le \int_{B_{r_h}(x_h)}(F+G)(Du+D\varphi)dx+\int_{(B_{r_h}(x_h)\setminus E)\cap \mathrm{supp}\varphi}G(Du)dx,
\end{aligned} \tag{5.15}$$

where we used that the last integral vanishes outside the support of φ and that $G\ge 0$. Using the change of variable $x=x_h+r_h y$ in the previous formula, we get

$$\begin{aligned}
\int_{B_1}(F+G)(Du(x_h+r_hy))dy \le& \int_{B_1}(F+G)(Du(x_h+r_hy)+D\varphi(x_h+r_hy))\,dy\\
&+\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi}G(Du(x_h+r_hy))dy,
\end{aligned}$$

or, equivalently, using the definitions of v_h,

$$\begin{aligned}
\int_{B_1}(F+G)(A_h+\lambda_h Dv_h)dy \le& \int_{B_1}(F+G)(A_h+\lambda_h(Dv_h+D\psi))\,dy\\
&+\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi}G(A_h+\lambda_h Dv_h)dy
\end{aligned}$$

where $\psi(y):=\frac{\varphi(x_h+r_hy)}{\lambda_h r_h}$, for $y\in B_1$. Therefore, setting

$$H_h := F_h + G_h,$$

by the definition of F_h and G_h in (5.11) and using the assumption $(G1)$, we have that

$$\begin{aligned}
\int_{B_1}H_h(Dv_h)dy &\le \int_{B_1}H_h(Dv_h+D\psi)dy+\frac{1}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi}G(A_h+\lambda_hDv_h)\,dy\\
&\le \int_{B_1}H_h(Dv_h+D\psi)\,dy+\frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi}\big(1+|A_h+\lambda_hDv_h|^2\big)^{\frac{p}{2}}\,dy,
\end{aligned} \tag{5.16}$$

i.e. (5.14).

Step 2. *A Caccioppoli type inequality.*

We claim that there exists a constant $c=c(n,p,\ell_1,\ell_2,L_2,M)>0$ such that for every $0<\rho<1$ there exists $h_0=h_0(n,\rho)\in\mathbb{N}$ such that for all $h>h_0$ we have

$$\int_{B_{\frac{\rho}{2}}}\big(1+\lambda_h^{p-2}|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^{p-2}\big)|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^2\,dy \tag{5.17}$$

$$\leq c\left[\int_{B_\rho} \left(\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y|^2}{\rho^2} + \lambda_h^{p-2} \frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y|^p}{\rho^p} \right) dy + \frac{P(E_h, B_1)^{\frac{n}{n-1}}}{\lambda_h^2} \right]$$

We divide the proof into two substeps.

Substep 2.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

Consider $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_s and $|\nabla \eta| \leq \frac{c}{t-s}$. Set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$, and set

$$w_h(y) := v_h(y) - b_h - B_h y,$$

for any $y \in B_1$. Proceeding similarly as in (5.6) let us rescale F and G around $A_h + \lambda_h B_h$,

$$\widetilde{F}_h(\xi) := \frac{F(A_h + \lambda_h B_h + \lambda_h \xi) - F(A_h + \lambda_h B_h) - DF(A_h + \lambda_h B_h)\lambda_h \xi}{\lambda_h^2}, \tag{5.18}$$
$$\widetilde{G}_h(\xi) := \frac{G(A_h + \lambda_h B_h + \lambda_h \xi) - G(A_h + \lambda_h B_h) - DG(A_h + \lambda_h B_h)\lambda_h \xi}{\lambda_h^2},$$

for any $\xi \in \mathbb{R}^{n\times N}$. It is easy to check that Lemma 2.3 applies to each $\widetilde{F}_h$ and $\widetilde{G}_h$, for some constants that depend on M (see (5.4)) and could also depend on ρ through $|\lambda_h B_h|$. However, given ρ we may choose $h_0 = h_0(n, \rho)$ large enough to have $|\lambda_h B_h| < \frac{\lambda_h \omega_n}{\rho^{\frac{n}{2}}} < 1$, for any $h \geq h_0$. Indeed, by (5.7) we have

$$|B_h| = \left| \fint_{B_{\frac{\rho}{2}}} Dv_h \, dy \right| \leq \left(\int_{B_{\frac{\rho}{2}}} |Dv_h|^2 \, dy \right)^{\frac{1}{2}} \frac{1}{|B_{\frac{\rho}{2}}|^{\frac{1}{2}}} \leq \frac{c(n)}{\rho^{\frac{n}{2}}},$$

and so the constant in (2.2) can be taken independently of ρ.
Set

$$\psi_{1,h} := \eta w_h \qquad \text{and} \qquad \psi_{2,h} := (1-\eta) w_h.$$

By the uniformly strict quasiconvexity of $\widetilde{F}_h$ we have

$$\begin{aligned}
&\ell_1 \int_{B_s} \left(1 + |\lambda_h D w_h|^2\right)^{\frac{p-2}{2}} |Dw_h|^2 \, dy \\
&\leq \ell_1 \int_{B_t} \left(1 + |\lambda_h D\psi_{1,h}|^2\right)^{\frac{p-2}{2}} |D\psi_{1,h}|^2 \, dy \leq \int_{B_t} \widetilde{F}_h(D\psi_{1,h}) \, dy = \int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h}) \, dy \\
&= \int_{B_t} \widetilde{F}_h(Dw_h) \, dy + \int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h}) \, dy - \int_{B_t} \widetilde{F}_h(Dw_h) \, dy \\
&= \int_{B_t} \widetilde{F}_h(Dw_h) \, dy - \int_{B_t} \int_0^1 D\widetilde{F}_h(Dw_h - \theta D\psi_{2,h}) D\psi_{2,h} \, d\theta \, dy.
\end{aligned} \tag{5.19}$$

We estimate separately the two addends in the right-hand side of the previous chain of inequalities. We deal with the first addend by means of a rescaling of the minimality condition of (u, E). Using the change of variable $x = x_h + r_h y$, the fact that $G \geq 0$ and the minimality of (u, E) with respect to $(u + \varphi, E)$ for $\varphi \in W_0^{1,p}(B_{r_h}(x_h); \mathbb{R}^N)$, we have

$$\int_{B_1} F(Du(x_h + r_h y)) dy \leq \int_{B_1} \left[F(Du(x_h + r_h y)) + \mathbb{1}_{E_h^*} G(Du(x_h + r_h y)) \right] dy$$

$$\leq \int_{B_1} \big[F(Du(x_h + r_h y) + D\varphi(x_h + r_h y)) + \mathbb{1}_{E_h^*} G(Du(x_h + r_h y) + D\varphi(x_h + r_h y))\big]\, dy,$$

i.e., by the definitions of v_h and w_h,

$$\begin{aligned}
&\int_{B_1} F(A_h + \lambda_h B_h + \lambda_h Dw_h) dy \\
&\leq \int_{B_1} \big[F(A_h + \lambda_h B_h + \lambda_h(Dw_h + D\psi)) + \mathbb{1}_{E_h^*} G(A_h + \lambda_h B_h + \lambda_h(Dw_h + D\psi))\big]\, dy,
\end{aligned}$$

for $\psi := \frac{\varphi(x_h + r_h y)}{\lambda_h r_h} \in W_0^{1,p}(B_1; \mathbb{R}^N)$. Therefore, recalling the definitions of $\widetilde{F}_h$ and $\widetilde{G}_h$ in (5.18), we have that

$$\begin{aligned}
\int_{B_1} \widetilde{F}_h(Dw_h) dy &\leq \int_{B_1} \big[\widetilde{F}_h(Dw_h + D\psi) + \mathbb{1}_{E_h^*} \widetilde{G}_h(Dw_h + D\psi)\big]\, dy \\
&+ \frac{1}{\lambda_h^2} \int_{B_1} \mathbb{1}_{E_h^*} \big[G(A_h + \lambda_h B_h) + DG(A_h + \lambda_h B_h)\lambda_h(Dw_h + D\psi)\big]\, dy.
\end{aligned}$$

Choosing $\psi = -\psi_{1,h}$ as test function in the previous inequality, we get

$$\begin{aligned}
&\int_{B_t} \widetilde{F}_h(Dw_h)\, dy \leq \int_{B_t} \big[\widetilde{F}_h\big(Dw_h - D\psi_{1,h}\big)\, dy + \mathbb{1}_{E_h^*} \widetilde{G}_h(Dw_h - D\psi_{1,h})\big]\, dy \qquad (5.20)\\
&+ \frac{1}{\lambda_h^2} \int_{B_1} \mathbb{1}_{E_h^*} \big[G(A_h + \lambda_h B_h) + DG(A_h + \lambda_h B_h)\lambda_h(Dw_h - D\psi_{1,h})\big]\, dy \\
&= \int_{B_t \setminus B_s} \big[\widetilde{F}_h(D\psi_{2,h}) + \mathbb{1}_{E_h^*} \widetilde{G}_h(D\psi_{2,h})\big]\, dy \\
&+ \frac{1}{\lambda_h^2} \int_{B_1} \mathbb{1}_{E_h^*} \big[G(A_h + \lambda_h B_h) + DG(A_h + \lambda_h B_h)\lambda_h D\psi_{2,h})\big]\, dy \\
&\leq c(M) \int_{B_t \setminus B_s} \big[|D\psi_{2,h}|^2 + \lambda_h^{p-2} |D\psi_{2,h}|^p|\big]\, dy + c(n,p,L_2,M) \left[\frac{|E_h^*|}{\lambda_h^2} + \frac{1}{\lambda_h} \int_{E_h^*} |D\psi_{2,h}|\, dy\right],
\end{aligned}$$

where we used Lemma 2.3 and the second estimate in (2.1), Hölder's inequality, and the fact that $|A_h + \lambda_h B_h| \leq M + 1$. Now we estimate the second addend in the right-hand side of (5.19). Using the upper bound on $D\widetilde{F}_h$ in Lemma 2.3, we obtain

$$\begin{aligned}
&\int_{B_t} \int_0^1 D\widetilde{F}_h(Dw_h - \theta D\psi_{2,h}) D\psi_{2,h}\, d\theta\, dy \qquad (5.21)\\
&\leq c(M) \int_{B_t \setminus B_s} \int_0^1 \big(|Dw_h - \theta D\psi_{2,h}| + \lambda_h^{p-2} |Dw_h - \theta D\psi_{2,h}|^{p-1}\big) |D\psi_{2,h}|\, d\theta\, dy \\
&\leq c(p,M) \int_{B_t \setminus B_s} \big(|Dw_h|^2 + \lambda_h^{p-2} |Dw_h|^p + |D\psi_{2,h}|^2 + \lambda_h^{p-2} |D\psi_{2,h}|^p\big)\, dy.
\end{aligned}$$

Hence, combining (5.19) with (5.20) and (5.21), using the properties of η, we obtain

$$\ell_1 \int_{B_s} \big(1 + |\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}} |Dw_h|^2\, dy$$

$$
\begin{aligned}
&\le c(p,M)\int_{B_t\setminus B_s}\left(|Dw_h|^2+\lambda_h^{p-2}|Dw_h|^p+|D\psi_{2,h}|^2+\lambda_h^{p-2}|D\psi_{2,h}|^p\right)dy\\
&+c(n,p,L_2,M)\left[\frac{1}{\lambda_h}\int_{E_h^*}|D\psi_{2,h}|\,dy+\frac{|E_h^*|}{\lambda_h^2}\right]\\
&\le c(p,M)\int_{B_t\setminus B_s}\left(|Dw_h|^2+\lambda_h^{p-2}|Dw_h|^p\right)dy+c(p,M)\int_{B_t\setminus B_s}\left[\frac{|w_h|^2}{(t-s)^2}+\lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right]dy\\
&+c(n,p,L_2,M)\left[\frac{1}{\lambda_h}\left(\int_{E_h^*}|D\psi_{2,h}(y)|^2\,dy\right)^{\frac12}|E_h^*|^{\frac12}+\frac{|E_h^*|}{\lambda_h^2}\right]\\
&\le c(p,M)\int_{B_t\setminus B_s}\left(1+\lambda_h^2|Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy+c(p,M)\int_{B_t\setminus B_s}\left[\frac{|w_h|^2}{(t-s)^2}+\lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right]dy\\
&+c(n,p,L_2,M)\left[\frac{1}{\lambda_h}\left(\int_{E_h^*}|D\psi_{2,h}(y)|^2\,dy\right)^{\frac12}|E_h^*|^{\frac12}+\frac{|E_h^*|}{\lambda_h^2}\right]\\
&\le c(n,p,L_2,M)\left[\int_{B_t\setminus B_s}\left(1+\lambda_h^2|Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy+\int_{B_\rho}\left[\frac{|w_h|^2}{(t-s)^2}+\lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right]dy\right.\\
&\left.+\frac{|E_h^*|}{\lambda_h^2}\right],
\end{aligned}
$$

where we used Young's inequality. By adding $c(n,p,L_2,M)\int_{B_s}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy$ to both sides of the previous estimate, dividing by $\ell_1+c(n,p,L_2,M)$, and thanks to the iteration Lemma 2.2, we deduce that

$$
\int_{B_{\frac{\varrho}{2}}}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy\le c(n,p,\ell_1,L_2,M)\left[\int_{B_\rho}\left(\frac{|w_h|^2}{\rho^2}+\lambda_h^{p-2}\frac{|w_h|^p}{\rho^p}\right)dy+\frac{|E_h^*|}{\lambda_h^2}\right].
$$

Therefore, by the definition of w_h, we conclude that

$$
\begin{aligned}
&\int_{B_{\frac{\varrho}{2}}}\left(1+\lambda_h^{p-2}|Dv_h-(Dv_h)_{\frac{\varrho}{2}}|^{p-2}\right)|Dv_h-(Dv_h)_{\frac{\varrho}{2}}|^2\,dy\\
&\le c(n,p,\ell_1,L_2,M)\left[\int_{B_\rho}\left(\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\varrho}{2}}y|^2}{\rho^2}+\lambda_h^{p-2}\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\varrho}{2}}y|^p}{\rho^p}\right)dy+\frac{|E_h^*|}{\lambda_h^2}\right],
\end{aligned}
\tag{5.22}
$$

which, by the relative isoperimetric inequality and the hypothesis of this substep, i.e. $\min\{|E_h^*|,|B_1\setminus E_h|\}=|E_h^*|$, yields the estimate (5.17).

Substep 2.b *The case* $\min\{|E_h^*|,|B_1\setminus E_h|\}=|B_1\setminus E_h|$.

As in the previous substep, we fix $0<\frac{\varrho}{2}<s<t<\rho<1$ and let $\eta\in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0\le\eta\le1$, $\eta\equiv1$ on B_s and $|\nabla\eta|\le\frac{c}{t-s}$. Also, we set $b_h:=(v_h)_{B_\rho}$, $B_h:=(Dv_h)_{B_{\frac{\varrho}{2}}}$ and define

$$
w_h(y):=v_h(y)-b_h-B_hy,
$$

for any $y \in B_1$ and

$$\widetilde{H}_h := \widetilde{F}_h + \widetilde{G}_h.$$

We remark that Lemma 2.3 applies to $\widetilde{H}_h$, that is

$$|\widetilde{H}_h(\xi)| \le c(M)\big(|\xi|^2 + \lambda_h^{p-2}|\xi|^p\big), \quad \forall \xi \in \mathbb{R}^{n\times N},$$

and, by the uniformly strict quasiconvexity conditions $(F1)$ and $(G2)$,

$$\int_{B_1} \widetilde{H}_h(\xi + D\psi)\,dx \ge \int_{B_t} \big[\widetilde{H}_h(\xi) + \widetilde{\ell}\big(\mu^2 + |\lambda_h D\psi_{1,h}|^2\big)^{\frac{p-2}{2}}|D\psi_{1,h}|^2\big]\,dy, \tag{5.23}$$

for all $\psi \in W_0^{1,p}(B_1;\mathbb{R}^N)$, where $\widetilde{\ell}$ is such that

$$\widetilde{\ell} \ge \ell_1 + \ell_2\,.$$

Set

$$\psi_{1,h} := \eta w_h \qquad \text{and} \qquad \psi_{2,h} := (1-\eta)w_h.$$

By (5.23) and since $\widetilde{H}_h(0) = 0$, we have

$$\begin{aligned}
&\widetilde{\ell}\int_{B_s} \big(1 + |\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\
&\le \widetilde{\ell}\int_{B_t} \big(1 + |\lambda_h D\psi_{1,h}|^2\big)^{\frac{p-2}{2}}|D\psi_{1,h}|^2\,dy \le \int_{B_t} \widetilde{H}_h(D\psi_{1,h})\,dy = \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy\\
&= \int_{B_t} \widetilde{H}_h(Dw_h)\,dy + \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy - \int_{B_t} \widetilde{H}_h(Dw_h)\,dy\\
&= \int_{B_t} \widetilde{H}_h(Dw_h)\,dy - \int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h - \theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy.
\end{aligned} \tag{5.24}$$

As in the previous step, we estimate separately the two addends in the right-hand side of the previous chain of inequalities. We deal with the first addend by means of a rescaling of the minimality condition of (u, E). By virtue of the minimality inequality in (5.16) and since $Dv_h = Dw_h + B_h$, we get

$$\begin{aligned}
\int_{B_1} H_h(Dw_h + B_h)dy \le& \int_{B_1} H_h(Dw_h + B_h + D\psi)\,dy\\
&+ \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\,dy,
\end{aligned}$$

or, equivalently, by the definition of $\widetilde{H}_h$,

$$\begin{aligned}
\int_{B_1} \widetilde{H}_h(Dw_h)dy \le& \int_{B_1} \widetilde{H}_h(Dw_h + D\psi)\,dy\\
&+ \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\,dy.
\end{aligned} \tag{5.25}$$

Choosing $\psi = -\psi_{1,h}$ as test function in (5.25) and using the fact that $\widetilde{H}_h(0) = 0$, we get

$$\begin{aligned}
&\int_{B_t} \widetilde{H}_h(Dw_h)\, dy \\
&\leq \int_{B_t} \widetilde{H}_h(Dw_h(y) - D\psi_{1,h})\, dy + \frac{L_2}{\lambda_h^2} \int_{B_t \setminus E_h} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy \\
&= \int_{B_t \setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\, dy + \frac{L_2}{\lambda_h^2} \int_{B_t \setminus E_h} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy \\
&\leq c(M) \int_{B_t \setminus B_s} \big(|D\psi_{2,h}|^2 + \lambda_h^{p-2}|D\psi_{2,h}|^p\big)\, dy + \frac{L_2}{\lambda_h^2} \int_{B_t \setminus E_h} \big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}}\, dy.
\end{aligned}$$

By remarking that

$$\begin{aligned}
&\big(1 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2\big)^{\frac{p}{2}} \leq \big(1 + |A_h + \lambda_h B_h|^2 + 2|A_h + \lambda_h B_h|\lambda_h |Dw_h| + \lambda_h^2 |Dw_h|^2\big)^{\frac{p}{2}} \\
&\leq \left[\left(1 + \frac{1}{\varepsilon}\right) c(M) + (1+\varepsilon)\lambda_h^2 |Dw_h|^2\right]^{\frac{p}{2}} \leq \left(1 + \frac{1}{\varepsilon}\right)^{\frac{p}{2}+1} c(M)^{\frac{p}{2}} + (1+\varepsilon)^{\frac{p}{2}+1} \lambda_h^p |Dw_h|^p,
\end{aligned}$$

for every $\varepsilon > 0$, we get

$$\begin{aligned}
&\int_{B_t} \widetilde{H}_h(Dw_h)\, dy \leq c(M) \int_{B_t \setminus B_s} \big(|D\psi_{2,h}|^2 + \lambda_h^{p-2}|D\psi_{2,h}|^p\big)\, dy \\
&+ (1+\varepsilon)^{\frac{p}{2}+1} L_2 \lambda_h^{p-2} \int_{B_t} |Dw_h|^p\, dy + c(p, L_2, M, \varepsilon) \frac{|B_1 \setminus E_h|}{\lambda_h^2}.
\end{aligned} \tag{5.26}$$

Now we estimate the second addend in the right-hand side of (5.24). Using the upper bound on $D\widetilde{H}_h$ in Lemma 2.3, we obtain

$$\begin{aligned}
&\int_{B_t} \int_0^1 D\widetilde{H}_h(Dw_h - \theta D\psi_{2,h}) D\psi_{2,h}\, d\theta\, dy \\
&\leq c(M) \int_{B_t \setminus B_s} \int_0^1 \big(|Dw_h - \theta D\psi_{2,h}| + \lambda_h^{p-2} |Dw_h - \theta D\psi_{2,h}|^{p-1}\big) |D\psi_{2,h}|\, d\theta\, dy \\
&\leq c(p, M) \int_{B_t \setminus B_s} \big(|Dw_h|^2 + \lambda_h^{p-2}|Dw_h|^p + |D\psi_{2,h}|^2 + \lambda_h^{p-2}|D\psi_{2,h}|^p\big)\, dy.
\end{aligned} \tag{5.27}$$

Inserting (5.26) and (5.27) in (5.24), we infer that

$$\begin{aligned}
&\tilde{\ell} \int_{B_s} \big(1 + |\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}} |Dw_h|^2\, dy \\
&\leq c(p, M) \int_{B_t \setminus B_s} \big(|Dw_h|^2 + \lambda_h^{p-2}|Dw_h|^p + |D\psi_{2,h}|^2 + \lambda_h^{p-2}|D\psi_{2,h}|^p\big)\, dy \\
&+ (1+\varepsilon)^{\frac{p}{2}+1} L_2 \lambda_h^{p-2} \int_{B_t} |Dw_h|^p\, dy + c(p, L_2, M, \varepsilon) \frac{|B_1 \setminus E_h|}{\lambda_h^2} \\
&\leq c(p, M) \int_{B_t \setminus B_s} \big(1 + |\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}} |Dw_h|^2\, dy + c(p, M) \int_{B_t \setminus B_s} \left(\frac{|w_h|^2}{(t-s)^2} + \lambda_h^{p-2} \frac{|w_h|^p}{(t-s)^p}\right) dy
\end{aligned}$$

$$+(1+\varepsilon)^{\frac{p}{2}+1}L_2\int_{B_t}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy+c(p,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}.$$

Using the hole filling technique as in the previous case, we obtain

$$\begin{aligned}&\int_{B_s}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\&\le\frac{(c(p,M)+(1+\varepsilon)^{\frac{p}{2}+1}L_2)}{(c(p,M)+\widetilde{\ell})}\int_{B_t}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\&+\int_{B_t\setminus B_s}\left(\frac{|w_h|^2}{(t-s)^2}+\lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right)dy+c(p,\ell_1,\ell_2,L_2,M,\varepsilon)\frac{|B_1\setminus E_h|}{\lambda_h^2}.\end{aligned}$$

The assumption (H) implies that there exists $\varepsilon=\varepsilon(p,\ell_1,\ell_2,L_2)>0$ such that $\frac{(1+\varepsilon)^{\frac{p}{2}+1}L_2}{\ell_1+\ell_2}<1$. Therefore we have

$$\frac{c+(1+\varepsilon)^{\frac{p}{2}+1}L_2}{c+\widetilde{\ell}}\le\frac{c+(1+\varepsilon)^{\frac{p}{2}+1}L_2}{c+\ell_1+\ell_2}<1.$$

So, by virtue of Lemma 2.2, from the previous estimate we deduce that

$$\int_{B_{\frac{\rho}{2}}}\left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy\le c(p,\ell_1,\ell_2,L_2,M)\left[\int_{B_\rho}\left(\frac{|w_h|^2}{\rho^2}+\lambda_h^{p-2}\frac{|w_h|^p}{\rho^p}\right)dy+\frac{|B_1\setminus E_h|}{\lambda_h^2}\right].$$

Therefore, by the definition of w_h, we conclude that

$$\begin{aligned}&\int_{B_{\frac{\rho}{2}}}\left(1+\lambda_h^{p-2}|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^{p-2}\right)|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^2\,dy\\&\le c(p,\ell_1,\ell_2,L_2,M)\left[\int_{B_\rho}\left(\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y|^2}{\rho^2}+\lambda_h^{p-2}\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y|^p}{\rho^p}\right)dy\right.\\&\left.+\frac{|B_1\setminus E_h|}{\lambda_h^2}\right],\end{aligned}$$

which, by the relative isoperimetric inequality and since we have $|B_1\setminus E_h|=\min\{|E_h^*|,|B_1\setminus E_h|\}$, gives the estimate (5.17).

Step 3. *We prove that there exists a constant $\tilde{c}=c(n,N,\ell_1,\ell_2,L_1,L_2)>0$ such that*

$$⨍_{B_{\frac{\tau}{2}}}|Dv-(Dv)_{\frac{\tau}{2}}|^2\le\tilde{c}\tau^2⨍_{B_\tau}|Dv-(Dv)_\tau|^2\,dx,\tag{5.28}$$

for any $\tau<1$.
It will follow that

$$⨍_{B_{\frac{\tau}{2}}}|Dv-(Dv)_{\frac{\tau}{2}}|^2\le\tilde{c}\tau^2⨍_{B_\tau}|Dv-(Dv)_\tau|^2\le\tilde{c}\tau^2,\tag{5.29}$$

since

$$\|Dv\|_{L^2(B_1)}\le\limsup_h\|Dv_h\|_{L^2(B_1)}\le c(n).$$

As before, we will divide the proof in two substeps.

Substep 3.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

We claim that v solves the linear system

$$\int_{B_1} D^2F(A)DvD\psi\, dy = 0,$$

for all $\psi \in C_0^1(B_1;\mathbb{R}^N)$. Since v_h satisfies (5.13), we have that

$$0 \leq \mathcal{I}_h(v_h + s\psi) - \mathcal{I}_h(v_h) + \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*} DG(A_h)sD\psi\, dy,$$

for every $\psi \in C_0^1(B_1;\mathbb{R}^N)$ and $s \in (0,1)$. By the definition of $\mathcal{I}_h$ we get

$$\begin{aligned}
0 &\leq \mathcal{I}_h(v_h + s\psi) - \mathcal{I}_h(v_h) + \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*} DG(A_h)sD\psi\, dy \\
&= \frac{1}{\lambda_h}\int_{B_1}\left(\int_0^1 \big[DF(A_h + \lambda_h(Dv_h + tsD\psi))\big]sD\psi\, dt - DF(A_h)sD\psi\right) dy \\
&+ \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*}\left(\int_0^1 DG(A_h + \lambda_h(Dv_h + tsD\psi))sD\psi\, dt - DG(A_h)sD\psi\right) dy \\
&+ \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*} DG(A_h)sD\psi(y)\, dy \\
&= \frac{1}{\lambda_h}\int_{B_1}\left(\int_0^1 \big[DF(A_h + \lambda_h(Dv_h + tsD\psi))\big]sD\psi\, dt - DF(A_h)sD\psi\right) dy \\
&+ \frac{1}{\lambda_h}\int_{B_1}\int_0^1 \mathbb{1}_{E_h^*} DG(A_h + \lambda_h(Dv_h + tsD\psi))sD\psi\, dt\, dy
\end{aligned}$$

We divide by s and do the limit as $s \to 0$, therefore we deduce that

$$\begin{aligned}
0 \leq\ & \frac{1}{\lambda_h}\int_{B_1} \left(DF(A_h + \lambda_h Dv_h) - DF(A_h)\right) D\psi\, dy \\
&+ \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*} DG(A_h + \lambda_h Dv_h)D\psi\, dy. \qquad (5.30)
\end{aligned}$$

We partition the unit ball as

$$B_1 = \mathbf{B}_h^+ \cup \mathbf{B}_h^- = \{y \in B_1 : \lambda_h |Dv_h| > 1\} \cup \{y \in B_1 : \lambda_h |Dv_h| \leq 1\}.$$

By (5.7), we get

$$|\mathbf{B}_h^+| \leq \int_{\mathbf{B}_h^+} \lambda_h^2 |Dv_h|^2\, dy \leq \lambda_h^2 \int_{\mathbf{B}_h^+} |Dv_h|^2\, dy \leq c(n)\lambda_h^2. \qquad (5.31)$$

We rewrite (5.30) as follows:

$$0 \leq \frac{1}{\lambda_h}\int_{B_1} (DF(A_h + \lambda_h Dv_h) - DF(A_h))D\psi\, dy + \frac{1}{\lambda_h}\int_{B_1} \mathbb{1}_{E_h^*} DG(A_h + \lambda_h Dv_h)D\psi\, dy$$

$$
\begin{aligned}
&= \frac{1}{\lambda_h}\int_{\mathbf{B}_h^+}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy+\frac{1}{\lambda_h}\int_{\mathbf{B}_h^-}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy\\
&+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h+\lambda_h Dv_h)D\psi\,dy\\
&=\frac{1}{\lambda_h}\int_{\mathbf{B}_h^+}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy\\
&+\int_{\mathbf{B}_h^-}\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dtDv_hD\psi\,dy\\
&+\int_{\mathbf{B}_h^-}D^2F(A)Dv_hD\psi\,dy+\frac{1}{\lambda_h}\int_{B_1}\mathbb{1}_{E_h^*}DG(A_h+\lambda_h Dv_h)D\psi\,dy. \qquad (5.32)
\end{aligned}
$$

By virtue of the first estimate in (2.1) and Hölder's inequality, we get

$$
\begin{aligned}
&\frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy\right|\\
&\le c(p,L_1,M,D\psi)\left[\frac{|\mathbf{B}_h^+|}{\lambda_h}+\lambda_h^{p-2}\int_{\mathbf{B}_h^+}|Dv_h|^{p-1}\,dy\right]\\
&\le c(n,p,L_1,M,D\psi)\left[\lambda_h+\lambda_h\left(\int_{\mathbf{B}_1}\lambda_h^{p-2}|Dv_h|^p\,dy\right)^{\frac{p-1}{p}}\left(\frac{|\mathbf{B}_h^+|}{\lambda_h^2}\right)^{\frac{1}{p}}\right]\le c(n,p,L_1,M,D\psi)\lambda_h,
\end{aligned}
$$

thanks to (5.4) (to bound $|A_h|\le M$), (5.7) and (5.31). Thus

$$
\lim_{h\to\infty}\frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+}(DF(A_h+\lambda_h Dv_h)-DF(A_h))D\psi\,dy\right|=0. \qquad (5.33)
$$

By (5.4) and the definition of $\mathbf{B}_h^-$ we have that $|A_h+\lambda_h Dv_h|\le M+1$ on $\mathbf{B}_h^-$. Hence the uniform continuity of D^2F on bounded sets implies

$$
\begin{aligned}
&\left|\int_{\mathbf{B}_h^-}\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dtDv_hD\psi\,dy\right|\\
&\le\int_{\mathbf{B}_h^-}\left|\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dt\right||Dv_h||D\psi|\,dy\\
&\le\left(\int_{\mathbf{B}_h^-}\left|\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dt\right|^2dy\right)^{\frac{1}{2}}\|Dv_h\|_{L^2(B_1)}\|D\psi\|_{L^\infty(B_1)}\\
&\le c(n,D\psi)\left(\int_{\mathbf{B}_h^-}\left|\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dt\right|^2dy\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used (5.7). Since by (5.8), $\lambda_h Dv_h\to 0$ a.e. in B_1, we deduce that

$$
\lim_{h\to\infty}\left|\int_{\mathbf{B}_h^-}\int_0^1\big(D^2F(A_h+t\lambda_h Dv_h)-D^2F(A)\big)\,dtDv_hD\psi\,dy\right|=0. \qquad (5.34)
$$

Note that (5.31) yields that $\mathbb{1}_{\mathbf{B}_h^-} \to \mathbb{1}_{B_1}$ in $L^r(B_1)$, for every $r < \infty$. Therefore, by (5.7),

$$
\begin{aligned}
&\left| \int_{\mathbf{B}_h^-} D^2F(A)Dv_hD\psi\,dy - \int_{B_1} D^2F(A)DvD\psi\,dy \right| \\
&\leq \left| \int_{\mathbf{B}_h^-} D^2F(A)Dv_hD\psi\,dy - \int_{B_1} D^2F(A)Dv_hD\psi\,dy \right| \\
&+ \left| \int_{B_1} D^2F(A)Dv_hD\psi\,dy - \int_{B_1} D^2F(A)DvD\psi\,dy \right| \\
&\leq c(D\psi)\|\mathbb{1}_{\mathbf{B}_h^-} - \mathbb{1}_{B_1}\|_{L^2(B_1)}\,\|Dv_h\|_{L^2(B_1)} \left| \int_0^1 D^2F(A)\,dt \right| + \left| \int_{B_1} D^2F(A)(Dv_h - Dv)D\psi\,dy \right|.
\end{aligned}
$$

Thus, by the weak convergence of Dv_h to Dv in $L^2(B_1)$, it follows that

$$
\lim_{h\to\infty} \int_{\mathbf{B}_h^-} D^2F(A)Dv_hD\psi\,dy = \int_{B_1} D^2F(A)DvD\psi\,dy. \tag{5.35}
$$

By the second estimate in (2.1), we deduce that

$$
\begin{aligned}
&\frac{1}{\lambda_h}\left| \int_{B_1} \mathbb{1}_{E_h^*}[D_\xi G(A_h + \lambda_h Dv_h)D\psi\,dy \right| \leq \frac{c(p,L_2)}{\lambda_h} \int_{B_1} \mathbb{1}_{E_h^*}\left(1 + |A_h + \lambda_h Dv_h|^2\right)^{\frac{p-1}{2}} |D\psi|\,dy \\
&\leq c(p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + \lambda_h^{p-2} \int_{E_h^*} |Dv_h|^{p-1}\,dy \right] \\
&\leq c(p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + \lambda_h \left(\int_{\mathbf{B}_1} \lambda_h^{p-2}|Dv_h|^p\,dy \right)^{\frac{p-1}{p}} \left(\frac{|E_h^*|}{\lambda_h^2} \right)^{\frac{1}{p}} \right] \\
&\leq c(n,p,L_2,M,D\psi)\left[\frac{1}{\lambda_h}|E_h^*| + c\lambda_h \left(\frac{|E_h^*|}{\lambda_h^2} \right)^{\frac{1}{p}} \right],
\end{aligned}
$$

thanks to (5.4) and (5.7). Since $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$, by (5.10) we have

$$
\lim_{h\to\infty} \frac{|E_h^*|}{\lambda_h^2} = 0,
$$

and so

$$
\lim_{h\to\infty} \frac{1}{\lambda_h} \int_{B_1} \mathbb{1}_{E_h^*} DG(A_h + \lambda_h Dv_h)D\psi\,dy = 0. \tag{5.36}
$$

By (5.33), (5.34), (5.35) and (5.36), passing to the limit as $h \to \infty$ in (5.32), we get

$$
\int_{B_1} DF(A)DvD\psi\,dy \geq 0,
$$

and with $-\psi$ in place of ψ we get

$$
\int_{B_1} DF(A)DvD\psi\,dy = 0,
$$

i.e., v solves a linear system with constant coefficients. By Proposition 2.1 we deduce that $v \in C^\infty$ and, for every $0 < \tau < 1$, we have

$$\fint_{B_{\frac{\tau}{2}}} |Dv - (Dv)_{\frac{\tau}{2}}|^2 \le c(n, N, \ell_1, L_1)\tau^2 \fint_{B_\tau} |Dv - (Dv)_\tau|^2 \, dx \le c(n, N, \ell_1, L_1)\tau^2,$$

since

$$\|Dv\|_{L^2(B_1)} \le \limsup_{h\to\infty} \|Dv_h\|_{L^2(B_1)} \le c(n).$$

Substep 3.b *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |B_1 \setminus E_h|$.

We claim that v solves the linear system

$$\int_{B_1} D^2(F+G)(A) Dv D\psi \, dy = 0,$$

for all $\psi \in C_0^1(B_1; \mathbb{R}^N)$. Arguing as (5.15) and rescaling, we have that

$$\begin{aligned}
&\int_{B_1} H_h(Dv_h) dy \le \int_{B_1} H_h(Dv_h + sD\psi) + \frac{1}{\lambda_h^2} \int_{B_1\setminus E_h} [G(A_h + \lambda_h Dv_h) - G(A_h + \lambda_h Dv_h + s\lambda_h D\psi)] dy \\
&= \int_{B_1} H_h(Dv_h + sD\psi) \, dy + \frac{1}{\lambda_h} \int_{B_1\setminus E_h} \int_0^1 DG(A_h + \lambda_h Dv_h + ts\lambda_h D\psi) sD\psi \, dt \, dy \\
&\le \int_{B_1} H_h(Dv_h + sD\psi) \, dy + \frac{c(p, L_2)}{\lambda_h} \int_{B_1\setminus E_h} \int_0^1 \left(1 + |A_h + \lambda_h Dv_h + ts\lambda_h D\psi|^2\right)^{\frac{p-1}{2}} s|D\psi| \, dt \, dy \\
&\le \int_{B_1} H_h(Dv_h + sD\psi) \, dy + c(p, L_2, M) \Bigg[\frac{1}{\lambda_h} \int_{B_1\setminus E_h} s|D\psi| \, dy \\
&+ \int_{B_1\setminus E_h} \int_0^1 \lambda_h^{p-2} |Dv_h + tsD\psi|^{p-1} s|D\psi| \, dt \, dy \Bigg],
\end{aligned}$$

for every $\psi \in C_0^1(B_1; \mathbb{R}^N)$ and for every $s \in (0,1)$. Therefore

$$\begin{aligned}
0 \le &\int_{B_1} \int_0^1 DH_h(Dv_h + s\theta D\psi) \, d\theta s D\psi \, dy \\
&+ c(p, L_2, M) \Bigg[\frac{1}{\lambda_h} \int_{B_1\setminus E_h} s|D\psi| \, dy + \int_{B_1\setminus E_h} \int_0^1 \lambda_h^{p-2} |Dv_h + tsD\psi|^{p-1} s|D\psi| \, dt \, dy \Bigg].
\end{aligned}$$

Dividing the previous inequality by s and taking the limit as $s \to 0$, we obtain that

$$0 \le \int_{B_1} DH_h(Dv_h) D\psi dy + c(p, L_2, M) \Bigg[\frac{1}{\lambda_h} \int_{B_1\setminus E_h} |D\psi| dy + \int_{B_1\setminus E_h} \lambda_h^{p-2} |Dv_h|^{p-1} |D\psi| \, dy \Bigg].$$

By the definition of H_h, we conclude that

$$0 \le \frac{1}{\lambda_h} \int_{B_1} \left[D(F+G)(A_h + \lambda_h Dv_h) D\psi - D(F+G)(A_h) D\psi \right] dy$$

$$+ c(p, L_2, M)\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi| dy + \int_{B_1\setminus E_h} \lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\, dy\right].$$

Just as before, we partition B_1 as

$$B_1 = \mathbf{B}_h^+ \cup \mathbf{B}_h^- = \{y \in B_1 : \lambda_h|Dv_h| > 1\} \cup \{y \in B_1 : \lambda_h|Dv_h| \leq 1\},$$

and we write

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_h}\int_{B_1} (D(F+G)(A_h + \lambda_h Dv_h) - D(F+G)(A_h))D\psi\, dy \qquad (5.37)\\
&+ c(p, L_2, M)\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi| dy + \int_{B_1\setminus E_h} \lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\, dy\right]\\
&= \frac{1}{\lambda_h}\int_{\mathbf{B}_h^+} (D(F+G)(A_h + \lambda_h Dv_h) - D(F+G)(A_h))D\psi\, dy\\
&+ \frac{1}{\lambda_h}\int_{\mathbf{B}_h^-} (D(F+G)(A_h + \lambda_h Dv_h) - D(F+G)(A_h))D\psi\, dy\\
&+ c(p, L_2, M)\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi| dy + \int_{B_1\setminus E_h} \lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\, dy\right].
\end{aligned}$$

Arguing as in (5.33) , we obtain that

$$\lim_{h\to\infty} \frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+} (D(F+G)(A_h + \lambda_h Dv_h) - D(F+G)(A_h))D\psi\, dy\right| = 0, \qquad (5.38)$$

and, as in (5.34) and (5.35),

$$\begin{aligned}
&\lim_{h\to\infty} \frac{1}{\lambda_h}\int_{\mathbf{B}_h^-} [D(F+G)(A_h + \lambda_h Dv_h) - D(F+G)(A_h)]D\psi\, dy\\
&= \int_{B_1} D(F+G)(A)DvD\psi\, dy. \qquad (5.39)
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
&\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi| dy + \int_{B_1\setminus E_h} \lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\, dy\\
&\leq c(p, D\psi)\left[\frac{|B_1 \setminus E_h|}{\lambda_h} + \lambda_h\left(\int_{\mathbf{B}_1} \lambda_h^{p-2}|Dv_h|^p\, dy\right)^{\frac{p-1}{p}}\left(\frac{|B_1 \setminus E_h|}{\lambda_h^2}\right)^{\frac{1}{p}}\right]\\
&\leq c(n, p, D\psi)\left[\frac{|B_1 \setminus E_h|}{\lambda_h} + \lambda_h\left(\frac{|B_1 \setminus E_h|}{\lambda_h^2}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where we used (5.7). Since $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |B_1 \setminus E_h|$, by (5.10), we have

$$\lim_{h\to\infty} \frac{|B_1 \setminus E_h|}{\lambda_h^2} = 0,$$

and we obtain

$$\lim_{h\to\infty}\left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy+\int_{B_1\setminus E_h}\lambda_h^{p-2}|Dv_h|^{p-1}|D\psi|\,dy\right]=0. \tag{5.40}$$

By (5.38), (5.39) and (5.40), passing to the limit as $h\to\infty$ in (5.37) we conclude that

$$\int_{B_1}D^2(F+G)(A)DvD\psi\,dy\geq 0$$

and with $-\psi$ in place of ψ we finally get

$$\int_{B_1}D^2(F+G)(A)DvD\psi\,dy=0,$$

asserting the claim. By Proposition 2.1, we deduce also in this case that $v\in C^\infty$ and for every $0<\tau<1$ satisfies estimate (5.28).

Step 4. *An estimate for the perimeters.*

Our aim is to show that there exists a constant $c=c(n,p,L_2,\Lambda,M)>0$ such that

$$P(E_h,B_\tau)\leq c\left[\frac{1}{\tau}P(E_h,B_1)^{\frac{n}{n-1}}+r_h\tau^n+r_h\lambda_h^2\right]. \tag{5.41}$$

By the minimality of (u,E) with respect to $(u,\widetilde{E})$, where $\widetilde{E}$ is a set of finite perimeter such that $\widetilde{E}\Delta E\Subset B_{r_h}(x_h)$ and $B_{r_h}(x_h)$ are the balls of the contradiction argument, we get

$$\int_{B_{r_h}(x_h)}\mathbb{1}_E G(Du)+\mathbf{\Phi}(E;B_{r_h}(x_h))\leq\int_{B_{r_h}(x_h)}\mathbb{1}_{\widetilde{E}}G(Du)+\mathbf{\Phi}(\widetilde{E};B_{r_h}(x_h)).$$

Using the change of variable $x=x_h+r_hy$ and dividing by r_h^{n-1}, we have

$$r_h\int_{B_1}\mathbb{1}_{E_h}G(A_h+\lambda_hDv_h)dy+\mathbf{\Phi}_h(E_h;B_1)\leq r_h\int_{B_1}\mathbb{1}_{\widetilde{E}_h}G(A_h+\lambda_hDv_h)dy+\mathbf{\Phi}_h(\widetilde{E}_h;B_1),$$

where

$$\mathbf{\Phi}_h(E_h;G):=\int_{G\cap\partial^*E_h}\Phi(x_h+r_hy,\nu_{E_h}(y))\,d\mathcal{H}^{n-1}(y),$$

for every Borel set $G\subset\Omega$. Assume first that $\min\{|B_1\setminus E_h|,|E_h^*|\}=|B_1\setminus E_h|$. Choosing $\widetilde{E}_h:=E_h\cup B_\rho$, we get

$$\mathbf{\Phi}_h(E_h;B_1)\leq r_h\int_{B_1}\mathbb{1}_{B_\rho}G(A_h+\lambda_hDv_h)dy+\mathbf{\Phi}_h(\widetilde{E}_h;B_1). \tag{5.42}$$

By coarea formula, the relative isoperimetric inequality, the choice of the representative $E_h^{(1)}$ of E_h, which is a Borel set, we get

$$\int_\tau^{2\tau}\mathcal{H}^{n-1}(\partial B_\rho\setminus E_h)\,d\rho\leq|B_1\setminus E_h|\leq c(n)P(E_h,B_1)^{\frac{n}{n-1}}.$$

Therefore, we may choose $\rho \in (\tau, 2\tau)$, independent of n, such that, up to subsequences, it holds

$$\mathcal{H}^{n-1}(\partial^* E_h \cap \partial B_\rho) = 0 \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h) \leq \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}}. \tag{5.43}$$

We remark that Proposition 2.8 holds also for $\mathbf{\Phi}_h$. Thus, thanks to the choice of ρ, being $\mathcal{H}^{n-1}(\partial^* E_h \cap \partial B_\rho) = 0$, we have that

$$\begin{aligned}
\mathbf{\Phi}_h(\widetilde{E}_h; B_1) &= \mathbf{\Phi}_h(E_h; B_\rho^{(0)}) + \mathbf{\Phi}_h(B_\rho; E_h^{(0)}) + \mathbf{\Phi}_h(E_h; \{\nu_{E_h} = \nu_{B_\rho}\}) \\
&= \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B_\rho}) + \mathbf{\Phi}_h(B_\rho; E_h^{(0)}).
\end{aligned}$$

By the choice of the representative of E_h (see Remark 2.4), taking into account (2.8) and the inequality in (5.43), it follows that

$$\begin{aligned}
\mathbf{\Phi}_h(\widetilde{E}_h; B_1) &\leq \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B_\rho}) + \Lambda \mathcal{H}^{n-1}(\partial B_\rho \cap E_h^{(0)}) \\
&\leq \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B_\rho}) + \Lambda \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h). \\
&\leq \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B_\rho}) + \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}}.
\end{aligned} \tag{5.44}$$

On the other hand, by (2.8) and the additivity of the measure $\mathbf{\Phi}_h(E_h, \cdot)$ it holds that

$$\frac{1}{\Lambda} P(E_h, B_\tau) \leq \mathbf{\Phi}_h(E_h; B_\tau) \leq \mathbf{\Phi}_h(E_h; B_1) - \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B}_\rho), \tag{5.45}$$

since $\rho > \tau$. Combining (5.42), (5.44) and (5.45), we obtain

$$\begin{aligned}
\frac{1}{\Lambda} P(E_h, B_\tau) &\leq \mathbf{\Phi}_h(E_h; B_1) - \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B}_\rho) \\
&\leq \mathbf{\Phi}_h(\widetilde{E}_h; B_1) + r_h \int_{B_1} \mathbb{1}_{B_\rho} G(A_h + \lambda_h D v_h) dy - \mathbf{\Phi}_h(E_h; B_1 \setminus \overline{B}_\rho) \\
&\leq \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + r_h \int_{B_1} \mathbb{1}_{B_\rho} G(A_h + \lambda_h D v_h) dy \\
&\leq \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + c(p, L_2) r_h \int_{B_{2\tau}} (1 + |A_h + \lambda_h D v_h|^2)^{\frac{p}{2}} \, dy \\
&\leq \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + c(n, p, L_2, M) r_h \tau^n + c(p, L_2) r_h \lambda_h^2 \int_{B_{2\tau}} \lambda_h^{p-2} |D v_h|^p \, dy \\
&\leq \Lambda \frac{c(n)}{\tau} P(E_h, B_1)^{\frac{n}{n-1}} + c(n, p, L_2, M) r_h \tau^n + c(n, p, L_2) r_h \lambda_h^2,
\end{aligned} \tag{5.46}$$

where we used (5.7). The previous estimate leads to (5.41). We reach the same conclusion if $\min\{|B_1 \setminus E_h|, |E_h^*|\} = |E_h^*|$, choosing $\widetilde{E}_h = E_h \setminus B_\rho$ as a competing set.

Step 5. *Conclusion.*

By the change of variable $x = x_h + r_h y$ and the Caccioppoli inequality in (5.17), for every $0 < \tau < \frac{1}{4}$ we have

$$\limsup_{h \to \infty} \frac{U_*(x_h, \tau r_h)}{\lambda_h^2}$$

$$
\begin{aligned}
&\leq \limsup_{h\to\infty} \frac{1}{\lambda_h^2} \fint_{B_{\tau r_h}(x_h)} \left[|Du(x) - (Du)_{x_h,\tau r_h}|^2 + |Du(x) - (Du)_{x_h,\tau r_h}|^p\right] dx \\
&+ \limsup_{h\to\infty} \frac{P(E, B_{\tau r_h}(x_h))}{\lambda_h^2 \tau^{n-1} r_h^{n-1}} + \limsup_{h\to\infty} \frac{\tau r_h}{\lambda_h^2} \\
&\leq \limsup_{h\to\infty} \fint_{B_\tau} \left[|Dv_h - (Dv_h)_\tau|^2 + \lambda_h^{p-2}|Dv_h - (Dv_h)_\tau|^p\right] dy + \limsup_{h\to\infty} \frac{P(E_h, B_\tau)}{\lambda_h^2 \tau^{n-1}} + \tau \\
&\leq c(n,p,\ell_1,\ell_2,L_2,,\Lambda,M) \Bigg\{ \limsup_{h\to\infty} \fint_{B_{2\tau}} \left[\frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^2}{\tau^2} + \lambda_h^{p-2} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^p}{\tau^p}\right] dy \\
&+ \frac{1}{\tau^n} \limsup_{h\to\infty} \frac{P(E_h, B_1)^{\frac{n}{n-1}}}{\lambda_h^2} + \frac{1}{\tau^{n-1}} \limsup_{h\to\infty} \left(\frac{r_h \tau^n}{\lambda_h^2} + r_h\right) + \tau \Bigg\},
\end{aligned}
$$

where we used (5.7) and estimate (5.46). We remark that

$$
\lim_{h\to\infty} \fint_{B_{2\tau}} \lambda_h^{p-2} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^p \, dy = 0, \tag{5.47}
$$

being $p > 2$. Indeed, fixed $r > p$, we consider

$$
p^\star := \begin{cases} \frac{np}{n-p}, & \text{if } 2 < p < n, \\ r & \text{if } p \geq n. \end{cases}
$$

There exists $\alpha \in (0,1)$ such that

$$
\frac{1}{p} = \frac{1-\alpha}{2} + \frac{\alpha}{p^\star}.
$$

Thus, we interpolate by Hölder inequality, and we get

$$
\begin{aligned}
&\fint_{B_{2\tau}} \lambda_h^{p-2} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^p \, dy \\
&= \lambda_h^{p-2} \left(\fint_{B_{2\tau}} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^{p^\star} dy\right)^{\frac{\alpha p}{p^\star}} \left(\fint_{B_{2\tau}} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^2 \, dy\right)^{\frac{(1-\alpha)p}{2}}.
\end{aligned}
$$

On one hand, by Poincaré-Wirtinger inequality and (5.29) we obtain

$$
\begin{aligned}
&\lim_{h\to\infty} \fint_{B_{2\tau}} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^2 \, dy = \fint_{B_{2\tau}} |v - v_{2\tau} - (Dv)_\tau y|^2 \, dy \\
&\leq c(n)\tau^2 \fint_{B_{2\tau}} |Dv - (Dv)_\tau|^2 \, dy \leq c(n,N,\ell_1,\ell_2,L_1,L_2)\tau^2.
\end{aligned}
$$

On the other hand, by Sobolev-Poincaré inequality, we infer

$$
\begin{aligned}
&\lambda_h^{p-2} \left(\fint_{B_{2\tau}} |v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^{p^\star} dy\right)^{\frac{\alpha p}{p^\star}} \leq c(n,p) \lambda_h^{p-2} \left(\fint_{B_{2\tau}} |Dv_h - (Dv_h)_\tau|^p \, dy\right)^\alpha \\
&\leq c(n,p)\lambda_h^{p-2} \left(\fint_{B_{2\tau}} |Dv_h|^p \, dy\right)^\alpha = c(n,p)\lambda_h^{(p-2)(1-\alpha)} \left(\fint_{B_{2\tau}} \lambda_h^{p-2}|Dv_h|^p \, dy\right)^\alpha \leq c(n,p)\lambda_h^{(p-2)(1-\alpha)},
\end{aligned}
$$

where we used (5.28) and (5.7). Therefore (5.47) follows at once.
By virtue of the strong convergence of $v_h \to v$ in $L^2(B_1)$, since $(Dv_h)_\tau \to (Dv)_\tau$ in $\mathbb{R}^{n\times N}$, by (5.8), (5.9), (5.10), (5.29), (5.47) and by the Poincaré-Wirtinger inequality, we get

$$\begin{aligned}\limsup_{h\to\infty} \frac{U_*(x_h,\tau r_h)}{\lambda_h^2} &\le c(n,p,\ell_1,\ell_2,L_2,\Lambda,M)\biggl\{ -\!\!\!\!\!\!\int_{B_{2\tau}} \frac{|v-(v)_{2\tau}-(Dv)_\tau y|^2}{\tau^2}\,dy + \tau\biggr\} \\ &\le c(n,p,\ell_1,\ell_2,L_2,\Lambda,M)\biggl\{ -\!\!\!\!\!\!\int_{B_{2\tau}} |Dv-(Dv)_\tau|^2\,dy + \tau\biggr\} \\ &\le c(n,N,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M)\bigl[\tau^2+\tau\bigr] \le C(n,N,p,\ell_1,\ell_2,L_1,L_2,\Lambda,M)\tau.\end{aligned}$$

The contradiction follows, by choosing C_* such that $C_* > C$, since, by (5.5),

$$\liminf_h \frac{U_*(x_h,\tau r_h)}{\lambda_h^2} \ge C_*\tau.$$

$\square$

Next, we prove a suitable decay estimate that allows us to prove Theorem 1.2 without the assumption (H). To this aim, we introduce a new “hybrid” excess as

$$U_{**}(x_0,r) := U(x_0,r) + \left(\frac{P(E,B_r(x_0))}{r^{n-1}}\right)^{\frac{\delta}{1+\delta}} + r^\beta, \tag{5.48}$$

where $U(x_0,r)$ is defined in (5.1), δ has been determined in Theorem 4.1 and $0<\beta<\frac{\delta}{1+\delta}$.
In the proof of the following Proposition 5.2, we will only elaborate the steps substantially different from the corresponding ones in the proof of Proposition 5.1.

Proposition 5.2. *Let (u,E) be a local minimizer of $\mathcal{I}$ under the assumptions $(F1)$, $(F2)$, $(G1)$ and $(G2)$. For every $M>0$ and $0<\tau<\frac{1}{4}$, there exist two positive constants $\varepsilon_0=\varepsilon_0(\tau,M)$ and $c_{**}=c_{**}(n,p,\ell_1,\ell_2,L_1,L_2,\Lambda,\delta,M)$ for which, whenever $B_r(x_0)\Subset\Omega$ verifies*

$$|(Du)_{x_0,r}| \le M \qquad \text{and} \qquad U_{**}(x_0,r)\le \epsilon_0,$$

then

$$U_{**}(x_0,\tau r) \le c_{**}\,\tau^\beta\, U_{**}(x_0,r). \tag{5.49}$$

Proof. In order to prove (5.49), we argue by contradiction. Let $M>0$ and $\tau\in(0,1/4)$ be such that for every $h\in\mathbb{N}$, $C_{**}>0$, there exists a ball $B_{r_h}(x_h)\Subset\Omega$ such that

$$|(Du)_{x_h,r_h}| \le M, \qquad U_{**}(x_h,r_h)\to 0 \tag{5.50}$$

and

$$U_{**}(x_h,\tau r_h) \ge C_{**}\tau^\beta U_{**}(x_h,r_h). \tag{5.51}$$

The constant C_{**} will be determined later. We remark that we can confine ourselves to the case $E\cap B_{r_h}(x_h)\neq\emptyset$, the case $B_{r_h}(x_h)\subset\Omega\setminus E$ being easier because $U=U_{**}-r^\beta$.

Step 1. *Blow-up.*

We set $\lambda_h^2 := U_{**}(x_h, r_h)$, $A_h := (Du)_{x_h,r_h}$, $a_h := (u)_{x_h,r_h}$ and we define as before

$$v_h(y) := \frac{u(x_h + r_h y) - a_h - r_h A_h y}{\lambda_h r_h}, \quad \forall y \in B_1.$$

One can easily check that $(Dv_h)_{0,1} = 0$ and $(v_h)_{0,1} = 0$. Again, as before, we set

$$E_h := \frac{E - x_h}{r_h}, \qquad E_h^* := \frac{E - x_h}{r_h} \cap B_1.$$

Let us note that

$$\begin{aligned} \lambda_h^2 = U_{**}(x_h, r_h) &= \fint_{B_1} \left[|Du(x_h + r_h y) - A_h|^2 + |Du(x_h + r_h y) - A_h|^p\right] dy \\ &+ \left(\frac{P(E, B_{r_h}(x_h))}{r_h^{n-1}}\right)^{\frac{\delta}{1+\delta}} + r_h^\beta \\ &= \fint_{B_1} \left[|\lambda_h Dv_h|^2 + |\lambda_h Dv_h|^p\right] dy + P(E_h, B_1)^{\frac{\delta}{1+\delta}} + r_h^\beta. \end{aligned} \tag{5.52}$$

It follows that

$$r_h \to 0, \quad P(E_h, B_1) \to 0, \quad \frac{r_h^\beta}{\lambda_h^2} \le 1, \quad \fint_{B_1} \left[|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p\right] \le 1, \quad \frac{P(E_h, B_1)^{\frac{\delta}{1+\delta}}}{\lambda_h^2} \le 1. \tag{5.53}$$

Therefore, by virtue of (5.50), (5.52) and (5.53), there exist a (not relabeled) subsequence $\{v_h\}_{h\in\mathbb{N}}$, $A \in \mathbb{R}^{n\times N}$ and $v \in W^{1,2}(B_1; \mathbb{R}^N)$, such that

$$\begin{aligned} &v_h \rightharpoonup v \text{ weakly in } W^{1,2}(B_1; \mathbb{R}^N), \quad v_h \to v \text{ strongly in } L^2(B_1; \mathbb{R}^N), \\ &A_h \to A, \qquad \lambda_h Dv_h \to 0 \text{ in } L^2(B_1; \mathbb{R}^{nN}) \text{ and pointwise a.e.,} \end{aligned} \tag{5.54}$$

where we used the fact that $(v_h)_{0,1} = 0$. We also note that

$$\frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2} = r_h^{\frac{\delta}{1+\delta} - \beta} \frac{r_h^\beta}{\lambda_h^2} \to 0, \tag{5.55}$$

since $0 < \beta < \frac{\delta}{1+\delta}$. Moreover, by (5.53), we deduce that

$$\lim_{h\to\infty} \frac{P(E_h, B_1)^{\frac{n}{n-1}\frac{\delta}{1+\delta}}}{\lambda_h^2} = \lim_{h\to\infty} P(E_h, B_1)^{\frac{\delta}{(n-1)(1+\delta)}} \limsup_{h\to\infty} \frac{P(E_h, B_1)^{\frac{\delta}{1+\delta}}}{\lambda_h^2} = 0. \tag{5.56}$$

Therefore, by the relative isoperimetric inequality,

$$\lim_{h\to\infty} \min\left\{\frac{|E_h^*|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}, \frac{|B_1 \setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\right\} \le c(n, \delta) \lim_{h\to\infty} \frac{P(E_h, B_1)^{\frac{n\delta}{(1+\delta)(n-1)}}}{\lambda_h^2} = 0. \tag{5.57}$$

Step 2. *A Caccioppoli type inequality.*

We claim that there exists a constant $c = c(n, p, \ell_1, L_1, L_2, M) > 0$ such that, for every $0 < \rho < 1$, there exists $h_0 \in \mathbb{N}$ such that for all $h > h_0$ we have

$$\begin{aligned}
&\int_{B_{\frac{\rho}{2}}} \left(1 + \lambda_h^{p-2}|Dv_h - (Dv_h)_{\frac{\rho}{2}}|^{p-2}\right)|Dv_h - (Dv_h)_{\frac{\rho}{2}}|^2\,dy \qquad (5.58)\\
&\le c\left[\int_{B_\rho}\left(\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^2}{\rho^2} + \lambda_h^{p-2}\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^p}{\rho^p}\right)dy + \frac{P(E_h, B_1)^{\frac{n\delta}{(n-1)(1+\delta)}}}{\rho^{\frac{n\delta}{1+\delta}}\lambda_h^2}\right].
\end{aligned}$$

We divide the proof into two substeps.

Substep 2.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

The proof of this substep goes exactly as that of Substep 2.a of Proposition 5.1 up to estimate (5.22). Next we observe that

$$\begin{aligned}
&\int_{B_{\frac{\rho}{2}}} \left(1 + \lambda_h^{p-2}|Dv_h - (Dv_h)_{\frac{\rho}{2}}|^{p-2}\right)|Dv_h - (Dv_h)_{\frac{\rho}{2}}|^2\,dy\\
&\le c\left[\int_{B_\rho}\left(\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^2}{\rho^2} + \lambda_h^{p-2}\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^p}{\rho^p}\right)dy + \frac{|E_h^*|}{\lambda_h^2}\right]\\
&\le c\left[\int_{B_\rho}\left(\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^2}{\rho^2} + \lambda_h^{p-2}\frac{|v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y|^p}{\rho^p}\right)dy + \frac{|E_h^*|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\right],
\end{aligned}$$

and this, by the relative isoperimetric inequality, yields the estimate (5.58).

Substep 2.b *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |B_1 \setminus E_h|$.

We fix $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s and $|\nabla\eta| \le \frac{c}{t-s}$. Furthermore, we set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$ and define

$$w_h(y) := v_h(y) - b_h - B_h y, \quad \psi_{1,h} := \eta w_h \quad \text{and} \quad \psi_{2,h} := (1-\eta)w_h,$$

for any $y \in B_1$. By (5.24) and (5.27), we obtain

$$\begin{aligned}
&\widetilde{\ell}\int_{B_s}\left(1 + |\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\
&\le \int_{B_t}\widetilde{H}_h(Dw_h)\,dy - \int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h - \theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy \qquad (5.59)\\
&\le \int_{B_t}\widetilde{H}_h(Dw_h)\,dy + c(p, M)\int_{B_t\setminus B_s}\left(|Dw_h|^2 + \lambda_h^{p-2}|Dw_h|^p + |D\psi_{2,h}|^2 + \lambda_h^{p-2}|D\psi_{2,h}|^p\right)dy.
\end{aligned}$$

In order to estimate the first addend of the right-hand side of the previous inequality we recall that

$$\int_{B_t}\widetilde{H}_h(Dw_h)\,dy \le \int_{B_t}\widetilde{H}_h(Dw_h(y) - D\psi_{1,h})\,dy + \frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h}\left(1 + |A_h + \lambda_h Dv_h|^2\right)^{\frac{p}{2}}dy$$

$$= \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy + \frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h}\big(1+|A_h+\lambda_h Dv_h|^2\big)^{\frac{p}{2}}\,dy. \tag{5.60}$$

We remark that the reverse Hölder inequality stated in Theorem 4.1, through the change of variable $x = x_h + r_h y$, can be rescaled in the following way:

$$\left(\fint_{B_t}|A_h+\lambda_h Dv_h|^{p(1+\delta)}\,dy\right)^{\frac{1}{1+\delta}} \le c(n,p,\ell_1,L_1,L_2)\left[\fint_{B_{2t}}|A_h+\lambda_h Dv_h|^p\,dy+1\right].$$

By Hölder's inequality and inserting the previous inequality in the estimate (5.60), we get

$$\begin{aligned}
&\int_{B_t}\widetilde{H}_h(Dw_h)\,dy\\
&\le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy + c(p)\frac{L_2}{\lambda_h^2}\left(\int_{B_t\setminus E_h}\big(1+|A_h+\lambda_h Dv_h|^{p(1+\delta)}\big)\,dy\right)^{\frac{1}{1+\delta}}|B_t\setminus E_h|^{\frac{\delta}{1+\delta}}\\
&\le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy + c(p)\frac{L_2}{\lambda_h^2}t^{\frac{n}{1+\delta}}\left(\fint_{B_t}\big(1+|A_h+\lambda_h Dv_h|^{p(1+\delta)}\big)\,dy\right)^{\frac{1}{1+\delta}}|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}\\
&\le \int_{B_t\setminus B_s}\widetilde{H}_h\big(D\psi_{2,h}\big)\,dy + c(n,p,\ell_1,L_1,L_2)\frac{t^{\frac{n}{1+\delta}}}{\lambda_h^2}\left(1+\fint_{B_{2t}}|A_h+\lambda_h Dv_h|^p\,dy\right)|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}\\
&\le \int_{B_t\setminus B_s}\widetilde{H}_h\big(D\psi_{2,h}\big)\,dy + \frac{c(n,p,\ell_1,L_1,L_2,M)}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2},
\end{aligned}$$

where we used the fact that $t > \frac{\varrho}{2}$. Hence, inserting the previous estimate in (5.59), we obtain

$$\begin{aligned}
\widetilde{\ell}\int_{B_s}\big(1+|\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}}|Dw_h|^2\,dy &\le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy + \frac{c(n,p,\ell_1,L_1,L_2,M)}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\\
&+ c(p,M)\int_{B_t\setminus B_s}(|Dw_h|+|D\psi_{2,h}|+\lambda_h^{p-2}|Dw_h|^{p-1}+\lambda_h^{p-2}|D\psi_{2,h}|^{p-1})|D\psi_{2,h}|\,dy.
\end{aligned}$$

Thanks to the Lemma 2.3, Young's inequality and the properties of η, we get

$$\begin{aligned}
&\widetilde{\ell}\int_{B_s}\big(1+|\lambda_h Dw_h|^2\big)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\
&\le c(n,p,\ell_1,L_1,L_2,M)\Bigg[\int_{B_t\setminus B_s}\big[|D\psi_{2,h}|^2+\lambda_h^{p-2}|D\psi_{2,h}|^p\big]\,dy + \frac{1}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\\
&+\int_{B_t\setminus B_s}\big[|Dw_h|^2+\lambda_h^{p-2}|Dw_h|^p\big]\,dy\Bigg]\\
&\le c(n,p,\ell_1,L_1,L_2,M)\Bigg[\int_{B_t\setminus B_s}\big(1+\lambda_h^2|Dw_h|^2\big)^{\frac{p-2}{2}}|Dw_h|^2\,dy\\
&+\int_{B_\rho}\left(\frac{|w_h|^2}{(t-s)^2}+\lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right)dy + \frac{1}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\Bigg].
\end{aligned}$$

By using the hole filling technique as in the proof of the previous theorem, we get

$$\int_{B_s} \left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}} |Dw_h|^2\,dy$$
$$\le \frac{c}{c+\tilde{\ell}} \int_{B_t} \left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}} |Dw_h|^2\,dy + \int_{B_\rho} \left(\frac{|w_h|^2}{(t-s)^2} + \lambda_h^{p-2}\frac{|w_h|^p}{(t-s)^p}\right) dy + \frac{1}{\rho^{\frac{n\delta}{1+\delta}}} \frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}$$

By virtue of the iteration Lemma 2.2, previous estimate gives

$$\int_{B_{\frac{\rho}{2}}} \left(1+|\lambda_h Dw_h|^2\right)^{\frac{p-2}{2}} |Dw_h|^2\,dy \le c\left[\int_{B_\rho} \left(\frac{|w_h|^2}{\rho^2} + \lambda_h^{p-2}\frac{|w_h|^p}{\rho^p}\right) dy + \frac{1}{\rho^{\frac{n\delta}{1+\delta}}} \frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\right],$$

where $c = c(n, p, \ell_1, \ell_2, L_1, L_2, M)$. Therefore, by the definition of w_h, we have

$$\int_{B_{\frac{\rho}{2}}} \left(1+\lambda_h^{p-2}|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^{p-2}\right)|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^2\,dy$$
$$\le c\left[\int_{B_\rho} \left(\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y|^2}{\rho^2} + \lambda_h^{p-2}\frac{|v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y|^p}{\rho^p}\right) dy + \frac{1}{\rho^{\frac{n\delta}{1+\delta}}} \frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\right],$$

which, by the relative isoperimetric inequality and since $|B_1\setminus E_h| = \min\{|E_h^*|, |B_1\setminus E_h|\}$, gives the estimate (5.58).

The proofs of Step 3 and Step 4 of Proposition 5.1 hold true also in this case.

Step 5. *Conclusion.*

The change of variable $x = x_h + r_h y$, the Caccioppoli inequality in (5.58) and (5.46), for every $0<\tau<\frac{1}{4}$, give

$$\limsup_{h\to\infty} \frac{U_{**}(x_h,\tau r_h)}{\lambda_h^2} \le \limsup_{h\to\infty} \frac{1}{\lambda_h^2} \fint_{B_{\tau r_h}(x_h)} \left[|Du-(Du)_{x_h,\tau r_h}|^2 + |Du-(Du)_{x_h,\tau r_h}|^p\right] dx$$
$$+ \limsup_{h\to\infty} \frac{1}{\lambda_h^2}\left(\frac{P(E,B_{\tau r_h})(x_h)}{\tau^{n-1} r_h^{n-1}}\right)^{\frac{\delta}{1+\delta}} + \limsup_{h\to\infty} \frac{\tau^\beta r_h^\beta}{\lambda_h^2}$$
$$\le \limsup_{h\to\infty} \fint_{B_\tau} \left[|Dv_h-(Dv_h)_\tau|^2 + \lambda_h^{p-2}|Dv_h-(Dv_h)_\tau|^p\right] dy + \limsup_{h\to\infty} \frac{1}{\lambda_h^2}\left(\frac{P(E_h,B_\tau)}{\tau^{n-1}}\right)^{\frac{\delta}{1+\delta}} + \tau^\beta$$
$$\le c\left[\limsup_{h\to\infty} \fint_{B_{2\tau}} \left(\frac{|v_h-(v_h)_{2\tau}-(Dv_h)_\tau y|^2}{\tau^2} + \lambda_h^{p-2}\frac{|v_h-(v_h)_{2\tau}-(Dv_h)_\tau y|^p}{\tau^p}\right) dy\right.$$
$$\left. + \limsup_{h\to\infty} \left(\frac{P(E_h,B_1)^{\frac{n\delta}{(n-1)(1+\delta)}}}{\lambda_h^2}\frac{1}{\tau^{\frac{n\delta}{1+\delta}}} + \frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\tau^{\frac{\delta}{1+\delta}} + \frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\frac{\lambda_h^{\frac{2\delta}{1+\delta}}}{\tau^{\frac{(n-1)\delta}{1+\delta}}}\right) + \tau^\beta\right]$$
$$\le c\left[\limsup_{h\to\infty} \fint_{B_{2\tau}} \left(\frac{|v_h-(v_h)_{2\tau}-(Dv_h)_\tau y|^2}{\tau^2} + \lambda_h^{p-2}\frac{|v_h-(v_h)_{2\tau}-(Dv_h)_\tau y|^p}{\tau^p}\right) dy\right.$$
$$\left. + \limsup_{h\to\infty} \left(\frac{P(E_h,B_1)^{\frac{n\delta}{(n-1)(1+\delta)}}}{\lambda_h^2}\frac{1}{\tau^{\frac{n\delta}{1+\delta}}} + \frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\tau^{\frac{\delta}{1+\delta}} + \frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\frac{\lambda_h^{\frac{2\delta}{1+\delta}}}{\tau^{\frac{(n-1)\delta}{1+\delta}}}\right) + \tau^\beta\right].$$

where $c = c(n, p, \ell_1, \ell_2, L_1, L_2, \Lambda, \delta, M)$. Proceeding exactly as in Step 5 of the previous Proposition, by virtue of the the strong convergence of $v_h \to v$ in $L^2(B_1)$, since $(Dv_h)_\tau \to (Dv)_\tau$ in $\mathbb{R}^{nN}$, by (5.29), (5.47), (5.53), (5.54), (5.55) (5.56), (5.57) and by the use of Poincaré-Wirtinger inequality, we get

$$\limsup_{h\to\infty} \frac{U_{**}(x_h, \tau r_h)}{\lambda_h^2} \leq C\tau^\beta,$$

where $C = C(n, p, \ell_1, \ell_2, L_1, L_2, \Lambda, \delta, M)$. The contradiction follows by choosing C_{**} such that $C_{**} > C$, since by (5.51)

$$\liminf_{h\to\infty} \frac{U_{**}(x_h, \tau r_h)}{\lambda_h^2} \geq C_{**}\tau^\beta.$$

□

6 Proof of the Main Theorem

Here we give the proof of Theorem 1.2 through a suitable iteration procedure. It is easy to get the following lemmata, arguing exactly in the same way as in [11, Lemma 6.1].

Lemma 6.1. *Let (u, E) be a minimizer of the functional $\mathcal{I}$. For every $M > 0$, $\alpha \in (0, 1)$ and $\vartheta \in (0, \vartheta_0)$, with $\vartheta_0 := \min\left\{c_*^{-\frac{1}{1-\alpha}}, \frac{1}{4}\right\}$, there exist $0 < \varepsilon_1 \leq \left(1 - \vartheta^{\frac{1}{2}}\right)^2 \vartheta^{n-1}$ and $R > 0$ such that, if $r < R$ and $x_0 \in \Omega$ satisfy*

$$B_r(x_0) \Subset \Omega, \quad |Du|_{x_0,r} < M \quad \text{and} \quad U_*(x_0, r) < \varepsilon_1,$$

where c_* is the constant introduced in Proposition 5.1, then

$$U_*(x_0, \vartheta^k r) \leq \vartheta^{k\alpha} U_*(x_0, r), \quad \forall k \in \mathbb{N}.$$

Lemma 6.2. *Let (u, E) be a minimizer of the functional $\mathcal{I}$ and let β be the exponent of Lemma 5.2. For every $M > 0$ and $\vartheta \in (0, \vartheta_0)$, with $\vartheta_0 < \min\left\{c_{**}, \frac{1}{4}\right\}$, there exist $\varepsilon_1 > 0$ and $R > 0$ such that, if $r < R$ and $x_0 \in \Omega$ satisfy*

$$B_r(x_0) \Subset \Omega, \qquad |Du|_{x_0,r} < M \qquad \text{and} \qquad U_{**}(x_0, r) < \varepsilon_1,$$

where c_{**} is the constant introduced in Proposition 5.2, then

$$U_{**}(x_0, \vartheta^k r) \leq \vartheta^{k\beta} U_{**}(x_0, r), \quad \forall k \in \mathbb{N}.$$

6.1 Proof of Theorem 1.2

Proof. We consider the set

$$\Omega_0 := \left\{ x \in \Omega : \limsup_{\rho \to 0} |(Du)_{x,\rho}| < \infty \text{ and } \limsup_{\rho \to 0} U_*(x,\rho) = 0 \right\}$$

and let $x_0 \in \Omega_0$. For every $M > 0$ and for ε_1 determined in Lemma 6.1 there exists a radius $R_{M,\varepsilon_1} > 0$ such that

$$|Du|_{x_0,r} < M \quad \text{and} \quad U_*(x_0,r) < \varepsilon_1,$$

for every $0 < r < R_{M,\varepsilon_1}$. If $0 < \rho < \vartheta r < R$, let $h \in \mathbb{N}$ be such that $\vartheta^{h+1} r < \rho < \vartheta^h r$, where $\vartheta = \frac{\vartheta_0}{2}$ and ϑ_0 is the same constant appearing in Lemma 6.1. By Lemma 6.1, we obtain

$$\begin{aligned}
U_*(x_0,\rho) &\le c(p) \fint_{B_\rho} \left[|Du - (Du)_{\vartheta^h r}|^2 + |Du - (Du)_{\vartheta^h r}|^p \right] dx \\
&\quad + c(p) \left[|(Du)_{\vartheta^h r} - (Du)_{x_0,\rho}|^2 + |(Du)_{\vartheta^h r} - (Du)_{x_0,\rho}|^p \right] + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\le c(p) \fint_{B_\rho} \left[|Du - (Du)_{\vartheta^h r}|^2 + |Du - (Du)_{\vartheta^h r}|^p \right] dx + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\le c(p) \left(\frac{\vartheta^{h+1} r}{\vartheta \rho} \right)^n \fint_{B_{\vartheta^h r}} \left[|Du - (Du)_{\vartheta^h r}|^2 + |Du - (Du)_{\vartheta^h r}|^p \right] dx + \frac{P(E, B_{\vartheta^h r}(x_0))}{(\vartheta^{h+1} r)^{n-1}} + \vartheta^h r \\
&\le \frac{c(p)}{\vartheta^n} \fint_{B_{\vartheta^h r}} \left[|Du - (Du)_{\vartheta^h r}|^2 + |Du - (Du)_{\vartheta^h r}|^p \right] dx + \frac{1}{\vartheta^{n-1}} \frac{P(E, B_{\vartheta^h r}(x_0))}{(\vartheta^h r)^{n-1}} + \vartheta^h r \\
&= \frac{c(n,p)}{\vartheta_0^n} \fint_{B_{\vartheta^h r}} \left[|Du - (Du)_{\vartheta^h r}|^2 + |Du - (Du)_{\vartheta^h r}|^p \right] dx + \frac{2^{n-1}}{\vartheta_0^{n-1}} \frac{P(E, B_{\vartheta^h r}(x_0))}{(\vartheta^h r)^{n-1}} + \vartheta^h r \\
&\le c(n,p,\vartheta_0) U_*(x_0, \vartheta^h r) \le c(n,p,\vartheta_0) c_* \vartheta^{h\alpha} U_*(x_0,r) \le c(n,p,\vartheta_0) c_* \left(\frac{\rho}{r} \right)^\alpha U_*(x_0,r),
\end{aligned}$$

where we used Jensen's inequality. The previous estimate implies that

$$U(x_0,\rho) \le C_* \left(\frac{\rho}{r} \right)^\alpha U_*(x_0,r),$$

where $C_* = C_*(n,p,\theta_0,c_*)$. Since $U_*(y,r)$ is continuous in y, we have that $U_*(y,r) < \varepsilon_1$ for all y in a suitable neighborhood I of x_0. Therefore, for every $y \in I$ we have that

$$U(y,\rho) \le C_* \left(\frac{\rho}{r} \right)^\alpha U_*(y,r).$$

The last inequality implies, by the Campanato characterization of Hölder continuous functions (see [32, Theorem 2.9]), that u is $C^{1,\alpha}$ in I for every $0 < \alpha < \frac{1}{2}$, and we can conclude that the set Ω_0 is open and the function u has Hölder continuous derivatives in Ω_0.

When the assumption (H) is not enforced, the proof goes exactly in the same way provided we use Lemma 6.2 in place of Lemma 6.1, with

$$\Omega_1 := \left\{ x \in \Omega : \limsup_{\rho \to 0} |(Du)_{x_0,\rho}| < \infty \text{ and } \limsup_{\rho \to 0} U_{**}(x_0,\rho) = 0 \right\}.$$

□

Acknowledgments. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). M. Carozza acknowledges support by GNAMPA.

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